Corneal topography and the Hirschberg test

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A simple trigonometric analysis of the Hirschberg test with the assumption that the corneal surface is spherical predicts a sinusoidal dependence of the corneal reflex displacement on the angle of ocular rotation. A comparison with corneal reflex photographs demonstrates that at angles larger than 50 prism diopters (26 deg) the reflex displacements are larger than predicted by the spherical model. This discrepancy may be accounted for by incorporating a more general description of the corneal topography into the geometric analysis. The linear Hirschberg relation that is seen in typical data is accounted for by a relative flattening of the peripheral cornea by ~20% of the apical curvature. This geometric analysis of the functional dependence of the Hirschberg relation on the corneal topography can be expressed as an integral equation. Differentiation yields a second-order differential equation for the corneal topography in terms of the Hirschberg data. If the Hirschberg relation is assumed to be linear, a quadratic dependence is found for the corneal curvature. A similar differential approach can be formulated for the Placido disk. In this sense the corneal topography problem given in terms of Placido disk data is shown to be well formulated. The relative simplicity of the Hirschberg geometry is seen to stem from the alignment of the light source with the eye of the observer.

The Hirschberg test constitutes an estimate of the angular position of a strabismic eye by measuring the apparent displacement of the image formed by the corneal reflecting surface of a hand-held light (the first Purkinje image) from the center of the entrance pupil of the eye. The estimation is ordinarily given in terms of a linear rule of thumb with a simple proportionality between the reflex displacement and the ocular rotation.[1] While traditional sources have suggested a conversion factor (Hirschberg coefficient) of 7 or 8 deg of rotation for each millimeter of reflex displacement,[1] recent studies of calibrated corneal reflex photographs have yielded a value of ~21 prism diopters (or ~12 deg) per millimeter.[1-6] This value has been confirmed by a geometric analysis of the optics of the corneal reflex under the assumption that the cornea is spherical by using nominal values for the dimensions of the human eye.[1] The same model has also been successfully applied to the somewhat smaller eye of the monkey.[7,8]

Under these assumptions the relationship between corneal reflex displacement δ and the angle of rotation ϕ is given by the simple equation

$$\delta = d \sin(\phi - \kappa),$$

where d is the distance between the center of curvature of the corneal surface and the entrance pupil and κ is the angle kappa, which is the angle between the direction of regard and the pupillary axis (the direction along which the corneal reflex appears centered on the entrance pupil). Comparisons of the measurements of corneal reflex displacement with this equation (Fig. 1) demonstrate an excellent fit for rotations up to 50 or 60 prism diopters (~25–30 deg). For larger rotations the measured reflex displacements are greater than those that are predicted by the spherical model, so that the reflex displacement curve closely resembles a straight line rather than the central portion of a sinusoid.[1,6,9] The most plausible explanation for this small but consistent discrepancy is that it is due to peripheral corneal flattening, which moves the corneal reflex further behind the corneal surface, effectively increasing the parameter d in the model.

The geometry for the general case, where the cornea is not assumed to be spherical, is illustrated in Fig. 2 (where we have for simplicity taken the angle κ as zero). The corneal topography along the meridian under examination is specified by the curvature function K or by its reciprocal, the local radius of curva-

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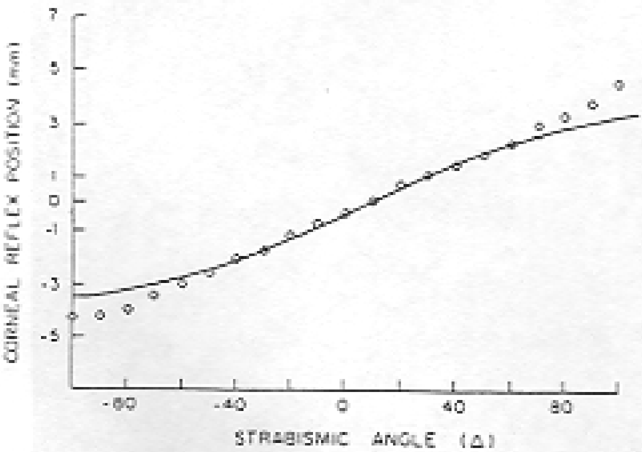

