Session 7: Tipping Points

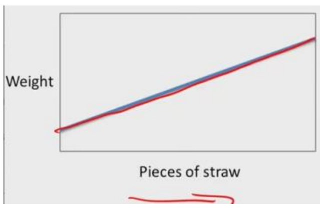
7.1 – Introduction to Linear Models

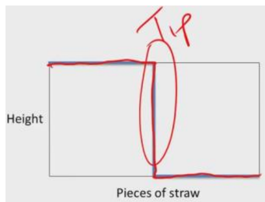
Overview of Tipping Points

- **Basic Idea:** A small change in input makes a very large change in output as seen in the graph. Note: Time charts can be misleading.

Examples:

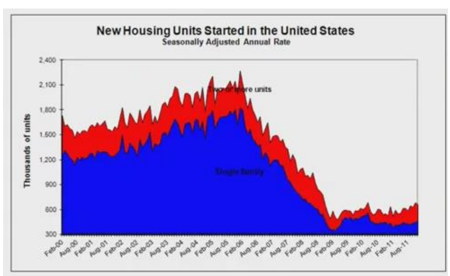
- **Straw that broke a camel's back**



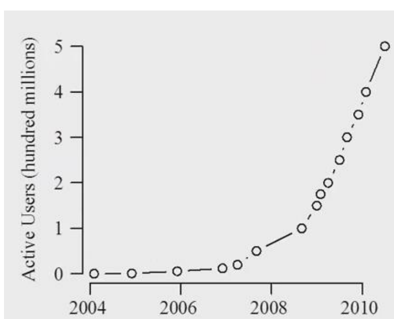


Weight of straw is linear, but at some point, just one more straw breaks the camel's back.

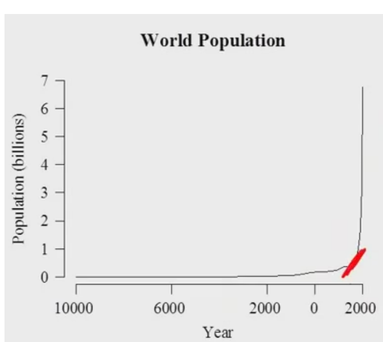
- **New Housing Units Started in US (TP)**



- **Facebook Active Users** (not TP: x)

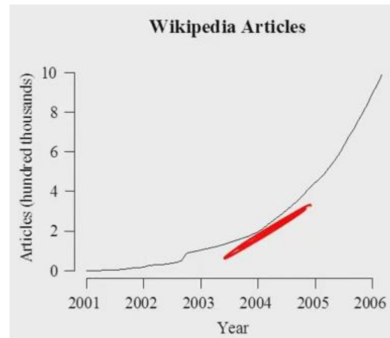


- **World Population** (not TP: x)



Wikipedia Articles

(not TP: exponential ➔ x)



Finding Tips:

- **Two Models:**
  (a) Percolation Model (e.g., water)

  (b) SIS Model (Susceptible Infected Susceptible)

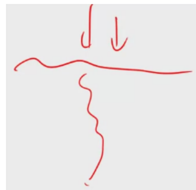
- **Types of Tips**
  (a) Direct Tips – Action tips, e.g. a battle in a war may mark a turning point in the war. A variable (action in the model) changes and causes an abrupt change in that same variable.

  (b) Contextual Tips – Something in the environment changes that makes it possible to change from one state to another.

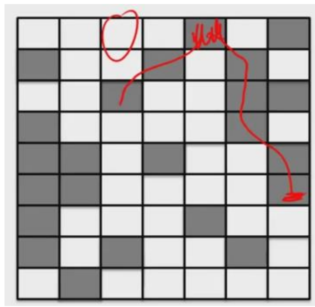
  (c) Classification of Tips – Changes between types of model behavior: i.e., ***Stable***, ***Periodic***, ***Random***, ***Complex***; OR <u>between equilibrium points</u> within one of the four classes of model behavior.

