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RESEARCH NOTE

On some implications of the Poisson relation

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SUMMARY

The potentialities of the so-called 'Poisson relation', which holds for uniformly magnetized bodies of constant density, for showing the connections between the gravitational and the magnetic fields of such bodies are considered. In particular, it is seen that the same characteristic ratios occur among the components of the magnetic field intensity and the components of the gradient tensors of the two fields, both locally and non-locally. It is also shown that along the magnetization axis of the body the gradient tensor of the magnetic field displays a 'tidal' structure.

Key words: gravity, magnetics, Poisson relation.

1 INTRODUCTION

The so-called 'Poisson relation' (Grant & West 1965) has been frequently used in the past as a means to evaluate the magnetic field of uniformly magnetized bodies via their gravitational field (e.g. Brüggemann *et al.* 1973); the reason is that, at least in principle, the potential field due to a distribution of monopoles is more easily handled than the potential field due to a distribution of dipoles; nevertheless, we are not concerned here with the applications of the Poisson relation but, instead, with its potential to make clear some more intrinsic relations between these two fields.

Since the magnetic field intensity $\mathbf{H}$ due to a magnetized body of volume V with a magnetic dipole moment per unit volume $\mathbf{M}$ is

$$\mathbf{H}(\mathbf{r}) = \nabla \int_V (\mathbf{M} \cdot \nabla) \frac{dV}{|\mathbf{r} - \mathbf{r}_0|}, \tag{1}$$

we can write, for the case in which the direction of magnetization α is the same throughout the body, so that in this case $\mathbf{M} \cdot \nabla = M\partial/\partial\alpha$,

$$\mathbf{H}(\mathbf{r}) = M\nabla \frac{\partial}{\partial\alpha} \int_V \frac{1}{|\mathbf{r} - \mathbf{r}_0|} dV. \tag{2}$$

Since the gravitational force $\mathbf{g}$ due to a body of constant density ρ is

$$\mathbf{g}(\mathbf{r}) = G\rho\nabla \int_V \frac{dV}{|\mathbf{r} - \mathbf{r}_0|}, \tag{3}$$

we finally have, for a uniformly magnetized body,

$$\mathbf{H}(\mathbf{r}) = \frac{M}{G\rho} \frac{\partial}{\partial\alpha} \mathbf{g}, \tag{4}$$

which is the so-called Poisson relation.

2 LOCAL AND NON-LOCAL IMPLICATIONS OF THE POISSON RELATION

We may write

$$\frac{\partial}{\partial\alpha} \mathbf{g} = U_{rs} l^r, \tag{5}$$

where U_{rs} is the gravitational gradients tensor of the body and l^r is the unit vector in the direction of magnetization; it thus follows that

$$H_s = \frac{\mathcal{M}}{Gm} U_{rs} l^r, \tag{6}$$

where H_s denotes the components of the magnetic field intensity, $\mathcal{M}$ the magnetic dipole moment of the body and m its mass. We see therefore that the magnetic field of the body is closely related to the properties of its gravitational gradients tensor U_{rs}; in this connection let us consider, for example, a spherical body having a constant density, which is uniformly magnetized along the z-axis of a Cartesian reference. Since

$$U_{rs} = \frac{Gm}{r^3} \begin{bmatrix} 3x^2 - r^2 & 3xy & 3xz \\ 3xy & 3y^2 - r^2 & 3yz \\ 3xz & 3yz & 3z^2 - r^2 \end{bmatrix} \tag{7}$$

at the points P_1 $(r, 0, 0)$, P_2 $(0, r, 0)$, P_3 $(0, 0, r)$, equidistant from the origin along the coordinate axes, we have

$$U_{rs}(P_1) = \frac{Gm}{r^3}\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$U_{rs}(P_2) = \frac{Gm}{r^3}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$U_{rs}(P_3) = \frac{Gm}{r^3}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \tag{8}$$

