CALIFORNIA STATE UNIVERSITY, NORTHRIDGE

PROTEIN FOLDING: PLANAR CONFIGURATION SPACES OF DISC ARRANGEMENTS AND HINGED POLYGONS

A thesis submitted in partial fulfillment of the requirements for the degree of
Master of Science in Applied Mathematics

by

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Table of Contents

ABSTRACT

PROTEIN FOLDING: PLANAR CONFIGURATION SPACES OF DISC ARRANGEMENTS AND HINGED POLYGONS

By

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Chapter 1

Realizability Problems for Weighted Trees

In this chapter our goal is to prove Theorem **??**: It is NP-Hard to decide whether a given tree with positive vertex weights is the contact graph of a disk arrangements with specified radii. This chapter's approach to proving Theorem **??** introduces an ordered weighted tree T, perturbed ordered weight tree T, the Hausdorff distance, and then prove a lemma which shows that hexagons can be approximated by an ordered disk contact graph corresponding to the weighted tree T.

1.1 Hausdorff Distance

Let A and B be sets in the plane. The *directed Hausdorff distance* is:

$$d\left(A,B\right) = \sup_{a\in A}\inf_{b\in B}||a-b|| \tag{1.1}$$

$d\left(A,B\right)$ finds the furthest point $a \in A$ from any point in B. *Hausdorff distance* is

$$D\left(A,B\right) = \max\left\{d\left(A,B\right), d\left(B,A\right)\right\} \tag{1.2}$$

In Figure 1.1, we have two sets X and Y and illustrate $d(X,Y)$ and $d(Y,X)$. From this, it is possible to calculate the Hausdorff distance between X and Y.

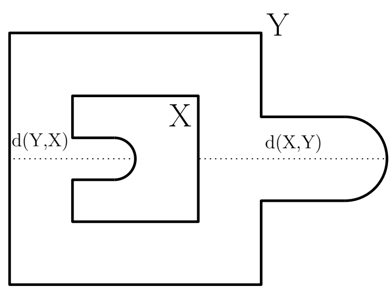

Figure 1.1: An illustrative example of $d(X,Y)$ and $d(Y,X)$ where X is the inner curve, and Y is the outer curve.

ϵ-approximation The weighted graph, G, is an ϵ-*approximation* of a polygon P if the Hausdorff distance between every realization of G as a contact graph of disks and a congruent copy of P is at most epsilon. A weighted graph G is said to be a $O\left(f\left(x\right)\right)$-*approximation* of a polygon P if there is a positive constant M such that for all sufficiently large values of x the Hausdorff distance between every realization such realization of G as a contact graph of disks and a congruent copy of P is at $M \cdot |f(x)|$. A weighted graph G is said to be a *stable* if it has the property that for every two such realizations of G, the distance between the centers of the corresponding disks is at most ϵ after a suitable rigid transformation.

Suppose we have a unit disk U and we have a grid overlayed on the disk with side length δ. Let $S_1(\delta)$ be the union of grid squares completely in the interior of U. Let $S_2(\delta)$ be the union of squares that intersect U. The Hausdorff distance of U and $S_1(\delta)$ is at most $H\left(S_1(\delta), U\right) \leq \sqrt{2}\delta$. Similarly, the Hausdorff distance of U and $S_2(\delta)$ is at most $H\left(S_2(\delta), U\right) \leq \sqrt{2}\delta$. For any $\epsilon > 0$, choose a δ such that $\sqrt{2}\delta \leq \epsilon$. Then the Hausdorff distance between U and $S_1(\delta)$, U and $S_2(\delta)$ is:

$$\begin{aligned} H\left(S_1(\delta), U\right) &\leq \sqrt{2}\delta \\ H\left(S_2(\delta), U\right) &\leq \sqrt{2}\delta \end{aligned}$$

Similiaryly, the Hausdorff distance of U and $S_2(\delta)$ is at most $H(S_2(\delta), U) = \sqrt{2}\delta$.

Lemma 1. *For every $\epsilon > 0$ and $x > 0$, there exists an ordered weighted tree T and regular hexagon h of side length x such that:*

1. *Every realization σ_i of T as an ordered disk contact graph where the radii of the disks equal the vertex weights, approximates the hexagon in the sense that:*

$$H(h, \sigma) \leq \epsilon$$

2. *The number of nodes in T and the weights are polynomial in ϵ and x, the weights $\frac{\epsilon}{10}$ and $\frac{\epsilon}{10} + \zeta$ are polynomial.*

Lemma 1 is rather restrictive; it begs the question of whether it can be generalized. Some ways this lemma could be generalized are: (1) if all weights are equal, (2) order does not matter, and (3) if we replace the regular hexagon to an arbitrary polygon.

