2.5 Manifolds with a structure

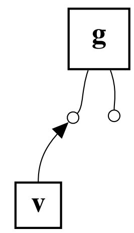
Preparation: read Chapter 1, Section 4.

In many applications of differential geometry one deals with a manifold on which a particular geometric object is fixed. Such an object is called a "structure" on M. The basic example is a manifold with an inner product, understood as an inner product defined on every tangent space:

$$g_p : \; T_pM \times T_pM \;\longrightarrow\; \mathbb{R} \tag{2.12}$$

It is smooth in the sense that for any pair of smooth vector fields X and Y their product $g(X,Y)$ is a smooth scalar function. Recall that the inner product may be viewed as a map from a space to the dual space

$$\mathring{g}_p : \; T_pM \;\longrightarrow\; T_p^*M \tag{2.13}$$



(in the case it is invertible, it is a particular identification of the two spaces.) Here is a guide how to define it in practice in the presence of a coordinate system $\{x,y\}$ on a 2-dimensional manifold (with the obvious generalization):

A. Direct specification. Via a map

$$\begin{aligned} \partial_x &\mapsto a\,dx + b\,dy \\ \partial_y &\mapsto c\,dx + d\,dy \end{aligned}$$

where a,b,c,d are some smooth function of x and y.

B. Matrix form. Via a matrix g and a map $\mathbf{v} \mapsto \mathbf{v}^T g$:

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \text{e.g.,} \qquad \partial_x \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix} \;\mapsto\; \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv a\,dx + b\,dy$$

where a,b,c,d as above.

C. Tensor form. Via a tensor g and a map $\mathbf{v} \mapsto \mathbf{v} \lrcorner g$. In coordinate system, $g = g_{ij}dx^i \otimes dx^j$ for some scalar functions g_{ij}. In the above example,

$$g = a\; dx \otimes dx + b\; dx \otimes dy + c\; dy \otimes dx + d\; dy \otimes dy \tag{2.14}$$

where a,b,c,d as above. In this formulation, the map $g : T_pM \to T_p^*$ may be realized simply as a contraction:

$$\mathbf{v} \mapsto \mathbf{v} \lrcorner g$$

which in the case of $\mathbf{v} = \partial_x$ and the tensor (2.14) becomes

$$\partial_x \lrcorner g = a\,dx + b\,dy\,.$$

Here are some special cases of manifolds with an inner product:

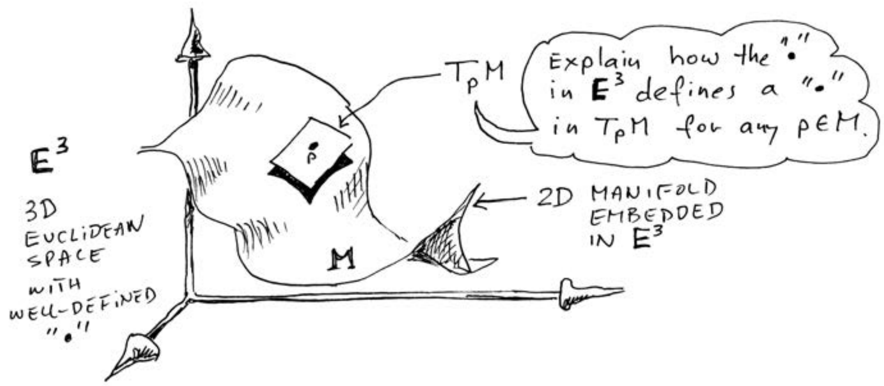
Flat Euclidean space. The simplest example is the Euclidean space $\mathbf{E^n} \cong \mathbb{R}^\mathbf{n}$ with the standard Euclidean structure (scalar product) at every point. It admits coordinates in which the matrix of g looks like he identity matrix; for the above example:

$$g = dx \otimes dx + dy \otimes dy \qquad \text{or, as a matrix:} \quad g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The identification is $\partial_x \mapsto dx$ and $\partial_y \mapsto dy$.

Riemannian manifold is a generalization of the above. It is a manifold on which there is an Euclidean structure chosen in every tangent space but which not necessarily admits "flat coordinate" in which $g_i j$ are constant. We demand only that at every tangent space T_pM, the product g is non-degenerate and symmetric.

A simple way to obtain such manifolds is by choosing a smooth submanifold in the flat Euclidean space, $M \subset \mathbf{E}$. The Riemannian structure is inherited from Euclidean $\mathbb{R}^3$ by simple reconsidering vectors $\mathbf{v} \in TM$ tangent to M as existing in the tangent space of E, on which the Euclidean product **is** defined. The standard sphere $S^2 \subset \mathbb{R}^3$ is an example. In general, a Riemannian manifold has a "shape"; one can calculate from g a *curvature*, define a *geodesic*, etc.



Premature problem: Suppose $M \subset \mathbf{E}^3$ is a 2-dimensional manifold in the standard Euclidean space, given by equation $z = f(x,y) \equiv x^2 + 3y$. By replacing dz in $g = dx \otimes dx + dy \otimes dy + dz \otimes dz$ by the differential $df = 2x\,dx + 3dy$, find the induced inner product on M to be $g = (1+4x^2)\,dx \otimes dx + 6x\,(dx \otimes dy + dy \otimes dx) + 10\,dy \otimes dy$. Write g as a matrix. This very informally obtained result will be made precise in Section 2.7.

Pseudo-Riemannian manifold $\{M, g\}$ is defined as above, except g is a pseudo-Euclidean product, i.e., not necessarily positive-definite. Space-time is modeled as a pseudo-Riemannian manifold of dimension 4 with a scalar product of signature (3,1). In the **general theory of relativity**, the gravitational forces are eliminated and are replaced by the concept of curvature of space-time, defined by g.

Symplectic manifold is a pair $\{M, \omega\}$ where $\omega \in \Lambda^2 M$ is a differential bi-form that is closed, $d\omega = 0$. It is also assumed to be non-degenerate in the sense that the map $T_pM \to T_p^*M : \mathbf{v} \mapsto i_{\mathbf{v}}\omega$ is a space isomorphism. It is a scalar product, with the difference that $\omega(\mathbf{v}, \mathbf{w})$ is skew-symmetric rather than symmetric). The "phase space" of classical mechanics is a symplectic manifold. Symplectic geometry is the heart of the Hamilton and Lagrange equations of motion. Also, symplectic geometry is central in thermodynamics.

Manifold with volume is a pair $\{M, \eta\}$ where $\eta \in \Lambda^n M$ is a differential of the highest possible degree $n = dimM$. It is called a "volume form" and is usually assumed to be non-vanishing at all points of the manifold.

2.6 Excursion to the 3D world: grad, div and curl

In a three dimensional manifold, like $M \cong \mathbb{R}^3$ with coordinate chart $\{x,y,z\}$, the most general differential exterior forms of each degree are

$$\begin{array}{rlr} f = & f(x,y,z) & f \in \Lambda^0 M = \mathcal{F}M \\ \alpha = & a\,dx + b\,dy + c\,dz & \alpha \in \Lambda^1 M \\ \beta = & A\,dy\wedge dz + B\,dz\wedge dx + C\,dx\wedge dy & \beta \in \Lambda^2 M \\ \eta = & h\,dx\wedge dy\wedge dz & \eta \in \Lambda^3 M \end{array}$$

where f, a, b, c, A, B, C and h are some smooth scalar functions on M. Calculating the exterior derivatives of the above differential forms (the last Exercise of Section 2.4.2) yields $d\eta = 0$ and:

$$\begin{array}{rl} d\,f = & \partial_x f\,dx + \partial_y f\,dy + \partial_z f\,dz \\ d\,\alpha = & (\partial_x b - \partial_y a)\,dx\wedge dy + (\partial_y c - \partial_z b)\,dy\wedge dz + (\partial_z a - \partial_x c)\,dz\wedge dx \\ d\,\beta = & (\partial_x A + \partial_y B + \partial_z C)\;dx\wedge dy\wedge dz \end{array} \tag{2.15}$$

Looks familiar?...

Recall the standard textbook definitions of 3D Calculus: The manifold is just the flat Euclidean space $\mathbf{E}^3$. The standard basis consists of three orthonormal vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and a general vector field is $\mathbf{X} = A\,\mathbf{e}_1 + B\,\mathbf{e}_2 + C\,\mathbf{e}_3$. Then one is heavy-handedly given these three derivatives:

$$\begin{array}{rl} \mathbf{grad}\,f = & \partial_x f\,\mathbf{e}_1 + \partial_y f\,\mathbf{e}_2 + \partial_z f\,\mathbf{e}_3 \\ \mathbf{curl}\,X = & (\partial_x b - \partial_y a)\,\mathbf{e}_3 + (\partial_y c - \partial_z b)\,\mathbf{e}_1 + (\partial_z a - \partial_x c)\,\mathbf{e}_2 \\ \mathrm{div}\;X = & \partial_x A + \partial_y B + \partial_z C \end{array} \tag{2.16}$$

They seem to be pretty arbitrary combinations of partial derivatives — but now the secret is out: these formulas look awfully similar to the results of exterior derivatives of exterior forms above!

