Benji_HW1

January 30, 2018

1a. Derive the formulas for $\hat{\beta_0}$ and $\hat{\beta_1}$ in a simple linear regression model.

$$Q = \sum_{i=1}^{n} \epsilon_i^2$$

$$\epsilon_i = Y_i - \beta_0 - \beta_1 X_i$$

First we will solve for $\hat{\beta_0}$:

$$\frac{\partial Q}{\partial \beta_0} = 0$$

$$\frac{\partial Q}{\partial \beta_0} = \sum_{i=1}^{n} [\frac{\partial}{\partial \beta_0} \epsilon_i^2] = \sum_{i=1}^{n} [2\epsilon_i \frac{\partial \epsilon_i}{\partial \beta_0}] = 2 \sum_{i=1}^{n} \epsilon_i(-1) = 0 \sum_{i=1}^{n} \epsilon_i = 0$$

$$\sum_{i=1}^{n} (Y_i - B_0 - B_1 X_i) = 0$$

$$\sum_{i=1}^{n} Y_i - n\beta_0 - \beta_1 \sum_{i=1}^{n} X_i = 0$$

$$n\beta_0 = \sum_{i=1}^{n} Y_i - \beta_1 \sum_{i=1}^{n} X_i$$

$$\beta_0 = \frac{1}{n} (\sum_{i=1}^{n} Y_i - \beta_1 \sum_{i=1}^{n} X_i)$$

$$\beta_0 = \overline{Y} - \beta_1 \overline{X}$$

Now we solve for $\hat{\beta_1}$:

$$\frac{\partial Q}{\partial \beta_1} = 0$$

$$\frac{\partial Q}{\partial \beta_1} = \sum_{i=1}^{n} [\frac{\partial}{\partial \beta_1} \epsilon_i^2] = \sum_{i=1}^{n} [2\epsilon_i \frac{\partial \epsilon_i}{\partial \beta_1}] = 2 \sum_{i=1}^{n} \epsilon_i(-X_i) = 0$$

$$\sum_{i=1}^{n} \epsilon_i = 0$$

$$\sum_{i=1}^{n} \epsilon_i X_i = 0 \sum_{i=1}^{n} (Y_i X_i - B_0 X_i - B_1 X_i^2) = 0$$

Plugging in β_0 we get:

$$\sum_{i=1}^{n} Y_i X_i - \frac{1}{n}(\sum_{i=1}^{n} Y_i - \beta_1 \sum_{i=1}^{n} X_i)(\sum_{i=1}^{n} X_i) - \beta_1 \sum_{i=1}^{n} X_i^2 = 0$$

$$\sum_{i=1}^{n} Y_i X_i - \frac{1}{n}(\sum_{i=1}^{n} X_i)(\sum_{i=1}^{n} X_i) + \frac{1}{n}\beta_1(\sum_{i=1}^{n} X_i)(\sum_{i=1}^{n} X_i) - \beta_1 \sum_{i=1}^{n} X_i^2 = 0$$

$$\beta_1 = \frac{\sum_{i=1}^{n} Y_i X_i - \frac{1}{n}(\sum_{i=1}^{n} X_i)(\sum_{i=1}^{n} Y_i)}{\sum_{i=1}^{n} X_i X_i - \frac{1}{n}(\sum_{i=1}^{n} X_i)(\sum_{i=1}^{n} X_i)}$$

$$\beta_1 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})^2}$$

Thus:

$$\hat{\beta_0} = \overline{Y} - \beta_1 \overline{X}$$

$$\hat{\beta_1} = \frac{XY}{XX}$$

1b. Derive the matrix forms of $\hat{\beta_0}$ and $\hat{\beta_1}$

Let X =

$$\begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix}$$

Let Y =

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix}$$

$X^T X =$

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} n & \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix}$$

$X^TY =$

$$\begin{bmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i y_i \end{bmatrix}$$

We know Y = X β + ϵ in matrix form. This implies that $\hat{\beta} = (X^TX)^{-1}X^TY$

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} n & \sum_{i=1}^n x_i \\ \sum_{i=1}^n x_i & \sum_{i=1}^n x_i^2 \end{bmatrix}^{-1} \begin{bmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_i y_i \end{bmatrix}$$

