

Structured input–output analysis of transitional wall-bounded flows

Chang Liu^{1†}, Dennice F. Gayme ¹

¹Department of Mechanical Engineering, Johns Hopkins University, Baltimore, MD 21218,
USA

(Received xx; revised xx; accepted xx)

Input–output analysis of transitional channel flows has proven to be a valuable analytical tool for identifying important flow structures and energetic motions. The traditional approach abstracts the nonlinear terms as forcing that is unstructured, in the sense that is not directly tied to the underlying nonlinearity in the dynamics. This paper instead employs a *structured* singular value-based approach that preserves certain input–output properties of the nonlinear forcing function in an effort to recover the larger range of key flow features identified through nonlinear analysis, experiments, and direct numerical simulation (DNS) of transitional channel flows. Application of this method to transitional plane Couette and plane Poiseuille flows leads to identification of not only the streamwise coherent structures predicted through traditional input–output approaches but also characterization of the oblique flow structures as those requiring the least energy to induce transition in agreement with DNS studies, and nonlinear optimal perturbation analysis. The proposed approach also captures the recently observed oblique turbulent bands that have been linked to transition in experiments and DNS with very large channel size. The ability to identify the larger amplification of the streamwise varying structures predicted from DNS and nonlinear analysis in both flow regimes suggests that the structured approach allows one to maintain the nonlinear effects associated with the saturation of the lift-up mechanism, which is known to dominate the linear operator. Capturing this key nonlinear mechanism enables the prediction of the wider range of known transitional flow structures within the analytical input–output modeling paradigm.

1. Introduction

Interest in transitional wall-bounded shear flow dates back to early studies by Reynolds (1883), who noted that the flow was sensitive to disturbances. Though much progress has been made, a full understanding of the phenomena has yet to be realized. One of the main challenges lies in the fact that linear stability analysis fails to accurately predict the Reynolds numbers at which flows are observed transition to turbulence. For example, plane Couette flow is linearly stable for any Reynolds number (Romanov 1973) yet is observed to transition to turbulence at Reynolds numbers as low as 360 ± 10 (Tillmark & Alfredsson 1992). This failure has led researchers to study the mechanisms underlying transition by instead analyzing energy growth. In particular, there has been an emphasis on characterizing the types of finite-amplitude perturbations that are most likely to lead to transition as well as the flow structures that dominate in this regime, see e.g. Schmid & Henningson (1992); Lundbladh *et al.* (1994); Reddy *et al.* (1998); Philip *et al.* (2007); Duguet *et al.* (2010a, 2013); Farano *et al.* (2015).

† Email address for correspondence: changliu@jhu.edu

Reddy *et al.* (1998) examined the relative effect of different transition-inducing flow perturbations in both plane Couette flow and Poiseuille flow through extensive direct numerical simulations (DNS). These authors observed that both streamwise vortices and oblique waves require less energy density than random noise to trigger transition (Reddy *et al.* 1998, figures 19 and 23) in both flows. They further showed that in Poiseuille flow even perturbations in the form of Tollmien–Schlichting (TS) waves, which are linearly unstable at $Re > 5772$ (Orszag 1971), require larger energy density to trigger transition than either streamwise vortices or oblique waves (Reddy *et al.* 1998, figure 19). Similar behavior has been observed in studies of the transient energy growth and input–output response of the linearized Navier–Stokes (LNS) equations (Reddy & Henningson 1993; Jovanović & Bamieh 2005). In fact, input–output analysis of channel flow suggests that streamwise constant structures have larger energy growth than the linearly unstable TS waves, even at supercritical Reynolds numbers (i.e. above the Reynolds number at which the laminar flow is no longer linearly stable) (Jovanović & Bamieh 2004; Jovanović 2004). Studies of the LNS have indicated that streamwise vortical structures represent both the initial condition (optimal perturbation) that leads to the largest energy growth (Gustavsson 1991; Butler & Farrell 1992; Reddy & Henningson 1993; Schmid & Henningson 2012) as well as the type of structures that sustains the highest energy growth, see e.g. (Farrell & Ioannou 1993; Bamieh & Dahleh 2001; Jovanović & Bamieh 2005). The importance of streamwise vortices was also confirmed by Bottin *et al.* (1998), who connected experimental results with this form of exact coherent structures in plane Couette flow.

On the other hand, the simulations of Schmid & Henningson (1992) and Reddy *et al.* (1998) as well as the experiments of Elofsson & Alfredsson (1998) indicate that perturbations of oblique waves require slightly less energy than streamwise vortices to initiate transition. Nonlinear optimal perturbations (NLOP) to plane Couette flow, i.e. the initial perturbations that require the least energy to transition the flow from laminar to turbulent, also take the form of oblique waves that are localized in the streamwise direction, see e.g., (Duguet *et al.* 2010a, 2013; Monokrousos *et al.* 2011; Rabin *et al.* 2012; Cherubini & De Palma 2013, 2015). For plane Poiseuille flow, hairpin vortices associated with the very short timescale of the Orr mechanism represent the NLOP (Farano *et al.* 2015, 2016). These results suggest that traditional linear analysis does not capture the full range of highly amplified structures in transitional flows.

Recent experiments and DNS of plane Couette flow with very large channel size ($\sim O(100)$ times the channel half-height) have also uncovered oblique turbulent bands (turbulent stripes) in the transitional flow regime of wall-bounded shear flows, see e.g. (Prigent *et al.* 2002, 2003; Duguet *et al.* 2010b; De Souza *et al.* 2020; Tuckerman *et al.* 2020). These turbulent-laminar patterns were also observed to appear in DNS of transitional plane Poiseuille flow when the channel size became large enough; see e.g., (Tsukahara *et al.* 2005; Xiong *et al.* 2015; Tao *et al.* 2018; Kanazawa 2018; Shimizu & Manneville 2019; Xiao & Song 2020; Song & Xiao 2020). The presence of such structures was later confirmed by experiments (Tsukahara *et al.* 2014; Paranjape 2019; Paranjape *et al.* 2020, figure 1). There is strong evidence that the mechanisms leading to the growth and maintenance of these oblique turbulent bands are nonlinear in both plane Couette (Barkley & Tuckerman 2007; Tuckerman & Barkley 2011; Duguet & Schlatter 2013) and plane Poiseuille flow (Tuckerman *et al.* 2014). That view has been further supported by analysis of exact equilibrium and traveling wave solutions of the nonlinear Navier–Stokes (NS) equations; see e.g., for plane Couette flow (Deguchi & Hall 2015; Reetz *et al.* 2019) and Poiseuille flow (Paranjape *et al.* 2020).

The literature described above points to the benefit of nonlinear methods in character-

izing the full range of flow structures in transitional channel flow. However, these methods have far larger computational costs than linear analysis methods; see e.g., (Kerswell *et al.* 2014; Kerswell 2018). This trade-off between obtaining a more comprehensive characterization of the phenomena and analysis that is computationally tractable is long-standing. However, there is significant evidence to suggest that insight can be gained through parametrizing or bounding the effect of the nonlinearity rather than undertaking the full computational burden of resolving it. For example, Kreiss *et al.* (1994) and Chapman (2002) employed a bound on the nonlinearity to derive a finite-amplitude permissible perturbation that a flow could sustain while remaining laminar. More recently, finite-amplitude stability analysis of transitional shear flows employed local componentwise (sector) bounds on the nonlinearity and exploited the passivity of the nonlinear operator to develop linear matrix inequality (LMI) based approaches to compute bounds on permissible perturbations for a range of shear flow models; see e.g., Kalur *et al.* (2020*a,b*, 2021); Liu & Gayme (2020*b*). The inclusion of information about the nonlinear behavior produced results that matched simulation data better than those derived through previous linear approaches. Data-driven methods to parametrize or color (in space or time) the forcing have also enabled nonlinear effects to be captured within the input–output framework leading to better prediction of flow statistics (Chevalier *et al.* 2006; Moarref *et al.* 2013, 2014; Zare *et al.* 2017; Towne *et al.* 2020; Morra *et al.* 2021; Nogueira *et al.* 2021). The effect of the nonlinearity in the Navier Stokes (NS) equations has also been incorporated directly into input–output and resolvent analysis through shaping or parametrizing the forcing, e.g. by including larger amplitude forcing in the near-wall region (Jovanović & Bamieh 2001; Höpffner *et al.* 2005).

In this work, we build on this notion of including the effect of the nonlinearity within a computationally tractable linear framework using the concept of a *structured* uncertainty, see e.g., (Packard & Doyle 1993; Zhou *et al.* 1996). In particular, we partition the NS equations as shown in Figure 1 and structure the feedback to enforce a block-diagonal structure (bottom block outlined by the blue dashed line $--$). The feedback defines the componentwise inputs to the linearized momentum equations, which are modeled in terms of an uncertain gain $-\boldsymbol{u}_\xi$ of an input-output mapping from each component ∇u , ∇v and ∇w to the respective forcings $f_{x,\xi}$, $f_{y,\xi}$ and $f_{z,\xi}$. We represent this gain using the *structured* singular value (Doyle 1982; Safonov 1982), μ , which we used to define the most destabilizing feedback gain under the structured forcing (Packard & Doyle 1993). Conceptually the approach allows us to develop a feedback interconnection between the LNS and a structured forcing that is explicitly constrained to preserve the componentwise structure of the nonlinearity in the NS equations.

Structured input–output analysis shares the advantages of all methods employing the spatio-temporal frequency response based analysis techniques upon which it is built, see e.g., (Farrell & Ioannou 1993; Bamieh & Dahleh 2001; Jovanović & Bamieh 2005; McKeon & Sharma 2010; McKeon *et al.* 2013; McKeon 2017; Illingworth *et al.* 2018; Vadarevu *et al.* 2019; Madhusudanan *et al.* 2019; Symon *et al.* 2021; Liu & Gayme 2019, 2020*a*). Of greatest interest in this work is its computational tractability versus nonlinear approaches and the lack of finite channel size effects that can plague both DNS and experimental studies. This approach is most closely related to the analysis of the largest singular value (\mathcal{H}_∞ norm) of the spatio-temporal frequency response of the linearized dynamics (top-block of figure 1), which measures the structure that sustains the highest input–output growth, see e.g., Jovanović (2004, chapter 8.1.2); Schmid (2007); Hwang & Cossu (2010*a,b*); Illingworth (2020). However, in that work, the forcing is assumed to excite the dynamics at all frequencies (e.g., delta-correlated spatio-temporal white noise); in this sense, it can be thought of as the open-loop response of the top-block in figure 1. The

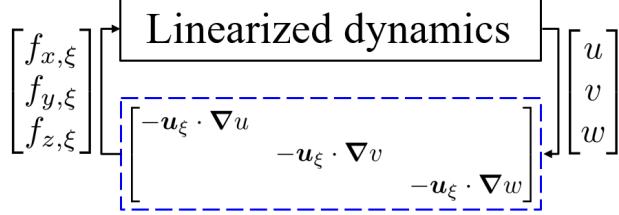


Figure 1: Block diagram representing structured input–output feedback interconnection between the linearized dynamics and (—) structured forcing (modeling the nonlinearity). In particular, each component of the forcing is modeled as an input–output mapping from the respective component of velocity gradient ∇u , ∇v , ∇w to the respective component forcings $f_{x,\xi}$, $f_{y,\xi}$, $f_{z,\xi}$ of the linearized dynamics with the gain $-u_\xi$ defined in terms of the structured singular value of a linearized closed-loop system response.

unstructured response provides an upper bound on the structured input–output response (Packard & Doyle 1993).

We apply the proposed *structured* input–output analysis to transitional plane Couette and plane Poiseuille flow. The results indicate that the addition of a structured feedback interconnection enables identification and analysis of the wider range of transition-inducing flow structures identified in the literature without the computational burden of nonlinear optimization or extensive simulations. More specifically, the results for transitional plane Couette flow reproduce results from DNS based analysis (Reddy *et al.* 1998) and predictions of NLOP approaches (Rabin *et al.* 2012), which both indicate that oblique waves require less energy to destabilize the flow than the streamwise elongated structures emphasized in traditional input–output analysis. In plane Poiseuille flow, these transition-inducing flow structures are consistent with DNS (Reddy *et al.* 1998) emphasizing oblique waves and NLOP analysis that highlights the importance of spatially localized structures with streamwise wavelengths larger than their spanwise extent (Farano *et al.* 2015). The proposed approach also reproduces the characteristic wavelengths and angle of the oblique turbulent band observed in very large channel studies of transitional plane Couette flow (Prigent *et al.* 2003). The wavelengths of oblique turbulent bands in transitional plane Poiseuille flow with very large channel size (Kanazawa 2018) also fall within the range of flow structures showing large structured input–output response.

