Problem 1: Orthogonal Matrices

Let $\mathbf{R}$ be an $n \times n$ matrix, and let $r_i \in \mathbb{R}^n$ be the i-th column of $\mathbf{R}$. $\mathbf{R}$ is said to be *orthogonal*, if for any $i \neq j$, the vectors r_i and r_j are orthogonal to each other, and each r_i is unit length. For this class, we then also say that the vectors $\{r_1, ..., r_n\}$ form an *orthonormal basis* for $\mathbb{R}^n$ if the determinant is +1 (following the right-hand rule).

(a) Show that a square matrix $\mathbf{A}$ is orthogonal if and only if $\mathbf{A}^T\mathbf{A} = \mathbf{I}$. *Hint: Consider writing the (i, j)th entry of $\mathbf{A}^T\mathbf{A}$ in terms of dot products of the columns of $\mathbf{A}$.*

(b) Let $\mathbf{R}$ be an orthogonal $n \times n$ matrix and let u be an $n-$dimensional vector. Show that $||\mathbf{R}u|| = ||u||$. In other words, show that $\mathbf{R}$ preserves norms when it acts on vectors. *Hint: Use the fact that for the standard euclidean norm, $||u||^2 = u^T u$.*

(c) Show that if $\mathbf{R}$ is an orthogonal matrix, then $\det(\mathbf{R}) = \pm 1$. (Although as mentioned above, in this class we will primarily work with orthonormal matrices, defined as a special case of orthogonal matrices having a determinant of +1.) *Hint: Take the determinant on both sides of the equation $\mathbf{R}^T\mathbf{R} = \mathbf{I}$.*

Problem 2: A Change of Coordinates

One linear algebra concept we heavily use is the *change of coordinates*. We will spend some time developing a mathematical framework for describing how rigid objects move relative to each other, the essence of rigid body motion (the kind we see in both robot arms and mobile robots!). This all starts from the ideas of a *basis*, a set of vectors which define a coordinate system. While the coordinate transforms introduced in this course may initially feel different than the change of basis you may have seen in earlier classes, the math is essentially the same. Consider this problem a refresher on basis concepts and a taste of what's to come.

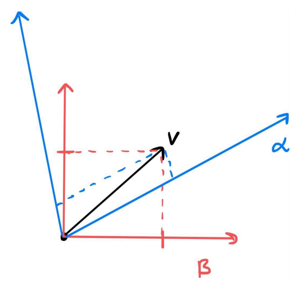

Figure 1: Vector v in two bases, α and β.

(a) Given vector v defined in terms of the standard basis and a set of basis vectors $\alpha = \{\alpha_1, \ldots, \alpha_n\}$, compute v_α, vector v in terms of the basis α.

(b) Define $G_{\alpha\beta}$ to be a change of basis matrix from basis β to basis α. (An easy way to think of this is to look at the order in the subscripts. For example, $G_{\alpha\beta}\, v_\beta$ will transform the v_β vector from the β frame to the α frame: $G_{\alpha\beta}\, v_\beta = v_\alpha$.) Given $G_{\alpha\beta}$, $G_{\gamma\beta}$ and vector v_α, compute v_γ.

(c) True or False: Orthogonality between vectors is independent of choice of basis for those vectors. That is, if a set of vectors are orthogonal in one basis, then they are orthogonal in any other basis. If true, provide a proof. If false, provide a counterexample.

(d) True or False: For any linearly independent set of vectors, we can pick a basis for those vectors such that the vectors are orthonormal in this new basis. If true, provide a proof. If false, provide a counterexample.

Problem 3: Algebraic Properties of the Matrix Exponential

Recall that for a scalar $a \in \mathbb{R}$, we can write its exponential e^a as a Taylor series that converges for any a:

$$e^a = \sum_{n=0}^{\infty} \frac{a^k}{k!} = 1 + a + \frac{a^2}{2!} + \cdots \tag{1}$$

We can similarly use an infinite series to *define* the exponential of a square real $n \times n$ *matrix* $\mathbf{A}$:

$$e^{\mathbf{A}} = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} = \mathbf{I} + \mathbf{A} + \frac{1}{2!}\mathbf{A}^2 + \cdots \tag{2}$$

Where by convention we take $\mathbf{A}^0$ to be the identity matrix for any square matrix $\mathbf{A}$. The result is also an $n \times n$ matrix. As it turns out, this infinite series converges absolutely for every matrix $\mathbf{A}$. So we use this series to define the *matrix exponential function* $e^{\mathbf{A}}$.

