CALIFORNIA STATE UNIVERSITY, NORTHRIDGE

PROTEIN FOLDING: PLANAR CONFIGURATION SPACES OF DISC ARRANGEMENTS AND HINGED POLYGONS

A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in Applied Mathematics

by

Clinton Bowen

August 2014

The thesis of Clinton Bowen is approved:

Dr. Silvia Fernandez	Date
Dr. John Dye	Date
Dr. Csaba Tóth, Chair	Date

California State University, Northridge

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ABSTRACT

PROTEIN FOLDING: PLANAR CONFIGURATION SPACES OF DISC ARRANGEMENTS AND HINGED POLYGONS

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Clinton Bowen

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Lemma 1. *Let P be a polygonal linkage obtained from the modified auxilary construction. In every realization of P, the obstacle polygons are close to canonical position such that*

Lemma 1 serves as assurance that once a boolean formula of P3SAT is encoded into an arbitrary realization of the modified auxilary construction, the information of the boolean formula is preserved regardless of the positioning of the gadgets and components in the construction. This quality shows that the information is stable and preserved in an arbitrary realization of the modified auxilary construction. In Figure **??**, we have a column of obstacle hexagons veering off ℓ. This is an example of extreme angular rotation that should not occur over a vertical stack of hexagons.

Proof. We need to show that the modified auxilary construction could not deform in such a way that any information the construction encodes is lost or modified and the functionality of the gadgets within the construction behave as stated in the description.

To help identify components of the construction for this proof, let's identify components in the canonical position:

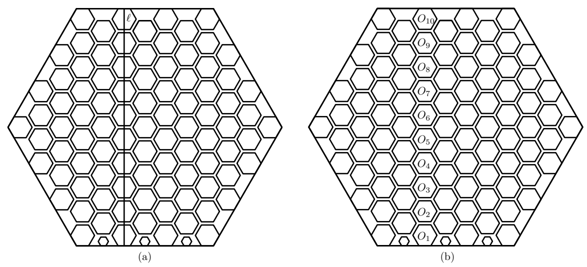


Figure 1: (a) depicts a column of obstacle hexagons O_1, ..., O_{10} along the vertical line ℓ; (b) identifies obstacle hexgons $O_1, \ldots, O_{10}$ in (a).

Without loss of generality, we can identify a column of obstacle hexagons O_i along a vertical line ℓ (See Figure 1). In this proof, unless otherwise specified, we assume that the argument refers to a column that starts and ends with an obstacle hexagon. In total there will be $u+1$ number of obstacle hexagons and u corridors in a column. Note that:

$$\begin{aligned} u &= \tfrac{J_h(z)}{2} - \tfrac{1}{2} \\ &= \tfrac{1}{2}(6z+1-1) \\ &= 3z \\ &= 12s \end{aligned}$$

where J_h is defined in Equation **??**.

The length of $H(n,m)$ (and ℓ in Figure 2(a)) can be expressed as a sum of the heights of the corridors and obstacle polygons. The width of a skinny rhombus in canonical position is $\frac{1}{100N}$. The obstacle hexagon has height of $2s(n,m)\cdot\sqrt{3}$, and the flag is of height $\sqrt{3}$.

$$H(n,m) = (12s(n,m)+1)\,2s(n,m)\sqrt{3} + 12s(n,m)\left(\frac{1}{100N} + \sqrt{3}\right) \tag{1}$$

The cross section of the corridor must have a minimum height of $\sqrt{3}$ everywhere. The height of the i^{th} obstacle polygon in noncanonical position is $2s(n,m)\sec(\alpha_i)\cdot\sqrt{3}$ (see Figure 3 for reference).

Angular Rotation α. First we show that the angular rotation of the obstacle hexagons with respect to canonical position is small. We first look at the relative angular difference between two adjacent obstacle polygons

$$|\alpha_i - \alpha_{i+1}|.$$

Given an arbitrary instance of a modified auxilary construction, consider O_i, O_{i+1}, and the corridor between O_i and O_{i+1}. The skinny rhombus has length $\sqrt{1+(100N)^{-2}}$.

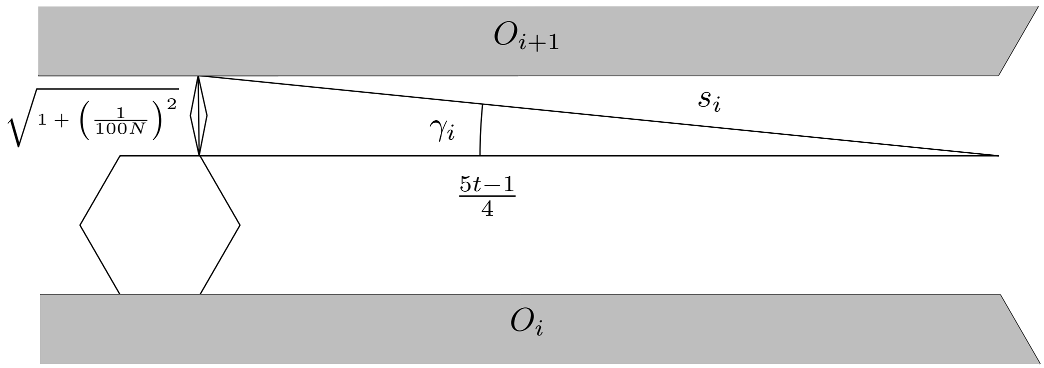


Figure 2: The obstacle hexagon here is in noncanonical position, and showing the side lengths adjacent to α_i.

