1. "The rank of a matrix is always less than or equal to the number of rows and columns, so $r \le m$ and $r \le n$."
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3. "These solutions make up the left nullspace, which has dimension $m - r > 0$ (that is, there are nonzero vectors in it)."
4. "It is solvable when d is in the row space, which consists of all vectors ATy."
5. "The solution y is unique when the left nullspace contains only the zero vector."
6. "I = P21 + P31 + P32 - P32P21 - P21P32."
7. "Thus, c1 = c2 = c3 = 0 along the diagonal, and c4 = c5 = 0 from the off-diagonal entries."
8. "Its nullspace has the basis: [1, 1, 1]^T."
9. "Since B is singular, det(B) = 0. Thus det(BTB) = det(BT) det(B) = 0."
10. "B has 0 as an eigenvalue and is therefore singular (not invertible). Since B is a three by three matrix, this means that its rank can be at most 2. Since B has two distinct nonzero eigenvalues, its rank is exactly 2."
11. "Since the eigenvalues of a triangular matrix are its diagonal entries, the eigenvalues of A are 1, 4, and 6."
12. "Hence the eigenvalues of B are $\pm\sqrt{3}$ and 2."
13. "The eigenvalues of C are 6, 0, and 0."
14. "One subtracts 3 times the first equation from the second equation in order to eliminate the 6x."
15. "We then apply elimination on matrix A. Using the first pivot (the number 2 in the upper left corner of A), we subtract three times the first row from the second row to get: 2 and 6."
16. "Using the first pivot (the number 2 in the upper left corner of A), we subtract three times the first row from the second row to get: [2 3; 0 6]."
17. "To solve our new equation, we use back substitution: y = -1/2."
18. "The matrix is E = [1 0 0 0; -1 1 0 0; 0 -1 1 0; 0 0 -1 1]."
19. "AX = 0 precisely when the columns of X are in the nullspace of A, i.e., when they are multiples of the basis of N(A). Therefore, if AX = 0, then X must have the form: [a b c; a b c; a b c]."
20. "A matrix has the form AX if and only if all of its columns individually sum to 0."
21. "The dimension of the 'nullspace' is 3, while the dimension of the 'column space' is 6. These add up to 9, which is the dimension of the space of 'inputs' M."
22. "Since B is singular, det(B) = 0. Thus det(BTB) = det(BT) det(B) = 0."
23. "Hence the eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = 2$."
24. "Since each of the columns of A sums to one, A is a Markov matrix and definitely has an eigenvalue $\lambda_1 = 1$."
25. "If the entries of every row of A sum to one, then the entries in every row of A - I sum to zero. Hence A - I has a non-zero nullspace and det(A - I) = 0."
26. "The eigenvalues of C are 6, 0, and 0."
27. "To make the basis orthonormal, divide by the lengths of the vectors."
28. "The most obvious choice of basis is: [[1, 0], [0, 0]], [[0, 1], [0, 0]], [[0, 0], [1, 0]], [[0, 0], [0, 1]]."
29. "The matrix is of rank 2 because it has 2 pivots."
30. "The special solutions to Ax = 0 are: [[-23/4], [-1/4], [1], [0]] and [[-1/4], [-7/4], [0], [1]]."
31. "The span of $S \cup T$ is the set of all combinations of vectors in this union of two lines. In particular, it contains all sums s + t of a vector s in S and a vector t in T, and these sums form S + T."
32. "The most general form of a four by four symmetric matrix is: [[a, e, f, g], [e, b, h, i], [f, h, c, j], [g, i, j, d]]. Therefore 10 entries can be chosen independently."
33. "The most general form of a four by four skew-symmetric matrix is: [[0, -a, -b, -c], [a, 0, -d, -e], [b, d, 0, -f], [c, e, f, 0]]. Therefore 6 entries can be chosen independently."
34. "$N(C) = N(A) \cap N(B)$ contains all vectors that are in both nullspaces: Ax = 0 and Bx = 0."
35. "We take the three matrices we used to perform each operation and multiply them to get E: [[1, 0, 0], [-2, 1, 0], [-2, 3, 1]]."
36. "Finally, multiply the second row by 5 and subtract it from the first row: [[1, 0, 23/4, 1/4], [0, 1, 1/4, 7/4], [0, 0, 0, 0]]."
37. "As $k \to \infty$, $\Lambda k \to$ [[1, 0], [0, 0]]. Thus, $S\Lambda\infty S^{-1}$ = [[9, 4], [9, 4]], and in the columns of this matrix you see the steady state vector."
38. "The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = -0.3$. Corresponding eigenvectors are [9, 4] for λ_1 and [1, -1] for λ_2."

39. "The equation $x = 12 + 3y + z$ says it all: $[x, y, z] = [12 + 3y + z, y, z]$."
40. "Permutation matrices have determinants ±1. Since $P^3 = I$ and $P \neq I$, $\det(P) = 1$."
41. "The pivots are the nonzero entries on the diagonal of U. So there are four pivots when these four conditions are satisfied: $a \neq 0$, $b \neq a$, $c \neq b$, and $d \neq c$."
42. "Since the matrix has 2 pivots, the variables x3 and x4 are free and can be chosen independently."
43. "$N(C) = N(A) \cap N(B)$ contains all vectors that are in both nullspaces: $Ax = 0$ and $Bx = 0$."