2. Show that:

$$\sum_{i=1}^n (\hat{Y}_i - \overline{Y})(Y_i - \hat{Y}_i) = 0$$

Let $(Y_i - \hat{Y}_i) = \epsilon_i$ and $\hat{Y}_i = \hat{\beta_0} + \hat{\beta_1} X_i$
Then:

$$\sum_{i=1}^n (\hat{Y}_i - \overline{Y})(Y_i - \hat{Y}_i) = \sum_{i=1}^n (\hat{Y}_i - \overline{Y})\epsilon_i$$

$$\sum_{i=1}^n (\hat{Y}_i - \overline{Y})(Y_i - \hat{Y}_i) = \sum_{i=1}^n (\hat{\beta_0} + \hat{\beta_1}\overline{X_i} - \overline{Y})\epsilon_i$$

$$\sum_{i=1}^n (\hat{Y}_i - \overline{Y})(Y_i - \hat{Y}_i) = \hat{\beta_0}\sum_{i=1}^n \epsilon_i + \hat{\beta_1}\sum_{i=1}^n \overline{X_i}\epsilon_i - \overline{Y}\sum_{i=1}^n \epsilon_i$$

Since

$$\sum_{i=1}^n \epsilon_i = 0$$

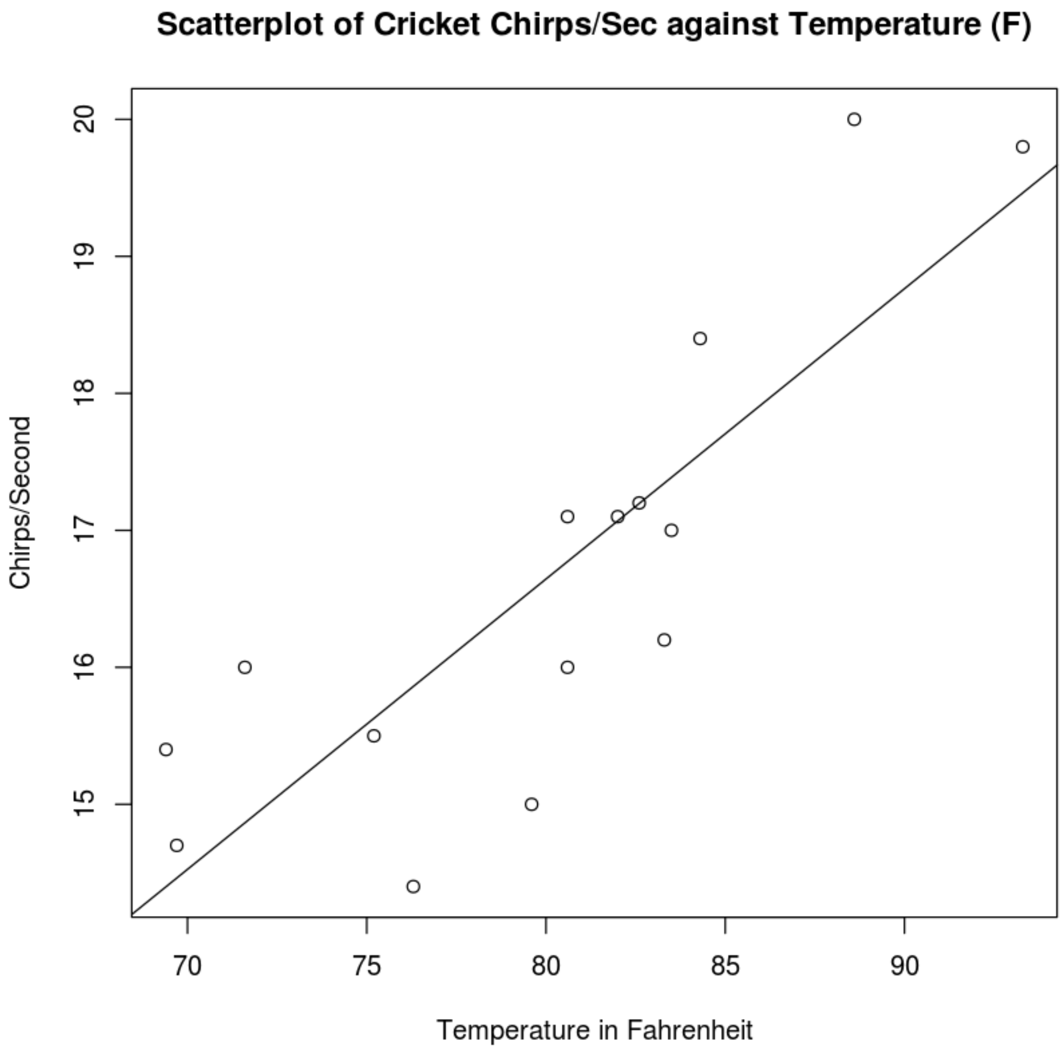
$$\sum_{i=1}^n (\hat{Y}_i - \overline{Y})(Y_i - \hat{Y}_i) = 0$$