In Figure 1(a) we see ℓ in the center of the column of obstacle hexagons. Our goal is to show that the column of hexagons cannot tilt in the manner shown in Figure **??** where the column veers greatly into the space occupied by other corridors and obstacle hexagons. The cross section of an arbitrary corridor must have a height of at least $\sqrt{3}$ everywhere. Otherwise, a flag would overlap with an obstacle hexagon; it would no longer remain a realization since the height of a flag is $\sqrt{3}$. In Figure 2, we illustrate an obstacle hexagon, its upper corridor with the flag that has the hinge to the skinny rhombus. The rhombus is hinged at the midpoint of the upper side of the corridor. The length from a corridor's midpoint to one end of the corridor is $\frac{5t-1}{4}$. γ_i is the angle between ζ_i and the horizontal axis at the height of the flag ($i = 1, 2, \ldots, u$). The bound of γ_i is:

$$\begin{aligned}
\gamma_i &\leq \tan^{-1}\left(\frac{\sqrt{1+\left(\frac{1}{100N}\right)^2}}{\frac{5t-1}{4}}\right) \\
&= \tan^{-1}\left(\frac{4\cdot\sqrt{1+\frac{1}{(100N)^2}}}{5t-1}\right) \\
&\leq \frac{4\cdot\sqrt{1+\frac{1}{(100N)^2}}}{5t-1} \\
&\leq \frac{\sqrt{16+\frac{16}{(100N)^2}}}{10N^3-6} \\
&\leq \frac{5}{10N^3-6} \\
&\leq \frac{5}{10N^3-10} \\
&\leq \frac{1}{2\left(N^3-1\right)} \\
&\leq \frac{1}{2\left(\frac{5t-1}{2}\right)^3-1} \\
&\leq \frac{4}{(5t-1)^3-4} \\
&\leq \frac{4}{\left(5s^k-1\right)^3-4}
\end{aligned} \tag{2}$$

Inequality 2 uses the first term Maclaurin series of $\tan^{-1}$. Thus the relative rotational difference between adjacent obstacle hexagons is

$$|\alpha_i - \alpha_{i+1}| \leq \frac{4}{(5s^k - 1)^3 - 4} \tag{3}$$

The relative difference between α_i and α_{i+1} is small. The bottom most obstacle hexagon is hinged to the frame (see Figure **??** for illustration). This implies that $\alpha_1 = 0$ and

$$|\alpha_1 - \alpha_2| = |\alpha_2| \leq \frac{4}{(5s^k - 1)^3 - 4}.$$

There are a total of u obstacle hexagons in a column with possibly up to $u-1$ nonzero obstcle hexagons rotations. We can derive 1) a bounded sum of rotational displacement over a column of obstacle hexagons:

$$\sum_{i=1}^{u-1} |\alpha_i - \alpha_{i+1}| \leq \frac{4(u-1)}{(5s^k - 1)^3 - 4} \tag{4}$$

and 2) derive the maximum rotational displacement at the i^{th} obstacle hexagon:

$$\begin{aligned}
\alpha_i &\leq \sum_{j=1}^{i} \frac{4j}{(5s^k - 1)^3 - 4} \\
&\leq \frac{4}{(5s^k - 1)^3 - 4} \sum_{j=1}^{i} j \\
&\leq \frac{4}{(5s^k - 1)^3 - 4} \cdot \frac{i^2 + i}{2} \\
&\leq \frac{4\left(u^2 + u\right)}{2\left((5s^k - 1)^3 - 4\right)} \\
&\leq \frac{4\left(144s^2 + 12s\right)}{2\left((5s^k - 1)^3 - 4\right)}
\end{aligned}$$

$$\alpha_i \leq \frac{288s^2 + 12s}{(5s^k - 1)^3 - 2} \tag{5}$$

The sum of total displacement in a given column is bounded by:

$$|\alpha_u| \leq \sum_{i=2}^{u-1} |\alpha_i - \alpha_{i+1}| \leq \frac{u-1}{2\,(N^3 - 1)} = \frac{12s - 1}{2\,((c_N s)^3 - 1)}$$

Vertical Displacement δ When an obstacle hexagon is rotated by α_i, the height of the hexagon becomes $2s(n,m)\sqrt{3}\sec\alpha_i$. Figure 3 shows the geometry of a rotated obstacle hexagon.

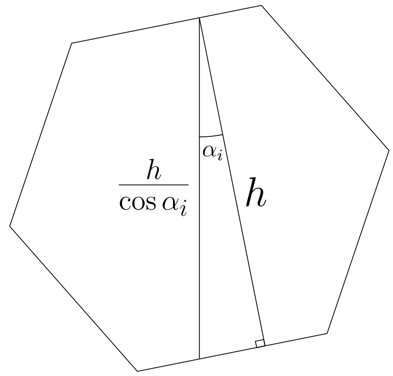


Figure 3: This figure shows a right triangle with angle α_i and sides of length h and $\frac{h}{\cos\alpha_i}$.

To show that the vertical displacement from canonical position is small, we first consider a column of obstacle hexagons in canonical position (see Figure 4). For canonical position, the j^{th} obstacle has $\delta_j = 0$.

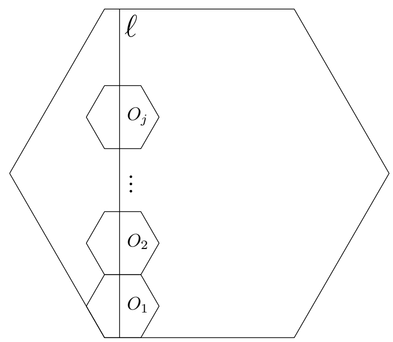


Figure 4: This illustration is of a column of obstacle hexagons in canonical position along a vertical line segment ℓ.

