Lista de exercicios 1
Convecçao
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Exercício 1

1.1

Derivar, a partir de um volume infinitesimal, as equações da conservação da massa, momento e energia na forma diferencial.

Conservação da massa:
Considerando um volume de controle fixo dentro do campo de fluxo (com $\Delta x \Delta y \Delta z \to 0$), infinitesimalmente pequeno, a abordagem para conservação de massa será:

$$\frac{\Delta m}{\Delta t} = \dot{m}_{entra} - \dot{m}_{sai} \tag{1}$$

OBSERVAÇÃO: Para facilidade, os diagramas de equilíbrio são desenhados em duas dimensões.

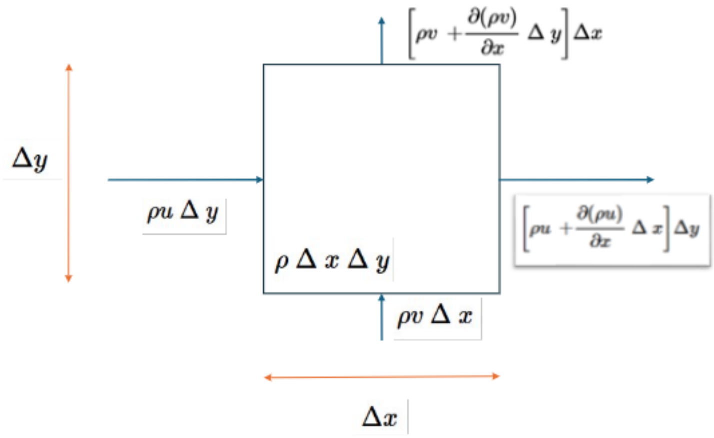

Figura 1: Conservação de massa em volume infinitesimal

Sua forma diferencial seria dada por

$$\begin{aligned}\frac{\partial}{\partial t}(\rho \Delta x \Delta y \Delta z) = \rho u \Delta y \Delta z + \rho v \Delta x \Delta z + \rho w \Delta y \Delta z - [\rho u + \frac{\partial}{\partial x}(\rho u)\Delta x]\Delta y \Delta z - \\ [\rho v + \frac{\partial}{\partial y}(\rho v)\Delta y]\Delta x \Delta z - [\rho w + \frac{\partial}{\partial z}(\rho w)\Delta z]\Delta x \Delta y\end{aligned} \tag{2}$$

E dividindo pelo volume constante $\Delta x \Delta y \Delta z$

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial t}(\rho z) = 0 \tag{3}$$

Usando a definição de derivada material e o operador divergencia

$$\frac{D}{Dt}\rho + \bigtriangledown(\rho V) = 0 \tag{4}$$

Conservação do momento:
Derivando a análise da segunda lei de Newton e tomando a velocidade como uma propriedade de transporte, o abordagem para conservação do momento dentro do volume de controle (vc) será

$$\frac{\Delta}{\Delta t}(mV)_{vc} = \sum F + \dot{m}V_{entra} - \dot{m}V_{sai} \tag{5}$$

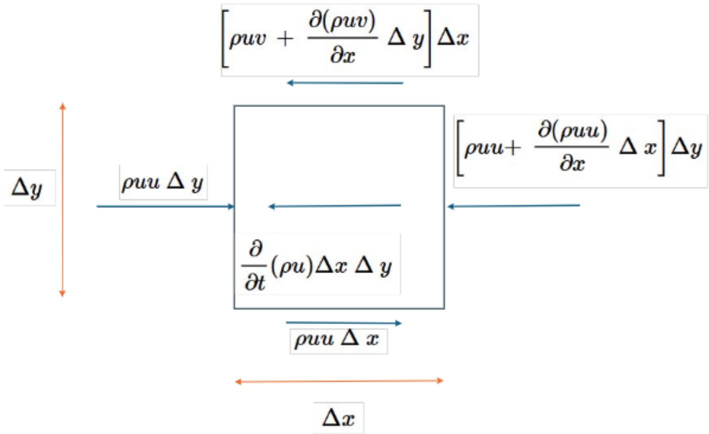

Figura 2: Balanço de forças devido ao fluxo de momento em volume infinitesimal

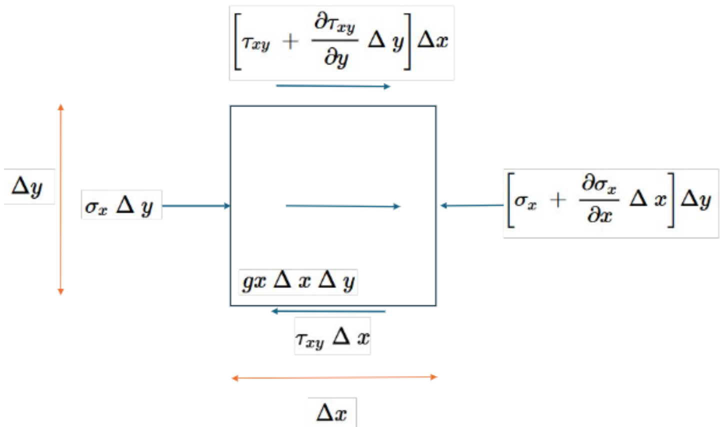

Figura 3: Balanço de forças devido a esforços normal e tangencial, e pelos terminos fonte

