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# Homework 5

**Exercise 1:** Using the general, coordinate-free definitions show that:

$$\text{(a)}\quad \mathrm{curl} \circ \mathrm{grad} = 0 \qquad \text{(b)}\quad \mathrm{div} \circ \mathrm{curl} = 0$$

(The first was done in class)

**Exercise 2:** Consider a three-dimensional manifold $M \cong \mathbb{R}^3$ with global coordinates $\{x, y, z\}$. Let a simple scalar function and a vector field be given as (c) For the two cases of manifold structure as above, repeat calculations for these very simple objects:

$$f(x, y, z) = x \qquad \mathbf{A} = \partial_x$$

(a) Assume a very "plain" structure on $M$, namely

$$\text{volume form} \quad \eta = dx \wedge dy \wedge dz$$

$$\text{inner product} \quad [g] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and calculate $\mathbf{grad}f$, $\mathbf{curlA}$, and div$\mathbf{A}$.

(b) Calculate the same for a slightly different structure on $M$, namely:

$$\text{volume form} \quad \eta = (x^2 + 1)\, dx \wedge dy \wedge dz$$

$$\text{inner product} \quad [g] = \begin{bmatrix} 1 & 0 & 1 + y^2 \\ 0 & 1 & 0 \\ 1 + y^2 & 0 & 0 \end{bmatrix}$$

(c) For the two cases of manifold structure as above, repeat calculations for these fields:

$$f(x, y, z) = 1 + x + y^2 + xyz \qquad \mathbf{A} = y\partial_x$$

**Problem 3:** Two expeditions to some remote regions, A and B, of the universe brought back some data on gravitational fields they detected. They claimed the fields to be (in their respective local coordinates)

$$(A) \qquad X = (x + y)\, \partial_x + (y + z)\, \partial_y - (z + x)\, \partial_z$$
$$(B) \qquad X = \arctan(x)\, \partial_x + y\, \partial_y + z\, \partial_z$$

One of these groups provided fake data. Which and why? For the other group: what distributions of masses would produce this field?

MATH450

HOMEWORK 5

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① USING THE GENERAL COORDINATE-FREE DEFINITIONS SHOW THAT:

MINI MANDALA

$$\begin{tikzcd}
\Lambda^0 M \arrow[r, "d"] & \Lambda^1 M \arrow[r, "d"] & \Lambda^2 M \arrow[r, "d"] & \Lambda^3 M \\
\mathcal{F}M \arrow[u, leftrightarrow, "id"] \arrow[r, "grad"'] & \mathfrak{X}M \arrow[u, "\lrcorner g"] \arrow[r, "curl"'] & \mathfrak{X}M \arrow[u, "\lrcorner \eta"] \arrow[r, "div"'] & \mathcal{F}M \arrow[u, "\cdot \eta"]
\end{tikzcd}$$

A $\text{curl} \circ \text{grad} = 0$

LET $f \in \mathcal{F}M$, $A \in \mathfrak{X}M$,

$\rightarrow \text{grad} f \lrcorner g = df$ (1.1)

$\rightarrow \text{curl} A \lrcorner \eta = d(A \lrcorner g)$ (1.2)

LET $A = \text{grad} f$, THEN (1.2) BECOMES

$\text{curl}(\text{grad} f \lrcorner g) = d(\text{grad} f \lrcorner g) = d(df) = 0$

$g \neq 0$, $f$ IS ARBITRARY $\therefore$ $\text{curl} \circ \text{grad} = 0$ ☒.

B $\text{div} \circ \text{curl} = 0$

LET $A, B \in \mathfrak{X}M$,

$\text{curl}(A) \lrcorner \eta = d(A \lrcorner g)$

$\text{div}(B) \cdot \eta = d(B \lrcorner \eta)$

Let $B = \text{curl} A$,

$\text{div}(\text{curl} A) \cdot \eta = d(\text{curl} A \lrcorner \eta) = d(d(A \lrcorner g)) = 0$

$\eta \neq 0$, $A$ IS ARBITRARY, $\therefore$ $\text{div} \circ \text{curl} = 0$ ☒.

② $M \cong \mathbb{R}^3$, $\{x, y, z\}$. $f(x,y,z) = x$; $A = \partial_x$

A ASSUME A EUCLIDEAN STRUCTURE ON $M$ ($[g] = I^3$, $\eta = dx \wedge dy \wedge dz$) AND CALCULATE grad$f$, curl$A$, AND div$A$.

$df = \frac{\partial}{\partial x}(x)dx = dx$ WHICH BY INSPECTION $\boxed{\text{grad} f = \partial_x}$ SINCE $\partial_x \lrcorner g = dx = df$.

