BIOS 705 HW1 Kuei-Yueh Ko

Q1

(a) (Slide 11) $\hat{\beta}_0$ $\hat{\beta}_1$ Derivation

(1)
$$Q = \sum_i \varepsilon_i^2$$
$$\varepsilon_i = Y_i - \beta_0 - \beta_1 X_i$$
$$\frac{\partial Q}{\partial \beta_0} = \sum_i \left[\frac{\partial}{\partial \beta_0}(\varepsilon_i^2)\right] = \sum_i 2\varepsilon_i \cdot \frac{\partial \varepsilon_i}{\partial \beta_0} = 2\sum_i \varepsilon_i(-1) = -2\sum_i \varepsilon_i$$
$$\frac{\partial Q}{\partial \beta_1} = \sum_i \left[\frac{\partial}{\partial \beta_1}(\varepsilon_i^2)\right] = \sum_i 2\varepsilon_i \cdot \frac{\partial \varepsilon_i}{\partial \beta_1} = 2\sum_i \varepsilon_i(-X_i) = -2\sum_i \varepsilon_i X_i$$

(2) let $\frac{\partial Q}{\partial \beta_0} = 0$, $\frac{\partial Q}{\partial \beta_1} = 0$
$$\Rightarrow \begin{cases} -2\sum_i \varepsilon_i = 0 \\ -2\sum_i \varepsilon_i X_i = 0 \end{cases} \Rightarrow \begin{cases} \sum_i \varepsilon_i = 0 \\ \sum_i \varepsilon_i X_i = 0 \end{cases}$$
$$\Rightarrow \begin{cases} \sum_i (Y_i - \beta_0 - \beta_1 X_i) = 0 \\ \sum_i (X_i Y_i - \beta_0 X_i - \beta_1 X_i^2) = 0 \end{cases} \Rightarrow \begin{cases} (\sum_i Y_i) - n\beta_0 - \beta_1(\sum_i X_i) = 0 & \cdots ① \\ (\sum_i X_i Y_i) - \beta_0(\sum_i X_i) - \beta_1(\sum X_i^2) & \cdots ② \end{cases}$$

(3) ① $\Rightarrow \beta_0 = \frac{1}{n}(\sum Y_i - \beta_1 \sum X_i)$, substitute β_0 in ②

② $\Rightarrow (\sum_i X_i Y_i) - \frac{1}{n}(\sum Y_i - \beta_1 \sum X_i)(\sum_i X_i) - \beta_1(\sum X_i^2) = 0$
$$\Rightarrow (\sum_i X_i Y_i) - \frac{1}{n}(\sum X_i)(\sum Y_i) + \frac{1}{n}\beta_1(\sum X_i)(\sum X_i) - \beta_1(\sum X_i^2) = 0$$
$$\Rightarrow \beta_1 = \frac{\sum X_i Y_i - \frac{1}{n}(\sum X_i)(\sum Y_i)}{\sum X_i X_i - \frac{1}{n}(\sum X_i)(\sum X_i)} = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})(X_i - \bar{X})}$$

where $\bar{X} = \frac{1}{n}\sum X_i$, $\bar{Y} = \frac{1}{n}\sum Y_i$

(4)
$$\hat{\beta}_0 = \frac{1}{n}(\sum Y_i - \beta_1 \sum X_i) = \bar{Y} - \beta_1 \bar{X}$$
$$\hat{\beta}_1 = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sum (X_i - \bar{X})(X_i - \bar{X})}$$

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