7.2 – Percolation Models

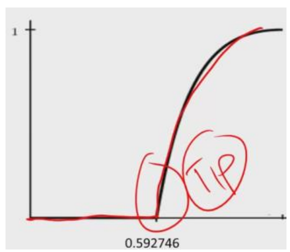
Percolation Concept:



- **Like Rain Soaking into Ground**
- **Model Structure**
  Simple rule for each grid. Can only move to an adjacent dark square as shown in red. Objective is to get to other side of the board.



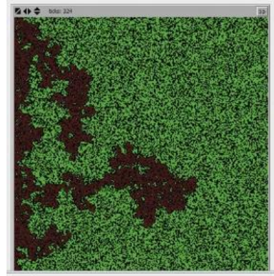
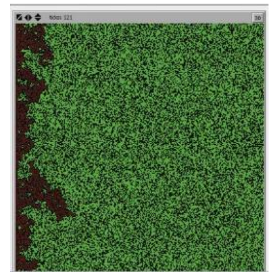
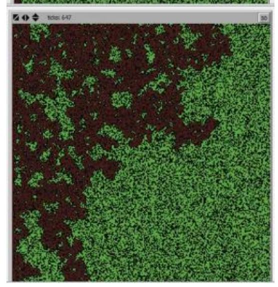
- **Model Math**
  Let P = probability a square is filled in (dark). Determine each square color with a flip of an unfair coin with $P(Heads) = P_H$.



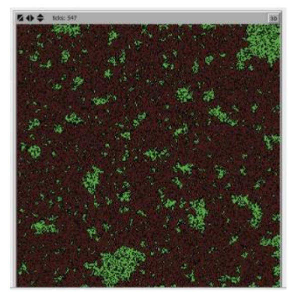
  Note the probability of finding a path to the opposite side is zero until $P_H = 0.592746$ where a Tip occurs.

Netlogo: Forest Fires and Percolation

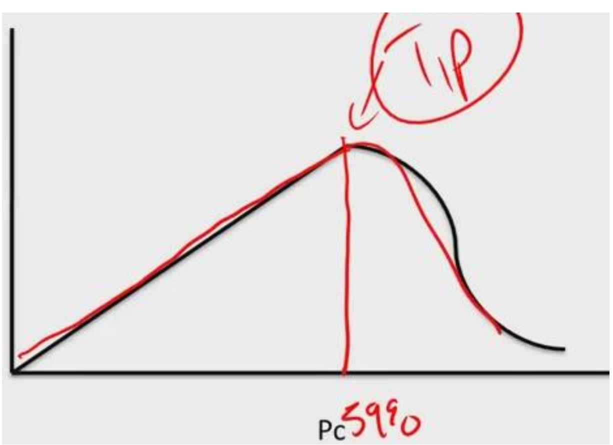
- **Density @ 57% ➔ 19% - 7.4% - 34.7% burned**

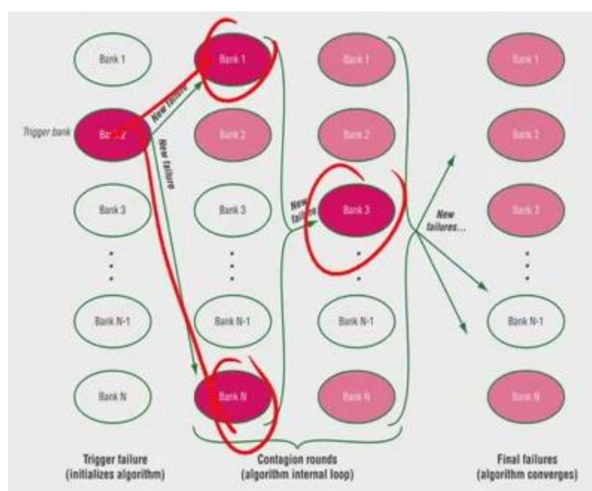
- **Density @ 61%:** above Tip Density ➔ largely burned, e.g., 84.2%
- **Density above 59.3% -** enables percolation to the other side.