From Equation 1, we know the exact height of ℓ in terms of the heights of the corridors and obstacle hexagons in canonical position. Consider the first j terms for the height of the column of obstacle hexagons and corridors for an arbitrary construction with angular rotation and vertical displacement for *one* obstacle hexagon $|\delta_v| > 0$, where $j = 2, \cdots, u+1$ and $1 < v \leq j$.

$$
\begin{aligned}
\sum_{i=1}^{j}\left(2\sqrt{3}s\sec(\alpha_i)\right) + \delta_v + (j-1)\left(\frac{1}{100N}+\sqrt{3}\right) &\leq j \cdot 2s\sqrt{3} + (j-1)\cdot\left(\frac{1}{100N}+\sqrt{3}\right) \\
2\sqrt{3}s\sum_{i=1}^{j}\sec(\alpha_i) + \delta_v &\leq j \cdot 2s\sqrt{3} \\
\sum_{i=1}^{j}\sec(\alpha_i) + \delta_v &\leq j \\
\delta_v &\leq j - \sum_{i=1}^{j}\sec(\alpha_i) \\
\delta_v &\leq j - \left(j - \sum_{i=1}^{j}\frac{\alpha_i^2}{2}\right) \\
\delta_v &\leq \sum_{i=1}^{j}\frac{\alpha_i^2}{2}
\end{aligned}
$$

Using Inequalities 4 and 5, we derive the following result:

$$\begin{aligned}
\Sigma_{i=1}^{j} \frac{\alpha_i^2}{2} &\leq \frac{\Sigma_{i=1}^{j}\left(\Sigma_{\kappa=1}^{i} \frac{\kappa}{2\left(N^3-1\right)}\right)^2}{2} \\
\Sigma_{i=1}^{j} \alpha_i^2 &\leq \Sigma_{i=1}^{j}\left(\Sigma_{\kappa=1}^{i} \frac{\kappa}{2\left(N^3-1\right)}\right)^2 \\
&\leq \Sigma_{i=1}^{j}\left(\frac{1}{2\left(N^3-1\right)} \Sigma_{\kappa=1}^{i} \kappa\right)^2 \\
&\leq \Sigma_{i=1}^{j}\left(\frac{1}{2\left(N^3-1\right)} \cdot \frac{i^2+i}{2}\right)^2 \\
&\leq \Sigma_{i=1}^{j}\left(\frac{i^2+i}{4\left(N^3-1\right)}\right)^2 \\
&\leq \Sigma_{i=1}^{j}\left(\frac{i^2+i}{\left(\frac{5t-1}{2}\right)^3-1}\right)^2 \\
&\leq \Sigma_{i=1}^{j}\left(\frac{8\left(i^2+i\right)}{\left((5t-1)^3-8\right)}\right) \\
&\leq \Sigma_{i=1}^{j}\left(\frac{8\left(i^2+i\right)}{\left(\left(5s^k-1\right)^3-8\right)}\right)
\end{aligned}$$

Thus we finally say that the bound for δ_v, where $1 < v \leq j$, is small:

$$\delta_v \leq \frac{\Sigma_{i=1}^{j}\left(\frac{i^2+i}{4\left(N^3-1\right)}\right)^2}{2} \tag{6}$$

Horizontal Displacement β We show the horizontal displacement of an obstacle hexagon is small. First we analyze the change of corridor's cross-section with respect to horizontal displacement.

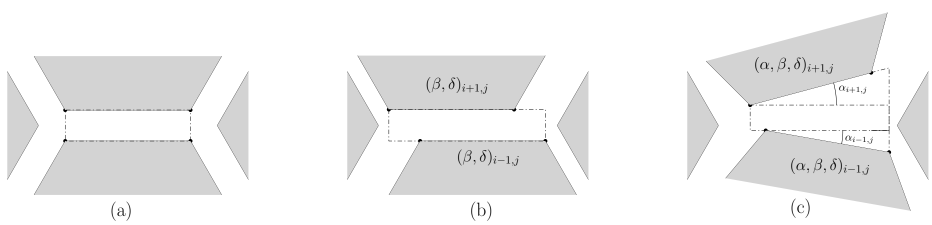


Figure 5: (a) A corridor and corresponding junctions in canoncial position with cross section $N(n,m) \times (\sqrt{3} + (100N)^{-1})$. (b) A corridor whose total horizontal displacement is $\beta_{i+1} + \beta_{i-1}$ on the top and total vertical displacement of $\delta_{i+1} + \delta_{i-1}$ on the bottom; its cross section is $(N(n,m) + \beta_{i+1} + \beta_{i-1}) \times (\sqrt{3} + (100N)^{-1}) + \delta_{i+1} + \delta_{i-1}$. (c) shows the cross sectional area of a corridor with full rotational, horizontal, and vertical displacement.

Figure 5 shows the same corridor and corresponding junctions in canonical position and non-canoncical positions. This figure is of a corridor and adjacent junctions formed by obstacle hexagons in counter-clockwise order: $O_{i-1,j}$, $O_{i,j-1}$, $O_{i+1,j}$, and $O_{i,j+1}$. The cross sectional areas for Figure 5(a) and Figure 5(b) are $N(n,m) \times (\sqrt{3} + (100N)^{-}1)$ and $(N(n,m) + \beta_{i+1} + \beta_{i-1}) \times (\sqrt{3} + (100N)^{-1}) + \delta_{i+1} + \delta_{i-1}$ respectively. Figure 5(c) shows the same corridor with rotational, horizontal, and vertical displacement. To find the crosss sectional area of the corridor in Figure 5(c), we decompose it into three parts, the upper tringle, the reectangle, and the lower triangle.