Rough comparison would suggest that "$df = \mathbf{grad}\,f$", "$d\alpha = \mathbf{curl}\,\alpha$", and "$d\beta = \mathrm{div}\;\beta$", as in this diagram:

$$\mathcal{F}M \;\xrightarrow[\mathbf{grad}\,?]{d}\; \Lambda^1 M \;\xrightarrow[\mathbf{curl}\,?]{d}\; \Lambda^2 M \;\xrightarrow[\mathrm{div}\,?]{d}\; \Lambda^3 M$$

But beware of such quick conclusions. The derivatives **grad**, **curl**, and div are about *vector fields* — not differential forms! We need to "transfer" the exterior derivatives d of (2.15) to the realm of vector fields $\mathcal{X}M$. To do it, we will need to fix some extra-structures on M. The main idea is to define the derivatives of 3D Calculus in a **coordinate-free** manner. Let us look at each case.

Figure 2.3: Mr. Natural, personification of naturality (from drawings of R. Crumb)

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1. Gradient. Consider a manifold $\{M, g\}$ with a nondegenerate inner product g (not necessarily symmetric). Gradient is defined as a map from scalar function to vector fields such that $\mathbf{grad}\, f$ satisfies

$$\mathbf{grad} : \mathcal{F}M \to \mathcal{X}M \qquad \boxed{(\mathbf{grad}\, f) \lrcorner g = df} \tag{2.17}$$

We use here the fact that a scalar product g may be viewed as a map $\mathring{g} : TQ \to T^*Q$, or, equivalently, $\mathring{g} : \mathcal{X}M \to \Lambda^1 M : \mathbf{X} \mapsto \mathbf{X} \lrcorner g$. One can think of gradient as defined by $\mathbf{grad}\, f = \mathring{g}^{-1} \circ df$. The formula in Equation (2.16) corresponds to the case of the orthonormal coordinate system.

2. Divergence. Divergence is defined on manifolds $\{M, \eta\}$ where η is a *volume form*, that is a non-vanishing differential form of the highest degree $n = \dim M$. Notice that it determines a map $\mathcal{X}M \to \Lambda^{n-1}M$, namely $\mathbf{X} \to \mathbf{X} \lrcorner \eta$. Divergence of a vector field $\mathbf{X}$ is a scalar function $\operatorname{div} X$ such that

$$\operatorname{div} : \mathcal{X}M \to \mathcal{F}M \qquad \boxed{d\,(\mathbf{X} \lrcorner \eta) = \operatorname{div} \mathbf{X} \cdot \eta} \tag{2.18}$$

The formula in Equation (2.16) corresponds to the case when the volume form is chosen to be $\eta = dx \wedge dy \wedge dz$ (check it).

3. Curl. The curl is defined on manifolds $\{M, g, \eta\}$ with an inner product g and a volume form η is a *volume form*. Curl of a vector field $\mathbf{X}$ is a vector field $\mathbf{curl}\,\mathbf{X}$ such that

$$\mathbf{curl} : \mathcal{X}M \to \mathcal{X}M \qquad \boxed{d\,(\mathbf{X} \lrcorner g) = (\mathbf{curl}\, \mathbf{X}) \lrcorner \eta} \tag{2.19}$$

As before, when the volume form is chosen to be $\eta = dx \wedge dy \wedge dz$ and the coordinate system is orthonormal, we can recover the standard 3D Calculus definitions. Note that to define *curl* one needs both η and g. In general, they do not need to be related.

Remark: Note that the operations **grad** and div may be defined on manifolds of any dimension. The first requires a nondegenerate scalar product (not necessarily symmetric!), the latter a nondegenerate volume form (thus the manifold must be orientable). The operator **curl** is limited to three-dimensional manifolds. Its generalization requires the notion of the "Hodge star" — we shall deal with it in a future section. The theory of electromagnetism, which requires a 4-dimensional manifold of space-time, is a the basic application of such generalizations.

Exercise 1: To get acquainted visually with different notations — rewrite the definitions of this section using "$i_{\mathbf{X}}$" instead of "$\mathbf{X} \lrcorner$".

Mandala of 3D calculus. Ponder upon the diagram given below. The definitions of the derivatives are turned into commutativity of this diagram. Commutativity of each cell corresponds to another "derivative" in the vector language.

$$
\begin{tikzcd}
\mathcal{F}M \arrow[r, "d"] & \Lambda^1 M \arrow[r, "d"] & \Lambda^2 M \arrow[r, "d"] & \Lambda^3 M & \text{“heavenly” realm} \\
\mathcal{F}M \arrow[r, "\mathbf{grad}"] \arrow[u, "\mathrm{id}"'] & \mathcal{X}M \arrow[r, "\mathbf{curl}"] \arrow[u, "\lrcorner g"'] & \mathcal{X}M \arrow[r, "\mathrm{div}"] \arrow[u, "\lrcorner \eta"'] & \mathcal{F}M \arrow[u, "\cdot \eta"'] & \text{“mundane” realm}
\end{tikzcd}
$$

We can easily transfer the differential-geometric property $d \circ d = 0$ to the realm of 3D Calculus.

Proposition 2.6.1: The derivatives of 3D Calculus satisfy the following identities for any $f \in \mathcal{F}M$ and $X \in \mathcal{X}M$:

$$
\begin{array}{rl}
(i) & \mathbf{curl}\ (\mathbf{grad}\, f) = \mathbf{0} \\
(ii) & \mathrm{div}\ (\mathbf{curl}\, \mathbf{X}) = 0
\end{array}
\tag{2.20}
$$

PROOF: We shall prove (i). Consider a scalar function $f \in \mathcal{F}M$, and apply the definitions of **curl** and then that of **grad**:

$$
\begin{array}{rcll}
(\mathbf{curl}\ (\mathbf{grad}\, f)) \lrcorner \eta & = & d\,((\mathbf{grad}\, f) \lrcorner g) & \text{by definition of } \mathbf{curl} \\
& = & d\, d\, f & \text{by definition of } \mathbf{grad} \\
& = & 0 & \text{by } d \circ d = 0
\end{array}
$$

The proof of (ii) is left as a simple exercise. □

Exercise 2: Prove (ii).

Remark. The above mandala contains the drama of the problems with identification of the objects occurring in the 3D calculus. Metaphorically speaking, the upper chain of maps and objects belongs to the realm of Platonic ideas: it is simple and transparent, very economic in structure use. But in the 19th century, the idea of exterior forms was not developed yet. Hence one was (unknowingly) forced to express various entities in terms of scalars and vectors only. This process of squeezing things to these two concepts introduces artificially the need of extra structures (like g and η) and results in in obfuscation of the nature of the objects. This is the mundane level.

A few things to notice: Mass density is not a scalar function, it is an exterior 3-form. Changing coordinates, say from from centimeters to inches, demonstrates it clearly. That is why density was called in the old times desperately a "pseudo-scalar". We shall see that, similarly, magnetic field is not a vector field but rather a 2-form (see Chapter xx). In the old textbooks it was called a "pseudo-vector".

Also, consider (2.20). Very natural to forms, it splits to two two different statements in the "mundane level". Such properties remain puzzling if expressed in terms of the standard 3D Calculus.

Practical calculations. Suppose $M \cong \mathbb{R}^3$ has the standard structure:

$$g = dx\otimes dx + dy\otimes dy + dz\otimes dz \qquad \text{and} \qquad \eta = dx\wedge dy\wedge dz$$

A. How to calculate the gradient of function $f = x^2 + yz$? Assume that $\mathbf{grad}\, f = A\partial_x + B\partial_y + C\partial_z$ for some A, B, and C to be found. Calculate:

$$\begin{array}{rl} \mathbf{grad}\, f \lrcorner g = & A\,dx + B\,dy + C\,dz \\ df = & 2x\,dx + z\,dy + y\,dz \end{array} \qquad \Rightarrow \qquad \begin{array}{ll} A & = 2x \\ B & = z \\ C & = y \end{array}$$

Thus $\mathbf{grad}\, f = 2x\partial_x + z\partial_y + y\partial_z$.

B. How to calculate the curl of vector field $\mathbf{X} = x\partial_y$? Assume that $\mathbf{curl} X = A\partial_x + B\partial_y + C\partial_z$ for some A, B, and C to be found. Calculate:

$$\begin{array}{rl} \mathbf{curl}\,\mathbf{X} \lrcorner \eta = & A\,dy\wedge dz + B\,dz\wedge dx + C\,dx\wedge dy \\ d(\mathbf{X}\lrcorner g) = & d\,(x\,dy) = dx\wedge dy \end{array} \qquad \Rightarrow \qquad \begin{array}{ll} A & = 0 \\ B & = 0 \\ C & = 1 \end{array}$$

Thus $\mathbf{curl}\,\mathbf{X} = \partial_z$.

C. How to calculate the divergence of vector field $\mathbf{X} = xy\,\partial_x$? Assume that $\operatorname{div}\mathbf{X} = f$ for some f to be found. Calculate:

$$d(\mathbf{X}\lrcorner\eta) = d\,(xy\,dy\wedge dz) = y\,dx\wedge dy\wedge dz \qquad \Rightarrow \qquad f = y$$

Thus $\operatorname{div}\mathbf{X} = y$.

Exercise 3: **(i)** Show that Equations (2.16) of the standard Calculus III correspond to the case of orthonormal coordinate system with the volume form chosen to be $\eta = dx\wedge dy\wedge dz$.
(ii) Suppose $M = \mathbf{R}^3$ is equipped with the following structure

$$\eta = (x^2+1)\,dx\wedge dy\wedge dz \qquad [g] = \begin{bmatrix} 1 & e^x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & y^2+z^2+1 \end{bmatrix}$$

For $f = x^2$ and $\mathbf{X} = x\partial_x$, calculate the derivatives **curl X**, $\mathbf{grad}\, f$ and $\operatorname{div}\mathbf{X}$.

