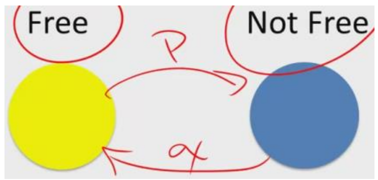
Session 10: Markov Processes

10.1 – Markov Models

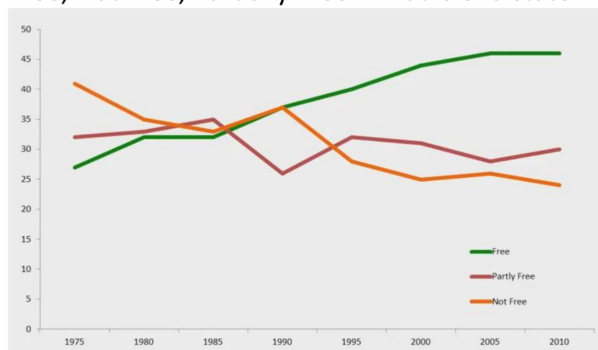
Idea: ***States*** and Entities move between those states according to ***transition probabilities.***

Markov Convergence Theorem:

- Finite # of states
- Fixed Transition Probabilities
- Any state is accessible.
- Markov is an example of a stable system type.



Future of Country as a Markov Model:
Free, Not Free, Partially Free. What is end state?



Use matrices to represent transition probabilities

0.5	0.3
0.3	04

Reason for Markov Model:

1. Stable System
2. Exaptation (model is fertile) Applicable to a large number of contexts

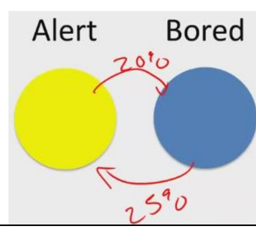
Quiz: Which of the following assumptions is required for the Markov Model to hold? 9a) Finite number of states, (b) Ability to get from any state to any other, (c) Fixed transition probabilities, (d) All of the above
Ans: (d) All of the above
Explanation: *The Markov Model assumes three principle things: There is a finite number of states. Things can move from any state to any other state. There are fixed transition probabilities.*

10.2 – Simple Markov Model

States: Entities move between the states according to transition probabilities.
Example: Alert and Bored students

- 20% alert students become Bored
- 25% of Bored Students become Alert

$_{AB} = 0.2,\ \ _{BA} = 0.25$
$_{AA} = 0.8,\ \ _{BB} = 0.75$



Scenario #1: 100 Alert

AB\State	1	2	3
A	100	80	64+5=69
B	0	20	16+15=31

4	5	6
69		
31		

Markov Transition Matrix:

	A_t	B_t
A_{t+1}	0.8	0.25
B_{t+1}	0.2	0.75

$$\begin{bmatrix} 0.8 & 0.25 \\ 0.2 & 0.75 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} (0.8)\cdot 1 + (0.25)\cdot 0 \\ (0.2)\cdot 1 + (0.75)\cdot 0 \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & 0.25 \\ 0.2 & 0.75 \end{bmatrix} \times \begin{bmatrix} 0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 0.8\cdot(0.8) + 0.25\cdot(0.2) \\ 0.2\cdot(0.8) + 0.75\cdot(0.2) \end{bmatrix} \begin{matrix} .69 \\ .31 \end{matrix}$$

Markov Transition Matrix (cont.):

$$\begin{bmatrix} 0.8 & 0.25 \\ 0.2 & 0.75 \end{bmatrix} \times \begin{bmatrix} .69 \\ .31 \end{bmatrix} = \begin{bmatrix} 0.8\cdot(.69) + 0.25\cdot(.31) \\ 0.2\cdot(.69) + 0.75\cdot(.31) \end{bmatrix}$$

Starting with all Alert students after six periods, 58% of the students were Alert.

