

CALIFORNIA STATE UNIVERSITY, NORTHRIDGE

PROTEIN FOLDING: PLANAR CONFIGURATION SPACES OF DISC  
ARRANGEMENTS AND HINGED POLYGONS

A thesis submitted in partial fulfillment of the requirements for the degree of  
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by

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ABSTRACT

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## Chapter 1

### Realizability Problems for Weighted Trees

In this chapter our goal is to prove Theorem ?? which states: “It is NP-Hard to decide whether a given tree with positive vertex weights is the contact graph of a disk arrangements with specified radii.” This chapter’s approach to proving Theorem ?? introduces an ordered weighted tree  $T$  and perturbed ordered weight tree  $T_\epsilon$ , the Hausdorff distance, and then prove the following lemma:

**Lemma 1.** *for ever  $\epsilon > 0$ , there exists an ordered weighted tree  $T_\epsilon$  such that every realization of  $T_\epsilon$  as an ordered disk contact graph where the radii of the disks equal the vertex weights.*

Using Lemma 1, we prove Theorem ?? by extending the modified auxiliary construction in Chapter ??.

We first cover the preliminary concepts of Hausdorff distance and the ordered weighted tree families of  $T$  and  $T_\epsilon$ . We then continue with the proof of Lemma 1 and Theorem ??.

#### 1.1 Hausdorff Distance

Let  $A$  and  $B$  be sets in the plane. The *directed Hausdorff distance* is:

$$d(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\| \quad (1.1)$$

$d(A, B)$  finds the furthest point  $a \in A$  from any point in  $B$ . *Hausdorff distance* is

$$D(A, B) = \max \{d(A, B), d(B, A)\} \quad (1.2)$$

In Figure 1.1, we have two sets  $X$  and  $Y$  and illustrate  $d(X, Y)$  and  $d(Y, X)$ . From this, it is possible to calculate the Hausdorff distance between  $X$  and  $Y$ .

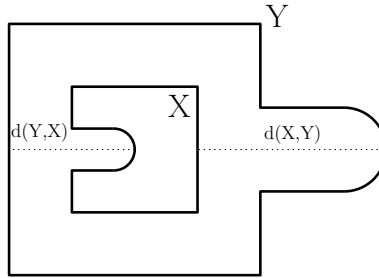


Figure 1.1: An illustrative example of  $d(X, Y)$  and  $d(Y, X)$  where  $X$  is the inner curve, and  $Y$  is the outer curve.

**$\epsilon$ -approximation** The weighted graph,  $G$ , is an  $\epsilon$ -approximation of a polygon  $P$  if the Hausdorff distance between every realization of  $G$  as a contact graph of disks and a congruent copy of  $P$  is at most epsilon. A weighted graph  $G$  is said to be a  $O(f(x))$ -approximation of a polygon  $P$  if there is a positive constant  $M$  such that for all sufficiently large values of  $x$  the Hausdorff distance between every realization such realization of  $G$  as a contact graph of disks and a congruent copy of  $P$  is at  $M \cdot |f(x)|$ . A weighted graph  $G$  is said to be a *stable* if it has the property that for every two such realizations of  $G$ , the distance between the centers of the corresponding disks is at most  $\epsilon$  after a suitable rigid transformation.

An example of an  $\epsilon$ -approximation .....

*Problem 1* (Approximating Polygonal Shapes with Contact Graphs). For every  $\varepsilon > 0$  and polygon  $P$ , there exists a contact graph  $G = (V, E)$  such that the Hausdorff distance  $d(P, G) < \varepsilon$

## 1.2 Weight Trees $T_k$

In this section we describe a particular family of unit weight trees and corresponding contact graphs disk arrangements called *snowflakes*. Note that we regard snowflakes with unit weight as a weight of  $\frac{1}{2}$ . For  $i \in \mathbb{N}$ , the construction of the snowflake tree,  $T_i$ , is as follows:

- Let  $v_0$  be a dvertex that has six paths attached to it:  $p_1, p_2, \dots, p_6$ . Each path has  $i$  vertices.
- For every other path  $p_1, p_3$ , and  $p_5$ :
  - Each vertex on that path has two paths attached, one path on each side of  $p_k$ .
  - The number of vertices that lie on a path attached to the  $j^{\text{th}}$  vertex of  $p_k$  is  $i - j$ .



Figure 1.2: The same contact graph as in figure 1.3 overlaid with the a perfectly weighted snowflake tree.

A *perfectly weighted snowflake tree* is a snowflake tree with all vertices having weight  $\frac{1}{2}$ . A *perturbed snowflake tree* is a snowflake tree with all vertices having weight of 1 with the exception of  $v_0$ ; in a perturbed snowflake tree,  $v_0$  will have a weight of  $\frac{1}{2} + \gamma$ . For our analysis, all realizations of any snowflake, perfect or perturbed, shall have  $v_0$  fixed at origin.

