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CS-225: Discrete Math for CS
Homework 5 Part 1
Exercise Set 5.2: Problem # 9, 14; Set 5.3: Problem # 10, 18, 23.b

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9.

Prove $4^3+4^4+4^5+....+4^n=\frac{4*(4^n-16)}{3}$ **for all integers n ≥ 3.**

Base Case: Show that P(3) is true.

$P(3)=\frac{4*(4^3-16)}{3}=\frac{4*(64-16)}{3}=\frac{4*(48)}{3}=\frac{192}{3}=64$ which is equal to the LHS, 4^3

Since our base case is true, we assume $P(k)=\frac{4*(4^k-16)}{3}$ is true for all integers k ≥ 3.

Induction: Show P(k+1) is true assuming P(k) is true.

$P(k+1)=4^3+4^4+4^5+....+4^k+4^{(k+1)}=\frac{4*(4^{(k+1)}-16)}{3}$ // induction hypothesis

LHS: This can be regrouped as $P(k+1)=(4^3+4^4+4^5+....+4^k)+4^{(k+1)}$. We know P(k) is true from our assumption, and $P(k)=(4^3+4^4+4^5+....+4^k)$, so $P(k+1)=P(k)+4^{(k+1)}$. Using substitution, $P(k+1)=\frac{4*(4^k-16)}{3}+4^{(k+1)}=\frac{4*(4^k-16)}{3}+\frac{3*4^{(k+1)}}{3}=\frac{(4*(4^k-16)+3*4^{(k+1)})}{3}$

$\frac{(4*4^k-4*16+3*4^{(k+1)})}{3}=\frac{(4^{(k+1)}-64+3*4^{(k+1)})}{3}=\frac{(4*4^{(k+1)}-64)}{3}=\frac{4*(4^{(k+1)}-16)}{3}$ The LHS matches the RHS proving our induction hypothesis is true. Since both the basis step and the inductive step have been proved, P(n) is true for all integers n ≥ 3.

14.

Prove $\sum_{i=1}^{n+1} i*2^i=n*2^{(n+2)}+2$ **for all integers n ≥ 0.**

Base Case: Show P(0) is true.

$P(0)=0*2^{(0+2)}+2=0*2^2+2=0*4+2=2$ Since this is equal to $\sum_{i=1}^{0+1} i*2^i=1*2^1=2$, our base case is true and we assume $P(k)=k*2^{(k+2)}+2$ for all integers k ≥ 0.

Induction: Show P(k+1) is true.

$P(k+1)=(k+1)*2^{((k+1)+2)}+2=k*2^{(k+3)}+2^{(k+3)}+2$ // induction hypothesis

Plugging (k+1) into the initial statement, we get $P(k+1)=\sum_{i=1}^{(k+1)+1} i*2^i=\sum_{i=1}^{(k+2)} i*2^i$. Since we assume P(k) is true, we can rewrite P(k+1) as the summation of the P(k) and (k+1), the next value after k.

So, $P(k+1)=[\sum_{i=1}^{k+1} i*2^i]+(k+2)*2^{(k+2)}$. Since we assume $P(k)=\sum_{i=1}^{k+1} i*2^i$ is equal to

$P(k)=k*2^{(k+2)}+2$, we can substitute.

Then, using algebra, $P(k+1)=k*2^{(k+2)}+2+(k+2)*2^{(k+2)} = k*2^{(k+2)}+2+k*2^{(k+2)}+2*2^{(k+2)}$

$= k*2^{(k+2)}+k*2^{(k+2)}+2*2^{(k+2)}+2 = k*2^{(k+3)}+2^{(k+3)}+2$. This matches our induction hypothesis and proves P(k+1) to be true. Since both the base step and the induction step have been proved, P(n) is true for all integers n ≥ 0.

10.

Prove n^3 – 7n + 3 is divisible by 3 for each integer n ≥ 0.

<u>Base Case</u>: Show P(0) is divisible by 3.