Forest Logging Yield Curve:
Suggests above 59% clear cutting, the yield would fall off. Not sure what details are being glossed over here. Example of fertility of models.



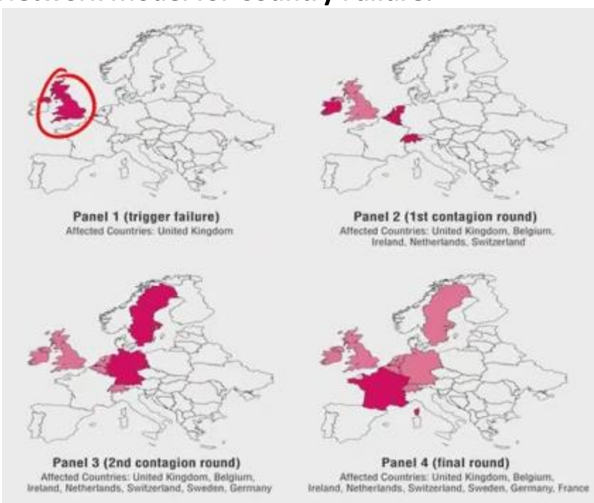
Bank Percolation: Bank Failure Propagation



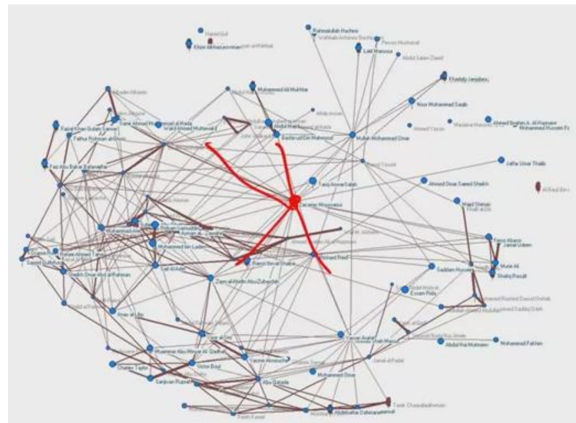
Complex model of assets, obligations, and relationships that can be investigated for tipping point behavior. Percolation model may or may not apply depending upon the details.

Quiz: Suppose that banks become more interconnected. Also suppose they increase the frequency of loans to other banks. Based on the percolation model, should these actions make the banking system more likely to have a massive failure, or less likely? (a) More Likely or (b) Less Likely
Ans: (a) more likely
Explanation: More likely. A failure from one bank would spread to more banks, and those banks would be more likely to fail because they have more loans out. Keep in mind, though, that this is only one model.

Network Model for Country Failure:



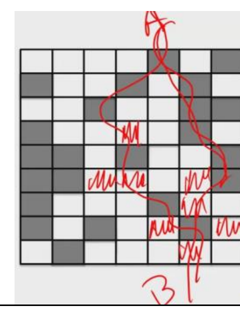
Information Percolation:
What is probability information will propagate across a social network? Interest as a metric.



Information Percolation:
Tipping Point likely depending upon value to network participants.

Fertility: Proofs and Innovations
Bursts of Scientific Creativity or Invention.

Percolation applies as critical parts of knowledge base are filled in.



Summary: Simple checker board concept of percolation phenomena as related to connections (also applies to graphical models) can be used to understand why some variable changes can lead to tipping phenomena.

7.3 – Contagion Models 1 - Diffusion

SIS Model (Susceptible Infected Susceptible)
aka SIR Model (Susceptible Infected Recovered)

- **Basic Reproduction Number**
  > 1 ➔ everyone gets the disease
  < 1 ➔ not everyone gets the disease
- **Diffusion Model Applies**
  (a) in pure diffusion model everyone gets it. Recall in this one once a square is colored, it never returns to original color.
  (b) Starting point for model.