We're given $N(n,m)$ and $(\alpha, \beta, \delta)_{i,j}$ for all $i, j = 1, \ldots, u$. The area of the upper triangle is

$$\frac{1}{2}\left(N + \beta_{i+1} + \beta_{i-1}\right)^2 \tan\left(\alpha_{i+1,j}\right). \tag{7}$$

The area of the rectangle is

$$(N+\beta_{i+1}+\beta_{i-1})\cdot\left(\left(\sqrt{3}+(100N)^{-1}\right)+\delta_{i+1}+\delta_{i-1}\right). \tag{8}$$

The area of the lower triangle is

$$\frac{1}{2}\left(N\cdot\cos\left(\alpha_{i-1,j}\right)\right)\cdot\left(N\cdot\sin\left(\alpha_{i-1,j}\right)\right)=N^2\sin\left(2\cdot\alpha_{i-1,j}\right). \tag{9}$$

We show by an induction argument that the horizontal displacement for any given modified construction is small. Consider a column of obstacle hexagons which has an obstacle hexagon O_1 hinged at the bottom of the frame J_z. The obstacle hexagon O_1 has no displacement; it is immobile. Consider the case where the column of hexagon has a half sized hexagon attached at the the bottom of the frame, the hinge points of the half sized hexagon immobilize the the first obstacle hexagon of that column (see Figure **??** for reference). In either case the obstacle hexagon O_2 has horizontal mobility by way of the range of motion from the skinny rhombi (see Figure 8 for reference) between O_1 and O_2. The diameter of the skinny rhombi are $\sqrt{1+(100N)^{-1}}$; the possible horizontal displacement O_2 could have is $\sqrt{1+(100N)^{-1}}$ plus the horizontal displacement contributed by $\alpha_{i,2}$ (see Figure **??**).

$$\beta_{i,2}\leq\sqrt{1+(100N)^{-1}}+\frac{h}{2}\sin\alpha_{i,2}\leq\sqrt{1+(100N)^{-1}}+\frac{h\alpha_{i,2}}{2} \tag{10}$$

The maximal horizontal displacement is shown in Inequality 10 where h is the height of an obstacle hexagon. The maximal horizontal displacement is $\beta_{i,2}=\sqrt{1+(100N)^{-1}}+\frac{h}{2}\sin\alpha_{i,2}$ however, this large value is infeasible. We will now show what the feasible upper bound of horizontal displacement is. Firstly, the range of motion of the skinny rhombus is limited by the vertical displacement of an obstacle hexagon. Secondly, we have bounded the angular displacement α of an obstacle hexagon.

The vertical displacement of the v^{th} obstacle hexagon is:

$$\delta_v\leq\frac{\sum_{i=1}^{v}\left(\frac{5\left(i^2+i\right)}{20N^3-12}\right)^2}{2}$$

Let the angular rotation of the skinny rhombus be ω_v, then corresponding horizontal dispalcement ϕ is:

$$\phi=\frac{\delta_v}{\tan\left(\omega_v\right)}=\frac{\delta_v}{\tan\left(\sin^{-1}\left(\frac{\delta_v}{\sqrt{1+\left(\frac{1}{100N}\right)^2}}\right)\right)}$$

using Maclaurin series, ϕ simplifies to:

$$\begin{aligned}\frac{\delta_v}{\tan\left(\sin^{-1}\left(\frac{\delta_v}{\sqrt{1+\left(\frac{1}{100N}\right)^2}}\right)\right)} &\leq \frac{\delta_v}{\tan\left(\frac{\delta_v}{\sqrt{1+\left(\frac{1}{100N}\right)^2}}+\frac{1}{6}\left(\frac{\delta_v}{\sqrt{1+\left(\frac{1}{100N}\right)^2}}\right)^3\right)}\\ &\leq \frac{\delta_v}{\frac{\delta_v}{\sqrt{1+\left(\frac{1}{100N}\right)^2}}+\frac{1}{6}\left(\frac{\delta_v}{\sqrt{1+\left(\frac{1}{100N}\right)^2}}\right)^3}\end{aligned}$$

The angluar displacement of the v^{th} obstacle hexagon is:

$$\alpha_v\leq\frac{\left(v^2+v\right)5}{2\left(10N^3-6\right)}$$

which implies:

$$\frac{h\alpha_2}{2} \leq \frac{15h}{2\,(10N^3-6)}$$

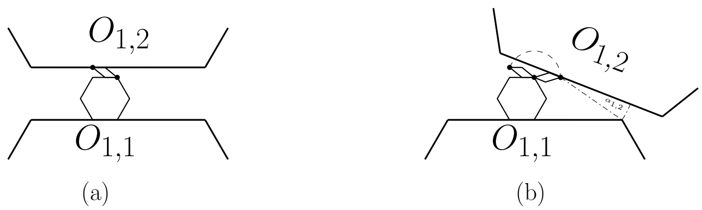


Figure 6: This figure illustrates the possible range of motion of the skinny rhombus between two obstacle hexagons and the feasible range of rotational displacement of $O_{1,2}$.

