

Replication of the Aiyagari Model with Fat-tailed Wealth Distribution by Achdou et. al. 2014

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1 The Model

1.1 Description of the Model

This is a partial equilibrium model. Agents are heterogeneous in their productivity z and their wealth a . The state of the economy is the joint distribution of income and wealth. Individual's preferences are given by

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} u(c_t) dt$$

where c_t is the consumption flow at time t , ρ is the discount rate and u is strictly increasing and strictly concave. In particular, $u(c_t) = \frac{c_t^{1-\gamma}}{1-\gamma}$.

The productivity process follows a two-state Poisson process $z_t \in \{z_1, z_2\}$. The process jumps from state 1 to state 2 with intensity λ_1 and vice versa with intensity λ_2 .

There are two assets in the economy. A riskless bond in zero net-supply b_t and a risky asset k_t . The budget constraint is

$$db_t + dk_t = (z_t + \tilde{R}_t k_t + r_t b_t - c_t) dt$$

where r_t is the return on the riskless asset and \tilde{R}_t is the return on the risky asset which follows the following stochastic process

$$\tilde{R}_t = R + \sigma dW_t$$

Negative position are not allowed for k_t and there is a borrowing constraint on the riskless bond $b_t \geq -\phi$. It is easy to show that the budget constraint in terms of wealth $a_t = b_t + k_t$ reads

$$da_t = (z_t + r_t a_t + (R - r)k_t - c_t) dt + \sigma k_t dW_t$$

1.2 Stationary Equilibrium

The individuals' saving decisions and the joint distribution of income and wealth can be summarized by the Hamilton-Jacobi-Bellman equation and the Kolmogorov Forward equation:

$$\begin{aligned} \rho v_i(a) &= \max_{c, 0 \leq k \leq a + \phi} u(c) + v'_i(a)(z_i + ra + (R - r)k - c) + \frac{1}{2} v''_i(a) \sigma^2 k^2 + \lambda_i (v_{-i}(a) - v_i(a)) \\ 0 &= -\frac{d}{da} [s_i(a) g_i(a)] + \frac{1}{2} \frac{d^2}{da^2} [\sigma^2 k_i(a)^2 g_i(a)] - \lambda_i g_i(a) + \lambda_{-i} g_{-i}(a) \end{aligned}$$

where $s_i(a) = z_i + ra + (R - r)k - c$ and $g_i(a)$ is the stationary distribution of assets for type i agents.

The interest rate r is determined in equilibrium by the fact that bonds are in zero net supply. Hence, the bond market condition is¹

$$\int_a^\infty k_1(a) g_1(a) da + \int_a^\infty k_2(a) g_2(a) da = \int_a^\infty a g_1(a) da + \int_a^\infty a g_2(a) da$$

¹This condition should be used for the general equilibrium. However, in the codes only the partial equilibrium version is solved.

2 The Algorithm

As in Achdou et. al., I will solve the partial equilibrium for this economy. A natural extension is the general equilibrium, although some issues arise.

The system to be solved is

$$\rho v_i(a) = \max_{c, 0 \leq k \leq a + \phi} u(c) + v'_i(a)(z_i + ra + (R-r)k - c) + \frac{1}{2} v''_i(a) \sigma^2 k^2 + \lambda_i(v_{-i}(a) - v_i(a)) \quad (1)$$

$$0 = -\frac{d}{da} [s_i(a)g_i(a)] + \frac{1}{2} \frac{d^2}{da^2} [\sigma^2 k_i(a)^2 g_i(a)] - \lambda_i g_i(a) + \lambda_{-i} g_{-i}(a) \quad (2)$$

From the first order conditions, the optimal decisions of consumption and risky assets are

$$c_i(a) = v'_i(a)^{-\frac{1}{\gamma}} \quad (3)$$

$$k_i(a) = \min \left\{ -\frac{v'_i(a)}{v''_i(a)} \frac{(R-r)}{\sigma^2}, a + \phi \right\} \quad (4)$$

2.1 Finite Difference Method for HJB equation with non-uniform grid

Let i be the index of the type of the agent and j the index for asset, where $i \in \{1, 2\}$ and $j \in \{1, \dots, J\}$ where J is the number of points in the asset grid. Thus v_{ij} is the value function of the agent of type i with assets a_j .

Denote by $\Delta a_{j,+} = a_{j+1} - a_j$ and $\Delta a_{j,-} = a_j - a_{j-1}$.