3. Consider crickets chirp frequency data presented in cricket data.csv posted on Sakai. There are n=15 bivariate measurements on striped ground crickets, where Y = chirps per seconds and X= temperture in Fahrenheit:

```
In [1]: crickets <- read.csv(file = 'cricket_data.csv', header = TRUE, sep = ',')
```

a. Obtain a scatterplot of these measurements.

```
In [2]: plot(x = crickets$x, y = crickets$y,
             main = "Scatterplot of Cricket Chirps/Sec against Temperature (F)",
             xlab = "Temperature in Fahrenheit", ylab = "Chirps/Second")
        fit <- lm(crickets$y ~ crickets$x)
        abline(fit)
```



b. Specify the simple linear regression model for these data. Identify all parameters in the model, providing interpretation of each.

```
In [3]: summary(lm(crickets$y ~ crickets$x))


Call:
lm(formula = crickets$y ~ crickets$x)

Residuals:
     Min       1Q   Median       3Q      Max
-1.56009 -0.57930  0.03129  0.59020  1.53259
```

```
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.30914    3.10858  -0.099 0.922299
crickets$x   0.21193    0.03871   5.475 0.000107 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.9715 on 13 degrees of freedom
Multiple R-squared:  0.6975,Adjusted R-squared:  0.6742
F-statistic: 29.97 on 1 and 13 DF,  p-value: 0.0001067
```

Looking at the summary, $\hat{\beta_0} = -0.30914$ and $\hat{\beta_1} = 0.21193$. This can be interpreted as for every 1-degree Fahrenheit increase in temperature, assuming everything else is held constant, the chirps per second will increase by 0.21193. Additionally, the $\hat{\beta_0}$ intercept is just a centering constant since it is not interpretable within the context of our data.

c. Explain how the interpretation (and the estimate) of the slope parameter changes if temperature is expressed in Celsius.

If temperature were to be expressed in Celsius, the interpretation would not change, but the slope parameter would change to accommodate the change to Celsius.

d. Estimate the mean chirp frequency among crickets in a temperature of 80 degrees F. Estimate the standard deviation among chirp frequency measurements made at this fixed temperature.

Using:

$$\hat{Y_i} = \hat{\beta_0} + \hat{\beta_1} X_i \hat{\sigma^2} = S^2_{y.x} = \frac{1}{n-2}\sum_{i=1}^{n}(Y_i - \hat{Y_i})^2$$

```
In [35]: (beta_0 <- lm(crickets$y ~ crickets$x)$coefficients[1])
         (beta_1 <- lm(crickets$y ~ crickets$x)$coefficients[2])
         s_yx <- sqrt(sum((crickets$y - predict(fit))^2) / (length(crickets$y) - 2))
```

(Intercept): -0.309144391026669
crickets\$x: 0.211925009049975

```
In [36]: cat("Estimated mean chirp frequency: ", beta_0 + beta_1*80, "Chirps/Second")
         cat("\nEstimated standard deviation among chirp frequencies: ", s_yx)
```

```
Estimated mean chirp frequency:  16.64486 Chirps/Second
Estimated standard deviation among chirp frequencies:  0.9715177
```

e. Estimate the mean chirp frequency among crickets in a temperature of 105 degrees F.

```
In [6]: cat("Estimated mean chirp frequency: ", beta_0 + beta_1*105, "Chirps/Second")
```

```
Estimated mean chirp frequency:  21.94298 Chirps/Second
```

If we extrapolated the model, the estimated mean chirp frequency would be ~22 chirps/second. However, since our measured temperatures do not extend to 105 degrees Fahrenheit, we cannot accurately estimate the mean chirp frequency at 105 degrees Fahrenheit.

f. Report the sum of squares deviations between the fitted values and the average chirp frequency $\overline{Y}$

