Two Basic Factions

- Sinusoidal Functions;
- Exponential Functions.

Sinusoidal Functions

A sinusoid function is a mathematical function that describes a smooth periodic oscillation.

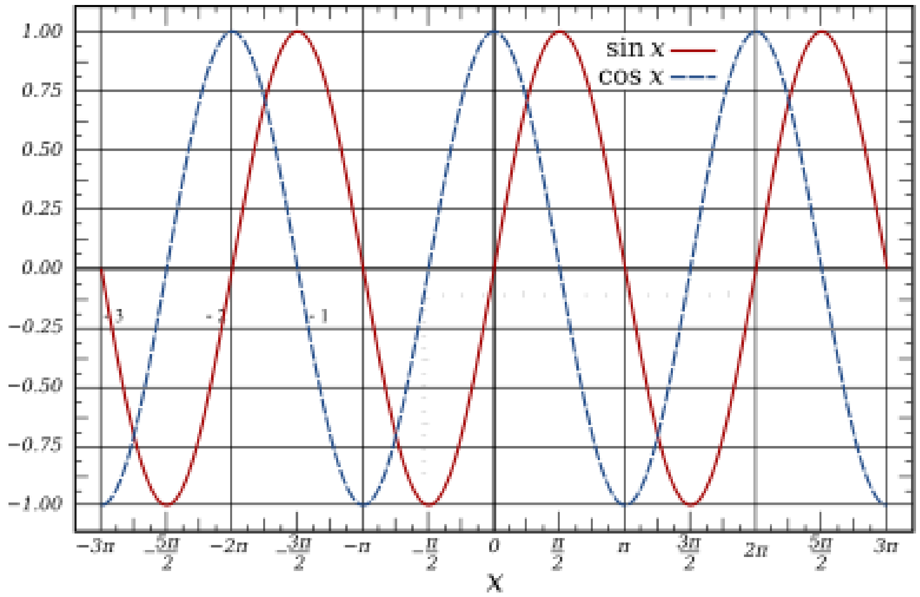
$$y(t) = A\sin(\omega t + \varphi)$$

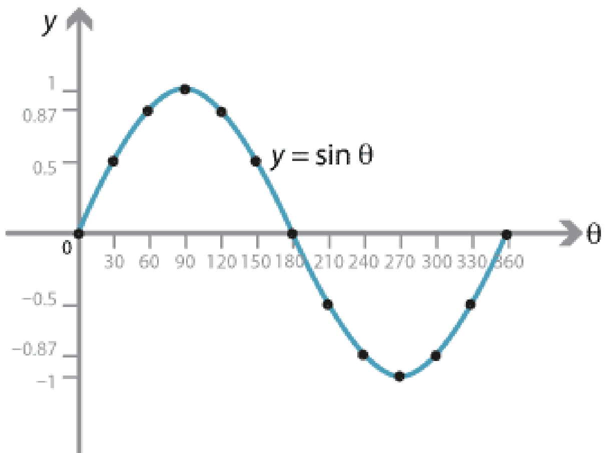
A: amplitude

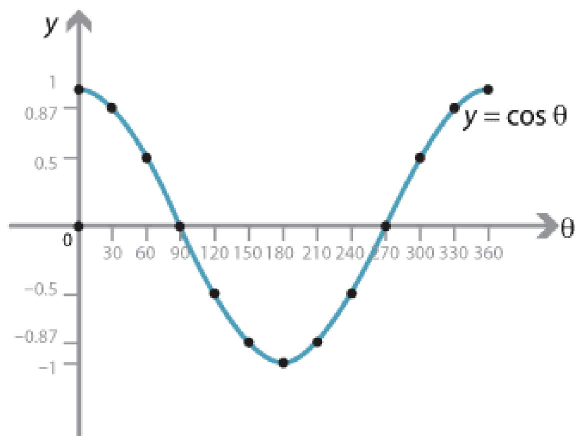
ω: angular frequency, rad/sec

$\omega = 2\pi f$ $\quad f$: frequency, Hz

φ: phase, radians, where in its cycle the oscillation is at t = 0.







