Chapter 2

Life on Manifolds

Preparation: read Chapter 1, Sections 1–3.

Manifold as an arena for geometric objects to exist. To describe mathematically any "life" like processes, functions, vector fields and other geometric objects, one starts with a set on which this life is happening, a sort of arena for the description. It may be as simple as a discrete graph or as specific as a Euclidean space, Hilbert space, or often the space of states of some process. For instance, in simple applications in physics or engineering one often deals with the 3-dimensional space $\mathbb{R}^3$ with the familiar coordinates $\{x, y, z\}$.

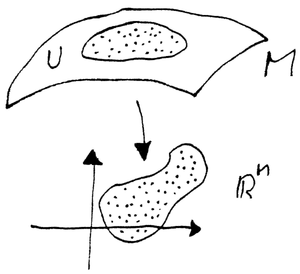
It may be a sphere (the surface of the Earth) or its subset. But the planet may be in the shape of torus. One may also be dealing with more abstract object, like the set of all circles in plane, or all configurations of three free particles, or location and orientation of a book in space.

Differential geometry offers a big family of such objects with these two goals in mind:

1. One wants to have **differentiability** to be well-defined. This requirement goes beyond the concept of topological space where one has only continuity.
2. One wants conceptual freedom from coordinates, and be able to define object without them, i.e., in a **coordinate-free** manner.

The idea of a **differential manifold** answers these requirements. It admits coordinates when they are needed, but none are special. In particular, we will discover how to define various things like vector, curve, covector, various derivatives **without** bringing up coordinates. With this, one discovers the inner beauty of life on manifolds. But the intellectual aesthetics is not the only benefit. It allows one to rethink many ideas in mathematics and physics, including even revealing the inner elegance of the 3d calculus.

2.1 Differential Manifolds

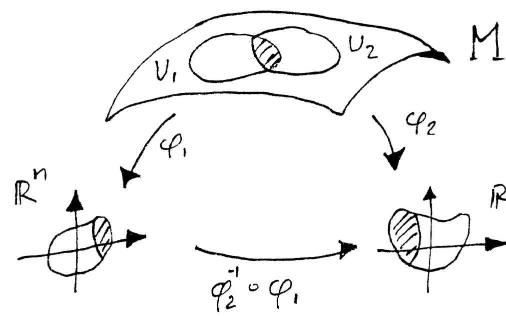


Consider a set M. In a few steps, we shall build a structure of differentiability on M.

1. **Chart** on a M is a pair (U,φ) where $U\subset M$ is a subset of M, and φ is a one-to-one map $\varphi:\ U\longrightarrow\mathbb{R}^n$.
2. Two charts (U_1,φ_1) and (U_2,φ_2) are **compatible** if the composition of maps

$$\varphi_2\circ\varphi_1^{-1}:\ \mathbb{R}^n\longrightarrow\mathbb{R}^n$$

is differentiable as a map $\mathbb{R}^n\to\mathbb{R}^n$. (This is a "change of variables"). Clearly $\varphi_2\circ\varphi_1^{-1}$ is well-defined only on a subset $\varphi_1(U_1\cup U_2)\subset\mathbb{R}^n$.



3. **Atlas** $\mathcal{A}$ on M is a collection of mutually compatible charts $\{U_i,\varphi_i\}$ such that every point $p\in M$ belongs to at least one U_i.
4. An atlas is **complete** if it contains all mutually compatible charts.
5. A **differentiable manifold** M is a set M with a complete atlas $\mathcal{A}$, such that
    1. one can choose a countable subset of charts $\{U_i,\varphi_i\}_{i\in\mathbb{N}}\subset\mathcal{A}$ that covers M, i.e., $\bigcup_{i\in\mathbb{N}}U_i=M$
    2. for any two points p and q in M there exist two charts such that $p\in U_1$, $q\in U_2$, and $U_1\cap U_2=\varnothing$.
    3. M is connected.

The dimension of M is well-defined as $\dim M=n(=\dim\operatorname{Im}\varphi)$.

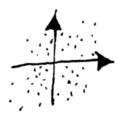
Terminology: By "differentiable" one means differentiable as many times as needed. We shall use term **smooth** to take care of this ambiguity.

Remark: Recall that a function is said to be C^k if it is k times differentiable with the k-th derivative continuous. Whitney has shown that every complete C^k-atlas contains a complete C^{k+1}-atlas, and — thus — a C^∞-atlas. Moreover, it contains a complete C^ω-atlas (analytic). If not stated otherwise, we shall assume C^ω-differentiability of charts and all maps. The word smooth will mean that.

Remark (for topologists): An alternative definition is often used: A differentiable manifold is a connected Hausdorff topological space with every neighborhood homeomorphic to Euclidean space.

Remark: Somewhat surprisingly, any manifold may be realized in $\mathbb{R}^m$ for some m sufficiently large (Whitney's Theorem) not greater then $2\dim M+1$. It has to be borne in mind however that such embedding is not unique, and often of little help.

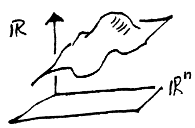
EXAMPLES:



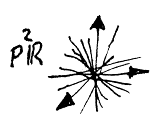
$M \cong \mathbb{R}^n$
linear space
$\dim M = n$

$M \cong S^k = \{v \in \mathbb{R}^{k+1} : |v|^2 = 1\}$
k-sphere in $\mathbb{R}^{k+1}$
$\dim M = k$

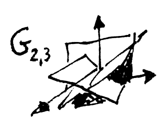
$M \cong T^k = S^1 \times \ldots \times S^1 \quad (k \text{ times})$
k-torus
$\dim M = k$



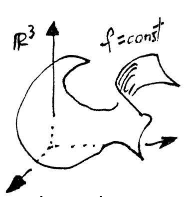
$M \cong \text{graph}\ \{f : \mathbb{R}^n \to \mathbb{R}\} \subset \mathbb{R}^{n+1}$
graph of a (differentiable) real function
$\dim M = n$



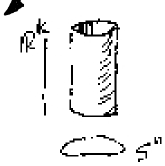
$M \cong P^n\mathbb{R}$
projective space
$\dim M = n$



$M \cong \mathcal{G}_{n,k}$
Grassmann manifold = set of all k-planes in $\mathbb{R}^n$ through $\mathbf{0}$
$\dim M = nk$



$M \cong \{x \in \mathbb{R}^n \mid f(x) = 0\}$
Level set of a scalar function $f : \mathbb{R}^n \to \mathbb{R}$
$\dim M = n-1$



$M \cong S^n \times \mathbb{R}^k$
Cylinder (generalized)
$\dim M = n+k$

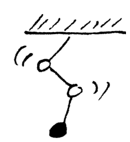
Exercise: Find an atlas for each of these examples.

Examples from mathematics: Any Lie group is a manifold by definition. It is defined as a manifold on which the group product is defined. Examples: the group of rotations in an n-dimensional space. Or unitary transformations of a complex space (Quantum mechanics).

For details, see Chapter X.

Examples from Physics: Configuration space Q is a manifold of all configurations that a physical system may assume. Examples of configuration spaces:

1. $Q = \mathbb{R}^3$ $\longleftarrow$ a single particle in 3D space
2. $Q = \mathbb{R}^3 \times \mathbb{R}^3 \cong \mathbb{R}^6$ $\longleftarrow$ two particles in 3D space
3. $Q = S^1$ $\longleftarrow$ a planar pendulum
4. $Q = T^3 \equiv S^1 \times S^1 \times S^1$ $\longleftarrow$ a triple planar pendulum
5. $Q = S^3$ $\longleftarrow$ spin of an electron; also light polarization.



Another source of manifolds as physical models provides **cosmology**: the universe as a whole is considered as a 3-dimensional manifold. Its overall shape remains to be determined. Including time, it is a 4-dimensional manifold.