1.2 Weighted Trees T_i

In this section we describe a particular family of unit weight trees and corresponding ordered weight contact graphs called *snowflakes*. The disks of a snowflake have weight r. For $i \in \mathbb{N}$, the construction of the snowflake tree, T_i, is as follows:

- Let v_0 be a vertex that has six paths attached to it: $p_1, p_2, \ldots, p_6$. Each path has i vertices. These paths are called *stems*.
- In botany, the stalk that attaches to a stem of a plant is called a *petiole*; petioles usually have leaves attached to their ends. Our snowflake will borrow the nomenclature from botany as the snowflake structure is analogous to a plant-like structure. Petioles will be paths that extend off the stems. *Leaves* will be paths of one vertex that extend off the petioles and stems. We will now attach paths (petioles) onto every other stem p_1, p_3, and p_5:
  - Every third vertex on that path has two petioles attached, one petiole on each side of p_k; otherwise the vertex on p_k will have two leaves, one on either side. The last node has no petioles.
  - The number of vertices that lie on a petiole attached to the j^{th} vertex of p_k is $i - j$.
  - The first and last vertices of the $i - j$ vertices has one leaf attached; the remaining $i - j - 2$ vertices has two leaves.
- Attach leaves to the vertices on paths p_2, p_4, and p_6.

A drawaing example of this snowflake description is shown in Figure 1.2

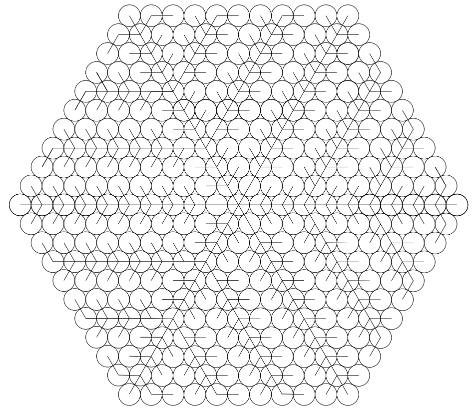

Figure 1.2: A contact graph that resembles the shape of concentric hexagons.

A *perfectly weighted snowflake tree* is a snowflake tree with all vertices having weight r. A *perturbed snowflake tree* is a snowflake tree with all vertices having weight of r with the exception of v_0; in a perturbed snowflake tree, v_0 will have a weight of $r + \zeta$. The value of ζ will be determined later on in the proof of the Lemma 1. For our analysis, all realizations of any snowflake, perfect or perturbed, shall have the disk corresponding to v_0 centered at the origin. We can assume that a neighbor of v_0 is on the positive x-axis. For the remainder of the thesis, denote S_i as the canonical realization of T_i and σ_i as an arbitrary realization of T_i.

Perfectly Weighted Snowflake Tree. Consider the graph of the triangular lattice with unit distant edges:

$$\begin{aligned} V &= \left\{ a \cdot (1,0) + b \cdot \left(\frac{1}{2}, \frac{\sqrt{3}}{2} \right) : a, b \in \mathbb{Z} \right\} \\ E &= \{\{u, v\} : ||u - v|| = 1 \text{ and } u, v \in V\} \end{aligned}$$

The graph, $G = (V, E)$ is said to be the *unit distance graph* of the triangular lattice. We can show that no two distinct edges cross. First suppose that there were two distinct edges that crossed, $\{u_1, v_1\}$ and $\{u_2, v_2\}$. By strict triangle inequality of the sides and diagonals of the convex quadrilateral (u_1, u_2, v_1, v_2), we have the following result:

$$|u_1, u_2| + |u_2, v_1| + |v_1, v_2| + |u_1, v_2| < 2\left(|u_1, u_2| + |u_1, u_2|\right) = 4.$$

On the lefthand side, one of the four terms is less than 1. No two points of the triangular lattice is less than 1 which is a contradiction.

The perfectly weighted snowflake tree that is a subgraph over the *unit distance graph*, $G = (V, E)$, of the triangular lattice. For the remainder of the thesis, a *snowflake*, the ordered contact graph of S_i contains the tree T_i. To show this, for any S_i, fix $v_0 = 0 \cdot (1,0) + 0 \cdot \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = (0,0) \in V$ at origin. Next consider the six paths attached from origin. Fix each consecutive path $\frac{\pi}{3}$ radians away from the next such that the following points like on the corresponding paths: $(1,0) \in p_1$, $\left(\frac{1}{2}, \frac{\sqrt{2}}{3}\right) \in p_2$, $\left(-\frac{1}{2}\mathsf{p}_4, \frac{\sqrt{3}}{2}\right) \in p_3$, $(-1,0) \in p4$, $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \in p_5$, $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) \in p_6$. For S_i, there are i vertices on each path.

