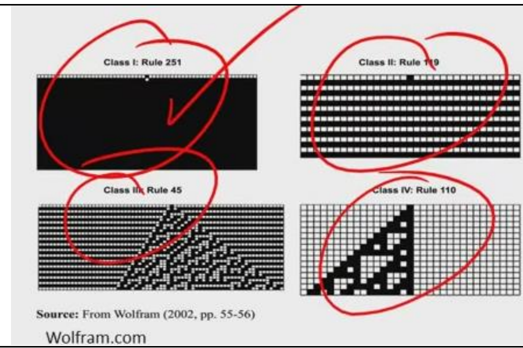
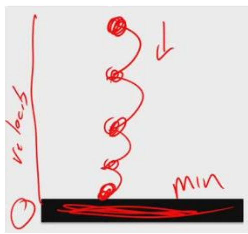
Session 11: Lyapunov Functions

11.1 – Lyapunov Functions

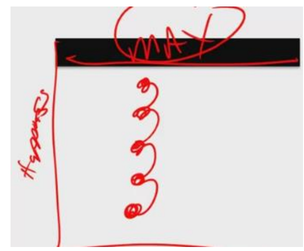
Functions to Model / Describe System:
- Tool to help determine if system goes to equilibrium or not
- Recall four system behavior types- looking to show if stable (equilibrium). If can't find function could be any of the four types.
- If can find a 'Lyapunov' function ➔ know system will find equilibrium and approximately (bound) on how fast.



Physics Example:
- Object with velocity starts high and continues to reduce, eventually going to zero where the object stops – and thus achieves equilibrium.
- Two conditions: (1) has to continually decrease , and (2) reach some minimum – stops.

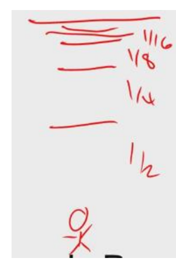


Economist Example:
- Happiness index continues to increase as agents makes trades. Each trade increases overall happiness.
- Eventually, a maximum in happiness occurs and does not increase further.
- Two conditions: (1) has to continue to increase, and (2) reach some maximum –stops.



Formally:
- F(x), a ***Lyapunov*** function
- A1: has a maximum (or minimum) value
- A2: there is a $\quad > 0$ such that if $x_{t+1} \neq x_t$, maximum, $F(x_{t+1}) > F(x_t) +$, or for a minimum, $F(x_{t+1}) < F(x_t) -$
- **Claim:** at some point $x_{t+1} = x_t$

Zeno's Paradox:
- Person trying to leave room, each step is half the length of the previous step. Never exits!
- Escape with constraint – have to move by at least k units/step at some point.



Bonus: How Fast
- Once k is known, then the number of steps to reach the bound is easily calculated by a simple division. This gives the convergence rate.

The Hard Part:
- Constructing the Lyapunov function
- Will discuss some examples below that illustrate how the Lyapunov function can be constructed and applied
- Will close and discuss Markov (statistical) and Lyapunov equilibrium differences.

11.2 – The Organization of Cities

Recall: F(x), a ***Lyapunov*** function
- A1: has a maximum (or minimum) value
- A2: there is a $\quad > 0$ such that if $x_{t+1} \neq x_t$, maximum, $F(x_{t+1}) > F(x_t) +$, or for a minimum, $F(x_{t+1}) < F(x_t) -$
- **Claim:** at some point $x_{t+1} = x_t$

Apparent City Self-Organization: number of restaurants, dry cleaners, groceries, train lines, etc. distributed usefully without overcrowding and without a micromanaging city planner.

Example – a weekly errand list:

- C: Cleaners, G: Grocery, D: Deli, B: Book Store, F: Fish Market
- Can choose which day to go to each.
  day of week: M, T, W, R Th, F
  errand: C, G, D, B, F ➔
  one schedule (by day) ➔ [C, G, D, B, F]
- Want to see if we can find a self-organizing insight via Lyapunov.

Example (cont.): Say we have five people with the same task list for the week and this is how they each schedule – each wants to avoid crowds.

