European Journal of Physics

PAPER

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To cite this article: Chang Liu and Lu Dong 2019 *Eur. J. Phys.* **40** 035006

IOP Publishing European Journal of Physics

Eur. J. Phys. **40** (2019) 035006 (15pp) https://doi.org/10.1088/1361-6404/ab03e8

Physics-based control education: energy, dissipation, and structure assignments

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Received 18 December 2018, revised 17 January 2019
Accepted for publication 31 January 2019
Published 22 March 2019

Abstract

Control theory usually finds no suitable place in the education of physics. The port-Hamiltonian framework, which generalizes the formalism of Hamiltonian mechanics, provides a physics-based control strategy. This framework is also promising in the education of physicists in control theory. In this paper, we use the port-Hamiltonian framework to reformulate a physics system and introduce the physics-based control strategy, including the energy, dissipation, and structure assignments. The closed-loop Hamiltonian is a candidate of the Lyapunov function, which guarantees the global stability of the closed-loop system. These physics-based control strategies are illustrated using the Duffing oscillator and the Lorenz system. We also provide port-Hamiltonian descriptions for two examples in celestial mechanics.

Keywords: physics-based control, port-Hamiltonian, control education in physics

(Some figures may appear in colour only in the online journal)

1. Introduction

Historically, the Hamiltonian equations of motion were born from analytical mechanics. It begins with Lagrangian mechanics, a previous reformulation of classical mechanics, and ends up as Hamiltonian equations through Legendre transform (Salmon 1988, Morrison 1998). Most of the analyses of physics systems have been performed within the Lagrangian and Hamiltonian framework. This was an important reformulation of classical mechanics, which later contributed to the formulation of statistical mechanics and quantum mechanics.

Stability analysis based on Hamiltonian formalism is usually somewhat touched upon at the end of Hamiltonian mechanics (Holm *et al* 1985), but control theory usually finds no suitable place in the education of physicists. Actually, control theory has recently been applied in diverse areas, including chaos and nonlinear dynamics, statistical mechanics, optics, and quantum computing (Bechhoefer 2005).

The introductory textbooks targeting the control engineer (Dutton *et al* 1997, Franklin *et al* 1998, Goodwin *et al* 2001) are usually disappointing places for a physicist to start. Their examples are tailored to the knowledge of an engineer instead of a physicist. They often use engineering jargon to describe concepts that is usually familiar to physicists, but the jargon is not adequate for physicists to understand them quickly and fully. Other more mathematically focused texts (Sontag 1998, Ozbay 1999, Doyle *et al* 2013) are concise but usually assume the reader already has some background in control theory and applied mathematics.

Furthermore, most of control theory is based on the linearization of the original dynamics and then uses mature linear control theory (Joseph and Tou 1961). Linear control theory or linear stability criteria may not be always reliable. For example, a system that is stable to infinitesimal perturbations may be unstable to finite perturbations.

Over a century ago, Lyapunov (Lyapunov 1992) generalized the notions of stability to nonlinear dynamics (Bhatia and Szegö 2002). Lyapunov's second method for stability considers the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ with an equilibrium $\boldsymbol{x} = \boldsymbol{x}^*$. If there is a Lyapunov function $V(\boldsymbol{x})$ satisfying

$$\begin{aligned}
&V(\boldsymbol{x}) = 0, \;\; \boldsymbol{x} = \boldsymbol{x}^* \\
&V(\boldsymbol{x}) > 0, \;\; \forall \boldsymbol{x} \neq \boldsymbol{x}^* \\
&\frac{\mathrm{d}V(\boldsymbol{x})}{\mathrm{d}t} = \nabla_{\boldsymbol{x}} V(\boldsymbol{x}) \dot{\boldsymbol{x}} = \nabla_{\boldsymbol{x}} V(\boldsymbol{x}) \boldsymbol{f}(\boldsymbol{x}) \leqslant 0, \;\; \forall \boldsymbol{x} \neq \boldsymbol{x}^*,
\end{aligned} \tag{1}$$

then the system is stable in the sense of Lyapunov. Furthermore, if $\frac{\mathrm{d}V(x)}{\mathrm{d}t} < 0, \;\; \forall \boldsymbol{x} \neq \boldsymbol{x}^*$, then the system is asymptotically stable. In the Lyapunov stability analysis and the Lyapunov-based control (De Queiroz *et al* 2012), the major difficulty is finding the Lyapunov function $V(\boldsymbol{x})$.

For Hamiltonian system, the Hamiltonian is a natural candidate of Lyapunov function. Hamiltonian formalism also provides a geometric description of nonlinear dynamics and it indicates the potential to be used in control. However, until recently, the physics literature and the control theory literature on nonlinear dynamics were still separate (Bechhoefer 2005).

Port-Hamiltonian framework (van der Schaft and Jeltsema *et al* 2014) extends the formalism of Hamiltonian system to include the input and dissipation, which are described as 'port'. This framework has potential to model, analyze and control complex physics systems and their interconnections (van der Schaft and Schumacher 2000).

The port-Hamiltonian framework also provides a physics-based control strategies (Ortega *et al* 2001, 2002, 2008), which are focused on shaping the closed-loop Hamiltonian as a candidate of the Lyapunov function. The control strategy for port-Hamiltonian framework is based on physics and the terminology is also familiar to physics students. As a result, it has potential to be used in physics class to introduce control theory.

In this paper, we describe the port-Hamiltonian framework and the physics-based control, including energy, dissipation and structure assignments. This framework is described with simple examples, hoping to facilitate the education of control theory in physics class. Section 2 describes the port-Hamiltonian formalism. In sections 3, 4 and 5, the control strategy based on energy, dissipation and structure assignments are described, respectively. In appendix (A–B), we also provide the port-Hamiltonian formalism for two examples in celestial mechanics.