Using:

$$SSR = \sum_{i=1}^{n}(\hat{Y}_i - \overline{Y})^2$$

```
In [7]: sum((predict(fit) - mean(crickets$y))^2)
```

28.2873263579725

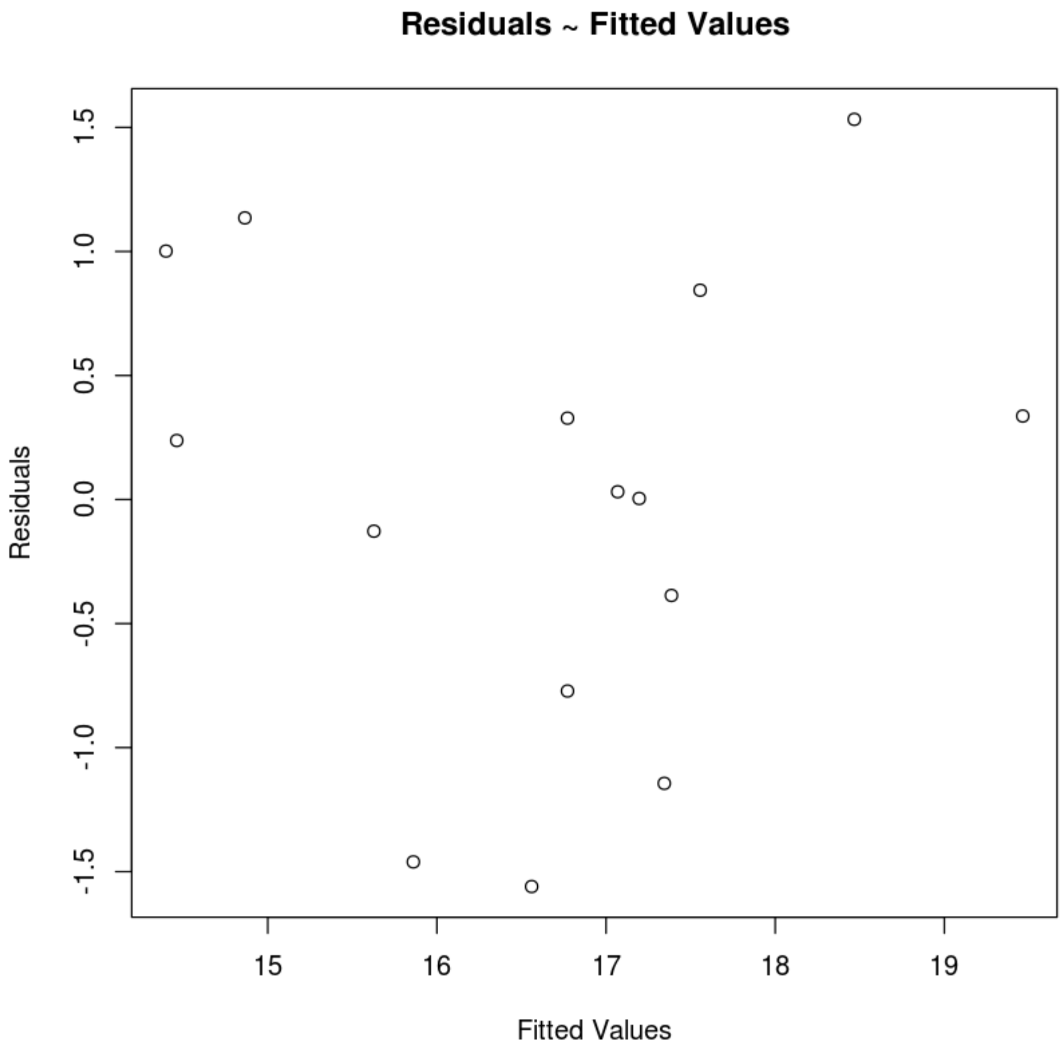
g. What proportion of variance in chirp frequencies is explained by the linear regression model?

```
In [8]: cat("The proportion of variance in chirp frequencies explained by the linear regression
        cor(crickets$x, crickets$y)^2)
```

```
The proportion of variance in chirp frequencies explained by the linear regression model is:
 0.6974651
```

