Modelagem de Escoamentos Turbulentos. Lista de Exercícios No. 3

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Questão 1

Obtenha a equação de transporte para o tensor de Reynolds

Desenvolvimento
Partindo da equação de conservação de movimimento linear

$$\frac{\partial U_i}{\partial t} + U_k \frac{\partial U_i}{\partial x_k} = \frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu\left(\frac{\partial^2 U_i}{\partial x_k \partial x_k}\right) \tag{1}$$

Aplicando o conceito da media de Reynolds $A = \overline{A} + a$ para cada termo:

$$\frac{\partial U_i}{\partial t} = \frac{\partial \overline{U_i}}{\partial t} + \frac{\partial u_i}{\partial t} \tag{2}$$

$$U_k \frac{\partial U_i}{\partial x_k} = \overline{U_k}\frac{\partial \overline{U_i}}{\partial x_k} + \overline{U_k}\frac{\partial u_i}{\partial x_k} + u_k \frac{\partial \overline{U_i}}{\partial x_k} + u_k \frac{\partial u_i}{\partial x_k} \tag{3}$$

$$\frac{\partial P}{\partial x_i} = \frac{\partial \overline{P}}{\partial x_i} + \frac{\partial p}{\partial x_i} \tag{4}$$

$$\frac{\partial^2 U_i}{\partial x_k \partial x_k} = \frac{\partial^2 \overline{U_i}}{\partial x_k \partial x_k} + \frac{\partial^2 u_i}{\partial x_k \partial x_k} \tag{5}$$

A equação completa fica então

$$\frac{\partial \overline{U_i}}{\partial t} + \frac{\partial u_i}{\partial t} + \overline{U_k}\frac{\partial \overline{U_i}}{\partial x_k} + \overline{U_k}\frac{\partial u_i}{\partial x_k} + u_k \frac{\partial \overline{U_i}}{\partial x_k} + u_k \frac{\partial u_i}{\partial x_k} = \frac{1}{\rho}\frac{\partial \overline{P}}{\partial x_i} + \frac{\partial p}{\partial x_i} + \frac{\partial^2 \overline{U_i}}{\partial x_k \partial x_k} + \frac{\partial^2 u_i}{\partial x_k \partial x_k} \tag{6}$$

Agora tomando só os termos da fluctuação turbulenta

$$\frac{\partial u_i}{\partial t} + \overline{U_k}\frac{\partial u_i}{\partial x_k} + u_k \frac{\partial \overline{U_i}}{\partial x_k} + u_k \frac{\partial u_i}{\partial x_k} = \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{\partial^2 u_i}{\partial x_k \partial x_k} \tag{7}$$

Multiplicando por u_j cada termo da equação

$$u_j \frac{\partial u_i}{\partial t} = \frac{\partial u_i u_j}{\partial t} - u_i \frac{\partial u_j}{\partial t} \tag{8}$$

$$u_j \overline{U_k}\frac{\partial u_i}{\partial x_k} = \overline{U_k}\frac{\partial u_i u_j}{\partial x_k} - \overline{U_k} u_i \frac{\partial u_i}{\partial x_k} \tag{9}$$

$$u_j u_k \frac{\partial \overline{U_i}}{\partial x_k} \tag{10}$$

$$u_j u_k \frac{\partial u_i}{\partial x_k} = u_k \frac{\partial u_i u_j}{\partial x_k} - u_i u_k \frac{\partial u_j}{\partial x_k} \tag{11}$$

$$u_j \frac{\partial^2 u_i}{\partial x_k \partial x_k} = \frac{\partial^2 u_i u_j}{\partial x_k \partial x_k} - u_i \frac{\partial^2 u_j}{\partial x_k \partial x_k} \tag{12}$$