$d(A \lrcorner g) = d(\partial_x \lrcorner (dx \otimes dx \cdots)) = d(dx) = 0$ (1)

$\rightarrow d(A \lrcorner g) = \text{curl}(A) \lrcorner \eta$ (2)

(1) + (2) $\rightarrow$ $\text{curl}(A) \lrcorner \eta = 0$ $\therefore$ $\boxed{\text{curl} A = 0}$

$d(A \lrcorner \eta) = d(\partial_x \lrcorner dx \wedge dy \wedge dz) = d(dy \wedge dz) = 0$ (3)

$\rightarrow d(A \lrcorner \eta) = \text{div}(A) \cdot \eta$ (4)

(3) + (4) $\rightarrow$ $\text{div}(A) \cdot \eta = 0$ $\therefore$ $\boxed{\text{div} A = 0}$

B CALCULATE THE SAME FOR A SLIGHTLY DIFFERENT STRUCTURE ON $M$.

$$\eta = (x^2+1)\, dx \wedge dy \wedge dz \qquad f = x \quad A = \partial_x$$

$$[g] = \begin{bmatrix} 1 & 0 & 1+y^2 \\ 0 & 1 & 0 \\ 1+y^2 & 0 & 0 \end{bmatrix}$$

**grad f**

$\operatorname{grad} f \lrcorner g = df = dx$

$$g^{-1} = \begin{pmatrix} 0 & 0 & \frac{1}{y^2+1} \\ 0 & 1 & 0 \\ \frac{1}{y^2+1} & 0 & \frac{-1}{(y^2+1)^2} \end{pmatrix} \quad \text{Wolfram.}$$

$g^{-1} \circlearrowleft \quad g^{-1}(g(\operatorname{grad} f)) = g^{-1} dx$

$$\operatorname{grad} f = \left[0(1) + 0(0) + \left(\tfrac{1}{y^2+1}\right)(0)\right]\partial_x + \left[0(1) + 1(0) + 0(0)\right]\partial_y + \left[\left(\tfrac{1}{y^2+1}\right)(1) + 0(0) + \tfrac{-1}{(y^2+1)^2}(0)\right]\partial_z$$

$$\Rightarrow \boxed{\operatorname{grad} f = \frac{1}{y^2+1}\partial_z}$$

THIS WORKS SINCE $g = dx \otimes dx + (1+y^2)\, dx \otimes dz + dy \otimes dy + (1+y^2)\, dz \otimes dx$

$$\Rightarrow \frac{1}{y^2+1}\partial_z \lrcorner g = \underbrace{\frac{1+y^2}{y^2+1}}_{=1} \overbrace{\langle dz | \partial_z \rangle}^{=1} \otimes dx = dx = df = \operatorname{grad} f \lrcorner g.$$

**curl A** $\operatorname{curl} A \lrcorner \eta = d(A \lrcorner g)$

$\rightsquigarrow d(A \lrcorner g) = d(\partial_x \lrcorner g) = d(dx + (1+y^2)dz) = 0 + d(1+y^2) \wedge dz + 0 = 2y\, dy \wedge dz$

$\rightsquigarrow \operatorname{curl} A \lrcorner \eta = \operatorname{curl} A \lrcorner (x^2+1) dx \wedge dy \wedge dz = 2y\, dy \wedge dz$

THIS IS TRUE IF $\boxed{\operatorname{curl} A = \frac{2y}{(x^2+1)}\partial_x}$

**div A** $\operatorname{div} A \cdot \eta = d(A \lrcorner \eta)$

$d(A \lrcorner \eta) = d(\partial_x \lrcorner (x^2+1)\, dx \wedge dy \wedge dz) = d((x^2+1)\, dy \wedge dz) = d(x^2+1)\, dy \wedge dz + 0 = 2x\, dx \wedge dy \wedge dz$

$\Rightarrow \operatorname{div} A \cdot \eta = \operatorname{div} A\, dx \wedge dy \wedge dz = 2x\, dx \wedge dy \wedge dz$

$$\Rightarrow \boxed{\operatorname{div} A = 2x}$$

C FOR BOTH STRUCTURES PERFORM CALCULATIONS FOR

$$f(x,y,z) = 1 + x + y^2 + xyz \qquad A = y\partial_x$$

**STRUCTURE I** $g, \eta$ standard.

$\operatorname{grad} f \lrcorner g = df$

$\rightsquigarrow df = (1+yz)dx + (2y + xz)dy + (xy)dz$

WORKING BACKWARDS...

