BIOS705 HW01

Q1

(b) (Slide 48) $\hat{\beta} = (X^TX)^{-1}(X^TY)$

(1)

$$X = \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$X^TX = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} = \begin{bmatrix} n & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix}$$

$$(X^TX)^{-1} = \frac{1}{n\Sigma x_i^2 - (\Sigma x_i)(\Sigma x_i)} \begin{bmatrix} \Sigma x_i^2 & -\Sigma x_i \\ -\Sigma x_i & n \end{bmatrix}$$

$$(X^TY) = \begin{bmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \end{bmatrix}$$

$$(X^TX)^{-1}(X^TY) = \frac{1}{n\Sigma x_i^2 - (\Sigma x_i)(\Sigma x_i)} \begin{bmatrix} \Sigma x_i^2 \Sigma y_i - \Sigma x_i \Sigma x_i y_i \\ -\Sigma x_i \Sigma y_i + n \Sigma x_i y_i \end{bmatrix}$$

$$\Rightarrow \begin{cases} \hat{\beta}_0 = \dfrac{\Sigma x_i^2 \Sigma y_i - \Sigma x_i \Sigma x_i y_i}{n\Sigma x_i^2 - \Sigma x_i \Sigma x_i} \\ \hat{\beta}_1 = \dfrac{n\Sigma x_i y_i - \Sigma x_i \Sigma y_i}{n\Sigma x_i^2 - \Sigma x_i \Sigma x_i} = \dfrac{\Sigma x_i y_i - \frac{1}{n}\Sigma x_i \Sigma y_i}{\Sigma x_i x_i - \frac{1}{n}\Sigma x_i \Sigma x_i} \end{cases} \#$$

(2) The $\hat{\beta}_0$ can be derived from $X^TX\hat{\beta} = X^TY$

$$\Rightarrow \begin{bmatrix} n & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{bmatrix} \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = \begin{bmatrix} \Sigma y_i \\ \Sigma x_i y_i \end{bmatrix}$$

$$\Rightarrow \begin{cases} n\hat{\beta}_0 + \hat{\beta}_1 \Sigma x_i = \Sigma y_i \longrightarrow \frac{1}{n}\Sigma y_i = \hat{\beta}_1 \frac{1}{n}\Sigma x_i + \hat{\beta}_0 \\ \hat{\beta}_0 \Sigma x_i + \hat{\beta}_1 \Sigma x_i^2 = \Sigma x_i y_i \end{cases}$$

$$\Rightarrow \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} \quad \#$$