Math 312 Lectures 6 and 7
More About Nondimensionalization

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In these notes, we first look at an example of nondimensionalizing a differential equation, and then we look at another application of dimensional analysis.

Nondimensionalizing the Logistic Equation

Recall the logistic equation:

$$\frac{dP}{dt} = r\left(1 - \frac{P}{K}\right)P, \quad P(0) = P_0, \tag{1}$$

where $r > 0$ and $K > 0$ are constants. We'll follow the steps outlined in the previous lectures to nondimensionalize this differential equation. Table 1 lists the variables and parameters.

To create the nondimensional time variable τ, we must divide t by something that has the dimension $\mathcal{T}$. The only choice here is $1/r$, so we define

$$\tau = \frac{t}{\left(\frac{1}{r}\right)} = rt. \tag{2}$$

To create the nondimensional dependent variable y, we must divide P by something that has the dimension $\mathcal{N}$. We have two choices here, K or P_0. I'll use K, and leave the choice of P_0 as an exercise. We have

$$y = \frac{P}{K} \tag{3}$$

Variable or Parameter	Meaning	Dimension
t	time	$\mathcal{T}$
P	size of the population	$\mathcal{N}$
r	*per capita* growth rate of a small population	$\mathcal{T}^{-1}$
K	carrying capacity	$\mathcal{N}$
P_0	initial size of the population	$\mathcal{N}$

Table 1: The list of variables and parameters for the logistic equation, along with their dimensions. $\mathcal{T}$ means *time* and $\mathcal{N}$ means an *amount* or *quantity*.

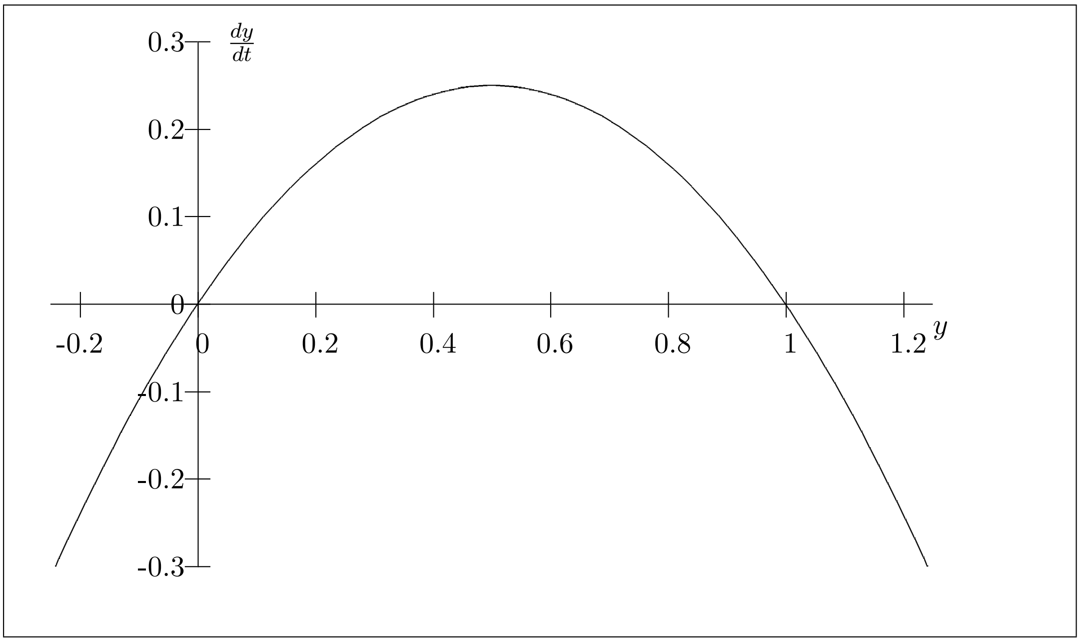

Figure 1: The plot of $\frac{dy}{dt}$ as a function of y for the nondimensional logistic equation (8).

