UAV Systems & Control
Lecture 4

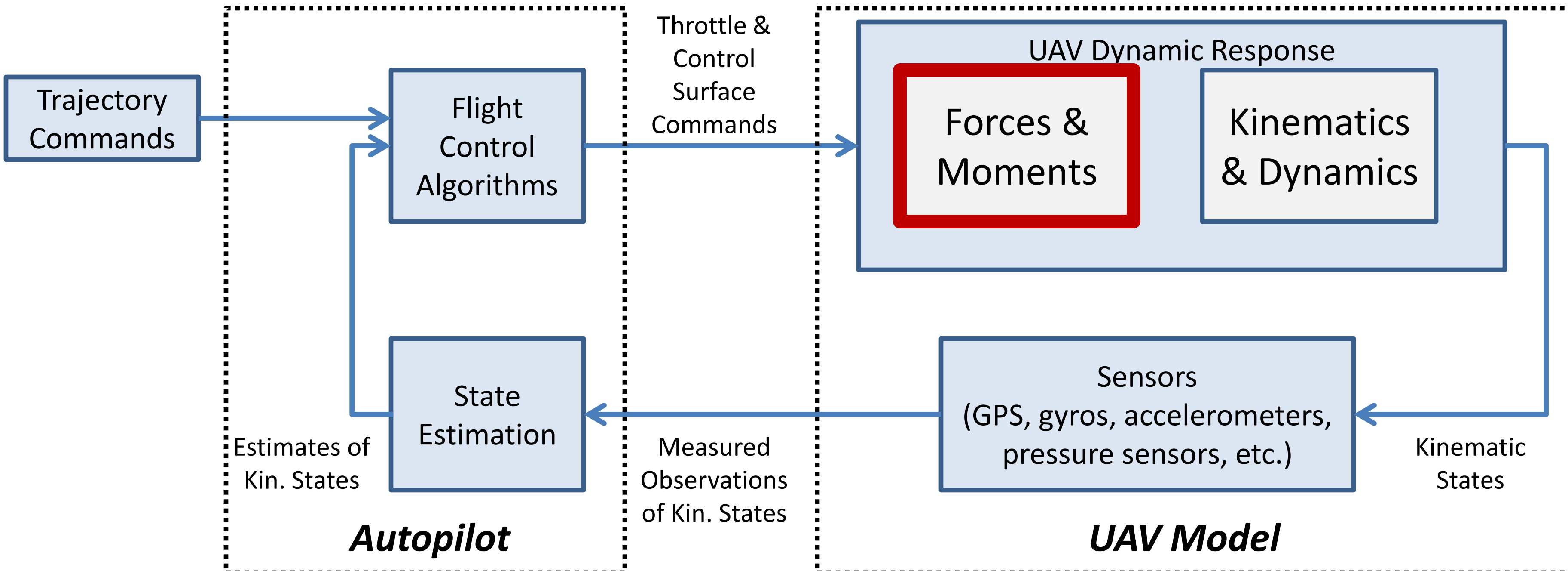
Forces & Moments

Gravity Force

Aerodynamic Forces and Moments

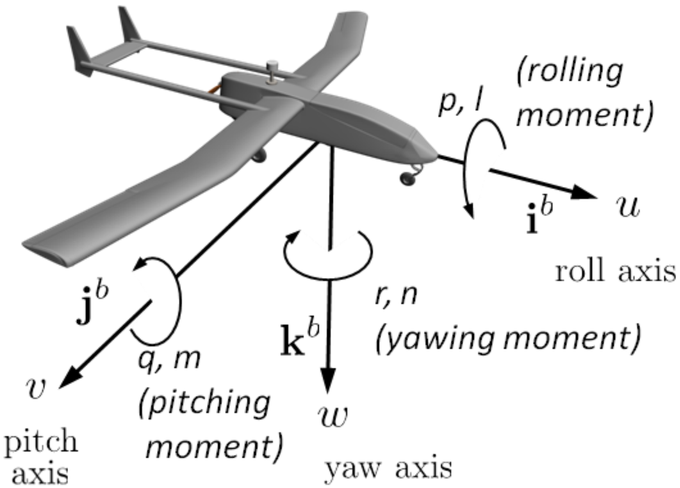
(Propeller described in next lecture)

UAV System



- In the previous lecture we developed the 12 equations of motion which propagate vehicle motion based on external forces and moments
- In this section we will develop equations for the *forces* and *moments* acting on an aerodynamic vehicle
  - **Forces due to**: gravity, aerodynamics, propulsion
  - **Moments due to**: aerodynamics, propulsion

Aircraft Variables



12 State Variables

Name	Description	Metric Units
p_n	Inertial North position of MAV expressed along $\mathbf{i}^i$ in $\mathcal{F}^i$.	m
p_e	Inertial East position of MAV expressed along $\mathbf{j}^i$ in $\mathcal{F}^i$.	m
p_d	Inertial Down position of MAV expressed along $\mathbf{k}^i$ in $\mathcal{F}^i$.	m
u	Ground velocity expressed along $\mathbf{i}^b$ in $\mathcal{F}^b$.	m/s
v	Ground velocity expressed along $\mathbf{j}^b$ in $\mathcal{F}^b$.	m/s
w	Ground velocity expressed along $\mathbf{k}^b$ in $\mathcal{F}^b$.	m/s
ϕ	Roll angle defined with respect to $\mathcal{F}^{v2}$.	rad
θ	Pitch angle defined with respect to $\mathcal{F}^{v1}$.	rad
ψ	Heading (yaw) angle defined with respect to $\mathcal{F}^v$.	rad
p	Body angular (roll) rate expressed along $\mathbf{i}^b$ in $\mathcal{F}^b$.	rad/s
q	Body angular (pitch) rate expressed along $\mathbf{j}^b$ in $\mathcal{F}^b$.	rad/s
r	Body angular (yaw) rate expressed along $\mathbf{k}^b$ in $\mathcal{F}^b$.	rad/s

Other Variables

mass	Vehicle mass, assumed constant	kg
J	3x3 Inertia matrix (Common simplifying assumption: J_{xy}=J_{yz}=0)	kg-m^2
f_x	Axial force along x-axis (e.g. majority of thrust and drag components)	N
f_y	Lateral force along y-axis (e.g. sideslip-induced force)	N
f_z	Normal force along z-axis (e.g. majority of lift and gravity compnts.)	N
l	Rolling moment, about x-axis	N-m
m	Pitching moment, about y-axis	N-m
n	Yawing moment, about z-axis	N-m

Vector relationships:

$$\underline{v}_g^b = u_g \hat{i}^b + v_g \hat{j}^b + w_g \hat{k}^b$$

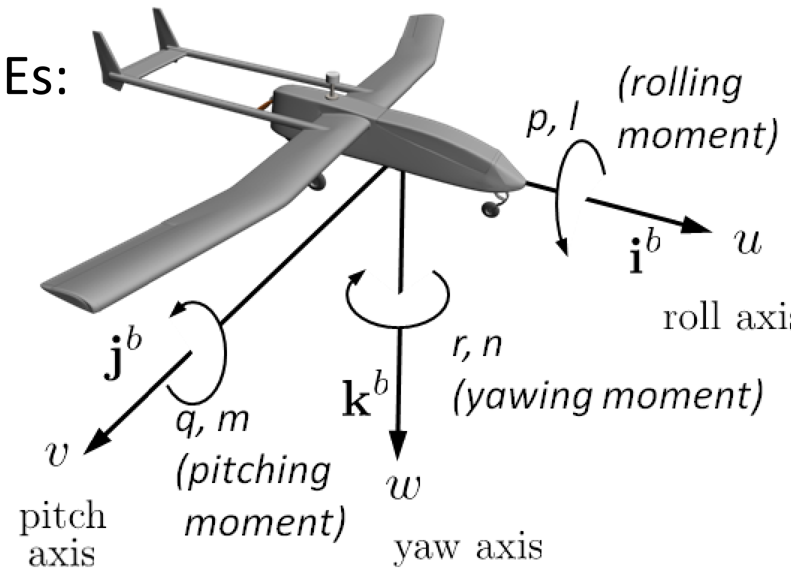
$$\underline{\omega}_{b/i}^b = p\,\hat{i}^b + q\,\hat{j}^b + r\,\hat{k}^b$$

$$\underline{\mathrm{f}}^b = f_x \hat{i}^b + f_y \hat{j}^b + f_z \hat{k}^b$$

$$\underline{\mathrm{m}}^b = l\,\hat{i}^b + m\,\hat{j}^b + n\,\hat{k}^b$$

Equations of Motion

- From Kinematics and Dynamics, the equations of motion are 12 ODEs:



$$\begin{bmatrix} \dot{p}_n \\ \dot{p}_e \\ \dot{p}_d \end{bmatrix} = R_b^{ned}\, \underline{v}_g^b = \left(R_{ned}^b\right)^T \underline{v}_g^b$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = -\underline{\omega}_{b/i}^b \times \underline{v}_g^b + \frac{1}{mass}\underline{\mathrm{f}}^b$$

$$\begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\sec\theta & \cos\phi\sec\theta \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$\begin{bmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{bmatrix} = \mathrm{J}^{-1}\left[\underbrace{-\underline{\omega}_{b/i}^b \times \left(\mathrm{J}\,\underline{\omega}_{b/i}^b\right)}_{\text{“Internal” moments induced by cross-coupling effects}} + \underline{\mathrm{m}}^b\right]$$

“Internal” moments induced by cross-coupling effects

External Moments

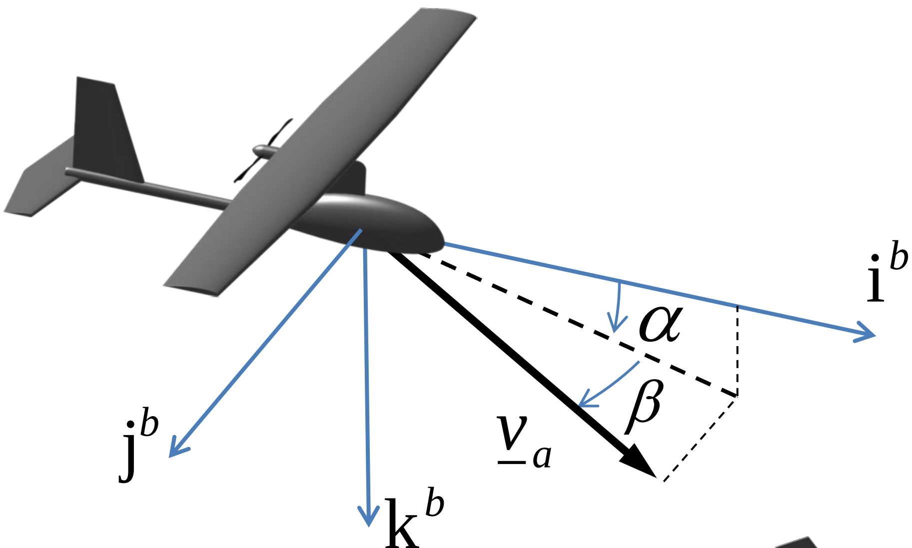
The objective of this lecture is to show how to compute the force and moment vectors:

$$\underline{\mathrm{f}}^b = f_x\hat{i}^b + f_y\hat{j}^b + f_z\hat{k}^b = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix}$$

$$\underline{\mathrm{m}}^b = l\,\hat{i}^b + m\,\hat{j}^b + n\,\hat{k}^b = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$$

Aerodynamic Angles

We will find that aerodynamic forces are highly dependent on the aerodynamic angles (α and β) which describe the direction of the airspeed vector relative to body frame.



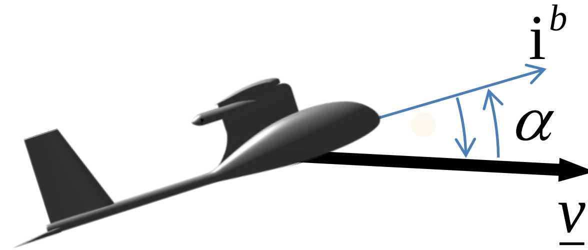
Wind-relative Airspeed vector:

$$\underline{v}_a = \underline{v}_g - \underline{v}_w$$

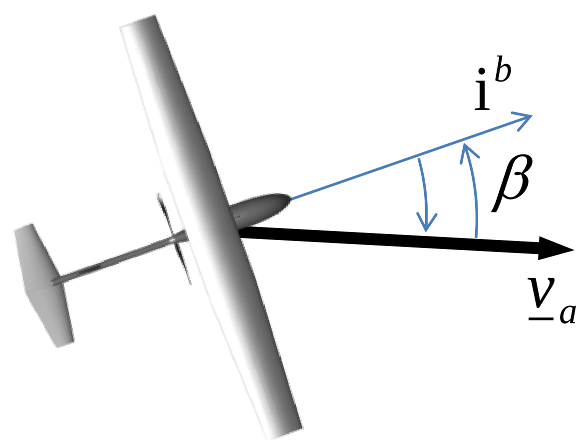
Angle-of-Attack: α

Sideslip: β

Airspeed: $V_a = |\underline{v}_a|$



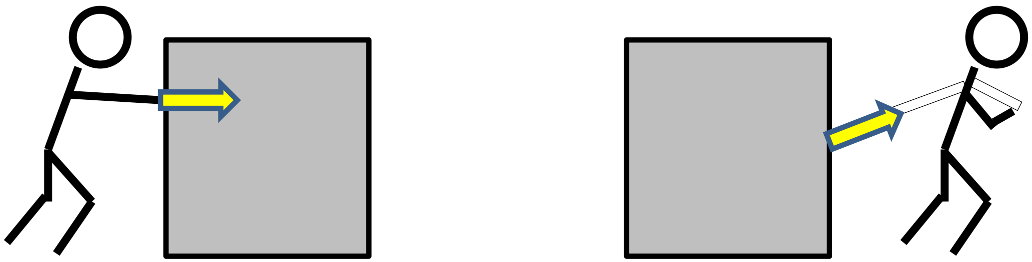
Equivalent ways of displaying Angle-of-Attack:
- α: *Angle of* $\underline{v}_a$ *below axial*
- α: *Angle of axial above* $\underline{v}_a$



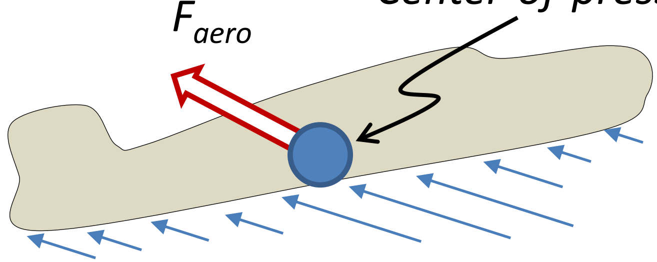
Equivalent ways of displaying Sideslip:
- β: *Angle of* $\underline{v}_a$ *right of axial*
- β : *Angle of axial left of* $\underline{v}_a$

What are Forces?

