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Abstract

One of the greatest scientists Albert Einstein laid the foundation of Special and General Relativity. The creation of General Relativity is considered one of the greatest discoveries in the 20th century. General Relativity is the representation of the physical universe in the form of a four-dimensional space-time manifold. The General Relativity, the seminal work of Einstein, is the basis of modern cosmology. Einstein used a cosmological constant Λ to represent the universe static. He has prepared the model of the universe on the basis that the universe is static, isotropic and homogeneous. In this study the steady state model of the Einstein universe is discussed in some details. The purpose of this article is to highlight the aspects of the Einstein's universe with easier mathematical analysis.

Keywords: General Relativity, cosmological constant, static universe.

1. INTRODUCTION

In 1915, famous scientist Albert Einstein, for the first time, introduces a new filed named General Relativity. It is considered one of the greatest physical and mathematical theories of the universe at large still today. It is a new theoretical description of gravity that is based on a completely new mathematical framework, and demanded a totally new way of thinking about the universe (1). In the late 1915, Albert Einstein submitted his completed General Theory of Relativity to the Prussian Academy of Sciences (2).

Albert Einstein in his universe assumed that the universe is static, and matter is evenly distributed throughout the universe. He introduced a cosmological constant $\Lambda > 0$ to represent the universe static. A static universe, also referred to as a *stationary* or *infinite* or *static infinite* universe, is a cosmological model in which the universe is both spatially and temporally infinite, and space is neither expanding nor contracting. Such a universe does not have spatial curvature; that is, to say that it is *flat* or *Euclidean*. A static infinite universe was first proposed by Thomas Digges in 1576 in his work *A Perfit Description of the Caelestial Orbes*, in which Digges presented and extended the Copernican system, suggesting that the universe was infinite (3).

Albert Einstein added the cosmological constant Λ to his equation of General Theory of Relativity to counteract the attractive effects of gravity on ordinary matter, which would otherwise cause a spatially finite universe to either collapse or expand forever. In an unpublished manuscript Einstein demonstrates that there is a possibility of a universe that expands but, remains essentially unchanged due to a continuous creation of matter from empty space. In 1931, Einstein compelled to admit the theoretical and observational reality of

the expansion of the universe. He took an extreme position and representation of a cosmological constant was the "biggest mistake" of his life (4). But, the cosmological constant has been a fruitful source of controversy ever since Einstein added it to his original field equation. Some researchers suggested that the introduction of the cosmological constant amounted to a redefinition of the vacuum state for the universe (5).

2. ELEMENTARY IDEAS OF GENERAL RELATIVITY

The final form of Albert Einstein's General Theory of Relativity was developed in 25 November 1915, which opens the door to think the structure of the universe. General Relativity models the physical universe as a 4-dimensional space-time manifold (M, g). Space is 3-dimensional and the fourth dimension is time. Tensors are geometric objects defined on a manifold M, which remain invariant under the change of coordinates. Tensor analysis is a generalization of vector analysis. The tensor formulation became very popular when Albert Einstein used it as an essential tool to present his General Theory of Relativity. The space-time manifold incorporates the observed continuity of space and time, the essential principle of General Theory of Relativity where the locally flat regions are glued together to obtain a globally curved continuum (6).

A contravariant vector A^μ $(\mu = 0,1,2,3)$ and a covariant vector (one form) A_μ in coordinates x^μ to x'^μ transform as;

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu, \qquad A'_\mu = \frac{\partial x'^\nu}{\partial x^\mu} A_\nu .$$ [1]

Similarly, a mixed tensor of rank three can be transformed as,

$$A'^\mu_{\nu\lambda} = \frac{\partial x'^\mu}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x'^\nu} \frac{\partial x^\gamma}{\partial x'^\lambda} A^\alpha_{\beta\gamma}$$ [2]

where we have used summation convention.