$= (1+yz)\partial_x\, dx \otimes dx + (2y+xz)\partial_y\, dy \otimes dy + (xy)\partial_z\, dz \otimes dz$

$= ((1+yz)\partial_x + (2y+xz)\partial_y + (xy)\partial_z) \lrcorner g$

$$\Rightarrow \boxed{\operatorname{grad} f = (1+yz)\partial_x + (2y+xz)\partial_y + (xy)\partial_z}$$

$\mathrm{curl}\,A \lrcorner \eta = d(A \lrcorner g) = d(y\partial_x \lrcorner g)$

$d(y\partial_x g) = d(y\,dx) = dy \wedge dx + 0 = dy \wedge dx$

$\mathrm{curl}\,A \lrcorner \eta = \underbrace{\mathrm{curl}\,A \lrcorner dx \wedge dy \wedge dz} = dy \wedge dx$

$\underbrace{-1 \cdot \partial_z} \lrcorner\, dx \wedge dy \wedge dz = -1 \cdot \partial_z \lrcorner - dz \wedge dy \wedge dx = (+1)\langle dz | \partial_z \rangle dy \wedge dx = dy \wedge dx$

$\Rightarrow \boxed{\mathrm{curl}\,A = -\partial_z}$

$\mathrm{div}\,A \cdot \eta = d(A \lrcorner \eta)$

$d(A \lrcorner \eta) = d(y\partial_x \lrcorner dx \wedge dy \wedge dz) = d(y\, dy \wedge dz) = dy \wedge dy \wedge dz = 0 = \mathrm{div}\,A \cdot \eta$

$\therefore \boxed{\mathrm{div}\,A = 0}$

STRUCTURE 2 $\eta = (x^2+1)\, dx \wedge dy \wedge dz,\quad g = \begin{bmatrix} 1 & 0 & 1+y^2 \\ 0 & 1 & 0 \\ 1+y^2 & 0 & 0 \end{bmatrix} = dx \otimes dx + (1+y^2)\, dx \otimes dz + dy \otimes dy + (1+y^2)\, dz \otimes dx$

grad f

$\mathrm{grad} f \lrcorner g = df,$

$\longrightarrow df = d(1 + x + y^2 + xyz) = (1+yz)\,dx + (2y+xz)\,dy + (xy)\,dz$

Let $A = \mathrm{grad} f$, where $A = A^x\partial_x + A^y\partial_y + A^z\partial_z$,

$\longrightarrow A^x\partial_x + A^y\partial_y + A^z\partial_z \lrcorner (dx \otimes dx + (1+y^2)\,dx \otimes dz + dy \otimes dy + (1+y^2)\,dz \otimes dx) = df$

$A^x dx + A^x(1+y^2)dz + A^y dy + A^z(1+y^2)\,dx = (1+yz)\,dx + (2y+xz)\,dy + xy\,dz$

$\hookrightarrow (A^x + A^z(1+y^2))\,dx + A^y dy + A^x(1+y^2)\,dz = (1+yz)\,dx + (2y+xz)\,dy + xy\,dz$

FROM THIS WE GET A SYSTEM OF EQUATIONS,

$$\begin{cases} A^x + A^z(1+y^2) = 1+yz & \longrightarrow A^z = \dfrac{1+yz - \frac{xy}{1+y^2}}{1+y^2} = \dfrac{1+yz}{1+y^2} - \dfrac{xy}{(1+y^2)^2} \\ A^y = 2y + xz & \\ A^x(1+y^2) = xy & \longrightarrow A^x = \dfrac{xy}{1+y^2} \end{cases}$$

$$\therefore \boxed{\mathrm{grad}\, f = \left(\frac{xy}{1+y^2}\right)\partial_x + (2y+xz)\partial_y + \left(\frac{1+yz}{1+y^2} - \frac{xy}{(1+y^2)^2}\right)\partial_z}$$

$\boxed{\text{curl}A}$ $A = y\partial_x$, $\eta = (x^2+1)\,dx\wedge dy\wedge dz$, $g = dx\otimes dx + (1+y^2)\,dx\otimes dz + dy\otimes dy + (1+y^2)\,dz\otimes dx$.

FROM MANDALA,

$$\text{curl}A \lrcorner \eta = d(A\lrcorner g)$$

FIRST FIND $A\lrcorner g$,

$$A\lrcorner g = y\langle dx|\partial_x\rangle dx + y(1+y^2)\langle dx|\partial_x\rangle dz$$
$$= y\,dx + (y+y^3)\,dz$$

$$\rightarrow d(A\lrcorner g) = dy\wedge dx + d(y+y^3)\wedge dz$$
$$= dy\wedge dx + (1+3y^2)\,dy\wedge dz \quad \text{—①}$$

Let $\text{curl}A = B$, where $B = B^x\partial_x + B^y\partial_y + B^z\partial_z$.

$$B\lrcorner\eta = (B^x\partial_x + B^y\partial_y + B^z\partial_z)\lrcorner(x^2+1)\,dx\wedge dy\wedge dz$$
$$= B^x(x^2+1)\,dy\wedge dz + B^y(x^2+1)\,dz\wedge dy + B^z(x^2+1)\,dx\wedge dy \quad \text{—②}$$