2. Port-Hamiltonian description

The port-Hamiltonian framework reformulates the original system as

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = [\boldsymbol{J} - \boldsymbol{R}]\nabla_x H(\boldsymbol{x}) + \boldsymbol{Bu}. \tag{2}$$

$\boldsymbol{x}$ represents the state variable and $\boldsymbol{u}$ the input. $\boldsymbol{J}$ is a skew-symmetric matrix, which represents the energy (Hamiltonian) conserving part of the system and $\boldsymbol{R}$ is a symmetric matrix, which represents the energy dissipation or production. In this paper, we assume that the Poisson bracket defined by the structure matrix $\boldsymbol{J}$: $\{F, G\}(\boldsymbol{x}) \coloneqq \nabla_x F(\boldsymbol{x}) \cdot \boldsymbol{J}\nabla_x G(\boldsymbol{x})$ satisfies the Jacobi identity, i.e. $\{F, \{G, K\}\} + \{G, \{K, F\}\} + \{K, \{F, G\}\} = 0$. It will be interesting to extend the current framework to a more general situation where $\boldsymbol{J}$ is skew-symmetric but does not satisfy the Jacobi identify; e.g, non-holonomic systems with homogeneous constraints (Kozlov 2002, Borisov and Mamaev 2003, 2015a, 2015b, Borisov *et al* 2017).

The port-Hamiltonian framework will become a Hamiltonian formalism if the system does not have any input and the Hamiltonian is conserved, i.e. $\boldsymbol{u} = \boldsymbol{0}$ and $\boldsymbol{R} = \boldsymbol{0}$. $\boldsymbol{u}$ is the input, which is designed to achieve a certain control target. $\boldsymbol{B}$ is the matrix that maps the inputs into the states and it is assumed as full column rank but not full row rank.

We will illustrate the port-Hamiltonian description using the Duffing oscillator:

$$\ddot{x} + \delta\dot{x} + \alpha x + \beta x^3 = u. \tag{3}$$

In physical terms, the Duffing oscillator models a spring pendulum whose spring's stiffness does not exactly obey Hooke's law ($\alpha > 0$). When $\beta < 0$, the spring is regarded as a softening spring, and, when $\beta > 0$ it is viewed as a hardening spring (Thompson *et al* 2002). For $\alpha < 0$, the Duffing oscillator describes the dynamics of a point mass in a double well potential, and it models a periodically forced steel beam which is deflected toward the two magnets (Guckenheimer and Holmes 2013). The Duffing oscillator is also a simple example of a dynamical system that exhibits chaotic behavior.

The undamped Duffing oscillator ($\delta = 0$):

$$\ddot{x} + \alpha x + \beta x^3 = 0 \tag{4}$$

is the Hamiltonian system with

$$H(\boldsymbol{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\alpha x^2 + \frac{1}{4}\beta x^4. \tag{5}$$

The damped Duffing oscillator with inputs in equation (3) can be written as the port-Hamiltonian system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = [\boldsymbol{J} - \boldsymbol{R}]\nabla_x H(\boldsymbol{x}) + \boldsymbol{B}u. \tag{6}$$

$\boldsymbol{x} = [x\ \dot{x}]^T$ are state variables and u is the input. $\boldsymbol{B} = [0\ 1]^T$ maps the input to the state variables. The skew-symmetric matrix is $\boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and the symmetric matrix is $\boldsymbol{R} = \begin{bmatrix} 0 & 0 \\ 0 & \delta \end{bmatrix}$. We also provide the port-Hamiltonian reformulation of celestial mechanics examples in appendix A.

For the damped Duffing oscillator, we have $\frac{\mathrm{d}H(x)}{\mathrm{d}t} = -\delta(\dot{x})^2 \leqslant 0$ considering $\delta \leqslant 0$. Without forcing, the damped Duffing oscillator will end up at one of its stable equilibrium points. The stable and unstable equilibrium points satisfy $\alpha x + \beta x^3 = 0$. Using the eigenvalue analysis, the equilibrium $x = 0$ is stable for $\alpha \geqslant 0$ and unstable for $\alpha < 0$. On the other hand, the equilibrium points $x = \pm\sqrt{-\frac{\alpha}{\beta}}$ are stable for $\alpha < 0$, $\beta > 0$ and unstable for $\alpha > 0$ and $\beta < 0$.

3. Energy assignment

For the Duffing oscillator described in equation (4), the coefficients $\alpha = 1$ and $\beta = -1$ render the stable equilibrium point $x = 0$ along with unstable equilibrium points $x = \pm\sqrt{-\frac{\alpha}{\beta}}$.

However, the equilibrium points or their stability in the open-loop system are sometimes not of interest. The energy assignment step is to control the Hamiltonian function to modify the equilibrium point and its stability.

