Problem 1.a.

Solution: If A is orthogonal, then $A^T A = I$. If A is orthogonal and $i \neq j$, then

$$A_i \cdot A_j = 0$$

, else

$$A_i \cdot A_j = 1$$

because they are unit vectors and each vector is orthogonal to each other in A but not to itself. Since they are unit vectors, the dot product of two columns is simply $1^2 = 1$. Therefore, it is I because the value is 1 when the row index = col index.

If $A^T A = I$, then A is orthogonal. The dot product of a vector with itself is its length squared, and since the value is 1 for all indices where row=col, then the length of the vectors in A are all unit length. Using the fact that $A^T A = A/cdotA = I$, $A_i \cdot A_j = 0, i \neq j$ because, which means that the columns of A are all orthogonal because $A^T A_{ij} = A_i \cdot A_j$, which is 0 for all indices when $i \neq j$. □

Problem 1.b.

Solution:

$$||Ru|| * ||Ru|| = (Ru)^T (Ru) = u^T R^T Ru$$

$R^T R = I$, so

$$= u^T Iu = u^T u = ||u|| * ||u||$$

norm is always non-negative, so therefore $||Ru|| = ||u||$

□

Problem 1.c.

Solution:

$$det(R^T R) = det(R^T) = det(R)$$

$R^T R = I$, $det(I) = 1$, and determinant of matrix and transpose are same. Therefore,

$$det(R)^2 = 1 => det(R) = \pm 1$$

□

Problem 2.a.

Solution: First, let's say

$$A = [\alpha_1, ..., \alpha_n]$$

Since the vector v is already in the standard basis, to convert the vector in terms of α, $v_\alpha = A^{-1} v$.

□

Problem 2.b.

Solution: Since $G_{\alpha\beta}$ converts a vector in terms of β to a vector in terms of α and we have v_α, we can first use the inverse to convert the vector in terms of α to a vector in terms of β. Then, we have $G_{\gamma\beta}$, which will convert a vector in terms of β to a vector in terms of γ. Therefore, we will get the vector in terms of γ, which is what we want.

$$v_\gamma = G_{\gamma\beta} G_{\alpha\beta}^{-1} v_\alpha$$

□

Problem 2.c.

Solution: False
Let's say the first basis is the standard basis in R^2

$$B_1 = \{(1,0),(0,1)\}$$

The orthogonal vectors in B_1 are $[1,0],[0,1]$. $[1,0] \cdot [0,1] = 0$

$$B_2 = \{(1,1),(0,1)\}$$

$P = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$ because we form a matrix from the vectors of B_2 and P is equal to the inverse of the matrix, which is the change of basis matrix from the standard basis to B_2. Now, the vectors are $P[1,0]$ and $P[0,1]$, which are $[1,-1]$ and $[0,1]$ respectively, which are no longer orthogonal, since their dot product is equal to -1 and not 0. For it to be true, we need the change of basis to be orthogonal. □

Problem 2.d.

Solution: True.
The original set of vectors are linearly independent, so if the number of vectors are equal to the number of dimensions (example:3 vectors in R^3), then there is an inverse, since they are linearly independent. This inverse will be the change of basis matrix we need to convert the vectors to the identity matrix vectors, which are orthonormal (orthogonal and all have unit length). If the number of vectors is less than the number of dimensions, then to find the change of basis matrix needed, we will first add vectors that are linearly independent to the other vectors in the set, by using the cross product of the manually-made linearly independent set. This is always possible. Then, we will do the same trick of using the inverse as the change of basis matrix, and that gives us the answer we need for the original set (we can eliminate the other vectors that were manually made for our task), since the original vectors will still be seen as unit length with a value of 1 in one index. □

Problem 3.a.

Solution: $0^k = 0, k > 1$, where 0 is zero matrix, so

$$e^0 = I + 0 + ... + 0 = I$$

□

Problem 3.b.

Solution: $(x+y)^T = x^T + y^T$, so

$$(e^A)^T = (I+A+(\frac{A^2}{2!})+(\frac{A^3}{3!})+...)^T = I+A^T+(\frac{A^2}{2!})^T+(\frac{A^3}{3!})^T+... = I+A^T+(\frac{(A^T)^2}{2!})+(\frac{(A^T)^3}{3!})+... = e^{(A^T)}$$

□

Problem 3.c.