**Perfectly Weighted Snowflake Tree.** Consider the graph of the triangular lattice with unit distant edges:

$$\begin{aligned} V &= \left\{ a \cdot (1, 0) + b \cdot \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) : a, b \in \mathbb{Z} \right\} \\ E &= \{ \{u, v\} : \|u - v\| = 1 \text{ and } u, v \in V \} \end{aligned}$$

The following graph,  $G = (V, E)$  is said to be the *unit distance graph* of the triangular lattice. We can show that no two distinct edges of this graph are non-crossing. First suppose that there were two distinct edges that crossed,  $\{u_1, v_1\}$  and  $\{u_2, v_2\}$ . With respect to  $u_1$ , there are 6 possible edges corresponding to it, with each edge  $\frac{\pi}{3}$  radians away from the next. Neither edge crosses another; and so we have a contradiction that there are no edge crossings with  $\{u_1, v_1\}$ .

The perfectly weighted snowflake tree that is a subgraph over the *unit distance graph*,  $G = (V, E)$ , of the triangular lattice. To show this, for any  $S_i$ , fix  $v_0 = 0 \cdot (1, 0) + 0 \cdot \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) = (0, 0) \in V$  at origin. Next

consider the six paths attached from origin. Fix each consecutive path  $\frac{\pi}{3}$  radians away from the next such that the following points lie on the corresponding paths:  $(1, 0) \in p_1$ ,  $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in p_2$ ,  $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \in p_3$ ,  $(-1, 0) \in p_4$ ,  $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \in p_5$ ,  $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \in p_6$ . For  $S_i$ , there are  $i$  vertices on each path.

We define the six paths from origin as follows:

$$\begin{aligned} p_1 &= \{a \cdot (1, 0) = \vec{v} \mid a \in \mathbb{R}^+\} \\ p_2 &= \left\{a \cdot \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \vec{v} \mid a \in \mathbb{R}^+\right\} \\ p_3 &= \left\{-a \cdot (1, 0) + a \cdot \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = a \cdot \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \vec{v} \mid a \in \mathbb{R}^+\right\} \\ p_4 &= \{a \cdot (-1, 0) = \vec{v} \mid a \in \mathbb{R}^+\} \\ p_5 &= \left\{a \cdot \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = \vec{v} \mid a \in \mathbb{R}^+\right\} \\ p_6 &= \left\{a \cdot (1, 0) - a \cdot \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = a \cdot \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \mid a \in \mathbb{R}^+\right\} \end{aligned}$$

For  $S_i$  there exists  $i$  vertices on each path. We shall denote the  $i^{\text{th}}$  vertex on the  $j^{\text{th}}$  path as  $v_{j,i}$ . For each path defined above, the paths are defined as a set of vectors,  $\vec{v} = a \cdot \vec{p}$  for some  $a \in \mathbb{R}^+$  and  $\vec{p} \in \mathbb{R}^2$ . By setting  $a = 1, 2, \dots, i$ , we obtain points that are contained in  $V$ . For  $j = 1, 3, 5$  and  $l = 1, \dots, i$ , there exists two paths attached to each vertex  $v_{j,l}$ . We borrow the term *petiole* from botany to describe the two paths attached to  $v_{j,l}$ . In botany, the stalk that attaches to a stem of a plant is called a petiole; petioles usually have leaves attached to their ends. For  $S_i$ , each petiole attached to the  $k^{\text{th}}$  vertex of  $p_j$ , there are  $i - k$  vertices. We will need to show that each of the  $i - k$  vertices on each corresponding path are also in  $V$ .

The triangular lattice is symmetric under rotation about  $v_0$  by  $\frac{\pi}{3}$  radians. For each vertex  $v_{1,l}$  for  $l = 1, 2, \dots, i - k$ , we place two petioles from it; the first petiole  $\frac{\pi}{3}$  above  $p_1$  at  $v_{1,l}$  and  $\frac{-\pi}{3}$  below  $p_1$  at  $v_{1,l}$  and call these petioles  $p_{1,l}^+$  and  $p_{1,l}^-$  respectively. With respect to  $v_{1,l}$ , one unit along  $p_{1,l}^+$  is a point on the triangular lattice and similarly so on  $p_{1,l}^-$ . Continuing the walk along these paths, unit distance-by-unit distance, we obtain the next point corresponding point on the the triangular lattice up to  $i - k$  distance away from  $v_{1,l}$ . This shows that each of the  $i - k$  vertices on  $p_{1,l}^-$  and  $p_{1,l}^+$  are in  $V$ . By rotating all of the paths along  $p_1$  by  $\frac{2\pi}{3}$  and  $\frac{4\pi}{3}$ , we obtain the the paths along  $p_3$  and  $p_5$  respectively, completing the construction.