h. Obtain a plot of the residuals against the fitted values

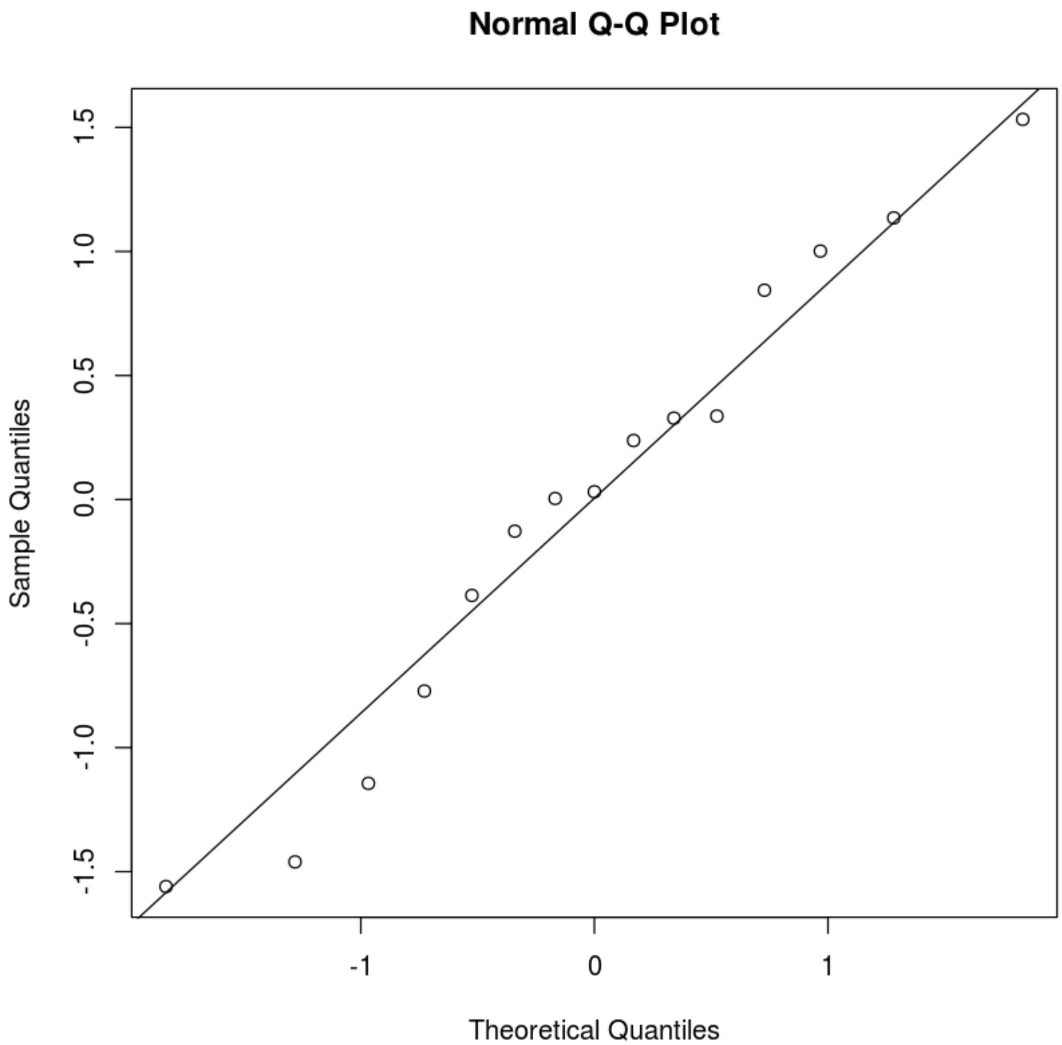
```
In [9]: plot(crickets$y - predict(fit) ~ predict(fit),
             xlab = "Fitted Values",
             ylab = "Residuals",
             main = "Residuals ~ Fitted Values")
```



i. Obtain a plot of the ordered residuals against the corresponding quantiles from the standard normal distribution.

```
In [10]: ordered_residuals <-(crickets$y - predict(fit))[order(crickets$y - predict(fit))]
```

```
In [11]: qqnorm(ordered_residuals)
         qqline(ordered_residuals)
```

