Brief Article

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1 Quasispin operators

We begin by defining the quasispin operators

$$J_+ = \sum_p a_{p+}^\dagger a_{p-} \tag{1}$$

$$J_- = \sum_p a_{p-}^\dagger a_{p+} \tag{2}$$

$$J_z = \sum_{p\sigma} \sigma a_{p\sigma}^\dagger a_{p\sigma} \tag{3}$$

$$J^2 = J_+ J_- + J_z^2 - J_z \tag{4}$$

We now set to show that these operators follow the canonical angular momenta commutator relations, which are as follows(setting hbar to 1):

1.1 Angular Momenta Commutator Relatons

$$[J_+, J_-] = 2J_z \tag{5}$$

$$[J_+, J_z] = -J_+ \tag{6}$$

$$[J_-, J_z] = J_- \tag{7}$$

$$\left[J^2, J_z\right] = 0 \tag{8}$$

1.2 Proof of commutator relations

1.2.1 $[J_+, J_-] = 2J_z$

First we consider the commutator relation (5)

$$[J_+, J_-] = \sum_p a^\dagger_{p+} a_{p-} \sum_k a^\dagger_{k-} a_{k+} - \sum_k a^\dagger_{k-} a_{k+} \sum_p a^\dagger_{p+} a_{p-} \tag{9}$$

We then immediately note that since for $k \neq p$

$$[a_{p\sigma}, a_{k\sigma'}] = 0 \tag{10}$$

It follows that all the terms where $p \neq k$ fall out of the commutator. Hence:

$$[J_+, J_-] = \sum_p a^\dagger_{p+} a_{p-} a^\dagger_{p-} a_{p+} - \sum_p a^\dagger_{p-} a_{p+} a^\dagger_{p+} a_{p-} \tag{11}$$

Now we note that because of the difference in spins, some of these terms commute and the sums can be merged and we can rewrite this as

$$[J_+, J_-] = \sum_p a^\dagger_{p+} a_{p+} a_{p-} a^\dagger_{p-} - a^\dagger_{p-} a_{p-} a_{p+} a^\dagger_{p+} \tag{12}$$

Finally we note that since

$$\left[a, a^\dagger\right] = 1 \tag{13}$$

This thus means that with further rearangement

$$[J_+, J_-] = \sum_p a^\dagger_{p+} a_{p+} (a^\dagger_{p-} a_{p-} + 1) - (a^\dagger_{p+} a_{p+} + 1) a^\dagger_{p-} a_{p-} \tag{14}$$

And finally simplifying this is

$$[J_+, J_-] = \sum_p a^\dagger_{p+} a_{p+} - a^\dagger_{p-} a_{p-} \tag{15}$$

Which we note is just

$$[J_+, J_-] = \sum_{p\sigma} a^\dagger_{p\sigma} a_{p\sigma} = 2J_z \tag{16}$$

And this is as expected in agreement with the relationship (5) above.

1.2.2 $[J_+, J_z] = -J_+$

Again we write this out in full using the creation and anhilation operators.

$$[J_+, J_z] = \frac{1}{2}\left(\sum_k a^\dagger_{k+}a_{k-}\sum_{p\sigma}\sigma a^\dagger_{p\sigma}a_{p\sigma} - \sum_{p\sigma}\sigma a^\dagger_{p\sigma}a_{p\sigma}\sum_k a^\dagger_{k+}a_{k-}\right) \tag{17}$$

From this it follows that if $k \neq p$ it follows that those terms cancel and similarly for the sigmas.

$$[J_+, J_z] = \frac{1}{2}\sum_{p\sigma}\sigma\left(a^\dagger_{p+}a_{p-}a^\dagger_{p\sigma}a_{p\sigma} - a^\dagger_{p\sigma}a_{p\sigma}a^\dagger_{p+}a_{p-}\right) \tag{18}$$

We then expand this sum over sigma as

$$[J_+, J_z] = \frac{1}{2}\sum_p\left(a^\dagger_{p+}a_{p-}a^\dagger_{p+}a_{p+} - a^\dagger_{p+}a_{p+}a^\dagger_{p+}a_{p-} - a^\dagger_{p+}a_{p-}a^\dagger_{p-}a_{p-} + a^\dagger_{p-}a_{p-}a^\dagger_{p+}a_{p-}\right) \tag{19}$$

Again taking advantage of the creation/annhilation operators' commutation relations it follows that

$$[J_+, J_z] = \frac{1}{2}\sum_p\left(a^\dagger_{p+}a^\dagger_{p+}a_{p+}a_{p-} - a^\dagger_{p+}(a^\dagger_{p+}a_{p+} + 1)a_{p-} - a^\dagger_{p+}(a^\dagger_{p-}a_{p-} + 1)a_{p-} + a^\dagger_{p+}a^\dagger_{p-}a_{p-}a_{p-}\right) \tag{20}$$

