MATLAB for 1st-Order Differential Equations & Systems

MATLAB has several numerical procedures for computing the solutions of first-order equations and systems of the form $y' = f(t, y)$; we shall concentrate on "ode45", which is a suped-up Runge-Kutta method. The *first step* is to enter the equation by creating a new function and giving it a name for reference, such as "diffeqn". The *second step* is to apply ode45 by using the syntax:

$$[\mathrm{t}, \mathrm{y}] = \mathrm{ode45}(@\mathrm{diffeqn}, [\mathrm{t_0}, \mathrm{t_f}], \mathrm{y_0}); \tag{1}$$

where @ tells MATLAB which function to use, $\mathrm{t_0}$ is the initial time, $\mathrm{t_f}$ is the final time, and $\mathrm{y_0}$ is the initial condition, i.e. $\mathrm{y(t_0) = y_0}$. (We have also used a semicolon at the end of the line to tell MATLAB to run the routine but not display the results.) The same syntax (1) works for equations and systems alike: for a system, y is a vector.

Example 1. Suppose we want to solve the nonlinear 1st-order equation $y' = y^2 - t$, $y(0) = 0$, for $0 \leq t \leq 4$. We first need to define the function associated with this equation. Start up MATLAB; the Command Window appears with the prompt >> awaiting instructions. However, first we select "New" and then "Function". This opens a text editor in which we can type

```
function ypr = example1(t,y)
ypr = y^2-t;
```

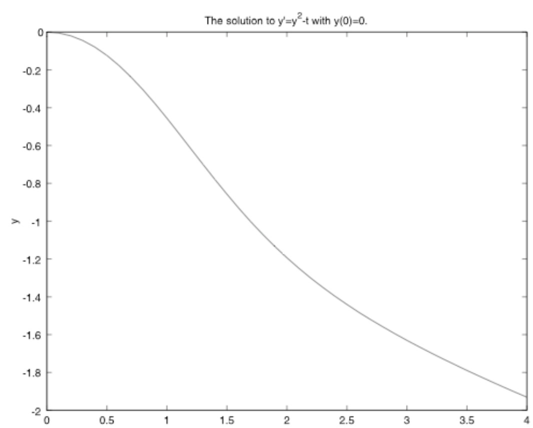
Return to the Command Window, and enter the following:

```
>> [t,y] = ode45(@example1,[0,4],0);
```

MATLAB will do its computing, then give you another prompt. You can plot the solution $y(t)$ by typing

```
>> plot(t,y)
```

and hitting the enter key. The following plot should appear on your screen:



To give your plot the axes labels in the picture above, type

```
>>xlabel('t')
>>ylabel('y')
```

You can also have MATLAB tabulate the t-values it has selected and the corresponding y-values that it has computed by entering

$$>> [\texttt{t}, \texttt{y}]$$

in the Command Window. This should produce a vertical column of numbers, the last of which is $t = 4.0000$ and $y = -1.9311$, i.e. $y(4) = -1.9311$ as appears in the plot.

Exercise 1. Consider the initial value problem $y' = t^2 + \cos y$, $y(0) = 0$. Use MATLAB to plot the solution for $0 \le t \le 1$, and find the approximate value of $y(1)$.

Hand In: A printout of your plot and the value of $y(1)$.

In the next example, we will consider a system of 1st-order differential equations. But let us observe that MATLAB is designed to handle vectors and matrices, so we can also use it to perform linear algebra.

Example 2. Let us recall the competitive whale model from the Exercises in Chapter 2 of the textbook: if x represents the population of blue whales, y represents the population of fin whales, and t is measured in years, then the model states

$$\begin{aligned} \frac{dx}{dt} &= r_1 x \left(1 - \frac{x}{K_1}\right) - \alpha_1 xy \\ \frac{dy}{dt} &= r_2 y \left(1 - \frac{y}{K_2}\right) - \alpha_2 xy, \end{aligned} \tag{2}$$

where the parameter r_i represent the intrinsic growth rate, the parameter K_i represents the carrying capacity of the environment, and the parameter α_i controls the effect of competition on the growth rate. Using the values of the parameters given in the textbook, this system takes the form

$$\begin{aligned} \frac{dx}{dt} &= .05\, x \left(1 - \frac{x}{1.5 \times 10^5}\right) - \frac{xy}{10^8} \\ \frac{dy}{dt} &= .08\, y \left(1 - \frac{y}{4 \times 10^5}\right) - \frac{xy}{10^8}. \end{aligned} \tag{3}$$

If we are given initial conditions x_0 and y_0, we want to be able to compute the solution $(x(t), y(t))$.

Before we use ode45 to compute solutions, let us find the equilibrium points for (3). These, of course, are the values of (x, y) for which

$$\begin{aligned} x\left(.05 - \frac{.05\, x}{1.5 \times 10^5} - \frac{y}{10^8}\right) &= 0 \\ y\left(.08 - \frac{.08\, y}{4 \times 10^5} - \frac{x}{10^8}\right) &= 0. \end{aligned} \tag{4}$$

It is easy to find three of the equilibrium points: $(0,0)$, $(0, 4\times 10^5)$, and $(1.5\times 10^5, 0)$. But these all correspond to at least one of the whale populations being extinct. However, there is a fourth equilibrium point that makes vanish both expressions in parentheses in (4), i.e.

$$\begin{aligned} \frac{x}{3} + \frac{y}{100} &= 5 \times 10^4 \\ \frac{x}{100} + \frac{y}{5} &= 8 \times 10^4. \end{aligned} \tag{5}$$

Of course, we can easily solve (5) by hand, but let us ask MATLAB to do the work for us. Let us do so by writing (5) in matrix form* as $Ax = b$ where the matrix A and vector b are

$$A = \begin{pmatrix} 1/3 & 1/100 \\ 1/100 & 1/5 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 5 \times 10^4 \\ 8 \times 10^4 \end{pmatrix}.$$

* Solving a system of $n \times n$ equations by inverting the coefficient matrix is not the most computationally efficient method. But this is not an issue for small n such as $n = 2$ or 3.