If $A_{\mu\nu}$ is symmetric then (7);

$$A_{\mu\nu} = A_{\nu\mu}$$ [3]

and if $A_{\mu\nu}$ is anti-symmetric then;

$$A_{\mu\nu} = -A_{\nu\mu} .$$ [4]

Hence, for a tensor with components $A_{\mu\nu}$, its symmetric and anti-symmetric parts are written respectively as;

$$A_{(\mu\nu)} = \frac{1}{2!}\left(A_{\mu\nu} + A_{\nu\mu}\right),$$ [5]

$$A_{[\mu\nu]} = \frac{1}{2!}\left(A_{\mu\nu} - A_{\nu\mu}\right).$$ [6]

Then, A is called symmetric if $A_{(\mu\nu)} = A_{\nu\mu}$ and it is called anti-symmetric if $A_{[\mu\nu]} = -A_{\nu\mu}$; we can express a four rank anti-symmetric tensor as;

$$A^{\mu}_{[\nu\lambda\sigma]} = \frac{1}{3!}\left(A^{\mu}_{\nu\lambda\sigma} + A^{\mu}_{\lambda\sigma\nu} + A^{\mu}_{\sigma\nu\lambda} - A^{\mu}_{\lambda\nu\sigma} - A^{\mu}_{\sigma\lambda\nu} - A^{\mu}_{\nu\sigma\lambda}\right).$$ [7]

The Kronecker delta is defined by;

$$g_{\mu\nu}g^{\nu\alpha} = g^{\alpha}_{\mu} = \delta^{\alpha}_{\mu} = \begin{cases} 1 \text{ if } \alpha = \mu \text{ (no summation)} \\ 0 \text{ if } \alpha \neq \mu. \end{cases}$$ [8]

The covariant differentiations of vectors are defined as;

$$A^{\mu}_{;\nu} = A^{\mu}_{,\nu} + \Gamma^{\mu}_{\nu\lambda}A^{\lambda}$$

[9]

$$A_{\mu;\nu} = A_{\mu,\nu} - \Gamma^{\lambda}_{\mu\nu}A_{\lambda}$$ [10]

where semi-colon denotes the covariant differentiation, and comma denotes the partial differentiation. Christoffel symbols of the first and second kind are defined respectively by;

$$[\nu\lambda, \mu] = \frac{1}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} - \frac{\partial g_{\nu\lambda}}{\partial x^{\mu}}\right), \quad \text{where } [\nu\lambda, \mu] = [\lambda\nu, \mu].$$ [11]

$$\Gamma^{\sigma}_{\nu\lambda} = \frac{1}{2}g^{\sigma\mu}\left(\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} + \frac{\partial g_{\mu\lambda}}{\partial x^{\nu}} - \frac{\partial g_{\nu\lambda}}{\partial x^{\mu}}\right), \text{ where } \Gamma^{\sigma}_{\nu\lambda} = \Gamma^{\sigma}_{\lambda\nu}.$$ [12]

By equation [10] we can write;

$$A_{\mu;\nu;\sigma} - A_{\mu;\sigma;\nu} = R^{\alpha}_{\mu\nu\sigma}A_{\alpha},$$ [13]

where $R^{\alpha}_{\mu\nu\sigma} = \Gamma^{\alpha}_{\mu\sigma;\nu} - \Gamma^{\alpha}_{\mu\nu;\sigma} + \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\sigma} - \Gamma^{\alpha}_{\beta\sigma}\Gamma^{\beta}_{\mu\nu}$

[14]

is a tensor of rank four and called Riemann curvature tensor. From equations [12] and [13] we observe that the curvature tensor components are expressed in terms of the metric tensor and its second derivatives. From equation [14] we get;

$$R^{\alpha}_{[\mu\nu\sigma]} = 0,$$

$$R^{\alpha}_{\mu\nu\sigma} + R^{\alpha}_{\nu\sigma\mu} + R^{\alpha}_{\sigma\mu\nu} = 0.$$ [15]