Exercise 4: Calculate Leibniz-like identities for the following derivatives of 3D calculus using the definitions via differential exterior forms:

$$\mathbf{grad}\,(fg), \qquad \mathbf{curl}\,(f\mathbf{X}), \qquad \mathrm{div}\,(f\mathbf{X}).$$

where $f, g \in \mathcal{F}M$ and $\mathbf{X} \in \mathcal{X}M$.

Exercise 5: Laplacian of a scalar function f is defined as div $\mathbf{grad}\, f$. Rewrite this definition using $\mathring{g}$ and $\mathring{\eta} \equiv \lrcorner\, \eta$. Is Laplacian native only to 3-dimensional manifolds?

Compatibility of structures. Till now the two structures, volume form η and the inner product g, where independent. But one may make them **compatible** by demanding that for any three vector fields $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{X}M$ we have

$$\eta(\mathbf{A},\mathbf{B},C)\ \cdot \eta = \mathring{g}(\mathbf{A}) \wedge \mathring{g}(\mathbf{B}) \wedge \mathring{g}(\mathbf{C}) \tag{2.21}$$

where $\mathring{g}(\mathbf{A}) \equiv \mathbf{A} \lrcorner g \equiv g(\mathbf{A},\,\cdot\,)$. The standard expositions of the 3D calculus tacitly assumes this compatibility. Our definitions are more general.

Exercise 6 (compatibility): Show that if for $\{M, g, \eta\}$ the scalar product g and a volume form η are compatible then in 3-dimensional case

$$\eta = \pm\sqrt{\det(g)}\; dx \wedge dy \wedge dz \tag{2.22}$$

in any particular coordinate system. Hint: Consider $\mathbf{A} = \partial_1$, $\mathbf{B} = \partial_2$ and $\mathbf{C} = \partial_3$). Conclusion: To define divergence and curl having at hand only g, one still needs to pick one of the two orientations (the $\pm$ sign in (2.21)). Generalize to n-dimensional manifold.

Exercise 7 (cross-product): Consider a three-dimensional manifold $\{M, g, \eta\}$ with a scalar product g and a volume form η. The question is to define the cross-product of two vector fields A and B as a vector field $A \times B$. We may consider two options:

$$\begin{array}{ll} (a) & (\mathbf{A}\times\mathbf{B}) \lrcorner \eta = \mathring{g}(\mathbf{A}) \wedge \mathring{g}(\mathbf{B}) \\ (b) & (\mathbf{A}\times\mathbf{B}) \cdot \mathbf{C} = \eta(\mathbf{A},\, \mathbf{B},\, \mathbf{C}) \end{array} \tag{2.23}$$

Condition (a) shows directly that the cross-product is a 3D representation of the wedge product. Condition (b) corresponds to the well-known formula for the volume of a parallelepiped spanned by vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$.

(i) Derive the formula for the cross product in each case, (a) and (b), for the case of orthonormal coordinate system $\{x, y, z\}$ with $\eta = dx \wedge dy \wedge dz$.

(ii) Are the two alternative definitions equivalent in the general case?

(iii) Rewrite condition (b) without use of $\mathbf{C}$ (use g).

2.7 Induced maps

A map $\varphi : M \to N$ between two manifolds M and N is called **differentiable**, if it is smooth in any chart. The manifolds may be of different dimension. Here are a few examples:

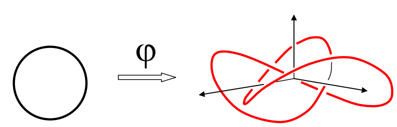
Example 1 [from 3D to 2D]: Let $M \cong \mathbb{R}^3$ have chart $\{x, y, z\}$ and $N \cong \mathbb{R}^2$ have chart $\{r, s\}$. A differential map $\varphi : M \to N$ can be given in the coordinates as

$$\begin{aligned} r \circ \varphi &= x + y \\ s \circ \varphi &= xyz \end{aligned}$$

It is well-defined in some appropriate open neighborhoods (find them).

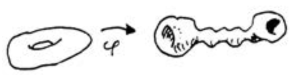
Example 2 [from 1D to 3D]: Let $M \cong S^1$ be a circle with coordinate t and $N \cong \mathbf{R}^3$ be the standard space with coordinates $\{x, y, z\}$. The following system of equations defines a map $\varphi : S^1 \to \mathbb{R}^3$, the image of which is a trefoil knot.

$$\begin{aligned} x(t) &= \cos t + 2\cos 2t \\ y(t) &= 2\sin 2t - \sin t \\ z(t) &= -\sin 3t \end{aligned} \tag{2.24}$$

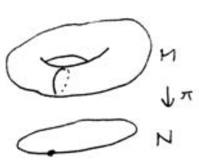


As usual, $x(t)$ is short for $(x \circ \varphi)(t) \equiv x(\varphi(t))$, etc.

A differentiable map $\varphi : M \to N$ is called:



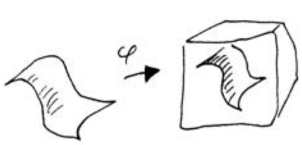
1. **diffeomorphism** if it is a bijection, i.e., *one-to-one* and *onto*. Clearly, both manifolds must be of the same dimension, $\dim M = \dim N$. As a special case, one may consider diffeomorphisms of a manifold to itself, $\varphi : M \to M$.



2. **projection** if it is *onto* (surjection) and $\dim M < \dim N$. For any point $m \in M$, its anti-image is a submanifold of N

$$\pi^{-1}(m) \subset N$$

   and will be called a *fiber* over m. In special regular cases, when fibers over different points are mutually diffeomorphic, the triple $\{M, \pi, N\}$ forms a *fiber bundle*.



3. **Embedding** if it is injective and $\dim N < \dim M$. In particular, the map understood as a map from M to its image in N is a diffeomorphism. Special case: if N is a submanifold of M, $N \subset M$, then one usually expresses this fact as an 'embedding', where the image of a point $p \in N$ is understood as the same point, now considered as a point of M.

4. **Immersion** if it is injective and $\dim N < \dim M$ but the image may intersect with itself. However, for every point $p \in M$ there exist a neighborhood small enough that the map becomes an embedding when restricted to it. For instance, the Klein bottle in $\mathbb{R}^3$ may be presented as an immersion but not as an embedding.

Every differentiable map determines a series of associated maps between other geometric objects living on the manifolds like vectors, covectors, scalar functions, curves, differential forms, etc. These maps are called **induced maps** and our goal is to define them unambiguously.

Interestingly, this operation divides aforementioned geometric objects into two categories: the so-called **covariant** objects and **contravariant**, depending whether the induced map goes *with* or *against* the direction of the original map φ. In mathematical folk-lore the maps are denoted by φ with a star located up or down, and called:

$$\varphi^* \rightarrowtail \quad \text{“pull-back” of covariant objects:} \quad \begin{cases} \text{covectors, functions,} \\ \text{exterior k-forms, etc} \end{cases}$$

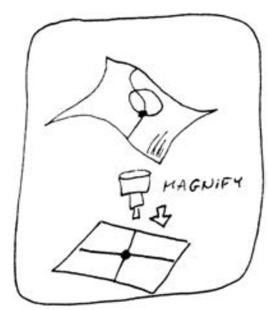
$$\varphi_* \rightarrowtail \quad \text{“push-forward” of contravariant objects:} \quad \begin{cases} \text{vectors, curves,} \\ \text{vector fields, etc} \end{cases}$$

The different directions of the induced maps may seem at first puzzling. Let us use a metaphor: When you walk from one room to another, besides transferring your body you also “induce” a transfer of other objects. For instance of a few items that you might have in your pocket — they go the same direction as you. But associated with your walk is the a motion of air – and this “induced transfer” goes in the *opposite* direction as you displace it by walking in.

2.7.1 Intuition

We shall start with an informal intuitive description.

Consider a diffeomorphism $\varphi : M \to M$. You may think of φ as a rubber deformation of M. It is clear that all curves on M become new curves on M, therefore there is no problem to see what happens with vectors, as they are vectors of velocity. If the map extends in some direction, the vectors are extended accordingly. If the neighborhood is rotated, so are the tangent vectors.

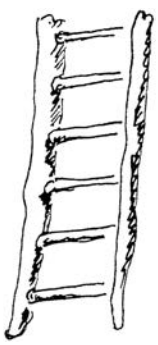


Another view on what is happening: In a sense, induced maps are linear approximations of the diffeomorphism. Pick a point and look at its very small neighborhood under a microscope. The curves look like straight lines. The level surfaces of scalar functions look like hyperplanes. The ‘curvilinearity’ disappears under magnification. Diffeomorphism φ becomes here almost “linear”. And that is the essence of the induced map: The tangent space may be identified with such neighborhood, and the linear approximation of φ is the induced map of tangent space (and cotangent space).

