

Brief Article

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1 Quasispin operators

We begin by defining the quasispin operators

$$J_+ = \sum_p a_{p+}^\dagger a_{p-} \quad (1)$$

$$J_- = \sum_p a_{p-}^\dagger a_{p+} \quad (2)$$

$$J_z = \sum_{p\sigma} \sigma a_{p\sigma}^\dagger a_{p\sigma} \quad (3)$$

$$J^2 = J_+ J_- + J_z^2 - J_z \quad (4)$$

We now set to show that these operators follow the canonical angular momenta commutator relations, which are as follows(setting \hbar to 1):

1.1 Angular Momenta Commutator Relations

$$[J_+, J_-] = 2J_z \quad (5)$$

$$[J_+, J_z] = -J_+ \quad (6)$$

$$[J_-, J_z] = J_- \quad (7)$$

$$[J^2, J_z] = 0 \quad (8)$$

1.2 Proof of commutator relations

1.2.1 $[J_+, J_-] = 2J_z$

First we consider the commutator relation (5)

$$[J_+, J_-] = \sum_p a_{p+}^\dagger a_{p-} \sum_k a_{k-}^\dagger a_{k+} - \sum_k a_{k-}^\dagger a_{k+} \sum_p a_{p+}^\dagger a_{p-} \quad (9)$$

We then immediately note that since for $k \neq p$

$$[a_{p\sigma}, a_{k\sigma'}] = 0 \quad (10)$$

It follows that all the terms where $p \neq k$ fall out of the commutator. Hence:

$$[J_+, J_-] = \sum_p a_{p+}^\dagger a_{p-} a_{p-}^\dagger a_{p+} - \sum_p a_{p-}^\dagger a_{p+} a_{p+}^\dagger a_{p-} \quad (11)$$

Now we note that because of the difference in spins, some of these terms commute and the sums can be merged and we can rewrite this as

$$[J_+, J_-] = \sum_p a_{p+}^\dagger a_{p+} a_{p-} a_{p-}^\dagger - a_{p-}^\dagger a_{p-} a_{p+} a_{p+}^\dagger \quad (12)$$

Finally we note that since

$$[a, a^\dagger] = 1 \quad (13)$$

This thus means that with further rearrangement

$$[J_+, J_-] = \sum_p a_{p+}^\dagger a_{p+} (a_{p-}^\dagger a_{p-} + 1) - (a_{p+}^\dagger a_{p+} + 1) a_{p-}^\dagger a_{p-} \quad (14)$$

And finally simplifying this is

$$[J_+, J_-] = \sum_p a_{p+}^\dagger a_{p+} - a_{p-}^\dagger a_{p-} \quad (15)$$

Which we note is just

$$[J_+, J_-] = \sum_{p\sigma} a_{p\sigma}^\dagger a_{p\sigma} = 2J_z \quad (16)$$

And this is as expected in agreement with the relationship (5) above.

1.2.2 $[J_+, J_z] = -J_+$

Again we write this out in full using the creation and annihilation operators.

$$[J_+, J_z] = \frac{1}{2} \left(\sum_k a_{k+}^\dagger a_{k-} \sum_{p\sigma} \sigma a_{p\sigma}^\dagger a_{p\sigma} - \sum_{p\sigma} \sigma a_{p\sigma}^\dagger a_{p\sigma} \sum_k a_{k+}^\dagger a_{k-} \right) \quad (17)$$

From this it follows that if $k \neq p$ it follows that those terms cancel and similarly for the sigmas.

$$[J_+, J_z] = \frac{1}{2} \sum_{p\sigma} \sigma \left(a_{p+}^\dagger a_{p-} a_{p\sigma}^\dagger a_{p\sigma} - a_{p\sigma}^\dagger a_{p\sigma} a_{p+}^\dagger a_{p-} \right) \quad (18)$$

We then expand this sum over sigma as

$$[J_+, J_z] = \frac{1}{2} \sum_p \left(a_{p+}^\dagger a_{p-} a_{p+}^\dagger a_{p+} - a_{p+}^\dagger a_{p+} a_{p+}^\dagger a_{p-} - a_{p+}^\dagger a_{p-} a_{p-}^\dagger a_{p-} + a_{p-}^\dagger a_{p-} a_{p+}^\dagger a_{p-} \right) \quad (19)$$

Again taking advantage of the creation/annihilation operators' commutation relations it follows that

$$[J_+, J_z] = \frac{1}{2} \sum_p \left(a_{p+}^\dagger a_{p+}^\dagger a_{p+} a_{p-} - a_{p+}^\dagger (a_{p+}^\dagger a_{p+} + 1) a_{p-} - a_{p+}^\dagger (a_{p-}^\dagger a_{p-} + 1) a_{p-} + a_{p+}^\dagger a_{p-}^\dagger a_{p-} a_{p-} \right) \quad (20)$$