The agreement between prediction from structured input–output analysis and observation in experiments, DNS, and NLOP show that this framework captures important nonlinear effects. In particular, the results suggest that restricting the feedback in a componentwise manner preserves the structure of the nonlinear mechanisms that weaken the streaks developed through the lift-up effect, in which cross-stream forcing amplifies streamwise streaks (Ellingsen & Palm 1975; Landahl 1975; Brandt 2014). This linear amplification of streamwise structures without a corresponding weakening effect likely leads to their dominance in traditional input–output analysis, see e.g. the discussion in Jovanović (2021). The Reynolds number dependence observed in our study further supports this idea that the change in response is associated with nonlinear saturation of the lift-up effect.

The remainder of this paper is organized as follows. Section 2 describes the flow configurations of interest and describes the details of the structured input–output analysis approach. Section 3 analyzes the results obtained from the application of structured

input–output analysis to both plane Couette and plane Poiseuille flows. This paper is concluded in § 4.

2. Formulating the structured input–output model

2.1. Governing Equations

We consider incompressible flow between two infinite parallel plates and employ x , y , and z to respectively denote the streamwise, wall-normal, and spanwise directions. The corresponding velocity components are denoted as u , v , and w . The coordinate frames and configurations for plane Couette and plane Poiseuille flows are shown in figure 2. We express the velocity field as a vector $\mathbf{u}_{\text{tot}} = [u_{\text{tot}} \ v_{\text{tot}} \ w_{\text{tot}}]^T$ with T indicating the transpose. We decompose the velocity field into the sum of a laminar base flow ($U(y) = y$ for plane Couette flow and $U(y) = 1 - y^2$ plane Poiseuille flow) and fluctuations about the base flow \mathbf{u} ; i.e., $\mathbf{u}_{\text{tot}} = U(y)\mathbf{e}_x + \mathbf{u}$ with \mathbf{e}_x denoting the x -direction (streamwise) unit vector. The pressure field is decomposed similarly as $p_{\text{tot}} = P + p$. The dynamics of the fluctuations \mathbf{u} and p are governed by the Navier-Stokes equations:

$$\partial_t \mathbf{u} + U \partial_x \mathbf{u} + v \frac{dU}{dy} \mathbf{e}_x + \nabla p - \frac{1}{Re} \nabla^2 \mathbf{u} = -\mathbf{u} \cdot \nabla \mathbf{u}, \quad (2.1a)$$

$$\nabla \cdot \mathbf{u} = 0. \quad (2.1b)$$

Here, the spatial variables are normalized by the channel half-height h : e.g., $y = y_*/h \in [-1, 1]$, where the subscript $*$ indicates dimensional quantities. The velocity is normalized by a nominal characteristic velocity U_n , where $\pm U_n$ is the velocity at channel walls for plane Couette flow, and U_n is channel centerline velocity for plane Poiseuille flow. Time and pressure are normalized by h/U_n and ρU_n^2 , respectively. The Reynolds number is defined as $Re = U_n h / \nu$, where ν is the kinematic viscosity. In equation (2.1), $\nabla := [\partial_x \ \partial_y \ \partial_z]^T$ represents the gradient operator, and $\nabla^2 := \partial_x^2 + \partial_y^2 + \partial_z^2$ represents the Laplacian operator. We impose no-slip boundary conditions at the wall; i.e., $\mathbf{u}(y = \pm 1) = \mathbf{0}$ for both flows. Finally, we write the nonlinear term in equation (2.1a) as

$$\mathbf{f} := -\mathbf{u} \cdot \nabla \mathbf{u} = [-\mathbf{u} \cdot \nabla u \ -\mathbf{u} \cdot \nabla v \ -\mathbf{u} \cdot \nabla w]^T =: [f_x \ f_y \ f_z]^T, \quad (2.2)$$

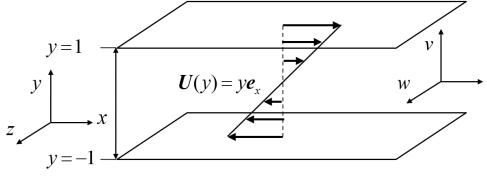
where $=:$ indicates that the right-hand side is defined by the left-hand side. We refer to f_x , f_y , and f_z as the respective streamwise, wall-normal, and spanwise components of the nonlinearity and collectively as the nonlinear components of (2.1). This expression of the nonlinearity as forcing terms makes (2.1) into a set of forced linear evolution equations.

We next construct the model of the nonlinearity that will allow us to build the feedback interconnection of figure 1. The velocity field $-\mathbf{u}$ in (2.2) associated with the forcing components can be viewed as the gain operator of an input–output system in which the velocity gradients ∇u , ∇v , ∇w act as the respective inputs and the forcing components f_x , f_y and f_z act as the respective output. It is this gain that we seek to model through $-\mathbf{u}_\xi$ in figure 1. This input–output model of the nonlinear components is given by

$$\mathbf{f}_\xi := -\mathbf{u}_\xi \cdot \nabla \mathbf{u} = [-\mathbf{u}_\xi \cdot \nabla u \ -\mathbf{u}_\xi \cdot \nabla v \ -\mathbf{u}_\xi \cdot \nabla w]^T =: [f_{x,\xi} \ f_{y,\xi} \ f_{z,\xi}]^T, \quad (2.3)$$

where $-\mathbf{u}_\xi$ maps the corresponding velocity gradient into each component of the modeled forcing driving linearized dynamics. The next subsection describes how we construct the input–output map so that it enables us to analyze the perturbations that are most likely to induce transition using the structured singular value formalism (Packard & Doyle 1993; Zhou *et al.* 1996).

(a)



(b)

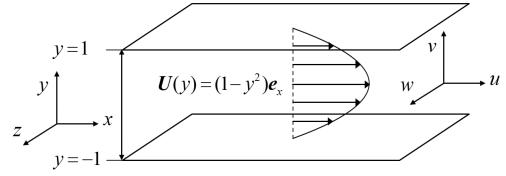


Figure 2: Illustrations of flows between two parallel flat plates: (a) plane Couette flow with laminar base flow $\mathbf{U}(y) = ye_x$ and (b) plane Poiseuille flow with laminar base flow $\mathbf{U}(y) = (1 - y^2)e_x$.

2.2. Structured input–output response

We now define the spatio-temporal frequency response $\mathcal{H}_\nabla(y; k_x, k_z, \omega)$ that will form the basis of structured input–output response. We first perform the standard transformation to express the dynamics (2.1) in terms of the wall-normal velocity v and wall-normal vorticity $\omega_y := \partial_z u - \partial_x w$ (Schmid & Henningson 2012), which enforces (2.1b) and eliminates the pressure dependence. We then exploit the shift-invariance in the (x, z) spatial directions of the two flow configurations of interest and assume invariance to shifts in t , which allows us to perform the following triple Fourier transform for a variable ψ .

$$\widehat{\psi}(y; k_x, k_z, \omega) := \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi(x, y, z, t) e^{-i(k_x x + k_z z + \omega t)} dx dz dt, \quad (2.4)$$

where $i = \sqrt{-1}$ is the imaginary unit and ω is the temporal frequency. $k_x = 2\pi/\lambda_x$ and $k_z = 2\pi/\lambda_z$ are the respective dimensionless x and z wavenumbers.

The resulting system of equations describing the transformed linearized equations subject to the modeled forcing \mathbf{f}_ξ is

$$i\omega \begin{bmatrix} \widehat{v} \\ \widehat{\omega}_y \end{bmatrix} = \widehat{\mathcal{A}} \begin{bmatrix} \widehat{v} \\ \widehat{\omega}_y \end{bmatrix} + \widehat{\mathcal{B}} \begin{bmatrix} \widehat{f}_{x,\xi} \\ \widehat{f}_{y,\xi} \\ \widehat{f}_{z,\xi} \end{bmatrix}, \quad \begin{bmatrix} \widehat{u} \\ \widehat{v} \\ \widehat{w} \end{bmatrix} = \widehat{\mathcal{C}} \begin{bmatrix} \widehat{v} \\ \widehat{\omega}_y \end{bmatrix}. \quad (2.5a,b)$$

The operators in equation (2.5) are defined following Jovanović & Bamieh (2005):

$$\widehat{\mathcal{A}}(k_x, k_z) := \begin{bmatrix} \widehat{\nabla}^2 & 0 \\ 0 & \mathcal{I} \end{bmatrix}^{-1} \begin{bmatrix} -ik_x U \widehat{\nabla}^2 + ik_x U'' + \widehat{\nabla}^4 / Re & 0 \\ -ik_z U' & -ik_x U + \widehat{\nabla}^2 / Re \end{bmatrix}, \quad (2.6a)$$

$$\widehat{\mathcal{B}}(k_x, k_z) := \begin{bmatrix} \widehat{\nabla}^2 & 0 \\ 0 & \mathcal{I} \end{bmatrix}^{-1} \begin{bmatrix} -ik_x \partial_y & -(k_x^2 + k_z^2) & -ik_z \partial_y \\ ik_z & 0 & -ik_x \end{bmatrix}, \quad (2.6b)$$

$$\widehat{\mathcal{C}}(k_x, k_z) := \frac{1}{k_x^2 + k_z^2} \begin{bmatrix} ik_x \partial_y & -ik_z \\ k_x^2 + k_z^2 & 0 \\ ik_z \partial_y & ik_x \end{bmatrix}, \quad (2.6c)$$

where $U' := dU(y)/dy$, $U'' := d^2U(y)/dy^2$, $\widehat{\nabla}^2 := \partial_{yy} - k_x^2 - k_z^2$, and $\widehat{\nabla}^4 := \partial_y^{(4)} - 2(k_x^2 + k_z^2)\partial_{yy} + (k_x^2 + k_z^2)^2$. The boundary conditions, which can be derived from the no-slip conditions are

$$\widehat{v}(y = \pm 1) = \frac{\partial \widehat{v}}{\partial y}(y = \pm 1) = \widehat{\omega}_y(y = \pm 1) = 0. \quad (2.7)$$

The spatio-temporal frequency response \mathcal{H} of the system in (2.5), which maps the input forcing $\widehat{\mathbf{f}}_\xi(y; k_x, k_z, \omega)$ to the velocity vector $\widehat{\mathbf{u}}(y; k_x, k_z, \omega)$ at the same spatial-temporal

wavenumber-frequency triplet; i.e., $\hat{\mathbf{u}}(y; k_x, k_z, \omega) = \mathcal{H}(y; k_x, k_z, \omega) \hat{\mathbf{f}}_\xi(y; k_x, k_z, \omega)$ is given by

$$\mathcal{H}(y; k_x, k_z, \omega) := \hat{\mathcal{C}} \left(i\omega \mathcal{I}_{2 \times 2} - \hat{\mathcal{A}} \right)^{-1} \hat{\mathcal{B}}. \quad (2.8)$$

Here $\mathcal{I}_{2 \times 2} := \text{diag}(\mathcal{I}, \mathcal{I})$, where \mathcal{I} is the identity operator and $\text{diag}(\cdot)$ indicates a block diagonal operation. Following the language in Jovanović (2021), we also refer to $\mathcal{H}(y; k_x, k_y, \omega)$ defined in (2.8) as the frequency response operator.

The linear form of (2.3) allows us to also perform the spatio-temporal Fourier transform (2.4) on the nonlinearity model to obtain

$$\hat{\mathbf{f}}_\xi = -\hat{\mathbf{u}}_\xi \cdot \hat{\nabla} \hat{\mathbf{u}} = \begin{bmatrix} \hat{f}_{x,\xi} \\ \hat{f}_{y,\xi} \\ \hat{f}_{z,\xi} \end{bmatrix} = \begin{bmatrix} -\hat{\mathbf{u}}_\xi^T & & \\ & -\hat{\mathbf{u}}_\xi^T & \\ & & -\hat{\mathbf{u}}_\xi^T \end{bmatrix} \begin{bmatrix} \hat{\nabla} \hat{u} \\ \hat{\nabla} \hat{v} \\ \hat{\nabla} \hat{w} \end{bmatrix}, \quad (2.9)$$

which can be decomposed as

$$\begin{bmatrix} \hat{f}_{x,\xi} \\ \hat{f}_{y,\xi} \\ \hat{f}_{z,\xi} \end{bmatrix} = \text{diag}(-\hat{\mathbf{u}}_\xi^T, -\hat{\mathbf{u}}_\xi^T, -\hat{\mathbf{u}}_\xi^T) \text{diag}(\hat{\nabla}, \hat{\nabla}, \hat{\nabla}) \begin{bmatrix} \hat{u} \\ \hat{v} \\ \hat{w} \end{bmatrix}. \quad (2.10)$$

This decomposition of the forcing function is illustrated in the two blocks inside of the blue dashed line (—) of figure 3(a), where the velocity field arising from the spatio-temporal frequency response \mathcal{H} is the input and the forcing is the output.