Fig. 1. Comparison of the measured corneal reflex displacement (open symbols) with the sinusoidal curve predicted by geometric analysis under the assumption that the cornea is spherical by using nominal dimensions for the human eye.[1]

ture R. This is in principle comparable with the curvature that is determined by commercially available corneal topography instruments, or even in an average sense, by ordinary keratometry. It is convenient, however, to take as the parameter of location on the corneal surface, not the horizontal coordinate x, but rather the angle ϕ between the reference direction (straight ahead) and the vector normal to the corneal surface. Because light from a point source is reflected along the normal vector, the angle ϕ also indicates the rotation of the globe as would be measured in a Hirschberg test.

We choose as the origin **O** the corneal vertex, that is, the point of the cornea whose normal lies in the reference direction. At a typical point **X** along the meridian of interest, let $\hat{t}$ denote the unit vector tangent to the cornea. Let s denote the arc length along the cornea. Then, according to the classical differential geometry,[10]

$$K = \mathrm{d}\phi/\mathrm{d}s,$$

$$R = \mathrm{d}s/\mathrm{d}\phi,$$

$$\hat{t} = (-\sin\phi, \cos\phi),$$

$$X = \int \hat{t}\mathrm{d}s = \int \hat{t}(\mathrm{d}s/\mathrm{d}\phi)\mathrm{d}\phi = \int \hat{t}R\mathrm{d}\phi$$

$$= \int (-R\sin\phi, R\cos\phi)\mathrm{d}\phi.$$

Let the center of the entrance pupil be located by the vector $\mathbf{p} = (-p, 0)$. Then, since the corneal reflex lies along the normal vector at **X**, the apparent displacement δ of the corneal reflex from the center of the entrance pupil is given by the projection of the vector $\mathbf{X} - \mathbf{p}$ from **P** to **X** onto the tangent vector $\hat{\mathbf{t}}$:

$$\delta = (\mathbf{X} - \mathbf{p})\cdot\hat{\mathbf{t}}$$

$$= \left\{\int (-\sin u)[R(u)]\mathrm{d}u + p\right\}(-\sin\phi)$$

$$+ \left\{\int (\cos u)[R(u)]\mathrm{d}u\right\}(\cos\phi)$$

$$= \int_0^{\phi} \cos(\phi - u)R(u)\mathrm{d}u - p\sin\phi \qquad (*).$$

This provides a complete description of the Hirschberg relationship between the angle of rotation and reflex displacement for the general case of aspheric corneas. If $R(u)$ is constant, say $R = R_0$ (i.e., the cornea is spherical), this reduces to

$$\delta(\phi) = (R_0 - p)\sin\phi,$$

which is in agreement with the simple trigonometric treatment that is discussed above.

Returning to the general case, if the relation (*) is differentiated twice with respect to ϕ, we obtain

$$\delta''(\phi) = R' - \delta(\phi),$$

or

$$R' = \delta'' + \delta,$$

with the initial condition

$$R(0) = \delta'(0) + p,$$

where the prime denotes differentiation with respect to ϕ. This is a straightforward expression for R (and hence the corneal topography **X**) as a first-order differential equation, which is completely specified by the Hirschberg function $\delta(\phi)$ and knowledge of the location of the entrance pupil. To be sure photographic corneal reflex measurements, let alone traditional clinical estimates of corneal reflex displacement, do not provide the continuous Hirschberg function $\delta(\phi)$, which is assumed in this approach to the corneal topography problem. However, with re-