And finally cancelling like terms

$$[J_+, J_z] = - \sum_p a_{p+}^\dagger a_{p-} = -J_+ \quad (21)$$

1.2.3 $[J_-, J_z] = J_-$

This is fundamentally the same as the previous argument and so I skip most of the explanations:

$$[J_-, J_z] = \frac{1}{2} \left(\sum_k a_{k-}^\dagger a_{k+} \sum_{p\sigma} \sigma a_{p\sigma}^\dagger a_{p\sigma} - \sum_{p\sigma} \sigma a_{p\sigma}^\dagger a_{p\sigma} \sum_k a_{k-}^\dagger a_{k+} \right) \quad (22)$$

$$[J_-, J_z] = \frac{1}{2} \sum_{p\sigma} \sigma \left(a_{p-}^\dagger a_{p+} a_{p\sigma}^\dagger a_{p\sigma} - a_{p\sigma}^\dagger a_{p\sigma} a_{p-}^\dagger a_{p+} \right) \quad (23)$$

$$[J_-, J_z] = \frac{1}{2} \sum_p \left(a_{p-}^\dagger a_{p+} a_{p+}^\dagger a_{p+} - a_{p+}^\dagger a_{p+} a_{p-}^\dagger a_{p+} - a_{p-}^\dagger a_{p+} a_{p-}^\dagger a_{p-} + a_{p-}^\dagger a_{p-} a_{p-}^\dagger a_{p+} \right) \quad (24)$$

Again taking advantage of the creation/annihilation operators' commutation relations it follows that

$$[J_-, J_z] = \frac{1}{2} \sum_p \left(a_{p+}^\dagger a_{p+}^\dagger a_{p-} a_{p+} - a_{p+}^\dagger (a_{p+}^\dagger a_{p+} - 1) a_{p-} - a_{p-}^\dagger (a_{p-}^\dagger a_{p-} - 1) a_{p+} + a_{p-}^\dagger a_{p-}^\dagger a_{p-} a_{p+} \right) \quad (25)$$

And finally cancelling like terms

$$[J_-, J_z] = \sum_p a_{p-}^\dagger a_p = J_- \quad (26)$$

1.2.4 $[J^2, J_z] = 0$

Here we take advantage of commutator algebra. Noting that

$$J^2 = J_+ J_- + J_z^2 - J_z \quad (27)$$

It is immediately clear that J_z commutes with the second and third terms.

Then

$$[J^2, J_z] = [J_+ J_-, J_z] = J_+ [J_-, J_z] + [J_+, J_z] J_- = J_+ J_- - J_+ J_- = 0 \quad (28)$$

2 Hamiltonian in Quasi Spin Form

We need to rewrite the hamiltonian operator in terms of these quasispin operators.

2.1 $H_0 = \frac{1}{2} \epsilon \sum_{\sigma,p} \sigma a_{\sigma,p}^\dagger a_{\sigma,p}$

We see immediately that this is just

$$H_0 = \epsilon \hat{J}_z \quad (29)$$

Which of course fits with the anticipated physical phenomenon.

$$2.2 \quad H_1 = \frac{V}{2} \sum_{\sigma,p,p'} a_{\sigma,p}^\dagger a_{\sigma,p'}^\dagger a_{-\sigma,p'} a_{-\sigma,p}$$

This derivation is fairly straightforward, as due to the fact that the annihilation and creation operators have different energy levels(i.e. the creation operators are all at energy σ and the annihilation at energy $-\sigma$), it follows that

$$H_1 = \frac{1}{2} V \sum_{\sigma,p,p'} a_{\sigma,p}^\dagger a_{-\sigma,p} a_{\sigma,p'}^\dagger a_{-\sigma,p} \quad (30)$$

This can then immediately be decomposed into a product over the two different σ states. Thus these become

$$\frac{V}{2} \left(\sum_p a_{+p}^\dagger a_{-,p} \sum_{p'} a_{+p'}^\dagger a_{-,p'} + \sum_p a_{-p}^\dagger a_{+,p} \sum_{p'} a_{-p'}^\dagger a_{+,p'} \right) \quad (31)$$

And it is then clear that this is just

$$\frac{V}{2} ((J^+)^2 + (J^-)^2) \quad (32)$$

$$2.3 \quad H_2 = \frac{W}{2} \sum_{\sigma,p,p'} a_{\sigma,p}^\dagger a_{-\sigma,p'}^\dagger a_{\sigma,p'} a_{-\sigma,p}$$

This derivation is similar to the above except in that if $p=p'$ there is need to make use of a commutation relation:

$$H_2 = \frac{W}{2} \sum_{\sigma,p,p'} a_{\sigma,p}^\dagger \left(a_{-\sigma,p} a_{-\sigma,p'}^\dagger - \delta_{p,p'} \right) a_{\sigma,p'} \quad (33)$$

This can thus be written as

$$\frac{W}{2} \sum_{\sigma,p,p'} a_{\sigma,p}^\dagger a_{-\sigma,p} a_{-\sigma,p'}^\dagger a_{\sigma,p'} - \sum_{\sigma,p} a_{\sigma,p}^\dagger a_{\sigma,p} \quad (34)$$