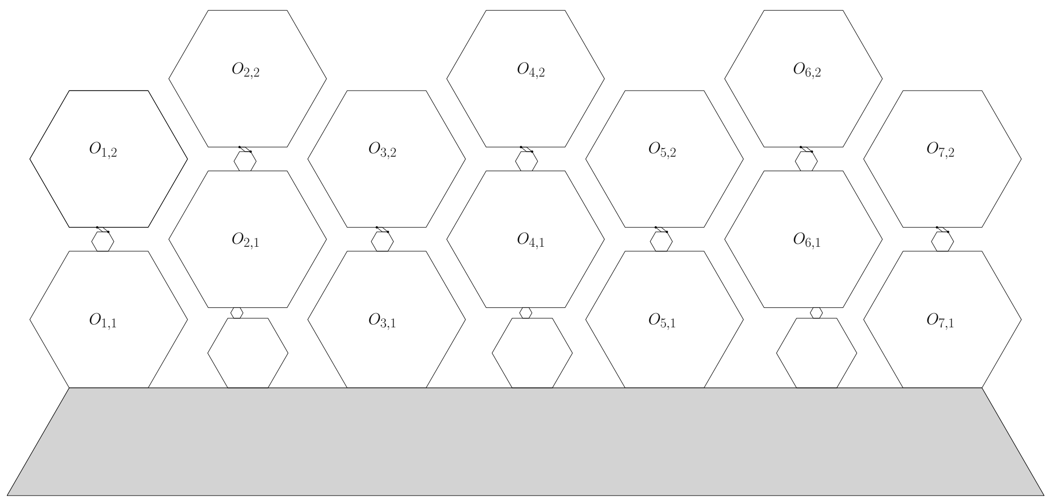


Figure 7: bottom

Note that the bottom most obstacle hexagon $O_{j,1}$ is hinged to the frame or hinged to some half sized hexagon such that $\beta_{j,1} = 0$. The range of motion horizontal motion that $O_{j,2}$ has is dictated by the range of possible motion from the skinny rhombus between $O_{j,1}$ and $O_{j,2}$. The diameter of the skinny rhombus is $\sqrt{1+(100N)^{-1}}$. If $\beta_{j,2} = \pm\sqrt{1+(100N)^{-1}}$, there would be a collision a corridor(s) adjacent to $O_{j,2}$.

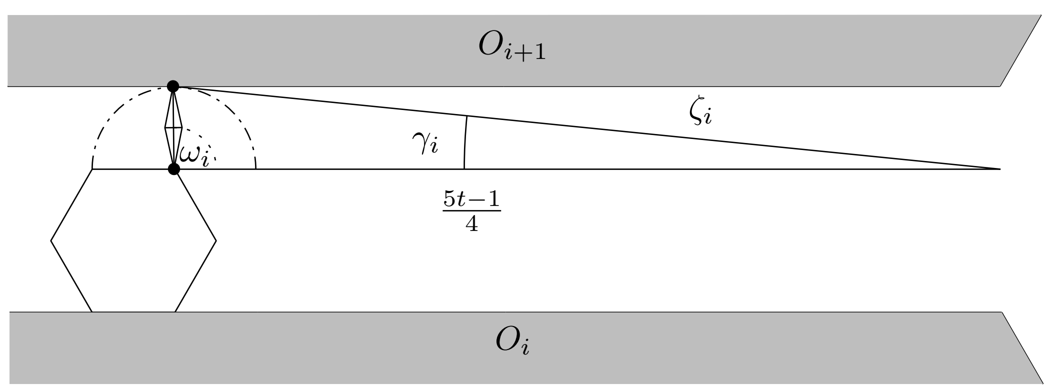


Figure 8: The full range of motion is shown dashed half circle about the diameter of the skinny rhombus.

Figure 8 shows the range of motion of the skinny rhombus. The angle formed between the diameter of the rhombus and the half length of the corridor is ω_i.

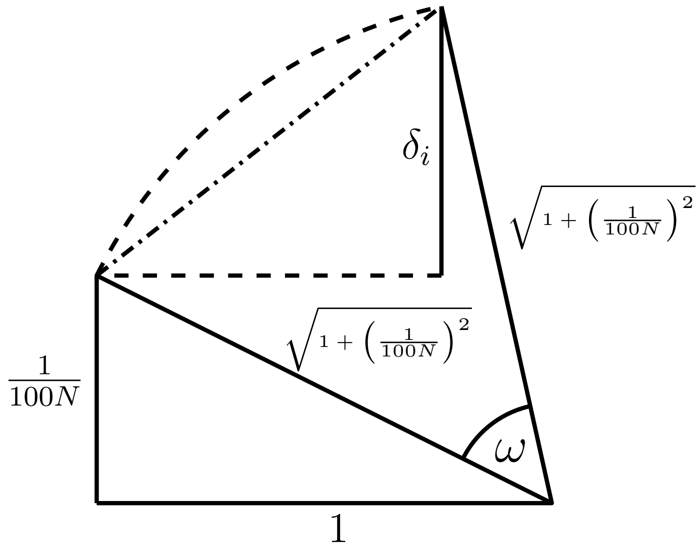


Figure 9: blah

$$|\beta_i - \beta_{i-1}| = \begin{cases} 1-\cos\frac{\omega}{2} \leq \frac{\omega}{2} \\ 2+(1-\cos\omega)+\frac{h}{2}(1-\cos\alpha) \end{cases} \tag{11}$$

$\square$

Chapter 1

Realizability Problems for Weighted Trees

1.1 Properties for Weighted Trees and Polygonal Linkages

In order to perform our analysis for weighted trees and polygonal linkages, we'll want to use a suitable metric. The usual Euclidian distance will not suffice for this analysis and so we turn to the Hausdorff distance.

Hausdorff Distance Let A and B be sets in the plane. The *directed Hausdorff distance* is

$$d(A,B) = \sup_{a \in A} \inf_{b \in B} ||a-b|| \tag{1.1}$$

$h(A,B)$ finds the furthest point $a \in A$ from any point in B. *Hausdorff distance* is

$$D(A,B) = \max\{d(A,B), d(B,A)\} \tag{1.2}$$

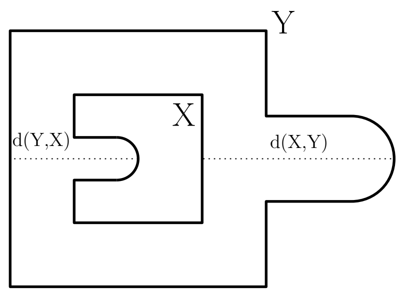


Figure 1.1: An illustrative example of $d(X,Y)$ and $d(Y,X)$ where X is the inner curve, and Y is the outer curve.