Form of Transition Matrix:

$$\begin{bmatrix} _{AA} & _{BA} \\ _{AB} & _{BB} \end{bmatrix} \begin{bmatrix} p \\ 1-p \end{bmatrix} = \begin{bmatrix} p \\ 1-p \end{bmatrix}$$

Scenario #2: 100 Bored students

$$\begin{bmatrix} 0.8 & 0.25 \\ 0.2 & 0.75 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} (0.8)\cdot 0 + (0.25)\cdot 1 \\ (0.2)\cdot 0 + (0.75)\cdot 1 \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & 0.25 \\ 0.2 & 0.75 \end{bmatrix} \times \begin{bmatrix} .25 \\ .75 \end{bmatrix} = \begin{bmatrix} .45 \\ .55 \end{bmatrix}$$

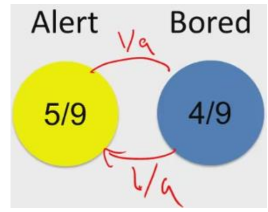
$$\begin{bmatrix} 0.8 & 0.25 \\ 0.2 & 0.75 \end{bmatrix} \times \begin{bmatrix} .45 \\ .55 \end{bmatrix} = \begin{bmatrix} .5 \\ .5 \end{bmatrix}$$

$$\begin{bmatrix} 0.8 & 0.25 \\ 0.2 & 0.75 \end{bmatrix} \times \begin{bmatrix} .5 \\ .5 \end{bmatrix} = \begin{bmatrix} .525 \\ .475 \end{bmatrix}$$

Equilibrium?:

$$\begin{bmatrix} 0.8 & 0.25 \\ 0.2 & 0.75 \end{bmatrix} \times \begin{bmatrix} p \\ (1-p) \end{bmatrix} = \begin{bmatrix} p \\ (1-p) \end{bmatrix}$$

$0.8P^* + 0.25(1 - \ ^*) = \ ^*$
$16P^* + 5(1 - \ ^*) = 20P^*$
$16P^* + 5 - 5P^* = 20P^*$
$^* = \frac{5}{9}$



Equilibrium Point:
Nothing changes
Statistical Equilibrium:
World keeps churning but the distribution of types stays the same.

Quiz: Every year, 25% of people with standard phones switch to smart phones and 5% of people with smart phones switch to standard phones. In equilibrium, which percentage of people will have STANDARD phones?
(a) 20.00%, (b) 16.67%, (c) 4.00%, (d) 15.60%

Solution: A=Standard Phone, B=Smart Phone, P^* is probability in steady state of those with a Standard Phone, and $(1 - P^*)$ is the probability that someone has a Smart Phone. Transition Matrix (note columns sum to 1). P_{AB} is the probability of transitioning between state A to state B. P_{AA} is the probability of remaining in state A.

$\begin{bmatrix} AA & BA \\ AB & BB \end{bmatrix} = \begin{bmatrix} 0.75 & 0.05 \\ 0.25 & 0.95 \end{bmatrix}$ so that $\begin{bmatrix} AA & BA \\ AB & BB \end{bmatrix}\begin{bmatrix} p_A \\ 1-p_A \end{bmatrix} = \begin{bmatrix} p_A \\ 1-p_A \end{bmatrix}$ and $\begin{bmatrix} 0.75 & 0.05 \\ 0.25 & 0.95 \end{bmatrix}\begin{bmatrix} p_A \\ 1-p_A \end{bmatrix} = \begin{bmatrix} p_A \\ 1-p_A \end{bmatrix}$

$0.75\ ^* + 0.05(1 - \ ^*) = \ ^*$ ➔ $15P^* + (1 - \ ^*) = 20P^*$ ➔ $1 = 6P^*$ ➔ $^* = \frac{1}{6} = 16.67\%$
Ans: (b) 16.67%

Explanation: *First, set up the Markov matrix. Next, solve: Percentage of people with SMART phones (p): p=.95p+.25(1−p), so p=0.833333... Percentage of people with STANDARD phones: 1−0.83333...=.1666... About 16.67% people will have standard phones in equilibrium.*

10.3 – Markov Model of Democratization

Country Democracy Model:
Each decade 5% of democracies become ditatorships and 20% of dictatorships become democracies.
Example: Start with 30% democracies (p_A) 70% not democracies ($p_B = 1 - p_A$).