Taking inner product of both sides of equation [13] with $g_{\rho\alpha}$ one gets covariant curvature tensor,

$$R_{\rho\mu\nu\sigma} = \frac{1}{2}\left(\frac{\partial^2 g_{\rho\sigma}}{\partial x^{\nu}\partial x^{\mu}} + \frac{\partial^2 g_{\mu\nu}}{\partial x^{\sigma}\partial x^{\rho}} - \frac{\partial^2 g_{\mu\sigma}}{\partial x^{\nu}\partial x^{\rho}} - \frac{\partial^2 g_{\rho\nu}}{\partial x^{\sigma}\partial x^{\mu}}\right) + g_{\alpha\lambda}\left(\Gamma^{\alpha}_{\mu\nu}\Gamma^{\lambda}_{\rho\sigma} - \Gamma^{\alpha}_{\mu\omega}\Gamma^{\lambda}_{\rho\nu}\right).$$ [16]

At the pole of geodesic, coordinates system both kinds of Christoffel symbols vanish but not their derivatives, hence, equation [14] takes the form,

$$R^{\alpha}_{\mu\nu\sigma} = \Gamma^{\alpha}_{\mu\sigma,\nu} - \Gamma^{\alpha}_{\mu\nu,\sigma}.$$

Taking covariant derivative with respect to ρ at the pole we get,

$$R^{\alpha}_{\mu\nu\sigma,\rho} = \frac{\partial^2 \Gamma^{\lambda}_{\mu\sigma}}{\partial x^{\rho} \partial x^{\nu}} - \frac{\partial^2 \Gamma^{\lambda}_{\mu\nu}}{\partial x^{\rho} \partial x^{\sigma}}. \tag{17}$$

From equation [17] we get the Bianchi identity as;

$$R^{\lambda}_{\mu[\nu\sigma;\rho]} = 0,$$

$$R^{\lambda}_{\mu\nu\sigma,\rho} + R^{\lambda}_{\mu\sigma\rho,\nu} + R^{\lambda}_{\mu\rho\nu,\sigma} = 0. \tag{18}$$

This holds for all coordinates systems. Contraction of curvature tensor equation [13] gives Ricci tensor;

$$R_{\mu\nu} = g^{\lambda\sigma} R_{\lambda\mu\sigma\nu}. \tag{19}$$

Further contraction of equation [19] gives Ricci scalar;

$$\hat{R} = g^{\lambda\sigma} R_{\lambda\sigma}. \tag{20}$$

The Ricci curvature scalar $\hat{R}$ has the property that it depends only on $g_{\lambda\sigma}$ and on their derivatives only up to the second order, and $\hat{R}$ is linear in the second derivatives of the metric components. From which one gets Einstein tensor as;

$$G^{\mu}_{\nu} = R^{\mu}_{\nu} - \frac{1}{2} \delta^{\mu}_{\nu} R \tag{21}$$

where $div(G^{\mu}_{\nu}) = G^{\mu}_{\nu;\mu} = 0$. The space-time (M, g) is said to have a flat connection if and only if;

$$R^{\mu}_{\nu\lambda\sigma} = 0. \tag{22}$$

This is necessary and sufficient condition for a vector at a point p to remain unaltered after parallel transported along an arbitrary closed curve through p. This is because all such curves can be shrunk to zero, in which case the space-time is simply connected.

In Euclidean 3-dimensional space the path of shortest distance between two fixed points is a straight line. But, the path of extremum (maximum or minimum) distance between any two points in Riemannian space is called the geodesic. Let, $\gamma(t): R \to M$ be a C^1-curve in M. If T is a $C^r (r \ge 0)$ tensor field on M then the covariant derivative of T along $\gamma(t)$ is defined as;

$$\frac{DT}{\partial t} = T^{a...b}_{c...d;e} X^e. \tag{23}$$

Here, X is a tangent vector to $\gamma(t)$. Then, γ is a geodesic if the tangent vector to γ is parallel along it. In a Riemannian manifold with a positive definite metric geodesic gives the curves of shortest distance between two points p and q. The arc length between these two points on a curve $x^{\mu} = x^{\mu}(t)$ is given by;

$$S = \int_p^q g_{\mu\nu} u^\mu u^\nu, \text{ where } u^\mu = X^\mu = \frac{dx^\mu}{dt}. \qquad [24]$$

In a space-time with a Lorentzian metric the non-spacelike geodesics maximize the distance between the points defined by equation [24]. If there is a timelike geodesic between these points p and q there is no shortest distance geodesics between these points because, introducing null geodesic pieces, one could join these points by curves of arbitrary small lengths. On the other hand, any maximal length curve between p and q must necessarily be timelike geodesic. If the distance between any two points on a geodesic is zero, then the geodesic is said to be null geodesic (8).