Diffusion Model

- $W_t = \#\, Wobblies\, @\ time\ t$
  $(N - W_t = \#without\ Wobblies)$
  $N = \#\ of\ people\ in\ society$
  $\tau = transmission\ rate$
- Two people meet: What is probability one gives Wobblies to the other? Only one case applies, one has Wobblies and the other does not. So
  $P(New\ Infection\ if\ 2\ meet) = \tau\left(\frac{W_t}{N}\right)\left(\frac{N - W_t}{N}\right)$

$infection\ rate * wobblie\ fr\ \ ction * non - Wobblie\ fr\ \ ction$

Add Contact rate: c

$P(infections\ @t) = W_t = \tau(Nc)\left(\frac{W_t}{N}\right)\left(\frac{N-W_t}{N}\right)$
$= \ (\#meetings) * (rate) * (has\ W) * (not\ have\ W)$

State Transition Eq.

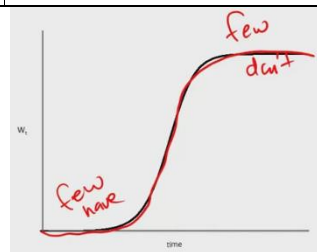
$$W_{t+1} = W_t + \tau(Nc)\left(\frac{W_t}{N}\right)\left(\frac{N-W_t}{N}\right)$$

Infections over Time:

Note: When Wt is 1, $\frac{1}{N}\left(\frac{N-1}{N}\right) \sim \frac{1}{N}$ and

$W_{t+1} = W_t + \tau(Nc)\frac{1}{N} = W_t + \tau c$

And at 50% infected: $\frac{\frac{N}{2}}{N}\left(\frac{\frac{N}{2}-1}{N}\right) \sim \frac{1}{4}$



Quiz: In a school with 800 students, a rumor spreads through a diffusion process. Is the rate that the rumor spreads faster when 50 people have heard or when 700 people have heard? (Hint: you can do this with only a little bit of math). (a) The rumor spreads faster when 50 people have heard. (b) The rumor spreads faster when 700 people have heard.

Analysis: Just compare ratios of $\left(\frac{W_t}{N}\right)\left(\frac{N-W_t}{N}\right)$. (a) $\left(\frac{50}{800}\right)\left(\frac{800-50}{800}\right) = 0.05859$ or (b) $\left(\frac{700}{800}\right)\left(\frac{800-700}{800}\right) = 0.10938$

Ans: (b)

Explanation: *The best way to answer this question is to realize that you can ignore most of the terms in the diffusion equation and look only at W(N-W). You can do this because all other terms are equal for both calculations. To solve, let W equal the number of students who have heard the rumor, and (N-W) equal the number of students who have not heard the rumor. Now we multiply and look for the higher result:*

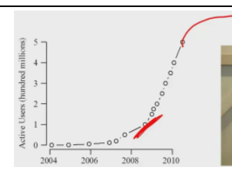
For W=50, W(N-W) = 50(750) = 37,500

For W=700, W(N-W) = 700(100) = 7,0000

Since 70,000 > 37,500, we can say that the rumor spreads at a faster rate when 700 people have heard than when 50 have heard.

Observation:
There is no tipping point in this equation. This is just a natural diffusion process. A kink (knee) in a curve does not mean at tip. The kinks are natural effects of diffusion.

Facebook:
A diffusion phenomena



7.4 – Contagion Models 2 – SIS Model

SIS model:

- Susceptible, Recover, and then become Susceptible again (mutation, not applicable to diseases like the measles).

$$W_{t+1} = W_t + \tau(Nc)\left(\frac{W_t}{N}\right)\left(\frac{N-W_t}{N}\right) - \quad W_t$$
$$W_{t+1} = W_t + W_t\left(c\tau\frac{N-W_t}{N} - a\right)$$
$$W_{t+1} = W_t\left(1 + \left(c\tau\frac{N-W_t}{N} - a\right)\right)$$

- W_t = the number of people that are getting cured.