O equilíbrio de forças devido ao fluxo de momento e à tensão normal e tangencial na direção x será

$$\begin{aligned}
-\frac{\partial}{\partial t}(\rho u \Delta x\Delta y\Delta z) + \rho uu\Delta y\Delta z - [\rho uu + \frac{\partial}{\partial x}(\rho uu)\Delta x]\Delta y\Delta z - \\
\rho uv\Delta x\Delta z - [\rho uv + \frac{\partial}{\partial y}(\rho uv)\Delta y]\Delta x\Delta z - \rho uw\Delta x\Delta y - [\rho uw + \frac{\partial}{\partial z}(\rho uw)\Delta z]\Delta x\Delta y \\
+\sigma_x\Delta x\Delta y - (\sigma_x + \frac{\partial}{\partial x}\sigma_x\Delta x)\Delta y\Delta z - \tau_{xy}\Delta x\Delta y - (\tau_{xy} + \frac{\partial}{\partial x}\tau_{xy}\Delta y)\Delta x\Delta z + gx\Delta x\Delta y\Delta z
\end{aligned} \tag{6}$$

Sendo gx o termo fonte. Dividindo pelo volume de controle quando $\Delta x\Delta y\Delta z \to 0$

$$\rho\frac{Du}{Dt} + u[\frac{D}{Dt}\rho + \rho(\frac{\partial}{\partial x}u\frac{\partial}{\partial y}v)] = -\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + gx \tag{7}$$

Lembrando que o segundo termo é igual a zero segundo a conservação da massa

$$\rho \frac{Du}{Dt} = -\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + gx \tag{8}$$

Com a definição do tensor de tensão dada por

$$\tau_{ij} = -p\delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) + \delta_{ij} \lambda \nabla \cdot \mathbf{V} \tag{9}$$

$$\lambda = -\frac{2}{3}\mu \tag{10}$$

Que combinado com a equação 8 dá origem à equação de Navier-Stokes em x

$$\begin{gathered} \frac{\partial}{\partial t}(\rho u) + \frac{\partial}{\partial x}(\rho uu) + \frac{\partial}{\partial y}(\rho vu) + \frac{\partial}{\partial z}(\rho wu) = \\ -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu \frac{\partial u}{\partial x} + \lambda \nabla \cdot \mathbf{V}\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right) + g_x \end{gathered} \tag{11}$$

E para y e z

$$\begin{gathered} \frac{\partial}{\partial t}(\rho v) + \frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho vv) + \frac{\partial}{\partial z}(\rho wv) = \\ -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right) + \frac{\partial}{\partial y}\left(2\mu \frac{\partial v}{\partial y} + \lambda \nabla \cdot \mathbf{V}\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right) + g_y \end{gathered} \tag{12}$$

$$\begin{gathered} \frac{\partial}{\partial t}(\rho w) + \frac{\partial}{\partial x}(\rho uw) + \frac{\partial}{\partial y}(\rho vw) + \frac{\partial}{\partial z}(\rho ww) = \\ -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right) + \frac{\partial}{\partial z}\left(2\mu \frac{\partial w}{\partial z} + \lambda \nabla \cdot \mathbf{V}\right) + g_z \end{gathered} \tag{13}$$

Conservação da energia:

Da mesma forma, o diagrama a seguir mostra a análise da primeira lei da termodinâmica para volume infinitesimal

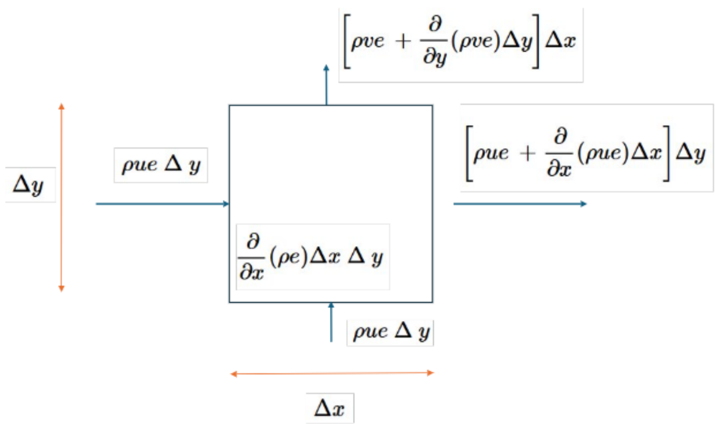

Figura 4: Equilíbrio da primeira lei da termodinâmica em volume infinitesimal pelo fluxo do fluido

