Planar Configuration Spaces

of Disk Arrangements and Hinged Polygons

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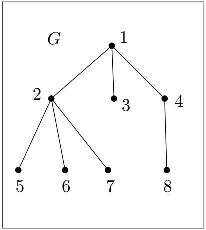
Motivation: Deciding Realizability of Polygonal Linkages

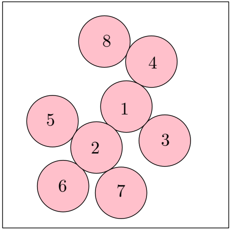
Problem

Decide whether a polygonal linkage whose hinge graph is a tree can be realized?

* A polygonal linkage is an ordered pair $(\mathcal{P}, \mathcal{H})$ where $\mathcal{P}$ is a finite set of polygons and $\mathcal{H}$ is a finite set of hinges.
* A *hinge* $h \in \mathcal{H}$ corresponds to two or more points on the boundary of distinct polygons in $\mathcal{P}$.



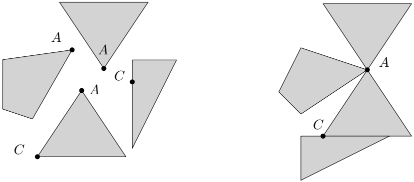
(a)

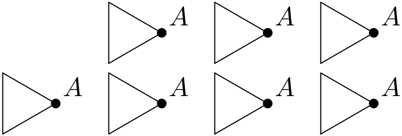


(b)

Motivation: Polygonal Linkages

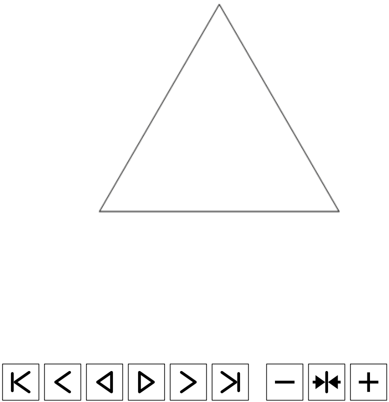
* The figure on the top shows a realizable polygonal linkage.
* The firgure on the bottom shows a polygonal linkage that is not realizable.





Motivation: Hinged Dissection

* *Haberdasher Problem*: Can a square and an equilateral triangle of the same area have a common dissection into four pieces?
* *Hilbert's Third Problem*: given any two polyhedra of equal volume, is it always possible to cut the first into finitely many polyhedral pieces which can be reassembled to yield the second?

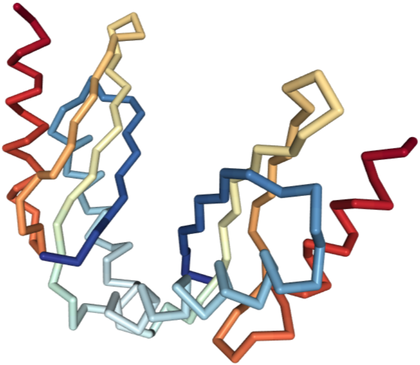


[0] Source: Wikipedia

Motivation: Protein Folding

Protein folding is the process in which a protein chain acquires its 3-dimensional structure.

* Proteins in an organism fold into a specific geometric pattern (sometimes referred as its *native state*).
* Geometric patterns can determine a protein's function and behavior.



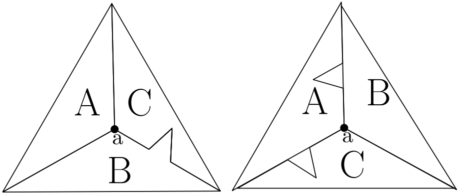
[0]Source: rcsb.org

Motivation: Deciding Realizability of Polygonal Linkages

Problem

Decide whether a polygonal linkage whose hinge graph is a tree can be realized with fixed orientation?

Here we have two realizations of a polygonal linkage with two different counter-clockwise order (C,B,A) and (B,C,A) respectively.

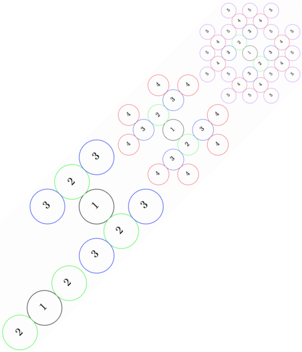


Motivation: Weighted Trees and Disk Arrangements

Problem

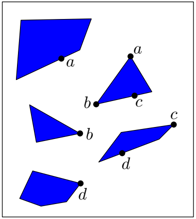
Decide whether a given ordered tree with positive vertex weights is the contact graph of a disk arrangements with specified radii?

* Consider the balanced binary trees of depth i $\{T_i\}_{i=1}^{\infty}$ with unit vertex weight.
* For $i \geq 8$, the corresponding disk arrangement is not realizable.