1: [C, G, D, B, F]
2: [G, C, D, B, F]
3: [C, D, G, F, B]
4: [C, B, F, G, D]
5: [C, F, D, B, G]

Observation:

- **Behavior –** people observe and adapt. If they find, say Mondays at the cleaners is always crowded, they switch going to the cleaners on Monday with some other task on their list to avoid the crowded cleaner on Monday.
- **Rule –** switch day to try to avoid crowd

Example (cont.):

1: [**C**, G, D, B, F]
2: [G, C, D, B, F]
3: [**C**, D, G, F, B]
4: [**C**, B, F, G, D]
5: [**C**, F, D, B, G]

1: observes crowded cleaners on Monday so switches with Fish Market, ➔ [F, G, D, B, C] Note also reduces crowding on Friday at the Fish Market.

Example (cont.): First attempt to find $F(x_t)$

Total number of people going to each location during the week. No luck, five people are out each day.

1: [**C**, G, D, B, F]
2: [G, C, D, B, F]
3: [**C**, D, G, F, B]
4: [**C**, B, F, G, D]
5: [**C**, F, D, B, G]

Example (cont.): Second attempt to find $F(x_t)$
Total number of people that (1) meets each week. Meets 3 at C on Monday, 0 at G on Tuesday, 2 at D on Wednesday, 2 at B on Thursday, and 1 at F on Friday equals 8 total.

1: [**C**, G, D, B, F]
2: [G, C, D, B, F]
3: [**C**, D, G, F, B]
4: [**C**, B, F, G, D]
5: [**C**, F, D, B, G]

Example (cont.): Switch F and C
Now with the switch of F and C, meet 0 at F on Monday, 0 at G on Tuesday, 2 at D on Wednesday, 2 at B on Thursday, and 0 at C on Friday equals 4 total. Note also have to add meetings of other four people

1: [**F**, G, D, B, C]
2: [G, C, D, B, F]
3: [**C**, D, G, F, B]
4: [**C**, B, F, G, D]
5: [**C**, F, D, B, G]

Counting total meets: Recall, when person one does not meet n people, they do not meet him so the total meetings reduced by (1) are multiplied by 2 to get the number of reduced meetings by all five people. That is: $\Delta mtg_{all} = 2(\Delta mtg_1)$

Analysis of Example:
A1: the minimum value is 0 (meeting of anyone at your errand store on the day you visit the store).
A2: Does a move cause fewer people to meet? Yes, if I meet fewer people then they also meet fewer people and the total number of meetings (at a store) for everyone during the week falls. Note that ***k=2*** because if (1) meets one less person, then that person also meets one less, so there are two fewer meetings in the week.

An Equilibrium from switching.

1: [F, G, D, B, C]
2: [B, C, G, D, F]
3: [G, D, B, F, C]
4: [C, B, F, G, D]
5: [D, F, C, B, G]

Quiz: What if we "did the opposite" - instead of wanting to avoid others, people want to meet others. Could we still have a Lyapunov Function? (a) Yes, (b) No
Analysis: A1: there is a maximum value (i.e., meet everyone), A2: $F(x_t) = switch\ to\ increase\ meets$
Ans: (a) Yes
Explanation: *The Lyapunov Function would be the total number of people who meet, with K=2, just like in our original example. Any time a person changed her route to meet more people, the total number of people who meet would increase. So this is an example of a Lyapunov Function with a maximum value as opposed to a minimum value.*

Summary: People adopt a behavior of trying to avoid crowds by way of adjusting their activities via switching that distributes activities efficiently. Real cities are more complicated of course with visitors, migrants (in and out), but there is an underlying trend that helps sort activities.

11.3 – Exchange Economies and Externalities

Quick Review:

- If Lyapunov function, system goes to equilibrium by step amounts k until stops.
- If no Lyapunov function is found, system may or may not go to equilibrium

Exchange Economies & Externalities:

- Will examine markets that don't go to equilibrium – examine what prevents us from finding a Lyapunov function
- Related to Chris Langton's lambda parameter. very abstract one-dimensional cellular automata models

Exchange Market:

- Flea market like with trading of goods and cash. Is it stable?