Escrevendo a equação completa:

$$\begin{aligned} \frac{\partial u_i u_j}{\partial t} - u_i \frac{\partial u_j}{\partial t} + \overline{U_k} \frac{\partial u_i u_j}{\partial x_k} - \overline{U_k} u_i \frac{\partial u_i}{\partial x_k} + u_j u_k \frac{\partial \overline{U_i}}{\partial x_k} + u_k \frac{\partial u_i u_j}{\partial x_k} - u_i u_k \frac{\partial u_j}{\partial x_k} = \\ \frac{u_j}{\rho} \frac{\partial p}{\partial x_i} + \nu \left(\frac{\partial^2 u_i u_j}{\partial x_k \partial x_k} - u_i \frac{\partial^2 u_j}{\partial x_k \partial x_k} \right) \end{aligned} \tag{13}$$

Da mesma forma, obtemos a equação para a flutuação u_j

$$\begin{aligned} \frac{\partial u_i u_j}{\partial t} - u_j \frac{\partial u_i}{\partial t} + \overline{U_k} \frac{\partial u_i u_j}{\partial x_k} - \overline{U_k} u_j \frac{\partial u_j}{\partial x_k} + u_i u_k \frac{\partial \overline{U_i}}{\partial x_k} + u_k \frac{\partial u_i u_j}{\partial x_k} - u_j u_k \frac{\partial u_j}{\partial x_k} = \\ \frac{u_i}{\rho} \frac{\partial p}{\partial x_i} + \nu \left(\frac{\partial^2 u_i u_j}{\partial x_k \partial x_k} - u_j \frac{\partial^2 u_j}{\partial x_k \partial x_k} \right) \end{aligned} \tag{14}$$

Somando a equação 14 da equação 13 obtemos

$$\begin{aligned} \frac{\partial u_i u_j}{\partial t} + \overline{U_k} \frac{\partial u_i u_j}{\partial x_k} + u_j u_k \frac{\partial \overline{U_i}}{\partial x_k} + u_i u_k \frac{\partial \overline{U_i}}{\partial x_k} + \frac{\partial u_i u_j u_k}{\partial x_k} = \\ \frac{1}{\rho} \left(u_i \frac{\partial p}{\partial x_i} + u_j \frac{\partial p}{\partial x_j} \right) + \nu \left(u_i \frac{\partial^2 u_j}{\partial x_k \partial x_k} + u_j \frac{\partial^2 u_i}{\partial x_k \partial x_k} \right) \end{aligned} \tag{15}$$

E aplicando os termos da media

$$\begin{aligned} \frac{\partial \overline{u_i u_j}}{\partial t} + \overline{U_k} \frac{\partial \overline{u_i u_j}}{\partial x_k} + \overline{u_j u_k} \frac{\partial \overline{U_i}}{\partial x_k} + \overline{u_i u_k} \frac{\partial \overline{U_i}}{\partial x_k} + \frac{\partial \overline{u_i u_j u_k}}{\partial x_k} = \\ \frac{1}{\rho} \left(\overline{u_i} \frac{\partial \overline{p}}{\partial x_i} + \overline{u_j} \frac{\partial \overline{p}}{\partial x_j} \right) + \nu \left(\overline{u_i} \frac{\partial^2 \overline{u_j}}{\partial x_k \partial x_k} + \overline{u_j} \frac{\partial^2 \overline{u_i}}{\partial x_k \partial x_k} \right) \end{aligned} \tag{16}$$

Questão 2

Simplifique a equação da energia cinética turbulenta $k = (u_i u_i)/2$ para o caso de um escoamento médio turbulento plenamente desenvolvido em um duto.

Desenvolvimento

Agora novamente da equação de transporte do tensor de Reynolds

$$\begin{aligned} \frac{\partial \overline{u_i u_j}}{\partial t} + \overline{U_k} \frac{\partial \overline{u_i u_j}}{\partial x_k} + \overline{u_j u_k} \frac{\partial \overline{U_i}}{\partial x_k} + \overline{u_i u_k} \frac{\partial \overline{U_i}}{\partial x_k} + \frac{\partial \overline{u_i u_j u_k}}{\partial x_k} = \\ \frac{1}{\rho} \left(\overline{u_i} \frac{\partial \overline{p}}{\partial x_i} + \overline{u_j} \frac{\partial \overline{p}}{\partial x_j} \right) + \nu \left(\overline{u_i} \frac{\partial^2 \overline{u_j}}{\partial x_k \partial x_k} + \overline{u_j} \frac{\partial^2 \overline{u_i}}{\partial x_k \partial x_k} \right) \end{aligned} \tag{17}$$