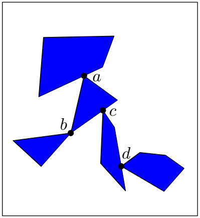


Problem: Weighted Trees and Disk Arrangements

* Is it NP-hard to decide whether a polygonal linkage whose hinge graph is a *tree* can be realized?
* Is it NP-Hard to decide whether a given ordered tree with positive vertex weights is the contact graph of a disk arrangements with specified radii?



(a)



(b)

Related Work

* Bhatt and Cosmadakis showed that deciding whether a polygonal linkage whose hinge graph is a *graph* is NP-Hard.
* Breu and Kirkpatrick showed that deciding whether a given ordered tree with unit vertex weights is the contact graph of a disk arrangements with specified radii.

Contributions

Theorem

It is NP-Hard to decide whether a polygonal linkage whose hinge graph is a tree can be realized.

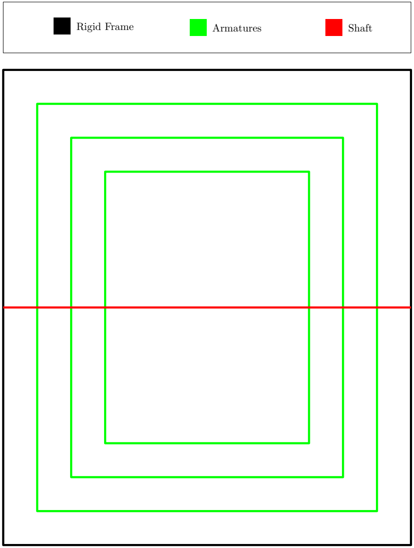
Theorem

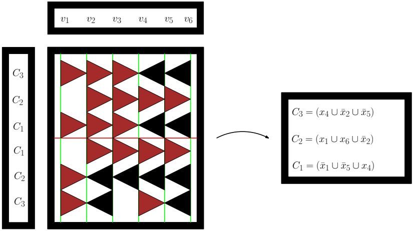
It is NP-Hard to decide whether a polygonal linkage whose hinge graph is a tree can be realized with fixed orientation.

Theorem

It is NP-Hard to decide whether a given ordered tree with positive vertex weights is the contact graph of a disk arrangements with specified radii.

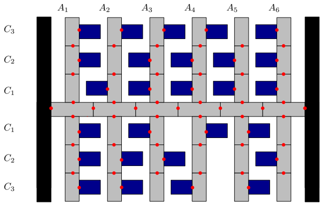
The Logic Engine





Logic Engine Realized as Hinged Polygons

* Suppose we are given an Boolean formula with m clauses and n variables in 3-CNF form, Φ, we construct the polygonal linkage similarly to the logic engine.

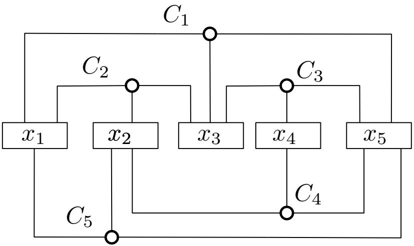


$C_1 = (x_1 \cup \bar{x}_2 \cup \bar{x}_6)$
$C_2 = (x_2 \cup \bar{x}_4 \cup \bar{x}_5)$
$C_3 = (x_5 \cup \bar{x}_3 \cup \bar{x}_6)$

Planar 3SAT

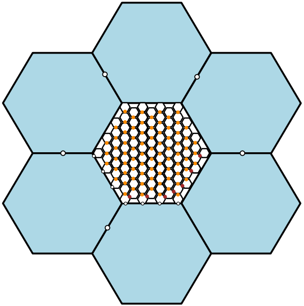
* Given a Boolean formula Φ in 3-CNF such that the associated graph is $A(\Phi)$, decide whether it is satisfiable is the *3-SAT problem*.

$$A(\Phi)$$



Modified Auxiliary Construction

* Define the *associated graph* $A(\Phi)$ as follows: the vertices correspond to the variables and clauses in Φ. We place an edge in the graph if variable x_i appears in clause C_j.
* Given a Boolean formula Φ in 3-CNF such that its associated graph is planar, decide whether it is satisfiable is a *3-SAT problem*.