In order to isolate the gain $-\mathbf{u}_\xi$ that we seek to model, it is analytically convenient to combine the linear gradient operator with the spatio-temporal frequency response. We denote the resulting modified frequency response operator as

$$\mathcal{H}_\nabla(y; k_x, k_z, \omega) := \text{diag}(\hat{\nabla}, \hat{\nabla}, \hat{\nabla}) \mathcal{H}(y; k_x, k_z, \omega), \quad (2.11)$$

and redraw the system as a feedback interconnection between this linear operator and the structured uncertainty

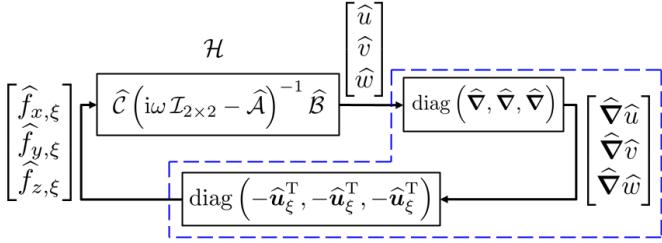
$$\hat{\mathbf{u}}_\Xi := \text{diag}(-\hat{\mathbf{u}}_\xi^T, -\hat{\mathbf{u}}_\xi^T, -\hat{\mathbf{u}}_\xi^T). \quad (2.12)$$

The structured uncertainty $\hat{\mathbf{u}}_\Xi$ in (2.12) has a block-diagonal structure such that the resulting feedback interconnection leads to a forcing model that retains the componentwise structure of the nonlinearity. Figure 3(b) describes the resulting feedback interconnection between the modified spatio-temporal frequency response and the structured uncertainty, where \mathbf{H}_∇ and $\hat{\mathbf{u}}_\Xi$ respectively represent the spatial discretizations (numerical approximations) of \mathcal{H}_∇ in (2.11) and $\hat{\mathbf{u}}_\Xi$ in (2.12).

We are interested in characterizing the perturbations that require the least energy to transition the flow from the laminar state (i.e. the optimal perturbations). These are precisely the perturbations that are most likely to destabilize the feedback loop, i.e., initiate transition from the laminar state. The form of the uncertainty associated with the smallest amplitude perturbation that can destabilize the feedback interconnection can be obtained in terms of the structured singular value of the modified frequency response operator \mathcal{H}_∇ ; see e.g., Packard & Doyle (1993, definition 3.1); Zhou *et al.* (1996, definition 11.1), which is defined as follows.

DEFINITION 1. *Given wavenumber and frequency pair (k_x, k_z, ω) , the structured sin-*

(a)



(b)

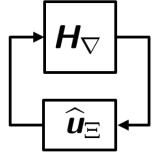


Figure 3: Illustration of structured input–output analysis: (a) a componentwise description, where blocks inside of (—, blue) represent the modeled forcing in equation (2.10) corresponding to the bottom block of figure 1; (b) a high-level description after discretization.

ingular value $\mu_{\hat{\mathbf{U}}_\Xi}[\mathbf{H}_\nabla(k_x, k_z, \omega)]$ is defined:

$$\mu_{\hat{\mathbf{U}}_\Xi}[\mathbf{H}_\nabla(k_x, k_z, \omega)] := \frac{1}{\min\{\bar{\sigma}[\hat{\mathbf{u}}_\Xi] : \hat{\mathbf{u}}_\Xi \in \hat{\mathbf{U}}_\Xi, \det[\mathbf{I} - \mathbf{H}_\nabla(k_x, k_z, \omega)\hat{\mathbf{u}}_\Xi] = 0\}}, \quad (2.13)$$

unless no $\hat{\mathbf{u}}_\Xi \in \hat{\mathbf{U}}_\Xi$ makes $\mathbf{I} - \mathbf{H}_\nabla\hat{\mathbf{u}}_\Xi$ singular, in which case $\mu_{\hat{\mathbf{U}}_\Xi}[\mathbf{H}_\nabla] := 0$.

Here, $\bar{\sigma}[\cdot]$ is the largest singular value, $\det[\cdot]$ is the determinant of the argument, and \mathbf{I} is the identity matrix. The subscript of μ in (2.13) is a set $\hat{\mathbf{U}}_\Xi$ containing all uncertainties having the same block-diagonal structure as $\hat{\mathbf{u}}_\Xi$; i.e.,

$$\hat{\mathbf{U}}_\Xi := \left\{ \text{diag}\left(-\hat{\mathbf{u}}_\xi^T, -\hat{\mathbf{u}}_\xi^T, -\hat{\mathbf{u}}_\xi^T\right) : -\hat{\mathbf{u}}_\xi^T \in \mathbb{C}^{N_y \times 3N_y} \right\}, \quad (2.14)$$

where N_y denotes the number of grid points in y .

We are interested in the structured singular values associated with a stable feedback interconnection. In particular, the perturbations that are most likely to destabilize the feedback interconnection in figure 3(b) across all frequencies. We can obtain this quantity through the application of the small gain theorem (Zhou *et al.* 1996, theorem 11.8).

PROPOSITION 2 (SMALL GAIN THEOREM). *Given $0 < \beta < \infty$ and wavenumber pair (k_x, k_z) . The loop shown in figure 3(b) is stable for all $\hat{\mathbf{u}}_\Xi \in \hat{\mathbf{U}}_\Xi$ with $\|\hat{\mathbf{u}}_\Xi\|_\infty := \sup_{\omega \in \mathbb{R}} \bar{\sigma}[\hat{\mathbf{u}}_\Xi] < \frac{1}{\beta}$ if and only if:*

$$\|\mathcal{H}_\nabla\|_\mu(k_x, k_z) := \sup_{\omega \in \mathbb{R}} \mu_{\hat{\mathbf{U}}_\Xi}[\mathbf{H}_\nabla(k_x, k_z, \omega)] \leq \beta. \quad (2.15)$$

Here, sup represents supremum (least upper bound) operation, and we abuse the notation by writing $\|\cdot\|_\mu$ (Packard & Doyle 1993), although μ is not a proper norm (i.e. it does not necessarily satisfy the triangle inequality). This value $\|\mathcal{H}_\nabla\|_\mu(k_x, k_z)$ directly quantifies flow structures that are most likely to lead to transition, in particular, a larger value indicates that the corresponding flow structures (associated k_x and k_z) require a lesser amplitude to destabilize the feedback interconnected system in figure 3(b), i.e. the magnitude of the structured uncertainty $\|\hat{\mathbf{u}}_\Xi\|_\infty$ is smaller.

3. Structured spatio-temporal frequency response

In this section, we use $\|\mathcal{H}_\nabla\|_\mu$ in equation (2.15) to characterize the flow structures that are most amplified in transitional plane Couette flow and plane Poiseuille flow. This suggests the characteristics of the most sensitive transition-inducing flow perturbations and flow structures prevalent in transitional regimes. In order to illustrate the relative effect of the feedback interconnection as well as the imposed structure we compare the results to

$$\|\mathcal{H}\|_\infty(k_x, k_z) := \sup_{\omega \in \mathbb{R}} \bar{\sigma} [\mathbf{H}(k_x, k_z, \omega)], \quad (3.1)$$

where \mathbf{H} is the discretization of spatio-temporal frequency response operator \mathcal{H} in (2.8). This quantity, which was previously analyzed in (Jovanović 2004; Schmid 2007; Illingworth 2020) for transitional flows, describes the maximum singular value of the frequency response operator \mathcal{H} . It represents the maximal gain of \mathcal{H} over all temporal frequencies; i.e., the worst-case amplification over harmonic inputs. Therefore the highest values of $\|\mathcal{H}\|_\infty$ correspond to structures that are most amplified but not those with the largest sustained energy density that is often reported in the literature, see e.g., Farrell & Ioannou (1993); Bamieh & Dahleh (2001); Jovanović & Bamieh (2005).

In order to isolate the effect of the structure imposed on the nonlinearity from the effect of imposing the closed-loop feedback interconnection, we also compute

$$\|\mathcal{H}_\nabla\|_\infty(k_x, k_z) := \sup_{\omega \in \mathbb{R}} \bar{\sigma} [\mathcal{H}_\nabla(k_x, k_z, \omega)]. \quad (3.2)$$

This quantity is the unstructured counterpart of $\|\mathcal{H}_\nabla\|_\mu$, which is obtained by replacing the uncertainty set $\hat{\mathbf{U}}_\Xi$ with the set of full complex matrices $\mathbb{C}^{3N_y \times 9N_y}$ (Packard & Doyle 1993; Zhou *et al.* 1996). In other words, the definition does not specify a particular feedback pathway associated with each component of forcing, which leads to an unstructured feedback interconnection and the relation $\|\mathcal{H}_\nabla\|_\mu \leq \|\mathcal{H}_\nabla\|_\infty$ (Packard & Doyle 1993, equation (3.4)). Comparisons between $\|\mathcal{H}_\nabla\|_\mu$ and $\|\mathcal{H}_\nabla\|_\infty$, therefore highlight the effect of structured uncertainty incorporating the componentwise structure of the nonlinearity.

The next subsection describes the numerical method we apply in computing each of these operators. We then analyze plane Couette flow at $Re = 358$ in § 3.2 and plane Poiseuille flow at $Re = 690$ in § 3.3. We discuss the role of the componentwise structure of nonlinearity that is preserved in the proposed structured input–output analysis in § 3.4. Finally, we study the influence of the Reynolds number at § 3.5.

3.1. Numerical Method

The operators in equation (2.6) are discretized using the Chebyshev differentiation matrices generated by the MATLAB routines of Weideman & Reddy (2000). The boundary conditions in equation (2.7) are implemented following Trefethen (2000, chapters 7 and 14). We employ the Clenshaw–Curtis quadrature weight (Trefethen 2000, chapter 12) in computing both singular and structured singular values resulting in these values being independent of the number of Chebyshev spaced wall-normal grid points. The numerical implementation of the operators is validated through comparison of the plane Poiseuille flow results for computations of the \mathcal{H}_∞ norm in Jovanović (2004, chapter 8.1.2) and Schmid (2007, figure 5), and plane Couette flow in Jovanović (2004, chapter 8.2). We use $N_y = 30$ collocation points (boundary not included) which is the same number employed in Jovanović & Bamieh (2005); Jovanović (2004) unless otherwise mentioned. We verified that doubling the number of collocation points in the wall-normal direction does not alter results, indicating grid convergence. We employ, respectively, 50 and 90 logarithmically

spaced points in the spectral range $k_x \in [10^{-4}, 10^{0.48}]$ and $k_z \in [10^{-2}, 10^{1.2}]$ similar to those employed in Jovanović & Bamieh (2005).

We compute $\|\mathcal{H}_\nabla\|_\mu$ in equation (2.15) for each wavenumber pair (k_x, k_z) using the `mussv` command with the ‘Uf’ option in the Robust Control Toolbox (Balas *et al.* 2005) of MATLAB R2020a. The algorithm is based on Fan *et al.* (1991) and Young *et al.* (1992). The average computation time for each wavenumber pair (k_x, k_z) is around 11s on a computer with a 3.4 GHz Intel Core i7-3770 CPU and 16GB RAM. The values $\|\mathcal{H}\|_\infty$ in (3.1) and $\|\mathcal{H}_\nabla\|_\infty$ in (3.2) are computed using `hinfnorm` command also in the Robust Control Toolbox (Balas *et al.* 2005). These computations can be easily parallelized over either k_x or k_z domain by e.g., using the `parfor` command in the Parallel Computing Toolbox in MATLAB.

3.2. Plane Couette flow at $Re = 358$

In this section, we use the proposed approach to analyze the perturbations that are most likely to trigger transition in plane Couette flow at $Re = 358$ using the $\|\mathcal{H}_\nabla\|_\mu$ formulation described in the previous section. Figure 4 shows this quantity (a) alongside results obtained using an input–output analysis based approach describing the most amplified flow structures in terms of $\|\mathcal{H}\|_\infty$ (b) and $\|\mathcal{H}_\nabla\|_\infty$ in equation (3.2) in panel (c). In all panels, we indicate the structures with $k_x \approx 0$ and $k_z = 2$ representing streamwise vortices using (*, blue) and indicate $k_x = 1$ and $k_z = 1$ representing the oblique waves that were observed as the structures requiring the least energy to trigger transition in the DNS of Reddy *et al.* (1998) using (○, red). The figure shows clear differences in the dominant structures identified using the structured input–output approach. The largest magnitudes of $\|\mathcal{H}_\nabla\|_\mu$ in figure 4(a) are associated with oblique waves with $k_x \in [10^{-2}, 1]$ and $k_z \in [10^{-1}, 1]$, while the streamwise vortices that are dominant in panels (b) and (c) have a lesser but still large magnitude. This result is consistent with findings of Reddy *et al.* (1998, figure 23) showing that oblique waves require less perturbation energy to trigger turbulence in plane Couette flow than streamwise vortices. A comparison of $\|\mathcal{H}_\nabla\|_\infty$ in figure 4(c) and $\|\mathcal{H}\|_\infty$ in figure 4(b) indicates that it is not the feedback interconnection that significantly changes the dominant flow structures but rather the imposition of the componentwise structure of the nonlinearity. In addition, we observe that the magnitude of $\|\mathcal{H}_\nabla\|_\mu$ in figure 4(a) is lower than $\|\mathcal{H}_\nabla\|_\infty$ in figure 4(c) for each (k_x, k_z) pair, which is consistent with the fact that the unstructured gain $\|\mathcal{H}_\nabla\|_\infty$ provides an upper bound on the structured one, $\|\mathcal{H}_\nabla\|_\mu$ (Packard & Doyle 1993).