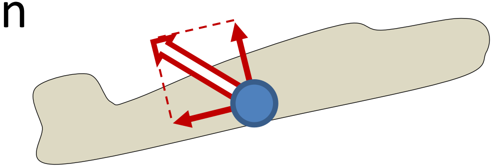
- **<u>Force:</u>** An action that causes a change in <u>translational</u> motion
  - A force can be the result of a discrete push or pull on a body…



  - … or a gradient of pressure acting on the body
    - To model, we often lump the pressure into an overall force acting at a "center of pressure"



  - A force <u>vector</u> can be broken into different components



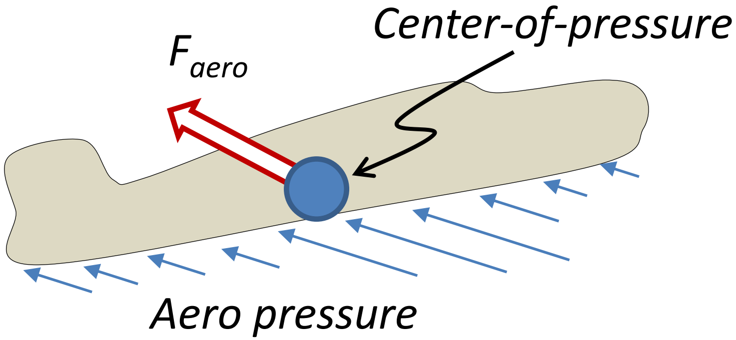
Units:

F: Newtons (N)

$$N = kg\frac{m}{s^2}$$

Center of pressure

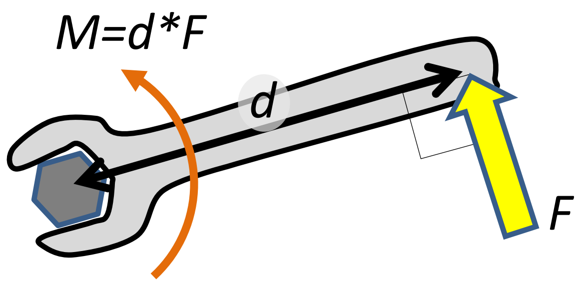
- An airfoil produces lift in an airstream. Different points along the airfoil produce different lift vectors. The center of pressure is the point at which the total sum of the pressure on a body causes a single aerodynamic force vector to act through that point



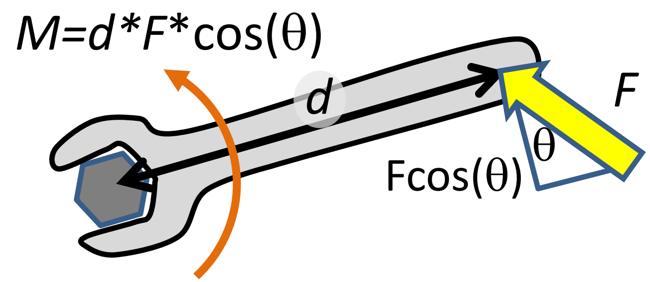
- The location of the center of pressure along the airfoil changes continuously with changes in angle of attack.

What are Moments?

- **<u>Moment:</u>** An action that causes a change in <u>rotational</u> motion
  - A moment (or torque) is caused by a force acting at a distance



If force is perpendicular to shortest distance, M=d*F



If force is at an angle, moment is proportional to the perpendicular component of force: M=d*F*cos(θ)

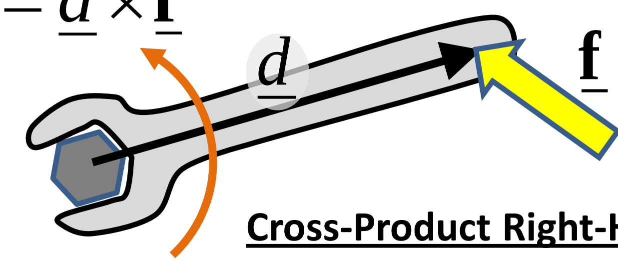
  - In vector form:

$$\underline{\mathbf{m}} = \underline{d} \times \underline{\mathbf{f}}$$

Units:

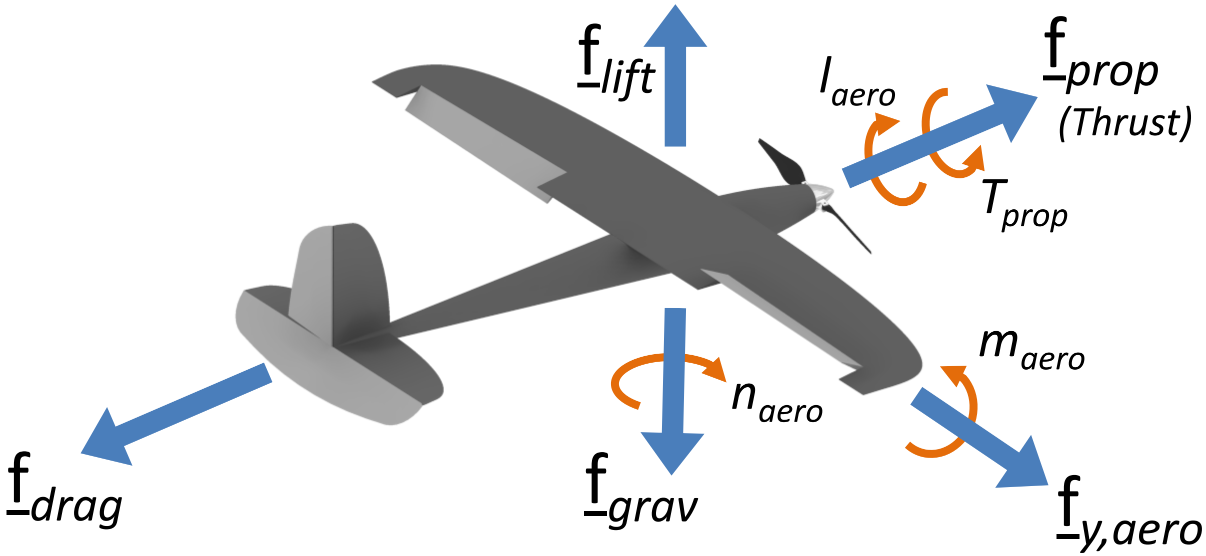
M: Newton-meters (N-m)

Resulting moment vector comes out of page



<u>Cross-Product Right-Hand Rule:</u> Curl fingers from first vector toward second. Thumb points toward cross product.

External Forces and Moments



Note 1: Modify $\underline{f}_{prop}$ and $\underline{m}_{prop}$ accordingly if propulsion/thrust source is not mounted along x-axis.

Note 2: $\underline{f}_{lift}$ and $\underline{f}_{drag}$ are actually in the stability frame to be discussed later.

Sum of Forces:

$$\Sigma\underline{f} = \underline{f}_{grav} + \underbrace{\underline{f}_{aero}}_{\underline{f}_{lift} + \underline{f}_{drag} + \underline{f}_{y,aero}} + \underline{f}_{prop}$$

Sum of Moments:

$$\Sigma\underline{m} = \underbrace{\underline{m}_{aero}}_{\begin{bmatrix} l_{aero} \\ m_{aero} \\ n_{aero} \end{bmatrix}^b} + \underbrace{\underline{m}_{prop}}_{\begin{bmatrix} T_{prop} \\ 0 \\ 0 \end{bmatrix}^b}$$

For “trim” flight, forces and moments are balanced.

Force due to Gravity

- Gravity attracts a vehicle toward the "down" direction.
  - Gravity (g) is an acceleration (9.8 m/s^2)
  - (mass * gravity) is a force [Units: *kg-m/s*^{2}, or *N*]
- The gravity force vector expressed in NED frame is:

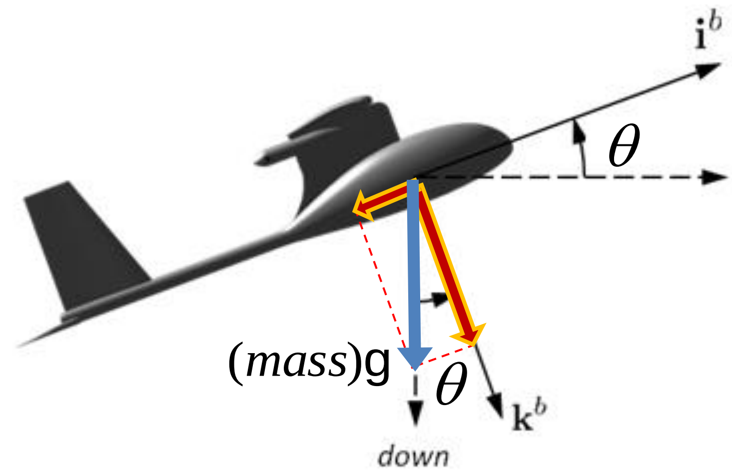
$$\underline{\mathrm{f}}_{grav}^{ned} = \begin{bmatrix} 0 \\ 0 \\ (mass)\mathrm{g} \end{bmatrix}$$

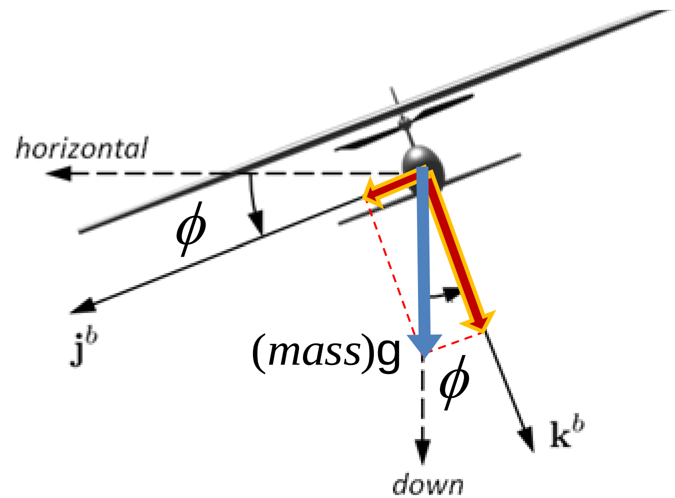
- The gravity force vector mapped onto body frame is:

$$\underline{\mathrm{f}}_{grav}^{b} = R_{ned}^{b}\,\underline{\mathrm{f}}_{grav}^{ned} = \begin{bmatrix} D.C. & D.C. & -\sin\theta \\ D.C. & D.C. & \cos\theta\sin\phi \\ D.C. & D.C. & \cos\theta\cos\phi \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ (mass)\mathrm{g} \end{bmatrix}$$

$$\underline{\mathrm{f}}_{grav}^{b} = (mass)\mathrm{g}\begin{bmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{bmatrix}$$

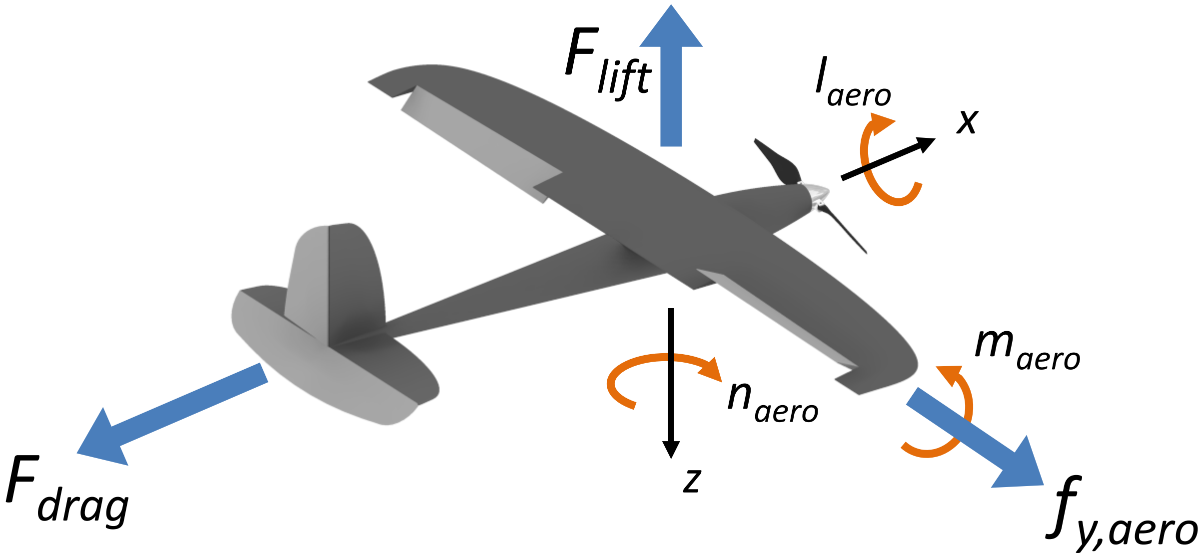
Force due to gravity expressed in body frame





$D.C.$: Don't Care

Aerodynamic Forces and Moments



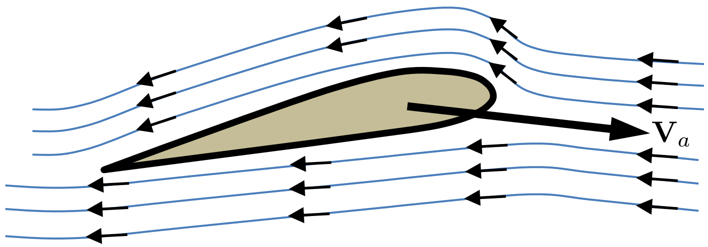
$\underline{f}_{aero}$ is comprised of: F_{lift}, F_{drag}, $f_{y,aero}$