Thus, we design a closed-loop Hamiltonian $H_d(\boldsymbol{x})$ with minimal position:

$$\boldsymbol{x}^* = \arg\min\{\boldsymbol{H}_d(\boldsymbol{x})\}. \tag{7}$$

Assuming the closed-loop system is also a port-Hamiltonian system, we have

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = [\boldsymbol{J} - \boldsymbol{R}]\nabla_x H_d(\boldsymbol{x}). \tag{8}$$

Comparing equation (8) with (6), we have the state feedback control law:

$$\boldsymbol{u}_{ES}(\boldsymbol{x}) = (\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T[\boldsymbol{J} - \boldsymbol{R}]\nabla_x H_a(\boldsymbol{x}) \tag{9}$$

$$H_a(\boldsymbol{x}) = H_d(\boldsymbol{x}) - H(\boldsymbol{x}), \tag{10}$$

and the difference between the closed-loop Hamiltonian and the open-loop Hamiltonian should satisfy

$$\boldsymbol{B}^{\perp}[\boldsymbol{J} - \boldsymbol{R}]\nabla_x H_a(\boldsymbol{x}) = 0, \tag{11}$$

where $\boldsymbol{B}^{\perp}$ is a left-annihilator of $\boldsymbol{B}$, i.e. $\boldsymbol{B}^{\perp}\boldsymbol{B} = \boldsymbol{0}$, of maximal rank. Under this control law, it can be shown that the closed-loop Hamiltonian $H_d(\boldsymbol{x})$ is a Lyapunov function satisfying

$$\begin{aligned} H_d(\boldsymbol{x}) &= 0, \ \ \boldsymbol{x} = \boldsymbol{x}^* \\ H_d(\boldsymbol{x}) &> 0, \ \ \forall \boldsymbol{x} \neq \boldsymbol{x}^* \\ \frac{\mathrm{d}H_d(\boldsymbol{x})}{\mathrm{d}t} &= \nabla_x H_d(\boldsymbol{x}) \cdot [-\boldsymbol{R}]\nabla_x H_d(\boldsymbol{x}) = 0, \ \ \forall \boldsymbol{x} \neq \boldsymbol{x}^*, \end{aligned} \tag{12}$$

for the undamped Duffing system ($\delta = 0$). According to the Lyapunov stability theorem in equation (1), the closed-loop system is stable. However, the obtained closed-loop system may not process asymptotic stability, which requires $\frac{\mathrm{d}H(x)}{\mathrm{d}t} < 0, \ \forall \boldsymbol{x} \neq \boldsymbol{x}^*$.

The framework of the energy assignment is summarized in figure 1. With the feedback control $\boldsymbol{u}_{\mathrm{ES}}$, the Hamiltonian of the open-loop system H is modified to that of the closed system H_d. To guarantee the stability of the closed system based on the Lyapunov theorem in equation (1), we choose a specific form of the closed-loop Hamiltonian H_d, which satisfies $\boldsymbol{x}^* = \arg\min\{\boldsymbol{H}_d(\boldsymbol{x})\}$. The feedback control of energy assignment $\boldsymbol{u}_{\mathrm{ES}}$ only modifies the Hamiltonian of the open-loop system, while the skew-symmetric matrix $\boldsymbol{J}$ and the symmetric matrix $\boldsymbol{R}$ of the system remain the same.

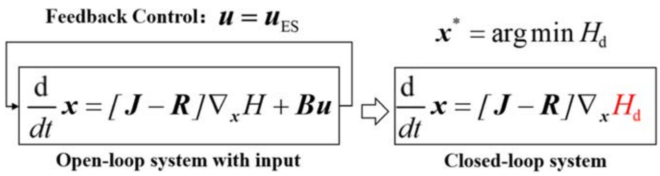

Figure 1. Summary of the energy assignment.

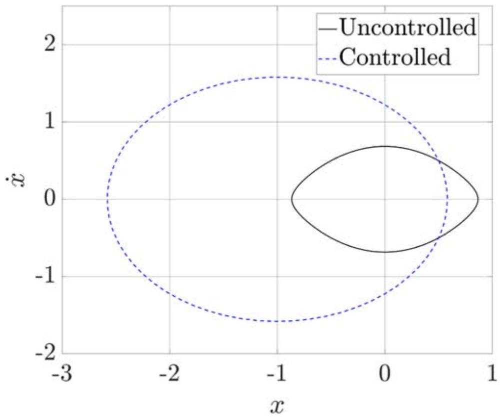

Figure 2. The $x - \dot{x}$ plane for the undamped Duffing oscillator with state feedback control $u = u_{\mathrm{ES}}$ (the blue dashed line) compared with the uncontrolled system (the black solid line).

We consider the example of an undamped Duffing oscillator ($\delta = 0$ and $\boldsymbol{R} = \boldsymbol{0}$) with control. The closed-loop Hamiltonian is designed as

$$H_d(\boldsymbol{x}) = \frac{1}{2}\dot{x}^2 + \frac{1}{2}(x - x^*)^2, \tag{13}$$

which reaches its minimal value at the equilibrium point $\boldsymbol{x} = [x^*\ 0]^T$. In this example, we set $x^* = -1$ and the initial conditions are set as $\boldsymbol{x} = [0.5\ 0.5]^T$. We implement the energy assignment control $\boldsymbol{u}_{\mathrm{ES}}$ as equation (9):

$$u_{\mathrm{ES}}(\boldsymbol{x}) = (\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T[\boldsymbol{J} - \boldsymbol{R}]\nabla_{\boldsymbol{x}} H_a(\boldsymbol{x}) = \alpha x + \beta x^3 - (x - x^*). \tag{14}$$

This feedback control $\boldsymbol{u}_{\mathrm{ES}}$ strategy actually modifies the potential of the Duffing system $\alpha x + \beta x^3$ into the potential of a spring with equilibrium x^*, i.e. $(x - x^*)$. The $x - \dot{x}$ phase planes of the controlled and uncontrolled systems are shown in figure 2. The results shown by the blue line show that the equilibrium of the closed-loop system has been modified as $\boldsymbol{x}^*$, which is the target of our control, while the open-loop system will evolve around its stable equilibrium $x^* = 0$.