Problem (Untnapishtim): This is a recurring and somewhat challenging problem. According to a Sumerian legend, after 40 days of rain the Earth was covered by water with only one boat on the surface with the hero Utnapishtim and his family and a good selection of animals. If the size of the boat is ignored one may consider the configuration space (manifold) to be S^2 (a sphere). But suppose we consider also the orientation of the boat (like East, West etc). Since the orientations form at every point of the planet a circle, S^1, one might guess that:

$$Q \equiv S^2 \times S^1 \qquad \text{Utnapishtim?}$$

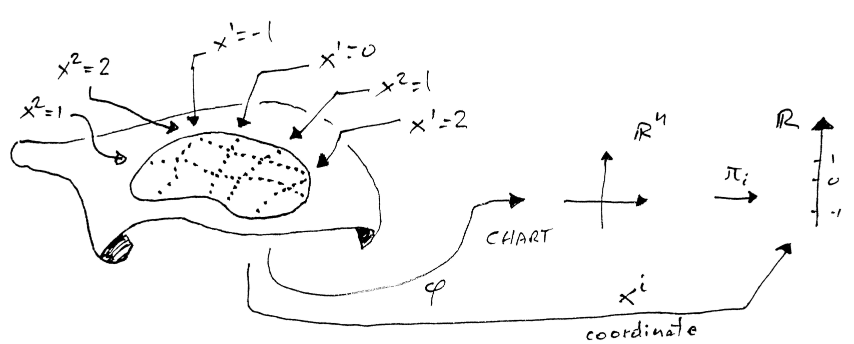
It however is **not** so. Why not? The problem will reappear later.

More on charts and coordinates

In $\mathbb{R}^n$, there is a well-defined map $\pi_i : \mathbb{R}^n \to \mathbb{R}$ of projection onto the i-th copy of $\mathbb{R}$ in $\mathbb{R}^n$. Consider a chart φ on a manifold M. A **coordinate** function is the composition of functions

$$x^i =: \pi^i \circ \varphi : M \to \mathbb{R}.$$

We shall often say that a "coordinate system $\{x^i\}$ is given" on a manifold M, which is equivalent to stating that "a chart given."



2.2 (Smooth) life on manifolds

Next thing is to see what kind of objects can "live" on manifolds. The basic are: curves, scalar functions, vectors, co-vectors, exterior forms and, in general, tensors. Later, we shall seek "natural" operations on these objects.

Terminology: We shall say and denote that there is given a **local** map from A to B when we mean that the map is well-defined on an open neighborhood of A. Thus

$$f : A \circ\!\!\longrightarrow B \qquad \text{actually means} \qquad f : A \supset U \;\to\; B$$

2.2.1 Curves and scalar functions

We shall start with a manifold M and a copy of real numbers $\mathbb{R}$. There are two possible maps that can be defined:

$$\begin{array}{rl} \text{Curve} & c : \mathbb{R} \longrightarrow M \\ \text{Scalar function} & f : M \longrightarrow \mathbb{R}. \end{array}$$

This simple twofoldness (a map *from* versus map *to*) is the source of the duality that permeates differential geometry. Let us consider the two maps in a chart φ:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "c"] \arrow[dr, "\varphi \circ c"'] & \mathrm{M} \arrow[r, "f"] \arrow[d, shift right, "\varphi"'] & \mathbb{R} \\
& \mathbb{R}^n \arrow[u, shift right, "\varphi^{-1}"'] \arrow[ur, "f \circ \varphi^{-1}"'] &
\end{tikzcd}$$

A curve or a scalar function is *smooth at point* $p \in M$, if it is C^ω-differentiable in any chart at this point. It is just *smooth* if it is smooth everywhere.

Exercise: Show that if a curve or a scalar function is smooth in one chart, then it is smooth in all charts of the atlas.

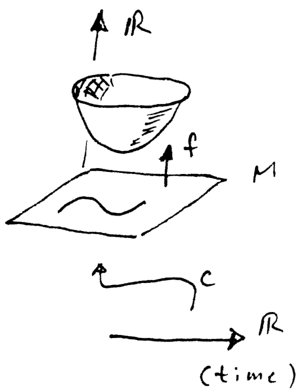
Notation: $\mathcal{F}M$ = the set of all smooth functions on M.
$\mathcal{C}M$ = the set of all smooth curves in M.

The set $\mathcal{F}M$ forms a *ring* with the usual multiplication of functions as the product.

Note that, in practice, we always describe curves and scalar functions in some particular chart, i.e., as functions

$$\bar{f} = f \circ \varphi^{-1} \qquad \text{and} \qquad \bar{c} = \varphi \circ c$$

although often for simplicity the component φ (or the *bar*) will be suppressed in writing.



Example: Consider a two dimensional manifold M with a chart $M \to \mathbb{R}^2$. Example of a scalar function:

$$\tilde{f} = f \circ \varphi^{-1} = x^2 + y^2 + 2$$

A curve c recorded in coordinates, $\tilde{c} = \varphi \circ c$, is may be reported in a few ways:

precise	acceptable	confusing
$\begin{cases} x \circ c(t) = t^2 \\ y \circ c(t) = 1 - t \end{cases}$	$\begin{cases} x(t) = t^2 \\ y(t) = 1 - t \end{cases}$	$\begin{cases} x = t^2 \\ y = 1 - t \end{cases}$

The first follows is accurate, the second is a reasonable shortcut, but the third, quite most frequently used, is rather sloppy and should be avoided. The composition $f \circ c : \mathbb{R} \to \mathbb{R}$ does not depend on the choice of a chart! That is, $\bar{f} \circ \bar{c} = f \circ c$. In our example, $(f \circ c)(t) = t^4 + (1-t)^2 + 2$.

In the following, when defining functions and curves on M in coordinates, for simplicity the tilde will be omitted.

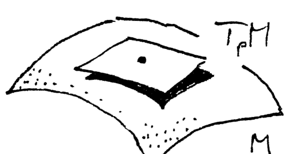
2.2.2 What is a (tangent) vector and tangent space.

Intuition: every vector is a vector of velocity. Velocity requires change; change requires comparison, comparison requires difference — this can be detected by a scalar function. Hence:

Definition 2.2.1. A tangent vector at point $p \in M$ is an operator $\mathbf{v}_p : \mathcal{F}M \to \mathbb{R}$ such that the following are satisfied:

$$\begin{aligned} (i) &\quad \mathbf{v}(f+g) = \mathbf{v}(f) + \mathbf{v}(g) \\ (ii) &\quad \mathbf{v}(cf) = c \cdot \mathbf{v}(f) \\ (iii) &\quad \mathbf{v}(f \cdot g) = \mathbf{v}(f) \cdot g(p) + f(p) \cdot \mathbf{v}(g) \qquad \text{(Leibniz' rule)} \end{aligned}$$

The set of tangent vectors at point $p \in M$ forms a *linear space* under addition and scaling defined as



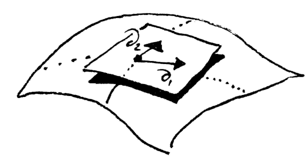
$$\begin{aligned} (i) &\quad (\mathbf{v} + \mathbf{w}) f = \mathbf{v}(f) + \mathbf{w}(f) \\ (ii) &\quad (c\mathbf{v}) f = c \cdot \mathbf{v}(f) \end{aligned}$$

We shall call this space the **tangent space** at point $p \in M$ and denote by T_pM. Tangent spaces at different points are independent entities.

The definition of a tangent vector might seem somewhat abstract and therefore not handy. But it turns out that in a chart every vector can be expressed as a linear combination of partial derivatives.

Theorem 2.2.1. *Let $\{x^i\}$ be a chart at $p \in M$. Denote partial derivatives at point p as*

$$\left.\frac{\partial}{\partial x^i}\right|_p , \qquad \text{or simply} \qquad \frac{\partial}{\partial x^i} \equiv \partial_{x^i} \equiv \partial_i \,.$$



The set $\{\partial/\partial x^i\}$ forms a basis in T_pM. Any vector $v \in T_pM$ may be represented as

$$\mathbf{v} = v^i\, \partial_i \tag{2.1}$$

(sum over i), where the i-th component of v is defined as

$$v^i = v(x^i)\,. \tag{2.2}$$

Proof: First, show that if f is a constant function, then $v(f) = 0$ for any vector v (Hint: start with applying the Leibniz rule to function $f = 1 = 1 \cdot 1$). Then consider Taylor's expansion of f in a chart around a point p and apply the properties listed in the definition of a vector.

$$f(\mathbf{x}) = f(\mathbf{p}) + \frac{\partial f(p)}{\partial x^i}(x^i - p^i) + \frac{1}{2}\frac{\partial^2 f(p)}{\partial x^i \partial x^j}(x^i - p^i)(x_j - p_j) + \ldots$$

Higher-order terms vanish when evaluated at p. Work out the details. □

Corollary: Tangent space has the same dimension as the manifold, $\dim T_pM = \dim M$.