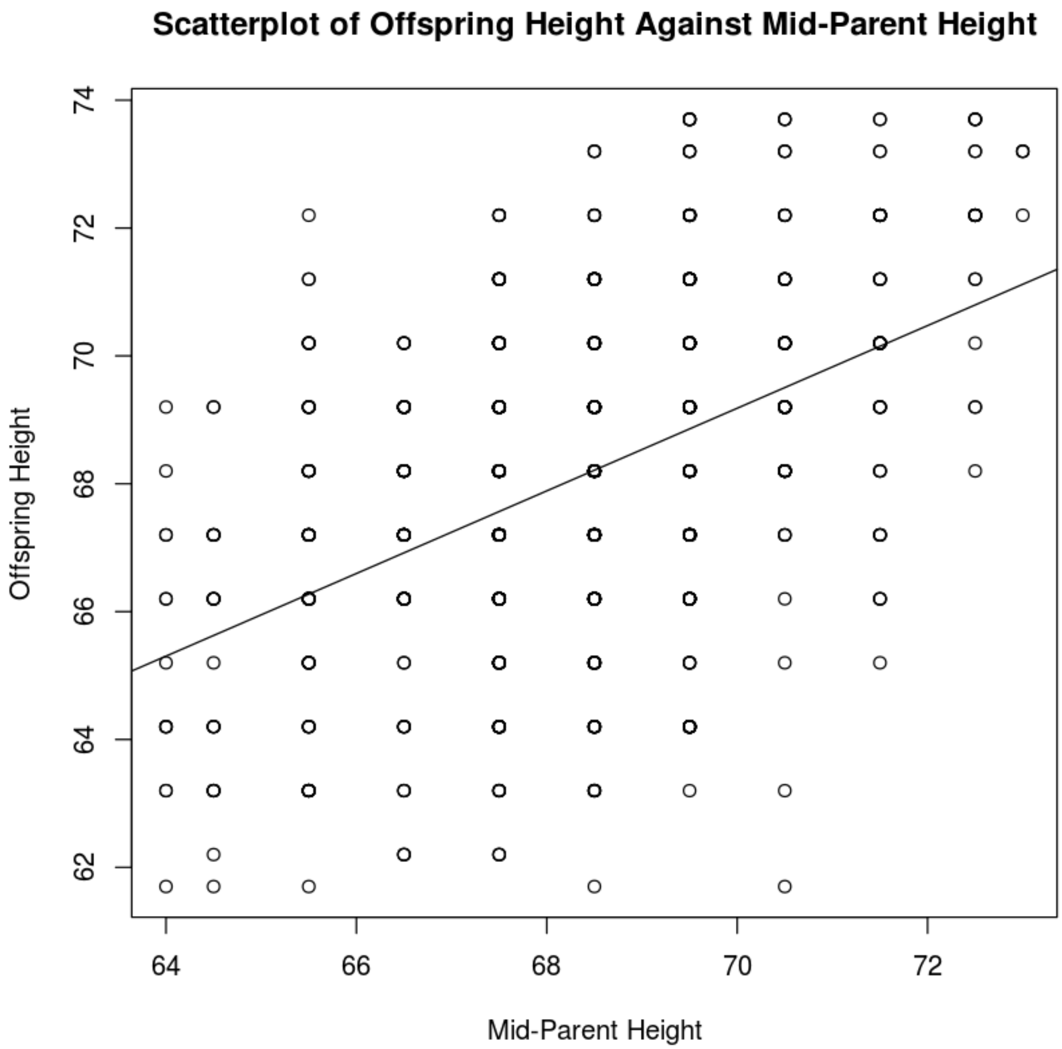


4. Stigler, History of Statistics pg. 285 gives Galton's famous data on heights of sons (Y in inches) and average parents height (X in inches) scaled to represent a male height (essentially sons' heights versus fathers' heights). Data are given in parents offsprings.csv on Sakai. Consider a statistical model for these data, randomly sampled from some population of interest. In particular, choose a model which accounts for the apparent linear dependence of the mean height of sons on midparent height X.

```
In [12]: heights <- read.csv(file = 'parents_offsprings.csv', header = TRUE)
```

