Brief Article

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1 Quasispin operators

We begin by defining the quasispin operators

$$J_+ = \sum_p a^\dagger_{p+} a_{p-} \tag{1}$$

$$J_- = \sum_p a^\dagger_{p-} a_{p+} \tag{2}$$

$$J_z = \sum_{p\sigma} \sigma a^\dagger_{p\sigma} a_{p\sigma} \tag{3}$$

$$J^2 = J_+ J_- + J_z^2 - J_z \tag{4}$$

We now set to show that these operators follow the canonical angular momenta commutator relations, which are as follows(setting hbar to 1):

1.1 Angular Momenta Commutator Relatons

$$[J_+, J_-] = 2J_z \tag{5}$$

$$[J_+, J_z] = -J_+ \tag{6}$$

$$[J_-, J_z] = J_- \tag{7}$$

$$\left[J^2, J_z\right] = 0 \tag{8}$$

1.2 Proof of commutator relations

1.2.1 $[J_+, J_-] = 2J_z$

First we consider the commutator relation (5)

$$[J_+, J_-] = \sum_p a_{p+}^\dagger a_{p-} \sum_k a_{k-}^\dagger a_{k+} - \sum_k a_{k-}^\dagger a_{k+} \sum_p a_{p+}^\dagger a_{p-} \tag{9}$$

We then immediately note that since for $k \neq p$

$$[a_{p\sigma}, a_{k\sigma'}] = 0 \tag{10}$$

It follows that all the terms where $p \neq k$ fall out of the commutator. Hence:

$$[J_+, J_-] = \sum_p a_{p+}^\dagger a_{p-} a_{p-}^\dagger a_{p+} - \sum_p a_{p-}^\dagger a_{p+} a_{p+}^\dagger a_{p-} \tag{11}$$

Now we note that because of the difference in spins, some of these terms commute and the sums can be merged and we can rewrite this as

$$[J_+, J_-] = \sum_p a_{p+}^\dagger a_{p+} a_{p-} a_{p-}^\dagger - a_{p-}^\dagger a_{p-} a_{p+} a_{p+}^\dagger \tag{12}$$

Finally we note that since

$$\left[a, a^\dagger\right] = 1 \tag{13}$$

This thus means that with further rearangement

$$[J_+, J_-] = \sum_p a_{p+}^\dagger a_{p+} (a_{p-}^\dagger a_{p-} + 1) - (a_{p+}^\dagger a_{p+} + 1) a_{p-}^\dagger a_{p-} \tag{14}$$

And finally simplifying this is

$$[J_+, J_-] = \sum_p a_{p+}^\dagger a_{p+} - a_{p-}^\dagger a_{p-} \tag{15}$$

Which we note is just

$$[J_+, J_-] = \sum_{p\sigma} a_{p\sigma}^\dagger a_{p\sigma} = 2J_z \tag{16}$$

And this is as expected in agreement with the relationship (5) above.

1.2.2 $[J_+, J_z] = -J_+$

Again we write this out in full using the creation and anhilation operators.

$$[J_+, J_z] = \frac{1}{2}\left(\sum_k a_{k+}^\dagger a_{k-} \sum_{p\sigma} \sigma a_{p\sigma}^\dagger a_{p\sigma} - \sum_{p\sigma} \sigma a_{p\sigma}^\dagger a_{p\sigma} \sum_k a_{k+}^\dagger a_{k-}\right) \tag{17}$$

From this it follows that if $k \neq p$ it follows that those terms cancel and similarly for the sigmas.

$$[J_+, J_z] = \frac{1}{2}\sum_{p\sigma} \sigma \left(a_{p+}^\dagger a_{p-} a_{p\sigma}^\dagger a_{p\sigma} - a_{p\sigma}^\dagger a_{p\sigma} a_{p+}^\dagger a_{p-}\right) \tag{18}$$

We then expand this sum over sigma as

$$[J_+, J_z] = \frac{1}{2}\sum_p \left(a_{p+}^\dagger a_{p-} a_{p+}^\dagger a_{p+} - a_{p+}^\dagger a_{p+} a_{p+}^\dagger a_{p-} - a_{p+}^\dagger a_{p-} a_{p-}^\dagger a_{p-} + a_{p-}^\dagger a_{p-} a_{p+}^\dagger a_{p-}\right) \tag{19}$$

Again taking advantage of the creation/annhilation operators' commutation relations it follows that

$$[J_+, J_z] = \frac{1}{2}\sum_p \left(a_{p+}^\dagger a_{p+}^\dagger a_{p+} a_{p-} - a_{p+}^\dagger (a_{p+}^\dagger a_{p+} + 1) a_{p-} - a_{p+}^\dagger (a_{p-}^\dagger a_{p-} + 1) a_{p-} + a_{p+}^\dagger a_{p-}^\dagger a_{p-} a_{p-}\right) \tag{20}$$

And finally cancelling like terms

$$[J_+, J_z] = -\sum_p a_{p+}^\dagger a_{p-} = -J_+ \tag{21}$$

1.2.3 $[J_-, J_z] = J_-$

This is fundamentally the same as the previous argument and so I skip most of the explanations:

$$[J_-, J_z] = \frac{1}{2}\left(\sum_k a_{k-}^\dagger a_{k+} \sum_{p\sigma} \sigma a_{p\sigma}^\dagger a_{p\sigma} - \sum_{p\sigma} \sigma a_{p\sigma}^\dagger a_{p\sigma} \sum_k a_{k-}^\dagger a_{k+}\right) \tag{22}$$

$$[J_-, J_z] = \frac{1}{2}\sum_{p\sigma} \sigma \left(a_{p-}^\dagger a_{p+} a_{p\sigma}^\dagger a_{p\sigma} - a_{p\sigma}^\dagger a_{p\sigma} a_{p-}^\dagger a_{p+}\right) \tag{23}$$

$$[J_-, J_z] = \frac{1}{2}\sum_p \left(a^\dagger_{p-}a_{p+}a^\dagger_{p+}a_{p+} - a^\dagger_{p+}a_{p+}a^\dagger_{p-}a_{p+} - a^\dagger_{p-}a_{p+}a^\dagger_{p-}a_{p-} + a^\dagger_{p-}a_{p-}a^\dagger_{p-}a_{p+} \right) \tag{24}$$

Again taking advantage of the creation/annhilation operators' commutation relations it follows that

$$[J_-, J_z] = \frac{1}{2}\sum_p \left(a^\dagger_{p+}a^\dagger_{p+}a_{p-}a_{p+} - a^\dagger_{p+}(a^\dagger_{p+}a_{p+} - 1)a_{p-} - a^\dagger_{p-}(a^\dagger_{p-}a_{p-} - 1)a_{p+} + a^\dagger_{p-}a^\dagger_{p-}a_{p-}a_{p+} \right) \tag{25}$$

And finally cancelling like terms

$$[J_-, J_z] = \sum_p a^\dagger_{p-}a_p = J_- \tag{26}$$

1.2.4 $[J^2, J_z] = 0$

Here we take advantage of commutator algebra. Noting that

$$J^2 = J_+J_- + J_z^2 - J_z \tag{27}$$

It is immediately clear that J_z commutes with the second and third terms.

Then

$$\left[J^2, J_z\right] = [J_+J_-, J_z] = J_+\,[J_-, J_z] + [J_+, J_z]\,J_- = J_+J_- - J_+J_- = 0 \tag{28}$$