W_t small:

$\frac{N-W_t}{N} \sim 1$ and

$W_{t+1} \sim W_t\left(1 + (c\tau - \quad)\right)$ and for disease to spread

$(c\tau - \quad) > 0$ or $(c\tau > \quad)$ or $\frac{c\tau}{a} > 1$

Basic Reproduction Number:

$R_0 = \frac{c\tau}{a}$ and if $R_0 > 1$ disease spreads

Note: This is a TIP. $R_0 > 1$ spreads, $R_0 < 1$ does not spread.

Quiz: Imagine a disease that people get from touching faucets, and then spread by touching other people. The disease has a recovery rate of 5% and its R0=0.3. A new strain of the disease appears; this one has twice the transmission rate. In other words, people who come in contact with one another are twice as likely to get the disease. Do we have reason to worry that this new disease will spread? (a) Yes (b) No

Analysis: Compare $R_{0,1} = 0.3 = \frac{c\tau_1}{a} = \frac{c\tau_1}{0.05}$ ➔ $c = {}^{0.015}/_{\tau_1}$ with $R_{0,2} = \frac{({}^{0.015}/_{\tau_1})2\tau_1}{0.05} = \frac{0.03}{0.05} = 0.6$

Ans: (b) NO

Explanation: *What matters is whether R0>1. Previously, R0=0.3. Remember our equation: R0=cta. If we double the transmission rate, t, then R0 doubles to 0.6.*

R_0 for Common Diseases: $R_0 = \frac{c\tau}{a}$

Measles ~ 15, (SIR Model)
Mumps ~ 5, (SIR Model)
Flu ~ 3, (SIS Model)

Models for Policy:

How to stop disease spread? How many to vaccinate?

SIR Model:

$V = \%\ vaccinated,$

this basically removes a fraction from the number who can catch the disease.

$R_0(1-V) = r_0$, since we want the new $r_0 \leq 1$ we find the vaccination fraction to be

$R_0(1-V) \leq r_0$ ➔ $V \geq 1 - \frac{1}{R_0}$

SIR Examples:

Measles @ 15 ➔ $V \geq 1 - \frac{1}{15} = \frac{14}{15}$

Mumps @ 5 ➔ $V \geq 1 - \frac{1}{5} = \frac{4}{5}$

Quiz: There exists a disease with an R0 of 2. To stop the spread of the disease, the State begins vaccinating. During vaccination, the State realizes that the R0 was calculated incorrectly, and its true value is 3. How much larger must the State make its vaccination program? (a) It must double the number of people it vaccinates. (b) It must increase the percentage by 13. (c) It must vaccinate 50% more people. (d) It must vaccinate everyone.

Analysis: Calculate both $V_{0,1}$ and $V_{0,2}$ and compare. $V_{0,1} \geq 1 - \frac{1}{R_{0,1}} = 1 - \frac{1}{2} = 0.5$ and

$V_{0,2} \geq 1 - \frac{1}{R_{0,2}} = 1 - \frac{1}{3} = 0.667$ and $\frac{V_{0,2}}{V_{0,1}} = \frac{0.66667}{0.5} = 1.333$ ➔ 33% or $\frac{1}{3}$ more.

Ans: (b) 1/3 more

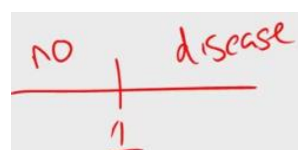
Explanation: *Initially it needed to vaccinate: 1−1/2=1/2. Now, it needs to vaccinate: 1−1/3=2/3*
So, the State needs to increase the number of people who it vaccinates by 1/3.

Observation: Unless you vaccinate above the minimum threshold for V, you will still get disease propagation.