4. Dissipation assignment

The previous energy assignment utilizes a state feedback control law to shape the closed-loop Hamiltonian (energy) with a stable equilibrium at $\boldsymbol{x}^*$. However, the closed-loop Hamiltonian also remains a constant if $\delta = 0$, i.e. $\frac{\mathrm{d}H_d}{\mathrm{d}t} = \nabla_{\boldsymbol{x}} H_d(\boldsymbol{x}) \cdot [-\boldsymbol{R}]\nabla_{\boldsymbol{x}} H_d(\boldsymbol{x}) = -\delta(\dot{x})^2 = 0$. We

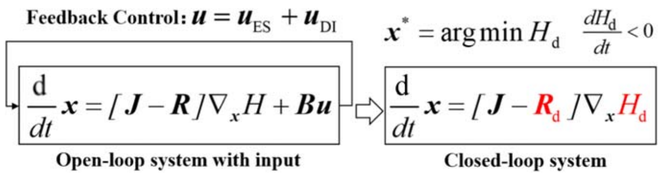

Figure 3. Summary of the energy and dissipation assignment.

may also design the equilibrium point as *asymptotically* stable. This can be achieved through dissipation assignment, which modifies the $\boldsymbol{R}$ matrix in the closed-loop system. Aside from the energy assignment control law, we also implement a state feedback control representing the dissipation assignment:

$$\boldsymbol{u}_{\mathrm{DI}}(\boldsymbol{x}) = -K_d \boldsymbol{B}^T \nabla_{\boldsymbol{x}} H_d(\boldsymbol{x}), \tag{15}$$

where K_d is a positive value. With $\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{u}_{\mathrm{ES}}(\boldsymbol{x}) + \boldsymbol{u}_{\mathrm{DI}}(\boldsymbol{x})$, we have the closed-loop system as

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = [\boldsymbol{J} - \boldsymbol{R}_d]\nabla_{\boldsymbol{x}} H_d(\boldsymbol{x}), \tag{16}$$

with the closed-loop dissipation matrix: $\boldsymbol{R}_d = \boldsymbol{R} + \boldsymbol{B}K_d\boldsymbol{B}^T$. Thus, the closed-loop Hamiltonian evolves like

$$\frac{\mathrm{d}H_d}{\mathrm{d}t} = \nabla_{\boldsymbol{x}} H_d(\boldsymbol{x}) \cdot [-\boldsymbol{R}_d]\nabla_{\boldsymbol{x}} H_d(\boldsymbol{x}) = -K_d(\dot{x})^2 < 0. \tag{17}$$

Under this control law, it is easy to show that the closed-loop Hamiltonian is a Lyapunov function satisfying

$$\begin{aligned} &H_d(\boldsymbol{x}) = 0, \ \ \boldsymbol{x} = \boldsymbol{x}^* \\ &H_d(\boldsymbol{x}) > 0, \ \ \forall \boldsymbol{x} \neq \boldsymbol{x}^* \\ &\frac{\mathrm{d}H_d(\boldsymbol{x})}{\mathrm{d}t} = \nabla_{\boldsymbol{x}} H_d(\boldsymbol{x}) \cdot [-\boldsymbol{R}_d]\nabla_{\boldsymbol{x}} H_d(\boldsymbol{x}) < 0, \ \ \forall \boldsymbol{x} \neq \boldsymbol{x}^*. \end{aligned} \tag{18}$$

According to the Lyapunov stability theorem in equation (1), the closed-loop system is *asymptotically* stable.

The framework of the energy and dissipation assignment is summarized in figure 3. The feedback control $\boldsymbol{u}_{\mathrm{ES}}$ and the modified Hamiltonian H_d are the same as that summarized in figure 1. Aside from these, we also introduce the dissipation assignment u_{DI}, which renders a different dissipation matrix $\boldsymbol{R}_d$ from the open-loop system. Thus, the closed-loop Hamiltonian satisfies $\frac{\mathrm{d}H_d}{\mathrm{d}t} < 0$, and we obtain the asymptotic stability of the closed-loop system based on the Lyapunov theorem in equation (1).

In the example of the undamped Duffing system, we implement the energy and dissipation assignment to this undamped system to make the equilibrium point $\boldsymbol{x}^*$ asymptotically stable. Using equation (15) with $K_d = 1$, we get

$$u_{\mathrm{DI}}(\boldsymbol{x}) = -\dot{x}. \tag{19}$$

This feedback control u_{DI} adds the damping into the closed-loop system, which is proportional to the velocity. The value of K_d will determine the convergence rate to the stable equilibrium. Actually, the dissipation assignment is not necessary to reach asymptotic stability, if the open-loop system already has the damping, i.e. $\delta > 0$.

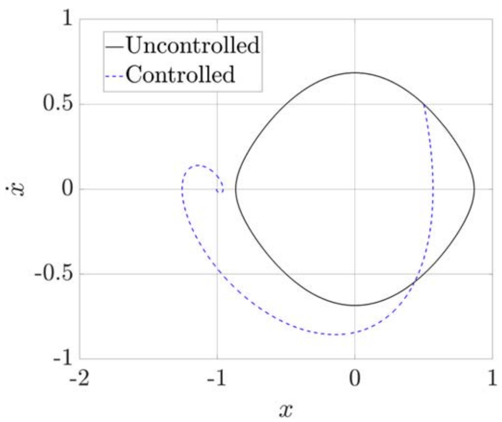

Figure 4. The $x - \dot{x}$ plane for the undamped Duffing oscillator with state feedback control $u = u_{\mathrm{ES}} + u_{\mathrm{DI}}$ (the blue dashed line) compared with the uncontrolled system (the black solid line).