Let, X be a vector field on M. The derivative operator ∇_X on M which gives the rate of change of vectors or tensor fields along the given vector field X at p for all points of M. If Y is another vector field at p then the operator ∇_X maps Y into a new vector field $Y \to \nabla_X Y$ such that the following conditions are satisfied:

- $\nabla_X (\alpha X + \beta Z) = \alpha \nabla_X Y + \beta \nabla_X Z; \; \alpha, \beta \in R$,

- $\nabla_{fX+gY} Z = f \nabla_X Z + g \nabla_Y Z \quad \forall$ functions f and g, and

- $\nabla_X (fY) = f \nabla_X Y + YX(f)$.

A connection ∇ at a point $p \in M$ is a rule which assigns to each vector field at p a different operator ∇_X which maps an arbitrary C^r vector field Y at p into a vector field $\nabla_X Y$ such that above conditions are satisfied. Now, if X denotes the tangent vector field along γ, then it is required that $\nabla_X X$ is proportional to X, i.e., there exists a function f such that;

$$\nabla_X X = fX,$$

$$\left(X^\mu_{;\nu} X^\nu\right) e_\mu = fX e_\mu, \qquad [25]$$

which holds for all e_μ. Hence, we can write equation [25] along the curve as;

$$X^\mu_{;\nu} X^\nu = fX^\nu. \qquad [26]$$

If $f = 0$ then equation for geodesic is;

$$X^\mu_{;\nu} X^\nu = 0. \qquad [27]$$

Let, $\{x^\mu\}$ is the local coordinate system, then $X^\mu = \frac{dx^\mu}{dt} = u^\mu$ are the components of the tangent vector to the geodesic. Here the parameter t is the affine parameter along γ, and such a situation γ is called the affinely parametrized geodesic. Now, the geodesic equation can be written as;

$$\frac{du^\mu}{dt} + \Gamma^\mu_{\nu\lambda} u^\mu u^\nu = 0. \qquad [28]$$

The energy-momentum tensor $T^{\mu\nu}$ is defined as;

$$T^{\mu\nu} = \rho_0 u^\mu u^\nu \tag{29}$$

where ρ_0 is the proper density of matter, and if there is no pressure. The component T^{00} of the energy-momentum tensor equation [29] is defined by (9),

$$T^{00} = \rho_0 \frac{dx^0}{d\tau} \frac{dx^0}{d\tau}. \tag{30}$$

A perfect fluid is characterized by pressure $p = p(x^\mu)$, then energy-momentum tensor is given by (7);

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + pg^{\mu\nu}. \tag{31}$$

The principle of local conservation of energy and momentum states that;

$$T^{\mu\nu}_{;\nu} = 0. \tag{32}$$

According to the Newton's law of gravitation, the field equation in the presence of matter is;

$$\nabla^2 \phi = 4\pi G\rho \tag{33}$$

where ϕ is the gravitational potential, ρ is the scalar density of matter, $G = 6.673 \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{sec}^{-2}$ is the Newton's gravitational constant.

If classical equation [33] is generalized for the relative theory of gravitation, then this must be expressed as a tensor equation satisfying following conditions (5);

i)   the tensor equation should not contain derivatives of $g_{\mu\nu}$ higher than the second order,

ii) it must be linear in the second differential coefficients, and

iii) its covariant divergence must vanish identically.