Solution:

$$(GAG^{-1})^n = GAG^{-1}GAG^{-1}... = GAA...G^{-1} = GAG^{-1}$$
$$e^{(GAG^{-1})} = I + GAG^{-1} + (GAG^{-1})^2/2! + (GAG^{-1})^3/3! + ...$$
$$= I + GAG^{-1} + (GA^2G^{-1})/2! + (GA^3G^{-1})/3! + ... = Ge^AG^{-1}$$

□

Problem 3.d.

Solution: $Av = \lambda v$ because λ is eigenvalue of A

$$e^A = I + A + A^2/2! + A^3/3! + ...$$
$$e^Av = Iv + Av + A^2v/2! + A^3v/3! + ...$$
$$Av = \lambda v = AAv = A\lambda v = \lambda Av = \lambda^2 v$$

Therefore,

$$e^Av = Iv + \lambda v + \lambda^2 v/2! + \lambda^3 v/3! + ...$$
$$= v(I + \lambda + \lambda^2/2! + \lambda^3/3! + ...)$$
$$= e^\lambda v$$
$$\therefore e^\lambda \textbf{ is an eigenvalue of } e^A$$

□

Problem 3.e.

Solution:

$$det(e^A) = e_1^\lambda * e_2^\lambda * e_3^\lambda * ... * e_n^\lambda \text{ because determinant is equal to product of eigenvalues}$$
$$det(e^A) = e^{\lambda_1+\lambda_2+\lambda_3+...+\lambda_n}$$
$$tr(A) = \lambda_1 + \lambda_2 + \lambda_3 + ... + \lambda_n$$

So,

$$det(e^A) = e^{tr(A)}$$

Since the determinant is exponential e^x, the determinant can never be 0, making e^A always invertible. □

Problem 4.a.

Solution:

$$\frac{dx}{dt} = ax(t)$$
$$\frac{dx}{x(t)dt} = a$$
$$ln(|x(t)|) = at + C$$
$$|x(t)| = e^{at+C}$$
$$x(t) = Ke^{at}$$

$x(0) = x_0$, so

$$x(t) = x_0 e^{at}$$

□

Problem 4.b.

Solution:

$$\frac{d}{dt}e^{At} = d/dt(I + At + (At)^2/2! + ...)$$
$$= A + A^2t + A^3t^2/2! + ... = A(I + At + A^2t^2/2! + ...) = Ae^{At}$$
$$e^{At}A = (I + At + (At)^2/2! + ...)A = A + A^2t + A^3t^2/2! + ... = A(I + At + A^2t^2/2! + ...) = Ae^{At}$$

□

Problem 4.c.

Solution:

$$dx(t)/dt = Ae^{A}tx_0 \textbf{ based on previous problem}$$
$$\therefore dx(t)/dt = Ax_t = Ax$$
$$x_0 = e^{A*0}x_0 = Ix_0 = x_0$$

Both differential equation and initial condition were satisfied, making $x(t)$ a solution. □

Problem 4.d.

Solution: We need eigenvalues and eigenvectors of A.

$$det(A - \lambda I) = 0$$
$$det(\begin{pmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{pmatrix})$$
$$(2-\lambda)(3-\lambda) - 2 = 0$$
$$4 - 5\lambda + \lambda^2 = (\lambda - 1)(\lambda - 4) = 0$$

$$\lambda = 1, 4$$

For eigenvectors,

$$(A - \lambda I)v = 0$$

For $\lambda_1 = 1$,

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} v_1 = 0$$

$$v_1 = [-2, 1]$$

For $\lambda_2 = 4$,

$$\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} v_2 = 0$$

$$v_2 = [1, 1]$$

For the general solution, it would be

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 = c_1 e^{t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

□

Problem 5.a.

Solution: mkdir Fa24 □

Problem 5.b.

Solution: mkdir Fa24/EECS_C106A □

Problem 5.c.

Solution: mkdir Lab Homework □

Problem 5.d.

Solution: pwd □

Problem 5.e.

Solution: vim SID.txt □

Problem 5.f.

Solution: cat SID.txt □

Problem 5.g.

Solution: rm Cleo_contact.txt □

Problem 5.h.

Solution: ls □

Problem 5.i.

Solution: cd ../..
mkdir English_R1B □