Example: In a certain coordinate system $\{x, y\}$ on a 2-dimensional manifold M a vector $v \in T_pM$ and a function f be given

$$\mathbf{v} = 2\partial_x + 7\partial_y \quad \text{at } (x(p), y(p)) = (2,5), \qquad f = x^3 + xy$$

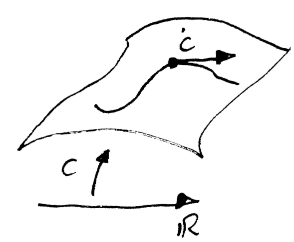
Then

$$\mathbf{v}f \;=\; 2\partial_x f + 7\partial_y \,\Big|_p \;=\; 2\cdot(3x^2 + y) + 7x\,\Big|_{(2,5)} p \;=\; 2\cdot(3\cdot 2^2 + 5) + 7\cdot 2 = 48$$

2.2.3 Vector of velocity

A natural source of vectors are curves. Every smooth curve c through p with, say, $c(t_0) = p$, defines a vector in T_pM denoted $\dot{\mathbf{c}}$, via this formula:

$$\dot{\mathbf{c}}(f) = \left.\frac{d}{dt}\right|_{t_0} f \circ c\,(t) \tag{2.3}$$



for any scalar function $f \in \mathcal{F}M$. It is easy to check that (2.3) is indeed linear in f and satisfies the Leibniz rule. Such a vector is called the **velocity vector** of the curve at time t_0. A smooth curve determines a vector (of velocity) at every point it passes.

Now, the question is how one expresses the velocity vector in terms of the basis $\{\partial_i\}$ corresponding to some coordinate system $\{x^i\}$. The answer lies in formula (2.2). If $\dot{\mathbf{c}} = \dot{c}^i \partial_i$ then

$$\dot{\mathbf{c}} \;=\; \dot{c}^i\,\partial_i = \;\dot{c}(x^i)\,\partial_i \;=\; \frac{d(x^i\circ c)\,(t)}{dt}\,\partial_i \qquad \text{in } T_{c(t)}M$$

(sum over i).

Example: Let the manifold be $M \cong \mathbb{R}^2$ with coordinates $\{x,t\}$, and a curve $c \in \mathcal{C}$ be described by $t \mapsto (t^2+2t+6,\, 2t-7)$. The velocity vector at $t=0$ is a vector at $p=(6,-7)$ of he form

$$\dot{\mathbf{c}} \;=\; \frac{d(t^2+2t+6)}{dt}\Big|_{t=0}\partial_x \;+\; \frac{d(2t-7)}{dt}\Big|_{t=0}\partial_y \;\;=\;\; 2\,\partial_x - 7\,\partial_y$$

Corollary: Any tangent vector may be viewed as velocity vector along some curve. Notice that the partial derivatives $\partial/\partial x^i$, the basis vectors, are also of this kind. For instance, for vector $\partial/\partial x^1$ at point $p=(x_0^1,\, x_0^2,\, x_0^3,\, \ldots)$, the corresponding curve is $t \mapsto (x_0^1+t,\, x_0^2,\, x_0^3,\, \ldots)$.

Proposition [chart change]: Let $\{x^1,\ldots,x^n\}$ and $\{y^1,\ldots,y^n\}$ represent two local charts with nonempty intersection U. The rule of change of chart for a vector $v \in T_pM$ is

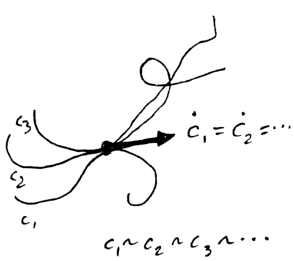
$$\mathbf{v} = v^i\,\partial_{x^i} = v^i\,\frac{\partial y^k(x)}{\partial x^i}\,\partial_{y^k}$$

Proof: Left as an exercise. [Hint: use the same trick as in the case of the velocity vector above.] $\square$

The matrix $J_i^k = \frac{\partial y^k(x)}{\partial x^i}$ (function of $p \in M$) is called the **Jacobian** of the chart change. It is often expressed shortly $J_i^k = \partial y^k/\partial x^i$. $\square$

Exercise: Consider two charts in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, the standard rectangular $\{x,y\}$ and the polar $\{r,\varphi\}$. Express the vector field $\mathbf{v} = \partial_y\,|_{(1,1)}$ in the latter chart..

A remark on an alternative definition of a tangent vector. Note that two curves through a point p may determine the same vector. This leads to another definition of a tangent vector. Consider the set C_pM of all smooth curves through $p \in M$, each at $t=0$. Two curves are called *equivalent*, $c_1 \sim c_2$, if



$$\frac{d\,x^i\circ c_1}{dt}\Big|_{t=0} = \frac{d\,x^i\circ c_2}{dt}\Big|_{t=0}$$

for all i in an (arbitrary) coordinate system around p. Check that $\sim$ defines an *equivalence relation*.

Proposition (Definition 2): There is one-to-one correspondence between **tangent vectors** at p and equivalence classes of smooth curves through p. That is the tangent space can be viewed as the quotient space

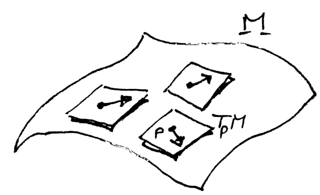
$$T_pM \cong C_pM/\sim$$

Conclusion: Every vector can be always modeled by a curve c that represents the corresponding equivalence class $[c]$. The above definition via curve equivalence – often evoked in textbooks – does not go well along our "coordinate-free philosophy". It may be improved as follows:

$$c_1 \sim c_2 \qquad \text{if} \qquad \frac{d\, f\circ c_1}{dt}\Big|_{t=0} = \frac{d\, f\circ c_2}{dt}\Big|_{t=0} \qquad \forall f \in \mathcal{F}M\,.$$

2.2.4 Vector fields

Definition 2.2.2. A **vector field** on M is a function A that associates to every point $p \in M$ a tangent vector at this point:



$$A:\ M \ni p \ \mapsto\ A_p \in T_pM$$

Vector field is **smooth**, if $Af \in \mathcal{F}M$ for any $f \in \mathcal{F}M$. The set of smooth vector fields on M will be denoted by $\mathcal{X}M$.

A vector field $X \in \mathcal{X}M$ may be also viewed a a different map, namely a linear map from the space of function into the same space:

$$A : \mathcal{F}M \ni f \ \mapsto\ Af \in \mathcal{F}M$$

so that the function Af takes at point $p \in M$ the value $(Af)(p) = A_p f$.

EXAMPLE: Let a vector field and a scalar function be given in a local coordinate system on a 2-dimensional manifold:

$$A := \sin y\, \partial_x + \cos^2 x\, \partial_y \qquad \text{and} \qquad f(x,y) := x^2 + y^3$$

Then $Af = 2x\sin y + 3y^2\cos^2 x$ (check).

Proposition: Show that the set $\mathcal{X}M$ is an infinite-dimensional space over $\mathbb{R}$ and also may be viewed as an $\mathcal{F}M$-module.