Summary:

- **Diffusion: No tip**

SIS: (Contagion) R_0
Tipping Point exists for no contagion



- **SIR:** $V \geq 1 - \frac{1}{R_0}$ Vaccination threshold to prevent disease propagation

7.5 – Classifying Tipping Points

Dynamical Systems:

- **Two models:** *percolation* and *diffusion/contagion*. Both can cause tips.
- **Formal Definition of Tipping Points:**
  (a) direct vs contextual
  (b) between vs within class

Direct versus Contextual

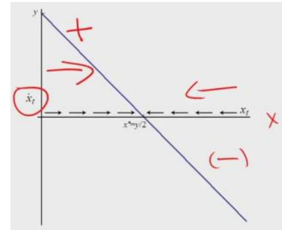
- **Direct –** a change in the variable it causes that variable to tip
- **Contextual –** the change in some parameter causes the system to change to a tip (e.g., forest density)

Between versus Within Class;

- **Between –** Systems can tip between: stable, pattern, random, complex.
- **Within –** Systems can tip within one of the classes: stable, pattern, random, complex.

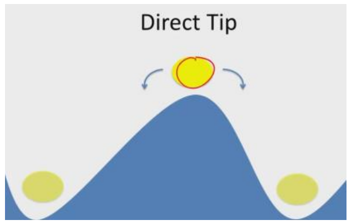
Dynamic System:

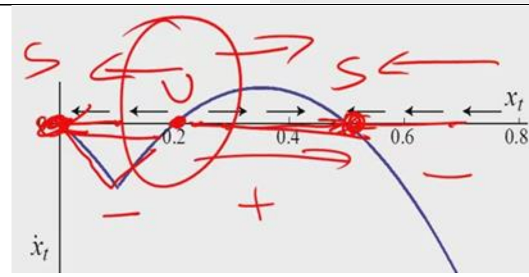
x' vs x chart showing a stable point at $x_t^* = \frac{y}{2}$



Direct Tip:

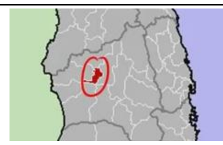
- *Direct Tip Concept* – a small action or event in a variable has a large effect on that end state
- Chart: x_t' vs x_t showing stable equilibrium at 0.0 and 0.5 but unstable at 0.2. WWI – Duke Ferdinand

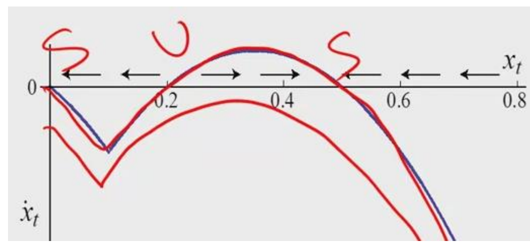




Contextual Tip:

- Vietnam: McGeorge Bundy - "Pleikus are like streetcars." Meaning that streetcars come down the road all the time, and "Pleikus", events that can escalate war also happen all the time. the context (environment) state ➔ bound to happen (percolation > 59.27% inevitable).
- Chart: 'environment' shift everything down and all equilibrium points disappear.
- Definition: A tiny change in the environment has a huge effect on the end state (percolation model)



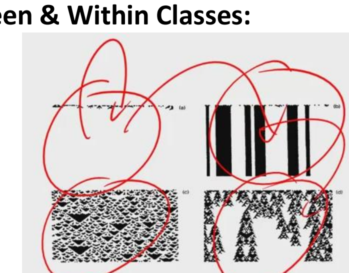


- Percolation model – checkerboard = filling more squares; forest = more tree density; social network = more connections

Context Change in Percolation Model:

- **0.592746 threshold** is change in context, i.e., the number of squares filled in.
- **SIS ($R_0 = \frac{c\tau}{a}$)** where a change in c (contact rate) or a (cure rate) is a change in context.

