Homework 2

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Problem 1. We repeat Tasks 1–2, so the problem is reproducible. We first consider task 1.

```
# set seed
set.seed(123)

# data
sum_x = 1
n = 30
# prior parameters
a = 0.05; b = 1
# posterior parameters
an = a + sum_x
bn = b + n - sum_x
th = seq(0,1,length.out = 100)
like = dbeta(th, sum_x+1,n-sum_x+1)
prior = dbeta(th,a,b)
post = dbeta(th,sum_x+a,n-sum_x+b)
```

We now consider the loss function.

```
# compute the loss given theta and c
loss_function = function(theta, c){
  if (c < theta){
    return(10*abs(theta - c))
  } else{
    return(l = abs(theta - c))
  }
}
```

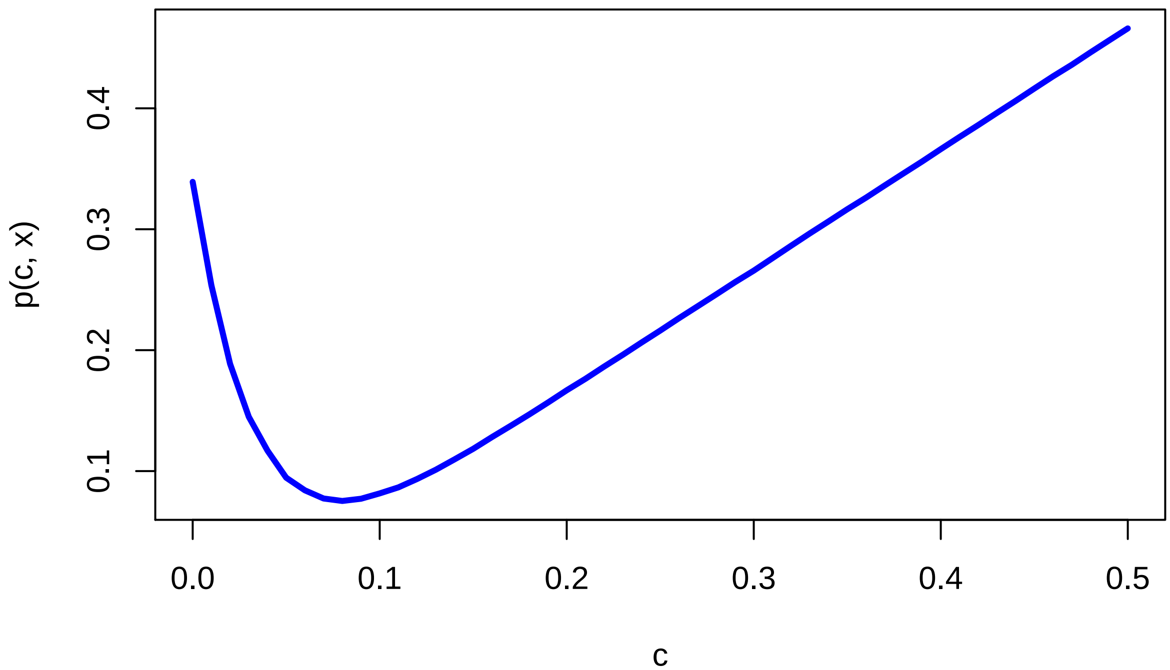
We now write a function **posterior__risk** which is a function of c, parameters a__prior and b__prior for the prior distribution of θ, the summation of x_i sum__x, the number of observations n, and also the number of random draws s.

```
# compute the posterior risk given c
# s is the number of random draws
posterior_risk = function(c, a_prior, b_prior, sum_x, n, s = 30000){
  # randow draws from beta distribution
  a_post = a_prior + sum_x
  b_post = b_prior + n - sum_x
  theta = rbeta(s, a_post, b_post)
  loss <- apply(as.matrix(theta),1,loss_function,c)
  # average values from the loss function
  risk = mean(loss)
}
# a sequence of c in [0, 0.5]
c = seq(0, 0.5, by = 0.01)
post_risk <- apply(as.matrix(c),1,posterior_risk, a, b, sum_x, n)
head(post_risk)
```

```
## [1] 0.33917940 0.25367603 0.18868962 0.14489894 0.11693106 0.09453471
```