And finally cancelling like terms

$$[J_+, J_z] = -\sum_p a^\dagger_{p+}a_{p-} = -J_+ \tag{21}$$

1.2.3 $[J_-, J_z] = J_-$

This is fundamentally the same as the previous argument and so I skip most of the explanations:

$$[J_-, J_z] = \frac{1}{2}\left(\sum_k a^\dagger_{k-}a_{k+}\sum_{p\sigma}\sigma a^\dagger_{p\sigma}a_{p\sigma} - \sum_{p\sigma}\sigma a^\dagger_{p\sigma}a_{p\sigma}\sum_k a^\dagger_{k-}a_{k+}\right) \tag{22}$$

$$[J_-, J_z] = \frac{1}{2}\sum_{p\sigma}\sigma\left(a^\dagger_{p-}a_{p+}a^\dagger_{p\sigma}a_{p\sigma} - a^\dagger_{p\sigma}a_{p\sigma}a^\dagger_{p-}a_{p+}\right) \tag{23}$$

$$[J_-, J_z] = \frac{1}{2}\sum_p \left(a^\dagger_{p-}a_{p+}a^\dagger_{p+}a_{p+} - a^\dagger_{p+}a_{p+}a^\dagger_{p-}a_{p+} - a^\dagger_{p-}a_{p+}a^\dagger_{p-}a_{p-} + a^\dagger_{p-}a_{p-}a^\dagger_{p-}a_{p+} \right) \tag{24}$$

Again taking advantage of the creation/annhilation operators' commutation relations it follows that

$$[J_-, J_z] = \frac{1}{2}\sum_p \left(a^\dagger_{p+}a^\dagger_{p+}a_{p-}a_{p+} - a^\dagger_{p+}(a^\dagger_{p+}a_{p+} - 1)a_{p-} - a^\dagger_{p-}(a^\dagger_{p-}a_{p-} - 1)a_{p+} + a^\dagger_{p-}a^\dagger_{p-}a_{p-}a_{p+} \right) \tag{25}$$

And finally cancelling like terms

$$[J_-, J_z] = \sum_p a^\dagger_{p-}a_p = J_- \tag{26}$$

1.2.4 $\left[J^2, J_z\right] = 0$

Here we take advantage of commutator algebra. Noting that

$$J^2 = J_+J_- + J_z^2 - J_z \tag{27}$$

It is immediately clear that J_z commutes with the second and third terms.

Then

$$\left[J^2, J_z\right] = [J_+J_-, J_z] = J_+[J_-, J_z] + [J_+, J_z]J_- = J_+J_- - J_+J_- = 0 \tag{28}$$

2 Hamiltonian in Quasi Spin Form

We need to rewrite the hamiltonian operator in terms of these quasispin operators.

2.1 $H_0 = \frac{1}{2}\epsilon\sum_{\sigma,p}\sigma a^\dagger_{\sigma,p}a_{\sigma,p}$

We see immediately that this is just

$$H_0 = \epsilon\hat{J}_z \tag{29}$$

Which of course fits with the anticpated physical phenomenon.

2.2 $H_1 = \frac{V}{2}\sum_{\sigma,p,p'} a^\dagger_{\sigma,p} a^\dagger_{\sigma,p'} a_{-\sigma,p'} a_{-\sigma,p}$

This derivation is fairly straightforward, as due to the fact that the anhiliation and creation operators have different energy levels(i.e. the creation operators are all at energy σ and the annhilation at energy -σ), it follows that

$$H_1 = \frac{1}{2} V \sum_{\sigma,p,p'} a^\dagger_{\sigma p} a_{-\sigma,p} a^\dagger_{\sigma p'} a_{-\sigma,p} \tag{30}$$

This can then immediately be decomposed into a product over the two different σ states. Thus these become

$$\frac{V}{2}\left(\sum_p a^\dagger_{+p} a_{-,p} \sum_{p'} a^\dagger_{+p'} a_{-,p'} + \sum_p a^\dagger_{-p} a_{+,p} \sum_{p'} a^\dagger_{-p'} a_{+,p'}\right) \tag{31}$$

And it is then clear that this is just

$$\frac{V}{2}\left((J^+)^2 + (J^-)^+\right) \tag{32}$$

2.3 $H_2 = \frac{W}{2}\sum_{\sigma,p,p'} a^\dagger_{\sigma,p} a^\dagger_{-\sigma,p'} a_{\sigma,p'} a_{-\sigma,p}$

This derivation is similar to the above except in that if p=p' there is need to make use of a commutation relation:

$$H_2 = \frac{W}{2} \sum_{\sigma,p,p'} a^\dagger_{\sigma,p}\left(a_{-\sigma,p} a^\dagger_{-\sigma,p'} - \delta_{p,p'}\right) a_{\sigma,p'} \tag{33}$$