State 1: $\begin{bmatrix}0.95 & 0.2\\0.05 & 0.8\end{bmatrix}\begin{bmatrix}0.3\\0.7\end{bmatrix} = \begin{bmatrix}0.425\\0.575\end{bmatrix}$

State 2: $\begin{bmatrix}0.95 & 0.2\\0.05 & 0.8\end{bmatrix}\begin{bmatrix}0.425\\0.575\end{bmatrix} = \begin{bmatrix}0.52\\0.48\end{bmatrix}$

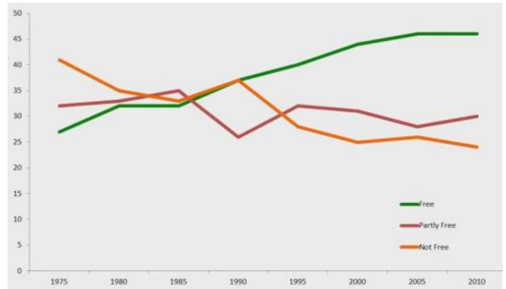
State Transition: $\begin{bmatrix}0.95 & 0.2\\0.05 & 0.8\end{bmatrix}\begin{bmatrix}p_A\\1-p_A\end{bmatrix} = \begin{bmatrix}p_A\\1-p_A\end{bmatrix}$

$0.95P^* + 0.2(1 - \quad ^*) = \quad ^*$ ➔
$95P^* + 20(1 - \quad ^*) = 100P^*$ ➔
$20 = 25P^*$ ➔ $^* = \frac{4}{5} = 80\%$

Democracies stabilize at 80% of countries.

Question/Observation *for 3x3 matrix to the right. Shouldn't each column add to 1.0 since each is the sum of possible transitions for the respective node in the model? That is $P_{F,F} = 0.95$ instead of 0? This makes all three equations consistent.*

Three Type Country Model:
Free, **Partly Free**, **Not Free**



Each decade:
Free to Partly Free = 5%
Not Free to Partly Free = 15%
Not Free to Free = 5%
Partly Free to Free = 10%
Partly Free to Not Free = 10%

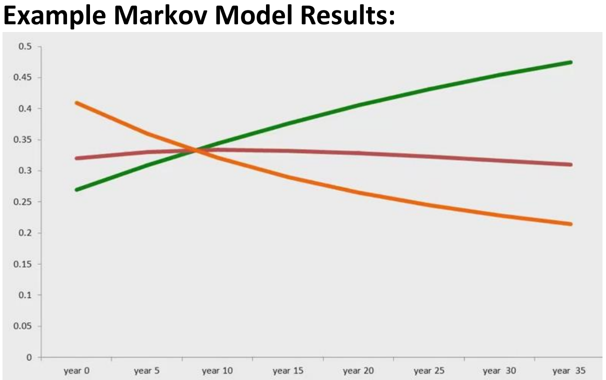
	F	PF	NF
F	0.9	0.1	.05
PF	.05	0.8	.15
NF	0	0.1	0.8

$$\begin{bmatrix}0.95 & 0.1 & 0.05\\0.05 & 0.8 & 0.15\\0 & 0.1 & 0.8\end{bmatrix}\begin{bmatrix}p\\q\\1-p-q\end{bmatrix} = \begin{bmatrix}p\\q\\1-p-q\end{bmatrix}$$

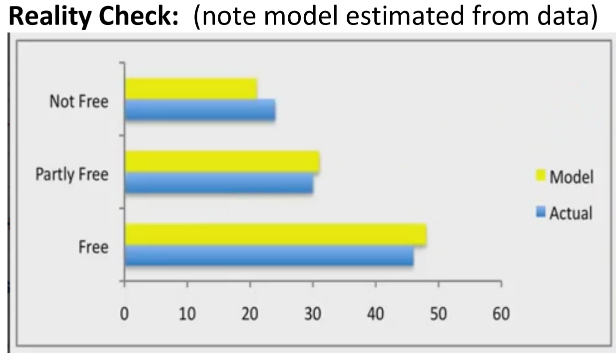
$(0.95)p + (0.1)q + .05(1 - p - q) = p$
$(0.05)p + (0.8)q + (0.15)(1 - p - q) = q$
$(0.0)p + (0.1)q + (0.8)(1 - p - q) = 1 - p - q$