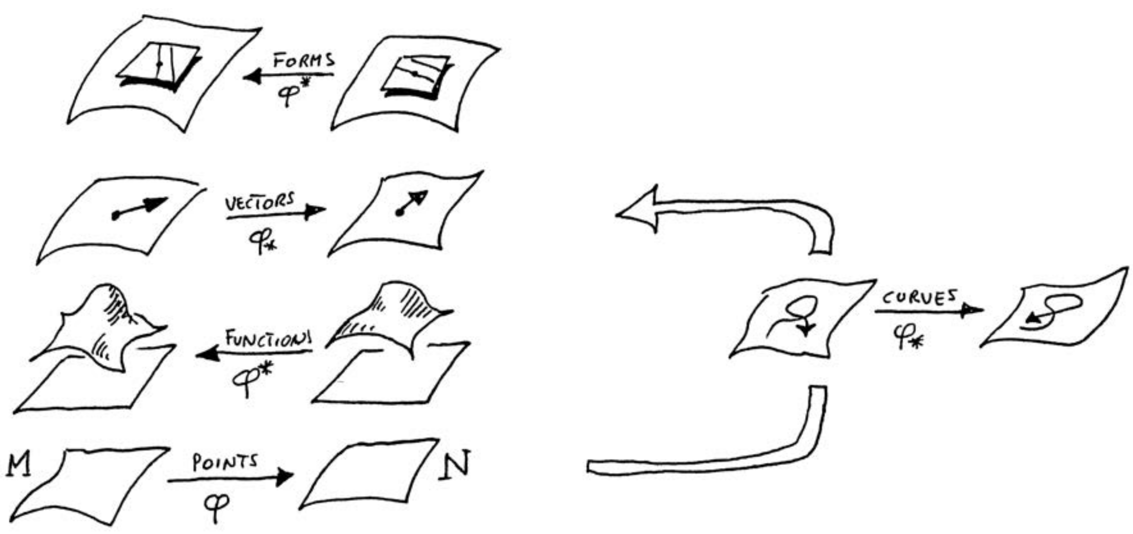
Diffeomorphisms are, by definition, bijections and therefore they carry with themselves all types of geometric objects. But in a more general case things become more subtle and one must pay attention to the direction of the induced maps.

For convenience, let us put upfront the overall situation: The tower of induced maps below summarizes the definitions of particular induced maps. It should be read from the bottom upwards; each level is defined by the level below it. Notice how the direction of these maps varies for different levels.

Mandala of induced maps



$\Lambda^n M$	$\xleftarrow{\varphi^*}$	$\Lambda^n N$	$\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta$
$\Lambda^1 M$	$\xleftarrow{\varphi^*}$	$\Lambda^1 N$	$\varphi^*(\alpha) = \alpha \circ \varphi_*$
$T_p^* M$	$\xleftarrow{\varphi^*}$	$T_{\varphi(p)} N$	$\varphi^*(\alpha) = \alpha \circ \varphi_*$
$T_p M$	$\xrightarrow{\varphi_*}$	$T^N_{\varphi(p)}$	$\varphi_*(v) = v \circ \varphi^*$
$\mathcal{F}M$	$\xleftarrow{\varphi^*}$	$\mathcal{F}N$	$\varphi^*(f) = f \circ \varphi$
M	$\xrightarrow{\varphi}$	N	*point to point*



2.7.2 Climbing the ladder of induced maps

Now, let us climb the “tower” and define each of the induced maps.

Floor 1: Pull-back of functions

The pull-back of a scalar function $f \in \mathcal{F}N$ is a function denoted $\varphi^*(f) \in \mathcal{F}M$ defined as

$$\varphi^*(f) = f \circ \varphi \tag{2.25}$$

This may be represented by the commutativity of the diagram of maps:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\mathrm{id}"] & \mathbb{R} \\
M \arrow[u, "\varphi^* f"'] \arrow[r, "\varphi"] & N \arrow[u, "f"']
\end{tikzcd}$$

Example 3: Let $M \cong \mathbb{R}^3$ have chart $\{x,y,z\}$ and $N \cong \mathbb{R}^2$ have chart $\{u,v\}$. Consider a differential map from Example 1, namely $\varphi : M \to N$ which in the coordinates is

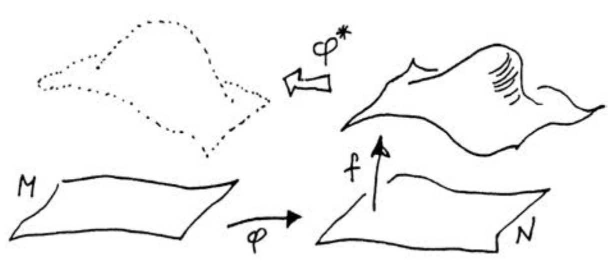
$$(u \circ \varphi)(x,y,z) = x + y$$
$$(v \circ \varphi)(x,y,z) = xyz$$

between some appropriate open neighborhoods. This can be reported also this way:

$$(x,y,z) \quad \mapsto \quad (u,v) = (x+y, xyz)$$

Now, consider a scalar function $f \in \mathcal{F}N$ on the second manifold, namely $f = u^2 + v^3$. The induced map $\varphi^* : \mathcal{F}N \to \mathcal{F}M$ sends f to a function $\varphi^*(f) \in \mathcal{F}M$ on the first manifold, where it becomes

$$\varphi^*(f) = f \circ \varphi = (x+y)^2 + (xyz)^3$$



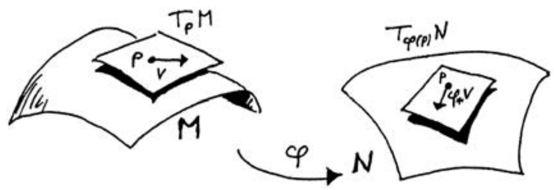
Floor 2: "Push-forward" of vectors

Differential map $\varphi : M \to N$ defines for any point $p \in M$ a linear map between the corresponding tangent spaces,

$$\varphi_* : T_pM \quad \to \quad T_{\varphi(p)}N$$

as follows: if $\mathbf{v} \in T_pM$, then its image $\varphi_*(\mathbf{v}) \in T_{\varphi(p)}N$ is defined via its action on functions $f \in \mathcal{F}N$:

$$\begin{aligned} \varphi_*(\mathbf{v})\,(f) &= \mathbf{v} \circ \varphi^*(f) \\ &= \mathbf{v}\,(f \circ \varphi) \end{aligned} \tag{2.26}$$



Example 4 Let M, N, and f be like in Example 1. If $\mathbf{v} = 4\,\partial_x$ is a vector at point $(1,1,1)$ in M, then $\varphi_*(\mathbf{v})$ is a vector tangent to N at point (2,1), which differentiates function f as follows:

$$(\varphi_*(4\,\partial_x))\,(u^2+v^3)\,\Big|_{u=2,\ v=1} = 4\,\partial_x\,((x+y)^2+(xyz)^3)\,\Big|_{x=y=z=1} = 28$$

Proposition 2.7.1. *Let $\{x^i\}$ be a chart on M, $i = 1,2,\ldots,m$, and $\{y^a\}$ be a chart on N, $a = 1,2,\ldots,n$. The image of the basis vectors at some $p \in M$ are in N:*

$$\varphi_*\Big(\frac{\partial}{\partial x^i}\Big) = \frac{\partial y^a}{\partial x^i}\,\frac{\partial}{\partial y^a}$$

PROOF: Since $\varphi_* :\ T_pM \to T_{\varphi(p)}N$ is a linear map, we should be able to express the image $\varphi_*(\partial_i)$ in terms of a linear combination of basis vectors of $T_{\varphi(p)}N$, that is

$$\varphi_*(\ \partial_i\ ) = J^a_i\,\partial_a \qquad \text{where} \quad \partial_i \equiv \partial/\partial x^i \qquad \partial_a \equiv \partial/\partial y^a$$

for some coefficients J^a_i (to be calculated). Express the map between manifolds in coordinates

$$y^a(x^1,\ldots,x^m)\ :=\ (y^a \circ \varphi)\,(x^1,\ldots,x^m)$$

Induced map may be understood as a geometric interpretation of the chain rule of calculus.

Recall that the a^{th} coefficient of a vector is the result of acting by this vector on the a^{th} coordinate function $y^a \in \mathcal{F}N$. Thus we have

$$J^a_i = \big((\varphi_*)(\partial_i)\big)\,y^a = \partial_{x^i}\,(\varphi^* y^a) = \partial_{x^i}\,(y^a \circ \varphi) = \frac{\partial y^a \circ \varphi}{\partial x^i}$$

$\square$

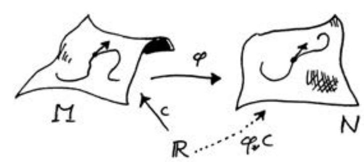
Corollary: The image of a vector $\mathbf{v} = v^i \partial_i$ in some T_pM is in N:

$$\mathbf{v} = v^i \partial_i \quad \xrightarrow{\varphi_*} \quad \varphi_*(\mathbf{v}) = v^i \, \frac{\partial y^a}{\partial x^i} \, \partial_a$$

Matrix J from the above proof is called the **Jacobian** of the map φ and it represents the matrix of the linear map between the corresponding tangent spaces. Map φ_* may be viewed as a local *linearization* of φ: it shows how 'points infinitesimally close to p' behave under the map φ.

Alternative definition: There is another way one could define the induced map of vectors. Notice that any differential map $\varphi : M \to N$ induces a natural map of curves in M into curves in N.

$$\begin{tikzcd} M \arrow[r, "\varphi"] & N \\ \mathbb{R} \arrow[u, "c"] \arrow[r, "\mathrm{id}"] & \mathbb{R} \arrow[u, "\varphi_* c"'] \end{tikzcd} \qquad \varphi_* c = c \circ \varphi$$



Since tangent vectors may be defined as vectors of velocity, this map of curves induces a map of vectors. Let c be a curve through p, say $p = c(0)$. If $[c]$ is the vector at p represented by c, then its image in N may be defined by

$$\varphi_\star \, [c] =: \, [\varphi_* c]$$

Problem: Show that the above definition of induced map of vectors is equivalent to Definition (2.26), i.e. that $\varphi_\star \equiv \varphi_*$.