It is clear that the second term is identical to

$$\hat{N} = \sum_{\sigma,p} a_{\sigma,p}^\dagger a_{\sigma,p} \quad (35)$$

It is furthermore clear that the first term can be decomposed as in the previous section so that

$$\sum_{\sigma} \left(\sum_p a_{\sigma,p}^\dagger a_{-\sigma,p} \sum_{p'} a_{-\sigma,p'}^\dagger a_{\sigma,p'} \right) \quad (36)$$

And we see this term is just

$$J_+J_- + J_-J_+ \quad (37)$$

Finally putting all these terms together we have that

$$H_2 = \frac{W}{2} (J_+J_- + J_-J_+ - N) \quad (38)$$

Finally taking advantage of the commutation relations in part one and the definition of J^2 it follows that

$$H_2 = W \left(J^2 - J_z^2 - \frac{N}{2} \right) \quad (39)$$

3 J^2 commutes with H

To prove J^2 commutes with H we show it commutes with each component of H.

3.1 H_0

Having already shown that $[J^2, J_z] = 0$ it is trivially clear that

$$[J^2, H_0] = [J^2, \epsilon J_z] = \epsilon [J^2, J_z] = 0 \quad (40)$$

3.2 H_1

Next we show that

$$[J^2, H_1] = 0 \quad (41)$$

We can show this by showing instead that

$$[J^2, J_+] = 0 \quad (42)$$

and

$$[J^2, J_-] = 0 \quad (43)$$

We prove these by taking advantage of the fact that

$$J^2 = J_+J_- + J_z^2 - J_z \quad (44)$$

3.2.1 $[J^2, J_+] = 0$

From this it follows that

$$[J^2, J_+] = [J_+ J_-, J_+] + [J_z^2, J_+] - [J_z, J_+] \quad (45)$$

By taking advantage of our previous commutation relations and basic commutator algebra we have

$$[J^2, J_+] = J_+ [J_-, J_+] + J_z [J_z, J_+] + [J_z, J_+] J_z - J_+ \quad (46)$$

And with further simplification

$$[J^2, J_+] = -2J_+ J_z + J_z J_+ + J_+ J_z - J_+ = 2J_+ J_z - 2J_+ J_z + J_+ - J_+ = 0 \quad (47)$$

3.2.2 $[J^2, J_-] = 0$

We can then do the same for the J_- case.

$$[J^2, J_-] = [J_+ J_-, J_-] + [J_z^2, J_-] - [J_z, J_-] \quad (48)$$

$$[J^2, J_-] = [J_+, J_-] J_- + J_z [J_z, J_-] + [J_z, J_-] J_z + J_- \quad (49)$$

And with further simplification

$$[J^2, J_-] = 2J_z J_- + J_z J_- + J_- J_z + J_- = 2J_z J_- - 2J_z J_- + J_- - J_- = 0 \quad (50)$$

From this it immediately follows that

$$[J^2, H_1] = \left[J^2, \frac{V}{2} ((J^+)^2 + (J^-)^2) \right] = 0 \quad (51)$$

3.3 $[J^2, H_2] = 0$

We prove this is also 0 by noting that as shown below the H_2 term has a term J^2 which trivially commutes with itself, and a term in powers of J_z which we have already shown commutes. Hence we only need to show that the remaining term commutes, or equivalently that $[J^2, N] = 0$

$$H_2 = W \left(J^2 - J_z^2 - \frac{N}{2} \right) \quad (52)$$

We prove this by first finding the commutators

$$[J_+, N], [J_-, N] \quad (53)$$

3.3.1 $[J_-, N]$

We can then have

$$[J_-, N] = \left[\sum_p a_{p-}^\dagger a_{p+}, \sum_p a_{p+}^\dagger a_{p+} + \sum_p a_{p-}^\dagger a_{p-} \right] \quad (54)$$

Noting that the only non zero terms of the commutator will be those when the p terms are identical we thus have

$$[J_-, N] = \sum_p \left[a_{p-}^\dagger a_{p+}, a_{p+}^\dagger a_{p+} + a_{p-}^\dagger a_{p-} \right] \quad (55)$$

Separating out the sums we have

$$[J_-, N] = \sum_p \left[a_{p-}^\dagger a_{p+}, a_{p+}^\dagger a_{p+} + a_{p-}^\dagger a_{p-} \right] \quad (56)$$

$$[J_-, N] = \sum_p \left[a_{p-}^\dagger a_{p+}, a_{p+}^\dagger a_{p+} \right] + \left[a_{p-}^\dagger a_{p+}, a_{p-}^\dagger a_{p-} \right] \quad (57)$$

Simplifying further we have

$$[J_-, N] = \sum_p \left(a_{p-}^\dagger \left[a_{p+}, a_{p+}^\dagger \right] a_{p+} + a_{p-}^\dagger \left[a_{p-}, a_{p-}^\dagger \right] a_{p-} \right) \quad (58)$$