Result: Free = 62.5%, Partly Free = 25.0%, Not Free = 12.5%

Example Markov Model Results:



Reality Check: (note model estimated from data)



Quiz: Let's assume there are three possible states of economies: Solvent, Brink (on the brink of bankruptcy) and Bankrupt. The transition probabilities are: <u>85% of Solvent stay Solvent</u>; <u>15% of Solvent become Brink</u>; <u>0% of Solvent become Bankrupt</u>. <u>40% of Brink become Solvent; 35% of Brink stay Brink</u>; <u>25% Brink go Bankrupt</u>. <u>0% of Bankrupt become Solvent</u>; <u>70% of Bankrupt become Brink</u>; <u>30% of Bankrupt stay Bankrupt</u>. In equilibrium, what percentage of economies are Solvent? (a) 64%, (b) 75%, (c) 45%, (d) 90%

Analysis: let A=Solvent, B=Brink, C=Bankrupt

$\begin{bmatrix}_{AA} & _{BA} & _{CA}\\P_{AB} & _{BB} & _{CB}\\_{AC} & _{BC} & _{CC}\end{bmatrix}\begin{bmatrix}p\\q\\1-p-q\end{bmatrix} = \begin{bmatrix}p\\q\\1-p-q\end{bmatrix}$ so that $\begin{bmatrix}0.85 & 0.4 & 0.0\\0.15 & 0.35 & 0.7\\0.0 & 0.25 & 0.3\end{bmatrix}\begin{bmatrix}p\\q\\1-p-q\end{bmatrix} = \begin{bmatrix}p\\q\\1-p-q\end{bmatrix}$
and

$(0.85)p + (0.4)q + 0.0(1 - p - q) = p$
$(0.85)p + (0.4)q = p$
$(0.4)q = (0.15)p$
$q = \left(\frac{0.15}{0.4}\right)p = \frac{3}{8}p = 0.375p$
$q = 0.375p = 0.375(0.6627) = 0.2485 = 24.9\%$
$1 - p - q = 0.8877 = 8.9\%$

$(0.15)p + (0.35)q + (0.7)(1 - p - q) = q$
$3p + 7q + 14(1 - p - q) = 20q$
$14 - 11p - 7q = 20q$
$14 - 11p = 27q = 27(0.375)p = 10.125p$
$14 = 21.125p$
$p = \frac{14}{21.125} = 0.6627 = 66.3\%$

Ans: (a) 64%

Explanation: *This is a 3x3 Markov matrix. We need to solve two equations for two unknowns (p=percentage solvent; q=percentage brink; 1-p-q=percentage bankrupt).*
0.85p+0.4q+0(1−p−q)=p; and 0p+0.25q+0.3(1−p−q)=1−p−q.
You should get (8/3)q=p from the first equation. Plug this result into the second equation. You should get q=0.24
Substitute that back into (8/3)q=p to get p=0.64 and 1−p−q=0.12.
The question asks for percentage solvent, which is p=0.64, or 64%.

Note on stochastic matrix for Markov Chain – Wikipedia
In mathematics, a stochastic matrix (also termed probability matrix, transition matrix, substitution matrix, or Markov matrix) is a matrix used to describe the transitions of a Markov chain. Each of its entries is a nonnegative real number representing a probability. A left stochastic matrix is a square matrix of nonnegative real numbers, with each column summing to 1.