We then look at the Posterior expected loss (posterior risk) for disease prevelance versus c.

```r
# plot posterior risk against c
plot(c, post_risk, type = 'l', col='blue',
    lwd = 3, ylab ='p(c, x)' )
```

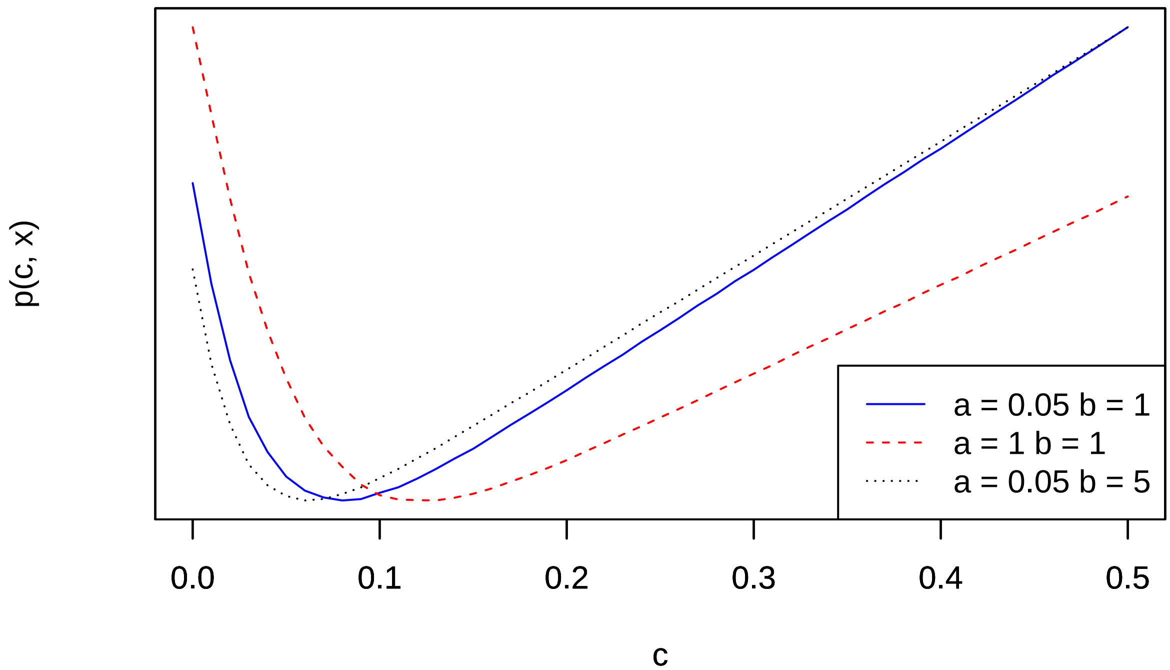



```r
# minimum of posterior risk occurs at c = 0.08
(c[which.min(post_risk)])
```

```
## [1] 0.08
```

We now consider task 2. We set $a = 0.05, 1, 0.05$ and $b = 1, 2, 10$. If we have different prior, the posterior risk is minimized at different c values. The optimal c depends on not only the data, but also the prior setting.

```r
# set prior
as = c(0.05, 1, 0.05); bs = c(1, 1, 10)
post_risk = matrix(NA, 3, length(c))

# for each pair of a and b, compute the posterior risks
for (i in 1:3){
  a_prior = as[i]
  b_prior = bs[i]

  post_risk[i,] = apply(as.matrix(c), 1, posterior_risk, a_prior, b_prior, sum_x, n)
}

plot(c, post_risk[1,], type = 'l', col='blue', lty = 1, yaxt = "n", ylab = "p(c, x)")
par(new = T)
plot(c, post_risk[2,], type = 'l', col='red', lty = 2, yaxt = "n", ylab = "")
par(new = T)
plot(c, post_risk[3,], type = 'l', lty = 3, yaxt = "n", ylab = "")
legend("bottomright", lty = c(1,2,3), col = c("blue", "red", "black"),
       legend = c("a = 0.05 b = 1", "a = 1 b = 1", "a = 0.05 b = 5"))
```