Taking the commutators we see that

$$[J_-, N] = \sum_p a_{p-}^\dagger a_{p+} - \sum_p a_{p-}^\dagger a_{p+} \quad (59)$$

And we see this is equal to

$$[J_-, N] = J_- - J_- = 0 \quad (60)$$

3.3.2 $[J_+, N]$

We now do the same for J_+

$$[J_+, N] = \left[\sum_p a_{p+}^\dagger a_{p-}, \sum_p a_{p+}^\dagger a_{p+} + \sum_p a_{p-}^\dagger a_{p-} \right] \quad (61)$$

Noting that the only non zero terms of the commutator will be those when the p terms are identical we thus have

$$[J_+, N] = \sum_p \left[a_{p+}^\dagger a_{p-}, a_{p+}^\dagger a_{p+} + a_{p-}^\dagger a_{p-} \right] \quad (62)$$

Separating out the sums we have

$$[J_+, N] = \sum_p \left[a_{p+}^\dagger a_{p-}, a_{p+}^\dagger a_{p+} \right] + \left[a_{p+}^\dagger a_{p-}, a_{p-}^\dagger a_{p-} \right] \quad (63)$$

Simplifying further we have

$$[J_+, N] = \sum_p \left(a_{p+}^\dagger \left[a_{p+}^\dagger, a_{p+} \right] a_{p-} + a_{p+}^\dagger \left[a_{p-}, a_{p-}^\dagger \right] a_{p-} \right) \quad (64)$$

Taking the commutators we see that

$$[J_+, N] = - \sum_p a_{p+}^\dagger a_{p-} - \sum_p a_{p+}^\dagger a_{p-} \quad (65)$$

And we see this is equal to

$$[J_+, N] = J_+ - J_+ = 0 \quad (66)$$

3.3.3 $[J^2, N]$

We note finally that if we define an operator

$$n_\sigma = \sum_p a_{p\sigma}^\dagger a_{p\sigma} \quad (67)$$

It follows that trivially

$$[n_\sigma, n_{\sigma'}] = 0 \quad (68)$$

It thereby follows that

$$J_z = \frac{1}{2} (n_+ - n_-) \quad (69)$$

and

$$N = n_+ + n_- \quad (70)$$

Now from this it therefore follows that N and J_z commute trivially.

Hence

$$[J^2, N] = 0 \quad (71)$$

4 Constructing the $J = 2$ states

4.1 $J_z = -1$

We begin by considering that the $J = -2$ state is

$$\left(\prod_p a_{-p}^\dagger \right) |0\rangle \quad (72)$$

4.2 $J_z = -1$

We then apply the J_+ operator knowing that

$$J_+ |J, J_z\rangle = \sqrt{J(J+1) - J_z(J_z+1)} |J, J_z+1\rangle \quad (73)$$

We can then directly apply this as

$$J_+ |2, -2\rangle = \sqrt{6-2} |2, -1\rangle = \sum_p a_{p+}^\dagger a_{p-} \left(\prod_p a_{p-}^\dagger \right) |0\rangle \quad (74)$$

We see that this last term is such that if the annihilation term comes first it will kill of the term. Hence we get a delta function in ps over the sum, leading to

$$\sqrt{4} |2, -1\rangle = \sum_p a_{p+}^\dagger a_{p-} a_{p-}^\dagger |0\rangle = \sum_p a_{p+}^\dagger \prod_{p'} \left(\left(a_{p'-}^\dagger a_{p-} + 1 \right) \delta_{p,p'} + a_{p'-}^\dagger (1 - \delta_{p,p'}) \right) |0\rangle \quad (75)$$

Now the term in the product on the far left clearly goes to zero so that this becomes

$$\sum_p a_{p+}^\dagger \prod_{p' \neq p} a_{p'-}^\dagger \quad (76)$$

We thus have

$$|2, -1\rangle = \frac{1}{2} \left(\sum_p a_{p+}^\dagger \prod_{p' \neq p} a_{p'-}^\dagger \right) \quad (77)$$

It is clear that this is just the normalized sum of all possible permutations of levels where one is $\sigma = +$ and all the others are -1.

4.3 $J_z = 0$

We can then apply the J_+ operator again.

$$J_+ |2, -1\rangle = \sqrt{6} |2, 0\rangle = \frac{1}{2} \left(\sum_{p''} a_{p''+}^\dagger a_{p''-}^\dagger \sum_p a_{p+}^\dagger \prod_{p' \neq p} a_{p'-}^\dagger \right) \quad (78)$$

We see clearly that this follows a similar pattern to the previous, with the only major difference being in that since the sum is over both p and p', that there is a factor of two when the delta functions are considered. We thereby wind up with

$$|2, 0\rangle = \frac{1}{\sqrt{6}} \left(\sum_{p, p' \neq p} a_{p+}^\dagger a_{p'+}^\dagger \prod_{p'' \neq p \neq p'} a_{p''-}^\dagger \right) \quad (79)$$

Which is to say this is just all the possibilities where all four degeneracies are occupied and two are up and two down normalized.