We want to compute A^{-1} and then compute $x = A^{-1}b$. In the Command Window, we first define the matrix A by typing

```
>> A = [ 1/3   1/100  ;  1/100   1/5 ]
```

Notice that the space between 1/3 and 1/100 tells MATLAB that these numbers are in different columns of the first row, while the semicolon tells MATLAB to go to the next row; MATLAB should correctly display A as a matrix. We define b as a column vector by entering

```
>> b = 10^4*[ 5 ; 8  ]
```

and MATLAB should correctly display b. Finally we enter

```
>> x = inv(A)*b
```

and the answer that MATLAB displays tells us that the fourth equilibrium point of (3) is $(x^*, y^*) = (1.3821 \times 10^5, 3.9309 \times 10^5)$. In other words, a population of 138,210 blue whales and 393,090 fin whales will remain in equilibrium.

Now suppose that a sudden event cuts the blue whale population roughy in half to 70,000, and we are interested to know how long it will take for the populations to return to their equilibrium values. In other words, we want to solve (3) with the initial values

$$x(0) = 7 \times 10^4 \quad \text{and} \quad y(0) = 3.93 \times 10^5.$$

We begin by defining a new function that represents the system (3). We change notation by replacing (x, y) by $y = (y_1, y_2)$, so that (3) becomes

$$\begin{aligned} \frac{dy_1}{dt} &= .05\, y_1 \left(1 - \frac{y_1}{1.5 \times 10^5}\right) - \frac{y_1\, y_2}{10^8} \\ \frac{dy_2}{dt} &= .08\, y_2 \left(1 - \frac{y_2}{4 \times 10^5}\right) - \frac{y_1\, y_2}{10^8}. \end{aligned} \tag{6}$$

MATLAB uses the notation $y(1)$ instead of y_1, so we create a new function that represents (6) by typing

```
function ypr = example2(t,y)
ypr = zeros(2,1);
ypr(1) = .05*y(1)*(1-y(1)/(1.5*10^5))-y(1)*y(2)/10^8;
ypr(2) = .08*y(2)*(1-y(2)/(4*10^5))-y(1)*y(2)/10^8;
```

The second line defines ypr as a column vector, and the next two lines define the components of the vector to express the equations in (6). Now we are ready to run ode45, say for $0 \le t \le 10$, by entering in the Command Window:

```
>> [t,y]=ode45(@example2,[0,10],[7*10^4,3.93*10^5]);
```

MATLAB has done its calculation, but we see no output because we ended with the semicolon. But we can see the numerical results by entering

```
>> [t,y]
```

Note that $y_1(10) = 85,590$ and $y_2(10) = 394,670$. This means that, even after ten years, the blue whale population has not yet gotten very close to its equilibrium population of 138,210, while the fin whale population has actually grown a bit from 393,090 to 394,670, due to the decreased competition for food.

Let us see what happens over a longer time scale, say $0 \le t \le 100$. We now enter

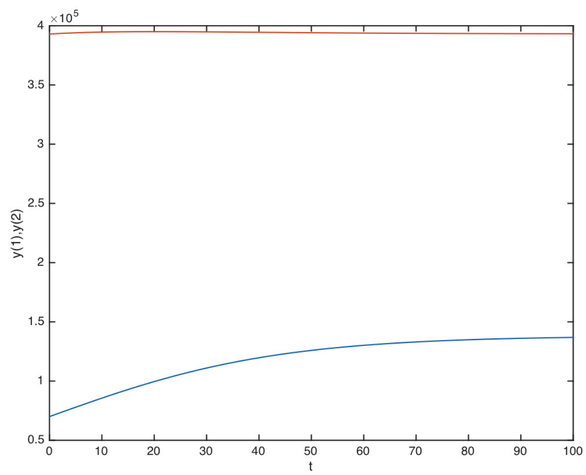
```
>> [t,y]=ode45(@example2,[0,100],[7*10^4,3.93*10^5]);
```

```
>> [t,y]
```

We see that $y_1(100) = 136,860$ and $y_2(100) = 393,240$, which means that both whale populations are finally within 1% of their equilibrium values. We can plot both functions $y_1(t)$ and $y_2(t)$ by entering

```
>>plot(t,y)
>>xlabel('t')
>>ylabel('y(1),y(2)')
```

The resultant plot appears below.



Exercise 2. The following system of equations is used to model species in a predator-prey relationship:

$$\begin{aligned} \frac{dx}{dt} &= a\,x - b\,xy \\ \frac{dy}{dt} &= -c\,y + d\,xy, \end{aligned} \tag{7}$$

where a, b, and c are positive constants.

(a) Does x represent the predator or the prey population?

(b) Let $a = b = c = d = 1$ and find the equilibrium points.

(c) For the initial conditions, $x(0) = 0.9$ and $y(0) = 1.1$, use MATLAB to plot the behavior of $x(t)$, $y(t)$ for $0 \le t \le 10$.

(d) Based upon your plot in (c), explain what happens to the two populations x and y and why.

Hand In: Answers to questions and a printout of your plot in (c).