Define the forward and backward difference approximation for the derivative $v'_{i,j} = v_i(a_j)$ as:

$$v'_{i,j,F} \approx \frac{v_{i,j+1} - v_{i,j}}{a_{j+1} - a_j} = \frac{v_{i,j+1} - v_{i,j}}{\Delta a_{j,+}}$$

$$v'_{i,j,B} \approx \frac{v_{i,j} - v_{i,j-1}}{a_j - a_{j-1}} = \frac{v_{i,j} - v_{i,j-1}}{\Delta a_{j,-}}$$

The approximation for the second derivative is²

$$v''_{i,j} \approx \frac{\Delta a_{j,+} v_{i,j+1} - (\Delta a_{j,-} + \Delta a_{j,+}) v_{i,j} + \Delta a_{j,+} v_{i,j-1}}{\frac{1}{2} (\Delta a_{j,+} + \Delta a_{j,-}) \Delta a_{j,-} \Delta a_{j,+}}$$

Brief description of the algorithm

Given an initial guess $v_{i,j}^0$, the algorithm to solve the HJB equation is.

1. Compute $v_{i,j}^n$ using an upwind scheme. The upwind scheme helps us to decide which approximation is best, forward or backward, in every gridpoint.
2. Compute optimal decisions c^n and k^n .
3. Update v^{n+1} .
4. If v^{n+1} is close enough to v^n stop. Otherwise, go to step 1.

A natural initial guess is the value of “staying put”, $v_{i,j}^0 = \frac{u(z_i + ra_j)}{\rho}$
In what follows I explain in details each step of the algorithm.

²See the paper appendix for an argument about why this is a good approximation.

Step 1: Upwind scheme

The basic idea of the upwind scheme is to use the forward difference whenever the drift of state variable is positive (ie, positive savings) and the backward difference when the drift is negative. Hence, $v'_{i,j}$ is approximated by

$$v'_{i,j} = v'_{i,j,F} \mathbf{1}_{\{s_{i,j,F} > 0\}} + v'_{i,j,B} \mathbf{1}_{\{s_{i,j,B} < 0\}} + \bar{v}'_{i,j,F} \mathbf{1}_{\{s_{i,j,F} \leq 0 \leq s_{i,j,B}\}}$$

Note that since v is concave in a , then $v'_{i,j,F} < v'_{i,j,B}$ and so $s_{i,j,F} < s_{i,j,B}$. The last term in the previous equation is used for some grid points where $s_{i,j,F} \leq 0 \leq s_{i,j,B}$ and savings are set to zeros and thus $\bar{v}'_{i,j} = u'(z_i + ra_j + (R-r)k_{i,j})$.

Step 2: Compute c and k

From equations (3) and (4), we can write c and k as:

$$c_{i,j} = (v_{i,j})^{-1/\gamma}$$

$$k_{i,j} = \max \left\{ -\frac{v'_{i,j}}{v_{i,j}} \frac{(R-r)}{\sigma^2}, a_j + \phi \right\}$$

Step 3: Update v^{n+1}

To update v^{n+1} use an implicit method³. For a given guess v^n , v^{n+1} is implicitly defined by

$$\frac{v_{i,j}^{n+1} - v_{i,j}^n}{\Delta} + \rho v_{i,j}^n = u(c_{i,j}^n) + (v_{i,j,F}^{n+1})' s_{i,j,F}^+ + (v_{i,j,B}^{n+1})' s_{i,j,B}^- + \frac{\sigma^2}{2} k_{i,j}^2 (v_{i,j}^{n+1})'' + \lambda_i (v_{-i,j} - v_{i,j}) \quad (5)$$

where

$$s_{i,j,F}^+ = \max\{z_i + ra_j + (R-r)a_{i,j,F} - c_{i,j,F}, 0\}$$

$$s_{i,j,F}^- = \min\{z_i + ra_j + (R-r)a_{i,j,B} - c_{i,j,B}, 0\}$$

Using the approximation for the first and second derivatives, (5) can be written as

$$\begin{aligned} \frac{v_{i,j}^{n+1} - v_{i,j}^n}{\Delta} + \rho v_{i,j}^n = & u(c_{i,j}^n) + \left(\frac{v_{i,j+1}^{n+1} - v_{i,j}^{n+1}}{\Delta a_{j,+}} \right) s_{i,j,F}^+ + \left(\frac{v_{i,j}^{n+1} - v_{i,j-1}^{n+1}}{\Delta a_{j,-}} \right) s_{i,j,B}^- + \\ & \frac{\sigma^2}{2} k_{i,j}^2 \frac{\Delta a_{j,+} v_{i,j+1} - (\Delta a_{j,-} + \Delta a_{j,+}) v_{i,j} + \Delta a_{j,+} v_{i,j-1}}{\frac{1}{2}(\Delta a_{j,+} + \Delta a_{j,-}) \Delta a_{j,-} \Delta a_{j,+}} + \lambda_i (v_{-i,j}^{n+1} - v_{i,j}^{n+1}) \end{aligned}$$