Façendo i=j e dividindo por 2 obtemos

$$\frac{\partial k}{\partial t}+\overline{U_k}\frac{\partial k}{\partial x_k}+k\frac{\partial \overline{U_i}}{\partial x_k}+k\frac{\partial \overline{U_i}}{\partial x_k}+u_k\frac{\partial k}{\partial x_k}=\frac{1}{\rho}\left(\overline{u_i}\frac{\partial \overline{p}}{\partial x_i}\right)+\nu\left(\overline{u_i}\frac{\partial^2\overline{u_i}}{\partial x_k\partial x_k}\right) \tag{18}$$

Para escoamento plenamente desenvolvido, a velocidade media na direçao i não muda com relaçao a j, então $\frac{\partial \overline{U_i}}{\partial x_k}=0$

$$\frac{\partial k}{\partial t}+\overline{U_k}\frac{\partial k}{\partial x_k}+u_k\frac{\partial k}{\partial x_k}=\frac{1}{\rho}\left(\overline{u_i}\frac{\partial \overline{p}}{\partial x_i}\right)+\nu\left(\overline{u_i}\frac{\partial^2\overline{u_i}}{\partial x_k\partial x_k}\right) \tag{19}$$

Questão 3

Demonstre (3.39) e mostre que (3.27) pode ser reescrita como

$$\overline{\varepsilon}=\nu\,\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}+\frac{\partial^2\overline{u_iu_j}}{\partial x_i\partial x_j} \tag{20}$$

<u>**Desenvolvimento**</u>
A equação 3.39 é

$$\overline{\frac{\partial u_i\partial u_j}{\partial x_j\partial x_j}}=\overline{\frac{\partial^2 u_iu_j}{\partial x_i\partial x_j}} \tag{21}$$

Abrindo o termo do lado dereito da equação 21

$$\frac{\partial^2 u_iu_j}{\partial x_i\partial x_j}=\frac{\partial}{\partial x_i}\left(\frac{\partial u_iu_j}{\partial x_j}\right)=\frac{\partial}{\partial x_i}\left(u_i\frac{\partial u_j}{\partial x_j}+u_j\frac{\partial u_i}{\partial x_j}\right) \tag{22}$$

Aplicando a igualdade da equação 3.6 $\frac{\partial u_i}{\partial x_i}=0$

$$\frac{\partial}{\partial x_i}\left(u_j\frac{\partial u_i}{\partial x_j}\right)=\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}+u_j\left(\frac{\partial}{\partial x_i}\frac{\partial u_i}{\partial x_j}\right) \tag{23}$$

Pela propiedade comutativa das derivadas parciais

$$\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}+u_j\left(\frac{\partial}{\partial x_j}\frac{\partial u_i}{\partial x_i}\right)=\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j} \tag{24}$$

Agora igualando e aplicando as medias

$$\overline{\frac{\partial u_i\partial u_j}{\partial x_j\partial x_j}}=\overline{\frac{\partial^2 u_iu_j}{\partial x_i\partial x_j}} \tag{25}$$

Agora para a equação 2.27

$$\overline{\varepsilon}=\frac{1}{2}\nu\overline{\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)} \tag{26}$$

Abrindo o producto dos termos

$$\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)=\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}+\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}+\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_i} \tag{27}$$

Agrupando os termos

$$\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\frac{\partial u_j}{\partial x_i}\right)+\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}+\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}\right)=\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}+\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right)+\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}+\frac{\partial u_j}{\partial x_i}\frac{\partial u_i}{\partial x_j}\right) \tag{28}$$

$$2\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right)+2\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\right)=2\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right)+2\left(\frac{\partial^2 u_i u_j}{\partial x_j \partial x_i}\right) \tag{29}$$

Agora sustituindo em eq. 26

$$\overline{\varepsilon}=\frac{1}{2}\nu\,\overline{2\left(\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}\right)+2\left(\frac{\partial^2 u_i u_j}{\partial x_j \partial x_i}\right)}=\nu\left[\left(\overline{\frac{\partial u_i}{\partial x_j}\frac{\partial u_i}{\partial x_j}}\right)+\left(\frac{\partial^2\overline{u_i u_j}}{\partial x_j \partial x_i}\right)\right] \tag{30}$$

Questão 4

Usando correlações para camada limite turbulenta sobre uma superfície plana disponíveis em livros texto da graduação, determine os valores de $U\infty/u^*$, Re_δ e Re* em x = 1 e 2 m.