We define the six paths from origin as follows:

$$\begin{aligned}
p_1 &= \{\, r \cdot (1,0) \,|r = 1,2,\ldots,i \,\} \\
p_2 &= \left\{ r \cdot \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) |r = 1,2,\ldots,i \right\} \\
p_3 &= \left\{ r \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) |r = 1,2,\ldots,i \right\} \\
p_4 &= \{\, r \cdot (-1,0) \,|r = 1,2,\ldots,i \,\} \\
p_5 &= \left\{ r \cdot \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) |r = 1,2,\ldots,i \right\} \\
p_6 &= \left\{ r \cdot \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) |r = 1,2,\ldots,i \right\}
\end{aligned}$$

For S_i there exists i vertices on each path. We shall denote the i^{th} vertex on the j^{th} path as $v_{j,i}$. For each path defined above, the paths are defined as a set of vectors, $\vec{v} = r \cdot \vec{p}$ for some $r \in \mathbb{N}$ and $\vec{p} \in \mathbb{R}^2$. By setting $a = 1, 2, \ldots, i$, we obtain points that are contained in V. For $j = 1$, 3, 5 and $\ell = 3b \leq i$ where $b \in \mathbb{N}$, there exists two paths attached to each vertex $v_{j,\ell}$. For S_i, each petiole attached to the ℓ^{th} vertex of p_j, there are $i - \ell$ vertices. For each vertex v on a petiole, which is not in the paths p_1, p_3, or p_5, there are two *leaves* on either side of the vertex; each leaf is a vertex that has an edge with v. The exceptions to the two leaves rule is on the first and last vertices of the petiole off of p_1, p_3, or p_5. In these exception, attach one leaf to the side of the vertex that is closest to center vertex v_0.

The triangular lattice is symmetric under rotation about v_0 by $\frac{\pi}{3}$ radians. For each vertex $v_{1,l}$ and $l = 3b \leq i$ where $b \in \mathbb{N}$, we place two petioles from it; the first petiole $\frac{\pi}{3}$ above p_1 at $v_{1,l}$ and $\frac{-\pi}{3}$ below p_1 at $v_{1,l}$ and call these petioles $p_{1,l}^{+}$ and $p_{1,l}^{-}$ respectively. With respect to $v_{1,l}$, one unit along $p_{1,l}^{+}$ is a point on the triangular lattice and similarly so on $p_{1,l}^{-}$. Without loss of generality, for each vertex v of the petiole which are not in p_1 has two associated leaf nodes v^+ and v^-; v^+ is placed $\frac{\pi}{3}$ and one unit above v and v^- is placed $\frac{-\pi}{3}$ and one unit below v. Thus all leaf nodes are in the triangular lattice. This shows that each of the $i - k$ vertices on $p_{1,l}^{-}$, $p_{1,l}^{+}$, and leaves are in V. By rotating all of the paths along p_1 by $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, we obtain the paths p_3 and p_5 respectively, completing the construction.

In Figure 1.2, we have a set of unit radii disks arranged in a manner that outlines the perfectly weighted snowflake description above.

1.2.1 Perturbed Weighted Trees T

In a perfect snowflake there are contacts in the disk arrangement that do not reflect the corresponding edge set of the tree. The contact graph is not a tree. By increasing the weight of the center disk corresponding to v_0, the disks around the central disk will have additional degrees of freedom and removes unintended contacts. This claim is shown as a result of Lemma 2 later. We refer to the positions of the disks of S_i as the *canonical arrangement* of S_i.

Given $\epsilon > 0$ and $x > 0$, we define T as follows: the tree T_i with weight $\frac{\epsilon}{10}$ for every vertex except v_0; v_0 has a weight $\frac{\epsilon}{10} + \zeta$ for some $\zeta > 0$ that is specified later. For T_i, let

$$r = \tfrac{\epsilon}{10} \quad i = \left\lceil \tfrac{10x}{\epsilon} \right\rceil \ .$$

A perturbed weighted tree T can be realized as an ordered disk contact graph (a disk arrangement). A perturbed snowflake realization has some distinct qualities from the perfect snowflake ordered contact graph of S_i. The angular relationships between adjacent vertices may vary.

Modification of σ_1. We show that for any realization of the ordered tree T the placement of vertices is close to canonical position. In order to show this, we show the components of a perturbed snowflake in arbitrary position are close to canonical position. The argument comprises of three parts: (1) Showing that the pertubation of the central disk and the six neighboring disks is small, (2) show that the displacement along the stems for all S_i is small, and (3) show that the displacement along the petioles is small. Given a instance of a perturbed snowflake with v_0 having weight $\epsilon + \zeta$ where $\epsilon > 0$, vertices neighboring v_0 each have a range of placement on the plane when realizaed as a disk arrangement.