Exercise: [chart change] Show that if $x = \{x^1,\dots,x^n\}$ and $y = \{y^1,\dots,y^n\}$ are two local charts with nonempty intersection, then the rule of the "change of basis" for a vector field $A \in \mathcal{X}M$ is

$$A = A^i\, \partial_{x^i} = A^i\, \frac{\partial y^k(x)}{\partial x^i}\, \partial_{y^k}$$

Exercise: Let $M \cong \mathbb{R}^2$ with the standard rectangular coordinates$\{x,y\}$. Draw the following vector fields:

$$(a)\ A = x\partial_y \qquad (b)\ B = y\partial_x \qquad (c)\ C = x\partial_x + y\partial_y$$

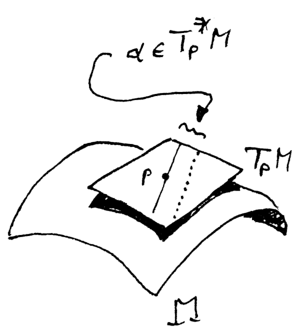
(pick a number of points on M). You may consider rescaling the vectors by a factor of 2 and see if it helps with clarity.

Exercise: Consider two charts in a plane with removed center, $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, namely: the standard rectangular $\{x,y\}$ and the angular $\{r,\varphi\}$. Express the vector field $X = x\partial_y - y\partial_x$ in the latter chart. Draw X.

2.2.5 What is a covector. Cotangent space

Definition 2.2.3. A cotangent space at point $p \in M$ is the space dual to the tangent space and is denoted

$$T_p^*M := (T_pM)^*$$



The elements of T_p^*M are called **co-vectors** at p, or just **1-forms** at p.

Similarly to the concept of vectors and vector fields, we can define a "**covector field**" on manifold M, however it is typically called rather a "differential 1-form". More precisely:

Definition 2.2.4. A **differential 1-form** is a function α that associates to every point $p \in M$ a linear form $\alpha_p \in T_p^*M$. Form α is **smooth** if $\alpha(A) \in \mathcal{F}M$ for any $A \in \mathcal{X}M$. The set of smooth 1-forms on M will be denoted by $\Lambda^1 M$.

Exercise: Show that the set the set $\Lambda^1 M$ is an infinite-dimensional space over $\mathbb{R}$ and also may be viewed as an $\mathcal{F}M$-module.

We have seen that a natural "source" for vectors are curves (via their velocities). Similarly, scalar functions form a natural source for differential forms:

Theorem 2.2.2. *Every function $f \in \mathcal{F}M$ determines at any point $p \in M$ a 1-form denoted $df : T_pM \to \mathbb{R}$ and defined*

$$df(\mathbf{v}) := \mathbf{v}f \tag{2.4}$$

for any vector $\mathbf{v} \in T_pM$. [Think of "d" as a decoration added to f.] Moreover, if $\{x^i\}$ is a local chart at $p \in M$, then the set

$$\{dx^i\} \qquad i = 1,\ldots,\dim M \tag{2.5}$$

forms a basis of the cotangent space T_p^*M.

PROOF: Linearity of df is easy to check via Eq.(2.4). Part (2.5) follows simply from

$$dx^i(\partial_j) = \partial_j x^i = \delta^i_j$$

□

Thus, any covector α can be written as a linear combination

$$\alpha = \alpha_i \, dx^i$$

where the coefficients can be determined by $\alpha_i = \alpha(\partial_i)$.

EXAMPLE: Here are examples of differential forms on $M \cong \mathbf{R}^2$ and $M \cong \mathbf{R}^3$, respectively:

$$\alpha = \sin y \, dx + (x-y)^2 \, dy, \qquad \beta = xyz \, dx - 7 \, dy + (x+y) \, dz$$

With this simple devise we have a natural map called **exterior derivative**

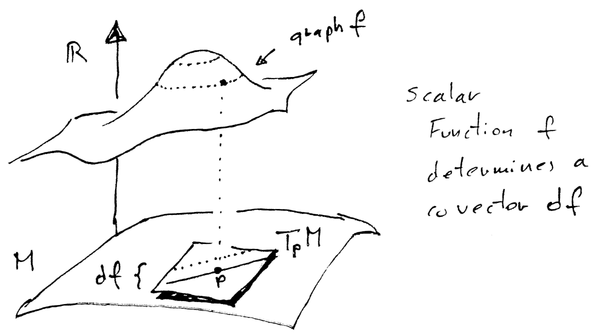
$$d : \mathcal{F}M \longrightarrow \Lambda^1 M \tag{2.6}$$

that associates to any smooth function f a differential 1-form df. The image df is called the **exterior derivative** of f, or simply **differential** of f. This is a linear map between linear spaces.

Problem: Show that if $\{x^1, ..., x^n\}$ represents a local chart, then for any $f \in \mathcal{F}M$, it is

$$df = \partial_i f \, dx^i \equiv \partial_1 f \, dx^1 + \partial_2 f \, dx^2 + ... + \partial_n f \, dx^n$$

It is important to bear in mind that not every differential 1-form is a differential (an image of a function through d).



Exercise: Show that differential 1-form α given as

$$\alpha = x \, dy - y \, dx$$

on a two-dimensional manifold M with coordinates $\{x, y\}$ cannot possibly be an exterior derivative of some $f \in \mathcal{F}M$.

Problem: [coordinate change] Show that if $\{x^1, ..., x^n\}$ and $\{x^1, ..., x^n\}$ represent two local charts with nonempty intersection, than rule of change of chart for any $\alpha \in \Lambda^1 M$ on this intersection is

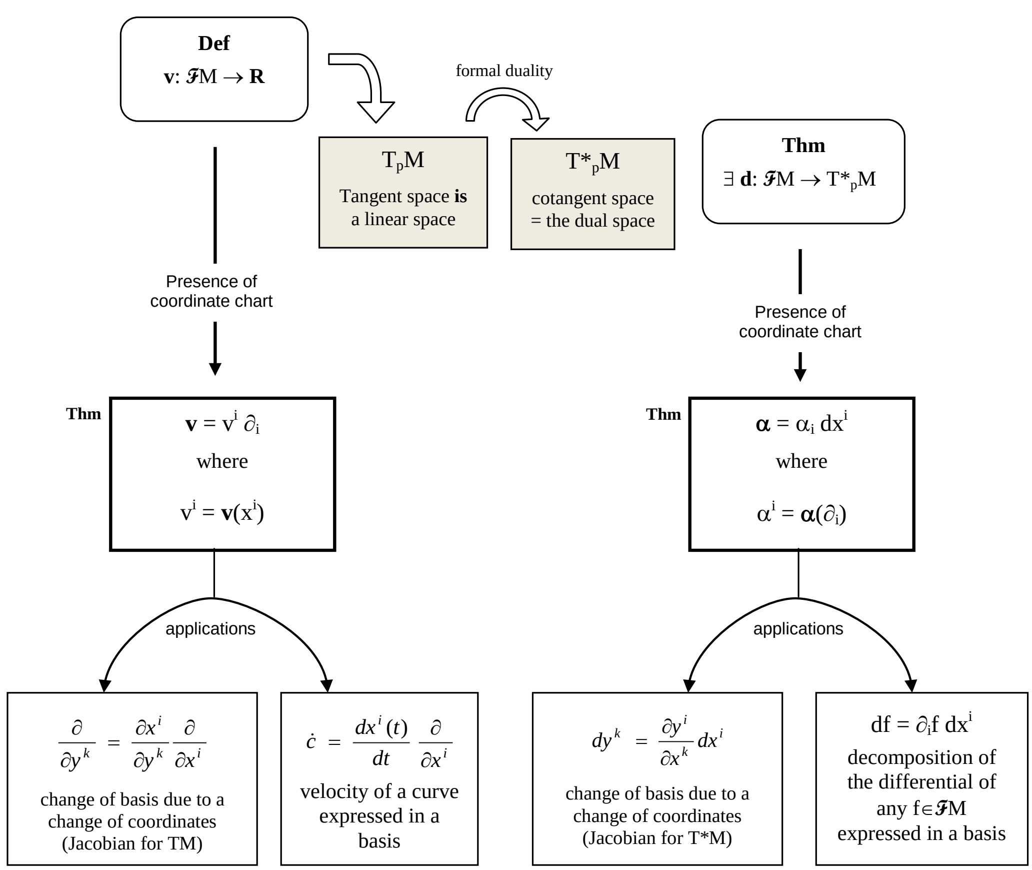
$$\alpha = \alpha_i \, dx^i = \alpha_j \, \frac{\partial x^j(y)}{\partial y^i} \, dy^i$$

Matrix $J_i^j = \frac{\partial x^j(y)}{\partial y^i}$ (function of $p \in M$) is the **Jacobian** of the chart change.