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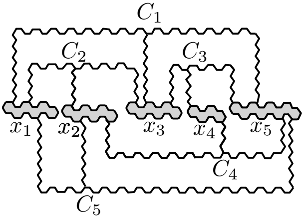
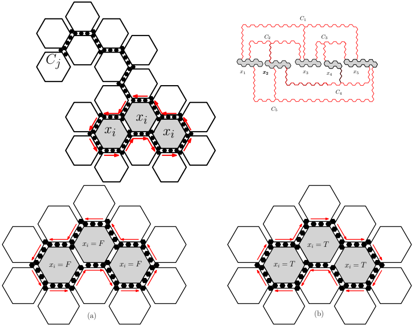


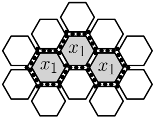
Transmitter Gadget

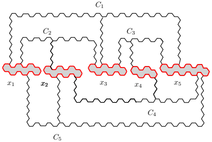
* A *transmitter gadget* is constructed for each edge $\{x_i, C_j\}$ of the graph $A(\Phi)$; it consists of a sequence of junctions and corridors from a variable gadget's junction to a clause junction.



Variable Gadget

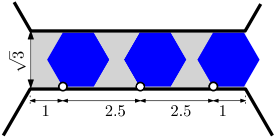
* Variable x_i corresponds to a cycle in the associated graph $\tilde{A}(\Phi)$.



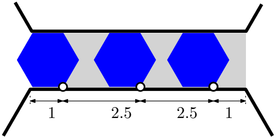


Variable Gadget

* Variable x_i corresponds to a cycle in the associated graph $\tilde{A}(\Phi)$.



(a)



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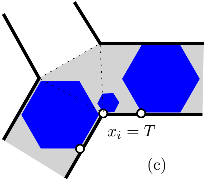
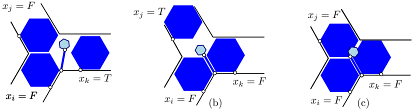
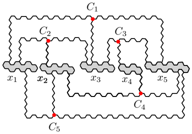


(c)

Clause Junction Gadget

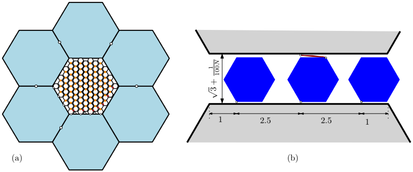
* The *clause gadget* lies at a junction adjacent to three transmitter gadgets.x





Modified Auxiliary Gadget

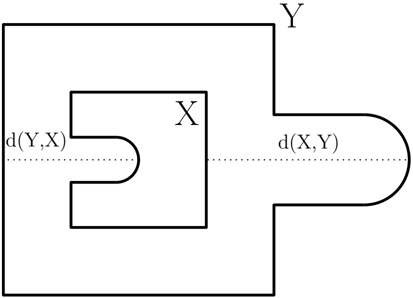
* The modified auxiliary gadget channels and junctions in a hexagonal grid enclosed by six frame hexagons.



(a) (b)

Approximation of Hexagon with a Disk Arrangement: Hausdorff Distance

* An illustrative example of $d(X, Y)$ and $d(Y, X)$ where X is the inner curve, and Y is the outer curve.



Approximation of Hexagon with a Disk Arrangement

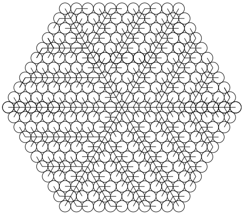
Lemma

For every $\epsilon > 0$ and $x > 0$, there exists an ordered weighted tree T and regular hexagon h of side length x such that:

- *T is realizable. Every realization σ_i of T as an ordered disk contact graph where the radii of the disks equal the vertex weights, approximates the hexagon in the sense that:*

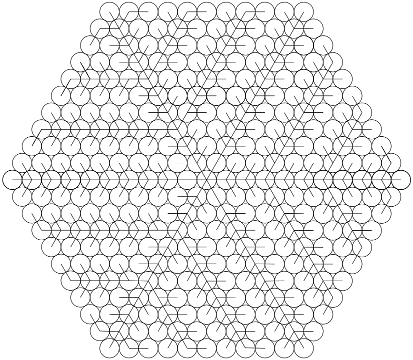
$$H(h, \sigma) \leq \epsilon$$

- *The number of nodes in T and the weights are polynomial in ϵ and x, the weights $\frac{\epsilon}{10}$ and $\frac{\epsilon}{10} + \zeta$ are polynomial.*



Approximation of Hexagon with a Disk Arrangement

* A drawing of a tree T overlayed with a corresponding disk arrangement, each disk with unit radius.
* The nodes of the tree are the centers of the disks.



Conclusion

Theorem

It is strongly NP-Hard to decide whether a polygonal linkage whose hinge graph is a tree can be realized.

Theorem

It is strongly NP-Hard to diskecide whether a polygonal linkage whose inge graph is a tree can be realized with fixed orientation.

Theorem

It is NP-Hard to decide whether a given ordered tree with positive vertex weights is the contact graph of a disk arrangements with pecified radii.

Thank You!