For convenience, we also rewrite the definitions of τ and y as

$$t = \frac{\tau}{r}, \qquad P = Ky. \tag{4}$$

By the chain rule, we have

$$\frac{dP}{dt} = \frac{d(Ky)}{d(\tau/r)} = rK\frac{dy}{d\tau}. \tag{5}$$

Then substituting τ and y into (1) gives

$$rK\frac{dy}{d\tau} = r(1-y)Ky, \quad y(0) = \frac{P_0}{K}. \tag{6}$$

In the differential equation, the rK factors cancel, so the only parameters left are in the initial condition. Note that the fraction P_0/K is nondimensional. We define the new nondimensional initial condition

$$y_0 = \frac{P_0}{K} \tag{7}$$

to arrive at the nondimensional version of the logistic equation:

$$\frac{dy}{d\tau} = (1-y)y, \quad y(0) = y_0. \tag{8}$$

Now, instead of three dimensional parameters, we have just one nondimensional parameter. This simplifies the analysis a bit. At the least, it makes the analysis less cluttered. Figure 1 shows the plot of $(1-y)y$, where we can see that there is a stable equilibrium at $y = 1$. Solutions to the nondimensional equation are shown in Figure 2.

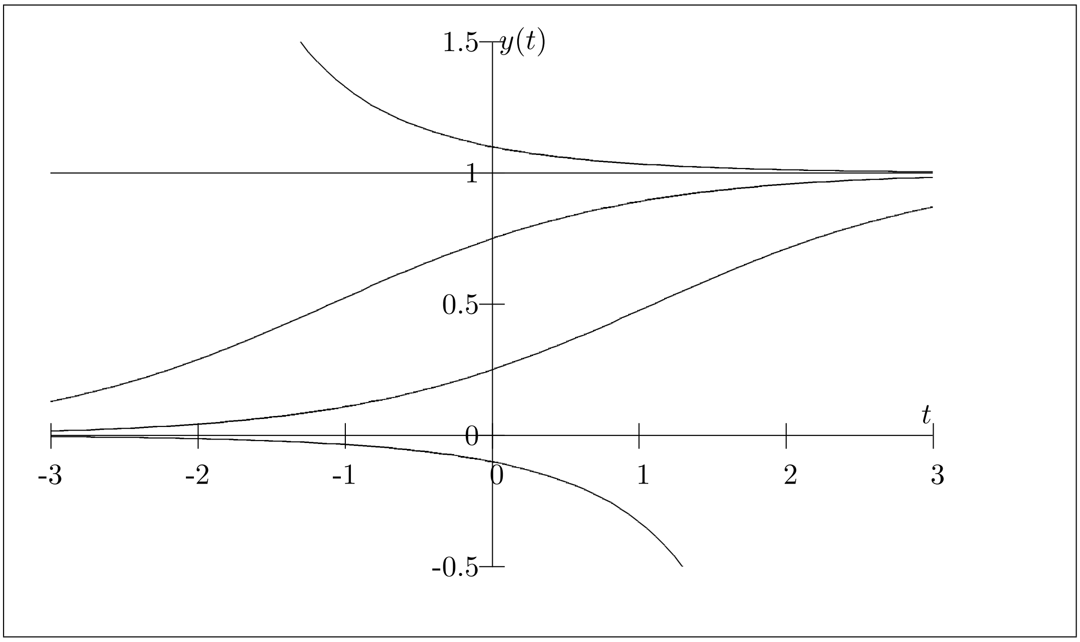

Figure 2: The plot of solutions to the nondimensional logistic equation (8) for several different initial conditions.

How do we interpret the meaning of the nondimensional variables? We defined $y = P/K$, so this one is clear. K is a natural unit for the size of the population; y represents the population as a fraction of the carrying capacity. The carrying capacity determines an intrinsic unit of measurement for the population.

Can we find a similar interpretation for τ? We formed τ by dividing t by $1/r$ because the dimension of $1/r$ is time. Is there some intrinsic "meaning" to $1/r$? Consider a small population, where $P/K \ll 1$. In this case, the differential equation is approximately

$$\frac{dP}{dt} = rP,$$

and the solution is $P(t) = P_0 e^{rt}$. Then $P(1/r) = P_0 e$. Thus we can interpret $1/r$ as the time required for a small population to increase by a factor of e. (This unit of time is similar to the "half-life" of radiactive materials. The half-life is time required for a given sample of the material to decay to half of the original amount.)

Dimensional Analysis

Here we see how dimensional analysis can be used to discover properties of a system without solving, or even writing down, any differential equations.

We begin with the notion of *dimensional homogeneity*. An equation is *dimensionally homogeneous* if it is true regardless of the system of units.