Floor 3: Pull-back of 1-forms

Let α be a covector at point $\varphi(p) \in N$. The *pull-back* of α is a covector $\varphi^*(\alpha)$ at $p \in M$ defined by

$$\langle \, \varphi^* \, \alpha \mid \mathbf{v} \, \rangle =: \langle \, \alpha \mid \varphi_* \, \mathbf{v} \, \rangle \qquad \forall \, \mathbf{v} \in T_pM \tag{2.27}$$

Ot, if you like brackets:

$$(\varphi^*(\alpha)) \, (\mathbf{v}) =: \alpha \, (\varphi_*(\mathbf{v})) \, .$$

Unlike the push-forward of vectors, the pull-back of 1-forms can be extended without problems to *differential* 1-forms (that is a "covector fields"):

$$\varphi^* \, : \Lambda^1 N \longrightarrow \Lambda^1 M$$

Justify, why.

Proposition 2.7.2. *Let $\{x^i\}$ be a chart on M, $i = 1, 2, \ldots, m$, and $\{y^a\}$ be a chart on N, $a = 1, 2, \ldots, n$. The image of a basis form dy^a is in M the following sum*

$$\varphi_*\big(dy^a\big) = \frac{\partial y^a \circ \varphi}{\partial x^i}\, dx^i$$

and therefore the pull-back of 1-forms is

$$\alpha = \alpha_a\, dy^a \xrightarrow{\varphi^*} \varphi^*(\alpha) = \alpha_a\, \frac{\partial y^a}{\partial x^i}\, dx_i$$

PROOF: Exercise (follow the steps of the analogous proposition for vectors). □

Proposition 2.7.3. *The exterior derivative commutes with the pull-back:*

$$\varphi^*(df) \;=\; d\,\varphi^*(f) \tag{2.28}$$

for any $f \in \mathcal{F}M$. Or, briefly:

$$\boxed{\varphi^* \circ d \;=\; d \circ \varphi^*} \tag{2.29}$$

Proof: Consider this chain of reformulations for any $\mathbf{v}$:

$$\begin{aligned}
\langle\, \varphi^*(df) \mid \mathbf{v} \rangle \;&=\; \langle\, df \mid \varphi_*\, \mathbf{v} \,\rangle && \text{(definition of } \varphi^*) \\
&=\; (\varphi_*(\mathbf{v}))f && \text{(definition of } d) \\
&=\; \mathbf{v}(\varphi^* f) && \text{(definition of } \varphi_*) \\
&=\; \langle\, d(\varphi^* f) \mid \mathbf{v} \,\rangle && \text{(definition of } d)
\end{aligned}$$

□

This leads to an equivalent definition the pull-back of one-forms: once one can pull-back functions, one may extend it to their differentials. Since -forms are combinations of terms like $f\,dg$, this defines a pull-back of one-forms: $\varphi^*(f\,dg) = \varphi^*(f)\,d(\varphi^* g)$

Floor 4: Pull-back of k-forms

Now we may define pull-back of forms of any degree by assuming linearity of φ^*, and

$$\varphi^*(\alpha \wedge \beta) = \varphi^*(\alpha) \wedge \varphi^*(\beta) \tag{2.30}$$

for any multi-forms $\alpha \in \Lambda^j N$ and $\beta \in \Lambda^k N$.

Since the (image of) pull-back is well-defined for differential exterior forms (not only a point map), the following makes sense:

Proposition 2.7.4. *Pull-back commutes with the exterior derivative*

$$\begin{tikzcd}
\Lambda N \arrow[r, "\varphi^*"] & \Lambda M \\
\Lambda N \arrow[r, "\varphi^*"] \arrow[u, "d"'] & \Lambda M \arrow[u, "d"']
\end{tikzcd}
\qquad \varphi^*(d\,\alpha) =: d\,\varphi^*(\alpha)$$

for any $\alpha \in \Lambda^k N$.

PROOF: Exercise. (Cf. Proposition 2.7.3). $\square$

Floor 5: Pull-back of covariant tensor fields

Now we may define pull-back of covariant tensor fields analogously to the exterior forms, namely assuming

$$\varphi^*(\alpha \otimes \beta) = \varphi^*(\alpha) \otimes \varphi^*(\beta) \tag{2.31}$$

for any covariant tensor fields $\alpha \in \mathcal{T}^{(0,j)}N$ and $\beta \in \mathcal{T}^{(0,k)}N$.

Since covariant tensors are products of 1-forms and tensor product is associative, this defines pull-back aof covariant tensors of arbitrary degree. Note also that we have well-defined pull-back of inner products.

2.7.3 Composition of maps

Problem: Let $\varphi : M \to N$ and $\gamma : N \to P$ are differential maps between manifolds. Clearly, their composition is a map $\gamma \circ \varphi :\; M \to N$ which induces a pull-back map $(\gamma \circ \varphi)^* :\; \Lambda M \to \Lambda N$. Show that

$$(\gamma \circ \varphi)^* = \varphi^* \circ \gamma^*$$

Hint: consider first pull-back of functions, using the following diagram

$$\begin{tikzcd}
\mathcal{F}M & \mathcal{F}N \arrow[l, "\varphi^*"'] & \mathcal{F}P \arrow[l, "\psi^*"'] \arrow[ll, bend right, "\varphi^* \circ \psi^*"'] \\
M \arrow[r, "\varphi"] \arrow[rr, bend right, "\psi \circ \varphi"'] & N \arrow[r, "\psi"] & P
\end{tikzcd}$$

Show also that in the case of the induced 'push-forward' maps (of vectors and curves), the composition preserves the order:

$$(\gamma \circ \varphi)_* = \gamma_* \circ \varphi_*$$

Draw the corresponding diagrams.

2.7.4 Velocity reinterpreted

The concept of velocity vector may be now reinterpreted in an interesting way. Recall a (parametrized) curve defines at its every point vector $\dot{c}(t)$ Using the new ideas of induced maps, we get:

$$\dot{c}(f) = \frac{d}{dt}\Big|_t f \circ c = \frac{\partial}{\partial t}\Big|_t f \circ c = \frac{\partial}{\partial t}\Big|_t c^*(f) = \left(c_* \left(\frac{\partial}{\partial t} \right) \right)(f)$$

Hence we have a new definition of

$$\dot{c} = c_* \left(\partial_t \right)$$

We can view it this way: "Time" is $\mathbb{R}$ viewed as a manifold. It possesses a natural vector field ∂_t (only because $\mathbb{R}$ has a natural variable: itself.) The velocity vectors along the image of c are just the images of this vector field via induced map c^*.

2.7.5 Three important cases of maps

We have seen that the induced maps differ in the direction for tangent and cotangent spaces:

$$\varphi_* : T_pM \longrightarrow T_qN \qquad \text{and} \qquad \varphi^* : T_q^*N \longrightarrow T_p^*M$$

where $q = \varphi(p)$. But these maps are well-defined at a point, and a question arises whether we can extend them to vector *fields* and *differential* forms on M or N, respectively. That is to say, are the maps

$$\varphi_* : \mathfrak{X}M \longrightarrow \mathfrak{X}N \qquad \text{and} \qquad \varphi^* : \Lambda N \longrightarrow \Lambda M$$

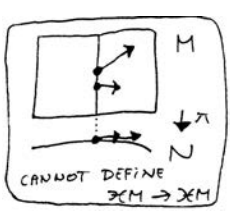
in general well-defined?

Special cases are discussed below.

1. Diffeomorphism. Diffeomorphism between two manifolds $\varphi : M \to N$ is a differential bijective map (that is, it is *one-to-one* and *onto*). Clearly, if φ is a diffeomorphism, there is no problem with extending point-wise defined induced maps to neighborhoods for both differential forms and vector fields. Moreover, the fact that φ^{-1} is well-defined differential map allows for inversion of both pull-back and the push-forward maps.

2. Projection. Let the dimensions of the manifolds be $\dim M < \dim N$ and a differential map

$$\pi : N \longrightarrow M$$

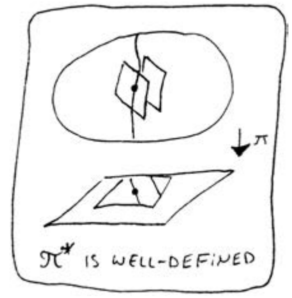


be *onto* (a surjection). For any point $m \in M$, its anti-image is a submanifold of N

$$\pi^{-1}(m) \subset N$$

which will be called a *fiber* over m.

Clearly, the induced map allows us to project any vector $\mathbf{v} \in T_n N$ down to a vector $\pi_*(\mathbf{v})$ tangent at point $\pi(n)$. But a vector *field* on some open set in M is in general 'un-projectable': vectors at different points from the same fiber do not need to have the same image under projection, and therefore projection of the whole vector field cannot be well defined.



As to the differential forms, there is no problem. Any differential k-form $\alpha \in \Lambda^k M$ may be 'lifted' to a k-form $\pi^*\alpha$ on N, which on any vector v takes value

$$\pi^*\alpha(v) = \alpha(\pi_*)$$

Problem: Explain why each of these maps (taken pointwise) cannot be uniquely inverted (give simple examples).