Displacement on σ_1 is small. Note that (1) the adjacent disks in a perfect snowflake cannot be adjacent in a given perturbed snowflake of S_1 and (2) $S_1 \subseteq S_i$ for any $i \in \mathbb{N}$ because every contact is encoded in the tree. Given a snowflake in arbirary position with i nodes per stem; the edge length of each segment is $2r$ except the edges having node v_0, these edges are length $2r + \zeta$. The stem has a length of at most $2ri + \zeta$. Figure 1.3 shows an stem of a tree in arbitrary position corresponds to a compression and shift of vertices. The stem realized in arbitrary position in Figure 1.3 is analgous to a tree realized in arbitrary position where vertices are in a different position than canonical. Our goal is to show that for any $\epsilon > 0$ and $x > 0$, the position of the center v of any disk in the disk arrangement in any arbitrary realization corresponding to T has a small displacement, i.e. v is found in an open ball referenced at the canonical position of v, v_c:

$$v \in b_{\psi_1}(v_c).$$

The definition of ψ_1 is shown later on.

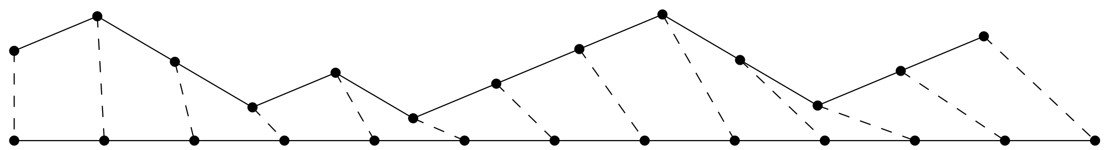

Figure 1.3: The polyline at the bottom represents a snowflake stem in canonical position. The polyline above represents a snowflake stem in non-canonical position.

In S_1 the six disks around the central disk kiss. The angle formed from the center of the central disk to the centers of any two adjacent disks is $\frac{\pi}{3}$. The side lengths of the equalateral triangle formed by the centers of three adjacent disks, one of which is the central disk, is $\frac{\epsilon}{5}$. For a perturbed S_1, the the central disk is weighted $\frac{\epsilon}{10} + \zeta$. This can yield a change of angular displacement $\frac{\pi}{3}$ to $\frac{\pi}{3} \pm 2\chi$. To find the bounds of how large or small χ can be, we show a trigonometric relation of the half angle of the triangle corresponding to three adjacent disks (See Figure 1.4).

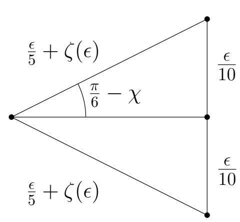

Figure 1.4: This figure depicts a triangle corresponding to the center of the central disk and two adjacent disks.

$$
\begin{aligned}
\sin\left(\frac{\pi}{6}-\chi\right) &= \frac{\frac{\epsilon}{10}}{\frac{\epsilon}{5}+\zeta} \\
\frac{1}{2}\cos\chi = \sin\frac{\pi}{6}\cos\chi &= \frac{\frac{\epsilon}{10}}{\frac{\epsilon}{5}+\zeta}+\cos\frac{\pi}{6}\sin\chi = \frac{\frac{\epsilon}{10}}{\frac{\epsilon}{5}+\zeta}+\frac{\sqrt{3}}{2}\sin\chi \\
&\Rightarrow \\
\frac{\frac{\epsilon}{10}}{\frac{\epsilon}{5}+\zeta}+\frac{\sqrt{3}}{2}\left(\chi-\frac{\chi^3}{6}\right) &\leq \frac{\frac{\epsilon}{10}}{\frac{\epsilon}{5}+\zeta}+\frac{\sqrt{3}}{2}\sin\chi = \frac{1}{2}\cos\chi \\
&\Rightarrow \\
\frac{\sqrt{3}}{2}\left(\chi-\frac{\chi^3}{6}\right) &\leq \frac{1}{2}-\frac{\frac{\epsilon}{10}}{\frac{\epsilon}{5}+\zeta} \qquad \text{if } \chi<1 \\
\frac{5\sqrt{3}}{12}\chi &\leq \frac{\zeta}{\frac{2\epsilon}{5}+2\zeta} \\
\chi &\leq \frac{12}{5\sqrt{3}}\frac{\zeta}{\frac{2\epsilon}{5}+2\zeta} = \frac{6\zeta}{(\epsilon+5\zeta)\sqrt{3}}
\end{aligned}
$$