With both the energy assignment and dissipation assignment $u = u_{\mathrm{ES}} + u_{\mathrm{DI}}$, the controlled system reaches the designed equilibrium points as shown in figure 4. The results shown by the blue line show that the equilibrium point $\boldsymbol{x}^* = [-1\ 0]^T$ is asymptotically stable and the system will evolve to this equilibrium position from any initial conditions. Without input, the open-loop system still moves around its stable equilibrium $x = 0$.

5. Structure assignment

The previous framework works well for the mechanical system, but it may fail for other complex systems without the assignment of $\boldsymbol{J}$ matrix. For example, for the Lorenz system (Bai and Lonngren 2000, Ho and Hung 2002, Jia 2007):

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \sigma y - m\sigma x + u \tag{20}$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = rx - xz - my \tag{21}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = xy - mbz, \tag{22}$$

with input u. The Lorenz system was initially developed as a simplified mathematical model for atmospheric convection, which describes the rate of change of three quantities with respect to time: x is proportional to the rate of convection, y to the horizontal temperature variation, and z to the vertical temperature variation. The constants σ, r, and b are system parameters proportional to the Prandtl number, Rayleigh number, and certain physical dimensions of the layer itself. The parameter m represents the system dissipation. For certain parameter values and initial conditions, the Lorenz system is notable for having chaotic solutions which resemble a butterfly or a figure eight.

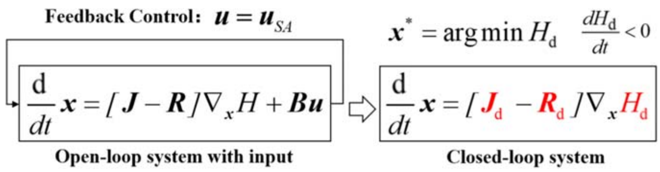

Figure 5. Summary of the structure assignment.

The Lorenz system with input u and dissipation ($m \neq 0$) can be reformulated as

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = [\boldsymbol{J}(\boldsymbol{x}) - \boldsymbol{R}]\nabla_{\boldsymbol{x}} H(\boldsymbol{x}) + \boldsymbol{B}u, \tag{23}$$

where $\boldsymbol{x} = [x\ y\ z]^T$ are state variables and $\boldsymbol{B} = [1\ 0\ 0]^T$ is the matrix that maps the input to the state variables. The skew-symmetric matrix $\boldsymbol{J}(\boldsymbol{x})$ and the symmetric matrix $\boldsymbol{R}$ are

$$\boldsymbol{J}(\boldsymbol{x}) = \begin{bmatrix} 0 & \sigma & 0 \\ -\sigma & 0 & -x \\ 0 & x & 0 \end{bmatrix}, \ \boldsymbol{R} = \begin{bmatrix} -m\sigma^2 r^{-1} & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & mb \end{bmatrix}, \tag{24}$$

respectively. The Hamiltonian is

$$H(\boldsymbol{x}) = \frac{1}{2}y^2 + \frac{1}{2}z^2 - \frac{r}{2\sigma}x^2, \tag{25}$$

which is a conserved quantity in the Lorenz system if $m = 0$.

The previous framework may fail for the Lorenz system in that it is difficult to shape the closed-loop Hamiltonian $H_d(\boldsymbol{x})$ and the dissipation matrix $\boldsymbol{R}_d$ together without modifying the skew-symmetric matrix $\boldsymbol{J}(\boldsymbol{x})$.

Here, the designed objective is to obtain a closed-loop system of the form

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = [\boldsymbol{J}_d(\boldsymbol{x}) - \boldsymbol{R}_d]\nabla_{\boldsymbol{x}} H_d(\boldsymbol{x}), \tag{26}$$

where all of the closed-loop skew-symmetry matrix $\boldsymbol{J}_d(\boldsymbol{x})$, the symmetric matrix $\boldsymbol{R}_d(\boldsymbol{x})$, and the Hamiltonian $H_d(\boldsymbol{x})$ are modified. If the closed-loop Hamiltonian $H_d(\boldsymbol{x})$ has an equilibrium point $\boldsymbol{x}^*$ and $\frac{\mathrm{d}H_d(\boldsymbol{x})}{\mathrm{d}t} = \nabla_{\boldsymbol{x}} H_d(\boldsymbol{x}) \cdot [-\boldsymbol{R}_d]\nabla_{\boldsymbol{x}} H_d(\boldsymbol{x}) < 0,\ \forall \boldsymbol{x} \neq \boldsymbol{x}^*$, the closed-loop system is asymptotically stable at the equilibrium point $\boldsymbol{x}^*$.

The inputs that achieve the closed-loop control objective are obtained through comparing equations (6) with (26):

$$\boldsymbol{u}_{\mathrm{SA}}(\boldsymbol{x}) = (\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T([\boldsymbol{J}_d(\boldsymbol{x}) - \boldsymbol{R}_d]\nabla_{\boldsymbol{x}} H_d(\boldsymbol{x}) - [\boldsymbol{J}(\boldsymbol{x}) - \boldsymbol{R}]\nabla_{\boldsymbol{x}} H(\boldsymbol{x})), \tag{27}$$

where the designed closed-loop Hamiltonian $H_d(x)$, the designed structure matrix $\boldsymbol{J}_d$, and damping matrix $\boldsymbol{R}_d$ are obtained by solving the matching condition:

$$\boldsymbol{B}^{\perp}[\boldsymbol{J}(x) - \boldsymbol{R}]\nabla_{\boldsymbol{x}} H(\boldsymbol{x}) = \boldsymbol{B}^{\perp}[\boldsymbol{J}_d(\boldsymbol{x}) - \boldsymbol{R}_d]\nabla_{\boldsymbol{x}} H_d(\boldsymbol{x}), \tag{28}$$

such that $H_d(x)$ has its minimum at the desired equilibrium $\boldsymbol{x}^*$. This matching condition is a generalized version of equation (11). Under this control law, it is easy to show that the closed-loop Hamiltonian $H_d(\boldsymbol{x})$ is a Lyapunov function satisfying equation (18). According to the Lyapunov stability theorem in equation (1), the closed-loop system is asymptotically stable.

