Homework 2

Exercise 1: The exercises below are trivial (apology) but this hands-on experience should provide some initial intuition on typical calculations. Consider a 2-dimensional manifold M with coordinate chart $\{x, y\}$. The following objects are given

$\mathbf{v} = 2\partial_x + \partial_y$	Vector at a point $p = (3, 1)$	$\mathbf{v} \in T_pM$
$f = x^2 + xy + 2$	scalar function	$f \in \mathcal{F}M$
$\mathbf{A} = 2x^2\partial_x + xy\partial_y$	vector field	$\mathbf{A} \in \mathcal{X}M$
$\mathbf{B} = y\partial_x$	vector field	$\mathbf{B} \in \mathcal{X}M$
$c(t) = (t^2 + t, 2\cos t)$	a curve	$c \in \mathcal{C}M$

Calculate the following:

(a) $\mathbf{v}f$ (b) $\mathbf{A}f$

(c) $f \circ c$ (d) $\frac{d}{dt}(f \circ c)$ at $t = 0$ (e) $\dot{\mathbf{c}} \equiv \dot{c}(0)$ (f) $\dot{\mathbf{c}}f$

(g) $[\mathbf{A}, \mathbf{B}]$

(h) Draw the vector field $\mathbf{B}$ in the neighborhood of $(0, 0)$.

Exercise 2: Consider $M = \mathbf{R}^3$ with the chart of rectangular coordinates (x, y, z). Express each vector of the basis associated with the chart of spherical coordinates (r, φ, θ), namely ∂_r , ∂_φ and ∂_θ, in terms of the standard basis $\{\partial_x, \partial_y, \partial_z\}$.
(First, recall the formulas tying the two charts.)

Problem 3: Simple but important) Show that the Lie bracket of vector fields satisfies the Jacobi identity:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

Problem 4: Let $C = [A, B]$ be the Lie bracket of two vector fields. In a chart, the vector fields are given as $A = A^i\partial_i$, $B = B^i\partial_i$ and $C = C^i\partial_i$. Express the coefficients C^i in terms of the coefficients of the other two vector fields.

Problem 5 (for grad students): Show that 2-dimensional projective space $P^2\mathbb{R}$ is a manifold. (You may start with $P^1\mathbb{R}$ as a warm-up). [ask in claass for a hint]