For any $\epsilon > 0$, the bounds for angular displacement formed at the center of the central disk and two adjacent disks is:

$$\frac{\pi}{3}-\frac{12\zeta}{(\epsilon+5\zeta)\sqrt{3}} \leq \frac{\pi}{3}-2\chi \leq 2\chi \leq \frac{\pi}{3}+2\chi \leq \frac{\pi}{3}+\frac{12\zeta}{(\epsilon+5\zeta)\sqrt{3}}$$

This shows the bounds of the angular displacement around the central disk. To translate this to a coordinate displacement of the centers of the disks around the central disk, we compute a the radius of the ball of radius ψ in which the centers may lie.

$$\psi = (2r+\zeta)\sin\chi$$

Using the Maclaurin series of $\sin\chi$:

$$
\begin{aligned}
\psi_i &= \left(\tfrac{\epsilon}{5}+\zeta\right)\sin\chi \\
&\leq \left(\tfrac{\epsilon}{5}+\zeta\right)\tfrac{6\zeta}{(\epsilon+5\zeta)\sqrt{3}} \\
&= \tfrac{6\zeta}{5\sqrt{3}}
\end{aligned}
$$

The coordinates of the centers of the disks are close to cannonical position following the angular displacement argument above. That is, if u is a center of a disk in a realization to T, u lies in a ball $b_{\psi_1}(u_c)$ where u_c is the cannonical position of the u.

Displacement on the stems is small. Without loss of generality, consider a path of vertices along a distinct stem, petiole, or leaf $(v_1, v_2, \ldots, v_n)$. In the case that there are two paths attached to each vertex, four different angles about the vertex are formed. For the i^{th} vertex, the counter clockwise order of the angles are $(\alpha_{i+1}, \gamma_i, \delta_i, \beta_{i+1})$ (See Figure 1.6 for reference). We want to establish an angular relationship between two consecutive vertices in a path.

Lemma 2. *Let (a, b, c, d) be a polygonal path in the plane such that the unit disks centered at a, b, c, and d are interior-disjoint. Then the sum of the two angles on each side at the two interioir vertices is greater than π. Then the sum of the two angles on each side at the two interior vertices is greater than π.*

Proof. Without loss of generality, consider the two angles on the left side at the two interioir vertices, $\angle abc$ and $\angle bcd$. We have $|ab| = |bc| = |cd| = 2$, since we have a disk arrangement of unit disks. If (a, b, c, d) is a rhombus, then $|ad| = 2$ and $\angle abc + \angle bcd = \pi$. Hence $|ad| > 2$ implies $\angle abc + \angle bcd > \pi$. ☐

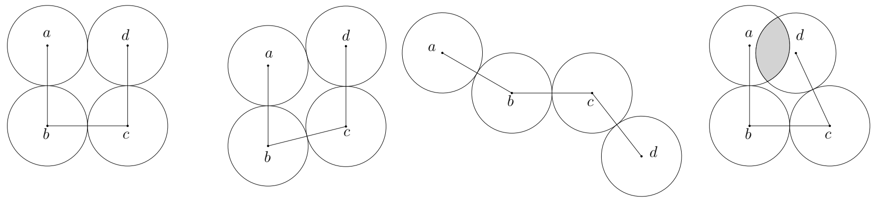

Figure 1.5: (a)Four disks centered along a polygonal path (a, b, c, d) in various drawings.

We can now say that for any two consecutive vertices (v_i, v_{i+1}) along a path, the following angular relationship from Lemma 2:

$$\begin{aligned} \pi &< \alpha_i + \gamma_i \\ \pi &< \beta_i + \delta_i \end{aligned}$$

This result shows that a perturbed snowflake removes the issue of having unintended constacts that do not reflect a give tree's edge relations that a perfect snowflake has.

In Figure 1.6(a), we have a parallelogram with four unit disks centered, one on each of the vertices of the parallelogram. If $\alpha_i + \gamma_{i+1} < \pi$, One of the disks will overlap with another. For any arbitrary realization of a perturbed snowflake, any two consecutive angles formed between two adjacent vertices along a path must satisfy the following constraint

$$\alpha_i + \gamma_{i+1} \geq \pi.$$

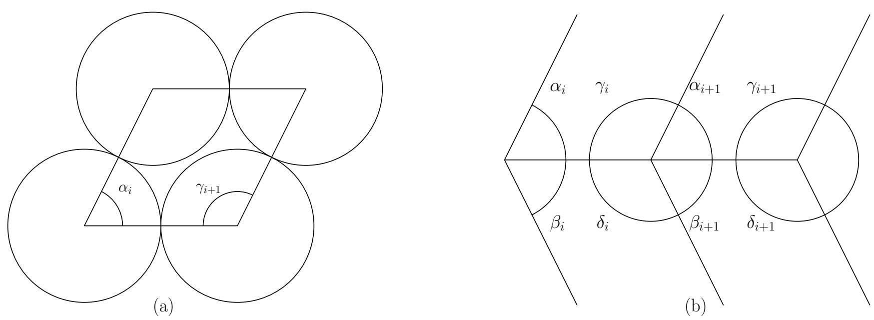

Figure 1.6: (a)Four disks of the snowflake shown where the top two disks can either be leaves off petioles or off the stems. (b)An stem depicted at the i^{th} and $(i+1)^{\text{st}}$ vertex.