3. Embedding. Embedding is an injective differential map

$$\iota : N \longrightarrow M$$

We assume that $\dim N \leq \dim M$. For example, if N is a submanifold of M, $N \subset M$, then one usually expresses this fact as an 'embedding' where the image of a point $p \in N$ is understood as the same point, now considered as a point of M. Any differential form

$$\alpha \in \Lambda^k M$$

on the 'bigger' manifold may be restricted to take values only on vectors tangent to N. But this is definition of form $\iota^*\alpha \in \Lambda^k N$! Indeed:

$$(\iota^*\alpha)\ (v, w, \ldots) = \alpha(\iota_* v, \iota_* w, \ldots)$$

where $\iota_* v$, $\iota_* w$, etc, are the same vectors but reconsidered as tangent to M. Hence double notation: the image of the form under the induced map is the original form **restricted** to the submanifold N:

$$\varphi^*(\alpha) \equiv \alpha\Big|_N$$

Thus, the pointwise induced map ι^* has a natural extension to *differential* exterior field defined on an open neighborhood. Clearly, it cannot be inverted uniquely.

As to the vector fields on N, the image of φ_* is well-defined, but the image of $\mathbf{X} \in \mathcal{X}N$, i.e. $\iota_*\mathbf{X}$, is hardly a vector field in $\mathcal{X}M$, since it has support (N) of measure zero if $\dim N < \dim M$. We shall call such a field a "vector field along N."

If the bigger manifold M is equipped with an inner product defined by a tensor field g, any submanifold $N \subset M$ will acquire an inner product via the induced map.

Example: Recall the problem from a previous section ("**Premature problem**"): Suppose $M \subset \mathbf{E}^3$ is a 2-dimensional manifold in the standard Euclidean space, given by equation $z = f(x,y) \equiv x^2 + 3y$. W view it as an embedding

$$\iota : M \ \rightarrow \ \mathbb{R}^3$$

Let us "borrow" coordinates on M from $\mathbb{R}^3$, namely for any $p \in M$

$$u(p) = x(\imath(p)) \qquad \text{and} \qquad v(p) = y(\imath(p))$$

In such coordinates, the embedding is

$$\begin{aligned} x \circ \imath(p) \quad &= u(p) \\ y \circ \imath(p) \quad &= v(p) \\ z \circ \imath(p) \quad &= u^2(p) + 3v(p) \end{aligned}$$

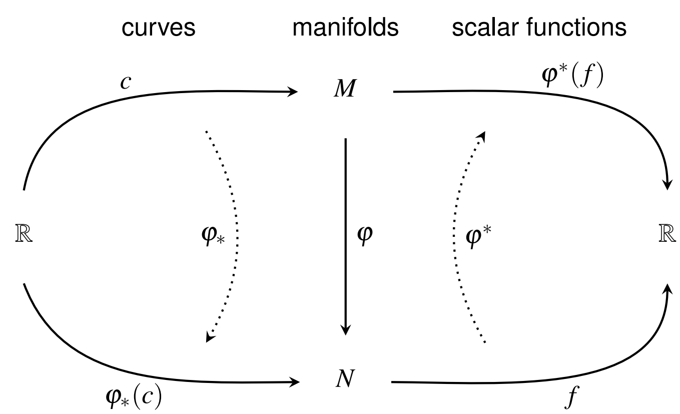
which is typically abbreviated to $x = u$, $y = v$ and $z = u^2 + 3v$. The induced inner product on M is thus:

$$\begin{aligned} g_M = \quad & \imath^*(dx \otimes dx + dy \otimes dy + dz \otimes dz) \\ = \quad & \imath^* dx \otimes \imath^* dx + \imath^* dy \otimes \imath^* dy + \imath^* dz \otimes \imath^* dz) \\ = \quad & du \otimes du + dv \otimes dv + d(u^2 + 3v) \otimes d(u^2 + 3dv) \\ = \quad & (1 + 4u^2)\, du \otimes du + 6u\,(du \otimes dv + dv \otimes du) + 10\, dv \otimes dv \end{aligned}$$

Fill details. As one can see, in practice this calculation reduces to replacing dz in the original $g = dx \otimes dx + dy \otimes dy + dz \otimes dz$ by the differential $df = 2x\, dx + 3dy$ and reinterpreting coordinates x and y as coordinates on M.

2.7.6 Coda

The concept of induced maps forms the conceptual basis for the well-known calculational tools of substitution and Jacobian. Here is the diagram of maps that started our story of "life on manifolds", extended by the induced maps:



The left side gives the rise to contravariant objects (vectors) while the right side to covariant objects. They behave differently under a differentiable map between manifolds: the first undergo "forward" maps while the later "pull-back" maps.

2.8 Differential forms, integration and Stokes' theorem

Till now we have viewed an exterior differential form through an evaluation on vectors. The next level of understanding a differential k-form —and perhaps its *raison d'être*— is the fact that it can be *integrated* along any k-dimensional "shape" in the manifold, like a curve, sphere or other surfaces and more-dimensional pieces. Such an integral does *not* need any additional structure on M — it is one of the most fundamental geometric concept.

In this new context, the exterior derivative d turns out to be dual to the operation of taking a boundary — this is the content of the Stokes' theorem:

$$\int_{\partial c} \omega = \int_{c} d\,\omega$$

This theorem of unique beauty could also be called a "generalized Newton-Lagrange-Ampere-Green-Ostrogradsky-Stokes-Voltera Theorem", for it encompasses the corresponding facts of geometry and mathematical physics, discovered and rediscovered at the end of the nineteenth century.

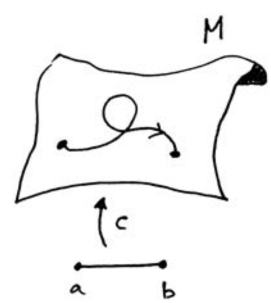
2.8.1 Integration of differential forms

In the following we shall construct in a few steps a definition of an "integral of a k-form over a k-chain."

Step 0: Integral along a curve. In order to acquire some intuition, let us see how one could define an integral of a differential 1-form along a curve. Consider manifold M and these two structures:

curve $c : I \to M$ where $I = [a,b] \subset \mathbb{R}$ is the "parameter of time";

1-form $\alpha \in \Lambda^1 M$ to be integrated;



We shall define the integral of α along curve c via a standard Riemann integral:

$$\int_c \alpha \;\stackrel{\text{def}}{=\!=}\; \int_I c^*\alpha \;=\; \int_a^b f(t)\,dt \tag{2.32}$$

What is going on: view curve c as a map between two manifolds (after all $\mathbb{R}$ *is* a manifold, too). The pull-back c^* brings forms from the manifold M into $\mathbb{R}$. In particular $c^*\alpha \in \Lambda^1\mathbb{R}$. But any 1-form on a one-dimensional manifold $\mathbb{R}$ must be proportional to dt, thus our 1-form must be $c^*\alpha = f(t)\,dt$ for some scalar function $f(t)$ (t is the natural coordinate on $\mathbb{R}$). Integration of this is defined by the usual Riemann integral.

Example: Let $I = [0,1]$ and $M \cong \mathbb{R}^2$ with the usual coordinates. Let $c : I \to M$ and $\alpha \in \Lambda^1 M$ be a curve and 1-form, respectively, given as:

$$\begin{cases} x \circ c\,(t) = t^2 \\ y \circ c\,(t) = t^7 \end{cases} \qquad \text{and 1-form} \quad \alpha = y\,dx + x^4\,dy$$

Now calculate

$$\int_c \alpha = \int_I c^* \alpha = \int_I t^7 \cdot 2t\,dt + t^8 \cdot 7t^6\,dt = \int_0^1 (2t^8 + 7t^{14}) dt = 31/45$$

Problem: Show that the value of (2.32) does not depend on particular parameterization of the curve, but only on its "shape", i.e. its image in M.

Remark: The integral of a differential 1-form may be viewed also as follows: the values of the 1-form α evaluated on the velocity vectors $\dot{c}$ define a scalar function along the curve $[a,b] \to M$. The integral of this function is equivalent to (2.32), i.e.:

$$\int_c \alpha = \int_a^b \langle \alpha \mid \dot{c}(t) \rangle\,dt \tag{2.33}$$

Step 1: Integration of a "volume" form. Let $\omega \in \Lambda^k \mathbb{R}^k$ be a form of maximal degree on the k-dimensional linear space $\mathbb{R}^k$. Let $(I^k,\ Ord)$ be a k-dimensional cube in this space,

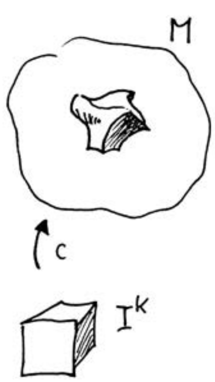
$$I^k = \{ (t_1, t_2, \dots, t_k) \in \mathbb{R}^k \mid a_i \le t_i \le b_i \}\ ,$$

where $a_i, b_i \in \mathbb{R}$ are some constants, and Ord denotes certain **orientation** of the cube defined as a choice of order of the coordinates t_i (here just $t_1, t_2, \dots, t_k$). Define the integral of ω over the cube as follows: since ω is a form of the highest degree, it may be expressed uniquely as

$$\omega = f(t_1, \dots, t_k)\,dt_1 \wedge dt_2 \wedge \dots\, dt_k$$

for some scalar function f, where the order of the differentials dt_i is made to correspond to the orientation Ord of the cube. Now, we define

$$\int_{I^k} \omega \ \stackrel{\text{def}}{=\!=}\ \int_{a_k}^{b_k} \cdots \int_{a_1}^{b_1} f(t_1, \dots, t_k)\ dt_1\,dt_2\,\dots\,dt_k \tag{2.34}$$



where the right side is understood as the usual multiple Riemann integral.