The framework of the structure assignment is summarized in figure 5. Aside from the previous procedure where the closed-loop Hamiltonian H_d and the closed-loop dissipation matrix $\boldsymbol{R}_d$ are different from the open-loop system, the closed-loop structure matrix $\boldsymbol{J}_d$ is also

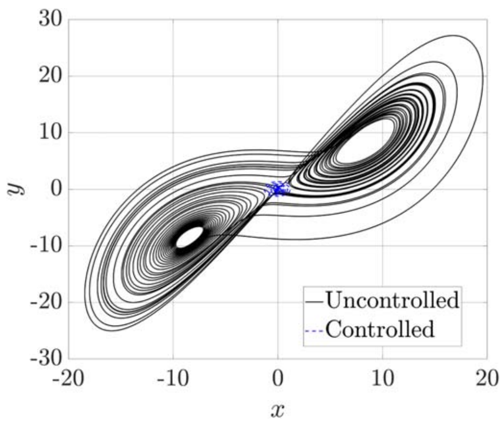

Figure 6. The $x - y$ plane for the Lorenz system with state feedback control $u = u_{SA}$ (the blue dashed line) compared with the uncontrolled system (the black solid line).

different from the open-loop system. Correspondingly, the feedback control $\boldsymbol{u}_{\mathrm{SA}}$ for the structure assignment might be more comprehensive, which serves to modify the closed-loop Hamiltonian, dissipation matrix and structure matrix together.

For example, we would like to make the origin $\boldsymbol{x}^* = [0\ 0\ 0]^T$ asymptotically stable and the closed-loop Hamiltonian

$$H_d(\boldsymbol{x}) = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2 \tag{29}$$

satisfies the requirements of the Lyapunov function in equation (1). Then, we need to solve the constraints in equation (28):

$$\begin{bmatrix} rx - xz - my \\ xy - mbz \end{bmatrix} = \boldsymbol{B}^{\perp}[\boldsymbol{J}_d(\boldsymbol{x}) - \boldsymbol{R}_d]\begin{bmatrix} x \\ y \\ z \end{bmatrix}, \boldsymbol{B}^{\perp} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{30}$$

to get $\boldsymbol{J}_d(\boldsymbol{x})$ and $\boldsymbol{R}_d$. The above constraints render many possible solutions for $\boldsymbol{J}_d(\boldsymbol{x})$ and $\boldsymbol{R}_d$, and we choose

$$\boldsymbol{J}_d(\boldsymbol{x}) = \begin{bmatrix} 0 & -r & 0 \\ r & 0 & -x \\ 0 & x & 0 \end{bmatrix}, \ \boldsymbol{R}_d = \begin{bmatrix} K_d & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & mb \end{bmatrix}, \ K_d > 0, \tag{31}$$

which satisfy the requirements in equation (30). According to the Lyapunov stability criteria in equation (1), the origin $\boldsymbol{x}^*$ is asymptotically stable in the closed-loop system.

Then, we have the control law according to equation (27):

$$u_{\mathrm{SA}}(\boldsymbol{x}) = -ry - K_d x - (\sigma y - m\sigma x). \tag{32}$$

In figures 6 and 7, we show the $x - y$ plane for the Lorenz system with state feedback control u_{SA} compared with the uncontrolled system and the $x - z$ plane results are shown in figures 8 and 9. The parameters in the Lorenz system are set as $\sigma = 10$, $r = 28$, $b = \frac{8}{3}$, and $m = 1$. The initial conditions are set as $\boldsymbol{x}_0 = [1\ 1\ 1]^T$.

These results show that the designed structure assignment state feedback control law allows us to make the origin asymptotically stable. Without input, the Lorenz system evolves like a 'butterfly' as shown by the black lines in figures 6–9. With the feedback control $\boldsymbol{u}_{\mathrm{SA}}$ in equation (32), the Lorenz system evolves to the origin as shown in the blue lines in figures 6–

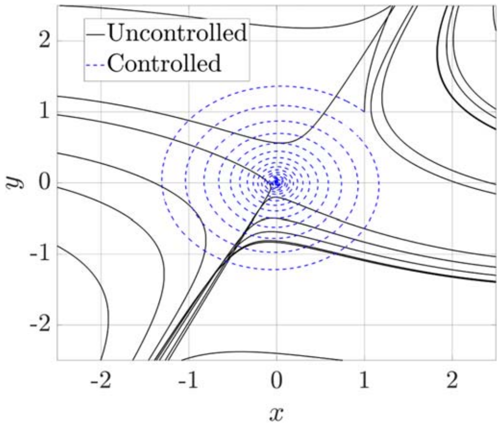

Figure 7. The locally enlarged $x - y$ plane for the Lorenz system with state feedback control $u = u_{\mathrm{SA}}$ (the blue dashed line) compared with the uncontrolled system (the black solid line).