In Figure 1.3, we have a set of unit radius disks arranged in a manner that outlines regular, concentric hexagons.



Figure 1.3: A contact graph that resembles the shape of concentric hexagons.

### 1.2.1 Perturbed Weighted Trees $T_\varepsilon$

A perturbed weighted tree  $T_\varepsilon$  is a weighted unit tree with unit weight on every vertex with the exception of the root vertex having weight  $\frac{1}{2} + \varepsilon$  where  $\varepsilon > 0$  can be realized as a disk touching graph (a disk arrangement).

The perturbed snowflake follows the construction of the perfect snowflake with the exception of  $v_0$  having weight  $\frac{1}{2} + \varepsilon$  where  $\varepsilon > 0$ . A perturbed snowflake realization has some distinct qualities from perfect snowflake realizations. The angular relationships between adjacent vertices may vary; the distance between adjacent and neighboring vertices may vary as well.

In general, the perturbation  $\varepsilon$  can modify the realization of a perfect snowflake  $S_i$  in the following ways:

#### 1. Modification of $S_1$ .

Given an instance of a perturbed snowflake with  $v_0$  having weight  $\frac{1}{2} + \varepsilon$  where  $\varepsilon > 0$ , vertices neighboring  $v_0$  each have a range of placement on the plane when realized as a disk arrangement. Figure ?? shows a realization of  $S_1$  and illustrates one such example of possible gaps,  $\varepsilon$ , that could be created between adjacent disks of  $S_1$  in a perfect snowflake.

Note that (1) the adjacent disks in a perfect snowflake may or may not be adjacent in a given perturbed snowflake of  $S_1$  and (2)  $S_1 \subseteq S_i$  for any  $i \in \mathbb{N}$ .

**Lemma 2.** *For any realized perturbed snowflake  $S_i$ , the gaps created in subset  $S_1 \subset S_i$  are small.*

2. **Modification of disk placement corresponding to vertices of  $p_k$ .** We've shown how the disks can be displaced in  $S_1$ ; for larger snowflakes, the displacement can propagate through the remaining disks of the arrangement. The disks can along the paths  $p_1$  through  $p_6$  may also have displacement as well. In canonical position, the disks along paths  $p_1$  through  $p_6$  will form angles  $\alpha_k = \frac{\pi}{3}$  and  $\beta_k = \frac{2\pi}{3}$  (see Figure 1.4 for example). In noncanonical position, the disks along paths  $p_1$  through  $p_6$  will form angles that may vary.



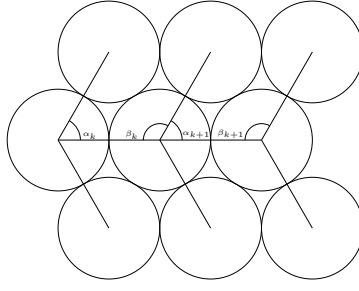


Figure 1.4: This figure shows a disk arrangement along a path  $p_k$  in canonical position. Note that perturbation in a snowflake and modify the placement of the disks in such a way that  $\alpha_k$  and  $\beta_k$  or  $\alpha_{k+1}$  and  $\beta_{k+1}$  may be of a noncanonical value.

Our goal here is to show that the change of the angular value of  $\alpha_k$  and  $\beta_k$  are small:

**Lemma 3.** *For any realized perturbed snowflake  $S_i$ , the angular value of  $\alpha_k$  and  $\beta_k$  are small.*

3. **Modification of disk placement along corresponding to the  $p_{k,j}^{\text{th}}$  vertex of  $S_i$ .** For the  $j^{\text{th}}$  disk along the  $k^{\text{th}}$  path  $p_{k,j}^+$  or  $p_{k,j}^-$ , perturbation can displace the position of the disk on the plane and the angular relationship of the neighboring disks (see Figure 1.5 for example).

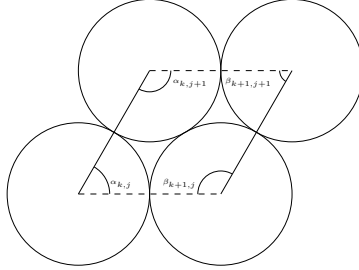


Figure 1.5: This illustration the relationship angular relationships between neighboring disks  $D_{k,j}$ ,  $D_{k+1,j}$ ,  $D_{k+1,j+1}$ , and  $D_{k,j+1}$ .

In this case we will show that the distances between the centers of neighboring disks are small relatively to canonical position of a perfect snowflake, i.e.

**Lemma 4.** *For any realized perturbed snowflake  $S_i$ , the distance between disks  $D_{k,j}$ ,  $D_{k+1,j}$ ,  $D_{k+1,j+1}$ , and  $D_{k,j+1}$  are relatively small with respect to the relative distance in a perfect snowflake where  $k = 1, \dots, 6$  and  $j = 2, \dots, i$ .*

We will show the proofs of Lemmas 2 through 4 later on.