1. Each person just brings a wagon full of stuff
2. You trade with someone only if you're happier with what you have now, than what you had before.
3. People trade with others but only if each gets an increase in happiness by some amount K. K is equivalent to the cost of trade.

Recall (maximum) Lyapunov Function:

- A1: has a maximum (or minimum) value
- A2: there is a > 0 such that if $x_{t+1} \neq x_t$, maximum, $F(x_{t+1}) > F(x_t) +$, or for a minimum, $F(x_{t+1}) < F(x_t) -$

Claim: at some point $x_{t+1} = x_t$

Attempt 1: Lyapunov Function

Sum of happiness of people. Does is satisfy the LF? (1) Yes. There is a fixed amount of stuff at the market ➔ there is a maximum limit to the happiness. (2) K. You will only trade if there is an increase in happiness. The Exchange market thus has a Lyapunov function.

Externality Example (North Korea, Iran, USA):

NK: trades nuclear weapons (NW) for oil (O)
I: trades oil (O) for nuclear weapons (NW)
U: not involved.

Result:

NK is happier, I is happier, USA is less happy.
Why? Externality of KN-I trading on USA feelings of security makes USA less happy. Consequence is total happiness may not have gone up (particularly if many other countries also less happy). And no Lyapunov function as A1 and A2 violated.

Other Externality Examples:

- Political Coalitions – Party A merges with party B may upset party C. Total happiness not increasing.
- Mergers – (LF possible metrics: profitability, security, happiness) other firms may be adversely affected so total metric does not go up.
- Political Alliances – Alliance $A \cup B$ could make other countries less secure ➔ LF unlikely
- Dating – Same with dating (happiness)

Quiz: Consider an exchange economy in which each of four people brings a different kind of fruit. Suppose that these people are altruistic - when someone becomes happier, everyone else also derives some happiness. Does trade in this environment create a Lyapunov Function? (Hint: altruism, here, is an externality - but instead of a negative externality, this a positive one). (a) Yes, (b) No

Analysis: A1 – happiness appears to always be increasing, A2 – trades occur only if generates happiness so > 0 exists.

Ans: (a) yes

Explanation: *Let the Lyapunov Function be total happiness. Every trade increases total happiness - even more than before. So we still have a Lyapunov Function. The difference between this example and the arms race is that here, the externalities are all positive, so total happiness still increases.*

Summary:
1. Exchange markets with happiness as a Lyapunov metric satisfy A1 and A2. However, when externalities are involved, a Lyapunov function may not exist, particularly for negative externalities. Positive externalities are reinforcing and do not negate A1 and A2.
2. Langdon's lambda (the binary representation of the rule) from the simple cellular automata model essentially says that a system whose behavior isn't influenced by others, tend to go to equilibrium. Conversely, where actions and behaviors are influenced by others tend to be more likely to be complex or random
3. Externalities materially affect other people either negatively or positively. Negative externalities tend to cause ongoing changes

11.4 – Time to Convergence and Optimality

Two Details:
- How long until LF reaches equilibrium?
- Does the process always stop at the min or max? That is with a step size K, does that last step complete if the distance remaining is less than K?

Recall (maximum) Lyapunov Function:
- A1: has a maximum (or minimum) value
- A2: there is a > 0 such that if $x_{t+1} \neq x_t$, maximum, $F(x_{t+1}) > F(x_t) +$, or for a minimum, $F(x_{t+1}) < F(x_t) -$

Claim: at some point $x_{t+1} = x_t$

How Long Until Equilibrium?:
- Example: $F(x_1) = 100, \quad = 2, max = 200$ implies $\# \, periods \leq 50$ from $\frac{100}{2} = 50$
- Find a 'bound' by using smallest maximum barrier and largest k step.

Does the process always stop at the min or max?:
- Short answer is NO. Process can get stuck on the way.
- Why? Rugged landscape – LF finds local max (or min).