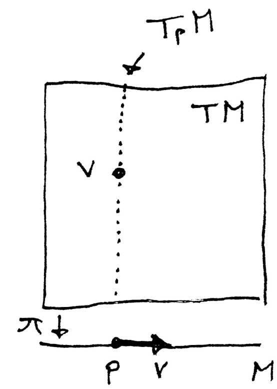
Exercise: Express the differential 1-form $\alpha = x\,dy - y\,dx$ and $\beta = x\,dy + y\,dx$ on the manifold $M \equiv \mathbb{R}^2 \setminus \{\mathbf{0}\}$ in the angular chart. Draw the results.

Exercise: Reason that α is smooth if in any chart the coefficients $\alpha_i(x)$ of $\alpha = \alpha_i\,dx^i$ are smooth functions of p.

Diagram of mutual relations
between the basic concepts
in Differential Geometry



2.3 Tangent and cotangent bundles



Given a manifold M, one defines the **tangent bundle** over M as the union of the tangent spaces

$$TM := \bigcup_{p \in M} T_pM$$

Note that the *points* of TM are *vectors* on M. The natural map of that associates to each vector the point of its "attachment" will be called **projection** and denoted $\pi : TM \to M$. That is

$$\pi : \; TM \ni v_p \longrightarrow p \in M$$

Sometimes for clarification the symbol π_M will be used instead of plain π.

Proposition: Tangent bundle can be given a structure of a manifold of dimension $\dim TM = 2n$, where $n = \dim M$

PROOF: Let $\{U, \varphi\}$ be a chart on M with local coordinates denoted $\{x^i\}$. Define an associated chart on $\overline{U} := \pi^{-1}U \subset TM$, where to a given *point* $v \in TM$ (which is simultaneously a tangent vector) a set of $2n$ numbers (coordinates) are prescribed:

$$\bar{\varphi} : v \longrightarrow (x^1(\pi(v)), \ldots, x^n(\pi(v)),\; v(x^1), \ldots, v(x^n))$$

Clearly, the map $\bar{\varphi} : \pi^{-1}(U) \to \mathbb{R}^{2n}$ is a good chart on TM. Any atlas on M will produce this way an atlas on TM. What remains to be proved is the compatibility of the so-produced charts (left as an exercise). □

Exercise: Finish the proof that the collection of the above-defined charts define an atlas on TM.

Similarly, the **cotangent bundle** over M is the union

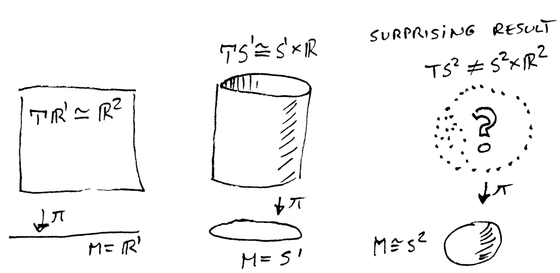
$$T^*M := \bigcup_{p \in M} T^*_pM$$

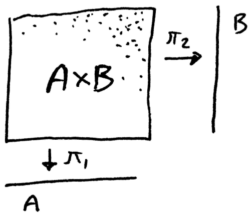
with the natural **projection** analogous to the case of the tangent bundle

$$\pi' : \; T^*M \ni \alpha_p \longrightarrow p \in M.$$

Exercise: Show that T^*M is a $2n$-dimensional manifold.

Examples of tangent bundles:





A Long Remark on Fiber Bundles

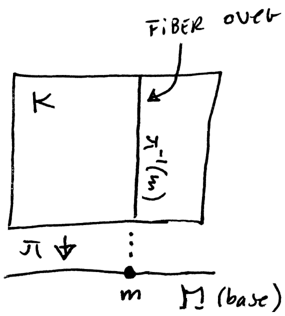
The tangent bundle and cotangent bundle are special cases of a more genereal construction, the so-called **fiber bundle**. We discuss it below.

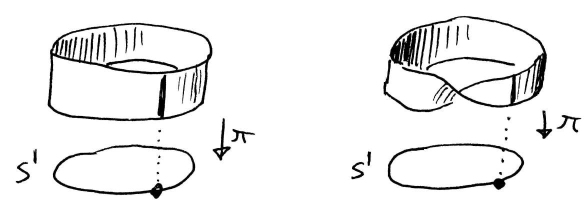
Recall that the Cartesian product $K = A \times B$ of two sets may be viewed as a set with two *projections*, a triplet:

$$\{K, \pi_1, \pi_2\} \qquad \text{where} \qquad \pi_1 : K \to A \quad \text{and} \quad \pi_2 : K \to B$$

Any $k \in K$ is mapped to a pair $k \to (\pi_1(k), \pi_2(k))$. A fiber bundle is a weaker version of Cartesian product. Suppose we have a set with only one projection, that is a pair $\{K,\ \pi : K \to A\}$ such that the sets $\pi^{-1}(a)$ for various $a \in A$, called **fibers**, are somehow "similar". Then such a structure will be called a fiber bundle over A.

Clearly, this needs to be formalized and the sets replaced by manifolds. Before we look at the formal definition, consider and compare a **cylinder** $S^1 \times I$ (here $I = (0,1)$ is an open segment), and a **Möbius strip**. While the first may be represented as a Cartesian product, the latter cannot. But both may be viewed as fiber bundles over S^1 with I as a "typical fiber".





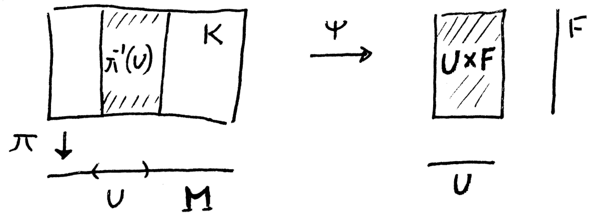
Definition: A **fiber bundle** K over M is a triple $\{E,\ M, \pi : E \to M\}$, where E and M are manifolds and π is a differentiable map such that for any open neighborhood $U \subset M$ there exists a diffeomorphism

$$\Psi : \pi^{-1}(U) \longrightarrow U \times F$$

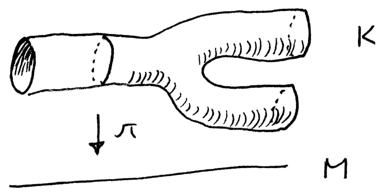
for some manifold F, such that

$$\forall m \in K \qquad \pi(m) = \pi_U \Psi(m)$$

where π_U is the standard projection onto the first component of the Cartesian product $U \times F$. The manifold F is called the **typical fiber**. Map Ψ is called **local trivialization** — note that it is not fixed but its mere *existence* is required in the definition.



Here is an example of a triplet $\{K, M, \pi : K \to M\}$ which is not a fiber bundle.



The reason is that the fibers over different points of M are not quite diffeomorphic: circles on the left side versus double circles on the right, and an "eight-figure" somewhere in between, which is not even a manifold...

However, the tangent bundle TM over a manifold M is indeed a fiber bundle. It can be trivial or not. Compare these two 4-dimensional fiber bundles:

> $T(T^2)$ – the tangent bundle over a torus has the typical fiber $F = \mathbb{R}^2$ and therefore we can represent it as a Cartesian product, namely $T(T^2) \cong T^2 \times \mathbb{R}^2$, but the representation is **not unique**. Only the projection onto T^2 is well defined, the other, onto $\mathbb{R}^2$, depends on your caprice.
>
> $T(S^2)$ – the tangent bundle over a sphere S^2, although a fiber bundle, it **cannot** be globally represented as a Cartesian product. TS^2 and $S^2 \times \mathbb{R}^2$ are quite different manifolds. Later we shall learn that there is no non vanishing vector field on 2-sphere. Explain how this fact proves that $TS^2 \neq S^2 \times \mathbb{R}^2$.