$\underline{m}_{aero}$ is comprised of: l_{aero}, m_{aero}, n_{aero}

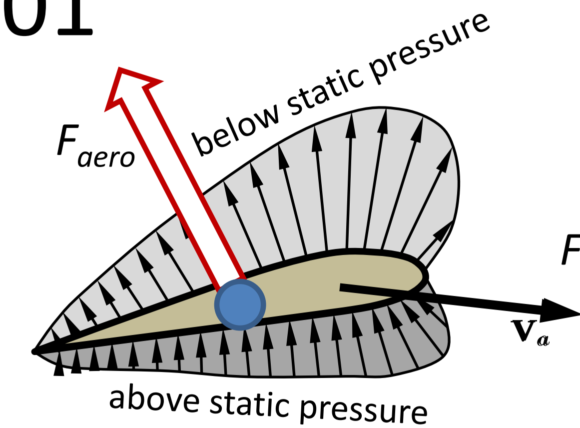
To derive aero forces and moments, we will:

- Discuss basic aerodynamic theory
- Uncouple quantities into two channels:
  - Longitudinal (pitching motion): F_{lift}, F_{drag}, m_{aero}
  - Lateral (side-to-side motion): $f_{y,aero}$, l_{aero}, n_{aero}

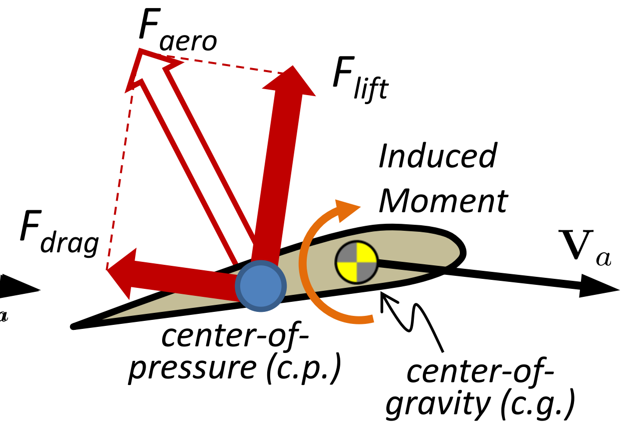
Aerodynamics 101



Airflow flows over and under an Airfoil. Airflow above airfoil travels farther and goes faster. Bernoulli's Principle states that faster airflow causes reduced pressure



Difference in pressure above and below Airfoil (due to Bernoulli's Principle) causes a net pressure upward, averaged to an F_{aero} about center-of-pressure



F_{aero} is broken into:

- F_{lift} (normal to V_a), and
- F_{drag} (opposite V_a).

Object rotates about c.g., not c.p., thus F_{aero} induces an aero moment about c.g.

- **Basic aerodynamics modeling (a bit simplified)**
  - Airflow over an airfoil causes a pressure differential resulting in a net aerodynamic force at a center-of-pressure
  - For convenience, we break up the aero force into perpendicular lift and drag components
  - An aerodynamic moment (about center-of-gravity) is induced because of the difference between center-of-pressure and center-of-gravity
  - Generally, a stable airframe has a center-of-pressure behind the center-of-gravity

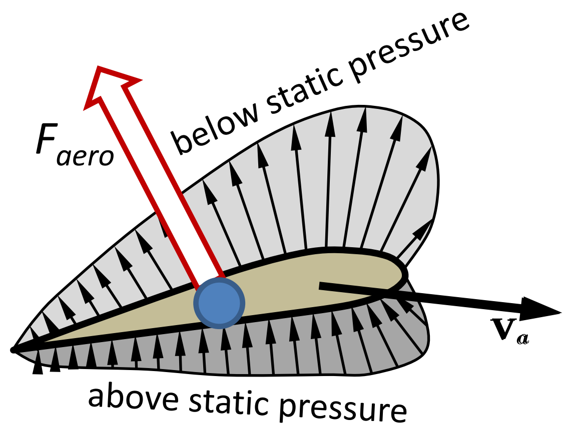
Aerodynamics Intuition #1

- Aerodynamic force should increase with airspeed (V_a)
  - Aero force is proportional to the <u>square</u> of airspeed

$$F_{aero} \propto V_a^2$$

  - Same applies to induced aero moments

$$M_{aero} \propto V_a^2$$



Aerodynamics Intuition #2

- Aero force should increase in "thicker" air
  - More air molecules pushing on an airfoil means a larger force
  - Aerodynamic force is proportional to air density (mass of air molecules in a unit volume)

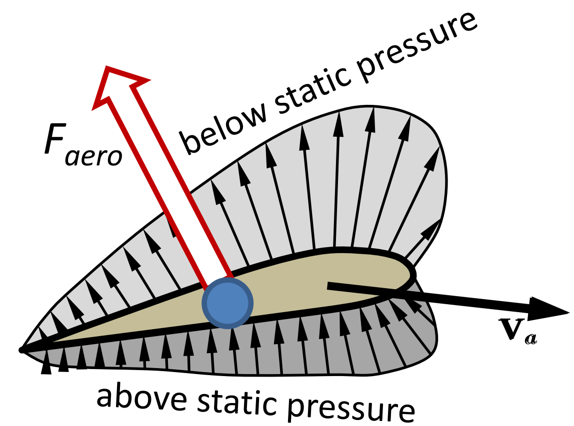
$$F_{aero} \propto \rho$$

where:

ρ : Air density, 1.2682 kg/m^3 at sea level, Standard day

  - Same applies to induced aero moments

$$M_{aero} \propto \rho$$



Air density (ρ) decreases with altitude

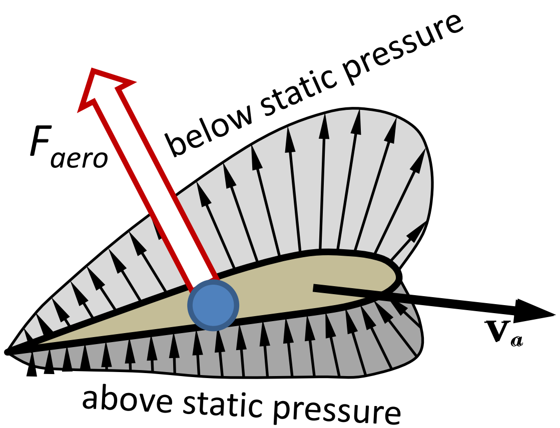
- Less aerodynamic force at really high altitudes

Intuition Corollary: Dynamic Pressure

- Because aero force is proportional to both air density (ρ) and V_a^2, we can say it is proportional to *dynamic pressure (Qbar)*, which is defined by:

$$\overline{Q} = \tfrac{1}{2}\rho V_a^2$$

ρ: Air density, kg/m³
V_a: Airspeed, m/s



  - Dynamic pressure is a measure of the pressure of air on a moving body

$$\left.\begin{aligned} F_{aero} &\propto \rho \\ F_{aero} &\propto V_a^2 \\ \overline{Q} &= \tfrac{1}{2}\rho V_a^2 \end{aligned}\right\} F_{aero} \propto \overline{Q}$$

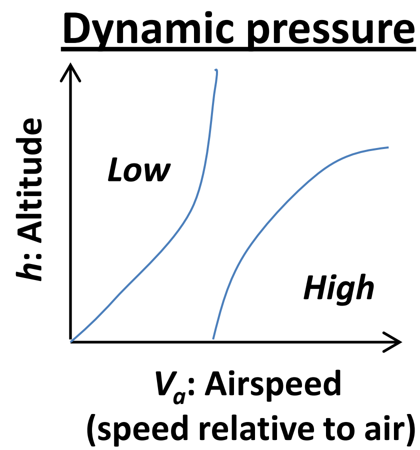
  - Same applies to induced aero moments

$$M_{aero} \propto \overline{Q}$$

  - Dynamic pressure has units of *N/m²*

$$\overline{Q} = \tfrac{1}{2} \; \rho \; V_a^2$$

Units: $\frac{N}{m^2} = \frac{kg}{m^3}\cdot\frac{m^2}{s^2} = kg\frac{m}{s^2}\cdot\frac{1}{m^2}$



ρ decreases with altitude

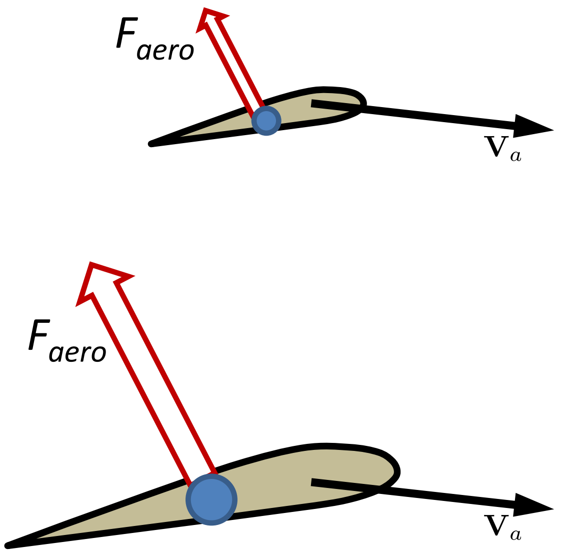
Aerodynamics Intuition #3

- Larger airfoils (e.g. wings) should create more aero force
  - Aero force is proportional to the exposed area of an airfoil:

$$F_{aero} \propto \text{Surface area of airfoil}$$

  - Same applies to induced aero moments

$$M_{aero} \propto \text{Surface area of airfoil}$$



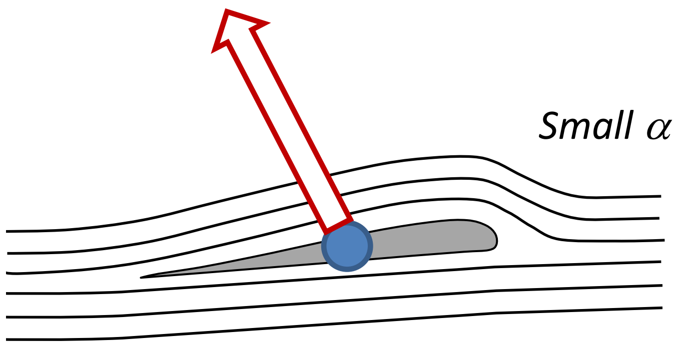
Aerodynamics Intuition #4

- Aero force should increase as the angle relative to the airflow increases
  - But… only up to a (stall) point
  - Aero force is *related to* the orientation (aerodynamic angle, α or β) of the airfoil relative to the air flow
    - For small angles, F_{aero} increases as the aerodynamic angle increases
    - At a certain critical angle, the airflow becomes detached and the airfoil "stalls" (i.e. loses lift)
  - In general, aero forces and moments have a non-linear functional relation with aerodynamic angles

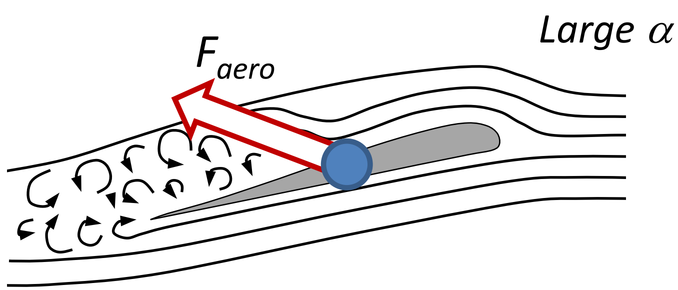
$$F_{aero} = f_F(\alpha, \beta, \text{etc.})$$
$$M_{aero} = f_M(\alpha, \beta, \text{etc.})$$

f_F(…), f_M(…): is a function of …

<u>Aerodynamic Angles:</u> Angles of airfoil relative to airflow: α, β



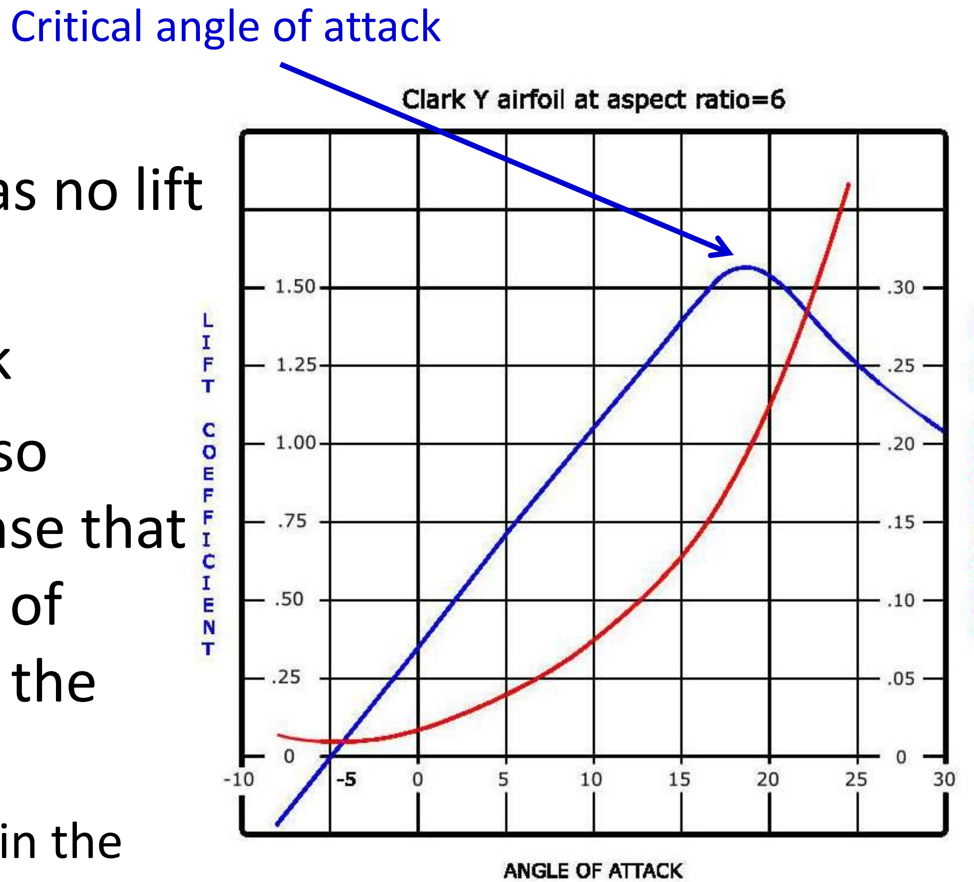
Small aero angles:
- Attached flow
- F_{aero} increases with aero angle



Larger aero angles:
- Detached flow
- Airfoil loses "lift". At a high enough angle, the airfoil will "stall".