To show that the angluar displacement along the stem is small, we extend the angular argument on the perturbed S_1 and by induction, show that it is small for all i. Denote the angles on the concave side of the i^{th} vertex as α_i and β_i and the convex side of the $(i+1)^{\text{st}}$ vertex as γ_i and δ_i respectively (see Figure 1.6(b) for reference). For any vertex, the sum of angles about the vertex is 2π, e.g.:

$$\gamma_i + \delta_i + \alpha_{i+1} + \beta_{i+1} = 2\pi$$

Suppose we numbered the disks about the central disk 1 through 6. Without loss of generality, the angles α_0 and β_0 correspond to the angles formed between the central angle, disks i and $i+1$ and disks $i+1$ and $i+2$ respectively, for $i = 1, 2, 3$. The bounds for α_0 and β_0 are the same as 2χ in the earlier argument, i.e.:

$$\begin{aligned} \tfrac{\pi}{3} - \tfrac{12\zeta}{(\epsilon+5\zeta)\sqrt{3}} \leq \tfrac{\pi}{3} - 2\chi \quad &\leq \quad \alpha_0 \quad \leq \quad \tfrac{\pi}{3} + 2\chi \leq \tfrac{\pi}{3} + \tfrac{12\zeta}{(\epsilon+5\zeta)\sqrt{3}} \\ \tfrac{\pi}{3} - \tfrac{12\zeta}{(\epsilon+5\zeta)\sqrt{3}} \leq \tfrac{\pi}{3} - 2\chi \quad &\leq \quad \beta_0 \quad \leq \quad \tfrac{\pi}{3} + 2\chi \leq \tfrac{\pi}{3} + \tfrac{12\zeta}{(\epsilon+5\zeta)\sqrt{3}} \end{aligned}$$

For $i = 0$ we know that $\alpha_0 + \beta_0 \leq \frac{2\pi}{3} + \frac{48\zeta}{5\sqrt{3}}$. For the inductive step, suppose the following:

$$\begin{aligned} \alpha_i + \beta_i &\leq \tfrac{2\pi}{3} + \tfrac{24\zeta}{(\epsilon+5\zeta)\sqrt{3}} \\ \pi &\leq \alpha_i + \gamma_i \\ \pi &\leq \beta_i + \delta_i \end{aligned}$$

$$\begin{aligned} \alpha_i + \beta_i &\leq \tfrac{2\pi}{3} + \tfrac{24\zeta}{(\epsilon+5\zeta)\sqrt{3}} \\ \pi &\leq \alpha_i + \gamma_i \\ \pi &\leq \beta_i + \delta_i \end{aligned}$$

Together, we have the following result:

$$\begin{aligned} 2\pi &= \alpha_i + \gamma_i + \beta_i + \delta_i \\ &= \alpha_i + \gamma_i + (2\pi - \alpha_{i+1} - \beta_{i+1}) \\ &\Longleftrightarrow \\ \alpha_{i+1} + \beta_{i+1} &= \alpha_i + \gamma_i \\ &\leq \frac{2\pi}{3} + \frac{24\zeta}{(\epsilon + 5\zeta)\sqrt{3}} \end{aligned}$$

And so the error bounds on $\frac{\pi}{3} + 2\chi$ hold in general for α_i and β_i for all i.

$$\alpha_i + \beta_i \leq \frac{2\pi}{3} + \frac{24\zeta}{(\epsilon + 5\zeta)\sqrt{3}}$$

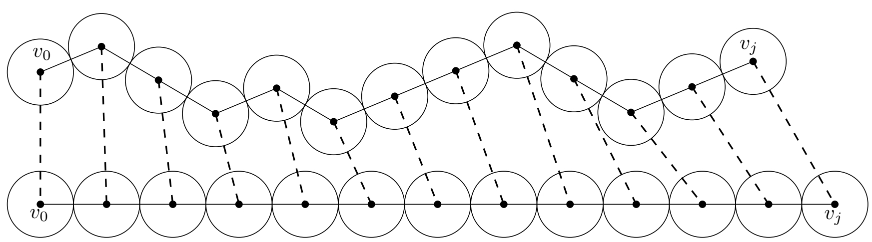

Figure 1.7: This figure illustrates the change of position of the centers of a disk in an arbitrary positin and in a straight, canoncial position.