This can thus be written as

$$\frac{W}{2} \sum_{\sigma,p,p'} a^\dagger_{\sigma,p} a_{-\sigma,p} a^\dagger_{-\sigma,p'} a_{\sigma,p'} - \sum_{\sigma,p} a^\dagger_{\sigma,p} a_{\sigma,p} \tag{34}$$

It is clear that the second term is identical to

$$\hat{N} = \sum_{\sigma,p} a^\dagger_{\sigma,p} a_{\sigma,p} \tag{35}$$

It is furthermore clear that the first term can be decomposed as in the previous section so that

$$\sum_\sigma \left(\sum_p a^\dagger_{\sigma,p} a_{-\sigma,p} \sum_{p'} a^\dagger_{-\sigma,p'} a_{\sigma,p'}\right) \tag{36}$$

And we see this term is just

$$J_+J_- + J_-J_+ \tag{37}$$

Finally putting all these terms together we have that

$$H_2 = \frac{W}{2}\left(J_+J_- + J_-J_+ - N\right) \tag{38}$$

Finally taking advantage of the commutation relations in part one and the definition of J^2 it follows that

$$H_2 = W\left(J^2 - J_z^2 - \frac{N}{2}\right) \tag{39}$$

3 J^2 commutes with H

To prove J^2 commutes with H we show it commutes with each component of H.

3.1 H_0

Having already shown that $\left[J^2, J_z\right] = 0$ it is trivially clear that

$$\left[J^2, H_0\right] = \left[J^2, \epsilon J_z\right] = \epsilon\left[J^2, J_z\right] = 0 \tag{40}$$

3.2 H_1

Next we show that

$$\left[J^2, H_1\right] = 0 \tag{41}$$

We can show this by showing instead that

$$\left[J^2, J_+\right] = 0 \tag{42}$$

and

$$\left[J^2, J_-\right] = 0 \tag{43}$$

We prove these by takng advantage of the fact that

$$J^2 = J_+J_- + J_z^2 - J_z \tag{44}$$

3.2.1 $\left[J^2, J_+\right] = 0$

From this it follows that

$$\left[J^2, J_+\right] = \left[J_+J_-, J_+\right] + \left[J_z^2, J_+\right] - \left[J_z, J_+\right] \tag{45}$$

By taking advantage of our previous commutation relations and basic commutator algebra we have

$$\left[J^2, J_+\right] = J_+\left[J_-, J_+\right] + J_z\left[J_z, J_+\right] + \left[J_z, J_+\right]J_z - J_+ \tag{46}$$

And with further simplification

$$\left[J^2, J_+\right] = -2J_+J_z + J_zJ_+ + J_+J_z - J_+ = 2J_+J_z - 2J_+J_z + J_+ - J_+ = 0 \tag{47}$$

3.2.2 $\left[J^2, J_-\right] = 0$

We can then do the same for the J_- case.

$$\left[J^2, J_-\right] = \left[J_+J_-, J_-\right] + \left[J_z^2, J_-\right] - \left[J_z, J_-\right] \tag{48}$$

$$\left[J^2, J_-\right] = \left[J_+, J_-\right]J_- + J_z\left[J_z, J_-\right] + \left[J_z, J_-\right]J_z + J_- \tag{49}$$

And with further simplification

$$\left[J^2, J_+\right] = 2J_zJ_- + J_zJ_- + J_-J_z + J_- = 2J_zJ_- - 2J_zJ_- + J_- - J_- = 0 \tag{50}$$

From this it immediately follows that

$$\left[J^2, H_1\right] = \left[J^2, \frac{V}{2}\left((J^+)^2 + (J^-)^2\right)\right] = 0 \tag{51}$$

3.3 $[J^2, H_2] = 0$

We prove this is also 0 by noting that as shown below the H_2 term has a term J^2 which trivially commutes with itself, and a term in powers of J_z which we have already shown commutes. Hence we only need to show that the remaining term commutes, or equivalently that $\left[J^2, N\right] = 0$

$$H_2 = W\left(J^2 - J_z^2 - \frac{N}{2}\right) \tag{52}$$

We prove this by first finding the commutators

$$[J_+, N], [J_-, N] \tag{53}$$

3.3.1 $[J_-, N]$

We can then have

$$[J_-, N] = \left[\sum_p a^\dagger_{p-} a_{p+}, \sum_p a^\dagger_{p+} a_{p+} + \sum_p a^\dagger_{p-} a_{p-}\right] \tag{54}$$

Noting that the only non zero terms of the commutator will be those when the p terms are identical we thus have

$$[J_-, N] = \sum_p \left[a^\dagger_{p-} a_{p+}, a^\dagger_{p+} a_{p+} + a^\dagger_{p-} a_{p-}\right] \tag{55}$$

Separating out the sums we have

$$[J_-, N] = \sum_p \left[a^\dagger_{p-} a_{p+}, a^\dagger_{p+} a_{p+} + a^\dagger_{p-} a_{p-}\right] \tag{56}$$

$$[J_-, N] = \sum_p \left[a^\dagger_{p-} a_{p+}, a^\dagger_{p+} a_{p+}\right] + \left[a^\dagger_{p-} a_{p+}, a^\dagger_{p-} a_{p-}\right] \tag{57}$$