The difference between the results in figure 4 mirrors the differences between the optimal perturbation structures predicted by linear and nonlinear optimal perturbation (NLOP) analysis. In particular, the structures predicted using $\|\mathcal{H}_\nabla\|_\mu$ are streamwise localized oblique waves reminiscent of those obtained as NLOP of plane Couette flow (Monokrousos *et al.* 2011; Duguet *et al.* 2010*a*, 2013; Rabin *et al.* 2012; Cherubini & De Palma 2013, 2015), whereas the results obtained using $\|\mathcal{H}\|_\infty$ in figure 4(b) indicate the dominance of the types of streamwise elongated flow structure predicted as linear optimal perturbations (Butler & Farrell 1992). Our results also reflect previous findings that the NLOP is wider in the spanwise direction than linear optimal perturbation, see Rabin *et al.* (2012, figure 11) showed the nonlinear optimal perturbation. The results in figure 4 therefore indicate that the current structured input–output framework provides closer agreement with both DNS and NLOP based predictions of perturbations to which the flow is most sensitive than traditional input–output methods focusing on the spatio-temporal frequency response \mathcal{H} . Moreover, the inclusion of a feedback loop for $\|\mathcal{H}_\nabla\|_\infty$ in figure 4(c) does lead to small improvements in the width of the structures predicted

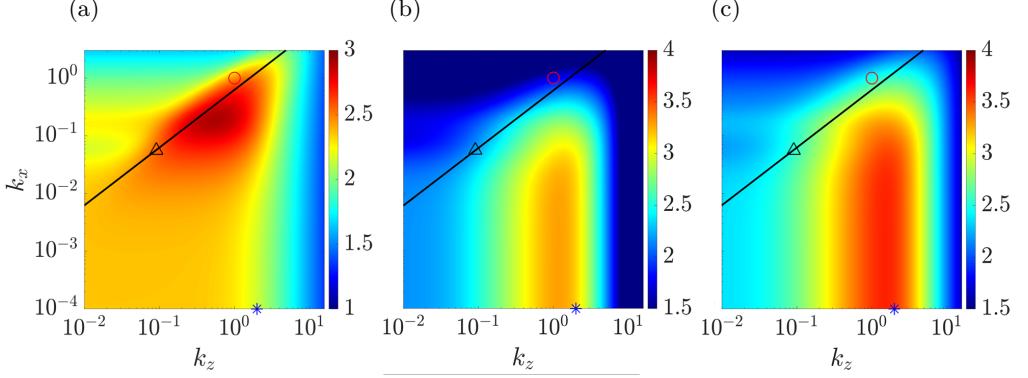


Figure 4: (a) $\log_{10}[\|\mathcal{H}_\nabla\|_\mu(k_x, k_z)]$, (b) $\log_{10}[\|\mathcal{H}\|_\infty(k_x, k_z)]$, and (c) $\log_{10}[\|\mathcal{H}_\nabla\|_\infty(k_x, k_z)]$ for plane Couette flow at $Re = 358$. Here the symbols (*, blue) indicates streamwise vortices with $k_x \approx 0$, $k_z = 2$; (○, red) marks oblique waves with $k_x = k_z = 1$; and (Δ , black) marks $\lambda_x = 113$, $\lambda_z = 69$ which are the observed wavelengths of the oblique turbulent band at $Re = 358$ (Prigent *et al.* 2003). The black solid line (—) is $\lambda_z = \lambda_x \tan(32^\circ)$ representing a 32° angle of the oblique turbulent band.

but does not lead to identification of the dominance of the oblique waves without the imposition of the componentwise structure from the nonlinear terms in (2.2).

Oblique turbulent bands have also been observed to be prominent structures in the transitional-regime of plane Couette flow with very large channel size (Prigent *et al.* 2003; Duguet *et al.* 2010b). Figure 4 indicates the wavelength pair $\lambda_x = 113$ and $\lambda_z = 69$ (Δ , black) associated with the oblique turbulent bands observed in very large channel studies of plane Couette flow at $Re = 358$, see (Prigent *et al.* 2003, figures 3(b) and 5). These structures are observed to have a characteristic inclination angle (measured from the streamwise direction in $x - z$ plane) of $\theta := \tan^{-1}(\lambda_z/\lambda_x) \approx 32^\circ$ represented by the black solid line (—): $\lambda_z = \lambda_x \tan(32^\circ)$ in figure 4. While there is some footprint of these structures and this angle in all three panels, the characteristic wavelengths and angle of the oblique turbulent bands reported in experiments (Prigent *et al.* 2003) also correspond to the region surrounding the peak value of $\|\mathcal{H}_\nabla\|_\mu$ in figure 4(a), and the line representing the angle of the structures is quite consistent with the shape of the peak region. This suggests that structured input–output analysis captures both wavelengths and angle of the oblique turbulent band in transitional plane Couette flow. The fact that these structures appear through this analysis suggests that these turbulent bands arise in transitional flows due to their large amplification (sensitivity to disturbances).

3.3. Plane Poiseuille flow at $Re = 690$

In this subsection, we apply the proposed structured input–output analysis to investigate transition-inducing flow structures in plane Poiseuille flow at $Re = 690$. Figure 5 compares (a) $\|\mathcal{H}_\nabla\|_\mu$, (b) $\|\mathcal{H}\|_\infty$, and (c) $\|\mathcal{H}_\nabla\|_\infty$ for this flow configuration. In each panel, we also indicate the streamwise vortices with $k_x \approx 0$ and $k_z = 2$ (*, blue), oblique waves with $k_x = k_z = 1$ (○, black), and TS waves with $k_x = 1$ and $k_z \approx 0$ (□, magenta) that were identified as transition-inducing perturbations in Reddy *et al.* (1998). Similar to the results for plane Couette flow in figure 4, the quantities $\|\mathcal{H}\|_\infty$ and $\|\mathcal{H}_\nabla\|_\infty$ show qualitatively similar behavior; the highest values for both correspond to streamwise streaks and vortices. The TS wave structure appears as a local peak which is about an order of magnitude smaller than the values associated with the streamwise vortices in $\|\mathcal{H}_\nabla\|_\infty$ and $\|\mathcal{H}\|_\infty$, with the latter showing a somewhat lesser magnitude relative to

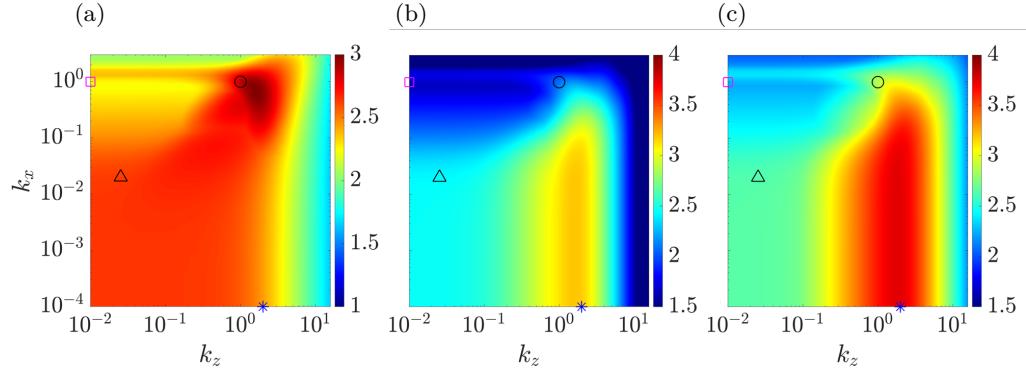


Figure 5: (a) $\log_{10}[\|\mathcal{H}_\nabla\|_\mu(k_x, k_z)]$, (b) $\log_{10}[\|\mathcal{H}\|_\infty(k_x, k_z)]$, and (c) $\log_{10}[\|\mathcal{H}_\nabla\|_\infty(k_x, k_z)]$ for plane Poiseuille flow at $Re = 690$. Here the symbols ($*$, blue): $k_x \approx 0, k_z = 2$ marks streamwise vortices; (\circ , black): $k_x = k_z = 1$ marks oblique waves; (\square , magenta): $k_x = 1, k_z \approx 0$ marks TS wave; and (\triangle , black): $\lambda_x = 314, \lambda_z = 248$ indicates the wavelengths of the oblique turbulent band at $Re = 690$ observed in Kanazawa (2018).

its peak value. The values for the oblique waves are of a similar order of magnitude as the peak corresponding to the TS waves in $\|\mathcal{H}\|_\infty$ and slightly higher in $\|\mathcal{H}_\nabla\|_\infty$. These findings agree with previous analyses of $\|\mathcal{H}\|_\infty$ in (Jovanović 2004; Schmid 2007) and the similarity of the $\|\mathcal{H}_\nabla\|_\infty$ results indicate that an unstructured feedback interconnection does not lead to substantial changes in most prominent structures.

The overall shape of $\|\mathcal{H}_\nabla\|_\mu$ is somewhat different than that of either $\|\mathcal{H}\|_\infty$ or $\|\mathcal{H}_\nabla\|_\infty$. The streamwise vortices that are dominant in panels (b) and (c) have a lesser but still large magnitude, while the peak value corresponds to oblique waves. The TS wave corresponds to a local peak in $\|\mathcal{H}_\nabla\|_\mu$, but the magnitudes are smaller than the peak values associated with oblique waves. This result is consistent with findings of Reddy *et al.* (1998, figure 19) showing that oblique waves require slightly less perturbation energy to trigger turbulence in plane Poiseuille flow than streamwise vortices. Both the peak region and the large region of very high values in the bottom left quadrant of the plot are consistent with the short-timescale NLOP of plane Poiseuille flow that was shown to be spatially localized with streamwise wavelength larger than spanwise wavelength (Farano *et al.* 2015). These results indicate that the inclusion of structured uncertainty uncovers a broader range of transition-inducing structures and correctly orders their relative amplification in the sense of transition-inducing potential.

There is evidence that the oblique turbulent bands that are observed in very large channel studies also play a role in transition. Their ability to trigger transition has been exploited in a number of studies that employ flow fields with a sustained oblique turbulent bands at a relatively high Re as the initial conditions to trigger the banded turbulent-laminar patterns associated with transitioning flows at a Reynolds number of interest; see e.g., (Tsukahara *et al.* 2005; Tuckerman *et al.* 2014; Tao *et al.* 2018; Xiao & Song 2020). The characteristic wavelength pair ($\lambda_x = 314, \lambda_z = 248$) associated with this structure in plane Poiseuille flow at $Re = 690$ (estimated from Kanazawa (2018, figure 5.1(b))) is indicated in each panel of figure 5 using (\triangle). These characteristic wavelengths are located within the range of large values of $\|\mathcal{H}_\nabla\|_\mu$. They are not associated with peak regions of $\|\mathcal{H}\|_\infty$ or $\|\mathcal{H}_\nabla\|_\infty$ in figures 5(b) or (c), although a footprint of these flow structures is visible in both. Figure 5(a) indicates that the flow structure associated with the oblique turbulent band has a similar destabilizing effect as streamwise elongated

structures, although both of their magnitudes are smaller than oblique waves. Further analysis of these structures and their role in transition is a topic of ongoing work.

3.4. Componentwise structure of nonlinearity: saturation of the lift-up mechanism

The results in the previous subsections, particularly the differences between $\|\mathcal{H}_\nabla\|_\mu$ and $\|\mathcal{H}_\nabla\|_\infty$ in figures 4 and 5 highlight the role of the feedback interconnection structure in the identification of the perturbations to which the flow is most sensitive. In particular, the imposition of the componentwise structure leads to lesser prominence of streamwise elongated structures ($k_x \approx 0, k_z = 2$) in $\|\mathcal{H}_\nabla\|_\mu$ versus $\|\mathcal{H}\|_\infty$ or $\|\mathcal{H}_\nabla\|_\infty$ in both plane Couette and Poiseuille flows (see figures 4 and 5).

