Clint Valentine
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First, we find the nullclines of the system:

$$\frac{dm_i}{dt} = -m_i + \frac{\alpha}{(1+p_j^n)} + \alpha_o = 0$$

$$\dot{p} = \dot{m}$$

$$\frac{dp_i}{dt} = -\beta(p_i - m_i) = 0$$

$$\dot{m} = \frac{\alpha}{(1+\dot{p}^n)} + \alpha_o$$

Next, we express the system in matrix form at the fixed point. This is the Jacobian matrix of the system at the non-trivial fixed point:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial f_1}{\partial m} & \frac{\partial f_1}{\partial p} \\ \frac{\partial f_2}{\partial m} & \frac{\partial f_2}{\partial p} \end{bmatrix} = \begin{bmatrix} -1 & X \\ \beta & -\beta \end{bmatrix}$$

$$X \equiv -\frac{\alpha n \dot{p}^{n-1}}{(1+\dot{p}^n)^2}$$

$$\dot{p} = \frac{\alpha}{(1+\dot{p}^n)} + \alpha_o$$

To derive the conditional for instability as expressed in the Elowitz and Leibler (2000) paper we must find the the condition when the eigenvalues have imaginary parts. The eigenvalues will have an imaginary part when the discriminant of the Jacobian is negative.

$$\tau^2 - 4\Delta < 0$$

$$(-1-\beta)^2 - 4(\beta - X\beta) < 0$$

$$\frac{(\beta+1)^2}{\beta} < 4(-X+1)$$

This expression is different than the Elowitz and Leibler (2000) expression found in Box 1:

$$\frac{(\beta+1)^2}{\beta} < \frac{3X^2}{4+2X}$$