Simplifying further we have

$$[J_-, N] = \sum_p \left(a^\dagger_{p-} \left[a_{p+}, a^\dagger_{p+}\right] a_{p+} + a^\dagger_{p-} \left[a^\dagger_{p-}, a_{p-}\right] a_{p+}\right) \tag{58}$$

Taking the commutators we see that

$$[J_-, N] = \sum_p a^\dagger_{p-} a_{p+} - \sum_p a^\dagger_{p-} a_{p+} \tag{59}$$

And we see this is equal to

$$[J_-, N] = J_- - J_- = 0 \tag{60}$$

3.3.2 $[J_+, N]$

We now do the same for J_+

$$[J_+, N] = \left[\sum_p a^\dagger_{p+} a_{p-}, \sum_p a^\dagger_{p+} a_{p+} + \sum_p a^\dagger_{p-} a_{p-}\right] \tag{61}$$

Noting that the only non zero terms of the commutator will be those when the p terms are identical we thus have

$$[J_+, N] = \sum_p \left[a^\dagger_{p+} a_{p-}, a^\dagger_{p+} a_{p+} + a^\dagger_{p-} a_{p-}\right] \tag{62}$$

Separating out the sums we have

$$[J_+, N] = \sum_p \left[a^\dagger_{p+} a_{p-}, a^\dagger_{p+} a_{p+}\right] + \left[a^\dagger_{p+} a_{p-}, a^\dagger_{p-} a_{p-}\right] \tag{63}$$

Simplifying further we have

$$[J_+, N] = \sum_p \left(a^\dagger_{p+} \left[a^\dagger_{p+}, a_{p+}\right] a_{p-} + a^\dagger_{p+} \left[a_{p-}, a^\dagger_{p-}\right] a_{p-}\right) \tag{64}$$

Taking the commutators we see that

$$[J_+, N] = -\sum_p a^\dagger_{p+} a_{p-} - \sum_p a^\dagger_{p+} a_{p-} \tag{65}$$

And we see this is equal to

$$[J_+, N] = J_+ - J_+ = 0 \tag{66}$$

3.3.3 $[J^2, N]$

We note finally that if we define an operator

$$n_\sigma = \sum_p a^\dagger_{p\sigma} a_{p\sigma} \tag{67}$$

It follows that trivially

$$[n_\sigma . n_{\sigma'}] = 0 \tag{68}$$

It thereby follows that

$$J_z = \frac{1}{2}\left(n_+ - n_-\right) \tag{69}$$

and

$$N = n_+ + n_- \tag{70}$$

Now from this it therefore follows that N and J_z commute trivially.

Hence

$$\left[J^2, N\right] = 0 \tag{71}$$

4 Constructing the $J = 2$ states

4.1 $J_z = -1$

We begin by considering that the $J = -2$ state is

$$\left(\prod_p a^\dagger_{-p}\right)|0\rangle \tag{72}$$

4.2 $J_z = -1$

We then apply the J_+ operator knowing that

$$J_+ |J, J_z\rangle = \sqrt{J(J+1) - J_z(J_z+1)}\,|J, J_z+1\rangle \tag{73}$$

We can then directly apply this as

$$J_+ |2,-2\rangle = \sqrt{6-2}\,|2,-1\rangle = \sum_p a^\dagger_{p+} a_{p-} \left(\prod_p a^\dagger_{p-}\right)|0\rangle \tag{74}$$

We see that this last term is such that if the anhiliation term comes first it will kill of the term. Hence we get a delta function in ps over the sum, leading to

$$\sqrt{4}\,|2,-1\rangle = \sum_p a^\dagger_{p+} a_{p-} a^\dagger_{p-}|0\rangle = \sum_p a^\dagger_{p+} \prod_{p'} \left(\left(a^\dagger_{p'-} a_{p-} + 1\right)\delta_{p,p'} + a^\dagger_{p'-}(1 - \delta_{p,p'})\right)|0\rangle \tag{75}$$

Now the term in the product on the far left clearly goes to zero so that this becomes

$$\sum_p a^\dagger_{p+} \prod_{p' \neq p} a^\dagger_{p'-} \tag{76}$$

We thus have

$$|2,-1\rangle = \frac{1}{2}\left(\sum_p a^\dagger_{p+} \prod_{p' \neq p} a^\dagger_{p'-}\right) \tag{77}$$

It is clear that this is just the normalized sum of all possible permutations of levels where one is $\sigma = +$ and all the others are -1.