The mechanisms underlying the differences in $\|\mathcal{H}_\nabla\|_\mu$ and $\|\mathcal{H}_\nabla\|_\infty$ can be analyzed by isolating the effect of forcing in each component of the momentum equation, i.e. f_x, f_y, f_z in equation (2.2) on the amplification of each velocity component u, v, w . These nine quantities are associated with $\|\mathcal{H}_{ij}\|_\infty$, where the spatio-temporal frequency response operator \mathcal{H}_{ij} from each forcing component ($j = x, y, z$) to each velocity component ($i = u, v, w$) is given by (Jovanović & Bamieh 2005)

$$\mathcal{H}_{ij} = \widehat{\mathcal{C}}_i \left(i\omega \mathcal{I}_{2 \times 2} - \widehat{\mathcal{A}} \right)^{-1} \widehat{\mathcal{B}}_j \quad (3.3)$$

with

$$\widehat{\mathcal{B}}_x := \widehat{\mathcal{B}} [\mathcal{I} \ 0 \ 0]^T, \quad \widehat{\mathcal{B}}_y := \widehat{\mathcal{B}} [0 \ \mathcal{I} \ 0]^T, \quad \widehat{\mathcal{B}}_z := \widehat{\mathcal{B}} [0 \ 0 \ \mathcal{I}]^T, \quad (3.4a)$$

$$\widehat{\mathcal{C}}_u := [\mathcal{I} \ 0 \ 0] \widehat{\mathcal{C}}, \quad \widehat{\mathcal{C}}_v := [0 \ \mathcal{I} \ 0] \widehat{\mathcal{C}}, \quad \widehat{\mathcal{C}}_w := [0 \ 0 \ \mathcal{I}] \widehat{\mathcal{C}}. \quad (3.4b)$$

These quantities were analyzed in (Jovanović 2004) and (Schmid 2007). Those results indicated that the most significant amplification was seen when forcing was applied in the cross-stream and the output was the streamwise velocity component; i.e., that associated with respective frequency response operators $\mathcal{H}_{uy}, \mathcal{H}_{uz}$ and input–output pathways $f_y \rightarrow u, f_z \rightarrow u$. Jovanović (2004, theorem 11) quantified the increased amplification for the case of streamwise constant ($k_x = 0$) flow structures by deriving Reynolds number scaling for each $\|\mathcal{H}_{ij}\|_\infty$. The results demonstrated that $\|\mathcal{H}_{uy}\|_\infty$ and $\|\mathcal{H}_{uz}\|_\infty$ (i.e., quantities associated with input–output pathways $f_y \rightarrow u$ and $f_z \rightarrow u$) scale as $\sim Re^2$ while the scalings of $\|\mathcal{H}_{ij}\|_\infty$ for other input–output pathways show at most $\sim Re$. Similar behavior occurs if we isolate \mathcal{H}_∇ by examining each input–output response pathways:

$$\mathcal{H}_{\nabla ij} := \widehat{\nabla} \mathcal{H}_{ij}. \quad (3.5)$$

In this prior work, the input forcing (applied either directly to the LNS (top-block in figure 1) or through a feedback interconnection) was unstructured in the sense that there was no restriction in terms of the permissible input–output pathways, therefore the behavior of the largest $\|\mathcal{H}_{ij}\|_\infty$ ($\|\mathcal{H}_{\nabla ij}\|_\infty$) response dominates the overall $\|\mathcal{H}\|_\infty$ ($\|\mathcal{H}_\nabla\|_\infty$) response.

The structured input–output analysis framework introduced here instead imposes a correlation between each component of the modeled forcing $f_{x,\xi}, f_{y,\xi}$, and $f_{z,\xi}$ and the respective velocity components u, v , and w by constraining the feedback interconnection to retain the componentwise structure of our input–output model of the forcing. This model of the forcing in terms of componentwise input–output relationships from $\nabla u, \nabla v, \nabla w$ to the respective components $f_{x,\xi} = -\mathbf{u}_\xi \cdot \nabla u, f_{y,\xi} = -\mathbf{u}_\xi \cdot \nabla v$, and $f_{z,\xi} = -\mathbf{u}_\xi \cdot \nabla w$ with the gain defined in terms of $-\mathbf{u}_\xi$ constrains the feedback relationships such that each component of the forcing is most strongly influenced by that component of the velocity field and velocity gradient. These constraints on the permissible feedback pathways within

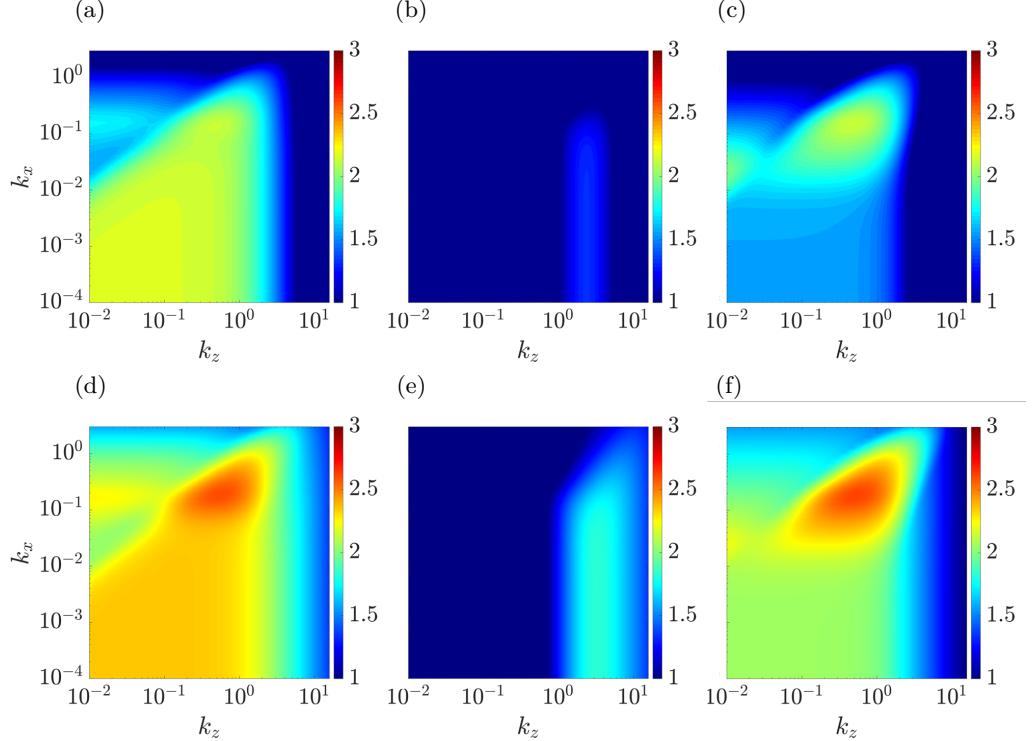


Figure 6: (a) $\log[\|\mathcal{H}_{ux}\|_\infty(k_x, k_z)]$, (b) $\log[\|\mathcal{H}_{vy}\|_\infty(k_x, k_z)]$, (c) $\log[\|\mathcal{H}_{wz}\|_\infty(k_x, k_z)]$, (d) $\log[\|\mathcal{H}_{\nabla ux}\|_\infty(k_x, k_z)]$, (e) $\log[\|\mathcal{H}_{\nabla vy}\|_\infty(k_x, k_z)]$, and (f) $\log[\|\mathcal{H}_{\nabla wz}\|_\infty(k_x, k_z)]$ for plane Couette flow at $Re = 358$.

our model of the nonlinear interactions limit the influence of the input–output pathways $f_y \rightarrow u$ and $f_z \rightarrow u$. This greater emphasis on the feedback pathways associated with $f_x \rightarrow u$, $f_y \rightarrow v$, and $f_z \rightarrow w$ as well as $f_x \rightarrow \nabla u$, $f_y \rightarrow \nabla v$, and $f_z \rightarrow \nabla w$ within the structured input–output analysis framework is illustrated in figures 6 and 7, which respectively plot (a) $\|\mathcal{H}_{ux}\|_\infty$, (b) $\|\mathcal{H}_{vy}\|_\infty$, (c) $\|\mathcal{H}_{wz}\|_\infty$, (d) $\|\mathcal{H}_{\nabla ux}\|_\infty$, (e) $\|\mathcal{H}_{\nabla vy}\|_\infty$, (f) $\|\mathcal{H}_{\nabla wz}\|_\infty$ for the plane Couette and Poiseuille cases respectively discussed in § 3.2 and § 3.3. Here, we can see that the results of structured input–output analysis $\|\mathcal{H}_\nabla\|_\mu$ for both of these flows in figures 4(a) and 5(a) resemble the combined effect of this limited set of input–output pathways.

The input–output pathways that dominate the overall unstructured response $\|\mathcal{H}\|_\infty$ ($\|\mathcal{H}_{uy}\|_\infty$ and $\|\mathcal{H}_{uz}\|_\infty$) emphasize amplification of streamwise streaks by cross-stream forcing; i.e., the lift-up mechanism, see e.g., the discussion in Jovanović (2021) for further details. The lift-up mechanism therefore appears to be weakened through the imposition of the componentwise structure of the nonlinearity, which is consistent with results suggesting that nonlinear mechanisms lead to streak breakdown, see e.g., Duguet *et al.* (2013); Brandt (2014). These results suggest that the preservation of the componentwise structure of nonlinearity within the proposed approach enables the method to capture important nonlinear effects, leading to better agreement with DNS and experimental studies and nonlinear analysis of the perturbations that require less energy to initiate transition, e.g. NLOP.

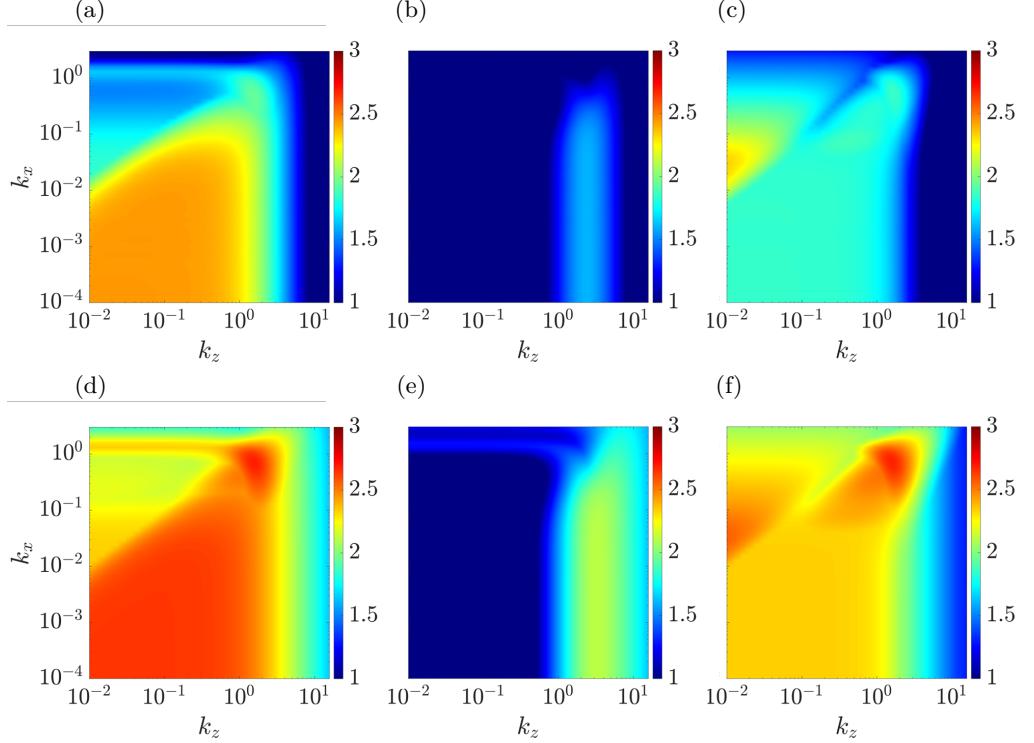


Figure 7: (a) $\log[\|\mathcal{H}_{ux}\|_\infty(k_x, k_z)]$, (b) $\log[\|\mathcal{H}_{vy}\|_\infty(k_x, k_z)]$, (c) $\log[\|\mathcal{H}_{wz}\|_\infty(k_x, k_z)]$, (d) $\log[\|\mathcal{H}_{\nabla ux}\|_\infty(k_x, k_z)]$, (e) $\log[\|\mathcal{H}_{\nabla vy}\|_\infty(k_x, k_z)]$, and (f) $\log[\|\mathcal{H}_{\nabla wz}\|_\infty(k_x, k_z)]$ for plane Poiseuille flow at $Re = 690$.

3.5. Reynolds number dependence

In this subsection, we aggregate results across a range of (k_x, k_z) scales to study the Reynolds number dependence and the associated scaling law of $\|\mathcal{H}_\nabla\|_\mu$ for both plane Couette flow and plane Poiseuille flows. In particular, we compute

$$\|\mathcal{H}_\nabla\|_\mu^M := \max_{k_z, k_x} \|\mathcal{H}_\nabla\|_\mu(k_x, k_z), \quad (3.6)$$

where \max corresponds to the maximum value over the wavenumber pair (k_x, k_z) in the computational range of $k_x \in [10^{-4}, 10^{0.48}]$ and $k_z \in [10^{-2}, 10^{1.2}]$.