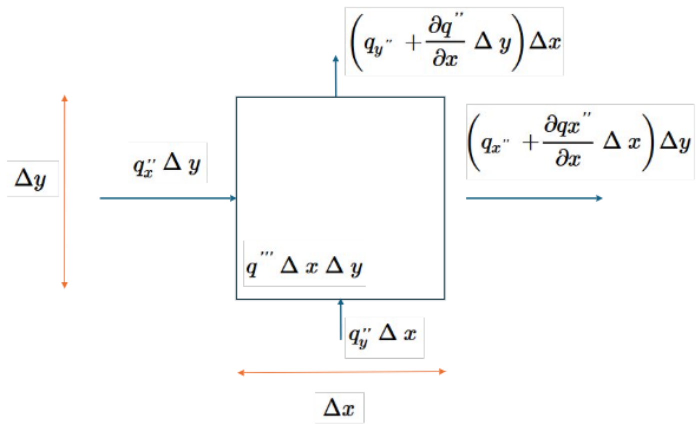

Figura 5: Equilíbrio da primeira lei da termodinâmica em volume infinitesimal pelo calor gerado e calor transferido

Do anterior, obtêm-se os seguintes termos de acumulação de energia no volume de controle, transferência de energia por fluxo, transferência de calor por condução, calor gerado e fluxo de trabalho, respectivamente

$$\Delta x \Delta y \Delta z \frac{\partial}{\partial t}(\rho e) \tag{14}$$

$$-(\Delta x \Delta y \Delta z)\left[\frac{\partial}{\partial x}(\rho u e) + \frac{\partial}{\partial y}(\rho v e)\frac{\partial}{\partial z}(\rho w e)\right] \tag{15}$$

$$-(\Delta x \Delta y \Delta z)\left(\frac{\partial q_x''}{\partial x} + \frac{\partial q_y''}{\partial y} + \frac{\partial q_z''}{\partial z}\right) \tag{16}$$

$$(\Delta x \Delta y \Delta z) q''' \tag{17}$$

$$\begin{aligned}(\Delta x \Delta y \Delta z)\left(\sigma_x \frac{\partial u}{\partial x} - \tau_{xy}\frac{\partial u}{\partial y} + \sigma_y \frac{\partial v}{\partial y} - \tau_{yx}\frac{\partial v}{\partial x} + \sigma_z \frac{\partial w}{\partial z} - \tau_{zx}\frac{\partial w}{\partial x}\right) + (\Delta x \Delta y \Delta z)\\ \left(u\frac{\partial \sigma_x}{\partial x} - u\frac{\partial \tau_{xy}}{\partial y} + v\frac{\partial \sigma_y}{\partial y} - v\frac{\partial \tau_{yx}}{\partial x} + w\frac{\partial \sigma_z}{\partial z} - w\frac{\partial \tau_{xz}}{\partial z}\right)\end{aligned} \tag{18}$$

Onde e é a energia específica, q'' o fluxo de calor na direção descrita e q''' a dissipação do calor interno gerado. Usando esses termos dentro da expressão de conservação de energia obtemos

$$\rho \frac{De}{Dt} + e(\frac{D}{Dt}\rho + \bigtriangledown(\rho V)) = -\bigtriangledown q'' + q''' - P \bigtriangledown V + \mu\phi \tag{19}$$

A função de dissipação viscosa sendo

$$\phi = 2[(\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial y})^2 + (\frac{\partial w}{\partial z})^2] + (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial z})^2 \tag{20}$$

Regime permanente:
Para conservação da massa

$$\bigtriangledown(\rho V) = 0 \tag{21}$$

De momento:

$$\begin{aligned}\frac{\partial}{\partial x}(\rho u u) + \frac{\partial}{\partial y}(\rho v u) + \frac{\partial}{\partial z}(\rho w u) =\\ -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x} + \lambda \nabla \cdot \mathbf{V}\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right) + g_x\end{aligned} \tag{22}$$

$$\frac{\partial}{\partial x}(\rho uv) + \frac{\partial}{\partial y}(\rho vv) + \frac{\partial}{\partial z}(\rho wv) = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right) + \frac{\partial}{\partial y}\left(2\mu\frac{\partial v}{\partial y} + \lambda\nabla\cdot\mathbf{V}\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right) + g_y \tag{23}$$

$$\frac{\partial}{\partial x}(\rho uw) + \frac{\partial}{\partial y}(\rho vw) + \frac{\partial}{\partial z}(\rho ww) = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right) + \frac{\partial}{\partial z}\left(2\mu\frac{\partial w}{\partial z} + \lambda\nabla\cdot\mathbf{V}\right) + g_z \tag{24}$$