The most appropriate tensor of the form required is the Einstein's tensor which is given by equation [19]; then Einstein's field equation can be written as;

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T^{\mu\nu} \tag{34}$$

where $G = 6.673 \times 10^{-11} \text{m}^3\text{kg}^{-1}\text{sec}^{-2}$ is the Newtonian gravitational constant and $c = 10^8$ m/s is the velocity of light. Einstein introduced a cosmological constant $\Lambda(\approx 0)$ for static universe solutions as (10);

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -\frac{8\pi G}{c^4} T^{\mu\nu}. \tag{35}$$

In relativistic unit $G = c = 1$, that is to say, we translate our units according to, 1 s = 3×10^{10} cm = 4×10^{38} g (11). Hence, in relativistic units for $\Lambda = 0$, equation [35] becomes;

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu}. \tag{36}$$

Equation [35] can be expressed as a statement about the relative acceleration of very close test free falling particles. It is clear that divergence of both sides of equations [35] and [36] is zero. For empty space $T_{\mu\nu} = 0$ then, $R_{\mu\nu} = \Lambda g_{\mu\nu}$, and hence;

$$R_{\mu\nu} = 0 \text{ for } \Lambda = 0 \qquad\qquad [37]$$

which is Einstein's law of gravitation for empty space.

3. EINSTEIN'S STATIC UNIVERSE

In 1917, General Relativity was first applied to the universe as a whole in two different papers; the first was by Albert Einstein (2) and the second was by Willem de Sitter (12).

The constant $\Lambda > 0$ in equation [35] is such that its effect is negligible for the phenomenon in the solar system or even in our own galaxy, but becomes important when the universe as a whole is considered. By combining various values of Λ with various possibilities of $T_{\mu\nu}$ different models of the universe may be constructed. The equation [35] represents the static cosmological model of the universe. Einstein's universe is constructed on the basis that the universe is static, isotropic and homogeneous (4). This solution is marked as 'the birth of modern cosmology'. The model is based on the following assumptions (2):

i) The universe is static, i.e., in a proper coordinate system matter is at rest, and the proper pressure P_0 and proper density ρ_0 are the same everywhere.

ii) The universe is isotropic, i.e., all the spatial directions are equivalent.

iii)   The universe is homogeneous, i.e., no part of the universe can be distinguished from the other.

iv) For small values of r the line element takes the form of special relativity of flat space-time, since local gravitational field can be neglected for small space-time.

Einstein in his model considered that the mean density of matter in an expanding universe remains constant due to a continuous creation of matter from empty space, a process he associated with the cosmological constant Λ (13).

The most general static, isotropic and homogeneous universe has the familiar form (14);

$$ds^2 = -e^{\lambda} dr^2 - r^2 d\Omega^2 + e^{\nu} dt^2 \qquad\qquad [38]$$

with $0 < r < \infty$, $0 < \theta < \pi$, $0 < \phi < 2\pi$ and $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$ defines the angular part of the metric in spherical polar coordinates. Here coordinate t is timelike and other coordinates r, θ, ϕ are spacelike, and $\nu = \nu(r)$, $\lambda = \lambda(r)$. For the universe containing perfect fluid the pressure P_0 and proper density ρ_0 are determined by the field equation [35] which are given by (14);

$$8\pi P_0 = e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda \; , \qquad\qquad [39a]$$

$$8\pi \rho_0 = e^{-\lambda}\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda \; , \qquad\qquad [39b]$$

$$\frac{dP_0}{dr} = -\frac{1}{2}(P_0 + \rho_0)v' \qquad\qquad [39c]$$

where v', λ' represent the differentiation with respect to r. By the assumption (i) $\frac{dP_0}{dr} = 0$, then equation [39c] gives;

$$(P_0 + \rho_0)v' = 0. \qquad\qquad [40]$$

Equation [40] satisfies any one of the following properties (15):

$$v' = 0, \qquad\qquad [41a]$$

$$P_0 + \rho_0 = 0, \qquad\qquad [41b]$$

$$P_0 + \rho_0 = 0 \text{ and } v' = 0. \qquad\qquad [41c]$$

Einstein's universe arises from the possibility,

$$v' = 0. \qquad\qquad [42]$$

Integrating equation [42] we get,

$$v = c_1 \qquad\qquad [43]$$

where is c_1 an arbitrary constant. By the assumption (iv) $\lambda = 0 = v$ at $r = 0$, we get,