4.3 $J_z = 0$

We can then apply the J_+ operator again.

$$J_+ |2,-1\rangle = \sqrt{6}\,|2,0\rangle = \frac{1}{2}\left(\sum_{p''} a^\dagger_{p''+} a_{p''-} \sum_p a^\dagger_{p+} \prod_{p' \neq p} a^\dagger_{p'-}\right) \tag{78}$$

We see clearly that this follows a similar pattern to the previous, with the only major difference being in that since the sum is over both p and p', that there is a factor of two when the delta functions are considered. We thereby wind up with

$$|2,0\rangle = \frac{1}{\sqrt{6}}\left(\sum_{p,p' \neq p} a^\dagger_{p+} a^\dagger_{p'+} \prod_{p''' \neq p \neq p'} a^\dagger_{p''-}\right) \tag{79}$$

Which is to say this is just all the possibilites where all four degeneraties are occupied and two are up and two down normalized.

4.4 $J_z = 1$

We here note that via the symmetry of the hamiltonian under $\epsilon \to -\epsilon$ and $\sigma \to -\sigma$ it follows that

$$|2,1\rangle = |2,-1\rangle\,|_{\sigma \to -\sigma} = \frac{1}{2}\left(\sum_p a^\dagger_{p-} \prod_{p' \neq p} a^\dagger_{p'+}\right) \tag{80}$$

It is clear that this is just the normalized sum of all possible permutations of levels where one is $\sigma = -$ and all the others are $+$.

4.5 $J_z = 2$

The same argument as previous holds and so we trivially have

$$|2,2\rangle = \sum_p a^\dagger_{p+} \tag{81}$$

5 Hamiltonian in 5d space

We can construct this Hamiltonian matrix in general by considering the application of J_+, J_-, J_z and N on the eigenstates $|J, J_z\rangle$.(so long as the hamiltonian is made up of these operators).

In this case we can see trivially from the way that the operators J_+, J_- and J_z operate that their matrix representations in general ought to be, where we take the basis to be

$$\vec{J} = \begin{pmatrix} |J, J_z = J\rangle \\ |J, J_z = J-1\rangle \\ \vdots \\ |J, J_z = -J\rangle \end{pmatrix} \tag{82}$$

$$J_- = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \sqrt{J(J+1)-(J-1)(J)} & 0 & 0 & \cdots & 0 \\ 0 & \sqrt{J(J+1)-(J-2)(J-1)} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \sqrt{J(J+1)-J(J-1)} & 0 \end{pmatrix} \tag{83}$$

Similarly for the J_- we have

$$J_+ = \begin{pmatrix} 0 & \sqrt{J(J+1)-J(J-1)} & 0 & \cdots & 0 \\ 0 & 0 & \sqrt{J(J+1)-(J-1)(J-2)} & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \sqrt{J(J+1)-(J-1)(J)} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{84}$$

And for J_z we have trivially that

$$J_z = \begin{pmatrix} J & 0 & 0 & \cdots & 0 \\ 0 & J-1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & -J+1 & 0 \\ 0 & 0 & 0 & 0 & -J \end{pmatrix} \tag{85}$$

Now finally we note that N returns the original state with an eigenvalue of how many particles are in the state. It's worth noting then that we can define J states for each of the possible numbers of particles N=(0,1,2,3,4) which map to $J = (0, \frac{1}{2}, 1, \frac{3}{2}, 2)$

From this it immediately follows that

$$N = 2 \begin{pmatrix} J & 0 & 0 & \cdots & 0 \\ 0 & J & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & J & 0 \\ 0 & 0 & 0 & 0 & J \end{pmatrix} \tag{86}$$

6 Hamiltonian

The Hamiltonian itself is thus easily constructed out of the J matricies by substitution, leading to:

$$H = H_0 + H_1 + H_2 = \epsilon \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} + V \begin{pmatrix} 0 & 0 & 2.45 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 2.45 & 0 & 0 & 0 & 2.45 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2.45 & 0 & 0 \end{pmatrix} + W \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{87}$$

Hence we have that

$$H = \begin{pmatrix} 2\epsilon & 0 & 2.45V & 0 & 0 \\ 0 & 3W+\epsilon & 0 & 3V & 0 \\ 2.45V & 0 & 4W & 0 & 2.45V \\ 0 & 3V & 0 & 3W-\epsilon & 0 \\ 0 & 0 & 2.45V & 0 & -2\epsilon \end{pmatrix} \tag{88}$$

7 Eigenvalues/Eigenvectors of the Hamiltonian

Now we consider both the case of $\epsilon = 2, V = -\frac{1}{3}, W = -\frac{1}{4}$ and $\epsilon = 2, V = -\frac{4}{3}, W = -1$. This is done using the script attached in the appendix.