Step 2: Integral over a simple chain A **simple chain** of degree k (shortly a k-chain) is a (differentiable) map

$$c:\ I^k \longrightarrow M$$

from an oriented k-cube into an n-dimensional manifold. Let $\omega \in \Lambda^k M$ be a differential k-form on M. We define its integral over the chain c by

$$\int_c \omega \stackrel{\text{def}}{=\!=} \int_{I^k} c^* \omega \tag{2.35}$$

Clearly, the degree of ω can be smaller than the dimension of the manifold, yet the pull-back $c^*\omega \in \Lambda^k I^k$ is always a volume form (that is of the highest degree), and therefore the right side of the above equation is well-defined in Step 1).

Example: In the manifold being the standard 3D space, define a 2-form $\omega = z\,dx \wedge dy$ and the simple 2-chain be $c : (t,s) \mapsto (ts, t+s, e^t)$ with $0 \le t,s \le 1$. The integral is calculated as follows

$$\int_c \omega = \int_{I^2} c^* \omega = \int_{I^2} = s\,d(t+s) \wedge d(ts) = \int (s^2 - st)\,dt \wedge ds = \int_0^1 \int_0^1 (s^2 - st)dt\,ds$$

Step 3: Integral over a chain A *chain* of degree k (shortly k-chain) is a formal sum of a finite number of maps

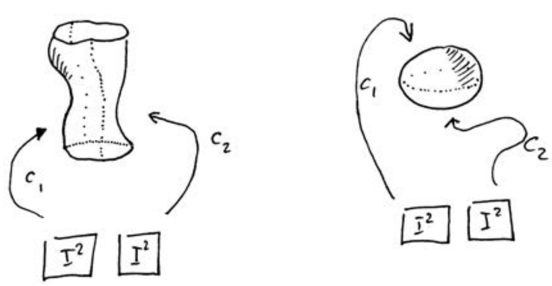
$$c = a_1\, c_1 + a_2\, c_2 + \ldots + a_m\, c_m \tag{2.36}$$

where c_i are simple k-chains and $a_i \in \mathbb{Z}$ are integer coefficients. You may think of (2.36) as a collection of maps. The meaning of this sum is determined by the following definition: An integral of a k-form $\omega \in \Lambda^k M$ along a k-chain c is

$$\int_c \omega = \sum_i a_i \int_{c_i} \omega \tag{2.37}$$

where each integral under the sum is defined in Step 2.

We need the concept of chains as formal sums (2.36) because some "shapes" do not need to be images of single cubes, but may require some "triangularization" (or — should we say — "cubization"). The figure below shows a cylinder and a sphere as 2-chains.



Exercise: How many squares do you need at least to represent projective space $P^2\mathbb{R}$ as a chain? (each square must be immersed in a 1-to-1 fashion.)

Special case: A 0-chain is a discrete set of points in M with some associated integer weights. A 0-form is a scalar function. The integral of a 0-form over a 0-chain is defined as the sum of the values of the function on the set of points, taken with the weights. That is, formally, if

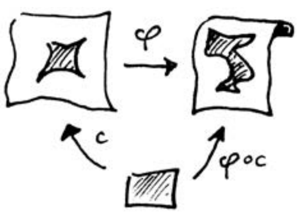
$$c = \sum_i a_i c_i$$

and $f \in \Lambda^0 M \cong \mathcal{F}M$, then

$$\int_c f \ \stackrel{\text{def}}{=\!=}\ f\big|_c \equiv \sum_i a_i \cdot f(c_i)$$

2.8.2 Properties of integrals

Any differential map $\varphi : M \to N$ induces a map of chains from manifold M to N. Namely, if $c : I^k \to M$ is a simple k-chain in M, then $c' = \varphi_*(c) \equiv \varphi \circ c$ is a simple k-chain in N.



$$\begin{tikzcd} M \arrow[r, "\varphi"] & N \\ I^k \arrow[u, "c"] \arrow[r, "\text{id}"] & I^k \arrow[u, "\varphi_*(c)"'] \end{tikzcd} \qquad \begin{array}{c} C_kM \xrightarrow{\ \varphi_*\ } C_kN \\[2em] M \xrightarrow{\ \varphi\ } N \end{array}$$

By formal linearity, this extends to arbitrary chains.

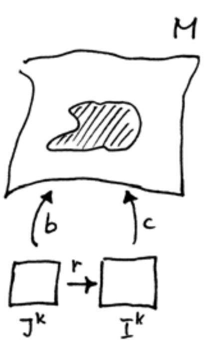
Theorem 2.8.1. *Let $\varphi : M \to N$ be a differential map. Let also $\omega \in \Lambda^k N$ be a differential k-form on N, and $c \in C_k M$ be k-chain in M. Then*

$$\int_{\varphi \circ c} \omega \ = \int_c \varphi^* \omega$$

PROOF: Starting from the left side

$$\int_{\varphi_*(c)} \omega \ = \int_{I^k} (\varphi \circ c)^* \omega = \int_{I^k} c^* \circ \varphi^* \ \omega = \int_c \varphi^* \omega$$

Thus the theorem is an immediate consequence of the property $(\varphi \circ c)^* = c^* \circ \varphi^*$. $\square$



Corollary 2.8.2. *The value of a form on a simple chain c does not depend on a particular parametrization, but only on the 'shape' of the image $\operatorname{Im} c \equiv c(I^k)$, and its orientation.*

PROOF: Consider a simple k-chain $c : I^k \to M$. Let map $r : J^k \to I^k$ be a one-to-one smooth map of re-parametrization of the cell. Thus we have two chains, c and $b = c \circ r$, which have the same image U in the manifold

$$U = c(I^k) = b(J^k) \ \subset M$$

Using Theorem 2.8.1 we get

$$\int_c \omega = \int_{I^k} c^*(\omega) = \int_{r(J^k)} c^*\omega = \int_{J^k} (r^* \circ c^*)\,(\omega) = \int_{J^k} (c \circ r)^*\,\omega = \int_{J^k} b^*\,\omega = \int_b \omega$$

□

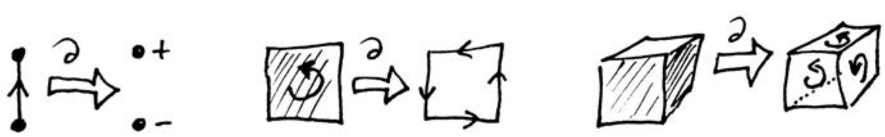
Remark: The above property is a well-known fact usually expressed as "integrals are invariant under a change of variables". Indeed, consider a volume k-form ω (see Step 1):

$$\omega = f(t_1,\ldots,t_k)\,dt_1 \wedge \ldots \wedge dt_k = f(s_1,\ldots,s_k)\,J\,ds_1 \wedge \ldots \wedge ds_k$$

where $f(s) = f(t(s))$, and $\mathbf{J}$ denotes the Jacobian of the coordinate transformation, i.e. the determinant of the matrix $J_{ij} = \partial t_i / \partial s_j$.

2.8.3 Boundary

The idea of boundary in small dimensions can be easily formulated. Here is an example of a segment, a square, and a cube. Note that the orientation is an important ingredient here. For each dimension it is marked differently. Taking a boundary is an operation ∂ that reduces dimension.



The orientation of the resulting object ∂A is determined by the orientation of the original object A. Orientation of walls is determined by the orientation of the whole k-cube: the parameters on a wall are oriented positively if the order — followed be the inward direction — re-establish the order of the cube.

We want to generalize this intuition to chains in higher dimension. All k-chains in a manifold M form a module over $\mathbb{Z}$ which is denoted by C_kM.

$$C_kM = \{\text{all k-chains in M}\} = \text{span}\,\{c : I^k \to M\}$$

We will define the *boundary* operator as a map

$$\partial\, : C_kM \longrightarrow C_{k-1}M\,.$$

Chain ∂c is called the boundary of c. We shall define it in two steps.

Step 1: First, we shall associate to each (oriented) k-dimensional cube I^k a collection of $(k-1)$-dimensional cubes forming its boundary. Let $(I^k,\, Ord)$ be a k-cube

$$I^k = \{(t_1, t_2, \ldots, t_k) \in \mathbb{R}^k \mid 0 \le t_i \le 1\,\}$$

for some constants $a_i, b_i \in \mathbb{R}$. The i^{th} **front** face and the i^{th} **back** face) are $(k-1)$-cubes defined

$$
\begin{aligned}
\text{(front)} \quad \overset{0}{F_i} &=: \{(t_1, t_2, \ldots, t_k) \in I^k \mid t_i = 0\,\} \\
\text{(back)} \quad \overset{1}{F_i} &=: \{(t_1, t_2, \ldots, t_k) \in I^k \mid t_i = 1\,\}\;,
\end{aligned}
$$

respectively. Essentially, the boundary of the cube I^k is the collection of the faces

$$\textit{boundary of } I^k = \bigcup_i \left(\overset{0}{F_i} \cup \overset{1}{F_i} \right)$$

But, keeping in mind the application in the integrals, we define each face not as a mere *set* but rather as a *map* from a $(k-1)$-dimensional "parameter cube" I^{k-1} with some $k-1$ variables s_i:

$$f_i : I^{k-1} \longrightarrow I^k$$

so that the images correspond to the faces, namely

$$
\begin{aligned}
\overset{0}{f_i} : \quad (s_1, s_2, \ldots, s_{k-1}) &\mapsto (s_1, s_2, \ldots, s_{i-1}, 0, s_i, \ldots, s_{k-1}) \\
\overset{1}{f_i} : \quad (s_1, s_2, \ldots, s_{k-1}) &\mapsto (s_1, s_2, \ldots, s_{i-1}, 1, s_i, \ldots, s_{k-1})
\end{aligned}
$$

We shall define the boundary of an k-cube, ∂I^k, as a $(k-1)$-chain in $\mathbb{R}^k$.

$$\partial I^k = \sum_i (-1)^i \left(\overset{0}{f_i} - \overset{1}{f_i} \right)$$

In the special case of $k = 0$ (a point), we define the boundary $\partial I^0 = 0$.