ε-approximation The weighted graph, G, is an *ε-approximation* of a polygon P if the Hausdorff distance between every realization such realization of G as a contact graph of disks and a congruent copy of P is at most epsilon. A weighted graph G is said to be a *$O(f(x))$-approximation* of a polygon P if there is a positive constant M such that for all sufficiently large values of x the Hausdorff distance between every realization such realization of G as a contact graph of disks and a congruent copy of P is at $M \cdot |f(x)|$. A weighted graph G is said to be a *stable* if it has the property that for every two such realizations of G, the distance between the centers of the corresponding disks is at most ε after a suitable rigid transformation.

1.2 Approximating Regular Hexagons with Snowflakes

In figure 1.2, we have a set of unit radius disks (circles) arranged in a manner that outlines regular, concentric hexagons.

Problem 1 (Appoximating Polygonal Shapes with Contact Graphs). For every $\varepsilon > 0$ and polygon P, there exists a contact graph $G = (V,E)$ such that the Hausdorff distance $d(P,G) < \varepsilon$

Recall problems (**??**) and (**??**): given a positive weighted tree, T, is T the (ordered) contact graph of some disk arrangement where the radii are equal to the vertex weights. For now, we'll focus on a particular family

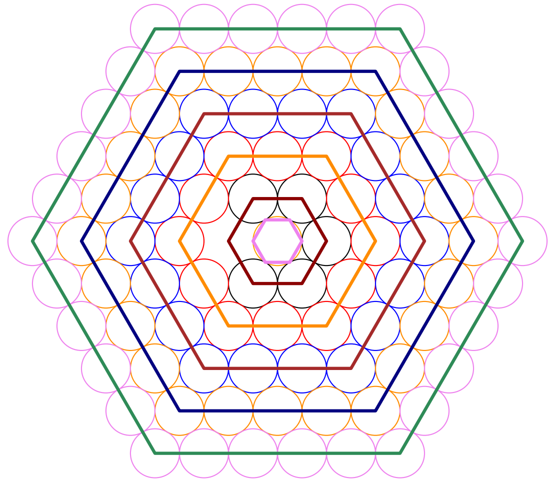


Figure 1.2: A contact graph that resembles the shape of concentric hexagons.

of this problem space where the weighted trees can be realized as a *snowflake*. For $i \in \mathbb{N}$, the construction of the snowflake tree, S_i, is as follows:

- Let v_0 be a vertex that has six paths attached to it: $p_1, p_2, \ldots, p_6$. Each path has i vertices.
- For every other path p_1, p_3, and p_5:
  - Each vertex on that path has two paths attached, one path on each side of p_k.
  - The number of vertices that lie on the path attached to the j^{th} vertex of p_k is $i - j$.

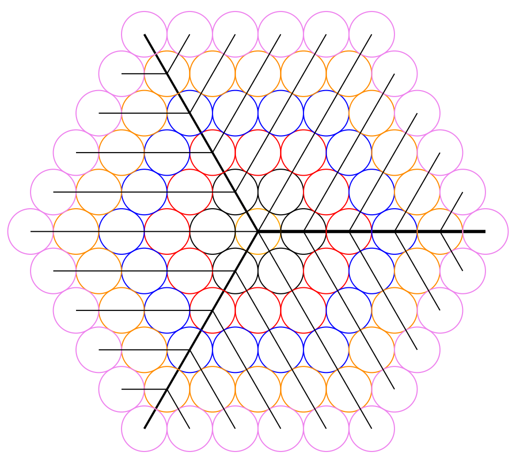


Figure 1.3: The same contact graph as in figure 1.2 overlayed with the a perfectly weighted snowflake tree.

A *perfectly weighted snowflake tree* is a snowflake tree with all vertices having weight $\frac{1}{2}$. A *perturbed snowflake tree* is a snowflake tree with all vertices having weight of 1 with the exception of v_0; in a perturbed snowflake tree, v_0 will have a weight of $\frac{1}{2} + \gamma$. For our analysis, all realizations of any snowflake, perfect or perturbed, shall have v_0 fixed at origin. This is said to be the canonical position under Hausdorff distance of the snowflake tree.

Perfectly Weighted Snowflake Tree. Consider the graph of the triangular lattice with unit distant edges:

$$
\begin{aligned}
V &= \left\{ a\cdot(1,0) + b\cdot\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right) : a,b\in\mathbb{Z} \right\} \\
E &= \{\{u,v\} : ||u-v|| = 1 \text{ and } u,v\in V\}
\end{aligned}
$$

The following graph, $G=(V,E)$ is said to be the *unit distance graph* of the triangular lattice. We can show that no two distinct edges of this graph are non-crossing. First suppose that there were two distinct edges that crossed, $\{u_1,v_1\}$ and $\{u_2,v_2\}$. With respect to u_1, there are 6 possible edges corresponding to it, with each edge $\frac{\pi}{3}$ radians away from the next. Neither edge crosses another; and so we have a contradiction that there are no edge crossings with $\{u_1,v_1\}$.