7.1 $\epsilon = 2, V = -\frac{1}{3}, W = -\frac{1}{4}$

Here we have the matrix of eigenvectors is

$$\tilde{\lambda} = \begin{vmatrix} 0.99 & 0.16 & 0.03 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & -0.23 & -0.97 \\ -0.16 & 0.95 & 0.25 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & -0.97 & 0.23 \\ 0.02 & -0.25 & 0.97 & 0.00 & 0.00 \end{vmatrix} \tag{89}$$

These correspond to the eigenvalues

$$\lambda = \begin{pmatrix} 4.132 & -0.919 & -4.213 & -2.986 & 1.486 \end{pmatrix} \tag{90}$$

We can see that in this case the ground state is that with energy -4.213, which is the state

$$\psi = 0.97\,|2,-2\rangle + 0.25\,|2,0\rangle + 0.03\,|2,2\rangle \tag{91}$$

To describe why this is, we first note that in this case the H_0 and H_2 both act on the same eigenstates(which would be the unmixed J_m states). What's more the H_1 term that does not have the same eigenvectors can only increase or decrease the J_m states by 2, which explains why the eigenvectors for the total H matrix are easily divided into odd and even catagories. In this case, the relative strength of the H_1 matrix is weak, and so the ground state is that where J_z is lowest, since the H_0 still dominates, and this strongly favors $J_z = -J$ states.

7.2 $\epsilon = 2, V = -\frac{4}{3}, W = -1$

Now we consider the same for the situation where the relative importance of H_1 and H_2 are important. We can thus expect a change in the eigenvalues due to H_2 and a change in the eigenvectors(and eigenvalues) due to H_1. As such we ought to expect far more mixing of the states. This is confirmed when we look at the matrix of eigenvectors

$$\tilde{\lambda} = \begin{vmatrix} -0.92 & -0.33 & 0.21 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & -0.53 & -0.85 \\ 0.37 & -0.57 & 0.74 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & -0.85 & 0.53 \\ -0.13 & 0.76 & 0.64 & 0.00 & 0.00 \end{vmatrix} \tag{92}$$

Which correspond to eigenvalues

$$\lambda = \begin{pmatrix} 5.31 & -1.56 & -7.751 & -7.472 & 1.472 \end{pmatrix} \tag{93}$$

Hence we see in this case that the ground state has energy -7.751 and corresponds to

$$\psi = 0.21\,|2,2\rangle + 0.74\,|2,0\rangle + 0.64\,|2,-2\rangle \tag{94}$$

This is a substantial admixture of states, and in fact has less of the $J_z = -2$ state than the $J_z = 0$ state, but because of the mixing term is the ground state. We can see the importance of the ixing term by considering that the odd term

$$\psi = -0.53\,|2,1\rangle - 0.85\,|2,-1\rangle \tag{95}$$

Has an energy of -7.472 which is very close to the ground state, and is clearly mostly dominated by the mixing term, though it also has less of the $J_z = 2$ term to factor in. That is, there is a competition between admixtures of even states that include both the high energy $|2,2\rangle$ state and the $|2,-2\rangle$ state which also has significant mixing, and the odd states that have less of the H_0 term and so less of the high energy states from it, but also consequently have somewhat less mixing.

7.3 Pure H_2

For a final comparison I note that if $W = \epsilon = 0$ and $V = 1$ it follows that the matrix of eigenvectors is

$$\tilde{\lambda} = \begin{vmatrix} 0.50 & -0.71 & 0.50 & 0.00 & 0.00 \\ -0.00 & 0.00 & 0.00 & -0.71 & 0.71 \\ 0.71 & -0.00 & -0.71 & 0.00 & 0.00 \\ -0.00 & 0.00 & 0.00 & 0.71 & 0.71 \\ 0.50 & 0.71 & 0.50 & 0.00 & 0.00 \end{vmatrix} \tag{96}$$

With corresponding eigenvalues

$$\lambda = \begin{pmatrix} -3.464 & 0 & 3.464 & 3 & -3 \end{pmatrix} \tag{97}$$

We see that this again splits the eigenvectors into even and odd solutions, and indeed the odd solutions have lower absolute values for energy than the even cases, except that where there is no admixture of the $J_z = 0$ term which has a balanced energy of 0. This fits what was observed in the Hamiltonian before.

8 Unitary Transformation

We now note that we can make a new single particle state out of the lipkin model by summing over sum mixture of Lipkin model particles

$$|\phi_{\alpha,\sigma}\rangle = \sum_{\sigma=\pm 1} C_{\alpha\sigma} |\mu_{\sigma,p}\rangle \tag{98}$$

We note further that these states will be orthonormal if we assume that $\tilde{\mathbf{C}}$ is unitary. This is because it would then follow that

$$\langle\phi_{\alpha',p'}| |\phi_{\alpha,p}\rangle = \left(\sum_{\sigma'=\pm 1} C^*_{\alpha'\sigma'} \langle\mu_{\sigma',p'}|\right) \left(\sum_{\sigma=\pm 1} C_{\alpha\sigma} |\mu_{\sigma,p}\rangle\right) \tag{99}$$