4.4 $J_z = 1$

We here note that via the symmetry of the hamiltonian under $\epsilon \rightarrow -\epsilon$ and $\sigma \rightarrow -\sigma$ it follows that

$$|2, 1\rangle = |2, -1\rangle |_{\sigma \rightarrow -\sigma} = \frac{1}{2} \left(\sum_p a_{p-}^\dagger \prod_{p' \neq p} a_{p'+}^\dagger \right) \quad (80)$$

It is clear that this is just the normalized sum of all possible permutations of levels where one is $\sigma = -$ and all the others are +.

4.5 $J_z = 2$

The same argument as previous holds and so we trivially have

$$|2, 2\rangle = \sum_p a_{p+}^\dagger \quad (81)$$

5 Hamiltonian in 5d space

We can construct this Hamiltonian matrix in general by considering the application of J_+ , J_- , J_z and N on the eigenstates $|J, J_z\rangle$. (so long as the hamiltonian is made up of these operators).

In this case we can see trivially from the way that the operators J_+ , J_- and J_z operate that their matrix representations in general ought to be, where we take the basis to be

$$\vec{J} = \begin{pmatrix} |J, J_z = J\rangle \\ |J, J_z = J-1\rangle \\ \vdots \\ |J, J_z = -J\rangle \end{pmatrix} \quad (82)$$

$$J_- = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ \sqrt{J(J+1) - (J-1)(J)} & 0 & 0 & \dots & 0 \\ 0 & \sqrt{J(J+1) - (J-2)(J-1)} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & \sqrt{J(J+1) - J(J-1)} & 0 \end{pmatrix} \quad (83)$$

Similarly for the J_- we have

$$J_+ = \begin{pmatrix} 0 & \sqrt{J(J+1) - J(J-1)} & 0 & \dots & 0 \\ 0 & 0 & \sqrt{J(J+1) - (J-1)(J-2)} & \dots & 0 \\ \vdots & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & 0 & \sqrt{J(J+1) - (J-1)(J)} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (84)$$

And for J_z we have trivially that

$$J_z = \begin{pmatrix} J & 0 & 0 & \dots & 0 \\ 0 & J-1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & -J+1 & 0 \\ 0 & 0 & 0 & 0 & -J \end{pmatrix} \quad (85)$$

Now finally we note that N returns the original state with an eigenvalue of how many particles are in the state. It's worth noting then that we can define J states for each of the possible numbers of particles $N=(0,1,2,3,4)$ which map to $J = (0, \frac{1}{2}, 1, \frac{3}{2}, 2)$

From this it immediately follows that

$$N = 2 \begin{pmatrix} J & 0 & 0 & \dots & 0 \\ 0 & J & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \dots & \vdots \\ 0 & 0 & 0 & J & 0 \\ 0 & 0 & 0 & 0 & J \end{pmatrix} \quad (86)$$

6 Hamiltonian

The Hamiltonian itself is thus easily constructed out of the J matrices by substitution, leading to:

$$H = H_0 + H_1 + H_2 = \epsilon \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -2 \end{pmatrix} + V \begin{pmatrix} 0 & 0 & 2.45 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 2.45 & 0 & 0 & 0 & 2.45 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2.45 & 0 & 0 \end{pmatrix} + W \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (87)$$

Hence we have that

$$H = \begin{pmatrix} 2\epsilon & 0 & 2.45V & 0 & 0 \\ 0 & 3W + \epsilon & 0 & 3V & 0 \\ 2.45V & 0 & 4W & 0 & 2.45V \\ 0 & 3V & 0 & 3W - \epsilon & 0 \\ 0 & 0 & 2.45V & 0 & -2\epsilon \end{pmatrix} \quad (88)$$

7 Eigenvalues/Eigenvectors of the Hamiltonian

Now we consider both the case of $\epsilon = 2, V = -\frac{1}{3}, W = -\frac{1}{4}$ and $\epsilon = 2, V = -\frac{4}{3}, W = -1$. This is done using the script attached in the appendix.

$$7.1 \quad \epsilon = 2, V = -\frac{1}{3}, W = -\frac{1}{4}$$

Here we have the matrix of eigenvectors is

$$\tilde{\lambda} = \begin{vmatrix} 0.99 & 0.16 & 0.03 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & -0.23 & -0.97 \\ -0.16 & 0.95 & 0.25 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & -0.97 & 0.23 \\ 0.02 & -0.25 & 0.97 & 0.00 & 0.00 \end{vmatrix} \quad (89)$$

These correspond to the eigenvalues

$$\lambda = (4.132 \quad -0.919 \quad -4.213 \quad -2.986 \quad 1.486) \quad (90)$$

