2.) multiple ways to solve this problem:

1: assume some initial distribution $T(x, y, t = 0)$, advance forward in time until system reaches steady state (when $\partial T/\partial t = 0$ across domain)

advection-diffusion equation: $$\frac{\partial T}{\partial t} = -v\frac{\partial T}{\partial x} + \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

separately, advection and diffusion equations can be solved

$$\frac{\partial T}{\partial t} = -v\frac{\partial T}{\partial x} \Rightarrow T^{n+1} = \mathcal{U}^a(T^n) \qquad \frac{\partial T}{\partial t} = \kappa\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \Rightarrow T^{n+1} = \mathcal{U}^d(T^n)$$

($\mathcal{U}^x$ is some scheme that updates T^n to T^{n+1})

strategy to solve combined advection-diffusion equation:

solve advection piece: $\widetilde{T} = \mathcal{U}^a(T^n)$

use perturbed $\widetilde{T}$ to solve diffusion piece: $T^{n+1} = \mathcal{U}^d(\widetilde{T})$

alternating direction implicit algorithm for diffusion equation:

$x = x_i : \ i = 1, ..., N \ ; \ y = y_j : \ j = 1, ..., K$

$$\frac{T^{n+1/2}_{i,j} - T^n_{i,j}}{\Delta t/2} = \frac{\kappa}{\Delta^2}\left(\delta^2_x T^{n+1/2}_{i,j} + \delta^2_y T^n_{i,j}\right); \qquad \frac{T^{n+1}_{i,j} - T^{n+1/2}_{i,j}}{\Delta t/2} = \frac{\kappa}{\Delta^2}\left(\delta^2_x T^{n+1/2}_{i,j} + \delta^2_y T^{n+1}_{i,j}\right)$$

where: $\delta^2_x T_{i,j} = T_{i+1,j} - 2T_{i,j} + T_{i-1,j}$; $\quad \delta^2_y T_{i,j} = T_{i,j+1} - 2T_{i,j} + T_{i,j-1}$

$\Delta x = \Delta y = \Delta$

let $\alpha = \kappa\Delta t/\Delta^2$

$$T^{n+1/2}_{i,j} - T^n_{i,j} = \frac{\alpha}{2}\left(T^{n+1/2}_{i+1,j} - 2T^{n+1/2}_{i,j} + T^{n+1/2}_{i-1,j} + T^n_{i,j+1} - 2T^n_{i,j} + T^n_{i,j-1}\right)$$

$$-\frac{\alpha}{2}T^{n+1/2}_{i+1,j} + (1+\alpha)T^{n+1/2}_{i,j} - \frac{\alpha}{2}T^{n+1/2}_{i-1,j} = \frac{\alpha}{2}T^n_{i,j+1} + (1-\alpha)T^n_{i,j} + \frac{\alpha}{2}T^n_{i,j-1}$$

let $l = (j-1)N + i$

$$-\frac{\alpha}{2}T^{n+1/2}_{l+1} + (1+\alpha)T^{n+1/2}_{l} - \frac{\alpha}{2}T^{n+1/2}_{l-1} = \frac{\alpha}{2}T^n_{l+N} + (1-\alpha)T^n_{l} + \frac{\alpha}{2}T^n_{l-N}$$

this is a tridiagonal system: $A \cdot x = d$

$$\begin{bmatrix} b_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ & & a_{L-1} & b_{L-1} & c_{L-1} \\ & & & a_L & b_L \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{L-1} \\ x_L \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_{L-1} \\ d_L \end{bmatrix}$$

where: $a_l = c_l = -\dfrac{\alpha}{2}$; $\quad b_l = 1+\alpha$; $\quad x_l = T^{n+1/2}_l$; $\quad d_l = \dfrac{\alpha}{2}T^n_{l+N} + (1-\alpha)T^n_l + \dfrac{\alpha}{2}T^n_{l-N}$

Dirichlet boundary conditions are straight forward to implement

$$
\begin{aligned}
\text{for } \; i = 1: \quad & T_l^{n+1/2} = T_l^n = 65 \\
& a_l = c_l = 0; \qquad\qquad b_l = 1; \qquad\qquad d_l = T_l^n \\
\text{for } \; i = N: \quad & T_l^{n+1/2} = T_l^n = 25 \\
& a_l = c_l = 0; \qquad\qquad b_l = 1; \qquad\qquad d_l = T_l^n \\
\text{for } \; j = K: \quad & T_l^{n+1/2} = T_l^n = 25 \\
& a_l = c_l = 0; \qquad\qquad b_l = 1; \qquad\qquad d_l = T_l^n
\end{aligned}
$$

for Neumann b.c. at $j = 1$, imagine a row of ghost zones exists outside the domain at y_j with $j = 0$

$$
\begin{aligned}
& \frac{\partial T}{\partial y} = 0 \;\rightarrow\; \frac{T_{i,j+1} - T_{i,j-1}}{2\Delta} = 0 \;\Rightarrow\; T_{i,0} = T_{i,2} \\
& -\frac{\alpha}{2} T_{i+1,1}^{n+1/2} + (1+\alpha) T_{i,1}^{n+1/2} - \frac{\alpha}{2} T_{i-1,1}^{n+1/2} \;=\; \alpha T_{i,2}^n + (1-\alpha) T_{i,1}^n
\end{aligned}
$$

$$
\text{therefore: } \; a_l = c_l = -\frac{\alpha}{2}; \qquad b_l = 1 + \alpha; \qquad d_l = \alpha T_{l+N}^n + (1-\alpha) T_l^n
$$

use Thomas algorithm to find $T^{n+1/2}$. a similar analysis can be performed to construct a tridiagonal equation for the second half of the timestep, mapping $T_{i,j} \rightarrow T_m$ with $m = (i-1)K + j$. integrate forward in time until $|T^{n+1} - T^n|$ is less than some tolerance at all points throughout the domain

\# 2: solve advective and diffusive terms both through ADI.

$$
v \frac{\partial T}{\partial x} = \kappa \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)
$$

during 1st half step treat the derivatives w.r.t. x as implicit and the derivatives w.r.t. y as explicit. let $\beta = 2\kappa / v\Delta$

$$
\begin{aligned}
T_{i+1,j}^{n+1/2} - T_{i-1,j}^{n+1/2} - \beta \left(T_{i+1,j}^{n+1/2} - 2T_{i,j}^{n+1/2} + T_{i-1,j}^{n+1/2} \right) &= \beta \left(T_{i,j+1}^n - 2T_{i,j}^n + T_{i,j-1}^n \right) \\
(1-\beta) T_{i+1,j}^{n+1/2} + 2\beta T_{i,j}^{n+1/2} - (1+\beta) T_{i-1,j}^{n+1/2} &= \beta \left(T_{i,j+1}^n - 2T_{i,j}^n + T_{i,j-1}^n \right)
\end{aligned}
$$

for second half step, exchange implicit and explicit variables

$$
\beta \left(T_{i,j+1}^{n+1} - 2T_{i,j}^{n+1} + T_{i,j-1}^{n+1} \right) = (1-\beta) T_{i+1,j}^{n+1/2} + 2\beta T_{i,j}^{n+1/2} - (1+\beta) T_{i-1,j}^{n+1/2}
$$

a tridiagonal matrix equation can be set up for each half step as above. repeat until $|T^{n+1} - T^n|$ is less than some tolerance at all points throughout the domain