A perturbed snowfloke σ_i is a realization of T. Let v be a vertex in T. There exists a unique path consisting of j edges from v_0 to v. We can compute a bound of coordinate displacement ψ_i. We want to establish a bound of coordinate displacement for every vertex in T. We extablish a bound for the path from v_0 to v by induction on j, the number of edges on the path from v_0 to v. The base case is v_0 which the displacement is zero suppose ψ_j is true.

$$\psi_j = (2j - 2)\left(\frac{\epsilon}{10} + \zeta\right)\sin\chi$$

Let $\zeta \leq \epsilon$. Using the Maclaurin series of $\sin \chi$:

$$\begin{aligned} \psi_j &= (2j-2)\left(\frac{\epsilon}{10}+\zeta\right)\sin\chi \\ &\leq (2j-2)\left(\frac{\epsilon}{10}+\zeta\right)\frac{6\zeta}{(\epsilon+5\zeta)\sqrt{3}} \\ &\leq \left(\frac{(j-1)\epsilon}{5}+\epsilon\right)\frac{6\zeta}{\epsilon\sqrt{3}} \\ &\leq \frac{6(i-1)\zeta+30\zeta}{5\sqrt{3}} = \frac{\zeta(6i-29)}{5\sqrt{3}} \end{aligned}$$

ϕ_i bound is true for almost all vertices but the outermost leaves of the petioles with additional freedom of movement (see Figure 1.8 for reference). By choosing $\zeta = \frac{5\epsilon\sqrt{3}}{2(6i-29)}$, we have that ζ is polynomial in x and ϵ since i is polynomial in x and ϵ.

Displacement on the petioles is small. Note that the petioles have the same geometric structure as the stems; the exception is the number of leaves on each side of the petioles. Since we've shown that the geometric shape in arbirary position is already close to canonical position for any $\epsilon > 0$, the same argument applies here for the petioles.

We have shown the displacements of all components of the perturbed snowflake are small for any $\epsilon > 0$. This shows that the structure has stability in preserving any information encoded with it.

We should note that the last leaves of the petioles have a freedom of motion (see Figure 1.8).

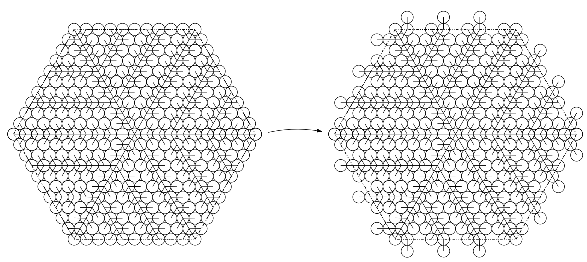

Figure 1.8: This illustrates how a perfect snowflake's outermost leaves on the petioles have a degree of freedom to move about the last vertex of the petiole.

Recall lemma 1 states:
For every $\epsilon > 0$ and $x > 0$, there exists an ordered weighted tree T and regular hexagon h of side length x such that:

1. Every realization σ_i of T as an ordered disk contact graph where the radii of the disks equal the vertex weights, approximates the hexagon in the sense that:

$$H(h, \sigma_i) \leq \epsilon$$

2. The number of nodes in T and the weights are polynomial in ϵ and x, the weights $\frac{\epsilon}{10}$ and $\frac{\epsilon}{10}+\zeta$ are polynomial.

We now begin the proof of Lemma 1.

Proof of Lemma 1. Given $\epsilon > 0$ and $x > 0$, we construct T in the following manner: define i and the perturbed weight of the central disk for T as:

$$\begin{aligned} i &= \left\lceil \frac{10x}{\epsilon} \right\rceil \\ r &= \frac{\epsilon}{10}. \end{aligned}$$

For any σ_i, overlay the hexagon h as the convex hull of the centers of the disks in the canonical arrangement S_i (see Figure 1.9 for reference).

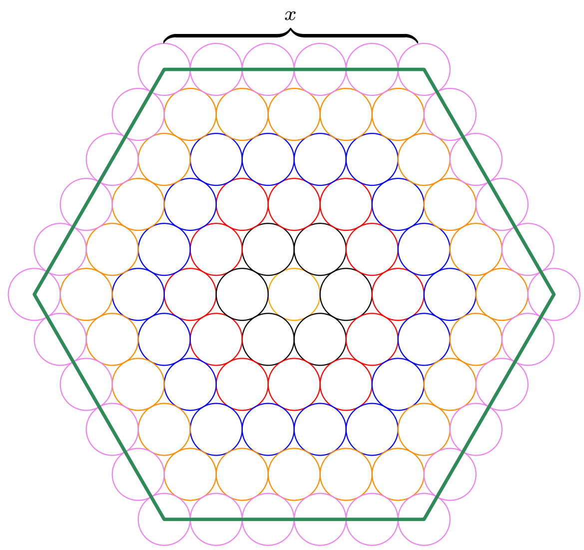

Figure 1.9: A regular hexagon of side length x as the convex hull of the centers of a disk arrangement in canonical position.