Aerodynamics Intuition #4.1

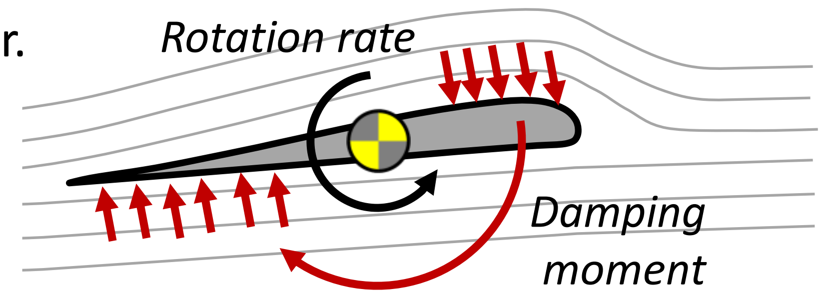
- An airfoil with zero airflow has no lift
- Technically, stalls are only dependent on angle of attack
- Indirectly, the stall point is also related to airspeed in the sense that at slow speeds a larger angle of attack is required to produce the necessary lift
  - The minimum speed to maintain the critical angle of attack is referred to as the “stall speed”



https://en.wikipedia.org/wiki/Airfoil

Aerodynamics Intuition #5

- Airflow should cause a "resistance" to airfoil rotations
  - Consider rotating your hand in still water.
    - Water will "resist" or *damp* the rotation
    - Damping will increase as you rotate faster.
  - As an airfoil rotates, air molecules cause opposing forces to resist rotation
    - Body rates (*p,q,r*) result in damping (opposing) forces and moments
      - e.g. a positive pitch rate (*q*) induces a negative pitching moment



Aerodynamic Damping:

- Rotation rate about center-of-gravity induces an opposing (damping) moment

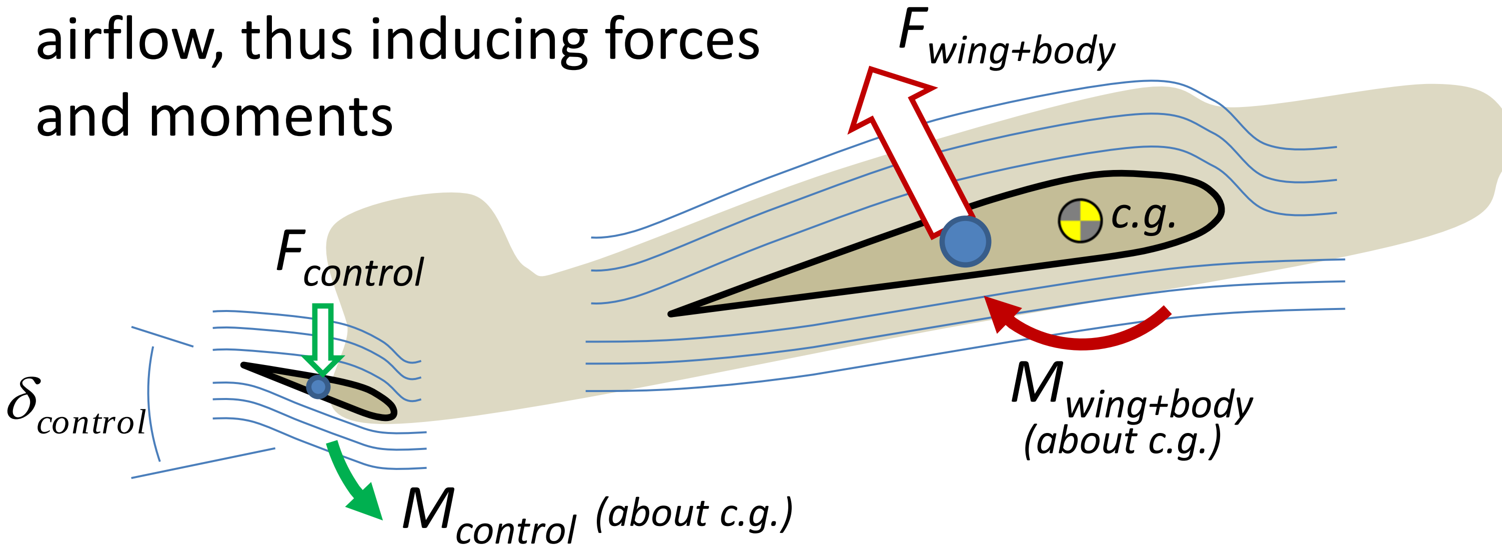
$$F_{aero} = f_F(p, q, r, \text{etc.})$$

$$M_{aero} = f_M(p, q, r, \text{etc.})$$

f_F(…), f_M(…): is a function of …

Aerodynamics Intuition #6

- Control surfaces disturb airflow, thus inducing forces and moments



– In general, aero forces and moments are non-linear functions of control surface angles

$$F_{aero} = F_{wing+body} + F_{control} = f_F\left(\delta_{control}, \text{etc.}\right)$$

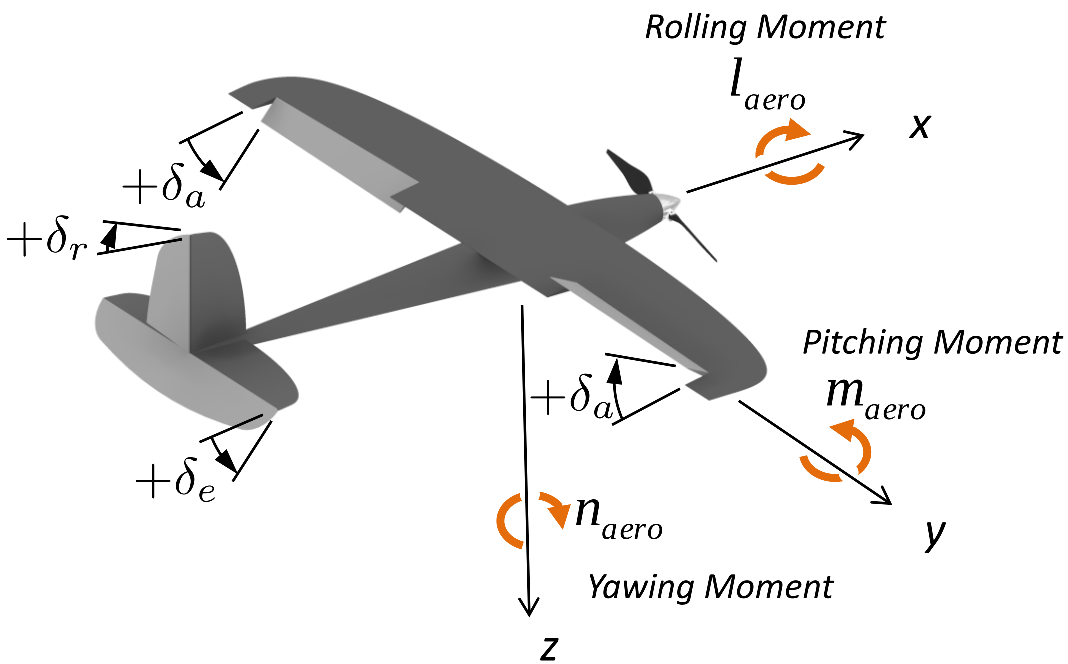
$$M_{aero} = M_{wing+body} + M_{control} = f_M\left(\delta_{control}, \text{etc.}\right)$$

Notes

- *Due to larger surface area, $F_{wing+body} >> F_{control}$*
- *Due to larger distance from c.g., small $F_{control}$ can counter-act moment from larger $F_{wing+body}$*
- *For "Trim" flight, moments must balance:*
  $M_{wing+body} + M_{control} = 0$

Control Surfaces

- We talk about 3 control surface channels:
  - Elevator (δ_e) affects pitching moment about body-y
  - Aileron (δ_a) affects rolling moment about body-x
    - δ_a is generally two opposing surfaces on wings or tail
  - Rudder (δ_r) affect yawing moment about body-z



In general, aero forces and moments are functions of control surface channels:

$$F_{aero} = f_F(\delta_e, \delta_a, \delta_r, \text{etc.})$$

$$M_{aero} = f_M(\delta_e, \delta_a, \delta_r, \text{etc.})$$

Using control surface deflections as shown here, the <u>primary</u> moment effects are:

$+\delta_a \Rightarrow +\Delta l_{aero}$ about x

$+\delta_e \Rightarrow -\Delta m_{aero}$ about y

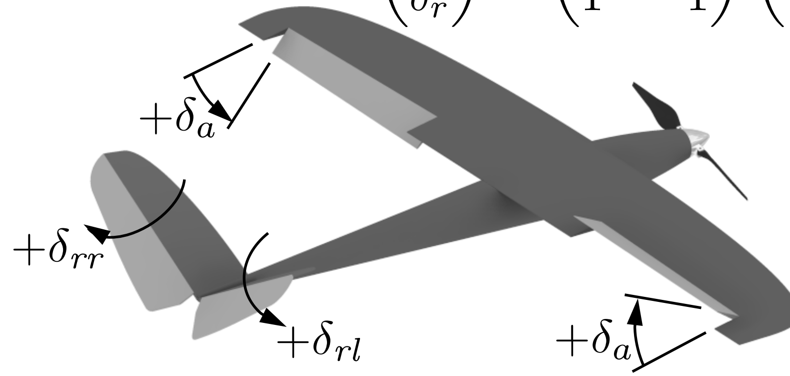
$+\delta_r \Rightarrow -\Delta n_{aero}$ about z

Control Surfaces

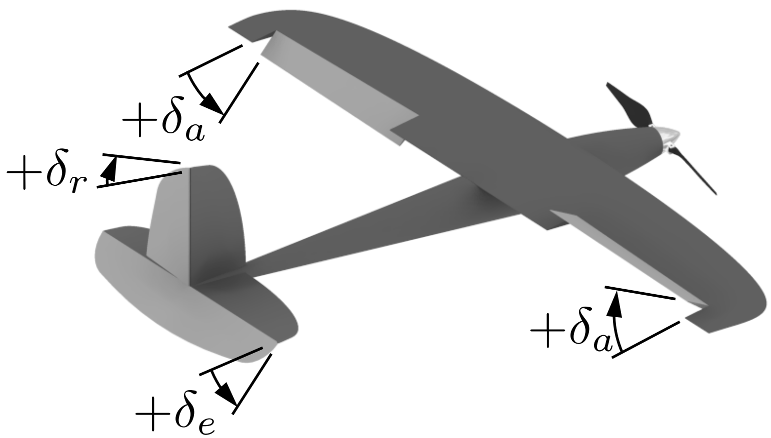
- In many airframes, physical control surface deflections must be mapped to 3 channels:
  - Elevator (δ_e), Aileron (δ_a), and Rudder (δ_r)

V-Tail:

$$\begin{pmatrix}\delta_e \\ \delta_r\end{pmatrix} = \begin{pmatrix}1 & 1 \\ 1 & -1\end{pmatrix}\begin{pmatrix}\delta_{rr} \\ \delta_{rl}\end{pmatrix}$$

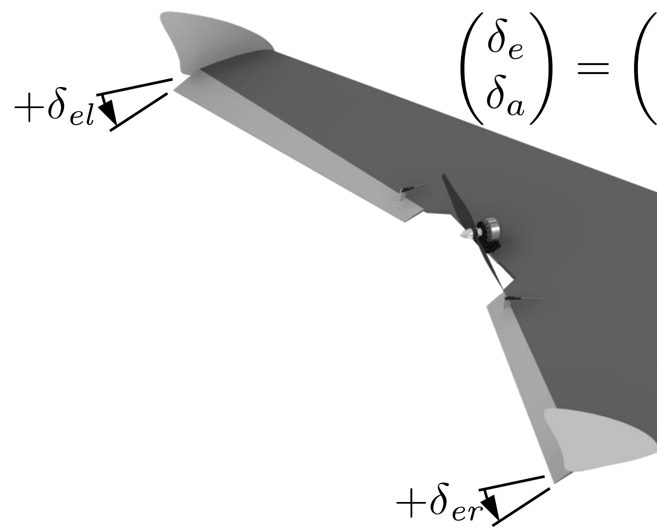


Conventional:



Flying Wing:

$$\begin{pmatrix}\delta_e \\ \delta_a\end{pmatrix} = \begin{pmatrix}1 & 1 \\ -1 & 1\end{pmatrix}\begin{pmatrix}\delta_{er} \\ \delta_{el}\end{pmatrix}$$

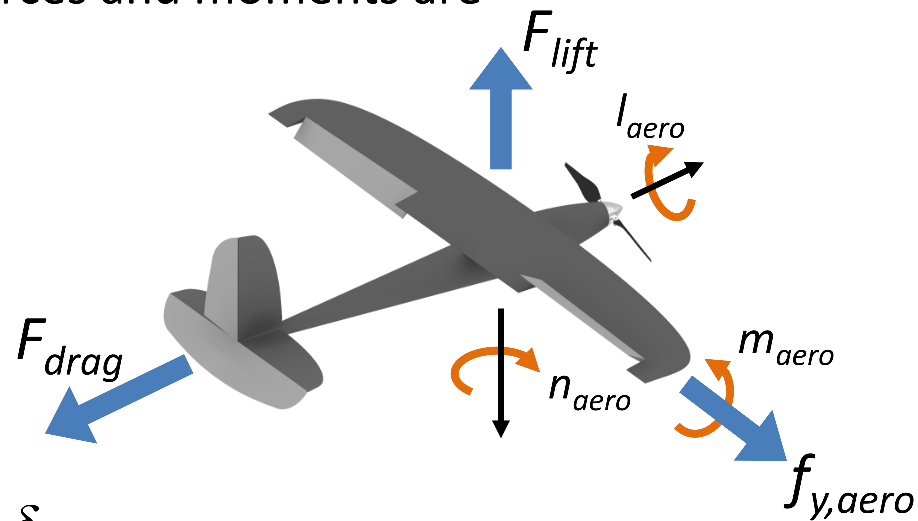


No Rudder Channel:
$\delta_r = 0$

Aerodynamic Intuitions Recap

From our intuitions we know that aero forces and moments are

- proportional to:
  - dynamic pressure, $\overline{Q} = \frac{1}{2}\rho V_a^2$
  - surface area, *S*
- functions of:
  - aero angles: α, β
  - body rates: *p, q, r*
  - control surface deflections: δ_e, δ_a, δ_r



From above, aerodynamicists convey aerodynamic forces and moments as non-dimensional (unit-less) coefficients (e.g. C_L, C_D, C_m, etc.).