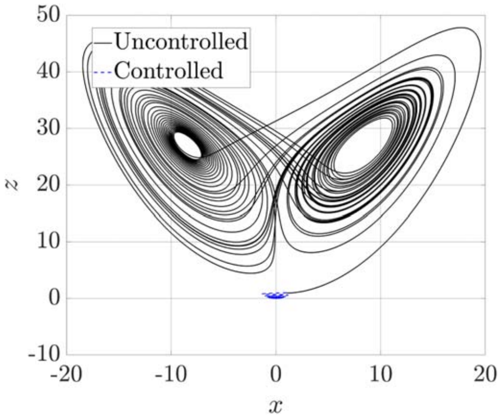

Figure 8. The $x - z$ plane for the Lorenz system with state feedback control $u = u_{\mathrm{SA}}$ (the blue dashed line) compared with the uncontrolled system (the black solid line).

9, i.e. the origin is asymptotically stable. It is remarkable that only using a linear controller in the x direction allows us to stabilize the origin point in the Lorenz system. As the Lorenz system is an example of a chaotic system, these results provide the potential for further control of chaos and the synchronization of a chaotic system (Bai and Lonngren 2000, Ho and Hung 2002, Jia 2007) using the port-Hamiltonian approach developed here.

6. Conclusion

In this paper, we use the port-Hamiltonian framework to reformulate a physics system and to introduce the physics-based control strategy, including the energy, dissipation, and structure assignments. The energy and dissipation assignments are illustrated by the Duffing oscillator, and the structure assignment is illustrated by the Lorenz system. The closed-loop Hamiltonian is a candidate of the Lyapunov function, which guarantees the global stability of the closed-

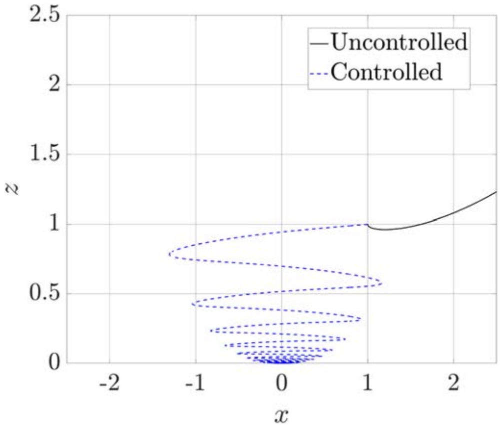

Figure 9. The locally enlarged $x - y$ plane for the Lorenz system with state feedback control $u = u_{\mathrm{SA}}$ (the blue dashed line) compared with the uncontrolled system (the black solid line).

loop system. We also provide port-Hamiltonian descriptions for two examples in celestial mechanics. This framework and examples are easy for physics students to understand and provide a promising way for control theory education in physics.

Appendix A. Port-Hamiltonian description of examples in celestial mechanics

In this appendix, we will provide two other examples from celestial mechanics that can be reformulated into the port-Hamiltonian system:

Two-body problem: The two-body problem (Topputo and Zhang 2014) governed by Newton's law of universal gravitation is a good model for a satellite moving around a low earth orbit, which is commonly seen in physics class. The two-body problem is also widely used in astronomy, e.g. describing the Earth-Moon system. In this problem, we neglect the air drag and other factors that result in energy dissipation. The governing equations in the polar coordinate are

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} r \\ v_r \\ \theta \\ v_t \end{bmatrix} = \begin{bmatrix} v_r \\ \frac{v_t^2}{r} - \frac{m_0}{r^2} \\ \frac{v_t}{r} \\ -\frac{v_r v_t}{r} \end{bmatrix}, \tag{A.1}$$

where r and v_r are the radius and radius velocity, respectively; θ and v_t are polar angle and tangential velocity, respectively. m_0 is the mass constant. We reformulate the governing equations choosing the total energy as the Hamiltonian:

$$H(\boldsymbol{x}) = \frac{1}{2}[v_r^2 + v_t^2] - \frac{m_0}{r} \tag{A.2}$$

$$= \frac{1}{2}\left[v_r^2 + \left(\frac{L}{r}\right)^2\right] - \frac{m_0}{r}, \tag{A.3}$$

where constant $L = rv_t$ is the angular momentum. Then, we have the port-Hamiltonian description of the two-body problem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = [\boldsymbol{J} - \boldsymbol{R}]\nabla_{\boldsymbol{x}} H(\boldsymbol{x}) + \boldsymbol{B}\boldsymbol{u}, \tag{A.4}$$

where $\boldsymbol{x} = [r\ v_r]^T$ is the state variable, $\boldsymbol{u}$ is the input, and $\boldsymbol{B}$ is the matrix that maps the inputs to the state variables. The skew-symmetric matrix $\boldsymbol{J}$ is

$$\boldsymbol{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{A.5}$$

and the symmetric matrix is $\boldsymbol{R} = \boldsymbol{0}_{2\times 2}$. A non-zero $\boldsymbol{R}$ matrix has the potential to consider the factors that result in energy dissipation, such as air drag, into the model.