We use the triangle inequality to show that every realization σ_i of T as an ordered disk contact graph where the radii of the disks equal the vertex approximates the hexagon h such that $H(\sigma_i, h) \leq \epsilon$:

$$\begin{aligned} H(h, \sigma_i) &\leq H(h, S_i) + H(S_i, \sigma_i) \\ &= \max\{d(h, S_i), d(S_i, h)\} + \max\{d(\sigma_i, S_i), d(S_i, \sigma_i)\} \end{aligned}$$

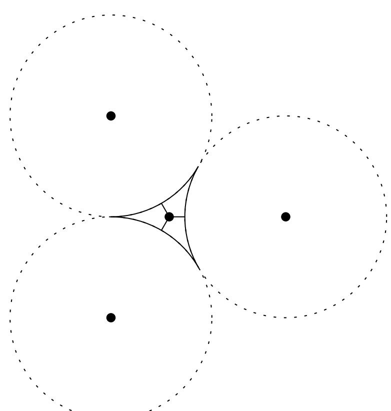

Figure 1.10: An illustration of the center point in the hexagon h that lies between three adjacent disks of S_i. The distance $d(h, S_i)$ is the length of the line segment from the center point to nearest point of either disk's boundary.

The distance $d(h, S_i) = \frac{(2-\sqrt{3})r}{\sqrt{3}}$ is from the center point in the hexagon h that lies between three adjacent disks of S_i to the nearest point of either disk's boundary (see Figure 1.10). The distance $d(S_i, h) = r$ is the distance from the furthest boundary point of any disk in S_i that lies on the exterior of the hexagon h (see the left illustration in Figure 1.8). The distance $d(S_i, \sigma_i) \leq \psi_i + 2r$ is the distance from the furthest point on the boundary of an outermost leaf in σ_i to the nearest boundary point of S_i, a distance of at most $2r$, and the additional coordinate displacement ψ_i. The distance $d(\sigma_i, S_i)$ is simply the coordinate displacement ψ_i. We can continue the inequality:

$$
\begin{aligned}
H(h, \sigma_i) &\leq H(h, S_i) + H(S_i, \sigma_i) \\
&= \max\{d(h, S_i), d(S_i, h)\} + \max\{d(\sigma_i, S_i), d(S_i, \sigma_i)\} \\
&\leq \max\left\{\frac{(2-\sqrt{3})r}{\sqrt{3}}, r\right\} + \max\{\psi_i, \psi_i + 2r\} \\
&\leq 3r + \psi_i \\
&= 3\frac{\epsilon}{10} + \frac{\zeta(6i-29)}{5\sqrt{3}} \\
&\leq \frac{3\epsilon}{10} + \frac{5\epsilon\sqrt{3}}{2(6i-29)}\frac{(6i-29)}{5\sqrt{3}} \\
&\leq \frac{8\epsilon}{10} \\
&\leq \epsilon
\end{aligned}
$$

We now show that the number of nodes in T is polynomial in ϵ and x. With $i = \left\lceil \frac{10x}{\epsilon} \right\rceil$, the longest path of nodes on a tree from v_0 is to the last leaf of any petiole. This path contains i edges. The number of nodes in the diameter and including the last leaf of a petiole of S_i corresponding to T is $2(i+1)$. By squaring the node count of the diameter, we have a generous upper bound of nodes in T, $4(i+1)^2$. This upper bound is a polynomial in x and ϵ since i is defined as a polynomial in x and ϵ.

□

Proof of Theorem ??

Proof. Given an instance of a P3SAT boolean formula, we can use the snowflake reduction of the modified auxiliary construction. Using Lemma 1, we can approximate any hexagon of a given side length with a tree T. In the modified auxiliary construction in Chapter 3, we had four types of hexagons with different side lengths and the skinny rhombus. We can scale the weights (radii) of the corresponding ordered weighted disk contact graph corresponding to T to the rigid frame, obstacle, flag, and half sized hexagons in a modified auxiliary contruction accordingly. The rhombus can be approximated by a chain of disks. The functionality of the gadgets in the modified auxiliary construction remain the same.

By approximating the polygons in the modified auxiliary construction with the snowflake, we show that Theorem **??** is a corollary by applying Lemma 1. □

References