We can see that in this case the ground state is that with energy -4.213, which is the state

$$\psi = 0.97 |2, -2\rangle + 0.25 |2, 0\rangle + 0.03 |2, 2\rangle \quad (91)$$

To describe why this is, we first note that in this case the H_0 and H_2 both act on the same eigenstates (which would be the unmixed J_m states). What's more the H_1 term that does not have the same eigenvectors can only increase or decrease the J_m states by 2, which explains why the eigenvectors for the total H matrix are easily divided into odd and even categories. In this case, the relative strength of the H_1 matrix is weak, and so the ground state is that where J_z is lowest, since the H_0 still dominates, and this strongly favors $J_z = -J$ states.

$$7.2 \quad \epsilon = 2, V = -\frac{4}{3}, W = -1$$

Now we consider the same for the situation where the relative importance of H_1 and H_2 are important. We can thus expect a change in the eigenvalues due to H_2 and a change in the eigenvectors (and eigenvalues) due to H_1 . As such we ought to expect far more mixing of the states. This is confirmed when we look at the matrix of eigenvectors

$$\tilde{\lambda} = \begin{vmatrix} -0.92 & -0.33 & 0.21 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & -0.53 & -0.85 \\ 0.37 & -0.57 & 0.74 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.00 & -0.85 & 0.53 \\ -0.13 & 0.76 & 0.64 & 0.00 & 0.00 \end{vmatrix} \quad (92)$$

Which correspond to eigenvalues

$$\lambda = (5.31 \quad -1.56 \quad -7.751 \quad -7.472 \quad 1.472) \quad (93)$$

Hence we see in this case that the ground state has energy -7.751 and corresponds to

$$\psi = 0.21 |2, 2\rangle + 0.74 |2, 0\rangle + 0.64 |2, -2\rangle \quad (94)$$

This is a substantial admixture of states, and in fact has less of the $J_z = -2$ state than the $J_z = 0$ state, but because of the mixing term is the ground state. We can see the importance of the mixing term by considering that the odd term

$$\psi = -0.53 |2, 1\rangle - 0.85 |2, -1\rangle \quad (95)$$

Has an energy of -7.472 which is very close to the ground state, and is clearly mostly dominated by the mixing term, though it also has less of the $J_z = 2$ term to factor in. That is, there is a competition between admixtures of even states that include both the high energy $|2, 2\rangle$ state and the $|2, -2\rangle$ state which also has significant mixing, and the odd states that have less of the H_0 term and so less of the high energy states from it, but also consequently have somewhat less mixing.

7.3 Pure H_2

For a final comparison I note that if $W = \epsilon = 0$ and $V = 1$ it follows that the matrix of eigenvectors is

$$\tilde{\lambda} = \begin{vmatrix} 0.50 & -0.71 & 0.50 & 0.00 & 0.00 \\ -0.00 & 0.00 & 0.00 & -0.71 & 0.71 \\ 0.71 & -0.00 & -0.71 & 0.00 & 0.00 \\ -0.00 & 0.00 & 0.00 & 0.71 & 0.71 \\ 0.50 & 0.71 & 0.50 & 0.00 & 0.00 \end{vmatrix} \quad (96)$$

With corresponding eigenvalues

$$\lambda = (-3.464 \quad 0 \quad 3.464 \quad 3 \quad -3) \quad (97)$$

We see that this again splits the eigenvectors into even and odd solutions, and indeed the odd solutions have lower absolute values for energy than the even cases, except that where there is no admixture of the $J_z = 0$ term which has a balanced energy of 0. This fits what was observed in the Hamiltonian before.

8 Unitary Transformation

We now note that we can make a new single particle state out of the lipkin model by summing over sum mixture of Lipkin model particles

$$|\phi_{\alpha,\sigma}\rangle = \sum_{\sigma=\pm 1} C_{\alpha\sigma} |\mu_{\sigma,p}\rangle \quad (98)$$

We note further that these states will be orthonormal if we assume that $\tilde{\mathbf{C}}$ is unitary. This is because it would then follow that

$$\langle \phi_{\alpha',p'} | \phi_{\alpha,p} \rangle = \left(\sum_{\sigma'=\pm 1} C_{\alpha'\sigma'}^* \langle \mu_{\sigma',p'} | \right) \left(\sum_{\sigma=\pm 1} C_{\alpha\sigma} |\mu_{\sigma,p}\rangle \right) \quad (99)$$

We realize that by the orthonormality of the Lipkin states this means that

$$\langle \phi_{\alpha',p'} | \phi_{\alpha,p} \rangle = \sum_{\sigma=\pm 1} C_{\alpha'\sigma}^* C_{\alpha\sigma} \langle \mu_{\sigma,p'} | \mu_{\sigma,p} \rangle = \sum_{\sigma=\pm 1} C_{\alpha'\sigma}^* C_{\alpha\sigma} \delta_{p,p'} \quad (100)$$

Now finally we note that because of the unitarity of the matrix that $C^\dagger C = 1$ and hence

$$\sum_a C_{ba}^\dagger C_{ca} = \delta_{bc} \quad (101)$$

From which it follows that

$$\langle \phi_{\alpha',p'} | \phi_{\alpha,p} \rangle = \delta_{p,p'} \delta_{\alpha,\alpha'} \quad (102)$$

And hence the new basis states are also orthonormal.