Force Example: $F_{lift} = \overline{Q} \cdot S \cdot C_L(\alpha, \beta, p, q, r, \delta_e, \delta_a, \delta_r, \ldots)$

Units: $N = \frac{N}{m^2} \cdot m^2$ *[C_L has no units]*

Moment Example: $m_{aero} = \overline{Q} \cdot S \cdot c \cdot C_m(\alpha, \beta, p, q, r, \delta_e, \delta_a, \delta_r, \ldots)$

Units: $N \cdot m = \frac{N}{m^2} \cdot m^2 \cdot m$ *[C_m has no units]*

S: Reference area (generally a wing area)

c: Reference length (often a wing chord)

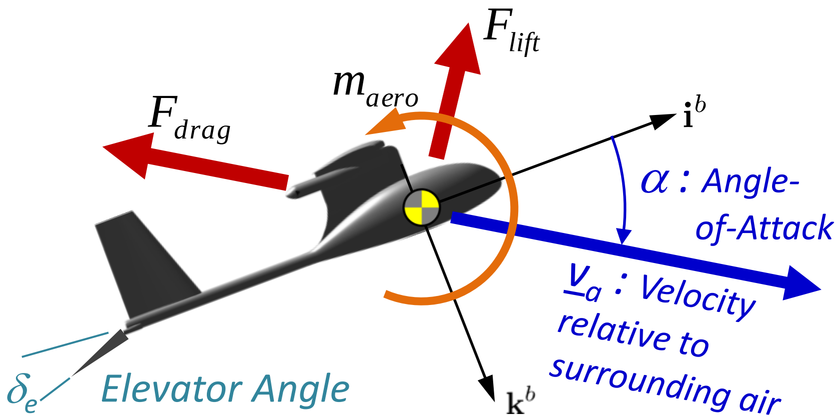
Note: Moments are scaled by reference area and a reference length

Aerodynamic Forces & Moments

Aircraft aerodynamics are commonly separated into two groups:

Longitudinal

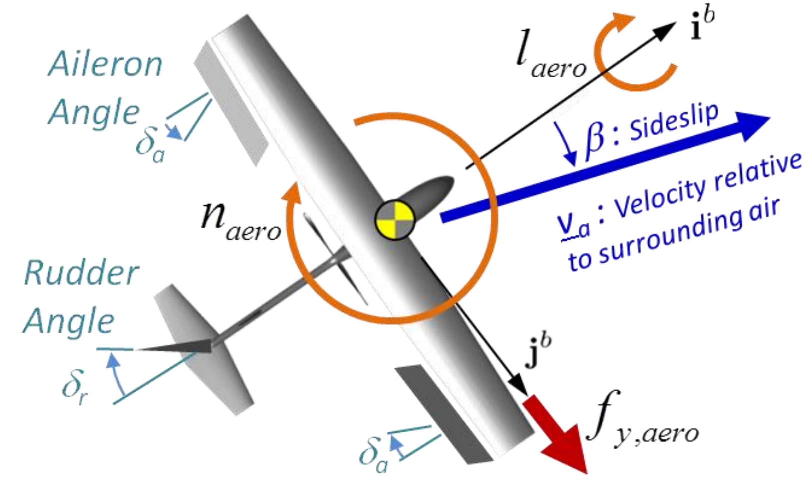
Up-Down, Pitch Plane, Pitch Motion



- Longitudinal aero comprised of:
  - Lift Force: F_{lift}
  - Drag Force: F_{drag}
  - Pitching Moment: m_{aero}
- Primary contributors:
  - Angle-of-Attack: α
  - Pitch Rate: q (about body-y)
  - Elevator: δ_e

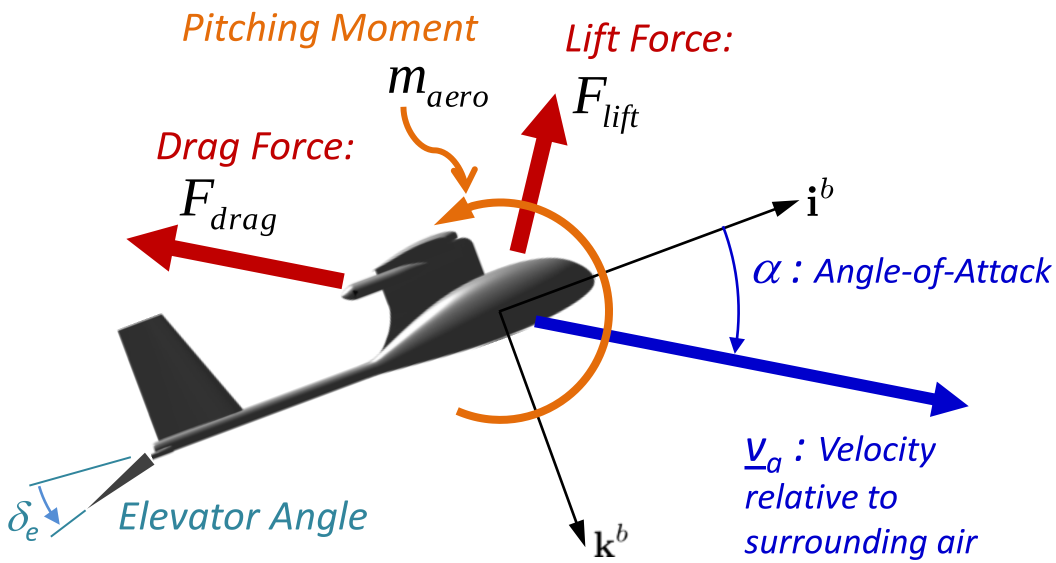
Lateral

Side-to-side, turning motions (roll and yaw)



- Lateral aero comprised of:
  - Side (Y) Force: $f_{y,aero}$
  - Rolling Moment: l_{aero}
  - Yawing Moment: n_{aero}
- Primary contributors:
  - Sideslip: β
  - Roll Rate: p (about body-x)
  - Yaw Rate: r (about body-z)
  - Aileron: δ_a
  - Rudder: δ_r

Longitudinal (Pitch-plane) Aerodynamics



$$F_{lift} = \tfrac{1}{2}\rho V_a^2 \cdot S \cdot C_L(\alpha, q, \delta_e, \ldots)$$

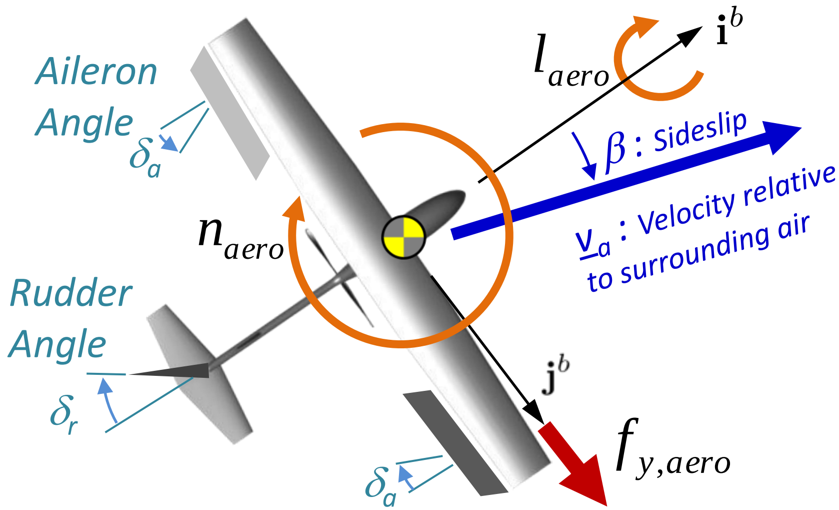
$$F_{drag} = \tfrac{1}{2}\rho V_a^2 \cdot S \cdot C_D(\alpha, q, \delta_e, \ldots)$$

$$m_{aero} = \tfrac{1}{2}\rho V_a^2 \cdot S \cdot c \cdot C_m(\alpha, q, \delta_e, \ldots)$$

S: Wing area
c: Wing chord (Longitudinal channel reference length)

- Longitudinal aero coefficients are primarily influenced by:
  - Angle-of-Attack, α
    - Pitch-plane angle between body axis and the wind-relative velocity vector
  - Pitch-plane body rate, *q*
  - Elevator channel control surface deflection: δ_e
- The longitudinal forces (F_{lift} and F_{drag}) are heavily influenced by α
- The pitching moment (m_{aero}) is heavily influenced by both α and δ_e

Lateral (side-to-side) Aerodynamics



$$f_{y,aero} = \tfrac{1}{2}\rho V_a^2 \cdot S \cdot C_Y(\beta, p, r, \delta_a, \delta_r, \ldots)$$

$$l_{aero} = \tfrac{1}{2}\rho V_a^2 \cdot S \cdot b \cdot C_l(\beta, p, r, \delta_a, \delta_r, \ldots)$$

$$n_{aero} = \tfrac{1}{2}\rho V_a^2 \cdot S \cdot b \cdot C_n(\beta, p, r, \delta_a, \delta_r, \ldots)$$

S: Wing area
b: Wing span (Lateral channel reference length)

- Lateral channel is yawing and rolling motion, which are <u>coupled</u>
- Lateral aero coefficients are primarily influenced by:
  - Sideslip: β
    - Yaw-plane angle between body axis and the wind-relative velocity vector
  - Yaw and roll body rates: *p, r*
  - Aileron and Rudder channel control surface deflections: δ_a, δ_r
- The side force ($f_{y,aero}$) is heavily influenced by β
- The yawing moment (n_{aero}) is heavily influenced by both β and δ_r
- The rolling moment (l_{aero}) is heavily influenced by δ_a and less so by δ_r

Aero Forces and Moments

Forces

$$F_{lift} = \tfrac{1}{2}\rho V_a^2 \cdot S \cdot C_L(\alpha, q, \delta_e, \ldots)$$

C_L: Lift Coefficient

$$F_{drag} = \tfrac{1}{2}\rho V_a^2 \cdot S \cdot C_D(\alpha, q, \delta_e, \ldots)$$

C_D: Drag Coefficient

$$f_{y,aero} = \tfrac{1}{2}\rho V_a^2 \cdot S \cdot C_Y(\beta, p, r, \delta_a, \delta_r, \ldots)$$

C_Y: Yaw Force Coefficient

Moments

$$l_{aero} = \tfrac{1}{2}\rho V_a^2 \cdot S \cdot b \cdot C_l(\beta, p, r, \delta_a, \delta_r, \ldots)$$

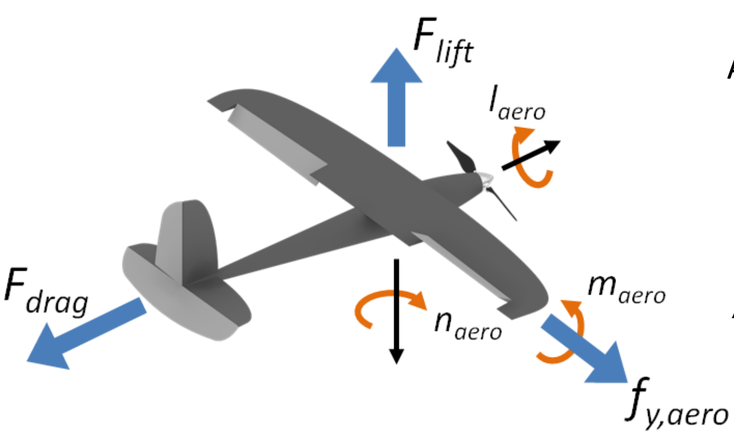
C_l: Rolling Moment Coefficient

$$m_{aero} = \tfrac{1}{2}\rho V_a^2 \cdot S \cdot c \cdot C_m(\alpha, q, \delta_e, \ldots)$$

C_m: Pitching Moment Coefficient

$$n_{aero} = \tfrac{1}{2}\rho V_a^2 \cdot S \cdot b \cdot C_n(\beta, p, r, \delta_a, \delta_r, \ldots)$$

C_n: Yawing Moment Coefficient



Aero Angles:
- α : Angle-of-Attack, º or rads
- β : Sideslip, º or rads

ρ : Air Density, kg/m^3
V_a: Airspeed, m/s

Body Rates:
- p : about x, º/s or rads/s
- q : about y, º/s or rads/s
- r : about z, º/s or rads/s

Control deflections:
- δ_e : Elevator, º or rads
- δ_a : Aileron , º or rads
- δ_r : Rudder , º or rads

S: Wing area, m^2
b: Wing span (Lateral channel ref. length), m
c: Wing mean chord (Longitudinal ref. length), m

Aerodynamic Linearization

- In general, aerodynamic forces and moments are non-linear multi-variable relationships, e.g.:

$$F_{lift} = \tfrac{1}{2}\rho V_a^2 \cdot S \cdot C_L(\alpha, q, \delta_e, \ldots)$$

- In order to develop autopilot control algorithms, we need to "linearize" the aerodynamics **about desired flight conditions**
  - Linearization of F_{lift} by expanding $C_L(\ldots)$ as a Taylor Series and keeping only the dominant linear terms

$$F_{lift} \approx \tfrac{1}{2}\rho V_a^2 \cdot S \cdot \left[C_{L_0} + \frac{\partial C_L}{\partial \alpha} \cdot \alpha + \frac{\partial C_L}{\partial q} \cdot q + \frac{\partial C_L}{\partial \delta_e} \cdot \delta_e \right]$$

  - Re-writing above using common aero derivative nomenclature:

$$F_{lift} \approx \tfrac{1}{2}\rho V_a^2 \cdot S \cdot \left[C_{L_0} + C_{L_\alpha}\alpha + C_{L_q}\left(\tfrac{c}{2V_a} q\right) + C_{L_{\delta e}}\delta_e \right]$$

Note: Body rate stability derivs. are actually taken wrt <u>scaled</u> body rates (scaled by $c/(2V_a)$) to maintain non-dimensionality

where:

$C_{L_0} =$ Base Lift Coef.

