CS230/561: Probability and Statistics for Computer Science

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August 28, 2024

Some slides/examples are motivated from *MIT 6-012 Introduction to Probability* by J. Tsitsiklis

Topic outline

1. Consequences of axioms
2. Probability calculation: examples
3. Conditional probability
4. The Bayes theorem
5. Independence of events

Probability axioms

Let Ω be the sample space and $A, B \subset \Omega$

1. Non negativity, $P(A) \geq 0$
2. Normalization, $P(\Omega) = 1$
3. Finite additivity: if A, B are disjoint (mutually exclusive), then

$$P(A \cup B) = P(A) + P(B)$$

Consequences of axioms

Probability axioms: consequences

A $P(\phi) = 0$

B $P(A) + P(A^{\complement}) = 1$

C $P(A) \leq 1$, for any $A \subset \Omega$

D $P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$, assuming $A_i \subset \Omega$ are disjoint

Finite additivity: a use case

- Probability of a finite set $\{s_1, s_2, \ldots, s_k\}$—a set which we can count and finish counting

$$P(\{s_1, s_2, \ldots, s_k\}) = P(\{s_1\} \cup \{s_2\} \cup \ldots \cup \{s_k\})$$

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$$P(\{2, 4, 6\}) = P(\{2\}) + P(\{4\}) + P(\{6\})$$

based on *finite additivity* axiom.

Union bound

- Let A, B are not disjoint events, i.e., $A \cap B \neq \phi$, then we have

$$P(A \cup B) \leq P(A) + P(B)$$

Finite additivity: another use case

$P(A \cup B \cup C) =$

Finite additivity: another use case

$$P(A \cup B \cup C) = P(A) + P(A^{\complement} \cap B) + P(A^{\complement} \cap B^{\complement} \cap C)$$

Identify partitions—i.e., disjoint sets; we can then apply the additivity rule.

Probability calculation: examples

Discrete example: rolling a dice

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- All outcomes are equally likely and have probability

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- What is the probability $P(X = 1)$?
- What is the probability $P(Z = 4), P(Z = 2)$, where $Z = \min(X, Y)$?

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- Assume finite set Ω consists of n equally likely elements, e.g., outcomes of dice rolling
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The probability of a finite set is the sum of the probabilities of finite elements of the set.

Continuous example: unit square

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- What is $P(\{3.1, 2.0\})$?

Probability computation: steps

- Define the sample space
- Define the probability law, e.g., the area of the set
- Identify an event of interest
- Compute

Additional axiom: countable additivity

If $A_1, A_2, \ldots$ is an infinite *sequence* of disjoint events,

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots$$

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- Additivity holds for *countable sequence of events*
- Unit square, real line[1], etc., are not countable—i.e., the elements cannot be arranged in a sequence

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Probability calculation: discrete but infinite Ω

- Experiment: toss a coin until you get the head
- Sample space

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- Check whether $P(\Omega) = 1$
- Event: the outcome is even, i.e., $E = \{2, 4, \ldots\}$, what is $P(E)$?—apply *countable additivity*

Conditional probability

Conditioning — intuition

- Consider we have equally likely outcomes in the sample space Ω
- Events $A, B \subset \Omega$, $A \cap B \neq \phi$

Conditioning — definition

- Let $P(B) > 0$
- We define the conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Example: two rolls of 4-sided dice

- Let B be the event $\min(X, Y)$
- Let A be the event $\max(X, Y)$
- What is $P(A = 1|B = 2)$?
- What is $P(A = 3|B = 2)$?

Properties of conditional probabilities

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  We can generalize this to finitely many or countable events

Conditional probability calculation: example

- Discrete events
- Event A: *airplane flying*
- Event R: *radar beacon*

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Multiplication rule

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3 events

$$\begin{aligned} P(A \cap B \cap C) &= P(A \cap B)P(C|A \cap B) \\ &= P(A)P(B|A)P(C|A \cap B) \end{aligned}$$

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Multiplication rule

We can generalize the rule for k events $A_1, A_2, \ldots, A_k$

$$P(A_1 \cap A_2 \cap \ldots \cap A_k) = P(A_1) \prod_{i=2}^{k} P(A_i | \underbrace{A_1 \cap A_2 \cap \ldots \cap A_{i-1}}_{\text{all the previous events}})$$