Example. Consider the equation

$$s = \frac{gt^2}{2} \tag{9}$$

This is equation for the distance s that an object will fall when released at $t = 0$ in a constant gravitational field. If we use units of feet for distance and seconds for time, then $g = 32$ ft/sec^2. Suppose we convert to the units miles and hours for distance and time, respectively. We'll use a bar to indicate variables in the new units. We have $s = 5280\bar{s}$ (there are 5280 feet per mile), and $t = 3600\bar{t}$ (3600 seconds per hour). Finally we express g in the new units: since $1\,\text{ft} = (1/5280)$ miles, and $1\,\text{sec} = (1/3600)\,\text{hr}$, we have $g = 32$ ft/sec^2 becomes $\bar{g} = 32(3600^2/5280)$ miles/hr^2. Let's substitute the new variables into (9):

$$\begin{aligned} 5280\bar{s} &= \frac{g\,(3600\bar{t})^2}{2} \\ \bar{s} &= \frac{3600^2}{5280}\frac{g\bar{t}^2}{2} \\ \bar{s} &= \frac{\bar{g}\bar{t}^2}{2} \end{aligned} \tag{10}$$

The new equation involving $\bar{s}$, $\bar{t}$ and $\bar{g}$ is the same as the original equation. This is an example of a dimensionally homogeneous equation. □

The power of dimensional analyis is based on the fundamental observation that *equations that arise from physical laws or real-world problems are dimensionally homogeneous.* The number of parameters in such an equation can generally be reduced, and this can lead to a better understanding of the system being studied.

Example: Period of a Pendulum. We consider a frictionless pendulum, as shown in Figure 3. Table 2 lists the parameters and their dimensions. (Note that angles, when expressed in radians, are actually dimensionless.) We consider an experiment in which we displace the pendulum by an angle θ_0, and release it with no initial velocity. Since we are ignoring friction, we expect the pendulum to oscillate. This oscillation will have some period T. (The periodic is the time required to complete one oscillation.) We would like to know how the period depends on the other parameters in the problem. First, we'll try to determine if all these parameters are really *independent.* To do this, we'll try to find all the nontrivial different ways that they can be combined to form dimensionless products. Our goal is to find the *dimensionless parameters.*

Consider the product

$$\pi = T^a l^b m^c g^d \theta_0^e \tag{11}$$

We want to choose a, b, c, d and e so that the new parameter π is dimensionless. (Note: π is the name of a parameter. We are not using $\pi = 3.1415\ldots$.) The dimensions of π are

$$\mathcal{T}^a\mathcal{L}^b\mathcal{M}^c\left(\mathcal{L}\mathcal{T}^{-2}\right)^d = \mathcal{T}^{a-2d}\mathcal{L}^{b+d}\mathcal{M}^c. \tag{12}$$

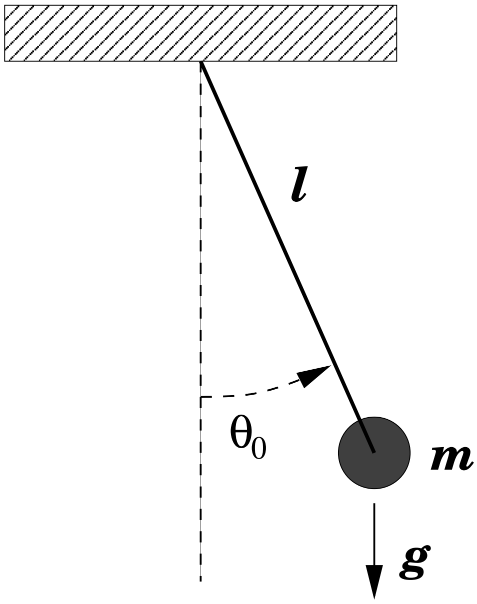

Figure 3: A pendulum of length l, mass m, acted on by gravity, released from the initial angle θ_0 with zero velocity.

Parameter	Meaning	Dimension
l	length of the pendulum	$\mathcal{L}$
m	mass of the pendulum bob	$\mathcal{M}$
g	gravitational acceleration	$\mathcal{L}\mathcal{T}^{-2}$
θ_0	initial angle	1
T	period of the oscillation	$\mathcal{T}$

Table 2: The list of variables and parameters for the logistic equation, along with their dimensions. $\mathcal{T}$ means *time* and $\mathcal{N}$ means an *amount* or *quantity*.

