Enumerative Combinatorics

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1 Advanced Topics in Enumeration

1.1 Set Partitions

In this section, we are interested in making Set partitions and their enumerations. For a finite set S that

$$|S| = m, S = \{1, 2, 3, ..., m\}$$

a partition of S is defined as collection of non - empty $A_j \subset S$ such that

$$A_i \cap A_j \neq \phi \text{ for all } i, j \tag{1}$$

$$A_1 \cup A_2 \cup A_3 \cdots \cup A_n = S \tag{2}$$

exercise In how many ways can you make partition of $S = \{1, 2, 3, 4\}$?
Definition: Stirling number of second kind

$$S(n, k) = S_{n,k} = \# \text{ of partitions of } S \text{ (with } |S| = \text{n) into } k \text{ parts}$$

It isn't enough to just make trivial cases with words, therefore:

$$S(0, 0) = 0$$
$$S(0, k) = 0$$
$$S(n, k) = 0 \text{ for } (k > n)$$

By simply brute force, one can make the following simple table for Sterling's 2nd number.

n, k	0	1	2	3	4
0	1				
1	0	1			
2	0	1	1		
3	0	1	3	1	
4	0	1	2	6	1

Meanwhile, for **ordinary** binomial coefficients, we have the **Pascal's Identity**

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

The analogous equation for Stirling's 2nd number is as follows;

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

The proof is pretty simple. Just casework either if element 1 is "alone" or not. Left as trivial

Meanwhile, it is valuable to ask the following question: How many functions from $X = [n]$ to $Y = [k]$ are there? The answers can vary as the condition of functions are undefined. If we simply count all the functions, it is obviously k^n. But how many injective functions are there? The answer is exactly same as the definition of **falling factorial**.

Falling Factorial $(k)_n$ **is :**

$$\begin{cases} k(k-1)(k-2)(k-3)\cdots(k-(n-1)) & \text{if } k \geq n \\ 0 & \text{if } k \leq n \end{cases}$$