Problem (Return of Utnapishtim): Reconsider your answer of the Problem of the configuration space of Utnapishtim from the beginning of the chapter.

Question: It is easy to draw a circle S^1 on sphere S^2, Could S^2 be represented as a fiber bundle with S^1 as the typical fiber?

2.4 Natural operations: $[\cdot,\cdot]$ and d

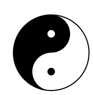
The nineteenth century geometry was preoccupied with with shapes and their parameterization. The main paradigm of the **modern differential geometry** is recognition of **natural operations**, that is operations that do not depend on the choice of a particular coordinate system.

This is paralleled by the program of "geometrization of physics", were it is believed that the fundamental concepts of physics should be represented by geometric objects, not by mere "numbers".

The first two families of such geometric objects are **vector fields** and **exterior differential forms**.

And here an interesting thing happens. The duality of vectors and covectors —so symmetric on the algebraic level as displayed by the isomorphism $(L^*)^* \cong L$— ends at the moment we generalize them to fields on manifolds. We will see that the vector fields admit a natural operation of the so-called **Lie bracket** of two vector fields. But there is no "Lie bracket" of differential forms – instead one discovers the **exterior derivative** and the need to extend the concept of 1-forms to exterior k-forms. Both operations are introduced in the following sections.

$$\mathbb{R} \xrightarrow{c} M \xrightarrow{f} \mathbb{R}$$

vectors  exterior forms

$\{\mathfrak{X}M,\ [\cdot,\cdot]\}$ *The end of naïve duality* $\{\Lambda M,\ d\}$

2.4.1 Lie bracket of vector fields

The set of smooth vector fields on manifold M will be denoted $\mathfrak{X}M$. It forms an infinite-dimensional *linear space* over real numbers $\mathbb{R}$. But it is also equipped with a natural product, which makes it an algebra.

Definition 2.4.1. A **Lie bracket** (or **commutator**) of two vector fields $A, B \in \mathfrak{X}M$ is a new vector field $[A,B] \in \mathfrak{X}M$, which for any $f \in \mathcal{F}M$ is defined by

$$[A,B]\,f := ABf - BAf \tag{2.7}$$

where for simplicity we suppress round brackets, e.g., $ABf \equiv A(Bf)$.

Exercise: Let $A = x\partial_x + xy\partial_y$ and $B = xyz\partial_x$ be two vector fields defined locally on some three-dimensional manifold. By calculating $[A,B]f$ for some unspecified scalar function f (in three variables x,y,z), find that their Lie bracket is a first order operator $[A,B] = x^2yz\partial_x - xy^2z\partial_y$. Observe that the second order derivatives vanish due to the general property $\partial_i\partial_j f = \partial_j\partial_i f$.

It is clear that for any function $f \in \mathcal{F}M$, $[A,B]f$ is also a scalar function, thus $[A,B]$ may be viewed as a map $[A,B] : \mathcal{F}M \to \mathcal{F}M$. But it does not make it automatically a vector field. Indeed, the definition (2.7) suggest that it is a *second order* operator, while to be a vector field, it would have to be a *first-order* differential operator. The following holds:

Theorem 2.4.1. *The Lie bracket of two vector fields is a vector field.*

PROOF: Typically, the proof is given in a chart notation by writing Equation (2.7) in coordinates and noticing that the second-order terms conveniently subtract each other. You should try it. A more elegant, coordinate-free, proof is also possible: just check that at any point $p \in M$ the object $[A,B]_p$ behaves like a vector, that is it satisfies the conditions of a definition of a vector, Def. 2.2.1. Left as an exercise. □

Problem. Clearly, given a function $f \in \mathcal{F}M$ and a vector field $B \in \mathcal{X}M$, one may compose a new vector field fY. Show that

$$[A, fB] = f[A,B] + A(f)B \qquad \text{and} \qquad [fA,B] = f[A,B] - B(f)A$$

Problem. As an exercise, draw the given vector fields, calculate their mutual bracket, and draw it as well:

$$A = x\partial_y \qquad B = y\partial_y$$

Theorem 2.4.2. *Lie bracket satisfies the following properties* $\forall A,B,C \in \mathcal{X}M$ *and* $\forall a,b \in \mathbb{R}$*:*

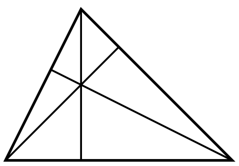
$$\begin{aligned} &(i) && [A,B] = -[B,A] && \text{anticommutativity} \\ &(ii) && [aA+bB,C] = a[A,C] + b[B,C] && \text{linearity} \\ &(iii) && [[A,B],C] + [[B,C],A] + [[C,A],B] = 0 && \text{Jacobi identity} \end{aligned} \tag{2.8}$$

PROOF: The first two are immediate. To see *(iii)*, act on a function $f \in \mathcal{F}M$ and "expand" every commutator via Eq. (2.7) to see that the terms cancel each other. Most of textbooks go however via a coordinate description. [Exercise: do both ways] □

Any vector space with a product that satisfies properties (i)–(iii) of (2.8) is called a **Lie algebra**.

Corollary 2.4.3. *The pair* $\{ \mathcal{X}M, [\,\cdot,\cdot\,] \}$ *forms an infinite-dimensional Lie algebra over the field* $\mathbb{R}$*.*

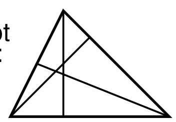
Examples of other Lie algebras:



1. $\{\mathbf{E}^3, \times\}$ — The standard three dimensional Euclidean space $\mathbf{E^3}$ with the usual cross-product $\times$ forms a 3-dimensional Lie algebra. The first two properties are obvious. Check that the Jacobi identity is also satisfied.

   Problem [V.I. Arnold's challenge]: Show that the Jacobi identity in $\{\mathbf{E}^3, \times\}$ is equivalent to the well-known fact in geometry that the altitudes in any triangle intersect in one point.

...and not like this:



2. $\{M_{n\times n}, [\ ,\ ]\}$ — the space of $n \times n$ matrices (over any field) with the commutator defined for two matrices as $[A,B] = AB - BA$ (a difference of matrix products) is an n^2-dimensional Lie algebra. This is a special case of a Lie algebra associated to a Lie group – for more see Chapter X.

   Exercise: Show tat the set of real skew-symmetric matrices, $A^T = -A$, forms a Lie algebra.

3. Any associative algebra $\{A, \circ\}$ can be turned into a Lie algebra with the product defined $[a,b] = x \circ y - y \circ a$. Prove (is associativity necessary?).

   Exercise: Consider the (associative) algebra of quaternions, $A = \mathbb{H}$.

Exercise 1: Show that Jacobi identity may be rewritten as

$$\begin{array}{ll} (i) & [[a,b],c] - [a,[b,c]] \ = \ [b,[c,a]] \\ (ii) & [a,[b,c]] \ = \ [[a,b],c] + [b,[a,c]] \qquad or \quad \mathrm{ad}_a\,[b,c] \ = \ [\mathrm{ad}_a\, b,\, c] + [b,\, [\mathrm{ad}_a\, c] \end{array}$$

where $\mathrm{ad}_a\, x = [a,\, x]$. Interesting interpretations emerge: (i) the right hand side may be viewed as a measure of the "degree" to which a given Lie algebra is nonassociative. Interpretation of (ii): ad_a satisfies the Leibniz rule ("product rule") with respect to the Lie product and therefore may be viewed as a *differentiation* in a Lie algebra.

Practical formula for Lie bracket of two vector fields. If

$$A = A^i\,\partial_i \qquad \text{and} \qquad B = B^i\,\partial_i$$

in some chart, then

$$[A,B] = \underbrace{\big(A(B^i) - B(A^i)\big)}_{[A,B]^i}\,\partial_i$$

or, more explicitely:

$$[A,B] = \big(A^j(\partial_j\, B^i) - B^j(\partial_j\, A^i)\big)\,\partial_i$$

Exercise: Prove the above formula. You could use the formula from Problem 2.4.1.