Remark: The boundary of a 1-cube (a segment), ∂I^1, is a formal difference two points: $\partial I = b - a$.

Exercise: Construct such a sum for the standard 2-dimensional, and 3-dimensional cube. Show in both cases that $\partial(\partial I^k) = 0$.

Step 2: Now we are ready to define a **boundary of a simple chain**. If $c \in C_k$ is a simple k-chain, $c : I^k \to M$, then define

$$\partial c = c \circ \partial$$

where the right side acts on the cube giving the "sum of walls." This means that for any simple k-chain c we have

$$\partial c = \sum_i \, (-)^i \left(c \circ \overset{0}{f_i} - c \circ \overset{1}{f_i} \right)$$

which is a collection of restrictions of map c to the walls of the original cube.

Step 3: For a **general chain** we assume linearity and define the boundary as:

$$\partial(ac+bc') = a\,\partial c + b\,\partial c' \qquad a,b\in\mathbb{Z} \quad c,c'\in C_kM$$

Theorem 2.8.3 (Law without a law). *The boundary of a boundary vanishes, that is, $\partial\circ\partial\, c = \varnothing$ for any chain c. We shall write it symbolically:*

$$\boxed{\partial\circ\partial \;=\; 0} \tag{2.38}$$

PROOF: Left as a exercise. □

Exercise: Inspect the examples at the start of this subsection (square and the cube) and see how the "Law without the law" works here.

The fact that the boundary of the boundary vanishes is of a profound topological importance. Archibald Wheeler sees in it one of the most fundamental "laws of physics" that come before any other, more specific, statements about reality. Hence his description "law without a law".

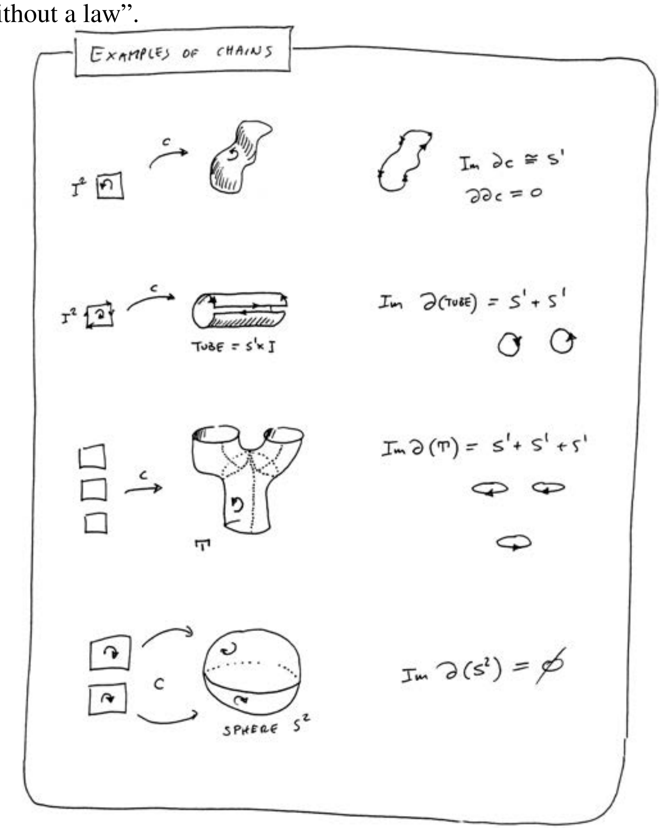
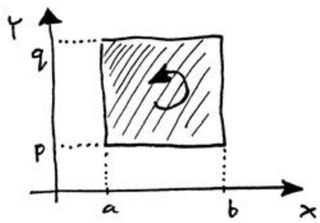


Illustration of the Law Without a Law (boundary of a boundary vanishes) for a few 2-dimensional chains.

2.8.4 Stokes' theorem

Theorem 2.8.4. *Let $\omega \in \Lambda^k M$ be a k-form on a manifold M, and $c \in C_{k+1}M$ be a $(k+1)$-chain in M. Then*

$$\int_{\partial c} \omega = \int_{c} d\,\omega$$



PROOF: Here we deal with the case of a 2-form and a 2-chain; the general case is similar — one just needs to write more indices. Let $I^2 = \{(x,y) \mid a \le x \le b,\ p \le y \le q\}$. Let $\omega = f(x,y)\,dx + g(x,y)\,dy$ be a 1-form on our $\mathbb{R}^2$. Its exterior derivative is $d\omega = (\partial_x g - \partial_y f)\,dx \wedge dy$. Let us calculate:

$$\begin{aligned}
\int_{I^2} d\omega &= \int_{I^2} (\partial_x g - \partial_y f)\,dx \wedge dy \\
&= \int_{I^2} \partial_x g\,dx \wedge dy - \int_{I^2} \partial_y f\,dx \wedge dy \\
&= \int_p^q \int_a^b \partial_x g\,dx\,dy - \int_p^q \int_a^b \partial_y f\,dx\,dy \\
&= \int_p^q g(b,y)\,dy - \int_p^q g(a,y)\,dy - \int_a^b f(x,q)\,dx + \int_a^b f(x,p)\,dx \\
&= \int_{\partial I^2} \omega
\end{aligned}$$

Since the integral of any 2-form over a simple 2-chain is equivalent to an integral of a volume-form over I^2 (via Step 2), this proves the theorem for any 2-chain. The general case for k-forms results just by breeding the indices. □

Remarks:

Calculus: Notice that an integral of a real *function* on $\mathbb{R}$ has always been an integral of a *form* on $\mathbb{R}$, now understood as a manifold, namely of the 1-form $f(x)\,dx$.

Newton: Notice that the fundamental theorem of calculus is a special case of Stokes' theorem:

$$\int_I df \quad = \quad \int_{\partial I} f \equiv f\,\big|_a^b \equiv f(b) - f(a)$$

where $I = [a,b]$.

Duality: Note that the property $d \circ d = 0$ for differential forms is dual to the topological property $\partial \circ \partial = 0$ and that Stokes' theorem provides a proof of one from the other:

$$\int_{\partial\partial c} \omega = \int_{\partial c} d\,\omega = \int_c dd\,\omega$$

for any ω and c, hence

$$\partial\partial = 0 \iff dd = 0$$

2.8.5 Summary

We associate to each pair of a k-form and a k-chain, $\alpha \in \Lambda^k M$ and $c \in C_k M$, a number $\langle \alpha \mid c \rangle$ defined by an integral

$$\langle\, \alpha \mid c \,\rangle =: \int_c \alpha$$

which satisfies the following properties:

$$\begin{aligned}
&(i) && \langle\, a\,\alpha + b\,\beta \mid c \,\rangle = a\,\langle\, \alpha \mid c \,\rangle + b\,\langle\, \beta \mid c \,\rangle \\
&(ii) && \langle\, \alpha \mid ac + bc' \,\rangle = a\,\langle\, \alpha \mid c \,\rangle + b\,\langle\, \alpha \mid c' \,\rangle \\
&(iii) && \langle\, \varphi^* \alpha \mid c \,\rangle = \langle\, \alpha \mid \varphi_* c \,\rangle \\
&(iv) && \langle\, d\,\alpha \mid c \,\rangle = \langle\, \alpha \mid \partial\, c \,\rangle
\end{aligned}$$

for any $c, c' \in CM$, $\alpha, \beta \in \Lambda M$, $a, b \in \mathbb{R}$, and $\varphi : M \to N$.

In the case of 3-dimensional manifold there are 4 types of integrals. The corresponding Mandala of these cases is given below. The vertical lines indicate the objects that can be paired in the integral. Identify the version of Stokes' theorem for each cell of the Mandala.

$$\begin{tikzcd}
\mathcal{F}M \arrow[r, "d"] \arrow[d, dash] & \Lambda^1 M \arrow[r, "d"] \arrow[d, dash] & \Lambda^2 M \arrow[r, "d"] \arrow[d, dash] & \Lambda^3 M \arrow[d, dash] \\
\mathcal{C}_0 M & \mathcal{C}_1 M \arrow[l, "\partial"'] & \mathcal{C}_2 M \arrow[l, "\partial"'] & \mathcal{C}_3 M \arrow[l, "\partial"']
\end{tikzcd}$$

$$\textstyle\int_{\partial c} \omega \quad = \quad \int_c d\,\omega \tag{2.39}$$

Invitation to Chapter 3: In order to see these concepts in action, the reader is encouraged to review the theory of Electromagnetism in the language of differential forms.

Exercise: One may think of a simple chain as a generalization of a curve to a "multi-dimensional time". This implies that at each point one has k independent velocities, each associated with one parameter of time. (Or better — velocity is a k-vector in the tangent space). Generalize the interpretation of the last remark in Step 0 to this case.