EQUATE ① + ②, GETTING A SYSTEM OF EQUATIONS,

$$\begin{cases} B^z(x^2+1) = -1 \longrightarrow B^z = \dfrac{-1}{x^2+1} \\ B^y(x^2+1) = 0 \longrightarrow B^y = 0 \\ B^x(x^2+1) = 1+3y^2 \longrightarrow B^x = \dfrac{1+3y^2}{x^2+1} \end{cases}$$

$$\boxed{\therefore\ \text{curl}A = \left(\frac{1+3y^2}{x^2+1}\right)\partial_x + \left(\frac{-1}{x^2+1}\right)\partial_z}$$

$\boxed{\text{div}A}$ $A = y\partial_x$

$$\text{div}A\cdot\eta = d(A\lrcorner\eta)$$

$$\rightarrow A\lrcorner\eta = y\partial_x\lrcorner(x^2+1)\,dx\wedge dy\wedge dz = y(x^2+1)\,dy\wedge dz = (x^2y+y)\,dy\wedge dz$$

$$\rightarrow d(A\lrcorner\eta) = (2xy)\,dx\wedge dy\wedge dz$$

Let $\text{div}A = f$, where $f\in\mathcal{F}M$.

$$\Rightarrow f\cdot\eta = f\,dx\wedge dy\wedge dz$$

$$\Rightarrow f\,dx\wedge dy\wedge dz = (2xy)\,dx\wedge dy\wedge dz$$

$$\hookrightarrow f = 2xy \Rightarrow \boxed{\text{div}A = 2xy}$$

GRAVITATIONAL FIELD DATA:

GROUP A: $X = (x+y)\partial_x + (y+z)\partial_y - (z+x)\partial_z$

GROUP B: $X = \arctan(x)\partial_x + y\partial_y + z\partial_z$

WHICH GROUP PROVIDED FAKE DATA? WHY? FOR THE OTHER GROUP, WHAT DISTRIBUTION OF MASSES PRODUCE THIS FIELD?

$\boxed{\text{GRAVITY MANDALA}}$

$$\begin{array}{ccccc} u & \xrightarrow{d} & E & \xrightarrow{d} & 0 \\ & & \downarrow * & & \\ & & E^* & \xrightarrow{d} & \rho \end{array}$$

$E \in \Lambda^1 M$

$E^* \in \Lambda^2 M$

$\rho \in \Lambda^3 M$

$u \in \Lambda^0 M = \mathcal{F}M$

$\text{WORK} = \int_C E$

(i)

Let's assume standard metric & volume for $(\mathbb{R}^3, g, \eta)$, where coordinates are $\{x, y, z\}$. To get the field $E$ from $X \in XM$, we need to contract $X$ into our metric ($X \lrcorner g \in \Lambda^1 M$). From there, if $dE = 0$, then the field is a gravitational field.

$\boxed{\text{GROUP A}}$ $X \lrcorner g = (x+y)dx + (y+z)dy - (z+x)dz$

$$d(X \lrcorner g) = (1) dy \wedge dx + (1) dz \wedge dy - dx \wedge dz$$
$$= -dy \wedge dz + dz \wedge dz - dx \wedge dy$$
$$\neq 0.$$

$\boxed{\text{GROUP B}}$ $X \lrcorner g = \arctan(x)dx + y\,dy + z\,dz$

$$d(X \lrcorner g) = \partial/\partial x(\arctan(x)) dx \wedge dx + dy \wedge dy + dz \wedge dz$$
$$= 0.$$

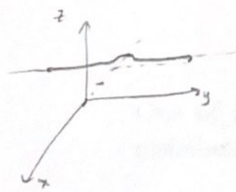
GROUP A PROVIDED FAKE DATA SINCE THEIR GRAVIATIONAL FIELD ISN'T CLOSED, WHICH MEANS THERE IS NOT A POTENTIAL $u$ THAT DESCRIBES THE FIELD & WORK IN THE FIELD IS NOT CONSERVED.

(ii)

To find the distribution of masses for Group B, we can contract our field into the manifold's volume, then take the exterior derivative.

$\boxed{\text{GROUP B}}$ $X \lrcorner \eta = \arctan(x)\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$

$$d(X \lrcorner \eta) = \partial/\partial x(\arctan(x)) dx \wedge dy \wedge dz + \partial/\partial y(y)\, dy \wedge dz \wedge dx + \partial/\partial z(z)\, dz \wedge dx \wedge dy$$
$$= \left(\frac{1}{1+x^2} + 2\right) dx \wedge dy \wedge dz$$




$$\rightarrow \boxed{\rho = \left(\frac{1}{1+x^2} + 2\right) \cdot \eta}$$