Total probability theorem

- Total probability of an event $B \subset \Omega$ that occur at different scenarios
- Partition the sample space into disjoint events $A_1, A_2, \ldots, A_k \subset \Omega$

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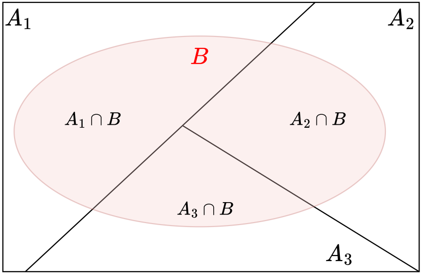

(Example with three disjoint sets)

Total probability theorem

- Total probability of an event $B \subset \Omega$ that occur at different scenarios
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- For every event/scenario i, we have
  - $P(A_i)$ — probability of the scenario
  - $P(B|A_i)$ — probability of the event under that scenario

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- We then have

$$P(B) = \sum_{i=1}^{k} P(A_i) P(B|A_i)$$

Total probability theorem

- The total probability of an outcome that can be achieved through multiple events is the sum of the probabilities of those other events in the sample space.
- This law relates marginal probabilities to conditional probabilities.

- One can see a marginal probability as a weighted average of conditional probabilities, where the weights are determined by the likelihood of each case.

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- We wish to revise the belief, given the observation B

$$
\begin{aligned}
P(A_i|B) &= \frac{P(A_i \cap B)}{P(B)} \\
&= \frac{P(A_i)P(B|A_i)}{\sum_{i=1}^{k} P(A_i)P(B|A_i)}
\end{aligned}
$$

Bayesian approach

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Gives intuition to update prior beliefs about A_i, observing data B

Bayes's rule — a use case

You plan to visit a beach on a Saturday afternoon; decide based on the possibility of rain[2].

[2] https://www.youtube.com/watch?v=cqTwHnNbc8g

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You plan to visit a beach on a Saturday afternoon; decide based on the possibility of rain[2].

- Chance of rain any day 15%
- On rainy afternoons, cloudy in the morning is 80% of the time.
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$$
\begin{aligned}
P(A = \text{Rain}) &= .15 && \text{prior} \\
P(B = \text{Cloudy}) &= .25 && \text{marginal probability} \\
P(B = \text{Cloudy} \mid A = \text{Rain}) &= .80 && \text{likelihood}
\end{aligned}
$$

$$
P(A = \text{Rain} \mid B = \text{Cloudy}) = \frac{.15 \times .80}{.25}
$$

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Independence of events

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$$P(A \cap B) = P(B)P(A),$$

- Symmetric with A, B
- Implies $P(B \mid A) = P(B)$
- Applies even when $P(A) = 0$ or $P(B) = 0$

More on independence of events

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- Disjoint and independence are different concepts—example

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Proposition

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Proof. Hint: $A = (A \cap B) \cup (A \cap B^{\complement})$

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- Formally, we write

$$P(A_1 \cap A_2 \cap \ldots \cap A_m) = P(A_1)P(A_2)\ldots P(A_m),$$

for any distinct choice of indices.

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Case $n = 3$: Three events A, B, C, are said to be independent, if we have

$$P(A \cap B \cap C) = P(A)P(B)P(C)$$

also, have pairwise independence.

$$P(A \cap B) = P(A)P(B)$$
$$P(B \cap C) = P(B)P(C)$$
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- If events A, B, C are independent, then A will be independent of any event formed from B, C.
- For instance, one can prove that A and $(B \cup C)$ are independent

Conditional independence

Definition

Given event C, the conditional independence is defined as the independence under the law $P(. \,|\, C)$.

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Consider the following scenario

- We know that $A \perp\!\!\!\perp B$

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Consider the following scenario

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- If we were told that C **occurred**, can we say that $A \perp\!\!\!\perp B$?
- Ans: it depends

Conditioning may affect independence — example

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- Are the coin tosses independent, given some additional information (e.g., based on an observation such as the number of heads in the previous n tosses)?