We realize that by the orthonormality of the Lipkin states this means that

$$\langle\phi_{\alpha',p'}| |\phi_{\alpha,p}\rangle = \sum_{\sigma=\pm 1} C^*_{\alpha'\sigma} C_{\alpha\sigma} \langle\mu_{\sigma,p'}| |\mu_{\sigma,p}\rangle = \sum_{\sigma=\pm 1} C^*_{\alpha'\sigma} C_{\alpha\sigma} \delta_{p,p'} \tag{100}$$

¡¡¡¡¡¡ HEAD Now finally we note that because of the unitarity of the matrix that $C^\dagger C = 1$ and hence

$$\sum_a C^\dagger_{ba} C_{ca} = \delta_{bc} \tag{101}$$

From which it follows that

$$\langle\phi_{\alpha',p'}| |\phi_{\alpha,p}\rangle = \delta_{p,p'}\delta_{\alpha,\alpha'} \tag{102}$$

And hence the new basis states are also orthonormal.

We note that in this transformation we have not changed p in constructing new particles. While in principle some mixing of p states could be considered, there is no need for this to create the relevant new particle states because the Hamiltonian is invariant under transformations that permute the ps. I.e. the p states are degenerate under H. This would not be possible if we added some $H'(p)$ term, in which case much of the machinery developed here would need to be changed substantially(or else treated as a perturbation to the previouly considered states).

9 Calculate the new Slater Determinant

We begin by noting that we can construct analogue anhilation and creation operators for the new single particle states so that

$$b^\dagger_{\alpha,p} |0\rangle = |\phi_{\alpha,p}\rangle \tag{103}$$

We can then realize that based on the description of the basis states previously considered it follows that it thus follows that

$$b_{\alpha,p}^{\dagger}|0\rangle = \left(C_{\alpha+}a_{+p}^{\dagger} + C_{\alpha-}a_{-p}^{\dagger}\right)|0\rangle \tag{104}$$

Now from this we see that we can easily construct J states by considering that the same pseudo spin can be made from $b^{\dagger}$ operators as was done previously. We hence have that for any state we have some sum of all products of combinations of creation operators that have a total sum of σ equal to the J_z state

Now I construct something akin to generalized operators A_+, A_- and B_+, B_- which are creation operators that are insensitive to p, and instead just add "1" more single particle state of the Lipkin model or the transformed model of $\sigma = \pm$ in a way that generates the proper J_z states. So, for example $A_+^2 A_-^2 |0\rangle = |J = 2, J_z = 0\rangle$.

We thus have trivially that because the matrix C does not depend on p that

$$B_+|0\rangle = (A_+C_{++} + A_-C_{+-})|0\rangle \tag{105}$$

$$B_-|0\rangle = (A_+C_{-+} + A_-C_{--})|0\rangle \tag{106}$$

Now we can in general construct any pseudo-angular momentum state. Let K represent the analogue of J for the modified particle states.

Then

$$|K, K_z\rangle = B_+^{K+K_z} B_-^{K-K_z}|0\rangle \tag{107}$$

It thus follows that

$$|K, K_z\rangle = (A_+C_{++} + A_-C_{+-})^{K+K_z}(A_+C_{-+} + A_-C_{--})^{K-K_z}|0\rangle \tag{108}$$

We see that this can be expanded as

$$|K, K_z\rangle = \left(\sum_{n=0}^{K+K_z}\binom{K+K_z}{n}(A_+C_{++})^n(A_-C_{+-})^{K+K_z-n}\right)\left(\sum_{n=0}^{K-K_z}\binom{K-K_z}{n}(A_+C_{-+})^n(A_-C_{--})^{K-K_z-n}\right)|0\rangle \tag{109}$$

Finally we realize that we can now construct a matrix U such that

$$\langle J| U^{\dagger}HU |J\rangle = \langle\psi| H |\psi\rangle \tag{110}$$

where

$$|J\rangle = \begin{pmatrix} |JJ_z\rangle \\ \vdots \\ |J-J_z\rangle \end{pmatrix} \tag{111}$$

We can find the mixing of each term by expanding out this product of sums and taking together all terms that have the same number of A_+s and $A_- s$.

This is explicitly

$$\langle J, J_z| \, |K, K_z\rangle = \delta_{J,K} \sum_{n+m=J+J_z} \binom{K+K_z}{n} C_{++}^{n} C_{+-}^{K+K_z-n} \binom{K-K_z}{m} C_{-+}^{m} C_{--}^{K-K_z-m} \tag{112}$$

Hence we have finally that we can construct the matrix U as (for J=K), noting that $\langle J, J_z|K, K_z\rangle = \langle K, K_z|J, J_z\rangle^\dagger$.

$$U = \begin{pmatrix} \langle K,K|J,J\rangle & \langle K,K|J,J-1\rangle & \cdots & \langle K,K|J,-J\rangle \\ \langle K,K-1|J,J\rangle & \langle K,K-1|J,J-1\rangle & \cdots & \langle K,K-1|J,-J\rangle \\ \vdots & \ddots & \ddots & \vdots \\ \langle K,-K|J,J\rangle & \langle K,-K|J,J-1\rangle & \cdots & \langle K,-K|J,-J\rangle \end{pmatrix} \tag{113}$$

