Projeect 2

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1 Hamiltonian for Pair Conservation

In this problem we consider a space of equally spaced single particle levels with a spin degeneracy(up and down spin). We can then consider a pair conserving Hamiltonian with a single particle part and a pairing term. We represent these as

$$H = H_0 + V \tag{1}$$

Where

$$H_0 = \chi \sum_{p\sigma} (p-1)\, a^{\dagger}_{p\sigma} a_{p\sigma} \tag{2}$$

$$\langle q_+ q_- | V | s_+ s_- \rangle = -g \rightarrow V = -g \sum_{pq} a^{\dagger}_{+p} a^{\dagger}_{p-} a_{q+} a_{q-} \tag{3}$$

Finally we define a pair creation and annhilation operator as

$$\hat{P}_p = a_{p+} a_{p-} \tag{4}$$

$$\hat{P}_p^{\dagger} = a^{\dagger}_{p+} a^{\dagger}_{p-} \tag{5}$$

From this it follows that

Now let us consider each of the commutation relations between the single particle and creation annhilation operators.

$$[P_p, P_q] = 0 \tag{6}$$

$$\left[P_p, P_q^{\dagger}\right] = \delta_{p,q} \tag{7}$$

$$[P_p, a_{q\sigma}] = 0 \tag{8}$$

$$\left[P_p^{\dagger}, a_{q\sigma}\right] = -\delta_{p,q} a^{\dagger}_{p-\sigma} \tag{9}$$

$$\left[P_p, a^\dagger_{q\sigma}\right] = \delta_{p,q} a_{p-\sigma} \tag{10}$$

$$\left[P^\dagger_p, a^\dagger_{q\sigma}\right] = 0 \tag{11}$$

$$\left[P^\dagger_q, P^\dagger_p\right] = 0 \tag{12}$$

We note that we can write V as

$$V = -g \sum_{pq} P^\dagger_q P_p \tag{13}$$

1.1 H_0, V commute withS_z, S^2

1.1.1 H_0 commutes with S_z

$$[H_0, S_z] = \epsilon \sum_{p'\sigma' p\sigma} (p-1)\,\sigma' \left[a^\dagger_{p\sigma} a_{p\sigma}, a^\dagger_{p'\sigma'} a_{p'\sigma'}\right] = A\left(\delta_{p,p'}\delta_{\sigma,\sigma'} - \delta_{p,p'}\delta_{\sigma,\sigma'}\right) = 0 \tag{14}$$

1.1.2 V commutes with S_z

$$[V, S_z] = A \sum_{p,q,p',\sigma} \left[P^\dagger_p P_q, a^\dagger_{p'\sigma} a_{p\sigma}\right] = \tag{15}$$

$$A \sum_{p,q,p',\sigma} \left(P^\dagger_p \left[P_p, a^\dagger_{p'\sigma}\right] a_{p'\sigma} + P^\dagger_p a^\dagger_{p'\sigma}\left[P_p, a_{p'\sigma}\right] + \left[P^\dagger_p a^\dagger_{p'\sigma}\right] P_p, a_{p'\sigma} + a^\dagger_{p'\sigma}\left[P^\dagger_p, a_{p'\sigma}\right] P_p\right) \tag{16}$$

$$= A \sum_{p,q,p'\sigma} \left(\delta_{p,p'} P^\dagger_p a_{p'-\sigma} a_{p'\sigma} - \delta_{p,p'} a^\dagger_{p',\sigma} a^\dagger_{p,-\sigma} P_p\right) = A \sum_{p\sigma}\left(P^\dagger_p P_p - P^\dagger_p P_p\right) = 0 \tag{17}$$

1.1.3 H_0 commutes with S^2

We note that

$$S^2 = S_z^2 + \frac{1}{2}\left(S_+ S_- + S_- S_+\right) \tag{18}$$

Knowing from above that H_0 commutes with S_z it thereby follows that we must only prove

$$\left[H_0, (S_+ S_- + S_- S_+)\right] = 0 \tag{19}$$

We note for any operator O

$$\left[O, (S_+ S_- + S_- S_+)\right] = \left[O, S_+\right] S_- + \left[O, S_-\right] S_+ + S_+\left[O, S_-\right] + S_-\left[O, S_+\right] \tag{20}$$

1.2 Hamiltonian Commutes with $P_p^+ P_p^-$

In order to show it keeps pairs together we must show that it conserves the product of the pair creation and annhilation operators.

Now we can then take advantage of some basic commutator algebra to solve that

$$\left[P_p^\dagger P_p, P_q^\dagger P_r\right] = P_p^\dagger \left[P_p, P_q^\dagger\right] P_r + P_q^\dagger P_q^\dagger \left[P_p, P_r\right] + \left[P_p^\dagger, P_q^\dagger\right] P_r P_p + P_q^\dagger \left[P_p^\dagger, P_r\right] P_p \tag{21}$$

$$= \delta_{p,q} P_p^\dagger P_r - \delta_{p,r} P_q^\dagger P_p = P_q^\dagger P_r - P_q^\dagger P_r = 0 \tag{22}$$

$$\left[P_p^\dagger P_p, a_{r\sigma}^\dagger a_{r\sigma}\right] = P_p^\dagger \left[P_p, a_{r\sigma}^\dagger\right] a_{r\sigma} + P_p^\dagger a_{r\sigma}^\dagger \left[P_p, a_{r\sigma}\right] + \left[P_p^\dagger, a_{r\sigma}^\dagger\right] a_{r\sigma} P_p + a_{r\sigma}^\dagger \left[P_p^\dagger, a_{r\sigma}\right] P_p \tag{23}$$

$$\left[P_p^\dagger P_p, a_{r\sigma}^\dagger a_{r\sigma}\right] = P_p^\dagger \delta_{p,r} a_{p-\sigma} a_{r\sigma} - \delta_{p,r} a_{r\sigma}^\dagger a_{p-\sigma}^\dagger P_p = P_p^\dagger P_p - P_p^\dagger P_p = 0 \tag{24}$$