Between & Within Classes:

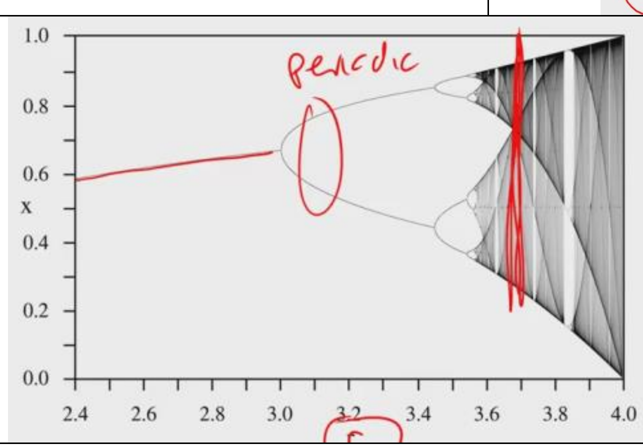


Example:

Stable->

Periodic ->

Complex



Review:

(a) direct
(b) contextual
(c) within class
(d) between classes

Quiz: Suppose that the temperature increases and the bees in a hive suddenly awaken from slumber and scurry randomly about to cool the hive. What type of tip would this be? (a) Direct, Within Class Tip (b) Contextual, Within Class Tip (c) Direct, Between Class Tip (d) Contextual, Between Class Tip
Analysis: Consider graph with percolation model. A change in temperature is a change in the environment which changes state from stable to complex. Hence, contextual between class tip.
Ans: (d) Contextual, Between Class Tip
Explanation: *It's contextual because of the environment -- the temperature changed -- and it's between class because the hive goes from equilibrium to random.*

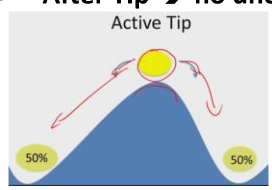
7.6 – Measuring Tips

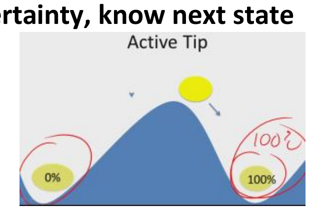
Measuring:

- A metric for determining the size and rareness of tipping
- Using these models with data so require some measures
  (a) diversity index (political, social, economic sciences)
  (b) entropy (physics, information theory)

Active Tip:

- **Before Tip ➔ uncertainty of next state**
- **After Tip ➔ no uncertainty, know next state**





Measure tippyness by reduction in uncertainty:

- **Measuring Possible Outcomes**
- **Example:** Say $P_a = P_b = P_c = P_d = \frac{1}{4}$
  Recall $P_a + P_b + P_c + P_d = 1$ and note that
  $P_a = P_b = P_c = P_d = \frac{1}{4}$ is more diverse than
  $P_a = P_b = \frac{1}{2}$ and $P_c = P_d = 0$

Diversity Index:

- **Formally:** $DI = \left(\sum P_i^2\right)^{-1}$ where P_i is the probability of the i-th outcome.

$P_A + P_B + P_C + P_D = 1$

$P_A \cdot P_A + P_B P_B + P_C P_C + P_D P_D$

$\sum_{i=A,B,C,D} P_i^2$

$P_A = P_B = P_C = P_D = 1/4$

$(1/4)^2 + (1/4)^2 + (1/4)^2 + (1/4)^2 = \frac{4}{16} = 1/4$

Diversity Index $= \frac{1}{\sum P_i^2}$

$DI = \left(\sum P_i^2\right)^{-1} = \left(\frac{1}{4}\right)^{-1} = 4$

Example:

- $P_a = \frac{1}{2}$, $P_b = \frac{1}{3}$, $P_c = \frac{1}{6}$ so that $\sum P_i^2 = {}^{14}/_{36}$ and $DI = \left(\sum P_i^2\right)^{-1} = {}^{36}/_{14} = 2\frac{4}{7}$
- If there was only one place it could go after the tip then it went from $DI = 2\frac{4}{7}$ to $DI = 1$