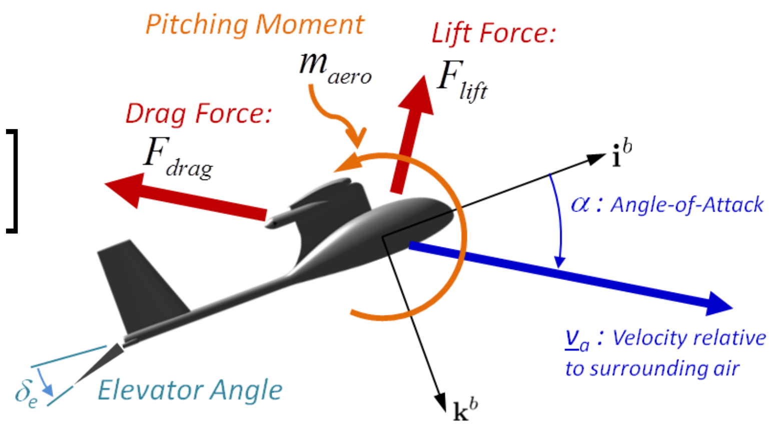
$$\underbrace{C_{L_\alpha} = \frac{\partial C_L}{\partial \alpha} \qquad C_{L_q} = \frac{\partial C_L}{\partial\left(\frac{c}{2V_a} q\right)}}_{\text{Stability Derivatives}} \qquad \underbrace{C_{L_{\delta e}} = \frac{\partial C_L}{\partial \delta_e}}_{\text{Control Derivative}}$$

Linearized Longitudinal Aerodynamics

$$F_{lift} \approx \tfrac{1}{2}\rho V_a^2 S\left[C_{L_0} + C_{L_\alpha}\alpha + C_{L_q}\left(\tfrac{c}{2V_a}q\right) + C_{L_{\delta e}}\delta_e\right]$$

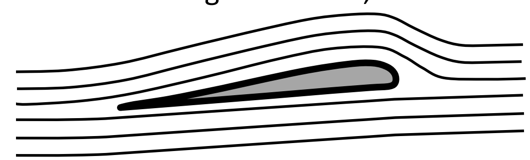
$$F_{drag} \approx \tfrac{1}{2}\rho V_a^2 S\left[C_{D_0} + \left|C_{D_\alpha}\alpha\right| + \left|C_{D_q}\left(\tfrac{c}{2V_a}q\right)\right| + \left|C_{D_{\delta e}}\delta_e\right|\right]$$

$$m_{aero} \approx \tfrac{1}{2}\rho V_a^2 Sc\left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\left(\tfrac{c}{2V_a}q\right) + C_{m_{\delta e}}\delta_e\right]$$



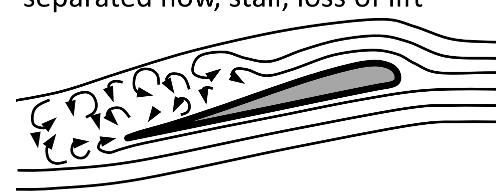
- Linear aerodynamics model is valid for moderate angles of attack
  - Flow remains attached over wing
- High angles of attack cause separated flow
  - Aircraft loses lift ("stalls") at high angles of attack
  - Linear model shown above isn't valid at high α
- Valid near trim V_a, above stall speed
- Book develops a modification to C_L and C_D equations to model stall characteristics.
  - For simplicity, we will use purely linear model
  - Assumption: Maintaining attached flow (e.g. α <30°)
- Note the "absolutes" in the drag equation
  - Drag should always be positive

Moderate angle of attack, attached flow



✓ Linear model valid

Moderate angle of attack, attached flow, high angle of attack, separated flow, stall, loss of lift



✗ Linear model <u>not</u> valid

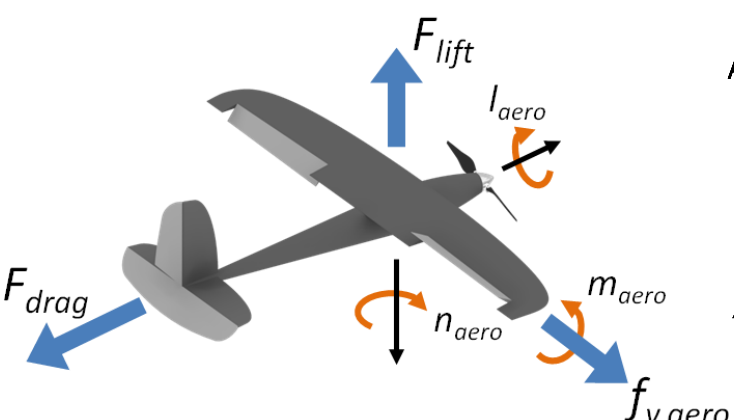
Linearized Aero Forces and Moments

Forces

$$\begin{cases} F_{lift} = \frac{1}{2}\rho V_a^2 S C_L(\ldots) \approx \frac{1}{2}\rho V_a^2 S\left[C_{L_0} + C_{L_\alpha}\alpha + C_{L_q}\left(\frac{c}{2V_a}q\right) + C_{L_{\delta e}}\delta_e\right] \\ F_{drag} = \frac{1}{2}\rho V_a^2 S C_D(\ldots) \approx \frac{1}{2}\rho V_a^2 S\left[C_{D_0} + \left|C_{D_\alpha}\alpha\right| + \left|C_{D_q}\left(\frac{c}{2V_a}q\right)\right| + \left|C_{D_{\delta e}}\delta_e\right|\right] \\ f_{y,aero} = \frac{1}{2}\rho V_a^2 S C_Y(\ldots) \approx \frac{1}{2}\rho V_a^2 S\left[C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\left(\frac{b}{2V_a}p\right) + C_{Y_r}\left(\frac{b}{2V_a}r\right) + C_{Y_{\delta a}}\delta_a + C_{Y_{\delta r}}\delta_r\right] \end{cases}$$

Moments

$$\begin{cases} l_{aero} = \frac{1}{2}\rho V_a^2 S b C_l(\ldots) \approx \frac{1}{2}\rho V_a^2 S b\left[C_{l_0} + C_{l_\beta}\beta + C_{l_p}\left(\frac{b}{2V_a}p\right) + C_{l_r}\left(\frac{b}{2V_a}r\right) + C_{l_{\delta a}}\delta_a + C_{l_{\delta r}}\delta_r\right] \\ m_{aero} = \frac{1}{2}\rho V_a^2 S c C_m(\ldots) \approx \frac{1}{2}\rho V_a^2 S c\left[C_{m_0} + C_{m_\alpha}\alpha + C_{m_q}\left(\frac{c}{2V_a}q\right) + C_{m_{\delta e}}\delta_e\right] \\ n_{aero} = \frac{1}{2}\rho V_a^2 S b C_n(\ldots) \approx \frac{1}{2}\rho V_a^2 S b\left[C_{n_0} + C_{n_\beta}\beta + C_{n_p}\left(\frac{b}{2V_a}p\right) + C_{n_r}\left(\frac{b}{2V_a}r\right) + C_{n_{\delta a}}\delta_a + C_{n_{\delta r}}\delta_r\right] \end{cases}$$



Aero Angles:

- α : Angle-of-Attack, º or rads
- β : Sideslip, º or rads

ρ : Air Density, kg/m^3
V_a: Airspeed, m/s

Body Rates:

- p : about x, º/s or rads/s
- q : about y, º/s or rads/s
- r : about z, º/s or rads/s

Control deflections:

- δ_e : Elevator, º or rads
- δ_a : Aileron , º or rads
- δ_r : Rudder , º or rads

S: Wing area, m^2
b: Wing span (Lateral channel ref. length), m
c: Wing mean chord (Longitudinal ref. length), m

Static Stability

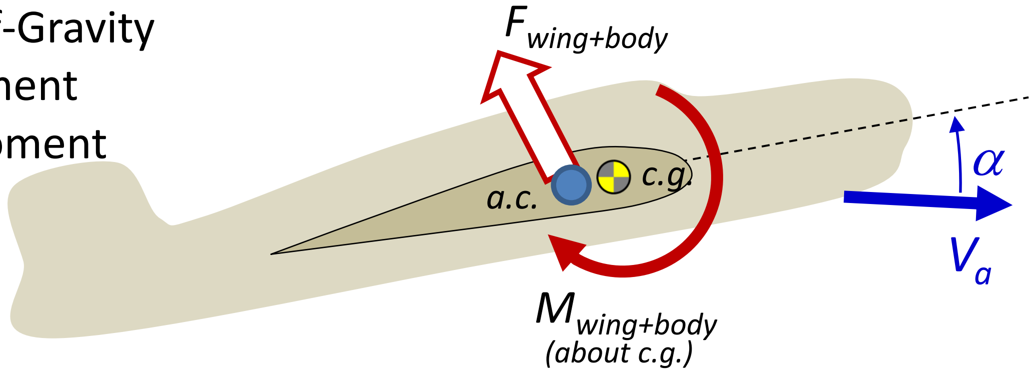
Longitudinal Static Stability

- An aircraft can be "statically stable" or "statically unstable"

Statically Stable:

- Aerodynamic Center is behind Center-of-Gravity
- Angle-of-Attack induces a restoring moment
    - An increase in α causes a pitching moment which decreases α
    - Airframe "wants" to return to zero α

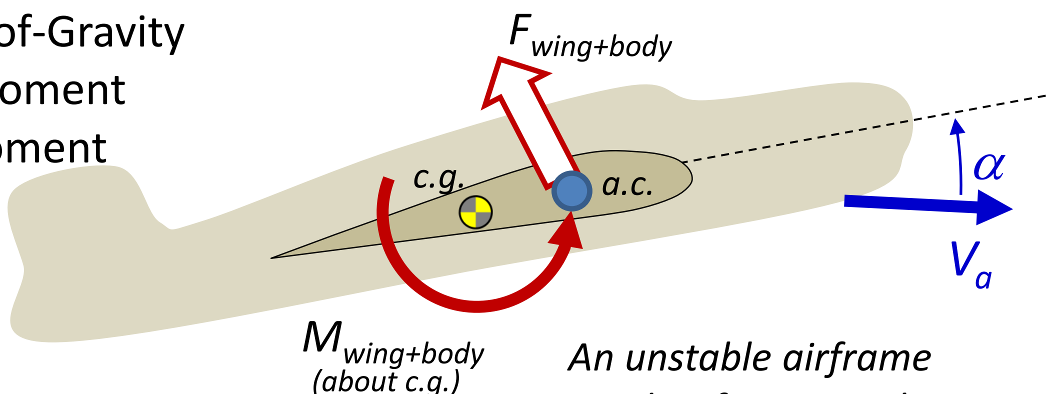
$$C_{m_\alpha} < 0$$



Statically Unstable:

- Aerodynamic center is ahead of Center-of-Gravity
- Angle-of-Attack induces an increasing moment
    - An increase in α causes a pitching moment which increases α
    - Airframe "wants" to flip over

$$C_{m_\alpha} > 0$$



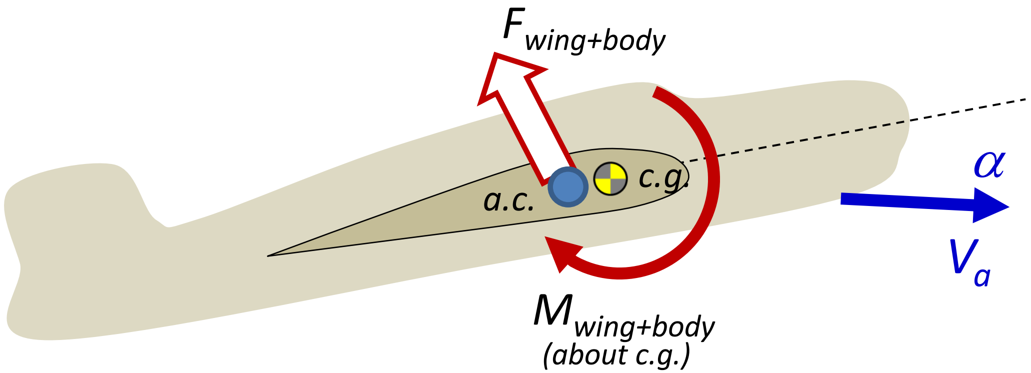
An unstable airframe needs a fast autopilot (or pilot) to keep it flying

Lateral Static Stability

- Via similar argument:, stable means $C_{l_\beta} < 0 \;\; \& \;\; C_{n_\beta} > 0$

Longitudinal Stability

- The location of the center of pressure along the airfoil changes continuously with changes in angle of attack. To determine longitudinal stability, we compute the mean aerodynamic center to evaluate longitudinal stability.
- Generally, a stable airframe has a aerodynamic center and behind the center-of-gravity.