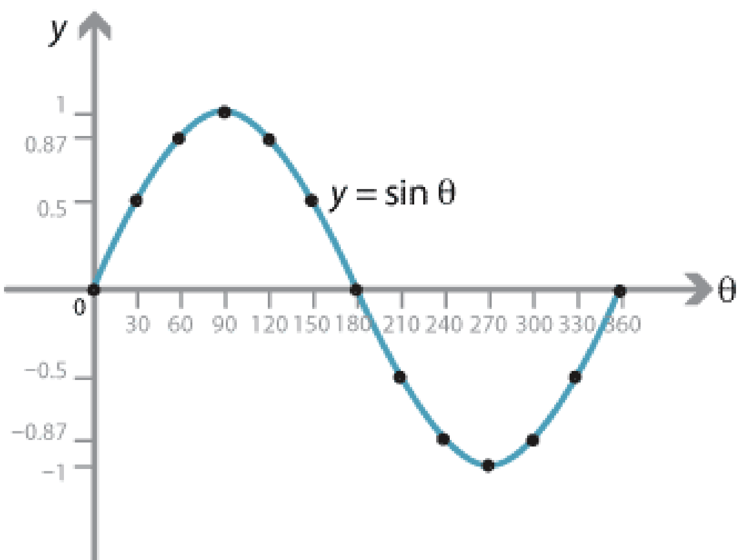
Sinusoidal Functions

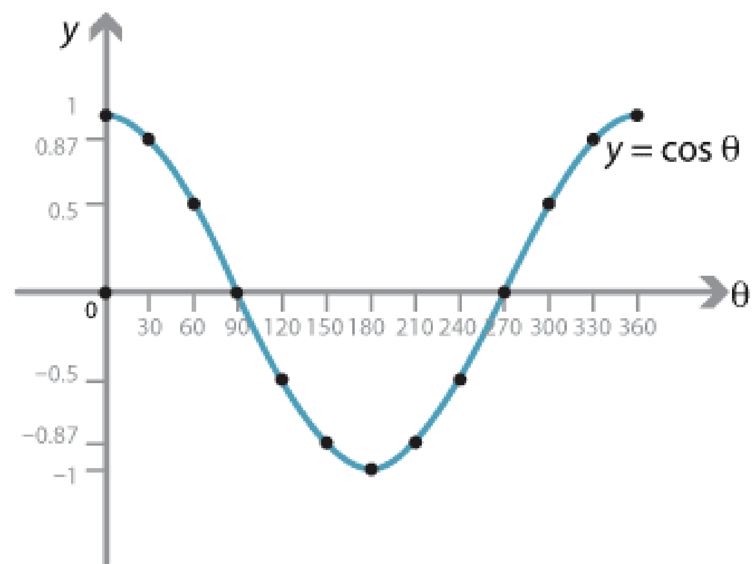
Some special values of the sin and cos functions:

1. $\theta = 2n\pi, \quad \sin\theta = 0, \quad \cos\theta = 1 \qquad n = 1,2,3,\ldots$

2. $\theta = (2n-1)\pi, \quad \sin\theta = 0, \quad \cos\theta = -1$

3. $\theta = \dfrac{(2n-1)\pi}{2}, \quad \sin\theta = (-1)^{n-1}, \quad \cos\theta = 0$





Frequency and Period of Sinusoidal Function

How to find frequency and period for a give sinusoid function?

$$y(t) = A\sin(\omega t + \varphi)$$

ω: angular frequency, rad/sec

Frequency: $f = \dfrac{\omega}{2\pi}$ Period: $T = \dfrac{1}{f} = \dfrac{2\pi}{\omega}$

Examples: $y(t) = 10\sin(50t + 10)$ What is its period?

$$\omega = 50\, rad / \sec$$

$$f = \frac{\omega}{2\pi} = \frac{25}{\pi} Hz$$

$$T = \frac{1}{f} = \frac{2\pi}{\omega} = \frac{\pi}{25}\sec$$

Frequency and Period of Sinusoidal Function

Example 2: $x(t) = \sin^2(20\pi t + 5)$ What is its period and frequency?

$$x(t) = \frac{1}{2}[1 - \cos 2(20\pi t + 5)] = \frac{1}{2} - \frac{1}{2}\cos(40\pi t + 10)$$

$$\omega = 40\pi \, rad/\sec \qquad f = \frac{\omega}{2\pi} = \frac{40\pi}{2\pi} = 20\,Hz$$

$$T = \frac{1}{f} = \frac{1}{20}\sec$$

Exponential Function e^{st}

One of the most important functions in this class is the exponential signal e^{st}, where s is complex in general, given by:

$$s = \sigma + j\omega$$

Therefore:

$$e^{st} = e^{(\sigma + j\omega)t} = e^{\sigma t} e^{j\omega t} = e^{\sigma t}(\cos \omega t + j \sin \omega t)$$

For the conjugate of s ($s^* = \sigma - j\omega$):

$$e^{s^* t} = e^{(\sigma - j\omega)t} = e^{\sigma t} e^{-j\omega t} = e^{\sigma t}(\cos \omega t - j \sin \omega t)$$

Moreover:

$$e^{\sigma t} \cos \omega t = \frac{1}{2}(e^{st} + e^{s^* t})$$

$$e^{\sigma t} \sin \omega t = \frac{1}{2j}(e^{st} - e^{s^* t})$$

Exponential Function e^{st}

Some special cases of the exponential function:

1. A constant $k = ke^{0t} \qquad (s = 0)$

2. A monotonic exponential $e^{\sigma t} \qquad (s = \sigma)$

3. A sinusoid function $\cos \omega t \qquad (s = \pm j\omega)$

4. An exponentially varying sinusoid function

$$e^{\sigma t} \cos \omega t \qquad (s = \sigma \pm j\omega)$$

Exponential Function e^{st}

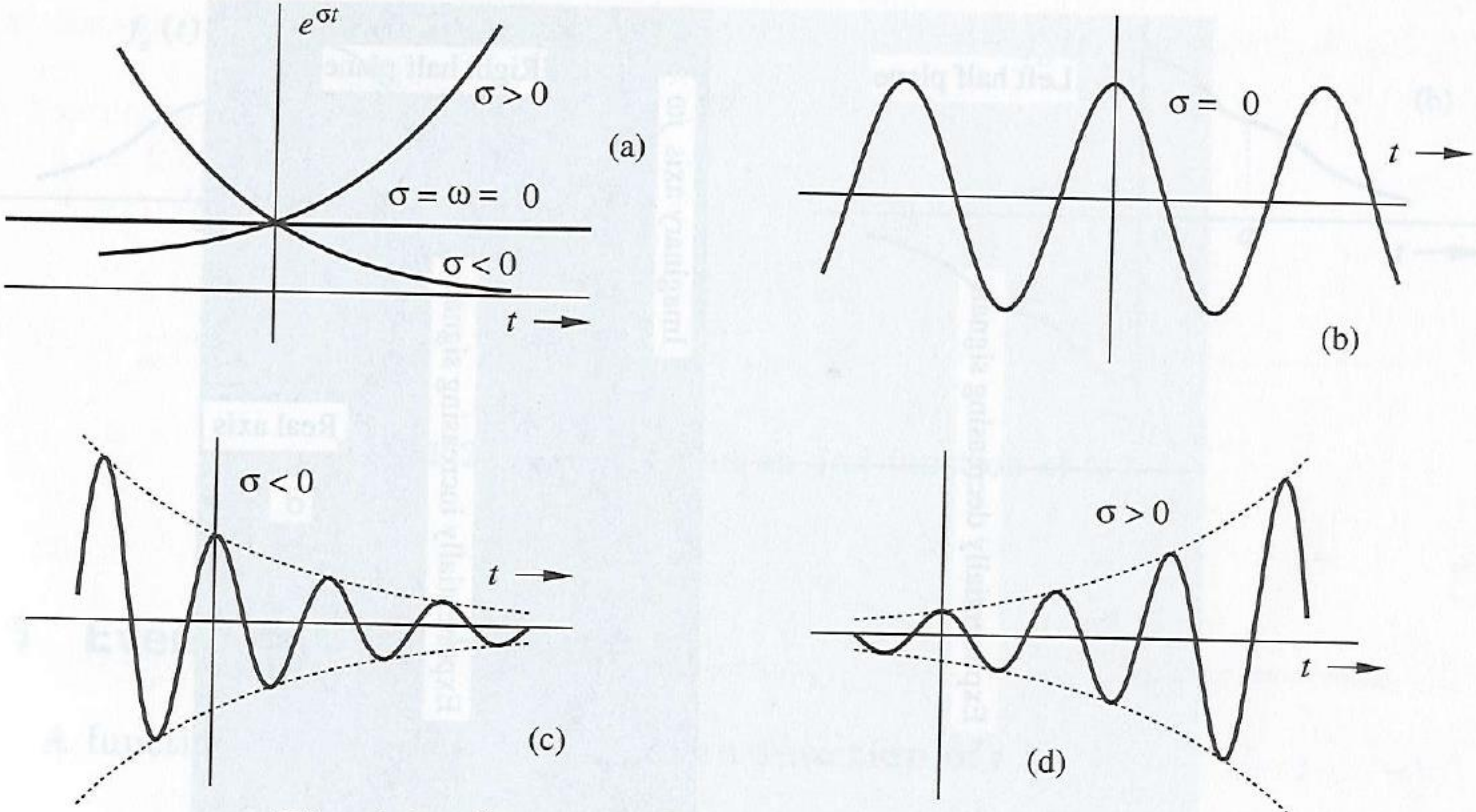


Fig. 1.21 Sinusoids of complex frequency $\sigma + j\omega$.

Euler's Formulas

Euler's formula states that, for any real number x,

$$e^{ix} = \cos x + i\sin x$$

$$e^{-ix} = \cos x - i\sin x$$

$$\cos x = \frac{1}{2}[e^{ix} + e^{-ix}]$$

$$\sin x = \frac{1}{2i}[e^{ix} - e^{-ix}]$$

$$z = r(\cos\theta + i\sin\theta)$$

$$\Rightarrow \qquad z = re^{i\theta}$$

Exponential Function

Example 1: Given exponential function $x(t) = je^{(5-j10\pi)t}$

(1) Find the real part and imaginary part of the function.

$$x(t) = je^{5t}e^{-j10\pi t} = je^{5t}[\cos(10\pi t) - j\sin(10\pi t)]$$

$$= e^{5t}[j\cos(10\pi t) + \sin(10\pi t)]$$

$$\operatorname{Re}[x(t)] = e^{5t}\sin(10\pi t) \qquad \operatorname{Im}[x(t)] = e^{5t}\cos(10\pi t)$$

(2) Find the frequency *f* of the function.

$$\omega = 10\pi\, rad/\sec \qquad f = \frac{\omega}{2\pi} = \frac{10\pi}{2\pi} = 5\,Hz$$