The perfectly weighted snowflake tree that is a subgraph over the *unit distance graph*, $G=(V,E)$, of the triangular lattice. To show this, for any S_i, fix $v_0 = 0\cdot\cdot(1,0)+0\cdot\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right) = (0,0)\in V$ at origin. Next consider the six paths attached from origin. Fix each consecutive path $\frac{\pi}{3}$ radians away from the next such that the following points like on the corresponding paths: $(1,0)\in p_1$, $\left(\frac{1}{2},\frac{\sqrt{2}}{3}\right)\in p_2$, $\left(-\frac{1}{2}\mathrm{p}4,\frac{\sqrt{3}}{2}\right)\in p_3$, $(-1,0)\in p4$, $\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right)\in p_5$, $\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)\in p_6$. For S_i, there are i vertices on each path.

We define the six paths from origin as follows:

$$
\begin{aligned}
p_1 &= \left\{ a\cdot(1,0) = \vec{v}\,\middle|\, a\in\mathbb{R}^+ \right\} \\
p_2 &= \left\{ a\cdot\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right) = \vec{v}\,\middle|\, a\in\mathbb{R}^+ \right\} \\
p_3 &= \left\{ -a\cdot(1,0) + a\cdot\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right) = a\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right) = \vec{v}\,\middle|\, a\in\mathbb{R}^+ \right\} \\
p_4 &= \left\{ a\cdot(-1,0) = \vec{v}\,\middle|\, a\in\mathbb{R}^+ \right\} \\
p_5 &= \left\{ a\cdot\left(-\frac{1}{2},-\frac{\sqrt{3}}{2}\right) = \vec{v}\,\middle|\, a\in\mathbb{R}^+ \right\} \\
p_6 &= \left\{ a\cdot(1,0) - a\cdot\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right) = a\cdot\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right) \,\middle|\, a\in\mathbb{R}^+ \right\}
\end{aligned}
$$

For S_i there exists i vertices on each path. We shall denote the i^{th} vertex on the j^{th} path as $v_{j,i}$. For each path defined above, the paths are defined as a set of vectors, $\vec{v} = a\cdot\vec{p}$ for some $a\in\mathbb{R}^+$ and $\vec{p}\in\mathbb{R}^2$. By setting $a = 1,2,\ldots,i$, we obtain points that are contained in V. For $j = 1,3,5$ and $l = 1,...,i$, there exists two paths attached to each vertex $v_{j,l}$. For S_i, each path attached to the k^{th} vertex of p_j, there are $i-k$ vertices. We will need to show that each of the $i-k$ vertices on each corresponding path are also in V.

The triangular lattice is symmetice under rotation about v_0 by $\frac{\pi}{3}$ radians. For each vertex $v_{1,l}$ for $l = 1,2,...,i-k$, we place two paths from it; the first path $\frac{\pi}{3}$ above p_1 at $v_{1,l}$ and $\frac{-\pi}{3}$ below p_1 at $v_{1,l}$ and call these paths $p^+_{1,l}$ and $p^-_{1,l}$ respectively. With respect to $v_{1,l}$, one unit along $p^+_{1,l}$ is a point on the triangular lattice and similarly so on $p^-_{1,l}$. Continuing the walk along these paths, unit distance-by-unit distance, we obtain the next point corresponding point on the the triangular lattice up to $i-k$ distance away from $v_{1,l}$. This shows that each of the $i-k$ vertices on $p^-_{1,l}$ and $p^+_{1,l}$ are in V. By rotating all of the paths along p_1 by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, we obtain the the paths along p_3 and p_5 respectively, completing the construction.

Perturbed Snowflake Tree. The perturbed snowflake follows the construction of the perfect snowflake with the exception of v_0 having weight $\frac{1}{2}+\gamma$ where $\gamma > 0$. A perturbed snowflake realization has some distinct qualities from perfect snowflake realizations. The angular relationships between adjacent vertices may vary; the distance between adjacent and neighboring vertices may vary as well. Note that we regard snowflakes with unit weight as a weight of $\frac{1}{2}$.

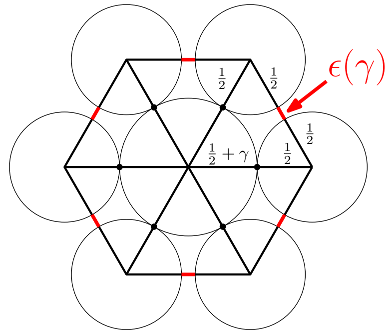


Figure 1.4: A canonical disk arrangement from a perturbed snowflake with 6 unit disks around a central disk with radius $\frac{1}{2}+\gamma$.

In Figure 1.4, we have a realization of disk arrangement from a perturbed snowflake. In a disk arrangement of a perfect snowflake, the disks around the central disk contact the adjacent disks. The disk arrangement from the perturbed snowflake does not have this quality. Figure 1.4 shows a gap $\varepsilon(\gamma)$ between adjacent disks around the central disk. This gap is formed from the perturbed weight $\frac{1}{2}+\gamma$ of the central disk.

$$\varepsilon(\gamma) = 2\gamma + \gamma^2 \tag{1.3}$$

As the perturbed snowflake grows outer layers, we can begin to define parts of the snowflake and the corresponding disk arrangement. Let the arms extending from the center of a snowflake be *dendrites* and the arms extending off of arms be *metadendrites*. In a perturbed snowflake, the dendrites and metadendrites have some freedom to about the plane. our

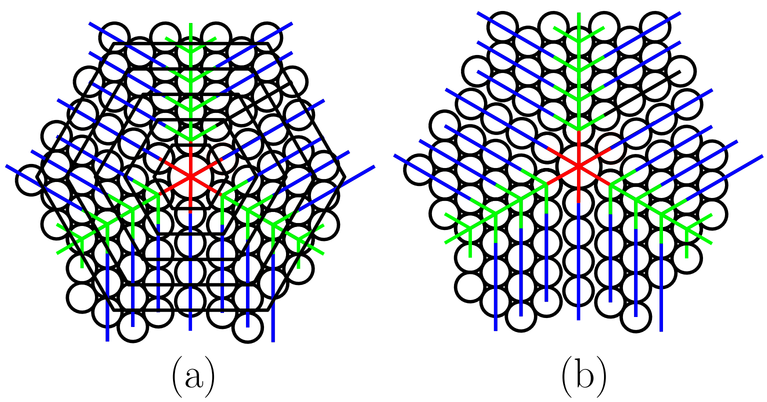


Figure 1.5

In Figure 1.5, we show an overlay of a realization of a perturbed snowflake, a corresponding disk arrangement, and concentric hexagons about the v_0.