Quiz: What is the diversity index in a situation in which there exist four outcomes: one that occurs with probability 1/2 and three that occur with probability 1/6? (a) 3, (b) 21/36, (c) 6/5, (d) 1 5/7
Analysis: $DI = (\sum P_i^2)^{-1} = \left(\frac{1}{4} + \frac{1}{36} + \frac{1}{36} + \frac{1}{36}\right)^{-1} = \left(\frac{12}{36}\right)^{-1} = 3$
Ans: (a) 3
Explanation: *See analysis.*

Entropy:

- **Formally:** $entropy = -\sum P_i log_2(P_i)$
  Recall $\log_2(2^x) = x$

Example: Given, $P_a = P_b = P_c = P_d = \frac{1}{4}$

Recall for these probabilities: $DI = \left(\sum P_i^2\right)^{-1} = 4$

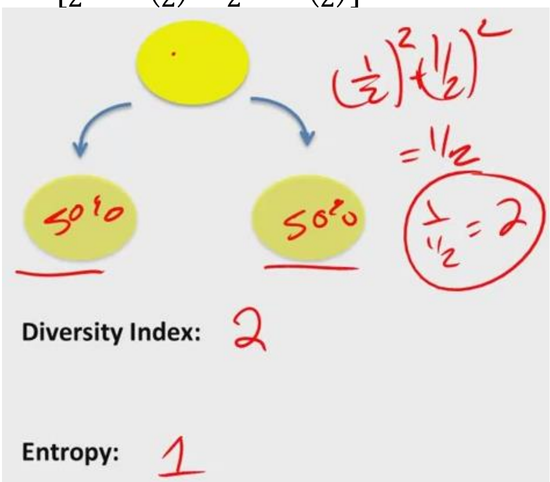
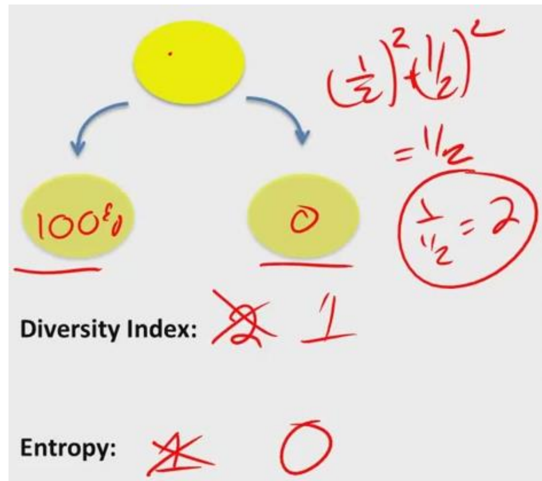
$$entropy = -\left[\frac{1}{4}\log_2\left(\frac{1}{4}\right) + \frac{1}{4}\log_2\left(\frac{1}{4}\right) + \frac{1}{4}\log_2\left(\frac{1}{4}\right) + \frac{1}{4}\log_2\left(\frac{1}{4}\right)\right] = -4\left(\frac{1}{4}(-2)\right) = 2$$

Possible Outcomes:
A
B
C
D

Q_1 (A or B) or (C or D)

Q_2 A?

- **Entropy tells us the number of bits needed to know in order to identify the outcome!**

<table>
<tr><td>Diversity Index: Number of types</td><td>Entropy: Amount of Information need to know to identify the outcome (type).</td></tr>
<tr><td>First Example: Pre Tip Diversity Index and Entropy
Pre tip calculations shown for DI.
$E = -\left[\frac{1}{2}\log_2\left(\frac{1}{2}\right)+\frac{1}{2}\log_2\left(\frac{1}{2}\right)\right] = 1$

</td><td>First Example: Post Tip Diversity Index and Entropy
Once tipped, DI=1, and zero (entropy) bits needed to count outcomes.

</td></tr>
<tr><td colspan="2">Tips: A change in the likelihood of outcomes is not kinks!</td></tr>
</table>