$$c_1 = 0. \qquad\qquad [44]$$

Hence, from equation [44] we get,

$$v = 0. \qquad\qquad [45]$$

Hence, equation [39a] becomes;

$$8\pi P_0 = \frac{e^{-\lambda}}{r^2} - \frac{1}{r^2} + \Lambda,$$

$$e^{-\lambda} = 1 - (\Lambda - 8\pi P_0)r^2 = 1 - \frac{r^2}{S_0^2} \qquad\qquad [46]$$

where $S_0(t)$ is the scale factor, and $\frac{1}{S_0^2} = \Lambda - 8\pi P_0$. $\qquad\qquad [47]$

The solution could correspond to a distribution of actual fluid, with $\rho \geq 0$ and $P \geq 0$, only with the cosmological constant satisfying the condition $\frac{3}{S^2} > \Lambda > \frac{1}{S^2}$ (14). Hence, equation [38] takes the form;

$$ds^2 = -\left(1 - \frac{r^2}{S_0^2}\right)^{-1} - r^2 d\Omega^2 + dt^2 \qquad\qquad [48]$$

which is the Einstein's line element for static, isotropic and homogeneous universe. In Einstein's universe we have, $\dot{S}(t) = \ddot{S}(t) = 0$ (16).

. GEOMETRICAL DESCRIPTION OF EINSTEIN'S UNIVERSE

Conformal Lorentzian geometry has played an important role in General Relativity since the work of German mathematician, physicist, and philosopher Hermann Weyl (17). In this section we discuss geometrical properties of Einstein universe (18).

Let us consider the transformation;

$$r = \frac{\rho}{1 + \frac{\rho^2}{4S_0^2}} \qquad \Rightarrow dr^2 = \frac{\left(1 - \frac{\rho^2}{4S_0^2}\right)^2}{\left(1 + \frac{\rho^2}{4S_0^2}\right)^4} d\rho^2.$$

Hence, we can write;

$$1 - \frac{r^2}{S_0^2} = 1 - \frac{\left(1 + \frac{\rho^2}{4S_0^2}\right)^2 - \frac{\rho^2}{S_0^2}}{\left(1 + \frac{\rho^2}{4S_0^2}\right)^2} \qquad \Rightarrow \left(1 - \frac{r^2}{S_0^2}\right)^{-1} dr^2 = \left(1 + \frac{\rho^2}{4S_0^2}\right)^{-2} d\rho^2.$$

Hence, equation [48] takes the form as (19);

$$ds^2 = -\left(1 + \frac{r^2}{4S_0^2}\right)^{-2}\left[d\rho^2 + \rho^2 d\Omega^2\right] + dt^2. \tag{49}$$

Let us consider,

$$x = \rho \sin\theta \cos\theta$$
$$y = \rho \sin\theta \sin\phi \tag{50}$$
$$z = \rho \cos\theta.$$

Now we can write;

$$dx^2 + dy^2 + dz^2 = d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta\, d\phi^2.$$

Hence, equation [49] takes the form;

$$ds^2 = -\left(1 + \frac{\rho^2}{4S_0^2}\right)^{-2}\left(dx^2 + dy^2 + dz^2\right) + dt^2. \tag{51}$$

Let us consider;

$$r = S_0 \sin\chi, \ 1 - \frac{r^2}{S_0^2} = \cos^2\chi \qquad \Rightarrow \left(1 - \frac{r^2}{S_0^2}\right)^{-1} dr^2 = S_0^2 d\chi^2.$$

Hence, line element [48] takes the form;

$$ds^2 = -S_0^2\left[d\chi^2 + \sin^2 \chi d\Omega^2\right] + dt^2 .$$ [52]

Let us consider;

$$z_1 = S_0\left(1 - \frac{r^2}{R_0^2}\right)^{1/2}$$

$$z_2 = r\sin\theta\cos\phi$$ [53]

$$z_3 = r\sin\theta\sin\phi$$

$$z_4 = r\cos\theta .$$

Hence, we can write after a detail calculation;