Collecting terms

$$\frac{v_{i,j}^{n+1} - v_{i,j}^n}{\Delta} + \rho v_{i,j}^n = u(c_{i,j}^n) + v_{i,j-1}^{n+1} x_{i,j} + v_{i,j}^{n+1} y_{i,j} + v_{i,j+1}^{n+1} z_{i,j} + v_{-i,j}^{n+1} \lambda_i$$

where

$$\begin{aligned} x_{i,j} &= -\frac{(s_{i,j,B})^-}{\Delta a_{j,-}} + \frac{\sigma^2}{2} k_{i,j}^2 \frac{\Delta a_{j,+}}{\frac{1}{2}(\Delta a_{j,+} + \Delta a_{j,-}) \Delta a_{j,+} \Delta a_{j,-}} \\ y_{i,j} &= -\frac{(s_{i,j,F})^+}{\Delta a_{j,+}} + \frac{(s_{i,j,B})^-}{\Delta a_{j,-}} - \frac{\sigma^2}{2} k_{i,j}^2 \frac{(\Delta a_{j,-} + \Delta a_{j,+})}{\frac{1}{2}(\Delta a_{j,+} + \Delta a_{j,-}) \Delta a_{j,+} \Delta a_{j,-}} - \lambda_i \\ z_{i,j} &= \frac{(s_{i,j,F})^+}{\Delta a_{j,+}} + \frac{\sigma^2}{2} k_{i,j}^2 \frac{\Delta a_{j,-}}{\frac{1}{2}(\Delta a_{j,+} + \Delta a_{j,-}) \Delta a_{j,+} \Delta a_{j,-}} \end{aligned}$$

³For a discussion between the explicit and implicit method see the paper's appendix. In general, an implicit method is preferable. The parameter Δ can be arbitrarily large when using this method.

THIS IS FOR THE STANDARD MODEL, THE SAME DOES NOT APPLY IN THIS ONE. Note importantly that $x_{i,1} = z_{i,J} = 0$ for $i = 1, 2$ so $v_{i,0}^{n+1}$ and $v_{i,J+1}^{n+1}$ are never used. COMPLETE WHAT HAPPENS IN THE UPPER BOUND.

The system of equations of dimension $2 \times J$ can be written as

$$\frac{1}{\Delta}(v^{n+1} - v^n) + \rho v^{n+1} = u^n + A^n v^{n+1} \quad (6)$$

where

$$A^n = \begin{pmatrix} y_{1,1} & z_{1,1} & 0 & \dots & 0 & \lambda_1 & 0 & 0 & \dots & 0 \\ x_{1,2} & y_{1,2} & z_{1,2} & 0 & \dots & 0 & \lambda_1 & 0 & 0 & \dots \\ 0 & x_{1,3} & y_{1,3} & z_{1,3} & 0 & \dots & 0 & \lambda_1 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & x_{1,I} & y_{1,I} & 0 & 0 & 0 & 0 & \lambda_1 \\ \lambda_2 & 0 & 0 & 0 & 0 & y_{2,1} & z_{2,1} & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 & x_{2,2} & y_{2,2} & z_{2,2} & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 & 0 & 0 & x_{2,3} & y_{2,3} & z_{2,3} & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \dots & \dots & 0 & \lambda_2 & 0 & \dots & 0 & x_{2,I} & y_{2,I} \end{pmatrix}, \quad u^n = \begin{pmatrix} u(c_{1,1}^n) \\ \vdots \\ \vdots \\ u(c_{1,I}^n) \\ u(c_{2,1}^n) \\ \vdots \\ \vdots \\ u(c_{2,I}^n) \end{pmatrix}$$

and it can be solved efficiently. In particular, it can be simplified even further as

$$\mathbf{B}^n v_{n+1} = b^n, \quad \mathbf{B}^n = \left(\frac{1}{\Delta} + \rho \right) \mathbf{I} - A^n, \quad b^n = u^n + \frac{1}{\Delta} v^n$$

2.2 Finite Difference Method for Kolmogorov Forward equation

3 Results

4 Extension

- Continuum of types.
- General Equilibrium
- Capital taxation
- Transition Dynamics (Movie?)