- A small increase in angle of attack will cause a pitching moment on the aircraft that will decrease the angle of attack. Similarly, a small increase in angle of attack will cause a pitching moment that increases angle of attack.
- Longitudinal stability of an aircraft is significantly dependent on the distance (moment arm or lever arm) between the CG and the aerodynamic center of the aircraft

Aerodynamic Derivatives

Static Stability Derivatives

- Change in moments due to changes in aero angles (α, β)
- For a "statically stable" vehicle, an increase in aero angle causes a restoring moment

C_{m_α} : *Longitudinal static stability derivative*
For stability, $C_{m\alpha} < 0$: An increase in α causes a pitching moment which decreases α
Generally, $C_{m\alpha} < 0$ if aerodynamic center is behind center-of-gravity.

C_{l_β} : *Roll static stability derivative*, associated with wing dihedral
For stability, $C_{l\beta} < 0$: An increase in β causes a rolling moment which decreases β

C_{n_β} : *Yaw static stability derivative*
For stability, $C_{n\beta} > 0$: An increase in β causes a yawing moment which decreases β
a.k.a. "Weathercock" stability derivative: Causes airframe to align with the wind vector

Dynamic Stability Derivatives

C_{mq} : Pitch damping derivative. C_{lp} : Roll damping derivative. C_{nr} : Yaw damping derivative.

Control Derivatives

$C_{m_{\delta e}}, C_{l_{\delta a}}, C_{n_{\delta r}}$: Primary control derivatives. Purpose of control surfaces is to "rotate" the aircraft.

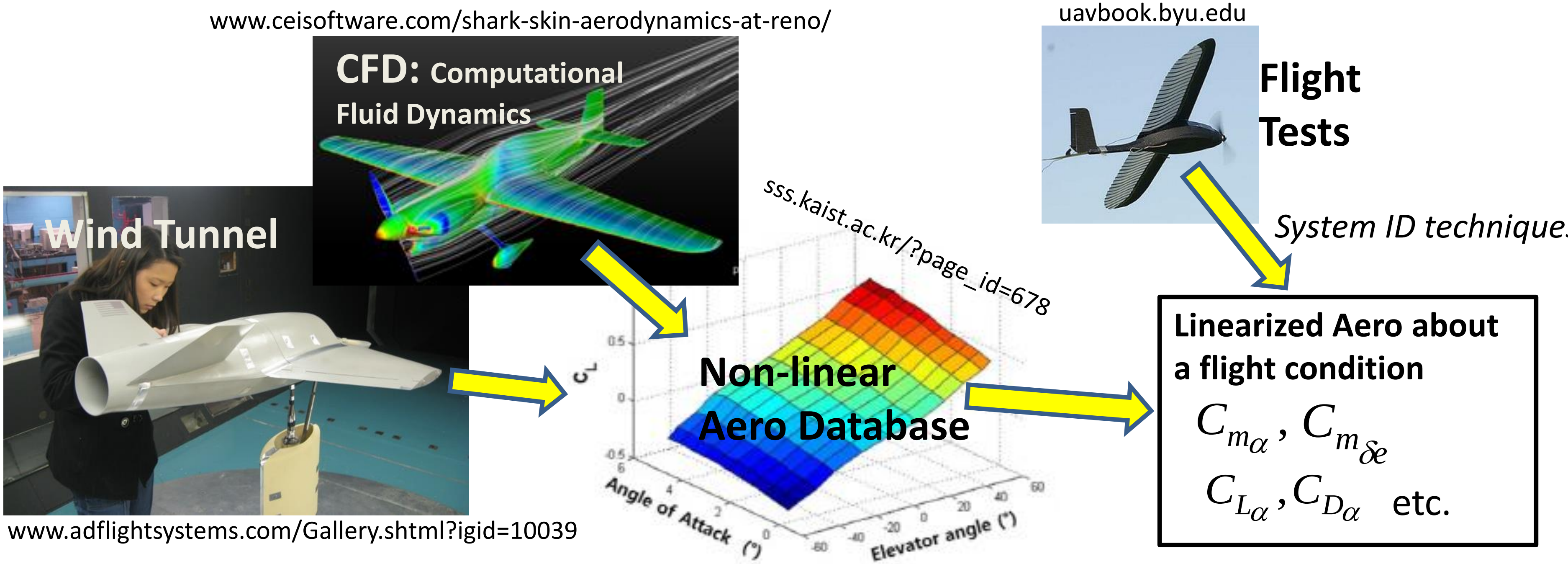
$C_{l_{\delta r}}, C_{n_{\delta a}}$: Cross-channel control derivatives

Force Derivatives

$C_{L_\alpha}, C_{D_\alpha}$: α affects Lift and Drag forces

C_{Y_β} : β affects Side force

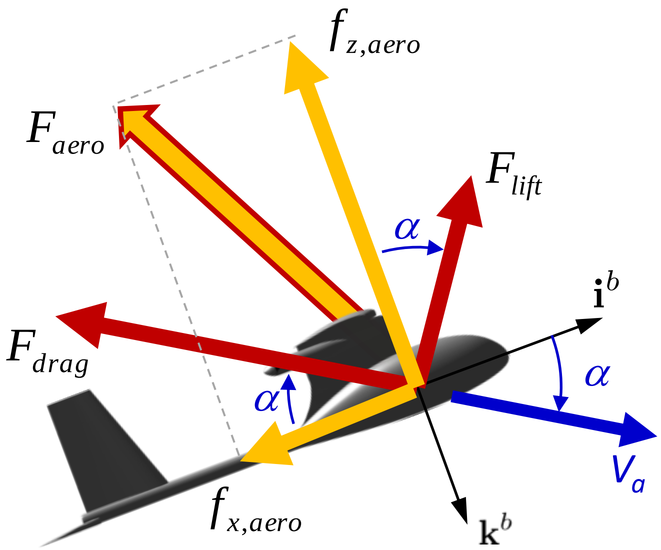
Where do Aero Models come from?



- Aero models can come from a Wind Tunnel, a CFD program, or flight testing.
- An aero model is comprised of:
  - Methods of computing 6 aero coefficients (C_L, C_D, C_Y, C_l, C_m, C_n)
    - May be multi-dimensional look-up-tables, linearized relationships, etc.
  - Reference area and lengths for scaling (S, b, and c)
- If linear relationships (e.g. $C_{L\alpha}$, and $C_{m\alpha}$) aren't provided, autopilot designer must "linearize" the full aero about a flight conditions to acquire the aero derivative coefs.
- In this class (and in book), a "linearized" aero model is provided and used

Longitudinal Forces in Body Frame

- For Equations of Motion, we need aero forces and moments in body frame
  - $f_{y,aero}$, l_{aero}, m_{aero} & n_{aero} are already along body axes
  - F_{lift} and F_{drag} are along "stability frame" axes (rotated by α)
- Must convert F_{lift} and F_{drag} into body frame ($\mathbf{i}^b$ and $\mathbf{k}^b$)



Generally for level flight:

$$f_{x,aero} < 0$$

$$f_{z,aero} < 0$$

Longitudinal Forces expressed in Stability Frame

$$F_{lift} = \tfrac{1}{2}\rho V_a^2 S C_L$$

$$F_{drag} = \tfrac{1}{2}\rho V_a^2 S C_D$$

$$\underbrace{\begin{bmatrix} f_{x,aero} \\ f_{z,aero} \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} -F_{drag} \\ -F_{lift} \end{bmatrix}}$$

Rotation of Stability Frame forces onto Body Frame

Longitudinal Forces expressed in Body Frame

$$f_{x,aero} = \tfrac{1}{2}\rho V_a^2 S\left[-C_D\cos(\alpha) + C_L\sin(\alpha)\right]$$

$$f_{z,aero} = \tfrac{1}{2}\rho V_a^2 S\left[-C_D\sin(\alpha) - C_L\cos(\alpha)\right]$$

Note: Functional dependencies of C_L and C_D suppressed for convenience.

Aero Forces and Moments, body frame

$$\underline{\mathrm{f}}^{b}_{aero} = \begin{bmatrix} f_{x,aero} \\ f_{y,aero} \\ f_{z,aero} \end{bmatrix} = \begin{bmatrix} -F_{drag}\cos(\alpha) + F_{lift}\sin(\alpha) \\ C_Y(\ldots) \\ -F_{drag}\sin(\alpha) - F_{lift}\cos(\alpha) \end{bmatrix}$$

$$= \tfrac{1}{2}\rho V_a^2 S \begin{bmatrix} -C_D(\ldots)\cos(\alpha) + C_L(\ldots)\sin(\alpha) \\ C_Y(\ldots) \\ -C_D(\ldots)\sin(\alpha) - C_L(\ldots)\cos(\alpha) \end{bmatrix}$$

$$\underline{\mathrm{m}}^{b}_{aero} = \begin{bmatrix} l_{aero} \\ m_{aero} \\ n_{aero} \end{bmatrix} = \tfrac{1}{2}\rho V_a^2 S \begin{bmatrix} bC_l(\ldots) \\ cC_m(\ldots) \\ bC_n(\ldots) \end{bmatrix}$$

C_L: Lift Coefficient

C_D: Drag Coefficient

C_Y: Yaw Force Coefficient

C_l: Rolling Moment Coefficient

C_m: Pitching Moment Coefficient

C_n: Yawing Moment Coefficient

α : Angle-of-Attack, ^{o} or rads

ρ: Air Density, kg/m^3

V_a: Airspeed, m/s

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b: Wing span (Lateral channel ref. length), m

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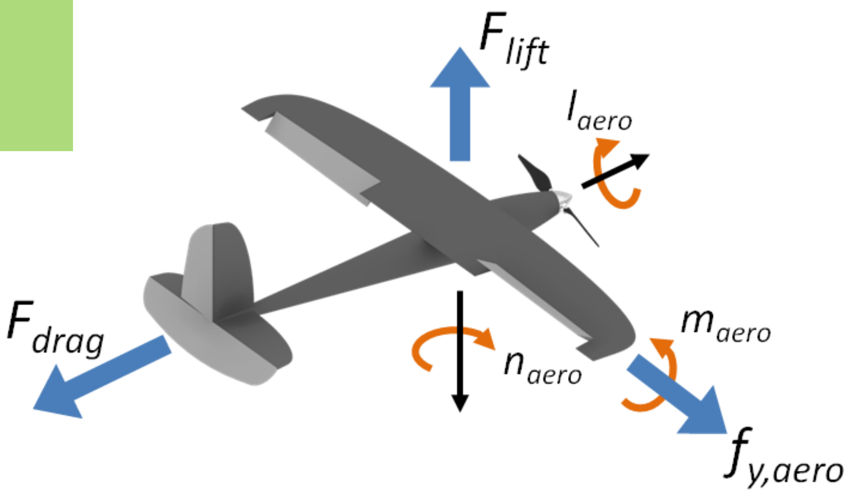
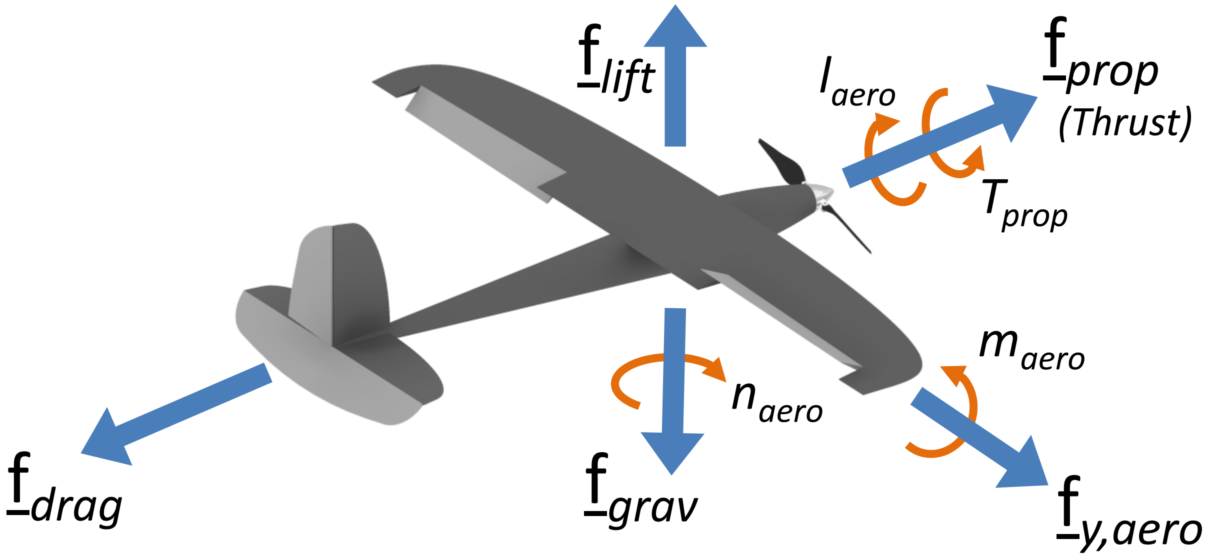


External Forces and Moments



Note 1: Modify $\underline{f}_{prop}$ and $\underline{m}_{prop}$ accordingly if propulsion/thrust source is not mounted along x-axis.

Note 2: $\underline{f}_{lift}$ and $\underline{f}_{drag}$ are actually in the stability frame to be discussed later.

Sum of Forces:

$$\Sigma \underline{f} = \underline{f}_{grav} + \underbrace{\underline{f}_{aero}}_{\underline{f}_{lift} + \underline{f}_{drag} + \underline{f}_{y,aero}} + \underline{f}_{prop}$$

Sum of Moments:

$$\Sigma \underline{m} = \underbrace{\underline{m}_{aero}}_{\begin{bmatrix} l_{aero} \\ m_{aero} \\ n_{aero} \end{bmatrix}^b} + \underbrace{\underline{m}_{prop}}_{\begin{bmatrix} T_{prop} \\ 0 \\ 0 \end{bmatrix}^b}$$

For “trim” flight, forces and moments are balanced.

