1 Combined contact response

This note discusses the problem of modeling contact between two surfaces, each of which exhibits a nonlinear elastic contact response defined by $p(d)$, where p is the contact pressure and d is the surface displacement distance. Since pressure is force per unit area, we can equivalently divide by the contact area and instead consider the localized restoring force $f(d)$.

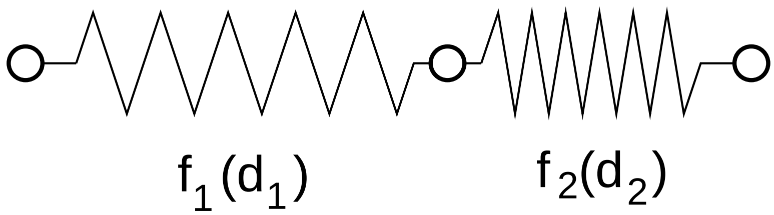

Figure 1: Two springs connected in series.

Let $f_1(d_1)$ and $f_2(d_2)$ be the contact force responses for each surface, with the combined surface displacement being $d_c \equiv d_1 + d_2$. The restoring forces act in tandem, and so can be thought of as comprising two nonlinear springs connected in series (Figure 1). As such, the restoring force for each spring must be equal:

$$f_1(d_1) = f_2(d_2). \tag{1}$$

In general, we may only know the combined surface displacement d_c, and so would like to find the equivalent spring force function $f_c(d_c)$ such that

$$f_c(d_c) = f_1(d_1) = f_2(d_2). \tag{2}$$

If we can determine d_1 as a function of d_c, such that $d_1 = g(d_c)$, then we have

$$f_c(d_c) = f_1(g(d_1)).$$

For the special case of linear springs, this is easy: the spring forces are given by $f_1 = K_1 d_1$ and $f_2 = K_2 d_2$, and with $d_2 = d_c - d_1$, we have

$$\begin{aligned} K_1 d_1 &= K_2(d_c - d_1) \\ \Longrightarrow (K_1 + K_2) d_1 &= K_2 d_c \\ \Longrightarrow d_1 &= \frac{K_2}{K_1 + K_2} d_c \end{aligned}$$

and so

$$f_c(d_c) \equiv \frac{K_1 K_2}{K_1 + K_2} d_c. \tag{3}$$

For nonlinear springs, the situation is more complex. In this note, we consider nonlinear springs where the restoring forces are given by

$$f_1 \equiv K_1 \ln(1 - \frac{d_1}{h_1}) \quad \text{and} \quad f_2 \equiv K_2 \ln(1 - \frac{d_2}{h_2}),$$

where K_i are the local spring constants at $d_i = 0$ and h_i are the maximum surface displacements, at which an infinite restoring force results. If $K_1 = K_2$, then we can solve for d_1 in terms of d_c using (1), where dividing out K_1 and taking inverse logs of both sides leads to

$$1 - \frac{d_1}{h_1} = 1 - \frac{d_2}{h_2} = 1 - \frac{d_c - d_1}{h_2}$$

which can be solved for d_1 in terms of d_c. However, if $K_1 \neq K_2$, then we obtain the more complex

$$\left(1 - \frac{d_1}{h_1}\right)^{K_1} = \left(1 - \frac{d_c - d_1}{h_2}\right)^{K_2}$$

for which I am not aware of a closed form solution for d_1. While an interative solution based on Newton's method would probably be easy and reliable to implement, it might be more profitable to simply assume an approximate solution for f_c of the form

$$f_c = K_c \ln(1 - \frac{d_c}{h_c})$$

and then determine appropriate values for K_c and h_c. h_c should obviously be $h_c = h_1 + h_2$, which gives the maximum total penetration allowed for both surfaces, while for K_c, we can take the combined linearized spring constant at $d_1 = d_2 = 0$. Since the derivatives of $f_1(d_1)$ and $f_2(d_2)$ at 0 are simply K_1 and K_2, then from (3) we have

$$K_c = \frac{K_1 K_2}{K_1 + K_2}.$$