We want π to be dimensionless, so we want

$$\begin{aligned} a - 2d &= 0 \\ b + d &= 0 \\ c &= 0 \end{aligned} \tag{13}$$

This is a linear equation for the unknown a, b, c, and d. Actually, e is also an unknown, but it only shows up in the exponent of θ_0, and θ_0 is already dimensionless, so we know e is arbitrary. It is not difficult to solve the above system of equations, but I will still rewrite in matrix form, and I'll include e in the system:

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

which we can write more concisely as

$$A\boldsymbol{\lambda} = \mathbf{0} \tag{15}$$

where A is the *dimension matrix* and $\boldsymbol{\lambda}$ is the vector of the powers a, b, c, d, e. Each nontrivial solution to the linear algebra problem provides a way to combine the dimensional parameters into a nondimensional product. Note, however, that if $\boldsymbol{\lambda} = [a,b,c,d,e]^{\mathsf{T}}$ is a solution, then so is $r\,[a,b,c,d,e]^{\mathsf{T}} = [ra,rb,rc,rd,re]^{\mathsf{T}}$ for any constant r. Since

$$T^{ra}l^{rb}m^{rc}g^{rd}\theta_0^{re} = \left(T^a l^b m^c g^d \theta_0^e\right)^r, \tag{16}$$

multiples of a solution to (14) do not really identify new combinations of parameters. Thus, all we need is a set of *linearly independent* solutions to (14). (To use the lingo from linear algebra, we need a *basis for the null-space of* A.) In this case, we see that the system (14) is already in reduced row echelon form, and the solution can be written

$$\boldsymbol{\lambda} = c_1 \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{17}$$

where c_1 and c_2 are arbitrary. Thus a basic for the null-space of A is given by the two vector in the above solution. So one nondimensional parameter is

$$\pi_1 = T^2 l^{-1} m^0 g^1 \theta_0^0 = \frac{gT^2}{l} \tag{18}$$

and another (which we already knew) is

$$\pi_2 = T^0 l^0 m^0 g^0 \theta_0^1 = \theta_0. \tag{19}$$

Presumably there *is* some relationship among l, m, g, θ_0, and the period T. We don't know what it is, so we'll just assume it can be written in the form

$$f(T,l,m,g,\theta_0) = 0, \tag{20}$$

where f is dimensionally homogeneous. The fundamental result that we will now use is known as the *Buckingham Pi Theorem*. It says that a dimensionally homogeneous relation is equivalent to another relation expressed in terms of only the independent nondimensional parameters π_i. For our example, the Buckingham Pi Theorem implies that there is a function F for which

$$F(\pi_1, \pi_2) = 0. \tag{21}$$

Thus it must be possible to express the relation assumed in (20) in the simpler form

$$F\left(\frac{gT^2}{l}, \theta_0\right) = 0. \tag{22}$$

Equation (21) is an implicit relation between π_1 and π_2. We expect that for most values of π_1 and π_2, we can solve for π_1 in terms of π_2. That is, we can write (21) as

$$\pi_1 = h(\pi_2) \tag{23}$$

where h is some function. (In principle, the function h exists, but the dimensional analysis performed here tells us nothing about the nature of h.) Substituting the formulas for π_1 and π_2 into (23) gives

$$\frac{gT^2}{l} = h(\theta_0), \tag{24}$$

or

$$T = \sqrt{\frac{l}{g} h(\theta_0)} = \sqrt{\frac{l}{g}}\, \hat{h}(\theta_0) \tag{25}$$

where $\hat{h}(\theta_0) = \sqrt{h(\theta_0)}$. With this result, we can predict how the period of the oscillation of the pendulum depends on the parameters g and l, without actually solving (or even writing down) the differential equations that describe the motion.

For example, suppose a pendulum of length l_1 has period T_1 when released from angle θ_0. If the length is doubled and the pendulum is released from the same angle, the new period must be

$$T_2 = \sqrt{\frac{l_2}{g}}\, \hat{h}(\theta_0) = \sqrt{\frac{2l_1}{g}}\, \hat{h}(\theta_0) = \sqrt{2}\sqrt{\frac{l_1}{g}}\, \hat{h}(\theta_0) = \sqrt{2}\, T_1. \tag{26}$$

Thus, doubling the length should cause the period to increase by a factor of $\sqrt{2}$.

We can also compare the behavior of a pendulum on Earth to its behavior on Mars. The gravitational constant g_M on Mars is roughly one-third that of Earth's gravitational consant g_E. If, for a given initial angle θ_0 and length l, the period of the oscillation on Earth is 4 seconds, then on Mars the period will be

$$T_M = \sqrt{\frac{l}{g_M}}\, \hat{h}(\theta_0) = \sqrt{\frac{l}{g_E/3}}\, \hat{h}(\theta_0) = \sqrt{3}\sqrt{\frac{l}{g_E}}\, \hat{h}(\theta_0) = \sqrt{3}\, T_E = \sqrt{3}\, 4 \approx 6.93 \text{ seconds}. \tag{27}$$