Total Fixed Wing Forces & Moments

$$\underline{\mathrm{f}}^b = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \underbrace{(mass)\mathrm{g}\begin{bmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{bmatrix}}_{\underline{\mathrm{f}}^b_{grav}} \underbrace{+\tfrac{1}{2}\rho V_a^2 S\begin{bmatrix} -C_D(\ldots)\cos(\alpha)+C_L(\ldots)\sin(\alpha) \\ C_Y(\ldots) \\ -C_D(\ldots)\sin(\alpha)-C_L(\ldots)\cos(\alpha) \end{bmatrix}}_{\underline{\mathrm{f}}^b_{aero}} + \underbrace{\begin{bmatrix} F_{prop} \\ 0 \\ 0 \end{bmatrix}}_{\underline{\mathrm{f}}^b_{prop}}$$

$$\underline{\mathrm{m}}^b = \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \underbrace{\tfrac{1}{2}\rho V_a^2 S\begin{bmatrix} bC_l(\ldots) \\ cC_m(\ldots) \\ bC_n(\ldots) \end{bmatrix}}_{\underline{\mathrm{m}}^b_{aero}} + \underbrace{\begin{bmatrix} T_{prop} \\ 0 \\ 0 \end{bmatrix}}_{\underline{\mathrm{m}}^b_{prop}}$$

$$F_{prop} = ?$$

$$T_{prop} = ?$$

$$C_L(\ldots) \approx C_{L_0} + C_{L\alpha}\alpha + C_{Lq}\left(\tfrac{c}{2V_a}q\right) + C_{L_{\delta e}}\delta_e$$

$$C_D(\ldots) \approx C_{D_0} + \left|C_{D\alpha}\alpha\right| + \left|C_{Dq}\left(\tfrac{c}{2V_a}q\right)\right| + \left|C_{D_{\delta e}}\delta_e\right|$$

$$C_Y(\ldots) \approx C_{Y_0} + C_{Y_\beta}\beta + C_{Y_p}\left(\tfrac{b}{2V_a}p\right) + C_{Y_r}\left(\tfrac{b}{2V_a}r\right) + C_{Y_{\delta a}}\delta_a + C_{Y_{\delta r}}\delta_r$$

$$C_l(\ldots) \approx C_{l_0} + C_{l_\beta}\beta + C_{l_p}\left(\tfrac{b}{2V_a}p\right) + C_{l_r}\left(\tfrac{b}{2V_a}r\right) + C_{l_{\delta a}}\delta_a + C_{l_{\delta r}}\delta_r$$

$$C_m(\ldots) \approx C_{m_0} + C_{m\alpha}\alpha + C_{mq}\left(\tfrac{c}{2V_a}q\right) + C_{m_{\delta e}}\delta_e$$

$$C_n(\ldots) \approx C_{n_0} + C_{n_\beta}\beta + C_{n_p}\left(\tfrac{b}{2V_a}p\right) + C_{n_r}\left(\tfrac{b}{2V_a}r\right) + C_{n_{\delta a}}\delta_a + C_{n_{\delta r}}\delta_r$$

Aero Vars:
- α : AoA, º or rads
- β : Sideslip, º or rads
- V_a: Airspeed, m/s

Controls:
- δ_e : Elevator, º or rads
- δ_a : Aileron , º or rads
- δ_r : Rudder , º or rads
- δ_t : Throttle , (0-1)

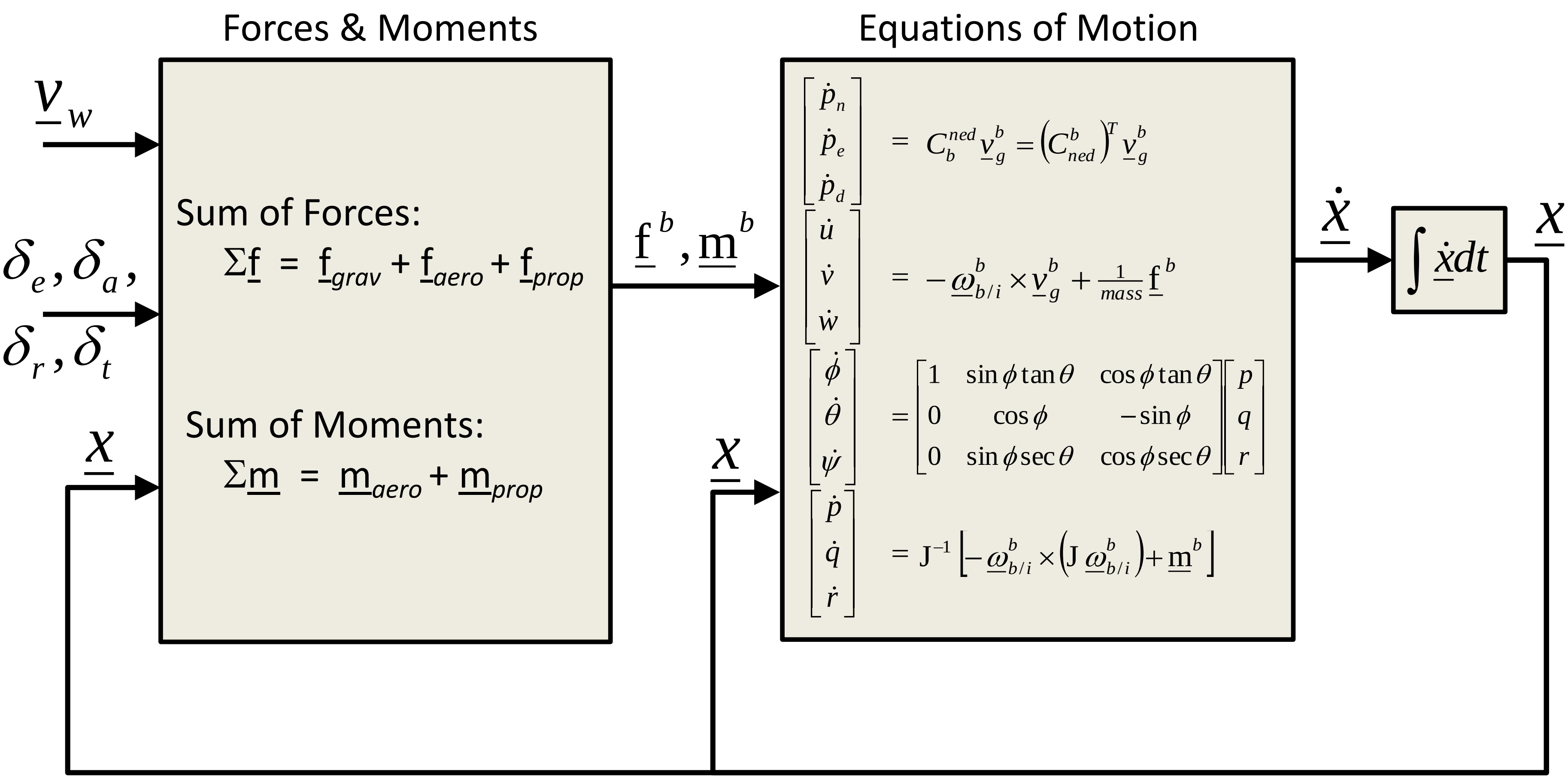
Propeller Params:
- C_{prop}: Efficiency Coef.
- S_{prop}: *Prop. area*, m²
- k_{motor}: *Motor constant, m/s*
- k_Ω: *Speed constant, m/s*
- k_{Tp}: *Torque constant, kg-m²*

Body Rates:
- p : about x, º/s or rads/s
- q : about y, º/s or rads/s
- r : about z, º/s or rads/s

Environment:
- g: Gravity, 9.81 m/s²
- ρ : Air Density, kg/m³

Aero Params:
- S: Wing area, m²
- b: Wing span, m
- c: Wing mean chord, m

Forces & Moments into EoMs



Assumptions/Simplifications we've made:

- NED (attached to ground) is an inertial reference frame
- Presuming a flat earth
- Vehicle has constant mass and inertia
- Vehicle is a rigid body
- Vehicle has symmetry about x-z plane
- Small angle-of-attack for linear, attached flow
- Aero moments provided about c.g.
- Provided linearized aero model is valid for reasonable deviations from nominal

Lecture 4 Homework, 1/3

1) **Implement aerodynamic forces and moments "uavsim_forces_moments.m" and combine with the force due to gravity (from last homework). Utilize necessary vehicle parameters via the "P" structure (e.g. P.C_L_alpha, etc.). (Note: All aero coefficients should be multiplied by angles in radians!)**

   Hint: Output of "`uavsim_forces_moments(ones(20,1),P)`**" is:**

   ```
   [-12.8897 6.9345 4.4475 0.0422 -0.0678 -0.0718]'
   ```

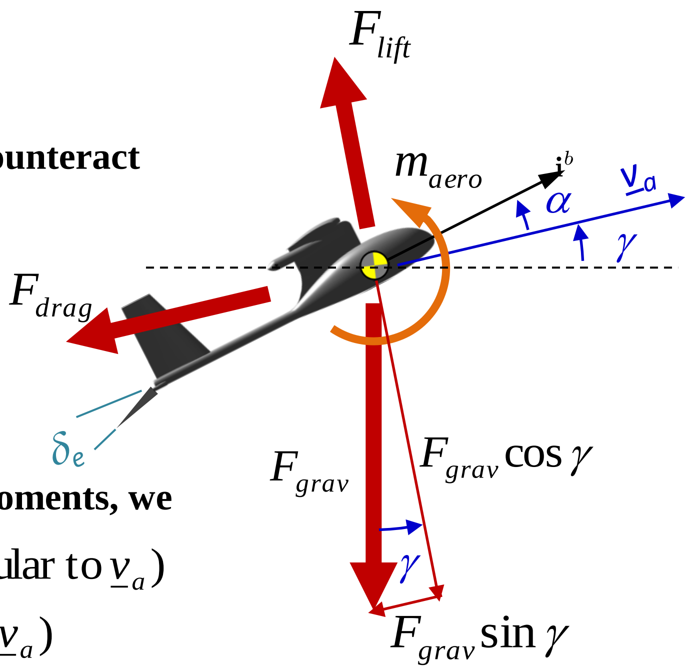
   a. **Print out your added code from uavsim_forces_moments.m**

   b. **Currently all control surfaces (delta_e, delta_a, delta_r) are zero, and the vehicle is initialized with zero pitch and angle-of-attack. Run the simulation and describe the vehicle motion. Why do you think it is behaving this way? (Hint: Think about speed, altitude and energy. How does speed affect aerodynamic lift and drag?)**

   c. **Plot the following as a function of time (plot angles in degrees):**
      altitude, airspeed, pitch, q (body rate), angle-of-attack

Recommended Reading: Beard & McLain 4.0, 4.1, 4.2
Book Errata:
- Page 42: $\delta a = .5*(\delta a_{left} + \delta a_{right})$ [incorrect sign]
- Page 51, top paragraph: beta=p=r= δa= δr=0
  [final variable is δr].
- Book does not include absolute values in linearized CD. Using absolutes is more representative if using the linearized aero equations for 6DOF propagation (vice using a non-linear model).

Lecture 4 Homework, 2/3



2) From our discussion, we know:

- **Aircraft need to fly at an angle-of-attack to generate lift to counteract gravity (but the angle-of-attack causes a pitching moment!)**
- **Aircraft need an elevator deflection to counteract the pitching moment induced by the angle-of-attack**
- **Without a thrust source, an aircraft will lose energy due to drag and will descend (γ < 0)**

We can draw a free-body diagram about the c.g. of an aircraft without thrust, shown here. Balancing forces and moments, we see:

$$F_{lift} - F_{grav}\cos\gamma = 0 \text{ (summing forces perpendicular to } \underline{v}_a\text{)}$$

$$F_{drag} + F_{grav}\sin\gamma = 0 \text{ (summing forces parallel to } \underline{v}_a\text{)}$$

$$m_{aero} = 0 \text{ (summing resulting moments)}$$

Note: Resulting γ^* will be negative, but it is helpful to draw free-body-diagrams with positive angles

a) Expand the above equations using the linear models we developed.

e.g. $$F_{lift} = \tfrac{1}{2}\rho V_a^2 S\left[C_{L_0} + C_{L_\alpha}\alpha + C_{L_q}\left(\tfrac{c}{2V_a}q\right) + C_{L_{\delta e}}\delta_e\right] \;\&\; F_{grav} = (mass)g$$

b) Using the default uavsim parameters (e.g. P.Va0, P.mass, P.C_L_alpha, etc.), and assuming p=q=r=β=0, numerically find the α*, δe*, and γ* values that solve these equations (The * notation denotes the "trim", or balancing, conditions). Provide the trim angles in degrees.

c) Recall that θ=γ+α, and that u=$V_a$cos(α) and w=$V_a$sin(α) (when β=ϕ=0). Using the trim conditions you found above, overwrite the aircraft initial conditions in the P structure:

```
P.theta0=gammaStar+alphaStar; P.delta_e0=delta_eStar;
P.u0=P.Va0*cos(alphaStar); P.w0=P.Va0*sin(alphaStar);
```

Run the simulation. Describe what happens? Are forces and moments balanced?

Ignore absolute values in drag equation in finding "trim" in 2b. (C_D_delta_e=0 and resulting alpha is positive.)

Lecture 4 Homework, 3/3

Do not continue until you have a 'trimmed' UAV in simulation. All states except for positions should be nearly constant.

For 2(b), if you don't know a means of solving a non-linear set of equations, use $\gamma^* = -4.980745*\pi/180$. Then, solve a linear set of 2 equations and 2 unknowns.

3) Using the "trim" condition found above, let's explore the UAVs open-loop response to small control surface deflections.

a) In "uavsim_control.m", add a small deflection <u>only</u> to the elevator surface:

delta_e = P.delta_e0 + 0.01*pi/180;

Plot the angle-of-attack, pitch-rate (q) and pitch, and describe the resulting responses. Look at both the initial response (t<1 second), and the longer-duration response (t>1 second).

b) In "uavsim_control.m" , add a small deflection <u>only</u> to the aileron surface:

delta_a = P.delta_a0 + 0.01*pi/180;

Plot the roll and roll-rate (p), and describe the resulting responses. Look at both the initial response (t<1 second), and the longer-duration response (t>1 second).

4) Now let's explore a larger deflection. In "uavsim_control.m", add a 1-degree deflection to the aileron channel (and only the aileron channel):

delta_a = P.delta_a0 + 1*pi/180;

Describe the resulting trajectory. The trajectory should impact the ground just before t=20 seconds. What is the roll angle at the time of impact?