A. Obtain a scatterplot of these data.

```
In [13]: plot(x = heights$midparent_height, y = heights$offspring_height,
              xlab = "Mid-Parent Height", ylab = "Offspring Height",
              main = "Scatterplot of Offspring Height Against Mid-Parent Height")
         abline(lm(heights$offspring_height ~ heights$midparent_height))
```




```
In [14]: beta_0 <- unname(lm(heights$offspring_height ~ heights$midparent_height)$coefficients[1
         beta_1 <- unname(lm(heights$offspring_height ~ heights$midparent_height)$coefficients[2

In [15]: summary(lm(heights$offspring_height ~ heights$midparent_height))
```

```
Call:
lm(formula = heights$offspring_height ~ heights$midparent_height)

Residuals:
    Min      1Q  Median      3Q     Max
-7.8050 -1.3661  0.0487  1.6339  5.9264
```

```
Coefficients:
                         Estimate Std. Error t value Pr(>|t|)
(Intercept)              23.94153    2.81088   8.517   <2e-16 ***
heights$midparent_height  0.64629    0.04114  15.711   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 2.239 on 926 degrees of freedom
Multiple R-squared:  0.2105,Adjusted R-squared:  0.2096
F-statistic: 246.8 on 1 and 926 DF,  p-value: < 2.2e-16
```

B. What is the meaning, in words, of β_1?

Looking at the summary, $\hat{\beta_1} = 0.64629$. This can be interpreted as when you hold everything else constant, for every 1-inch increase of midparent height, the estimated offspring height increases by 0.64629 inches.

C. What is the observed value of $\hat{\beta_1}$?

```
In [16]: cat("The observed value of beta_1 is: ", beta_1)
```

```
The observed value of beta_1 is:  0.6462906
```

D. How much does $\hat{\beta_1}$ vary about β_1 from sample to sample? (Provide an estimate of the standard error, as well as an expression indicating how it was computed)

Using:

$$\widehat{\text{var}(\hat{\beta_1})} = \frac{S^2_{y.x}}{\sum_{i=1}^{n}(X_i - \overline{X})^2}$$

and

$$\sigma^2 = S^2_{y.x} = \frac{1}{n-2}\sum_{i=1}^{n}(Y_i - \hat{Y_i})^2$$

```
In [17]: residuals <- lm(heights$offspring_height~heights$midparent_height)$residuals
         var_b1 <- (sum(residuals^2)/(length(heights$midparent_height) - 2))/
                     sum((heights$midparent_height - mean(heights$midparent_height))^2)
         cat("The estimate of the standard error is: ", var_b1^0.5)
```

```
The estimate of the standard error is:  0.04113588
```

E. What is a region of plausible values for β_1 suggested by the data?

Using a 95% confidence interval for β_1, we can use the following formula:

$$\hat{\beta_1} \pm t_{0.975,n-2} SE(be\hat{t}a_1)$$

```
In [18]: cat("The region of plausible values for beta_1 suggested by the data is : (",
             beta_1 - qt(p = 0.975, df = length(heights$midparent_height) - 2) * var_b1^0.5,
             ", ",
             beta_1 + qt(p = 0.975, df = length(heights$midparent_height) - 2) * var_b1^0.5,
             ")"
             )
```

```
The region of plausible values for beta_1 suggested by the data is : ( 0.5655602 ,  0.7270209 )
```

F. What is the line that best fits these data, using criterion that smallest sum of squared residuals is "best?"

The line that fits best is the one we used above in the linear model: $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i = 23.94153 + 0.64629 X_i$

G. How much of the observed variation in the heights of sons (the y-axis) is explained by this "best" line?

We can measure the observed variation in the heights of sons explained by the best-fit line using the R^2 value:

$$R^2 = 1 - \frac{SSE}{SST}$$

```
In [19]: SSR <- sum((predict(lm(heights$offspring_height ~ heights$midparent_height)) -
                     mean(heights$offspring_height))^2)
         SSE <- sum((heights$offspring_height -
                     predict(lm(heights$offspring_height ~ heights$midparent_height)))^2)
         SST <- sum((heights$offspring_height - mean(heights$offspring_height))^2)
```

```
In [20]: cat("Check for equivalency:\nSSR + SST = ", SSR + SSE, "\nSST = ", SST)
```

```
Check for equivalency:
SSR + SST =  5877.207
SST =  5877.207
```

R^2 value is:

```
In [21]: 1 - SSE/SST
```

0.210462910561639

H. What is the estimated average height of sons whose midparent height is x = 68?

```
In [22]: cat("The estimated average height of sons whose midparent height is 68 is: ",
             beta_0 + beta_1*68)
```

```
The estimated average height of sons whose midparent height is 68 is:  67.88929
```

I. Is this the true average height in the whole population of sons whose midparent height is x = 68?

No, the *true* average height in the whole population of sons whose midparent height is 68 inches is what we are trying to approximate with our model, but the true average is unknown.