In order to compare our results to the scaling relationships of $\|\mathcal{H}\|_\infty$ previously described in the literature and to isolate the effect of the structure in the feedback loop, we analogously define

$$\|\mathcal{H}\|_\infty^M := \max_{k_z, k_x} \|\mathcal{H}\|_\infty(k_x, k_z), \quad (3.7a)$$

$$\|\mathcal{H}_\nabla\|_\infty^M := \max_{k_z, k_x} \|\mathcal{H}_\nabla\|_\infty(k_x, k_z). \quad (3.7b)$$

The scaling of quantities related to $\|\mathcal{H}\|_\infty$ and $\|\mathcal{H}\|_\infty^M$ with different input–output pathways, i.e. different $\widehat{\mathcal{B}}$ and $\widehat{\mathcal{C}}$ matrices in equation (3.4), has been widely studied. For example, Trefethen *et al.* (1993, table 1) showed that $\sup_{\omega \in \mathbb{R}} \|(\iota\omega \mathcal{I} - \widehat{\mathcal{A}})^{-1}\| \sim Re^2$ for both plane Couette and plane Poiseuille flows, where the operator norm $\|\cdot\|$ renders the value

$\sup_{\omega \in \mathbb{R}} \|\cdot\|$ equivalent to the definition of $\|\cdot\|_\infty$ employed in (3.1). Kreiss *et al.* (1994) showed that the related quantity

$$\sup_{\operatorname{Re}[s] \geq 0} \|(s\mathcal{I} - \widehat{\mathcal{A}})^{-1}\| \sim Re^2$$

for plane Couette flow, where $\operatorname{Re}[s]$ denotes the real part of Laplace variable s . Jovanović (2004, theorem 11) analytically derived the same $\sim Re^2$ scaling for the special case of $\|\mathcal{H}\|_\infty$ restricted to $k_x = 0$ for both plane Couette and Poiseuille flows.

Figure 8 plots the quantities in equations (3.6)-(3.7) as a function of Reynolds number ($Re \in [300, 4000]$) for (a) plane Couette flow and (b) plane Poiseuille flow. The upper bound of $Re = 4000$ was selected to remain below the known linear stability limit for plane Poiseuille flow of $Re \simeq 5772$ (Orszag 1971). As expected all of these quantities increase with the Reynolds number and the values of $\|\mathcal{H}_\nabla\|_\infty^M$ are larger than those of $\|\mathcal{H}_\nabla\|_\mu^M$. We obtain a Reynolds number scaling of each quantity by fitting the lines in figure 8 to $c_0 Re^\eta$, where c_0 is a constant scalar. The results show that $\|\mathcal{H}\|_\infty^M$ and $\|\mathcal{H}_\nabla\|_\infty^M$ scale as $\sim Re^2$ in the range $Re \in [300, 4000]$ for both plane Couette and plane Poiseuille flows. This scaling is consistent with the results in Trefethen *et al.* (1993) for the frequency response operator with identity operators for $\widehat{\mathcal{C}}$ and $\widehat{\mathcal{B}}$ as well as the related quantity in Kreiss *et al.* (1994). The fact that the scaling of this quantity for the modified frequency response operator \mathcal{H}_∇ is the same as that of \mathcal{H} suggests that adding an unstructured uncertainty in the feedback loop to represent the nonlinear interactions does not change the Reynolds number scaling.

The quantity $\|\mathcal{H}_\nabla\|_\mu^M$ in figure 8 instead shows scalings of $\sim Re^{1.1}$ over $Re \in [300, 4000]$ for plane Couette flow and $\sim Re^{1.5}$ for plane Poiseuille flow in the range $Re \in [500, 4000]$. The difference between this scaling and the $\sim Re^2$ scaling associated with either $\|\mathcal{H}\|_\infty^M$ or $\|\mathcal{H}_\nabla\|_\infty^M$ again arises through the imposition of the componentwise structure of nonlinearity. As discussed in § 3.4, the reduced scaling is related to the smaller amplification in the input–output pathways $f_x \rightarrow u$, $f_y \rightarrow v$, and $f_z \rightarrow w$ and associated $f_x \rightarrow \nabla u$, $f_y \rightarrow \nabla v$, and $f_z \rightarrow \nabla w$. The scaling for these input–output pathways can be evaluated directly through

$$\|\mathcal{H}_{\nabla ij}\|_\infty^M := \max_{k_z, k_x} \|\mathcal{H}_{\nabla ij}\|_\infty(k_x, k_z) \quad (3.8)$$

which is plotted in figure 9 alongside $\|\mathcal{H}_\nabla\|_\mu^M$. Performing a similar fit we find that both $\|\mathcal{H}_{\nabla ux}\|_\infty^M$ and $\|\mathcal{H}_{\nabla wz}\|_\infty^M$ respectively scale as $\sim Re^{1.1}$ for plane Couette flow and $\sim Re^{1.5}$ for plane Poiseuille flow, which matches the scaling of $\|\mathcal{H}_\nabla\|_\mu$. On the other hand, the quantity $\|\mathcal{H}_{\nabla vy}\|_\infty^M$ is much smaller than its counterpart associated with other input–output pathways, and it scales as $\|\mathcal{H}_{\nabla vy}\|_\infty^M \sim Re$ for both plane Couette and Poiseuille flows.

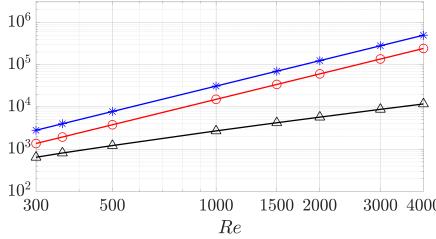
The observed importance of streamwise elongated structures has motivated previous analysis of the streamwise constant ($k_x = 0$) component of the frequency response operator. We evaluate this behavior by computing analogous quantities

$$\|\mathcal{H}_\nabla\|_\mu^{sc} := \max_{k_z, k_x=10^{-4}} \|\mathcal{H}_\nabla\|_\mu(k_x, k_z), \quad (3.9a)$$

$$\|\mathcal{H}_{\nabla ij}\|_\infty^{sc} := \max_{k_z, k_x=10^{-4}} \|\mathcal{H}_{\nabla ij}\|_\infty(k_x, k_z), \quad (3.9b)$$

which restricts the streamwise wavenumber to $k_x = 10^{-4}$ to approximate the streamwise constant modes. In figure 10, we replot $\|\mathcal{H}_\nabla\|_\mu^M$ alongside $\|\mathcal{H}_\nabla\|_\mu^{sc}$ ($\Delta-$, black), and observe that $\|\mathcal{H}_\nabla\|_\mu^{sc} \sim Re$ for both plane Couette and Poiseuille flows. Figure 10 also shows $\|\mathcal{H}_{\nabla ux}\|_\infty^{sc}$, $\|\mathcal{H}_{\nabla vy}\|_\infty^{sc}$, and $\|\mathcal{H}_{\nabla wz}\|_\infty^{sc}$. Here, we find that $\|\mathcal{H}_\nabla\|_\mu^{sc}$ overlap with

(a)



(b)

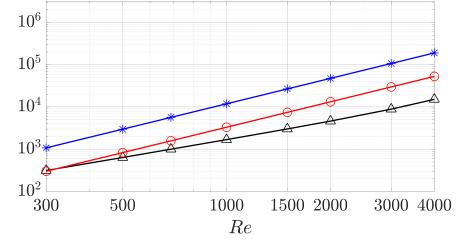
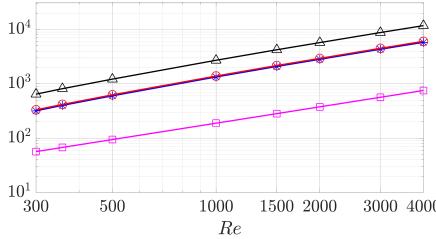


Figure 8: The Reynolds number dependence of $\|\mathcal{H}_\nabla\|_\mu^M$ (\triangle —, black); $\|\mathcal{H}\|_\infty^M$ (\circ —, red); $\|\mathcal{H}_\nabla\|_\infty^M$ ($*$ —, blue). Here, panel (a) is plane Couette flow and panel (b) is plane Poiseuille flow.

(a)



(b)

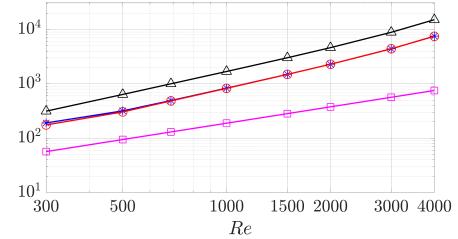
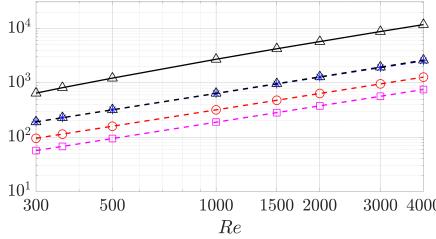


Figure 9: The Reynolds number dependence of $\|\mathcal{H}_\nabla\|_\mu^M$ (\triangle —, black); $\|\mathcal{H}_{\nabla ux}\|_\infty^M$ ($*$ —, blue); $\|\mathcal{H}_{\nabla vy}\|_\infty^M$ (\square —, magenta); $\|\mathcal{H}_{\nabla wz}\|_\infty^M$ (\circ —, red). Here, panel (a) is plane Couette flow and panel (b) is plane Poiseuille flow.

(a)



(b)

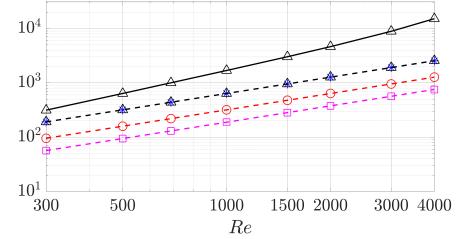
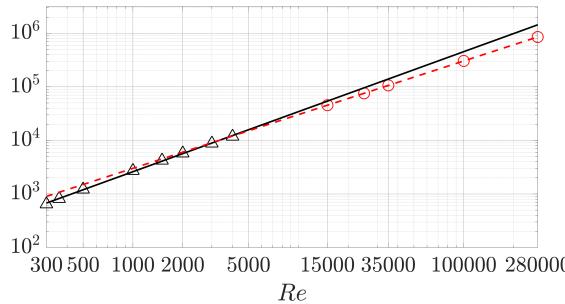


Figure 10: The Reynolds number dependence of $\|\mathcal{H}_\nabla\|_\mu^M$ (\triangle —, black); $\|\mathcal{H}_\nabla\|_\mu^{sc}$ (\triangle —, black); $\|\mathcal{H}_{\nabla ux}\|_\infty^{sc}$ ($*$ —, blue); $\|\mathcal{H}_{\nabla vy}\|_\infty^{sc}$ (\square —, magenta); $\|\mathcal{H}_{\nabla wz}\|_\infty^{sc}$ (\circ —, red). Here, panel (a) is plane Couette flow and panel (b) is plane Poiseuille flow.

$\|\mathcal{H}_{\nabla ux}\|_\infty^{sc}$ and thus shows the same scaling $\sim Re$. These three input–output pathways $\|\mathcal{H}_{\nabla ij}\|_\infty^{sc}$ ($ij = ux, vy, wz$) scale as $\sim Re$ for both plane Couette and Poiseuille flows. This behavior is consistent with the results in Jovanović (2004, theorem 11) showed that $\|\mathcal{H}_{ij}\|_\infty \sim Re$ ($ij = ux, vy, wz$) when it is restricted to $k_x = 0$ for both plane Couette and plane Poiseuille flows. The reduced scaling exponent η of the largest structured gain $\|\mathcal{H}_\nabla\|_\mu^M \sim Re^\eta$ observed here compared with $\eta = 2$ for unstructured gain (Trefethen *et al.* 1993; Kreiss *et al.* 1994; Jovanović 2004) further highlights the importance of the

(a)



(b)

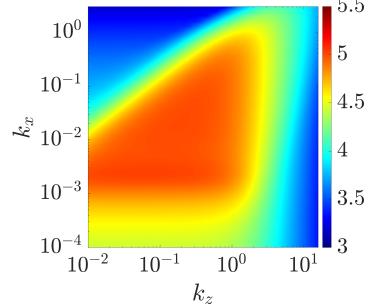


Figure 11: (a) The Reynolds number dependence of $\|\mathcal{H}_\nabla\|_\mu^M$ for plane Couette flow. The line (—, black) fits through result at $Re \in [300, 4000]$ (Δ , black) scales as $\sim Re^{1.1}$, and (—, red) fits through result at $Re \in [15000, 280000]$ (\circ , red) scales as $\sim Re$. (b) $\log_{10}[\|\mathcal{H}_\nabla\|_\mu(k_x, k_z)]$ for plane Couette flow at $Re = 35000$.

componentwise structure of nonlinearity imposed in this framework, which appears to saturate the large amplification associated with the lift-up mechanism.