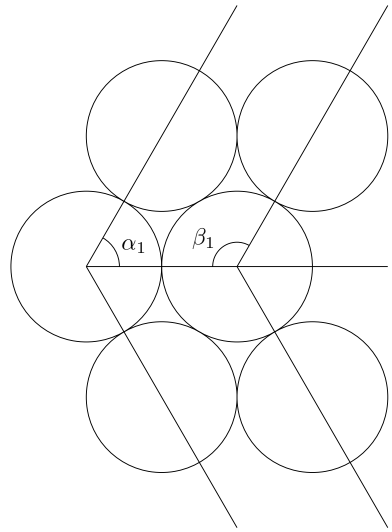


Figure 1.6: ?A?S

In Figure 1.6, we have a perturbed spine

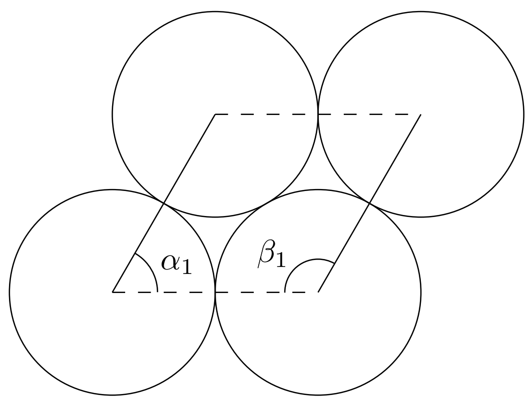


Figure 1.7: ?A?S

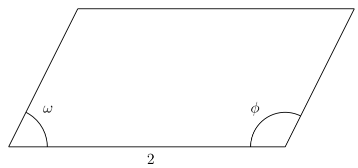
In Figure 1.7, we have four an arrangement of disks on the snowflake, off the spine and away from the central disk. We call this a vertebrae.

Lemma 2. *Given any realization of a perturbed snowflake of 7 weighted vertices, with the central vertex v_0 weighted $\frac{1}{2}+\gamma$ and the others weighted $\frac{1}{2}$, the total additional distance between all vertices is $6\varepsilon(\gamma)$ compared to a perfect snowflake of 7 unit weight vertices.*

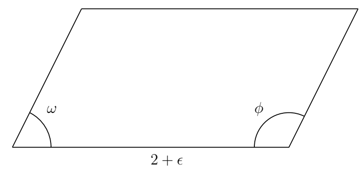
Proof. Consider a canonical disk arrangement of a perturbed snowflake of 7 weighted vertices (see Figure 1.4). The side length of the sides formed between the center of the central disk and two adjacent disks around the central disk is $1+\gamma$. Let the distance between the two adjacent disks be $1+\varepsilon(\gamma)$. There are a total of $6\varepsilon(\gamma)$ between adjacent centers of disks. The total perimeter of the hexagon formed about the centers of the disks in contact with the central disk is $6+6\varepsilon(\gamma)$. Note that 1) the total perimeter of the hexagon formed on a perfect snowflake of 7 weighted vertices is 6 and 2) the canonical disk arrangement can be transformed to any other disk arrangement corresponding to the perturbed snowflake of 7 weighted vertices by pushing the the ring of disks around the central disk together such that all adjacent disks are in contact with each other with the exception of the disks at the end. $\square$

1.3 On the Decidability of Problem ??

Proof. Consider a $k \times (\sqrt{3}k)$ rectangle section of a triangular lattice, and place disks of radius 1 at each grid point as in Fig. ?????. The contact graph of these disks contains 2-cycles. Consider the spanning tree T of the contact graph indicated in Fig. ????. The tree T decomposes into paths of collinear edges: T contains two paths along the two main diagonals, each containing $2k-1$ vertices; all other paths have an endpoint on a main diagonal. We now modify the disk arrangement to ensure that its contact graph is T. The disks along the main diagonal do not change. We reduce the radii of all other disks by a factor of $1-k^{-3}$ (as a result, they lose contact with other disks), and then successively translate them parallel in the direction of the shortest path in T to the main diagonal until the contact with the adjacent disk is reestablished. The Hausdorff distance between the union of these disks and the initial $k \times (\sqrt{3}k)$ rectangle is clearly less than 1. However, the contact tree T with these radii no longer has a unique realization (small perturbations are possible). To show stability, we argue by induction on the hop distance from the central disk. There are $O(i)$ disks at i hops from the central disk, most one which have radius $(1-k^{-3})\frac{1}{2}$. Since all radii are 1 or $(1-k^{-3})\frac{1}{2}$,the six neighbors of the central disk can differ from the regular hexagon by at most $O(k)$. Similarly, the disks at i hops from the center be off from the triangular grid pattern by $O(i2^{k-3})$, for $i = 1, 2, \ldots, k$.



(a)



(b)

Figure 1.8

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Bibliography