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = S_0^2 .$$

Hence, line element [52] takes the form (20);

$$ds^2 = -\left(dz_1^2 + dz_2^2 + dz_3^2 + dz_4^2\right) + dt^2 .$$ [54]

5. SPHERICAL SPACE OF EINSTEIN'S UNIVERSE

Now we consider the space is spherical. The line element [52] takes the form (21);

$$d\sigma^2 = dt^2 - ds^2 = S_0^2\left[d\chi^2 + \sin^2 \chi d\Omega^2\right].$$ [55]

The metric [55] has a singularity at $\chi = 0$. But, none make any a priori comment on the nature of such singularities without studying the behavior of relevant scalar quantities. Now we can write the total proper volume of the Einstein's spherical universe as (6);

$$V_0 = \int_{\chi=0}^{\pi}\int_{\theta=0}^{\pi}\int_{\phi}^{2\pi} S_0 d\chi S_0 \sin\chi d\theta . S_0 \sin\chi \sin\theta d\phi$$

$$V_0 = S_0^3 \int_{\chi=0}^{\pi} \sin^2 \chi d\chi \int_{\theta=0}^{\pi} \sin\theta d\theta \int_{\phi}^{2\pi} d\phi = 2\pi^2 S_0^3 .$$ [56]

The total proper distance of Einstein's spherical universe is;

$$l_0 = 2\int_0^{\pi} S_0 d\chi = 2\pi S_0 .$$ [57]

6. ELLIPTICAL SPACE OF EINSTEIN'S UNIVERSE

Let us assume that the space is elliptical, and then equation [48] becomes (6);

$$d\sigma^2 = dt^2 - ds^2 = \frac{dr^2}{1 - \dfrac{r^2}{S_0^2}} + r^2 d\Omega^2 .$$

Hence, the total proper volume of Einstein's elliptical universe is;

$$V_0 = \int_{r=0}^{R_0} \frac{dr}{\left(1 - \frac{r^2}{S_0^2}\right)^{1/2}} \int_{\theta=0}^{\pi} r d\theta \int_{\phi}^{2\pi} r \sin\theta d\phi$$

$$V_0 = \int_{r=0}^{R_0} \frac{r^2 dr}{\left(1 - \frac{r^2}{S_0^2}\right)^{1/2}} \int_{\theta=0}^{\pi} \sin\theta d\theta \int_{\phi}^{2\pi} d\phi. \quad\quad [58]$$

Let us consider, $r = S_0 \sin A$ then,

$$\int_{r=0}^{R_0} \frac{r^2 dr}{\left(1 - \frac{r^2}{S_0^2}\right)^{1/2}} = \int_0^{\pi/2} \frac{S_0^2 \sin^2 A}{\sqrt{1 - \sin^2 A}} . S_0 \cos A dA = \frac{\pi S_0^3}{4} .$$

Hence, equation [58] becomes,

$$V_0 = \pi^2 S_0^3. \quad\quad [59]$$

The total proper distance of Einstein's elliptical universe is;

$$l_0 = 2 \int_0^{R_0} \frac{dr}{\left(1 - \frac{r^2}{S_0^2}\right)^{1/2}} = \pi S_0. \quad\quad [60]$$

Hence, we see that the total proper volume and the total proper distance of the elliptic space is just half of the corresponding quantities of the spherical space.

. DENSITY AND PRESSURE IN EINSTEIN'S UNIVERSE

Using equations [46] and [47], respectively, in equations [39a] and [39b] we get (2);

$$P_0 = \frac{1}{8\pi}\left(\Lambda - \frac{1}{S_0^2}\right) \quad\quad [61]$$

$$8\pi\rho_0 = \frac{3}{S_0^2} - \Lambda$$

$$\rho_0 = \frac{1}{8\pi}\left(\frac{3}{S_0^2} - \Lambda\right). \quad\quad [62]$$

Equations [61] and [62] represent respectively the pressure and density of the Einstein's universe.