Finally we do exactly this for the case of $J = K = 2$. we can incorporate this to find E as

$$E = \langle \vec{J}|U^\dagger H U|\vec{J}\rangle = \langle \vec{\psi}|U H U^\dagger|\vec{\psi}\rangle \tag{114}$$

This has been explicitly incorporated into the pseudospin.py shown at the end. In the case that

$$|\psi\rangle = \prod_{p=1}^{4} b^\dagger_{\alpha,p} \, |0\rangle \tag{115}$$

It follows that we take the following values

$$E = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}^\dagger \begin{pmatrix} \langle 2,2|2,2\rangle & \langle 2,2|2,1\rangle & \cdots & \langle 2,2|2,-2\rangle \\ \langle 2,1|2,2\rangle & \langle 2,1|2,1\rangle & \cdots & \langle 2,1|2,-2\rangle \\ \vdots & \ddots & \ddots & \vdots \\ \langle 2,-1|2,2\rangle & \langle 2,-1|2,1\rangle & \cdots & \langle 2,-1|2,-2\rangle \\ \langle 2,-2|2,2\rangle & \langle 2,-2|2,1\rangle & \cdots & \langle 2,-2|2,-2\rangle \end{pmatrix} H \begin{pmatrix} \langle 2,2|2,2\rangle & \langle 2,2|2,1\rangle & \cdots & \langle 2,2|2,-2\rangle \\ \langle 2,1|2,2\rangle & \langle 2,1|2,1\rangle & \cdots & \langle 2,1|2,-2\rangle \\ \vdots & \ddots & \ddots & \vdots \\ \langle 2,-1|2,2\rangle & \langle 2,-1|2,1\rangle & \cdots & \langle 2,-1|2,-2\rangle \\ \langle 2,-2|2,2\rangle & \langle 2,-2|2,1\rangle & \cdots & \langle 2,-2|2,-2\rangle \end{pmatrix} \tag{116}$$

From the attached python file we see that

$$U = \begin{bmatrix} 1.0C_{++}^{4} & 4.0C_{++}^{3}C_{+-} & 6.0C_{++}^{2}C_{+-}^{2} & 4.0C_{++}C_{+-}^{3} & 1.0C_{+-}^{4} \\ -1.0C_{++}^{3}\overline{C_{+-}} & 1.0C_{++}^{3}C_{--}-3.0C_{++}^{2}C_{+-}\overline{C_{+-}} & 3.0C_{++}^{2}C_{+-}C_{--}-3.0C_{++}C_{+-}^{2}\overline{C_{+-}} & 3.0C_{++}C_{+-}^{2}C_{--}-1.0C_{+-}^{3}\overline{C_{+-}} & 1.0C_{+-}^{3}C_{--} \\ 1.0C_{++}^{2}\overline{C_{+-}}^{2} & -2.0C_{++}^{2}C_{--}\overline{C_{+-}}+2.0C_{++}C_{+-}\overline{C_{+-}}^{2} & 1.0C_{++}^{2}C_{--}^{2}-4.0C_{++}C_{+-}C_{--}\overline{C_{+-}}+1.0C_{+-}^{2}\overline{C_{+-}}^{2} & 2.0C_{++}C_{+-}C_{--}^{2}-2.0C_{+-}^{2}C_{--}\overline{C_{+-}} & 1.0C_{+-}^{2}C_{--}^{2} \\ -1.0C_{++}\overline{C_{+-}}^{3} & 3.0C_{++}C_{--}\overline{C_{+-}}^{2}-1.0C_{+-}\overline{C_{+-}}^{3} & -3.0C_{++}C_{--}^{2}\overline{C_{+-}}+3.0C_{+-}C_{--}\overline{C_{+-}}^{2} & 1.0C_{++}C_{--}^{3}-3.0C_{+-}C_{--}^{2}\overline{C_{+-}} & 1.0C_{+-}C_{--}^{3} \\ 1.0\overline{C_{+-}}^{4} & -4.0C_{--}\overline{C_{+-}}^{3} & 6.0C_{--}^{2}\overline{C_{+-}}^{2} & -4.0C_{--}^{3}\overline{C_{+-}} & 1.0C_{--}^{4} \end{bmatrix} \tag{117}$$

Returning to the previous we see then that this is equal to

$$\begin{pmatrix} \langle 2,2|2,2\rangle \\ \langle 2,2|2,1\rangle \\ \vdots \\ \langle 2,2|2,-2\rangle \end{pmatrix}^{\dagger} H \begin{pmatrix} \langle 2,2|2,2\rangle \\ \langle 2,1|2,2\rangle \\ \vdots \\ \langle 2,-2|2,2\rangle \end{pmatrix} \tag{118}$$