Quiz: There are 100 people divided between Waiting Room A and Waiting Room B. These people will switch rooms if one is too busy. They have varying thresholds for "too busy", but no one has a threshold below 58. In other words, so long as 58 or fewer people are in a room, no one will want to leave that room. In the first time period, there are 87 people in Room A and only 13 in Room B. What is the maximum number of time periods that it could take for this system to reach equilibrium? (a) 29, (b) 13, (c) 87, (d) 20

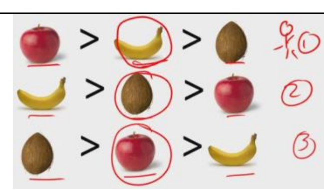
Analysis: Step size is $k = 1$ person at a time. Room A has a lower bound of 58. Room B has threshold of 58 before anyone would move. 29 people move from Room A to Room B emptying room A to 58 people and filling Room B to 42 people (still below move threshold).

Ans: (a) 29

Explanation: *We want to assume that one person moves each time period - in terms of our function, K=1 (since we're looking for the maximum amount of time periods, we want K to be as small as possible). Since there are 87 people in Room A, it will take 87−58=29 time periods until there are 58 people in Room A, and no one wants to switch.*

You might have noticed that there may be an externality in this example - that is, when someone switches rooms, he is happier, and so are the people in the room he just left, but the people in the new, emptier room may have a decrease in happiness. But this doesn't change our answer. After our 29 time periods, there are 13+29=42 people in Room B (and 58 in Room A). Since no one in either waiting room has a threshold below 58, no one in Room B will move. So the externalities don't change the answer here.

Process always stops at the min or max? (cont.):
- Preference model example: Assume preferences as shown. Can they trade to improve happiness? Pairwise trading blocks the path to the clear higher happiness possible (column 1) as all pairwise trades are rejected. Similar to the 'rugged landscape' trap at a local maximum.

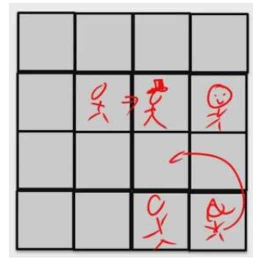


Summary:
- Possible to have a Lyapunov function that stops at some intermediate point (local maximum or minimum).
- Possible to find a bound on how fast with better bound on maximum (or minimum) and largest k step value.
- Process can get stuck at local maximum (or minimum).

11.5 – Lyapunov Fun and Deep

Can we always tell if goes to equilibrium?:
Example – Chairs and Offices
- Give everyone a chair at random and then let them trade (pairwise). Will trading churn or not?
- LF (happiness) suggests that will trade to get preferred chair until everyone is sufficiently happy ($\Delta happiness < k$). Note there is a maximum happiness (everyone has their preferred chair). Isn't this like a LF that gets stuck at a local maximum?

Contrary Example (Offices):
- Office swapping faces ***externalities***!
- Behavior of people around you influences your happiness – just like the externalities examples in 11.3. So each move may make some happy and some unhappy meaning no clear LF. May converge or not.



How can you decide if Equilibrium?:
Simple problem to show how difficult it is to determine.

Collatz Problem
- HOTPO (half or three plus 1)
  pick a number
  if even: divide by two
  if odd: times three plus one
  Stop: if you reach one
  Start at 19, go to $3(19) + 1 = 58$
  At 58 divide by 2 to get 29

HOTPO Examples:

