Exercise 1: Consider a 2-dimensional manifold M with coordinate chart $\{x, y\}$. The following objects are given:

- $\mathbf{v} = 2\partial_x + \partial_y$; at point $p = (3, 1)$; $\mathbf{v} \in T_pM$
- $f = x^2 + xy + 2$; scalar function; $f \in \mathcal{F}M$
- $\mathbf{A} = 2x^2\partial_x + xy\partial_y$; vector field; $\mathbf{A} \in \mathcal{X}M$
- $\mathbf{B} = y\partial_x$; vector field; $\mathbf{B} \in \mathcal{X}M$
- $c(t) = (t^2 + t, 2\cos t)$; a curve; $\mathbf{c} \in \mathcal{C}M$

Calculate the following:

(a) $\mathbf{v}f$
(b) $\mathbf{A}f$
(c) $f \circ c$
(d) $\frac{\mathrm{d}}{\mathrm{d}t}(f \circ c)$ at $t = 0$
(e) $\dot{\mathbf{c}} \equiv \dot{c}(0)$
(f) $\dot{\mathbf{c}}f$
(g) $[\mathbf{A}, \mathbf{B}]$
(h) Draw the vector field $\mathbf{B}$ in the neighborhood of $(0, 0)$

Solution.

(a)

$$\begin{aligned}
\mathbf{v}f &= (2\partial_x + \partial_y)(x^2 + xy + 2) \\
&= 2\partial_x(x^2 + xy + 2) + \partial_y(x^2 + xy + 2) \\
&= (2(2x + y) + x)\Big|_{(x,y)=(3,1)} \\
&= 2(2(3) + 1) + 3 \\
&= 17
\end{aligned}$$

(b)

$$\begin{aligned}
\mathbf{A}f &= (2x^2\partial_x + xy\partial_y)(x^2 + xy + 2) \\
&= 2x^2\partial_x(x^2 + xy + 2) + xy\partial_y(x^2 + xy + 2) \\
&= 2x^2(2x + y) + xy(x) \\
&= 4x^3 + 3x^2y
\end{aligned}$$

(c)

$$\begin{aligned}
f \circ c &= f(c(t)) \\
&= (t^2 + t)^2 + (t^2 + t)(2\cos t) + 2
\end{aligned}$$

(d)

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}(f\circ c) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}[(t^2+t)^2+(t^2+t)(2\cos t)+2] \\
&= 2(t^2+t)(2t+1)+2(2t+1)\cos t-\sin t(t^2+t)\Big|_{t=0} \\
&= 2(0)(1)+2(1)(1)-0(0) \\
&= 2
\end{aligned}$$

(e) *Remember* $\dot{\mathbf{c}} = \dot{c}^i\partial_i = \dot{c}^i(x^i)\partial_i = \frac{\mathrm{d}(x^i\circ c)(t)}{\mathrm{d}t}\partial_i$. Operating on a function f,

$$\dot{\mathbf{c}}(f) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t_0} f\circ c(t)$$

Here,

$$\begin{aligned}
\dot{\mathbf{c}} &= \frac{\mathrm{d}(t^2+t)}{\mathrm{d}t}\partial_x + \frac{\mathrm{d}(2\cos t)}{\mathrm{d}t}\partial_y \\
&= (2t+1)\partial_x - (2\sin t)\partial_y\Big|_{t=0} \\
&= \partial_x
\end{aligned}$$

(f)

From (c), we know that

$$(f\circ c)(t) = (t^2+t)^2+(t^2+t)(2\cos t)+2$$

So,

$$\begin{aligned}
\dot{\mathbf{c}}f &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0}(t^2+t)^2+(t^2+t)(2\cos t)+2 \\
&= 2(t^2+t)(2t+1)+(2t+1)(2\cos t)-(t^2+t)(2\sin t)\Big|_{t=t_0} \\
&= 2(t_0^2+t_0)(2t_0+1)+(2t_0+1)(2\cos t_0)-(t_0^2+t_0)(2\sin t_0)
\end{aligned}$$

(g) *Remember* the Lie bracket of two vector fields $\mathbf{A}$ and $\mathbf{B}$ is

$$[\mathbf{A},\mathbf{B}] = \left(A^j(\partial_j B^i) - B^j(\partial_j A^i)\right)\partial_i$$

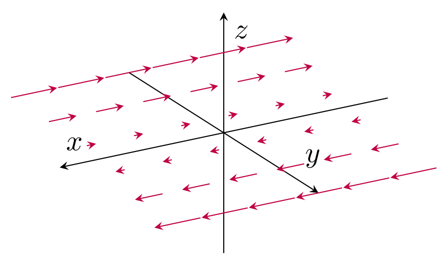
Plugging in our vector fields,

$$\begin{aligned}
[\mathbf{A},\mathbf{B}] &= [A^j(\partial_j y) - B^j(\partial_j 2x^2)]\partial_x + [A^j(\partial_j 0) - B^j(\partial_j xy)]\partial_y \\
&= [A^j(\partial_j y)]\partial_x - [B^j(\partial_j 2x^2)]\partial_x - [B^j(\partial_j xy)]\partial_y \\
&= [2x^2(\partial_x y) + xy(\partial_y y)]\partial_x - [y(\partial_x 2x^2)]\partial_x - [y(\partial_x xy)]\partial_y \\
&= (0 + xy)\partial_x - 4xy\partial_x - y^2\partial_y \\
&= -3xy\partial_x - y^2\partial_y
\end{aligned}$$

(h)

In the neighborhood of $(0,0)$,

$$\mathbf{B} = y\partial_x$$



Exercise 2: Consider $M = \mathbb{R}^3$ with the chart of rectangular coordinates (x, y, z). Express each vector of the basis associated with the chart of spherical coordinates (r, φ, θ), namely $\{\, \partial_r, \partial_\varphi, \partial_\theta \,\}$ in terms of the standard basis $\{\, \partial_z, \partial_y, \partial_z \,\}$.

Solution.

Exercise 3: Show that the Lie bracket of vector fields satisfies the Jacobi identity:

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

Solution.

Exercise 4: Let $C = [A, B]$ be the Lie bracket of two vector fields. In a chart, the vector fields are given as $A = A^i\partial_i$, $B = B^i\partial_i$ and $C = C^i\partial_i$. Express the coefficients C^i in terms of the coefficients of the other two vector fields.

Solution.