P(0)= 0^3 – 7 (0) + 3 = 3, and 3 is divisible by 3. Because the base case is true, we assume that

P(k) = k^3 – 7k + 3 is divisible by 3 for all integers k ≥ 0. It can also be stated that P(k) = 3r for some integer r by the properties of integer division.

<u>Induction:</u> Show P(k+1) is true.

P(k+1) = $(k+1)^3$ – 7(k+1) + 3, is divisible by 3 // induction hypothesis

= k^3 + $3k^2$ + 3k + 1 – 7k – 7 + 3 //expand using algebra

=(k^3 – 7k + 3) + $3k^2$ + 3k + 1 – 7 //by commutative law

We assume P(k) = (k^3 – 7k + 3) is true, so we can use substitution,

= P(k) + $3k^2$ + 3k – 6 //substitution and combine like terms

Since P(k) is divisible by three, P(k) = 3r for some integer r by the properties of integer division.

= 3r + $3k^2$ + 3k – 6 //using substitution

= 3(r + k^2 + k – 2) //distributive law

Since (r + k^2 + k – 2) is an integer by multiplication and addition of integers, this proves P(k+1) is divisible by 3 (property of integer division).

Since both the basis step and the induction step have been proved, P(n) is true for all integers n ≥ 0.

18.

Prove 5^n + 9 < 6^n, for all integers n ≥ 2.

<u>Base Case:</u> Show P(2) is true.

P(2) = 5^2 + 9 < 6^2 == 25 + 9 < 36 == 34 < 36

The base case is true, so we assume $P(k) = 5^k + 9 < 6^k$ for all integers $k \geq 2$.

<u>Induction:</u> Show P(k+1) is true.

$P(k+1) = 5^{k+1} + 9 < 6^{k+1}$ //induction hypothesis

Starting with P(k), we can arrive at P(k+1).

$P(k) = 5^k + 9 < 6^k$ //assume true

$6(5^k + 9) < 6^k * 6$ //multiply both sides by 6

$(5+1)(5^k+ 9) < 6^k * 6$ //re-write 6 as 5+1

$5*5^k + 45 + 5^k + 9 < 6^{k+1}$ //expand using distributive law and algebra

$(5^{k+1} + 9) + 5^k + 45 < 6^{k+1}$ //proves P(k+1) is true

Because we assume P(k) is true, the truth value of the inequality does not change when both sides are multiplied by 6. After several steps of algebraic manipulation, we arrive at

$(5^{k+1} + 9) + 5^k + 45 < 6^{k+1}$. Since we know this inequality is true, we know that $P(k+1) = 5^{k+1} + 9 < 6^{k+1}$ is true. The left hand side of P(k+1), $5^{k+1} + 9$, is actually less than the left hand side of the derived inequality, $(5^{k+1} + 9) + 5^k + 45$, while the right side on both is equivalent. This proves P(k+1) to be true.

Since both the basis step and the induction step have been proven, P(n) is true for all integers $n \geq 2$.

###Help from Eddie Woo videos on Youtube. This took me FOREVER###

23.b

Prove $n! > n^2$, for all integers $n \geq 4$.

<u>Base Case:</u> Show P(4) is true.

$P(4) = 4! > 4^2$ == $4 * 3 * 2 * 1 > 16$ == $24 > 16$

The base case is true, so we assume $P(k) = k! > k^2$ for all integers $k \geq 4$.

<u>Induction:</u> Show P(k+1) is true.

$P(k+1) = (k+1)! > (k+1)^2$

LHS: We can write (k+1)! as k!(k+1)

RHS: We can write $(k+1)^2$ as (k+1)(k+1)

So $P(k+1) = k!(k+1) > (k+1)(k+1)$

$= k! > k+1$ //divide both sides by (k+1)

We already assume $k! > k^2$, so we need to evaluate (k+1) against k^2. We are restricted to $k \geq 4$, so it is correct to assume that any number multiplied by itself (k^2) is greater than 1 added to itself (k+1). Since $k^2 > (k+1)$, we know by the transitive property that $k! > (k+1)$. This shows P(k+1) is true. Since both the basis step and the induction step have been proved, P(n) is true for all integers $n \geq 4$.