Enumerative Combinatorics

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1 Advanced Topics in Enumeration

1.1 Set Partitions

In this section, we are interested in making Set partitions and their enumerations. For a finite set S that

$$|S| = m, S = \{1, 2, 3, ..., m\}$$

a partition of S is defined as collection of non - empty $A_j \subset S$ such that

$$A_i \cap A_j \neq \phi \text{ for all } i, j \tag{1}$$

$$A_1 \cup A_2 \cup A_3 \cdots \cup A_n = S \tag{2}$$

exercise In how many ways can you make partition of $S = \{1, 2, 3, 4\}$?
Definition: Stirling number of second kind

$$S(n, k) = S_{n,k} = \# \text{ of partitions of } S \text{ (with } |S| = \text{n) into } k \text{ parts}$$

It isn't enough to just make trivial cases with words, therefore:

$$S(0, 0) = 0$$
$$S(0, k) = 0$$
$$S(n, k) = 0 \text{ for } (k > n)$$

By simply brute force, one can make the following simple table for Sterling's 2nd number.

n, k	0	1	2	3	4
0	1				
1	0	1			
2	0	1	1		
3	0	1	3	1	
4	0	1	2	6	1

Meanwhile, for **ordinary** binomial coefficients, we have the **Pascal's Identity**

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

The analogous equation for Stirling's 2nd number is as follows;

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

The proof is pretty simple. Just casework either if element 1 is "alone" or not. Left as trivial

Meanwhile, it is valuable to ask the following question: How many functions from $X = [n]$ to $Y = [k]$ are there? The answers can vary as the condition of functions are undefined. If we simply count all the functions, it is obviously k^n. But how many injective functions are there? The answer is exactly same as the definition of **falling factorial**.

Falling Factorial $(k)_n$ **is :**

$$\begin{cases} k(k-1)(k-2)(k-3)\cdots(k-(n-1)) & \text{if } k \geq n \\ 0 & \text{if } k \leq n \end{cases}$$

It is pretty clear that injective function is as above, bijective function is non - existence (unless $k = n$, in that case, the answer is $k!$), so it is our interest to calculate the number of surjective functions.

The idea is to consider the inverse image of each element in Y. It can be seen as set partition of X, and allowing permutation within the partitions. Just like that, we have the following theorem.

Theorem 1.1 (Number of Surjective Functions)**.** *The number of surjective functions from $X = [n]$ to $Y = [k]$ is $k!S(n,k)$.*

Again, proof is talaaeftr. (trivial and left as an exercise for the reader) More theorems are here.

Theorem 1.2. *We have for all $n, m \in \mathbb{N}$,*

$$S(m,n) = \sum_{j=0}^{m-1} \binom{m-1}{j} S(j, n-1)$$

Proof. We will view the partition of $[m]$ by considering the position of element m and the size of the set that has it. Without loss of generality, let $m \in A_1$ and $|A_1| = m-j-1$. But now see that the left case for the specific assumption that we had above is simply partitioning j elements into $n-1$ sets. It is quite obvious that therefore it is left to choose $m-j-1$ elements from $m-1$ elements.

□

As one can see, when talking about functions, surjective or injective, one has to talk about **falling factorials, exponentials and Stirling's 2nd numbers**. This all comes down to following beautiful theorem:

Theorem 1.3.

$$n^m = \sum_{j=0}^{n} S(m,j)(n)_j$$

Example (case for 2, 3)**.**

$$x^2 = S(2,0) + S(2,1)x + S(2,2)x(x-1)$$
$$x^3 = S(3,0) + S(3,1)x + S(3,2)x(x-1) + S(3,3)x(x-1)(x-2)$$

We will prove the case for $m \geq n$ As we've already mentioned, analogy of functions are necessary to prove this formula.

Proof. $LHS :=$ number of functions from [m] to [n]

RHS:

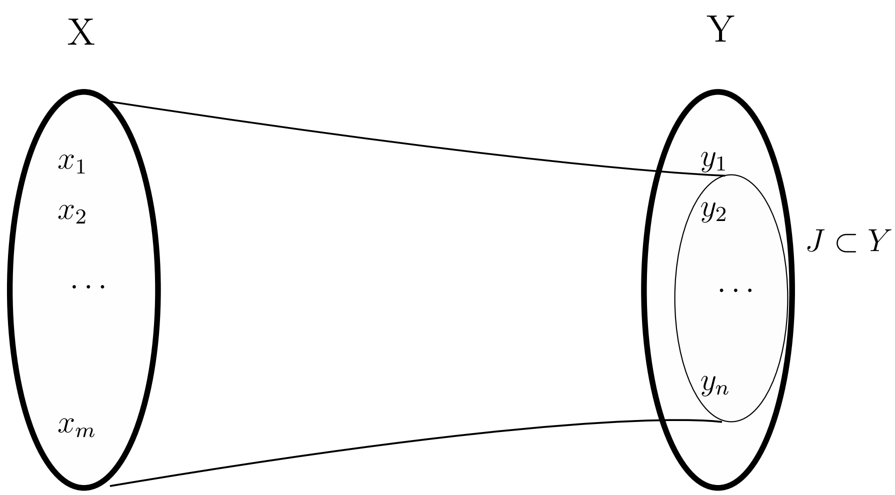

Figure 1: Function from $[m]$ to $[n]$ goes to $J \subset Y$

Here, $|J| = j$, and per every choice of j, there are $j!\binom{n}{j}$ choices. Per every set J, there are $S(m, j)$ choices of surjective functions. Therefore per every j, there are $S(m, j)(n)_j$ many functions. $\square$