We now explore scaling of $\|\mathcal{H}_\nabla\|_\mu$ at higher Reynolds numbers for plane Couette flow, which is known to be linearly stable for $Re \rightarrow \infty$ (Romanov 1973). The scaling of permissible perturbation amplitude versus Reynolds number in the asymptotic limit $Re \rightarrow \infty$ is also widely studied (Kreiss *et al.* 1994; Chapman 2002; Bedrossian *et al.* 2015, 2019; Wei & Zhang 2020). Here we analyze this behavior by investigating computing results at $Re = 15000, 25000, 35000, 100000, 280000$, with associated increased wall-normal resolution including $N_y = 60, 80, 100, 120, 140$ wall-normal grid points, respectively. Figure 11(a) replots $\|\mathcal{H}_\nabla\|_\mu^M$ at $Re \in [300, 4000]$ (Δ , black) along with $\|\mathcal{H}_\nabla\|_\mu^M$ for plane Couette flow at $Re \in [15000, 280000]$ (\circ , red), and we can see that scaling law changes to the $\sim Re$ for the higher Reynolds numbers, which matches the scaling law for $\|\mathcal{H}_\nabla\|_\infty^{sc}$ restricted to $k_x = 10^{-4}$. This modification of scaling law at high Reynolds to better match that of the streamwise constant operator can be further understood by plotting $\|\mathcal{H}_\nabla\|_\mu(k_x, k_z)$, which is shown in figure 11(b) for plane Couette flow at $Re = 35000$. Here, we observe that the peak of $\|\mathcal{H}_\nabla\|_\mu$ is associated with a wider range of (k_x, k_z) scales and moves towards smaller k_x compared with the same quantity at $Re = 358$ in figure 4(a). This trend is also observed for Reynolds number $Re = 280000$ (not shown here). This suggests that the flow structures associated with the smallest destabilizing velocity amplitude (the largest $\|\mathcal{H}_\nabla\|_\mu$) are becoming more streamwise aligned (k_x smaller) with increasing Reynolds number.

4. Conclusions and future work

This work proposed a *structured* input–output analysis that augments the traditional spatio-temporal frequency response with a structured uncertainty. The structure preserves the componentwise input–output structure of the nonlinearity in NS equations. We then analyze the spatio-temporal response of the resulting feedback interconnection between the linearized Navier-Stokes (NS) and the structured forcing in terms of the structured singular value of the associated spatio-temporal frequency response operator.

We apply the structured input–output analysis to transitional plane Couette and plane Poiseuille flows. Comparisons of the results to those of traditional analysis and an unstructured feedback interconnection indicate that the addition of a structured feedback

interconnection enables the prediction of a wider range of known dominant flow structures to be identified without the computational burden of nonlinear optimization or extensive simulations. More specifically, the results for transitional plane Couette flow reproduce the findings from direct numerical simulation (DNS) (Reddy *et al.* 1998) and nonlinear optimal perturbation (NLOP) (Rabin *et al.* 2012) in showing that oblique waves require less energy to destabilize the flow than the streamwise elongated structures emphasized in traditional input–output analysis. In plane Poiseuille flow the results again predict the oblique wave structure as in DNS (Reddy *et al.* 1998). They also highlight the importance of spatially localized structures with a streamwise wavelength larger than spanwise (Farano *et al.* 2015). The framework also reproduces the oblique turbulent bands (Prigent *et al.* 2003; Kanazawa 2018) that have been associated with transitioning flows with very large channel sizes ($\sim O(100)$ times the channel half-height) in both experiments and DNS.

The agreement between the predictions from structured input–output analysis and observation in experiments, DNS, and NLOP indicate that the structured feedback interconnection reproduces important nonlinear effects. Our analysis suggests that restricting the feedback pathways preserves the structure of the nonlinear mechanisms that weaken the streaks developed through the lift-up effect, in which cross-stream forcing amplifies streamwise streaks (Ellingsen & Palm 1975; Landahl 1975; Brandt 2014). This linear amplification of streamwise structures without a corresponding weakening effect likely leads to their dominance in traditional input–output analysis; see e.g. the discussion in Jovanović (2021). The Reynolds number dependence observed in our studies further supports this saturation of the lift-up effect.

The results here suggest the promise of this computationally tractable approach and opens up many directions for future work. A natural direction is an extension to pipe flow, where the subcritical transition is also widely studied; see e.g., (Hof *et al.* 2003; Peixinho & Mullin 2007; Eckhardt *et al.* 2007; Mellibovsky & Meseguer 2009; Mullin 2011; Pringle & Kerswell 2010; Pringle *et al.* 2012; Barkley 2016). Adaptions of this approach to the fully turbulent regime, where the resolvent framework and input–output analysis have provided important insights (McKeon & Sharma 2010; McKeon *et al.* 2013; McKeon 2017; Hwang & Cossu 2010a,b; Moarref *et al.* 2013, 2014; Zare *et al.* 2017; Illingworth *et al.* 2018; Vadarevu *et al.* 2019; Madhusudanan *et al.* 2019; Symon *et al.* 2021; Liu & Gayme 2019, 2020a; Towne *et al.* 2020; Morra *et al.* 2021; Nogueira *et al.* 2021) is another direction of ongoing study.

Acknowledgment

The authors gratefully acknowledge support from the US National Science Foundation (NSF) through grant number CBET 1652244 and the Office of Naval Research (ONR) through grant number N00014-18-1-2534. C.L. greatly appreciates the support from the Chinese Scholarship Council.

Declaration of Interests

The authors report no conflict of interest.

REFERENCES

- BALAS, G., CHIANG, R., PACKARD, A. & SAFONOV, M. 2005 Robust control toolbox. *For Use with Matlab. User's Guide, Version 3.*

- BAMIEH, B. & DAHLEH, M. 2001 Energy amplification in channel flows with stochastic excitation. *Phys. Fluids* **13** (11), 3258–3269.
- BARKLEY, D. 2016 Theoretical perspective on the route to turbulence in a pipe. *J. Fluid Mech.* **803**, P1.
- BARKLEY, D. & TUCKERMAN, L. S. 2007 Mean flow of turbulent-laminar patterns in plane Couette flow. *J. Fluid Mech.* **576**, 109–137.
- BEDROSSIAN, J., GERMAIN, P. & MASMOUDI, N. 2015 Dynamics near the subcritical transition of the 3D Couette flow I: Below threshold case. *arXiv preprint arXiv:1506.03720*.
- BEDROSSIAN, J., GERMAIN, P. & MASMOUDI, N. 2019 Stability of the Couette flow at high Reynolds numbers in two dimensions and three dimensions. *Bull. Am. Math. Soc.* **56** (3), 373–414.
- BOTTIN, S., DAUCHOT, O., DAVIAUD, F. & MANNEVILLE, P. 1998 Experimental evidence of streamwise vortices as finite amplitude solutions in transitional plane Couette flow. *Phys. Fluids* **10** (10), 2597–2607.
- BRANDT, L. 2014 The lift-up effect: The linear mechanism behind transition and turbulence in shear flows. *Eur. J. Mech. B. Fluids* **47**, 80–96.
- BUTLER, K. M. & FARRELL, B. F. 1992 Three-dimensional optimal perturbations in viscous shear flow. *Phys. Fluids A* **4** (8), 1637–1650.
- CHAPMAN, S. J. 2002 Subcritical transition in channel flows. *J. Fluid Mech.* **451**, 35–97.
- CHERUBINI, S. & DE PALMA, P. 2013 Nonlinear optimal perturbations in a Couette flow: bursting and transition. *J. Fluid Mech.* **716**, 251–279.
- CHERUBINI, S. & DE PALMA, P. 2015 Minimal-energy perturbations rapidly approaching the edge state in Couette flow. *J. Fluid Mech.* **764**, 572–598.
- CHEVALIER, M., HÆPFFNER, J., BEWLEY, T. R. & HENNINGSON, D. S. 2006 State estimation in wall-bounded flow systems. Part 2. Turbulent flows. *J. Fluid Mech.* **552**, 167–187.
- DE SOUZA, D., BERGIER, T. & MONCHAUX, R. 2020 Transient states in plane Couette flow. *J. Fluid Mech.* **903**, A33.
- DEGUCHI, K. & HALL, P. 2015 Asymptotic descriptions of oblique coherent structures in shear flows. *J. Fluid Mech.* **782**, 356–367.
- DOYLE, J. 1982 Analysis of feedback systems with structured uncertainties. *IEE Proceedings D-Control Theory and Applications* **129** (6), 242–250.
- DUGUET, Y., BRANDT, L. & LARSSON, B. R. J. 2010a Towards minimal perturbations in transitional plane Couette flow. *Phys. Rev. E* **82** (2), 026316.
- DUGUET, Y., MONOKROUSOS, A., BRANDT, L. & HENNINGSON, D. S. 2013 Minimal transition thresholds in plane Couette flow. *Phys. Fluids* **25** (8), 084103.
- DUGUET, Y. & SCHLATTER, P. 2013 Oblique laminar-turbulent interfaces in plane shear flows. *Phys. Rev. Lett.* **110** (3), 034502.
- DUGUET, Y., SCHLATTER, P. & HENNINGSON, D. S. 2010b Formation of turbulent patterns near the onset of transition in plane Couette flow. *J. Fluid Mech.* **650**, 119–129.
- ECKHARDT, B., SCHNEIDER, T. M., HOF, B. & WESTERWEEL, J. 2007 Turbulence transition in pipe flow. *Annu. Rev. Fluid Mech.* **39** (1), 447–468.
- ELLINGSEN, T. & PALM, E. 1975 Stability of linear flow. *Phys. Fluids* **18** (4), 487–488.
- ELOFSSON, P. A. & ALFREDSSON, P. H. 1998 An experimental study of oblique transition in plane Poiseuille flow. *J. Fluid Mech.* **358**, 177–202.
- FAN, M. H., TITS, A. & DOYLE, J. 1991 Robustness in the presence of mixed parametric uncertainty and unmodeled dynamics. *IEEE Trans. Autom. Control* **36** (1), 25–38.
- FARANO, M., CHERUBINI, S., ROBINET, J. C. & DE PALMA, P. 2015 Hairpin-like optimal perturbations in plane Poiseuille flow. *J. Fluid Mech.* **775**, R2.
- FARANO, M., CHERUBINI, S., ROBINET, J. C. & DE PALMA, P. 2016 Subcritical transition scenarios via linear and nonlinear localized optimal perturbations in plane Poiseuille flow. *Fluid Dyn. Res.* **48** (6), 061409.
- FARRELL, B. F. & IOANNOU, P. J. 1993 Stochastic forcing of the linearized Navier–Stokes equations. *Phys. Fluids A* **5** (11), 2600–2609.
- GUSTAVSSON, L. H. 1991 Energy growth of three-dimensional disturbances in plane Poiseuille flow. *J. Fluid Mech.* **224**, 241–260.
- HÆPFFNER, J., CHEVALIER, M., BEWLEY, T. R. & HENNINGSON, D. S. 2005 State estimation