### 1.3 Proofs of Lemmas 2 through 4

We now prove the Lemmas 2 through 4.

#### 1.3.1 Proof of Lemma 2

*Proof.* Recall that we are to show that for any realized perturbed snowflake  $S_i$ , the gaps created in subset  $S_1 \subset S_i$  are small.



Figure 1.6: A canonical disk arrangement from a perturbed snowflake with 6 unit disks around a central disk with radius  $\frac{1}{2} + \gamma$ .

One way to do this is to demonstrate that the sum of gaps for any realization of a contact graph of a perturbed snowflake  $S_1$  is small. Denote the vertices around  $v_0$  as  $v_1$  through  $v_6$  in a clockwise pattern about  $v_0$ . Without loss of generality, given a realization denote  $\epsilon_{k,k+1}(\gamma) \geq 0$  as the gap created between adjacent disks corresponding to  $v_1$  through  $v_6$ .

Consider the realization where  $\epsilon_{k,k+1}(\gamma) = 0$  with the exception of  $\epsilon_{1,6} > 0$ . That is, every consecutive pair of disks about the central disk is in contact with each other with the exception of  $D_1$  and  $D_6$ . The realization provides 5 congruent triangles between the centers of  $D_0$  and  $(D_1, D_2)$ ,  $(D_2, D_3)$ ,  $(D_3, D_4)$ ,  $(D_4, D_5)$ , and  $(D_5, D_6)$ . Given perturbation  $\gamma > 0$ , the side lengths between  $(D_0, D_i)$  are  $1 + \gamma$  and the side length of  $(D_i, D_{i+1})$  is 1. Using law of cosine the angle formed between  $(D_0, D_i)$  and  $(D_0, D_{i+1})$  is

$$2 \tan^{-1} \frac{1}{2(1+\gamma)}.$$

The angle between  $(D_6, D_1)$  is

$$y = 2\pi - 5 \cdot \left( 2 \tan^{-1} \frac{1}{2(1+\gamma)} \right).$$

The side length of  $(D_6, D_1)$  is

$$\sqrt{-2(\gamma+1)^2 \cos \left( -10 \arctan \left( \frac{1}{2(\gamma+1)} \right) \right) + 2(\gamma+1)^2}.$$

Note that as  $\gamma \rightarrow 0$ , the side length of  $(D_6, D_1)$  is approximately  $1 + .466861$ , where  $\epsilon(\gamma) \approx .466861$  as  $\gamma \rightarrow 0$ . This establishes an upperbound on the maximal displacement about  $S_1$  with respect to the side lengths between the centers of disks about  $D_0$ .

The lower bound is established using the configuration found in Figure 1.6. The realization provides 6 congruent triangles between the centers of  $D_0$  and each disk about  $D_0$ . Without loss of generality, to find the side length of between neighboring disks about  $D_0$ , we find need to  $\epsilon(\gamma)$ . The angle between  $(D_0, D_i)$  and  $(D_0, D_{i+1})$  is  $\frac{\pi}{3}$ ; using the law of cosine, we can determine the side length of  $(D_i, D_{i+1})$  is

$$\sqrt{1 + 2\gamma + \gamma^2}.$$

Thus the perturbation about  $S_1$  in and configuration is bounded and small.  $\square$

### 1.3.2 Proof of Lemma 3

*Proof.* Recall that we are to show that for any realized perturbed snowflake  $S_i$ , the angular value of  $\alpha_k$  and  $\beta_k$  are small. In canonical position, the angle between  $p_{k,j}^+$  and  $p_{k,j}^-$  is  $\frac{\pi}{3}$ . In a non-canonical position, we

define the change in angle to be  $f(\varepsilon)$ . About a We prove this with induction.



Figure 1.7

$$\begin{aligned}\alpha_i + \beta_i &\leq 120 + f(\varepsilon) \\ 2\pi &\leq \gamma_i + \delta_i + \frac{2\pi}{3} + f(\varepsilon) \\ 2\pi - (\gamma_i + \delta_i) &\leq \frac{2\pi}{3} + f(\varepsilon) \\ \alpha_{i+1} + \beta_{i+1} &\leq \frac{2\pi}{3} + f(\varepsilon)\end{aligned}$$

□

### 1.3.3 Proof of Lemma 4

*Proof.* Recall that we are to show that for any realized perturbed snowflake  $S_i$ , the distance between disks  $D_{k,j}$ ,  $D_{k+1,j}$ ,  $D_{k+1,j+1}$ , and  $D_{k,j+1}$  are relatively small with respect to the relative distance in a perfect snowflake where  $k = 1, \dots, 6$  and  $j = 2, \dots, i$ .



Figure 1.8

□

## **Bibliography**