Geometric interpretation of the Lie bracket (your class notes)

2.4.2 Differential forms and exterior derivative

Preparation: read Chapter 1, Sections 6.

Now we move from vectors to covectors. Similarly to covector fields introduced two sections ago, one may consider fields of *exterior multi-forms*.

Definition: A **differential exterior k-form** on M is a map α that associates with every point of the manifold an exterior k-form on its tangent space:

$$\alpha : M \ni p \longrightarrow \alpha_p \in \wedge^k T_p^* M$$

in the sense that, at every point, we have a skew-symmetric multi-linear map

$$\alpha_p : T_pM \times T_pM \times \ldots \times T_pM \longrightarrow \mathbb{R}$$

This may be viewed *globally* as a map from k vector fields to smooth scalar functions

$$\alpha : \mathfrak{X}M \times \mathfrak{X}M \times \ldots \times \mathfrak{X}M \longrightarrow \mathcal{F}M$$

from which we demand that $\alpha(X_1, \ldots, X_k)$ is a *smooth* function for any selection of smooth vector fields $X_1, \ldots, X_k \in \mathfrak{X}M$.

The space of differential k-forms will be denoted $\Lambda^k M$. They generalize the concept of differential 1-forms. Differential **zero-forms** coincide with the space of smooth scalar functions: $\Lambda^0 M \equiv \mathcal{F}M$. If no degree is indicated at ΛM, we mean the direct product of spaces

$$\Lambda M = \Lambda^0 M \oplus \Lambda^1 M \oplus \ldots \oplus \Lambda^n M$$

($n = \dim M$). It is a *linear space* over $\mathbb{R}$, but it also forms an $\mathcal{F}M$-module.

Given a chart in an n-dimensional manifold M — a basis of k-forms can be chosen as:

$$\{\, dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k} \}$$

where the multi-index $(i_1, i_2, \ldots, i_k)$ runs over all ordered selections from $(1, 2, \ldots, n)$, that is $i_1 < i_2 < \ldots < i_k$.

Exercise: Here are some differential forms on $M \cong \mathbb{R}^4$:

$$\begin{aligned}
\alpha &= xy\,dx + y\,dy \\
\beta &= \sin x\,dx \wedge dy + z\,dx \wedge dz + x\,dz \wedge dt \\
\gamma &= x^2\,dx \wedge dy \wedge dz
\end{aligned}$$

Calculate $\alpha \wedge \beta$, $\beta \wedge \beta$, and $\gamma \wedge \alpha$.

Definition 2.4.2. An n-dimensional manifold M is called **orientable**, if it admits a global non-vanishing n-form $\eta \in \Lambda^n M$ (a volume form). An n-dimensional manifold M is called **oriented**, if such a form has been chosen.

We have seen that there is a natural map scalar functions to one-forms, $d : \mathcal{F}M \to \Lambda^1 M$. This map can be extended to all differential forms.

Definition 2.4.3. The exterior derivative is the natural map

$$d_k : \Lambda^k M \to \Lambda^{k+1} M\,, \tag{2.9}$$

which for every k associates to every k-form α a $(k+1)$-form $d\alpha$ so that

$$\begin{array}{rl} (i) & d_k\,(\alpha+\beta) = d\alpha + d\beta \\ (ii) & d_k\,(f\alpha) = df\wedge\alpha + f\,d_k\alpha \\ (iii) & d_{k+1}\,d_k\;\alpha = 0 \\ (iv) & d_0\,f = df \end{array}$$

for any $\alpha,\,\beta \in \Lambda^k M$ and $f \in \mathcal{F}M$.

We shall customarily omit the subscript k and denote the operator of exterior derivative by just d.

Proposition 2.4.4. *The exterior derivative d is uniquely defined.*

PROOF: First, notice behavior of the exterior derivative for simple forms: if $f, g, h \in \mathcal{F}M$ are scalar functions, then the conditions of the definition determine that

$$\begin{array}{rclcrcl} f & \xrightarrow{\;d\;} & df & \Rightarrow & df & \xrightarrow{\;d\;} & 0 \\ g\,df & \xrightarrow{\;d\;} & dg\wedge df & \Rightarrow & dg\wedge df & \xrightarrow{\;d\;} & 0 \\ h\,dg\wedge df & \xrightarrow{\;d\;} & dh\wedge dg\wedge df & \Rightarrow & dh\wedge dg\wedge df & \xrightarrow{\;d\;} & 0 \end{array}$$

and so on (by virtue of properties (ii)–(iv)). Since any differential k-form can be represented in a chart as a sum of such terms:

$$\omega = \omega_{i_1 i_2 \ldots i_k}\, dx^{i_1}\wedge dx^{i_2}\wedge\ldots\wedge dx^{i_k} \tag{2.10}$$

where $\omega_{i_1 i_2 \ldots i_k} \in \mathcal{F}M$ are scalar functions, we see that the axioms 2.4.3 determine uniquely the exterior derivative $(k+1)$-form $d\omega$:

$$d\omega = d\omega_{i_1 i_2 \ldots i_k}\wedge dx^{i_1}\wedge dx^{i_2}\wedge\ldots\wedge dx^{i_k} \tag{2.11}$$

As for the *existence*, assume (2.11), and show that all axioms of the definition are satisfied. □

Remark 2.4.5. Easy calculations show that the property $d\circ d = 0$ is directly relate to $\partial_i\partial_j f = \partial_j\partial_i f$ in calculus. The latter is sometimes called Clairaut's theorem (after Alexis Clairaut (1713-1765)).

Proposition 2.4.6. *The exterior derivative satisfies the skewed version of the Leibniz rule:*

$$d\,(\alpha\wedge\beta) = d\alpha\wedge\beta + (-1)^{\deg\alpha}\,\alpha\wedge d\beta$$

PROOF: Exercise. Hint: due to linearity, it is sufficient to consider only monomials like $f\,df_1\wedge\ldots\wedge df_k$ for some functions $f, f_1, \ldots, f_k \in \mathcal{F}M$. Finish. □

Exercise: (Important) For each of the following differential forms on manifold $M \cong \mathbb{R}^3$ with the standard coordinates $\{x, y, z\}$ calculate its exterior derivative. Calculate also the second exterior derivative to see that it indeed vanishes.

scalar function:	$f = f(x,y,z)$,	$f \in \mathcal{F}M$
1-form:	$\alpha = f\,dx + g\,dy + h\,dz$,	$f, g, h \in \mathcal{F}M$
2-form:	$\beta = A\,dx \wedge dy + B\,dy \wedge dz + C\,dz \wedge dx$,	$A, B, C \in \mathcal{F}M$
3-form:	$\eta = f\,dx \wedge dy \wedge dz$	$f \in \mathcal{F}M$

Can you recognize the resulting formulas...

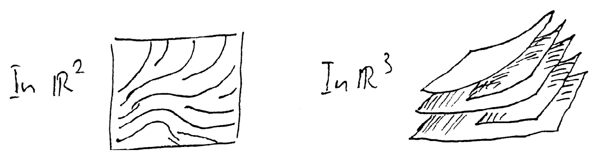


Figure 2.1: Forms that are not closed

To sum up, we have a **chain** of maps

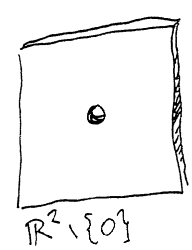
$$\mathcal{F}M \xrightarrow{\ d\ } \Lambda^1 M \xrightarrow{\ d\ } \Lambda^2 M \xrightarrow{\ d\ } \cdots \xrightarrow{\ d\ } \Lambda^{n-1} M \xrightarrow{\ d\ } \Lambda^n M$$

We may think of a pair $\{\Lambda M,\ d\}$ as a linear space with a distinguished *linear* transformation d. The additional property $dd = 0$ makes it a *chain*.

Definition 2.4.4. A form $\omega \in \Lambda M$ is called:

closed if $d\omega = 0$
exact if there exist a form α such that$d\alpha = \omega$.