8. TOTAL MASS IN EINSTEIN'S UNIVERSE

Let us assume that the universe is filled with incoherent matter exerting no pressure then equations [61] and [62] give (2);

$$\Lambda = \frac{1}{S_0^2},$$ [63]

$$\rho_0 = \frac{1}{4\pi S_0^2}.$$ [64]

The total mass of the of the spherical Einstein's universe is;

$$M = V_0 \rho_0 = 2\pi^2 S_0^3 \cdot \frac{1}{4\pi S_0^2} = \frac{1}{2}\pi S_0.$$ [65]

The total mass of the of the elliptical Einstein's universe is;

$$M = V_0 \rho_0 = \pi^2 S_0^3 \cdot \frac{1}{4\pi S_0^2} = \frac{1}{4}\pi S_0.$$ [66]

9. RADIATION THE EINSTEIN'S UNIVERSE

Let us assume that the matter fluid filling the universe is the radiation only. For this case $\rho_0 = 3P_0$, so that the parameters Λ and S_0^2 are given by (2);

$$\Lambda = 8\pi\rho_0,$$ [67]

$$\frac{1}{R_0^2} = 16\pi P_0,$$ [68]

$$\rho_0 = 3P_0 = \frac{3}{16\pi S_0^2}.$$ [69]

The total mass of the of the spherical Einstein's radiation universe is;

$$M = V_0 \rho_0 = 2\pi^2 S_0^3 \cdot \frac{3}{16\pi S_0^2} = \frac{3}{8}\pi S_0.$$ [70]

The total mass of the of the elliptical Einstein's radiation universe is;

$$M = V_0 \rho_0 = \pi^2 S_0^3 \cdot \frac{3}{16\pi S_0^2} = \frac{3}{16}\pi S_0.$$ [71]

0. EINSTEIN'S EMPTY UNIVERSE

Let us assume that the universe is entirely empty. Then, $P_0 = 0 = \rho_0$; so that $\Lambda = 0$ and $\dfrac{1}{S_0^2} = 0$. For this case, $e^{-\lambda} = 1 - \dfrac{r^2}{S_0^2} = 1$. Thus, in this case the line element takes the form of special relativity of flat space-time (6).

11. MOTION OF A TEST PARTICLE IN EINSTEIN'S UNIVERSE

The geodesics equation [28] can be written as;

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0$$

where $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, and $x^4 = t$. Let us assume that the test particle is initially at rest, then we get;

$$\frac{dx^1}{ds} = \frac{dx^2}{ds} = \frac{dx^3}{ds} = 0, \text{ so that; } \frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{44} \left(\frac{dt}{ds} \right)^2 = 0.$$

Now, $\Gamma^\mu_{44} = \dfrac{1}{2} \left(g_{4\mu,4} + g_{4\mu,4} - g_{44,\mu} \right) = -\dfrac{1}{2} \dfrac{\partial g_{44}}{\partial r} = 0$. Then we get,

$$\frac{d^2 r}{ds^2} = \frac{d^2 \theta}{ds^2} = \frac{d^2 \phi}{ds^2} = 0. \tag{72}$$

That is, the particle has zero acceleration. Hence, in Einstein universe a rest particle would remain permanently at rest. We observe that, Einstein's universe is full of matter, but it does not predict the observed recession of nebulae. Hence, Einstein's universe does not represent a true model of the actual universe (2).

12. CONCLUSIONS

In this study we have introduced Riemann curvature tensor, covariant curvature tensor, Bianchi identity, Ricci tensor, Ricci curvature scalar, the geodesic equation, the energy-momentum tensor, Einstein's field equation, etc., in brief. Then, we have derived the Einstein's model for static, isotropic and homogeneous universe. Further, we have discussed geometrical description, spherical and elliptical spaces, density and pressure, total mass, radiation and motion of a test particle in Einstein's universe. This article will be benefited for the beginners who need the elementary ideas in General Relativity. We hope, the readers feel the usefulness of the Einstein universe in the foundation of the modern cosmology.

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