Circular Restricted Three-Body Problem (CRTBP): In the CRTBP, two massive bodies move in a circular orbit around their common center of mass. The third mass is negligible with respect to the other two, and thus we neglect the force from the third mass acting on the other two larger masses. CRTBP is used to model the Earth-Moon-Satellite system for the determination of spacecraft trajectories for satellites (Musielak and Quarles 2014). With respect to a rotating reference frame, the two co-orbiting bodies are considered stationary, and the ratios of mass and distance are considered instead of their actual values because their relative values are of the most importance. The governing equations of the CRTBP (Musielak and Quarles 2014, Zhang *et al* 2015, Liu and Dong 2019a) are

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - 2\frac{\mathrm{d}y}{\mathrm{d}t} = x - (1-\mu)\frac{x+\mu}{r_1^3} - \mu\frac{x-1+\mu}{r_2^3} \tag{A.6}$$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}x}{\mathrm{d}t} = y - (1-\mu)\frac{y}{r_1^3} - \mu\frac{y}{r_2^3}, \tag{A.7}$$

where x, y are the positions parallel and perpendicular to the line of two larger bodies, respectively. Here, μ is the mass ratio between the second largest body and the two largest bodies. For example, we set $\mu = 0.012\ 155\ 085$ in the Earth-Moon-Satellite three-body system, where μ is the mass ratio between the Moon and the Earth-Moon system. The r_1 and r_2 in above equations

$$r_1 = \sqrt{(x+\mu)^2 + y^2} \tag{A.8}$$

$$r_2 = \sqrt{(x-1+\mu)^2 + y^2} \tag{A.9}$$

are the distance from the third body to the two other larger bodies, which are located at $(-\mu, 0)$ and $(1-\mu, 0)$, respectively. Here, we use the Hamiltonian:

$$H(\boldsymbol{x}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - \frac{1}{2}[(x^2 + y^2) + 2(\frac{1-\mu}{r_1} + \frac{\mu}{r_2})], \tag{A.10}$$

to reformulate the CRTBP problem into the port-Hamiltonian system (Liu and Dong 2019b):

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{x} = [\boldsymbol{J} - \boldsymbol{R}]\nabla_{\boldsymbol{x}} H(\boldsymbol{x}) + \boldsymbol{B}\boldsymbol{u}. \tag{A.11}$$

$\boldsymbol{x} = [x\ y\ \dot{x}\ \dot{y}]^T$ are state variables and $\boldsymbol{u} = [u_x\ u_y]^T$ are inputs, which represent the thrust force acting in the x and y directions. These inputs may be related to the state variables $\boldsymbol{x}$, which represent the state feedback control. $\boldsymbol{B} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^T$ is the matrix that maps the inputs to the state variables. $\boldsymbol{J}$ is a skew-symmetric matrix, which represents the energy (Hamiltonian) conserving part, and $\boldsymbol{R}$ is a symmetric matrix, which represents the energy dissipation, i.e.

$$\boldsymbol{J} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 2 \\ 0 & -1 & -2 & 0 \end{bmatrix}, \ \boldsymbol{R} = \boldsymbol{0}_{4\times 4}. \tag{A.12}$$

The circular restricted three-body problem described above does not consider the energy dissipation, and a non-zero $\boldsymbol{R}$ matrix provides further potential to take the factors that result in energy dissipation into consideration, such as air drag. The proposed framework in this paper also works for the two-body problem and CRTBP. We refer to Liu and Dong (2019b) for further information.

Appendix B. Derivation of the feedback control law

In this appendix, we provide the details of deriving the feedback control law in sections 3–5, including the energy assignment $\boldsymbol{u}_{\mathrm{ES}}$, the dissipation assignment $\boldsymbol{u}_{\mathrm{DI}}$, and the structure assignment $\boldsymbol{u}_{\mathrm{SA}}$.

We start from the most general case, i.e. the structural assignment in section 5. Comparing the right-hand sides of equation (6) with (26) or comparing the open-loop system with the closed-loop system illustrated in figure 5, we obtain

$$[\boldsymbol{J} - \boldsymbol{R}]\nabla_x H + \boldsymbol{B}\boldsymbol{u}_{\mathrm{SA}} = [\boldsymbol{J}_d - \boldsymbol{R}_d]\nabla_x H_d. \tag{B.1}$$

With the assumption that $\boldsymbol{B}$ is a full column rank, we have $\boldsymbol{B}^T\boldsymbol{B}$ as invertible. Through multiplying equation (B.1) with $(\boldsymbol{B}^T\boldsymbol{B})^{-1}\boldsymbol{B}^T$, we can solve the input $\boldsymbol{u}_{\mathrm{SA}}$ as equation (27). However, the closed-loop structure matrix $\boldsymbol{J}_d$ and dissipation matrix $\boldsymbol{R}_d$ should satisfy certain constraints in that $\boldsymbol{B}$ is usually not full row rank. Multiplying equation (B.1) by $\boldsymbol{B}^{\perp}$; i.e. the left-annihilator of $\boldsymbol{B}$, it gives the constraint in equation (28) in that $\boldsymbol{B}^{\perp}\boldsymbol{B} = \boldsymbol{0}$.

For the energy assignment in section 3, it is a special case of structure assignment where $\boldsymbol{J}_d = \boldsymbol{J}$ and $\boldsymbol{R}_d = \boldsymbol{R}$. Setting $\boldsymbol{J}_d = \boldsymbol{J}$ and $\boldsymbol{R}_d = \boldsymbol{R}$, equations (27) and (28) end up as equations (9) and (11) with $H_a(\boldsymbol{x}) = H_d(\boldsymbol{x}) - H(\boldsymbol{x})$ for ease of exposition.

With respect to the dissipation assignment in equation (15), $\boldsymbol{u}_{\mathrm{DI}}$ modifies the evolution of the closed-loop Hamiltonian, i.e. $\frac{\mathrm{d}H_d}{\mathrm{d}t}$. The $\boldsymbol{B}^T$ appears in equation (15) and is used to guarantee the symmetricity of the $\boldsymbol{B}\boldsymbol{u}_{\mathrm{DI}}$. The physical meaning of $\boldsymbol{u}_{\mathrm{DI}}$ is dissipation assignment, which provides the potential to ensure asymptotic stability. If the original system already has dissipation, the dissipation assignment is not necessary.

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