Assessment : individual coursework 1

1

a Let A, B and C be sets. Show that $A \times (B \cap C) = (A \times B) \cap (A \times C)$

LHS: $x \in A \times x \in (B \cap C)$

$= x \in A \times ((x \in B) \cap (x \in C))$

$= ((x \in A) \times (x \in B)) \cap ((x \in A) \times (x \in C))$

$= x \in (A \times B) \cap x \in (A \times C)$

$= (A \times B) \cap (A \times C) = \text{RHS}$

LHS = RHS

b. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions such that $f(x) = 5x - 2$ and $g(x) = x^2 - 4$. Give a definition for composite function $f \circ g$

$F \circ g$

$F \circ g(x) = f(g(x))$

$= 5(x^2 - 4) - 2$

$= 5x^2 - 20 - 2$

$= 5x^2 - 22$

c. A Graph $G = (V, E)$ is said to be isomorphic to a graph $H = (V', E')$ if there exists a bijective function f from the vertices of G to the vertices of H such that for two adjacent vertices in G, u and v the corresponding vertices in H, $f(u)$ and $f(v)$ are adjacent. Given the two graphs $G = (\{1,2,3\}, \{(1,2), (2,3)\})$ and $H = (\{-2,0,2\}, \{(-2,0), (0,2)\})$, show that G is isomorphic to H

$f: G \rightarrow H$

The function $f: G \rightarrow H$ is bijective if and only if H is invertible. Therefore to show that f has an inverse, it suffices that f is bijective.

Assumption: $f(u) = f(v)$ for any $u, v \in G$
[in order to find the inverse]

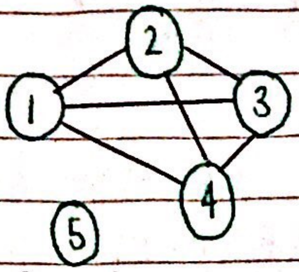
Apply f^{-1} to both sides:

$f^{-1}(f(u)) = f^{-1}(f(v))$

Therefore we can assume that $u = v$ as for any x, $f^{-1}(f(x)) = x$.
As we have proved that f has an inverse, we can therefore conclude that f is bijective.

2.

a. Given the graph G which can be illustrated below. formally define the graph G' which is the largest possible subgraph of G where every vertex in G' has degree 3



V = {v | v is a vertex}
E = {$e_{i,j}$ | $e_{i,j}$ is the edge joining vertex i, j ∈ V}

G' = (V, E)
V = {1, 2, 3, 4}
E = {e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}}

b. Construct a well-formed formula using appropriate propositional variables for the expression:
"If Paul and Jay go to the shops then alex will not go to the shops"

P = Paul goes to the shop
J = Jay goes to the shop
A = Alex goes to the shop

$(P \wedge J) \Rightarrow (\neg A)$

c. Using the same propositional variables defined in b. construct WFF for the expression:
"Either Alex or Paul go to the shops with Jay. which is the same as saying, Alex and Jay go to the shops and not Paul or Paul and Jay go to the shops and not Alex."

$(A \wedge J) \vee (P \wedge J) = ((A \wedge J) \wedge (\neg P)) \vee ((P \wedge J) \wedge (\neg A))$
$(A \vee P) \wedge J = ((A \wedge J) \wedge (\neg P)) \vee ((P \wedge J) \wedge (\neg A))$

3. Let A be an array of 15 integers and B be an array of 10 integers. The arrays A and B are indexed from 1 to 15 and 1 to 10 respectively. Construct predicates to assert that:

(a.) The third element of A is larger than all elements of B.
$\forall i, 1 \leq i \leq 10, A[3] > B[i]$

(b.) The elements of B form a consecutive sequence within A
$\forall i, 1 \leq i \leq 10, \exists j, 1 \leq j \leq 10, B[i] = A[j] \wedge B[i+1] = A[j+1]$

(c.) if x is an element of A then $-x$ is an element of B
$\exists i, 1 \leq i \leq 10, \exists j \; 1 \leq j \leq 15, A[j] = x \Rightarrow B[i] = -x$

(d.) The value 7 only occurs once in A
$\exists i, 1 \leq i \leq 15, \forall j \; 1 \leq j \leq 15, A[i] = 7, A[j] \neq 7, i \neq j$

4.
a. Let $x, y \in \mathbb{Z}$ Prove by construction that the sum of $x + y$ is even when:
(i) x and y are both even
Assumption: Let x and y both be even
$x = 2K$ $\qquad$ $x = 2n$
therefore $x + y = 2K + 2n$
$= \underbrace{2(K+n)}$

K+n is any number.
Two times any number is even
∴ therefore $x+y$ is even

(ii) x and y are both odd

Assumption: Let x and y both be odd

$x = 2k+1$ $y = 2n+1$

therefore $x+y = (2k+1)+(2n+1)$

$= 2k+2n+2$

$= 2(k+n+1)$ ∴ Any number multiplied by 2 is even.

As $k+n$ can be written as 2 multiplied by something, $x+y$ = even

5.

a. Give a recursive definition for the following sequence which is defined over the natural numbers: -2, 6, -18, 54, -162, 486

0	1	2	3	4	5
-2	6	-18	54	-162	486

$$f(n) = \begin{cases} -2 \text{ when } n=0 \\ -3f(n-1) \text{ otherwise} \end{cases}$$

b. Use the pseudo code below to formulate an equivalent recursive definition:

```
Function f(n)
  if n=0:
    return 1;
  else:
    return f(n-1)+3n+2;
  end
end
```

$$f(n) = \begin{cases} 1 \text{ when } n=0 \\ f(n-1)+3n+2 \text{ otherwise} \end{cases}$$

c.

0	1	2	3
1	6	14	25

$f(3) = 25$

d. Find a closed-form solution to the function:

$$f(n) = \begin{cases} 0 & \text{if } n=0 \\ f(n-1)+2n & \text{otherwise} \end{cases}$$

$f(0) = 0$

$f(1) = 0+2 = 2$

$f(2) = f(1)+2n$

$= 2+2(2) = 6$

$f(3) = f(2)+2n$

$= 6+6 = 12$

$f(4) = f(3)+2n$

$= 12+8 = 20$

0	1	2	3	4
0	2	6	12	20

+2 +4 +6 +8

$f(n) = n(n+1)$

e. Prove by induction that your closed-form solution in d is correct

$$f(n) = \begin{cases} 0 & \text{if } n=0 \\ f(n-1)+2n & \text{otherwise} \end{cases}$$

My closed-form sol = $f(n) = n(n+1) = f(n-1)+2n$

BASECASE: $f(0) = 0 = 0(1) = 0$

ASSUMPTION: $f(K) = K(K+1)$ for $K \geq 0$ consider $n = K \geq 0$

INDUCTIVE STEP: Consider $n = K+1$

$f(K) = K(K+1)$

$f(K+1) = f(K+1-1) + 2(K+1)$

$= f(K) + 2(K+1)$

$= K(K+1) + 2(K+1)$

$= (K+2)(K+1)$

$= K^2+2K+K+2$

$= K^2+3K+2$

$(K+1)((K+1)+1)$

$= (K+1)(K+2)$

$= K^2+2K+K+2$

$= K^2+3K+2$

LHS=RHS: The claim holds by induction

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