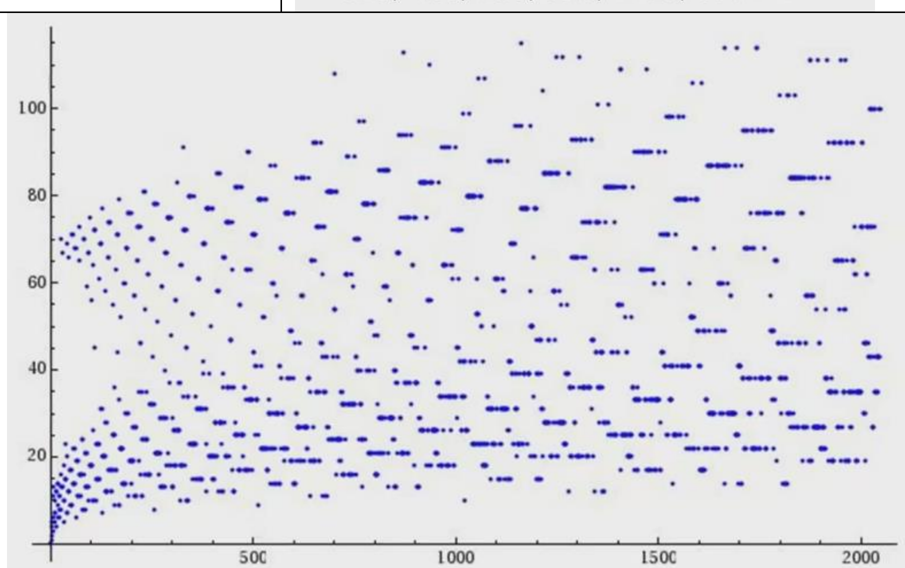
5: 3x5+1=16 8 4 2 1

7 3x7+1=22 11 34 17 52 26 13 40
20 10 5 16 8 4 2 1

27, 82, 41, 124, 62, 31, 94, 47, 142, 71, 214, 107, 322, 161, 484, 242, 121, 364, 182, 91, 274, 137, 412, 206, 103, 310, 155, 466, 233, 700, 350, 175, 526, 263, 790, 395, 1186, 593 ….

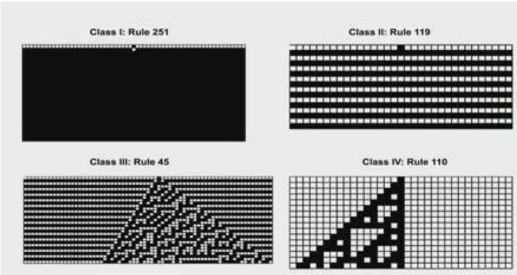
Chart of Periods to converge versis starting number. Note the complexity.

- **Bottom line:** some problems are tractable and some not. Pay attention to externalities.



11.6 – Lyapunov or Markov

Recall:



Source: From Wolfram (2002, pp. 55-56)

Markov and Lyapunov:

- Both describe conditions under which we say the system can go to equilibrium (fixed and statistical).

Markov and Lyapunov:

- **Lyapunov**
  A1: has a maximum (or minimum) value
  A2: there is a > 0 such that if $x_{t+1} \neq x_t$, maximum, $F(x_{t+1}) > F(x_t) +$, or for a minimum, $F(x_{t+1}) < F(x_t) -$
  Claim: at some point $x_{t+1} = x_t$
- **Markov** (statistically ***unique*** convergence)
  A1- Finite **States**
  A**2-** Fixed **Transition probabilities**
  A3- Can eventually get from any one state to any other state
  A4- Not a **simple cycle.**

Lyapunov Differences:

- Could be highly path dependent.
- Depends on the initial conditions
- Possibly many equilibrium
- Not a stochastic equilibrium, it's fixed.

Lyapunov:

1. If you can construct a Lyapunov function, then the system goes to equilibrium.
2. You can compute a maximum time to equilibrium.
3. The equilibrium need not be unique or efficient
4. Externalities are a reason systems don't go to equilibrium – if they ***counteract behavior and preferred*** actions, check closely.

Quiz: A supervisor for a firm is deciding whether to let employees make their own decisions about a number of things: desks, office paint, vacation dates, and membership on committees. Select any option(s) for which you think a Lyapunov Function would work. (a) Desks, (b) Vacation Dates, (c) Committee Membership, (d) Paint Color in Individual's Office

Analysis: Look for externalities that influence negatively.

Ans: (a) desks, (d) paint color

Explanation: *Desks and office paint are typically personal choices - who cares about someone else's desk or wall color? - so a Lyapunov Function is likely to work. On the other hand, vacation dates and committee memberships are likely to include negative externalities, so I would say Lyapunov Functions are no good there.*

Summary:

1. Externalities are a reason systems don't go to equilibrium. Recall Langdon's Lambda (λ) for cellular automata.*
2. Multiple models add diversity to thinking and understanding.
3. Understand source of equilibrium: Stochastic from Markov processes and in exchange market the Lyapunov function of happiness.

* *Automata Theory is the study of self-operating virtual machines to help in logical understanding of input and output process. Wikipedia*