10.4 – Markov Convergence Theorem

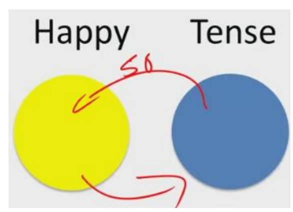
Markov Statistical Equilibrium Conditions:

1. Finite **States**
2. Fixed **Transition probabilities**
3. Can eventually get from any one state to any other state
4. Not a **simple cycle.**

Markov Convergence Theorem:
Given the four equilibrium conditions, a Markov process converges to an equilibrium distribution which is **unique.**
That is, there is only one solution for the following Markov process. $\begin{bmatrix} .9 & .1 & 0 \\ .1 & .6 & .2 \\ 0 & .3 & .8 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$

Convergence Facts:
- **Initial State** Doesn't Matter (save for how many steps it takes to converge?)
- **History** doesn't Matter
- **Intervening** to change the state doesn't matter (note implication in international policy such as fostering democratic states)

Relationship Example:
Moving a bunch of people from Tense to Happy will not change the end convergence state. ***As long as the transition probabilities stay fixed***, the system **will converge**.



Interventions and Policy:
- Does this mean that interventions have no effect?
- Does this mean that we should not have redistribution policies?
- It could take a long time to reach the equilibrium – the intervening years could be worth it.
- **Key Assumption*: transition probabilities stay fixed***. Given that people adapt, this is the weakest assumption in the policy arena.

Observations:
- Changing State: temporary effect
- Changing Transition Probabilities: permanent effect

Transition Probabilities and Tipping Points:
- Tipping points will come from changing transition probabilities
- The convergence state will change with a change in transition probabilities.

Quiz: The number of Alert and Bored students in a classroom is a Markov process with the following transition probabilities: 80% of Alert stay Alert; 20% of Alert become Bored. 25% of Bored become Alert; 75% of Bored stay Bored. The equilibrium percentage of Alert students in this scenario is about 55.56%. The professor would like to have the equilibrium percentage of Alert students be 80%. Assume she can change her lecture style in order to change the transition probabilities. Which of the following sets of transition probabilities will allow her to reach her goal of 80% Alert in Equilibrium? (a) 85% of Alert stay Alert (15% become Bored); 40% of Bored become Alert (60% stay Bored). (b) 90% of Alert stay Alert (10% become Bored); 40% of Bored become Alert (60% stay Bored). (c) 90% of Alert stay Alert (10% become Bored); 30% of Bored become Alert (70% stay Bored). (d) It will never be possible, even with changing transition probabilities, to achieve 80% of students Alert in equilibrium.

Analysis: $\begin{bmatrix} _{AA} & _{BA} \\ _{AB} & _{BB} \end{bmatrix}\begin{bmatrix} p_A \\ 1-p_A \end{bmatrix} = \begin{bmatrix} p_A \\ 1-p_A \end{bmatrix}$ so $\begin{bmatrix} .8 & .25 \\ .2 & .75 \end{bmatrix}\begin{bmatrix} p_A \\ 1-p_A \end{bmatrix} = \begin{bmatrix} p_A \\ 1-p_A \end{bmatrix}$ and

$0.8\ ^* + 0.25(1 - \ ^*) = \ ^*$ ➔ $16P^* + 5(1 - \ ^*) = 20P^*$ ➔$5 = 9P^*$ ➔ $^* = \frac{5}{9} = 55.56\%$ which is given.

Compare $\begin{bmatrix} x & y \\ 1-x & 1-y \end{bmatrix}\begin{bmatrix} ^* \\ 1- \ ^* \end{bmatrix} = \begin{bmatrix} ^* \\ 1- \ ^* \end{bmatrix}$ where $^* = 0.8$ so we have

(a) $xP^* + y(1 - \ ^*) = \ ^*$ ➔ $0.8x + 0.2y = 0.8$ ➔$4x + y = 4$ ➔ $4(.85) + 0.4 = 3.8 \neq 4$ ➔ does not satisfy
(b) $4x + y = 4$ ➔$4(0.9) + 0.4 = 4.0$ ➔ does satisfy Markov equilibrium
(c) $4x + y = 4$ ➔ $4(0.9) + 0.3 = 3.9 \neq 4$ ➔ does not satisfy Markov equilibrium