Desenvolvimento

A expressão para o comprimento da camada limite turbulenta é definida como

$$\delta = 0,382xRe_x^{-1/5} \tag{31}$$

Calculando os valores para $x = 1m$

$$Re_x=\frac{U_\infty x}{\nu}=\frac{25\frac{m}{s}\cdot 1m}{1,5x10^{-5}m^2/s}=1,66x10^6 \tag{32}$$

$$\delta_{1m}=0,382xRe_x^{-1/5}=0,382\cdot 1m\cdot(1,66x10^6)^{-1/5}=0,021m \tag{33}$$

$$Re_\delta=\frac{U_\infty \delta}{\nu}=\frac{25\frac{m}{s}\cdot 0,021m}{1,5x10^{-5}m^2/s}=3.5x10^4 \tag{34}$$

Calculando os valores para $x = 2m$

$$Re_x=\frac{U_\infty x}{\nu}=\frac{25\frac{m}{s}\cdot 2m}{1,5x10^{-5}m^2/s}=3,33x10^6 \tag{35}$$

$$\delta_{1m}=0,382xRe_x^{-1/5}=0,382\cdot 2m\cdot(3,33x10^6)^{-1/5}=0,036m \tag{36}$$

$$Re_\delta=\frac{U_\infty \delta}{\nu}=\frac{25\frac{m}{s}\cdot 0,036m}{1,5x10^{-5}m^2/s}=6x10^4 \tag{37}$$

Considerando o modelamiento da velocidade dado pela expressão de la lei logarítmica (eq 3.210 de [1])

$$\frac{U_\infty}{u_*}+\frac{1}{\kappa}\ln\left(\frac{U_\infty}{u_*}\right)=\frac{1}{\kappa}\ln Re_\delta + a - b \tag{38}$$

Onde (de [1])

$$Re_\delta = \frac{U_\infty}{u^*} Re_* \tag{39}$$

Da eq. 38, encontra-se o valor de u_* para $x = 1m$, com os valores de $k \approx 0,41; a = 5,2; b = -2,5$

$$\frac{25}{u_*} + \frac{1}{0,41} \ln\left(\frac{25}{u_*}\right) = 44,3 \;\rightarrow\; u_* = 0,70m/s \tag{40}$$

$$\frac{U_\infty}{u_*} = \frac{25m/s}{0,70m/s} = 35,7 \tag{41}$$

E para $x = 2m$

$$\frac{25}{u_*} + \frac{1}{0,41} \ln\left(\frac{25}{u_*}\right) = 35,5 \;\rightarrow\; u_* = 0,95m/s \tag{42}$$

$$\frac{U_\infty}{u_*} = \frac{25m/s}{0,95m/s} = 26,1 \tag{43}$$

Questão 5

Considere o escoamento turbulento plenamente desenvolvido de água em uma tubulação circular de parede lisa com raio R.
a) Assumindo que a espessura da subcamada limite viscosa δ_v é equivalente a y+ = 5, mostre em um gráfico log-log, a razão δ_v/R para números de Reynolds = (UD/ν) iguais a 10^4, 10^5 e 10^6. Use alguma correlação, como a de Blasius, para determinar a tensão na parede.
b) Represente em um gráfico a distribuição de velocidade $U(r)/U$ para cada um dos números de Reynolds.
c) Avalie o valor local da velocidade média em y+ = 5 e 50.

References

[1] Cesar Deschamps, Escalas da turbulencia Cap 2. UFSC Florianopolis, SC, Notas de aula, 2025.