The matrix exponential shows up all over the place in the study of rigid body motion and dynamical systems, especially in the solutions to vector differential equations, as we shall see. We will make heavy use of the matrix exponential in this class. In this problem, you will use the infinite series representation in equation (2) to derive some of the fundamental algebraic properties of this function, which will prove very useful in our study of rigid body kinematics.

(a) Show that $e^{\mathbf{0}} = \mathbf{I}$. i.e. the exponential of the zero matrix is the identity matrix.

(b) Show that $(e^{\mathbf{A}})^T = e^{(\mathbf{A}^T)}$.

(c) Let G be any invertible square matrix of the same size as $\mathbf{A}$. Show that $e^{G\mathbf{A}G^{-1}} = Ge^{\mathbf{A}}G^{-1}$.

Hint: Start by showing that for all n, $\left(G\mathbf{A}G^{-1}\right)^n = G\mathbf{A}^n G^{-1}$.

(d) Show that if λ is an eigenvalue of $\mathbf{A}$ then e^{λ} is an eigenvalue of $e^{\mathbf{A}}$.

Hint: Use the series expansion. Show that if v is an eigenvector of $\mathbf{A}$ with eigenvalue λ then it is also an eigenvector of $e^{\mathbf{A}}$ with eigenvalue e^{λ}. i.e. show that $e^{\mathbf{A}}v = e^{\lambda}v$.

Remark: In fact, a suitable converse of the above statement is also true, though more difficult to prove. We can conclude that if the eigenvalues of $\mathbf{A}$ (possibly repeated) are $\lambda_1, ..., \lambda_n$ then the eigenvalues of $e^{\mathbf{A}}$ are exactly $e^{\lambda_1}, ..., e^{\lambda_n}$.

(e) Using the previous part, show that $\det(e^{\mathbf{A}}) = e^{\operatorname{tr}\mathbf{A}}$. Conclude that the exponential of any matrix is always invertible.

Hint: What is the relationship between the eigenvalues of a matrix, its determinant and its trace? Also use the remark from the previous part.

Remark: In fact, the inverse of $e^{\mathbf{A}}$ is simply $e^{-\mathbf{A}}$.

Problem 4: Enter the Matrix

In this problem, we'll review the solution to an important class of ordinary differential equations. In next week's lecture, we'll see the importance of these equations in describing *rotations of rigid bodies.*

(a) Solve for the solution to the ordinary differential equation $\frac{dx}{dt} = ax(t)$, for $t \geq 0$, $a \in \mathbb{R}$ and $a \neq 0$, assuming the initial condition $x(0) = x_0$. You may use the separation of variables method or a more rigorous method.

(b) For a scalar $a \in \mathbb{R}$, we know that $\frac{d}{dt}(e^{at}) = ae^{at}$. Let's examine how this property scales to the matrix exponential. The matrix exponential of At, for $A \in \mathbb{R}^{n\times n}$ and $t \in \mathbb{R}$ is defined:

$$e^{At} = I + At + \frac{1}{2!}(At)^2 + ... = \sum_{k=0}^{\infty} \frac{(At)^k}{k!} \tag{3}$$

Using this definition, show that $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$.

(c) Show that the solution to the differential equation:

$$\frac{dx}{dt} = Ax \tag{4}$$

Where $A \in \mathbb{R}^{n\times n}, x \in \mathbb{R}^n$ with initial condition $x(0) = x_0$ is given by:

$$x(t) = e^{At}x_0 \tag{5}$$

(d) **[Bonus]** Using any method you like, find the general solution to the following homogeneous system of linear differential equations:

$$\frac{dx}{dt} = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} x(t) \tag{6}$$

Your solution should be of the form $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$, for $v_1, v_2 \in \mathbb{R}^2$. *Hint: How can diagonalizing the matrix help us find a solution?*

Problem 5: Luna Masters Linux

Luna, being a dog, has never worked with Linux before. She is, however, excited for the new academic term! She wants to get organized and ready, so she wants to build some directories for all of her classes and make some files. Below is a graphic to demonstrate the hierarchy she imagines for her directories and files.