Ans: (b) $_{AA} = 0.9$ and $_{BA} = 0.4$

Explanation: *Now that you are given p=.8 (and 1−p=.2), you know the percentages you're looking for in equilibrium. You should just set up and solve the transition matrices for the options given and see which set 'works'.*

Summary:
- **In a world where the Markov assumptions are true - history doesn't matter, inventions don't matter, initial conditions don't matter.**
- In the real world, finite states are likely to be true. Interventions are likely to be only temporary unless state transition probabilities change or perhaps not all states can be reached from any state.
- Tipping points are likely when transition probabilities change. Most likely invalid assumption is that transition probabilities can change, and interventions that really matter, interventions that tip, histories that matter, are events that change those transition probabilities. Not everything is a Markov process. What we see is that this Markov model helps us understand why are some results inevitable, because they satisfy those assumptions, and why are some results not.

10.5 – Exapting the Markov Model

Exapting Topics.
- Applying Markov model to other things
- Applying the Markov Transition Matrix

Recall States: Entities move between those states according to **transition probabilities. Markov** model convergence means initial condition don't matter, history doesn't matter.

exaptation *(gzp-tshn), n. Biology*
The utilization of a structure or feature for a function other than that for which it was developed through natural selection.

Applying Markov Model to other Things:
- **Voter Turnout**

	V_t	N_t
V_{t+1}	.8	.4
V_{t+1}	.2	.6

- **School Enrollment**

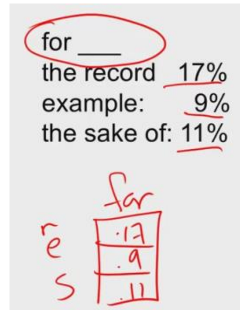
	Go	NO
G_{t+1} G	.9	.5
N	.1	.5

Markov Transition Matrix Insights:

	A(t)	B(t)
A(t+1)	0.8	0.25
B(t+1)	0.2	0.75

Applications:

- **Identify Writers** - how they tend to use words and phrases. Then compare the transition matrices (TMs) for different likely authors.
- **Arlene Saxonhouse** - Found potential Thomas Hobbs essays. How to prove? Used TMs to show 3 of 4 likely by Hobbs.



Applications:

- **Medical Diagnosis –** Can use TMs. Consider the following state-to-state paths, one leading to success, the other to failed diagnosis and treatment.
  Success: pain, depression, pain, success
  Failure: depression, mild pain, no pain, failure

- **Transition matrix:** Used to help determine which is the most likely path the patient is following.

Application:

- **Lead Up To War** – two countries with some tension between them. This leads to political statements (PS) followed by changes in trade embargos (TE) followed by military buildup.
  PS ➔ TE ➔ MB

- **Historically** – when these three events occur in sequence what is the probability of war or not.

Summary (quoted from lecture video):

- Don't have to use the full power of the Markov model. These transition probabilities do not necessarily stay fixed. Don't have to focus on solving for the equilibrium. Instead, can use this matrix to organize the data to think more clearly about what's likely to happen.
- Markov process is a fixed set of states, fixed transition probabilities. You can get from any one state to any other, and then you get an equilbrium. So that equilibrium doesn't depend on where you start, it doesn't depend on interventions, and it doesn't depend on history in any way.
- The model is really powerful. And so if you want to argue history matters. Or if you want to argue interventions matter. If someone is going to argue that this isn't a transition, that this isn't a Markov process. Or that you've got to argue that you're changing the transition probabilities. Now that isn't impossible. And in fact, policies that really make a difference, interventions that really make a difference, do change the transition probabilities.
- We don't even need to use the full Markov process model, just the transition probabilities. Just that idea, the matrix of transition probabilities, and we can find out all sorts of interesting things, like who wrote a book? Is there likely to be more? Or is this medical treatment working? So that framework, the transition probability framework and that matrix of transition probabilities is a really powerful tool to keep in your pocket when you confront some sort of dynamic process and you're trying to figure out, what do I think is likely to happen.