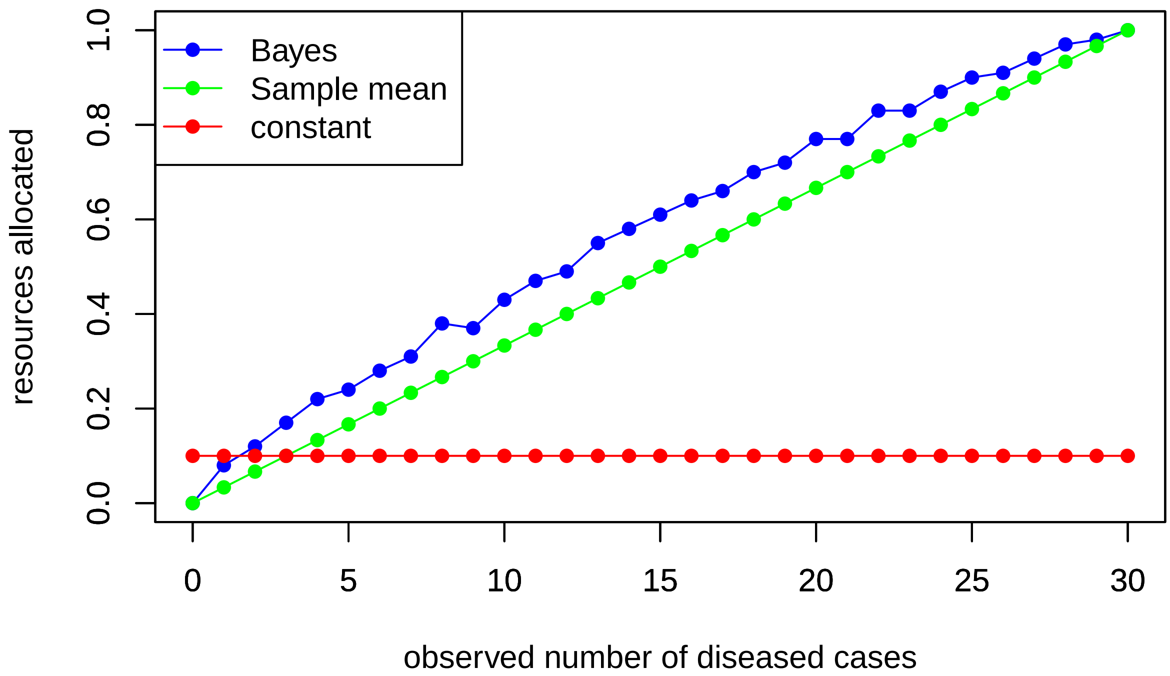
Note there is a more automated solution but this is the most simple one and is completely correct.

Task 3.