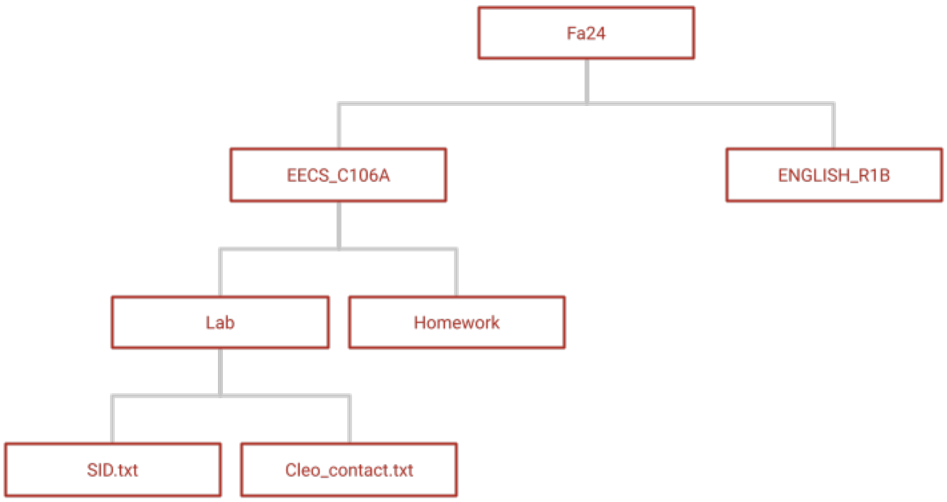

Figure 2: Luna's ideal file hierarchy

Note: If you are unfamiliar with Linux, please refer to the course website's Resources page and consult the Linux resources available there. '

(a) Luna first wants to make the Fa24 folder. She's opened Terminal and is where she wants to make the directory. What command should she run to make a directory titled "Fa24"?

(b) Luna now wants to make the EECS_C106A directory within the Fa24 directory. What command(s) should she perform to create this directory in the correct place?

(c) Luna directs herself into the EECS_C106A directory. She now wants to make two subdirectories: one for Lab, and one for Homework. Typing is hard for her (she has paws, not fingers), so she wants to make the directories in one line in Terminal. How can she do that?

(d) Wait! We're so many folders deep that Luna doesn't actually know where she is in the tree of directories. What command can she run to figure out which folder she is currently in?

(e) Whew! Now that Luna knows where she is in the tree of directories, she changes into the Lab directory. Here, she wants to make the SID.txt file that she will need to submit for part of Lab 1. How might she do that? *Hint: There are many ways to do this. Luna has heard one way to do it is to use some commands that share the same names as her cat brothers: Vim and Nano.*

(f) Now that Luna has finished making the SID.txt file, she wants to double check its contents to make sure she didn't make a typo. How can she check this within the Terminal without opening a text editor?

(g) It looks like we escaped any typos! After working on that, Luna wanted to get the contact information for Cleo, her *lab* (haha) partner. She added it to Cleo_contact.txt, but after thinking about it, realized she would rather save that information in her phone and not in this directory. What command can Luna run from the Terminal to delete this file?

(h) Luna wants to make sure that the only thing in her Lab directory is now SID.txt, so what command can she run from the Terminal to see all the contents of her current directory?

(i) Now that Luna is finished in her Lab directory, she wants to change directories to the Fa24 directory so she can make her English_R1B subdirectory. (She's taking a class offered this semester called "Good Boy: Dogs, Race, Sex, and Ethics". No, seriously, it's a real class being offered this term!) In one line, how can Luna change directories so she is in the right place? Then, how can she make a folder for her English class?

Luna is grateful for all your help!

Figure 3: a thankful puppy