E de energia:

$$e(\rho + \bigtriangledown(\rho V)) = -\bigtriangledown q'' + q''' - P\bigtriangledown V + \mu\phi \tag{25}$$

Laminar: Para fluxos laminares, os efeitos advectivos são negligenciados em números de Reynolds baixos. Portanto, há maiores efeitos dos fenômenos viscosos

$$\frac{\partial}{\partial t}(\rho u) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x} + \lambda\nabla\cdot\mathbf{V}\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right) + g_x \tag{26}$$

$$\frac{\partial}{\partial t}(\rho v) = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right) + \frac{\partial}{\partial y}\left(2\mu\frac{\partial v}{\partial y} + \lambda\nabla\cdot\mathbf{V}\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right) + g_y \tag{27}$$

$$\frac{\partial}{\partial t}(\rho w) = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right) + \frac{\partial}{\partial z}\left(2\mu\frac{\partial w}{\partial z} + \lambda\nabla\cdot\mathbf{V}\right) + g_z \tag{28}$$

Incompressivel:

Para conservação da massa

$$\bigtriangledown(\rho V) = 0 \tag{29}$$

Para conservação de momento:

$$\rho(\frac{\partial}{\partial x}(uu) + \frac{\partial}{\partial y}(vu) + \frac{\partial}{\partial z}(wu)) = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x} + \lambda\nabla\cdot\mathbf{V}\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right) + g_x \tag{30}$$

E para y e z

$$\rho(\frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(vv) + \frac{\partial}{\partial z}(wv)) = -\frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right) + \frac{\partial}{\partial y}\left(2\mu\frac{\partial v}{\partial y} + \lambda\nabla\cdot\mathbf{V}\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right) + g_y \tag{31}$$

$$\rho(\frac{\partial}{\partial x}(uw) + \frac{\partial}{\partial y}(vw) + \frac{\partial}{\partial z}(ww)) = -\frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left(\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right) + \frac{\partial}{\partial y}\left(\mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\right) + \frac{\partial}{\partial z}\left(2\mu\frac{\partial w}{\partial z} + \lambda\nabla\cdot\mathbf{V}\right) + g_z \tag{32}$$

E para energia:

$$\rho\frac{De}{Dt}+e\rho(\bigtriangledown(\rho V)=-\bigtriangledown q''+q'''-P\bigtriangledown V+\mu\phi \tag{33}$$

Propiedades constantes:
Para conservação da massa

$$\bigtriangledown(\rho V)=0 \tag{34}$$

Para conservacao de momento:

$$\begin{aligned}\rho(\frac{\partial}{\partial x}(uu)+\frac{\partial}{\partial y}(vu)+\frac{\partial}{\partial z}(wu)) = \\ -\frac{\partial p}{\partial x}+2\mu\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\lambda\nabla\cdot\mathbf{V}\right)+\mu\frac{\partial}{\partial y}\left(\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\right)+\mu\frac{\partial}{\partial z}\left(\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right)+g_x\end{aligned} \tag{35}$$

E para y e z

$$\begin{aligned}\rho(\frac{\partial}{\partial x}(uv)+\frac{\partial}{\partial y}(vv)+\frac{\partial}{\partial z}(wv)) = \\ -\frac{\partial p}{\partial y}+\mu\frac{\partial}{\partial x}\left(\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right)+2\mu\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}+\lambda\nabla\cdot\mathbf{V}\right)+\mu\frac{\partial}{\partial z}\left(\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\right)+g_y\end{aligned} \tag{36}$$

$$\begin{aligned}\rho(\frac{\partial}{\partial x}(uw)+\frac{\partial}{\partial y}(vw)+\frac{\partial}{\partial z}(ww)) = \\ -\frac{\partial p}{\partial z}+\mu\frac{\partial}{\partial x}\left(\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right)+\mu\frac{\partial}{\partial y}\left(\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)\right)+2\mu\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial z}+\lambda\nabla\cdot\mathbf{V}\right)+g_z\end{aligned} \tag{37}$$

E para energia, na lei de Fourier fican Cp e k constantes
Dissipação viscosa desprezível: Na ecuação de energia elimina-se ϕ:

$$\rho\frac{De}{Dt}+e(\frac{D}{Dt}\rho+\bigtriangledown(\rho V))=-\bigtriangledown q''+q'''-P\bigtriangledown V \tag{38}$$