The Bayes procedure always picks c to be a little bigger than $\bar{x}$.

```
# prior parameters
#a = 0.05
#b = 1

#n = 30
sum_xs = seq(0, 30)

min_c = matrix(NA, 3, length(sum_xs))

# find_optimal_C finds the optimal c under Bayes procedure
# function of sum of x, parameters for prior, number of observations, and number of random draws
find_optimal_C = function(sum_x, a_prior, b_prior, n, s = 500){
  c = seq(0, 1, by = 0.01)
  post_risk =  apply(as.matrix(c), 1, posterior_risk, a_prior, b_prior, sum_x, n, s)
  c[which.min(post_risk)]
}

min_c[1,] = apply(as.matrix(sum_xs), 1, find_optimal_C, a, b, n)

# find optimal c under sample mean
min_c[2,] = (sum_xs)/n
# constant c
min_c[3,] = 0.1

# plot
plot(sum_xs, min_c[1,], col='blue',type = 'o',pch = 16,
     ylab = "resources allocated", xlab = 'observed number of diseased cases',
     ylim = c(0,1))
```

```r
par(new = T)
plot(sum_xs, min_c[2,], type = 'o', col='green',
     pch = 16, ylab = "", xlab = '', ylim = c(0,1))
par(new = T)
plot(sum_xs, min_c[3,], type = 'o',col = 'red',
     pch = 16, ylab = "", xlab = '', ylim = c(0,1))
legend("topleft", lty = c(1,1,1), pch = c(16,16,16),
       col = c("blue", "green", "red"),
       legend = c("Bayes", "Sample mean", "constant"))
```

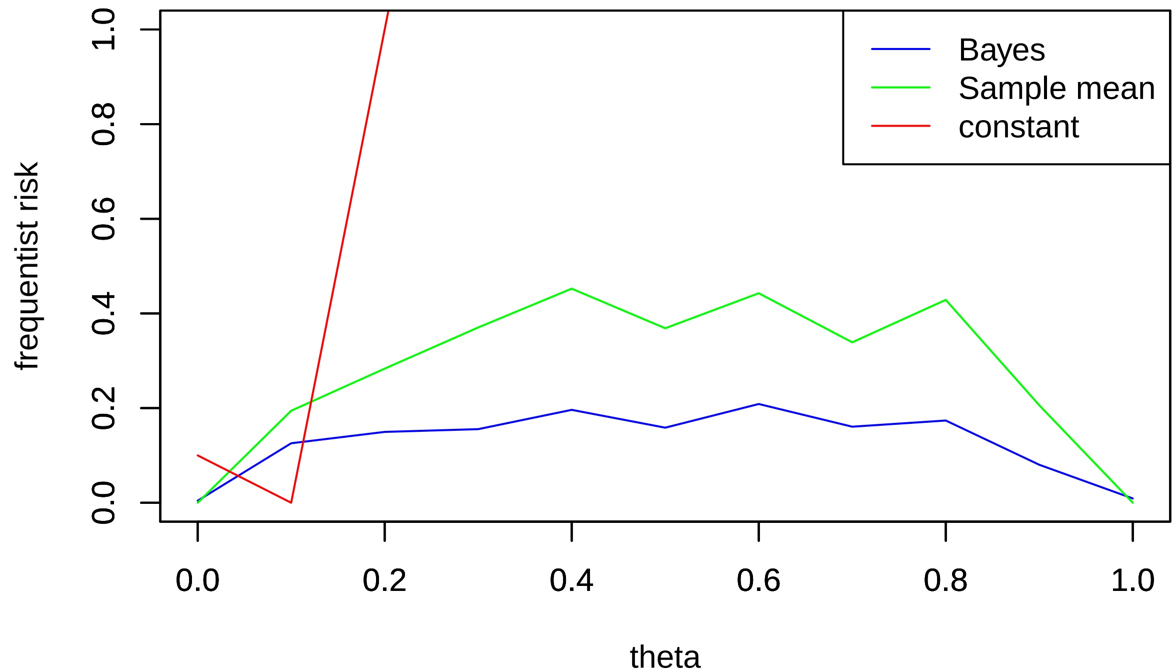


Task 4.