We note that in this transformation we have not changed p in constructing new particles. While in principle some mixing of p states could be considered, there is no need for this to create the relevant new particle states because the Hamiltonian is invariant under transformations that permute the p s. I.e. the p states are degenerate under H . This would not be possible if we added some $H'(p)$ term, in which case much of the machinery developed here would need to be changed substantially (or else treated as a perturbation to the previously considered states).

9 Calculate the new Slater Determinant

In this section we calculate the new Slater Determinant using the original creation and annihilation operators. We then have that

$$|\psi\rangle = \prod_p \left(C_{\alpha+} a_{p+}^\dagger + C_{\alpha-} a_{p-}^\dagger \right) |0\rangle \quad (103)$$

We note that

$$H_0 = \frac{1}{2}\epsilon \sum_{\sigma,p} \sigma a_{\sigma,p}^\dagger a_{\sigma,p} \quad (104)$$

So we have that

$$\langle \psi | H_0 | \psi \rangle = \langle 0 | \prod_p (C_{\alpha+}^* a_{p+} + C_{\alpha-}^* a_{p-}) \frac{1}{2}\epsilon \sum_{\sigma,p} \sigma a_{\sigma,p}^\dagger a_{\sigma,p} \prod_p (C_{\alpha+} a_{p+}^\dagger + C_{\alpha-} a_{p-}^\dagger) | 0 \rangle \quad (105)$$

Let us now consider an individual term in the sum. We thus have, letting i be the index of the chosen term and σ the spin state:

$$\langle \psi | H_0 | \psi \rangle = \langle 0 | \prod_{p \neq i} (C_{\alpha+}^* a_{p+} + C_{\alpha-}^* a_{p-}) (C_{\alpha+}^* a_{i+} + C_{\alpha-}^* a_{i-}) \frac{1}{2}\epsilon \sigma a_{\sigma,i}^\dagger a_{\sigma,i} (C_{\alpha+} a_{i+}^\dagger + C_{\alpha-} a_{i-}^\dagger) \prod_{p \neq i} (C_{\alpha+} a_{p+}^\dagger + C_{\alpha-} a_{p-}^\dagger) | 0 \rangle \quad (106)$$

Now we can immediately note that for the terms not equal to i we wind up with their products multiplying together, and that any term where the annihilation operator does not meet the creation operator will be destroyed. As such we must have

$$\prod_p (C_{\alpha+}^* C_{\alpha+} a_{p+} a_{p+}^\dagger + C_{\alpha-}^* C_{\alpha-} a_{p-} a_{p-}^\dagger) \quad (107)$$

But from the unitarity it follows that, if the terms are real

$$C_{\alpha+}^2 + C_{\alpha-}^2 = 1 \quad (108)$$

And so this term simply becomes 1. We are thus left only with

$$\langle \psi | H'_0 | \psi \rangle = \langle 0 | (C_{\alpha+}^* a_{i+} + C_{\alpha-}^* a_{i-}) \frac{1}{2}\epsilon \sigma a_{\sigma,i}^\dagger a_{\sigma,i} (C_{\alpha+} a_{i+}^\dagger + C_{\alpha-} a_{i-}^\dagger) | 0 \rangle \quad (109)$$

We note here that the result will depend on σ . For $\sigma = +$ the minus terms will commute and so will annihilate unless matched. But in this case it would lead to the annihilation operator from the hamiltonian acting on the zero state, so the minus terms do not contribute.

$$\langle \psi | H'_0 | \psi \rangle = \langle 0 | C_{\alpha+}^* a_{i+} + \frac{1}{2}\epsilon a_{\sigma,i}^\dagger a_{\sigma,i} C_{\alpha+} a_{i+}^\dagger | 0 \rangle \quad (110)$$

But this is easily seen to just be equal to

$$\langle \psi | H'_0 | \psi \rangle = \frac{1}{2} \langle 0 | C_{\alpha+}^* \epsilon C_{\alpha+} | 0 \rangle = \frac{\epsilon}{2} |C_{\alpha+}|^2 \quad (111)$$

Now we see that adding the minus term in we will have

$$\langle \psi | H'_0 | \psi \rangle = \frac{\epsilon}{2} (|C_{\alpha+}|^2 - |C_{\alpha-}|^2) \quad (112)$$

Finally we see that adding each of the four terms

$$\langle \psi | H_0 | \psi \rangle = 2\epsilon (|C_{\alpha+}|^2 - |C_{\alpha-}|^2) \quad (113)$$