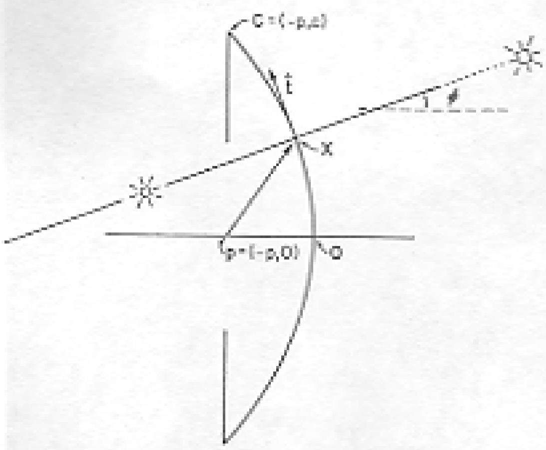

Fig. 2. Geometry for calculation of the Hirschberg relation for aspheric corneas. A distant point source of light is indicated by the larger square. The smaller square represents the image of the point source that is formed by the corneal reflecting surface. This image is viewed or photographed from directly behind the light source.

cent advances in digital frame-grabber technology and inexpensive video cameras, high-resolution measurements of the entire Hirschberg function made with a moving light source are no longer implausible.[5]

It is instructive to consider the special case where the displacement δ is proportional to the ocular rotation ϕ, which is a reasonable simplification of the experimental data:

$$\delta = C \cdot \phi.$$

Then $\delta'' = 0$ and

$$R' = C \cdot \phi.$$

Thus

$$R = C \cdot \phi^2/2 + R(0).$$

The initial condition $\delta'(0) = R(0) - p$ implies that $C = R(0) - p$, whence

$$R(\phi) = [R(0) - p] \cdot \phi^2/2 + R(0).$$

That is, the corneal radius of curvature varies as a simple quadratic function of the angle of rotation.

In the human eye[11] $R(0)$ is nominally 7.7 mm, p is ~2.9 mm, and ϕ ranges from 0 (straight ahead) to ~$\pi/4$ (so that 45 deg of rotation place the corneal light reflex at the limbus). If indeed the Hirschberg relation is generally linear, we are led to predict a flattening of the peripheral cornea of ~20%, i.e., a peripheral power of ~35 diopters versus an apical power of 44 diopters. While there is little published data concerning the curvature of the extreme peripheral cornea, this degree of flattening is consonant with that described in some classical studies[12] but is less than is commonly observed with commercial corneal topography instruments, even if one extrapolates the measurements to the far periphery. A recent reanalysis of corneal topography algorithms suggests that the peripheral flattening is greater than had been previously determined[13] and may be compatible with the prediction that we have obtained here. Better data for the topography of the peripheral cornea than have been published to date will be needed to confirm these equations.

The apparent linearity of the Hirschberg relation thus appears to result from a subtle variation in the corneal curvature. It is tempting to speculate that this effect facilitates our ability to discern the direction of gaze of our companions. For example, the depiction of corneal reflexes is well known as a technique that artists employ to indicate the attention and alertness of their subjects even in highly simplified, schematic drawings (Fig. 3).

It is also instructive to compare these simple consequences of the Hirschberg geometry with the traditional Placido disk geometry, which has been the basis of most corneal topography instruments (Fig. 4). In these devices the image that is formed by the corneal reflecting surface of a sequence of alternating bright and dark rings is photographed, and the corneal topography is reconstructed from the spacing of the ring images. The uncertainty that encumbers the Placido geometry as to the actual corneal location giving rise to the reflection that is perceived as the mire image is eliminated in the Hirschberg arrangement by the strategy of sighting (or photographing) the corneal reflex along the same ray as traversed by the incident light.

Nevertheless the strategy of computing the corneal topography as the solution to a differential equation (rather than as a discrete problem in geometric ray tracing) may be carried out in the Placido geometry also. As above we analyze each meridian separately. We choose as the origin O the nodal point of the corneascope camera. Suppose that the mire on the Placido disk is located at the point $T = (t, 0)$ and that the radius of the image of the mire as recorded on the film is s, where the film plane lies at a distance a from the origin. Thus the corneascope data are understood to be available in terms of a function $s = s(t)$, where the function s is assumed to be differentiable. The cornea to be measured is aligned so that the normal ray through the corneal vertex passes through the origin. We presume that the distance x_0 from the corneal vertex to the origin is known. The condition of reflection, that the angle of incidence equals the angle of reflection, implicitly constrains the point $\mathbf{x} = (x, y)$ on the corneal surface where the mire image is formed. (There is no simple connection between this point x on the corneal surface and the virtual image of the mire that is formed by the corneal surface acting

Fig. 3. Stylized renditions of corneal reflexes in depictions of cartoon characters as children's toys. (Courtesy of Cathy Kaplan, Esq.)