it follows that

$$H_z(P_1) = \frac{\mathcal{M}}{Gm} U_{zz}(P_1),$$

$$H_z(P_2) = \frac{\mathcal{M}}{Gm} U_{zz}(P_2),$$

$$H_z(P_3) = \frac{\mathcal{M}}{Gm} U_{zz}(P_3). \tag{9}$$

Since

$$U_{zz}(P_1) = U_{xx}(P_3) = \frac{Gm}{\mathcal{M}} H_z(P_1),$$

$$U_{zz}(P_2) = U_{yy}(P_3) = \frac{Gm}{\mathcal{M}} H_z(P_2), \tag{10}$$

we obtain, taking account of the Laplace condition for U_{rs}, the following relation:

$$H_z(P_1) + H_z(P_2) + H_z(P_3) = 0. \tag{11}$$

It can also be seen, because of rotational symmetry around the z-axis, that

$$H_z(P_1) = H_z(P_2) = -H_z(P_3)/2. \tag{12}$$

The interest of the relations (11) and (12) is mainly in the way they have been obtained here and in the characteristic ratios -1, -1, 2 among the components of $\mathbf{H}$ at the space positions P_1, P_2, P_3.

In a Cartesian reference system we obtain from eq. (6)

$$H_{ij} = \frac{\mathcal{M}}{Gm} U_{ij/s} l^s, \tag{13}$$

where H_{ij} is the gradients tensor for the magnetic field intensity. If a spherical shape for the body is assumed,

$$H_{ij} = -\frac{3\mathcal{M}}{r^5}\begin{bmatrix} \left(\frac{5x^2}{r^2} - 1\right)z & \frac{5xyz}{r^2} & \left(\frac{5z^2}{r^2} - 1\right)x \\ \frac{5xyz}{r^2} & \left(\frac{5y^2}{r^2} - 1\right)z & \left(\frac{5z^2}{r^2} - 1\right)y \\ \left(\frac{5z^2}{r^2} - 1\right)x & \left(\frac{5z^2}{r^2} - 1\right)y & \left(\frac{5z^2}{r^2} - 3\right)z \end{bmatrix}. \tag{14}$$

By comparison of eqs (7) and (14), it follows that, along the magnetization axis,

$$H_{ij}(0, 0, r) = -\frac{3\mathcal{M}}{Gmr} U_{ij}(0, 0, r). \tag{15}$$

We therefore see that, apart from the steeper decrease of H_{ij} with distance compared with U_{ij}, the gradient tensors for the gravitational and magnetic fields of the body display the same structure along the magnetization axis. Since the gravitational field of the body, at the points $(\Delta r, 0, r)$, $(0, \Delta, r, r)$, $(0, 0, r + \Delta r)$ on a small sphere of radius Δr centred at P_3, gives rise to the tidal forces

$$\Delta \mathbf{g}' = -\frac{Gm\Delta r}{r^3}(1, 0, 0),$$

$$\Delta \mathbf{g}'' = -\frac{Gm\Delta r}{r^3}(0, 1, 0),$$

$$\Delta \mathbf{g}''' = \frac{2Gm\Delta r}{r^3}(0, 0, 1), \tag{16}$$

we obtain, using eq. (15), for the magnetic field at the same points,

$$\Delta \mathbf{H}' = -\frac{3\mathcal{M}\Delta r}{r^4}(1, 0, 0),$$

$$\Delta \mathbf{H}'' = -\frac{3\mathcal{M}\Delta r}{r^4}(0, 1, 0),$$

$$\Delta \mathbf{H}''' = \frac{6\mathcal{M}\Delta r}{r^4}(0, 0, 1). \tag{17}$$

We therefore see that the magnetic field has a 'tidal' structure at points on the magnetization axis; it also follows that

$$U_{xx}(P_3){:}\ U_{yy}(P_3){:}\ U_{zz}(P_3) = -1{:}\ -1{:}\ 2, \tag{18}$$

$$H_{xx}(P_3){:}\ H_{yy}(P_3){:}\ H_{zz}(P_3) = -1{:}\ -1{:}\ 2. \tag{19}$$

Considering also eq. (12), we can thus conclude that the same characteristic ratios occur among components of the magnetic field intensity and components of the gradients tensors H_{ij} and U_{ij}, both in local and in non-local relations; this rather unusual behaviour seems not to take place by chance and deserves further investigation. A possible explanation could perhaps be found within the frame of a topological approach to the properties of gravitational and magnetic fields (Bocchio 1989, 1990; Hide, Barraclough & MacMillan 1997).

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