For all θ, the Bayes procedure has the lower frequentist risk than the sample mean.

```r
thetas = seq(0, 1, 0.1)

# frequentist risk for the 3 estimators given a theta
frequentist_risk = function(theta){
  sum_xs = rbinom(100, 30, theta)
  Bayes_optimal = apply(as.matrix(sum_xs), 1, find_optimal_C, a, b, n, s = 100)
  mean_c = sum_xs / 30

  loss1 = apply(as.matrix(Bayes_optimal), 1, loss_function, theta = theta)
  loss2 = apply(as.matrix(mean_c), 1, loss_function, theta = theta )
  risk1 = mean(loss1)
  risk2 = mean(loss2)
  risk3 = loss_function(theta, 0.1)
  return(c(risk1, risk2, risk3))
}

# given a sequance a theta, compute frequentist risk for each theta
R = apply(as.matrix(thetas), 1, frequentist_risk)
```

```r
# plot
plot(thetas, R[1,], col='blue', type = "l",
     ylab = "frequentist risk", xlab = 'theta',ylim = c(0,1))
par(new = T)
plot(thetas, R[2,], type = 'l', col='green',
     ylab = "", xlab = '', ylim = c(0,1))
par(new = T)
plot(thetas, R[3,], type = 'l',col = 'red',
     ylab = "", xlab = '', ylim = c(0,1))
legend("topright", lty = c(1,1,1), col = c("blue", "green", "red"),
       legend = c("Bayes", "Sample mean", "constant"))
```



Task 5.

The Bayes procedure and the constant are both admissble. The mean is not admissble because the Bayesian procedure, for example, is strictly better than it.

Suppose we want to compute the integrated risk for the Bayes rule. We first want to compute the frequentist risk for the Bayes rule.

```r
thetas = rbeta(100, a, b)

# given a sequance a theta, compute frequentist risk for each theta
R = apply(as.matrix(thetas), 1, frequentist_risk)

# compute integrated risk
IR = apply(R, 1, mean)
```

Problem 2.

$$
\begin{aligned}
p(\theta|x) &\propto \frac{1}{\theta} I_{(0,\theta)}(x) \frac{\alpha\beta^\alpha}{\theta^{\alpha+1}} I_{(\beta,\infty)}(\theta) && (1)\\
&\propto \frac{1}{\theta} I_{(x,\infty)}(\theta) \frac{1}{\theta^{\alpha+1}} I_{(\beta,\infty)}(\theta) && (2)\\
&= \frac{1}{\theta^{\alpha+2}} I_{(\max\{x,\beta\},\infty)}(\theta) && (3)
\end{aligned}
$$

This implies that $\theta|x \sim \text{Pareto}(\alpha+1, \max\{x,\beta\})$.

Problem 3.

When $L(\theta,\delta) = c(\theta-\delta)^2$, the Bayes estimator is simply the posterior mean.

When $L(\theta,\delta) = w(\theta)(g(\theta)-\delta(x))^2$, we want to minimize

$$\rho = E[w(\theta)\{g(\theta)-\delta(x)\}^2 \mid X] = E[w(\theta)\{g(\theta)^2+\delta(x)^2-2\delta(x)g(\theta)\} \mid X].$$

Then

$$\frac{\partial\rho}{\partial\delta(x)} = 2E[w(\theta)\delta(x) \mid X] - 2E[w(\theta)g(\theta) \mid X] = 0 \implies \delta(x) = \frac{E[w(\theta)g(\theta) \mid X]}{E[w(\theta) \mid X]}$$

Then verify that this is indeed a minimum.

Problem 4.

This is what we will call a **no data decision problem** since we do not observe $X|\theta$. Thus, simply find the $\delta = a_i$ such that $E_\theta[L(\theta,a_i)]$ is minimized.

Recall that the posterior risk $\rho(\pi,a_i) = \sum_i L(\theta,a_i)\pi(\theta_i)$. Then

$$
\begin{aligned}
&\rho(\pi,a_1) = 4/5,\\
&\rho(\pi,a_2) = 4/5\times 3 + 1/5\times 6 = 18/5,\\
&\rho(\pi,a_3) = 4/5\times 1 = 4/5,\\
&\rho(\pi,a_4) = 4/5\times 3 = 12/5,\\
&\rho(\pi,a_5) = 4/5\times 4 + 1/5\times 1 = 17/5.
\end{aligned}
$$

Thus, a_1 and a_3 are Bayes actions.