Clearly, every exact form is closed, but not necessarily it vice versa! This discrepancy allows one to measure k-dimensional holes in the manifold — see Section 2.10.



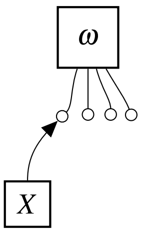
Example 2.4.1. Consider a manifold $M \cong \mathbb{R}^2 \setminus \{\mathbf{0}\}$. Show that the form

$$\omega = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

is closed, $d\omega = 0$, yet it is not exact. [Hint: change the chart to the polar coordinate system].

2.4.3 Contractions of exterior forms with vector fields

If in a function of a few variables, say $f(x,y,z) = x^2 + xyz + 1$, one of the variables becomes fixed at some value, say $x = 2$, the function becomes a function of two variables: $f(2,y,z) = 5 + 2yz$. "Contraction" is exactly such a simple idea in the realm of exterior forms.



Definition 2.4.5. A **contraction** – or **interior product** – of a vector field $X \in \mathfrak{X}M$ and a k-form $\omega \in \Lambda^k M$ is a $(k-1)$-form denoted $X \lrcorner \omega$ and defined

$$X \lrcorner \omega = \omega(X, \cdot\,, \cdot\,, \cdots, \cdot\,)$$

in the sense that this $(k-1)$-form evaluated on any set of $k-1$ vector fields $Y_1, ..., Y_{k-1}$ obtains value

$$\big(X \lrcorner \omega\big)\, (Y_1, ..., Y_{k-1}) := \omega(X, Y_1, ..., Y_{k-1}).$$

Examples of calculations: For a simple 2-form and 3-form composed from 1-forms α, β, and γ, it is

$$\begin{aligned} X \lrcorner\, (\alpha \wedge \beta) \quad &= \alpha(X) \wedge \beta - \alpha \wedge \beta(X) \\ &= \langle \alpha | X \rangle\, \beta - \langle \beta | X \rangle\, \alpha \\ X \lrcorner\, (\alpha \wedge \beta \wedge \gamma) \quad &= \alpha(X) \wedge \beta \wedge \gamma - \alpha \wedge \beta(X) \wedge \gamma + \alpha \wedge \beta \wedge \gamma(X) \\ &= \langle \alpha | X \rangle\, \beta \wedge \gamma - \langle \beta | X \rangle\, \alpha \wedge \gamma + \langle \gamma | X \rangle\, \alpha \wedge \beta \end{aligned}$$

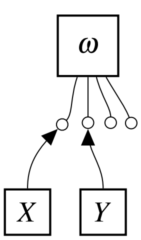
Exercise: Following the above, notice that if $X = X^1 \partial_1 + X^2 \partial_2 + ... + X^n \partial_n$, then

$$\begin{aligned} X \lrcorner (dx^1 \wedge dx^2 \wedge ... \wedge dx^k) = &\qquad X^1\, dx^2 \wedge dx^3 \wedge dx^4 \wedge ... \wedge dx^k \\ = &\quad - \; X^2\, dx^1 \wedge dx^3 \wedge dx^4 \wedge ... \wedge dx^k \\ = &\quad + \; X^3\, dx^1 \wedge dx^2 \wedge dx^4 \wedge ... \wedge dx^k \\ = &\qquad \ldots \\ = &\quad \pm \; X^k\, dx^1 \wedge dx^2 \wedge dx^3 \wedge ... \wedge dx^{k-1} \end{aligned}$$

This can be written briefly as

$$X \lrcorner (dx^1 \wedge dx^2 \wedge ... \wedge dx^k) = (-1)^{k-1}\, X^i\, dx^1 \wedge dx^2 \wedge ... \wedge \widehat{dx^i} \wedge ... \wedge dx^k$$

where the hat denotes omission of the term, and where sum over i is understood.



Notation: Another symbol to indicate a contraction is "i", so that for any exterior form ω and vector field:

$$X \lrcorner \omega \equiv i_X \omega$$

Composition of contractions may be denoted:

$$Y \lrcorner (X \lrcorner \omega) = (i_Y \circ i_X)\, \omega = \omega(X, Y, \cdots)$$

In the case of a 1-form $\alpha \in \Lambda^1 M$ we are free to use any of these four notations:

$$\langle \alpha, X \rangle \equiv \alpha(X) \equiv X \lrcorner \alpha \equiv i_X \alpha$$

Note that for a fixed vector field $X \in \mathcal{X}M$ we can form a chain $\{\Lambda, i_X\}$ analogous to the chain $\{\Lambda M, d\}$, namely

$$\mathcal{F}M \xleftarrow{i_X} \Lambda^1 M \xleftarrow{i_X} \Lambda^2 M \xleftarrow{i_X} \dots \xleftarrow{i_X} \Lambda^{n-1} M \xleftarrow{i_X} \Lambda^n M$$

In particular, note that

$$i_X \circ i_X = 0,$$

a property quite similar to that of exterior derivative.

2.4.4 The rank of a differential 1-form

Definition: The rank of a 1-form $\alpha \in \Lambda^1 M$, denoted $\operatorname{rank}\alpha$, is the smallest number of functions necessary to express it.

Clearly, the notion has a local character. Now, to acquire some insight, consider this sequence of 1-forms of increasing rank:

$$\begin{aligned}
\alpha &= dg & &\longrightarrow \operatorname{rank}\alpha = 1\\
\alpha &= f\,dg & &\longrightarrow \operatorname{rank}\alpha = 2\\
\alpha &= f\,dg + dh & &\longrightarrow \operatorname{rank}\alpha = 3\\
\alpha &= f\,dg + k\,dh & &\longrightarrow \operatorname{rank}\alpha = 4 \quad etc...
\end{aligned}$$

for scalar functions $f, g, h, k \in \mathcal{F}M$. We assume that the functions are *independent*, by which we mean that the covectors df, dg, dh, ..., *etc.*, are linearly independent at every point of M.

Algorithm. The question is: given $\alpha \in \Lambda^1 M$, find its rank. Here is the algorithm. Calculate this sequence of forms:

$$\begin{aligned}
A_1 &= \alpha\\
A_2 &= d\alpha\\
A_3 &= \alpha \wedge d\alpha\\
A_4 &= d\alpha \wedge d\alpha\\
A_5 &= \alpha \wedge d\alpha \wedge d\alpha \quad etc...
\end{aligned}$$

If $A_{k+1} = 0$ and $A_k \neq 0$, then $\operatorname{rank}\alpha = k$.

Example: In a three-dimensional manifold with coordinates $\{x, y, z\}$ the following 1-form

$$\alpha = 2x\,dx + 2y\,dy + 2z\,dz$$

seem to be of rank 3, but actually is of rank 1, since it can be written with a use of a single function, namely as

$$\alpha = df, \qquad \text{where} \qquad f = x^2 + y^2 + z^2$$

This is in agreement with $d\alpha = 0$.

Exercise: Consider 4-dimensional manifold with coordinates $\{x, y, z, t\}$. Find rank α for

$$\alpha = 2xt\,dx + yt\,dt + t^2\,dy$$

Can you reduce α to the form that exhibits its rank?

Problem: If $\dim M = n$, then the highest possible rank of a 1-form is n.

Question: Would an analogous definition of *rank* work for vector fields? What is the rank of any vector field? [1]

2.4.5 Orientation and orientability

Definition 2.4.6. A **volume form** on a manifold M is a nowhere vanishing form of the highest possible degree, i.e., a form $\eta \in \Lambda^n M$, where $n = \dim M$.

Definition 2.4.7. A manifold M is **orientable** if it admits a volume form.

Definition 2.4.8. A manifold is **oriented** if a volume form has been chosen.

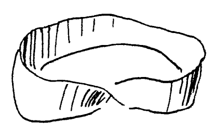


Figure 2.2: Möbius band does not admit any non-vanishing bi-form.

[1]**Answer:** According to the Fundamental Theorem of ODE, every smooth vector field X is locally integrable. That is it admits local coordinates $\{x^1, \ldots, x^n\}$ in which it is $X = \partial/\partial_1$. Thus the question of rank is meaningless.