Next we consider

$$H_1 = \frac{1}{2} V \sum_{\sigma, p, p'} a_{\sigma, p}^\dagger a_{\sigma, p'}^\dagger a_{-\sigma, p'} a_{-\sigma, p} \quad (114)$$

As before we consider

$$\langle \psi | H_0 | \psi \rangle = \langle 0 | \prod_p (C_{\alpha+}^* a_{p+} + C_{\alpha-}^* a_{p-}) \frac{1}{2} V \sum_{\sigma, p, p'} a_{\sigma, p}^\dagger a_{\sigma, p'}^\dagger a_{-\sigma, p'} a_{-\sigma, p} \prod_p (C_{\alpha+} a_{p+}^\dagger + C_{\alpha-} a_{p-}^\dagger) | 0 \rangle \quad (115)$$

And again we look at a single value of p and p'. In this case, as before, any terms not involving p or p' will multiply out to 1, and so we have for one term of p, p' and σ

$$\langle \psi | H_0 | \psi \rangle = \langle 0 | \prod_{p, p'} (C_{\alpha+}^* a_{p+} + C_{\alpha-}^* a_{p-}) \frac{1}{2} V a_{\sigma, p}^\dagger a_{\sigma, p'}^\dagger a_{-\sigma, p'} a_{-\sigma, p} \prod_{p, p'} (C_{\alpha+} a_{p+}^\dagger + C_{\alpha-} a_{p-}^\dagger) | 0 \rangle \quad (116)$$

Noting that if p is equal to p' the two annihilation multipliers in the hamiltonian will lead to annihilation of the state vector and so we know that $p \neq p'$. We then recognize that since the two annihilation parameters have the same spin, it follows that only the annihilation parameters and the coefficients to the power of two will have an effect. This happens exactly the same on the bra side by the symmetry of the problem, except that the spins are different. There are two for each because we can pick the positive spin on the ket or bra side. We would thereby have

$$\langle \psi | H'_1 | \psi \rangle = V \sum_{\sigma} \langle 0 | C_{\alpha\sigma}^2 C_{\alpha-\sigma}^2 a_p a_{p'} a_p^\dagger a_{p'}^\dagger a_p a_{p'} a_p^\dagger a_{p'}^\dagger | 0 \rangle = 2V C_{\alpha-}^2 C_{\alpha+}^2 \quad (117)$$

Finally we note that there is such term as this for each combination of p, p'. There are $\binom{4}{2} = 6$ and so

$$\langle \psi | H_1 | \psi \rangle = 12V C_{\alpha+}^2 C_{\alpha-}^2 \quad (118)$$

(Again assuming real unitary matrix coefficients.)

Finally we consider the

$$H_2 = \frac{W}{2} \sum_{\sigma, p, p'} a_{\sigma, p}^\dagger a_{-\sigma, p'}^\dagger a_{\sigma, p'} a_{-\sigma, p} \quad (119)$$

However, we note immediately that the same arguments as before hold, that p must not equal p' . From that it follows that the operators will commute into the form of H_1 and as such we can immediately have that

$$\langle \psi | H_2 | \psi \rangle = 12W C_{\alpha+}^2 C_{\alpha-}^2 \quad (120)$$

From which finally we have that

$$\langle \psi | H | \psi \rangle = 2\epsilon (C_{\alpha+}^2 - C_{\alpha-}^2) + 12(W + V) C_{\alpha+}^2 C_{\alpha-}^2 \quad (121)$$

10 Energy minimization

In order to minimize the energy we need to take the derivative with respect to $C_{\alpha+}$ or $C_{\alpha-}$. From this it follows that we can substitute $C_{\alpha-}^2 = 1 - C_{\alpha+}^2$.

As such we have

$$E = 2\epsilon (1 + 2C_{\alpha+}^2) + 12(W + V) (C_{\alpha+}^2 - C_{\alpha+}^4) \quad (122)$$

We then have that

$$8\epsilon C_{\alpha+} + 12(W + V) (2C_{\alpha+} - 4C_{\alpha+}^3) \quad (123)$$

Then setting that to zero we have

$$\epsilon = 3(W + V) (-1 + 2C_{\alpha+}) \quad (124)$$

Finally because this takes its minimum value when $C_{\alpha+} = 0$ it follows that

$$\frac{\epsilon}{3} \geq (V + W) \quad (125)$$

11 Hartree Fock

In the two programs(one a script, one a set of class definitions) in the github, I have created a program to do the Hatree fock approximation to the solutions. In doing so I plug in the hamiltonians with the values from before with an initial C matrix that has all values as equally likely.. First I consider $\epsilon = 2, V = \frac{-1}{3}, W = -\frac{1}{4}$

This results in a ground state energy of -4.23 , which is very close to the exact value of -4.213.

We then check the
 $\epsilon = 2, V = \frac{-4}{3}, W = -1$

This however leads to a value of -16.06 , which is much lower than the actual ground state of -7.751 . This is likely because in this case the original assumption of H0 being dominant with the other terms being perturbations is less valid.