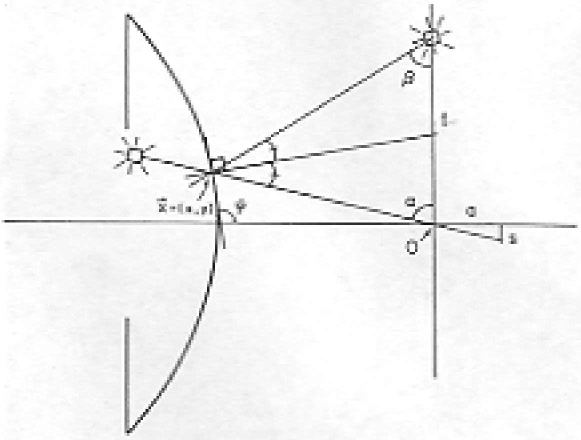

Fig. 4 Geometry for the localization of mire images for a Placido-disk-type corneascope. The radius of the mire ring being imaged is indicated by t; the nodal point of the corneascope camera is at O.

as a reflector, which lies behind the cornea and at a somewhat greater radius.)

We focus on the as yet undetermined triangle XTO. Denote the angle XTO by β, the angle XTO by α. For any choice of the point **x** we may construct the bisector of the angle XTO and the line (tangent to the corneal contour) through **x** perpendicular to this angle bisector. Denote the angle between this tangent line and the x axis by Ψ. The slope of this line depends only on the triangle XTO; we indicate this dependence by the function F:

$$F(x, y, t) = \tan \Psi = \frac{dy}{dx}.$$

It may be shown trigonometrically that

$$\Psi = \frac{\pi}{2} + \frac{\alpha - \beta}{2}$$

and that

$$\tan \Psi = \frac{\tan \beta - \tan \alpha}{\sec \alpha \sec \beta - 1 - \tan \alpha \tan \beta}.$$

Since $\tan \alpha = x/y$, $\tan \beta = x/(t - y)$, and $\sec u = (1 + \tan^2 u)^{1/2}$, we find that

$$F(x, y, t) = \frac{x(2y - t)}{(x^2 + y^2)^{1/2}[x^2 + (t - y)^2]^{1/2} - y(t - y) - x^2}.$$

Now, since $y = sx/a$, we have

$$\frac{dy}{dt} = \frac{1}{a}\left(\frac{ds}{dt}x + s\frac{dx}{dt}\right).$$

Alternatively,

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = F(x, y, t)\frac{dx}{dt}.$$

Comparing these two expressions for dy/dt, we deduce that

$$\frac{dx}{dt}\left[F\left(x, \frac{sx}{a}, t\right) - \frac{s}{a}\right] = \frac{x}{a}\frac{ds}{dt},$$

or

$$\frac{dx}{dt} = \frac{\frac{x}{a}\frac{ds}{dt}}{\left[F\left(x, \frac{sx}{a}, t\right) - \frac{s}{a}\right]},$$

with the initial condition $x(t = 0) = x_0$. Since F is a known function of its three arguments, a is a known constant of the camera geometry, and $s = s(t)$ is the known differentiable function that describes the raw corneascope data, the last differential equation together with the relation $y = sx/a$ completely specifies the corneal geometry.

To the extent that we have exhibited an explicit formulation of the corneascope problem as an ordinary first-order differential equation, it is possible to conclude that, at least in the limit of continuous data, the problem is well determined; that is, each corneascope data function $s(t)$ uniquely determines a single compatible corneal contour. Any practical realization of this analysis will ultimately require discretization of the corneascope data and most likely some iterative scheme for the solution of the differential equation, which is similar to the computational techniques that are currently in use in some corneal topography instruments.[14,15] Thus it remains to be proved whether the differential approach that is derived here will ultimately yield significant gains in computational facility or accuracy.

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