- in wall-bounded flow systems. Part 1. Perturbed laminar flows. *J. Fluid Mech.* **534**, 263–294.
- HOF, B., JUEL, A. & MULLIN, T. 2003 Scaling of the turbulence transition threshold in a pipe. *Phys. Rev. Lett.* **91** (24), 244502.
- HWANG, Y. & COSSU, C. 2010a Amplification of coherent streaks in the turbulent Couette flow: an input–output analysis at low Reynolds number. *J. Fluid Mech.* **643**, 333–348.
- HWANG, Y. & COSSU, C. 2010b Linear non-normal energy amplification of harmonic and stochastic forcing in the turbulent channel flow. *J. Fluid Mech.* **664**, 51–73.
- ILLINGWORTH, S. J. 2020 Streamwise-constant large-scale structures in Couette and Poiseuille flows. *J. Fluid Mech.* **889**, A13.
- ILLINGWORTH, S. J., MONTY, J. P. & MARUSIC, I. 2018 Estimating large-scale structures in wall turbulence using linear models. *J. Fluid Mech.* **842**, 146–162.
- JOVANOVIĆ, M. & BAMIEH, B. 2001 Modeling flow statistics using the linearized Navier-Stokes equations. In *Proceedings of the 40th IEEE Conference on Decision and Control*, pp. 4944–4949. IEEE.
- JOVANOVIĆ, M. R. 2004 Modeling, analysis, and control of spatially distributed systems. PhD thesis, University of California at Santa Barbara.
- JOVANOVIĆ, M. R. 2021 From bypass transition to flow control and data-driven turbulence modeling: An input-output viewpoint. *Annu. Rev. Fluid Mech.* **53** (1), 311–345.
- JOVANOVIĆ, M. R. & BAMIEH, B. 2004 Unstable modes versus non-normal modes in supercritical channel flows. In *Proceedings of the 2004 American Control Conference*, pp. 2245–2250. IEEE.
- JOVANOVIĆ, M. R. & BAMIEH, B. 2005 Componentwise energy amplification in channel flows. *J. Fluid Mech.* **534**, 145–183.
- KALUR, A., MUSHTAQ, T., SEILER, P. & HEMATI, M. S. 2021 Estimating regions of attraction for transitional flows using quadratic constraints. *arXiv preprint arXiv:2103.05426*.
- KALUR, A., SEILER, P. & HEMATI, M. S. 2020a Nonlinear stability analysis of transitional flows using quadratic constraints. *arXiv preprint arXiv:2004.05440*.
- KALUR, A., SEILER, P. & HEMATI, M. S. 2020b Stability and performance analysis of nonlinear and non-normal systems using quadratic constraints. In *AIAA Scitech 2020 Forum*, p. 0833.
- KANAZAWA, T. 2018 Lifetime and growing process of localized turbulence in plane channel flow. PhD thesis, Osaka University.
- KERSWELL, R. R. 2018 Nonlinear nonmodal stability theory. *Annu. Rev. Fluid Mech.* **50** (1), 319–345.
- KERSWELL, R. R., PRINGLE, C. C. & WILLIS, A. P. 2014 An optimization approach for analysing nonlinear stability with transition to turbulence in fluids as an exemplar. *Rep. Prog. Phys.* **77** (8), 085901.
- KREISS, G., LUNDBLADH, A. & HENNINGSON, D. S. 1994 Bounds for threshold amplitudes in subcritical shear flows. *J. Fluid Mech.* **270**, 175–198.
- LANDAHL, M. T. 1975 Wave breakdown and turbulence. *SIAM J. Appl. Math.* **28** (4), 735–756.
- LIU, C. & GAYME, D. F. 2019 Convective velocities of vorticity fluctuations in turbulent channel flows: an input–output approach. In *Proceedings of the Eleventh International Symposium on Turbulence and Shear Flow Phenomenon*. Southampton, UK.
- LIU, C. & GAYME, D. F. 2020a An input–output based analysis of convective velocity in turbulent channels. *J. Fluid Mech.* **888**, A32.
- LIU, C. & GAYME, D. F. 2020b Input–output inspired method for permissible perturbation amplitude of transitional wall-bounded shear flows. *Phys. Rev. E* **102** (6), 063108.
- LUNDBLADH, A., HENNINGSON, D. S. & REDDY, S. C. 1994 Threshold amplitudes for transition in channel flows. In *Transition, turbulence and combustion*, pp. 309–318. Springer.
- MADHUSUDANAN, A., ILLINGWORTH, S. J. & MARUSIC, I. 2019 Coherent large-scale structures from the linearized Navier–Stokes equations. *J. Fluid Mech.* **873**, 89–109.
- MCKEON, B. 2017 The engine behind (wall) turbulence: perspectives on scale interactions. *J. Fluid Mech.* **817**, P1.
- MCKEON, B. J. & SHARMA, A. S. 2010 A critical-layer framework for turbulent pipe flow. *J. Fluid Mech.* **658**, 336–382.

- MCKEON, B. J., SHARMA, A. S. & JACOBI, I. 2013 Experimental manipulation of wall turbulence: a systems approach. *Phys. Fluids* **25** (3), 031301.
- MELLIBOVSKY, F. & MESEGURER, A. 2009 Critical threshold in pipe flow transition. *Philos. Trans. R. Soc. A Math. Phys. Eng. Sci.* **367** (1888), 545–560.
- MOARREF, R., JOVANOVIĆ, M. R., TROPP, J. A., SHARMA, A. S. & MCKEON, B. J. 2014 A low-order decomposition of turbulent channel flow via resolvent analysis and convex optimization. *Phys. Fluids* **26** (5), 051701.
- MOARREF, R., SHARMA, A. S., TROPP, J. A. & MCKEON, B. J. 2013 Model-based scaling of the streamwise energy density in high-Reynolds-number turbulent channels. *J. Fluid Mech.* **734**, 275–316.
- MONOKROUSOS, A., BOTTARO, A., BRANDT, L., DI VITA, A. & HENNINGSON, D. S. 2011 Nonequilibrium thermodynamics and the optimal path to turbulence in shear flows. *Phys. Rev. Lett.* **106** (13), 1–4.
- MORRA, P., NOGUEIRA, P. A. S., CAVALIERI, A. V. G. & HENNINGSON, D. S. 2021 The colour of forcing statistics in resolvent analyses of turbulent channel flows. *J. Fluid Mech.* **907**, A24.
- MULLIN, T. 2011 Experimental studies of transition to turbulence in a pipe. *Annu. Rev. Fluid Mech.* **43** (1), 1–24.
- NOGUEIRA, P. A. S., MORRA, P., MARTINI, E., CAVALIERI, A. V. G. & HENNINGSON, D. S. 2021 Forcing statistics in resolvent analysis: application in minimal turbulent Couette flow. *J. Fluid Mech.* **908**, A32.
- ORSZAG, S. A. 1971 Accurate solution of the Orr–Sommerfeld stability equation. *J. Fluid Mech.* **50** (4), 689–703.
- PACKARD, A. & DOYLE, J. 1993 The complex structured singular value. *Automatica* **29** (1), 71–109.
- PARANJAPE, C. 2019 Onset of turbulence in plane Poiseuille flow. PhD thesis, Institute of Science and Technology Austria.
- PARANJAPE, C. S., DUGUET, Y. & HOF, B. 2020 Oblique stripe solutions of channel flow. *J. Fluid Mech.* **897**, A7.
- PEIXINHO, J. & MULLIN, T. 2007 Finite-amplitude thresholds for transition in pipe flow. *J. Fluid Mech.* **582**, 169–178.
- PHILIP, J., SVIZHER, A. & COHEN, J. 2007 Scaling law for a subcritical transition in plane Poiseuille flow. *Phys. Rev. Lett.* **98** (15), 154502.
- PRIGENT, A., GRÉGOIRE, G., CHATÉ, H. & DAUCHOT, O. 2003 Long-wavelength modulation of turbulent shear flows. *Physica D* **174** (1–4), 100–113.
- PRIGENT, A., GRÉGOIRE, G., CHATÉ, H., DAUCHOT, O. & VAN SAARLOOS, W. 2002 Large-scale finite-wavelength modulation within turbulent shear flows. *Phys. Rev. Lett.* **89** (1), 014501.
- PRINGLE, C. C. T. & KERSWELL, R. R. 2010 Using nonlinear transient growth to construct the minimal seed for shear flow turbulence. *Phys. Rev. Lett.* **105** (15), 1–4.
- PRINGLE, C. C. T., WILLIS, A. P. & KERSWELL, R. R. 2012 Minimal seeds for shear flow turbulence: using nonlinear transient growth to touch the edge of chaos. *J. Fluid Mech.* **702**, 415–443.
- RABIN, S. M. E., CAULFIELD, C. P. & KERSWELL, R. R. 2012 Triggering turbulence efficiently in plane Couette flow. *J. Fluid Mech.* **712**, 244–272.
- REDDY, S. C. & HENNINGSON, D. S. 1993 Energy growth in viscous channel flows. *J. Fluid Mech.* **252**, 209–238.
- REDDY, S. C., SCHMID, P. J., BAGGETT, J. S. & HENNINGSON, D. S. 1998 On stability of streamwise streaks and transition thresholds in plane channel flows. *J. Fluid Mech.* **365**, 269–303.
- REETZ, F., KREILOS, T. & SCHNEIDER, T. M. 2019 Exact invariant solution reveals the origin of self-organized oblique turbulent-laminar stripes. *Nat. Commun.* **10** (1), 1–6.
- REYNOLDS, O. 1883 XXIX. An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels. *Philos. Trans. R. Soc. London.* **174**, 935–982.
- ROMANOV, V. A. 1973 Stability of plane-parallel Couette flow. *Funct. Anal. Appl.* **7** (2), 137–146.

- SAFONOV, M. G. 1982 Stability margins of diagonally perturbed multivariable feedback systems. *IEE Proceedings D (Control Theory and Applications)* **129** (6), 251–256.
- SCHMID, P. & HENNINGSON, D. 1992 A new mechanism for rapid transition involving a pair of oblique waves. *Phys. Fluids A* **4** (9), 1986–1989.
- SCHMID, P. J. 2007 Nonmodal stability theory. *Annu. Rev. Fluid Mech.* **39**, 129–162.
- SCHMID, P. J. & HENNINGSON, D. S. 2012 *Stability and transition in shear flows*. Springer Science & Business Media.
- SHIMIZU, M. & MANNEVILLE, P. 2019 Bifurcations to turbulence in transitional channel flow. *Phys. Rev. Fluids* **4** (11), 113903.
- SONG, B. & XIAO, X. 2020 Trigger turbulent bands directly at low Reynolds numbers in channel flow using a moving-force technique. *J. Fluid Mech.* **903**, A43.
- SYMON, S., ILLINGWORTH, S. J. & MARUSIC, I. 2021 Energy transfer in turbulent channel flows and implications for resolvent modelling. *J. Fluid Mech.* **911**, A3.
- TAO, J., ECKHARDT, B. & XIONG, X. 2018 Extended localized structures and the onset of turbulence in channel flow. *Phys. Rev. Fluids* **3** (1), 011902.
- TILLMARK, N. & ALFREDSSON, P. H. 1992 Experiments on transition in plane Couette flow. *J. Fluid Mech.* **235**, 89–102.
- TOWNE, A., LOZANO-DURÁN, A. & YANG, X. 2020 Resolvent-based estimation of space–time flow statistics. *J. Fluid Mech.* **883**, A17.
- TREFETHEN, L. N. 2000 *Spectral methods in MATLAB*. SIAM.
- TREFETHEN, L. N., TREFETHEN, A. E., REDDY, S. C. & DRISCOLL, T. A. 1993 Hydrodynamic stability without eigenvalues. *Science* **261** (5121), 578–584.
- TSUKAHARA, T., KAWAGUCHI, Y. & KAWAMURA, H. 2014 An experimental study on turbulent-stripe structure in transitional channel flow. *arXiv preprint arXiv:1406.1378*.
- TSUKAHARA, T., SEKI, Y., KAWAMURA, H. & TOCHIO, D. 2005 DNS of turbulent channel flow at very low Reynolds numbers. In *Proceedings of the Fourth International Symposium on Turbulence and Shear Flow Phenomena*. Williamsburg, USA.
- TUCKERMAN, L. S. & BARKLEY, D. 2011 Patterns and dynamics in transitional plane Couette flow. *Phys. Fluids* **23** (4), 041301.
- TUCKERMAN, L. S., CHANTRY, M. & BARKLEY, D. 2020 Patterns in wall-bounded shear flows. *Annu. Rev. Fluid Mech.* **52** (1), 343–367.
- TUCKERMAN, L. S., KREILOS, T., SCHROBSDORFF, H., SCHNEIDER, T. M. & GIBSON, J. F. 2014 Turbulent-laminar patterns in plane Poiseuille flow. *Phys. Fluids* **26** (11), 114103.
- VADAREVU, S. B., SYMON, S. & ILLINGWORTH, S. J. 2019 Coherent structures in the linearized impulse response of turbulent channel flow. *J. Fluid Mech.* **911**, 1190–1203.
- WEI, D. & ZHANG, Z. 2020 Transition threshold for the 3D Couette flow in Sobolev space. *Commun. Pure Appl. Math.* pp. 0001–0082.
- WEIDEMAN, J. A. C. & REDDY, S. C. 2000 A MATLAB differentiation matrix suite. *ACM Trans. Math. Softw.* **26** (4), 465–519.
- XIAO, X. & SONG, B. 2020 The growth mechanism of turbulent bands in channel flow at low Reynolds numbers. *J. Fluid Mech.* **883**, R1.
- XIONG, X., TAO, J., CHEN, S. & BRANDT, L. 2015 Turbulent bands in plane-Poiseuille flow at moderate Reynolds numbers. *Phys. Fluids* **27** (4), 041702.
- YOUNG, P. M., NEWLIN, M. P. & DOYLE, J. C. 1992 Practical computation of the mixed μ problem. In *1992 American Control Conference*, pp. 2190–2194. IEEE.
- ZARE, A., JOVANOVIĆ, M. R. & GEORGIOU, T. T. 2017 Colour of turbulence. *J. Fluid Mech.* **812**, 636–680.
- ZHOU, K., DOYLE, J. C. & GLOVER, K. 1996 *Robust and optimal control*. Prentice hall.