Proof of $\mathbf{v}_p(1) = 0$.

Chase A. Lotito

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1 Definiton of a (tangent) vector

Def 1. *A (tangent) vector at a point $p \in M$, where M is a smooth manifold, is an operator:*

$$\mathbf{v}_p : \mathcal{F}M \to \mathbb{R}$$

Our vector $\mathbf{v}_p$, the operator, takes a smooth function (which itself maps our manifold to a scalar quantity) and provides a real number, given these conditions are satisfied:

i. $\mathbf{v}_p(f+g) = \mathbf{v}_p(f) + \mathbf{v}_p(g)$

ii. $\mathbf{v}_p(cf) = c(\mathbf{v}_p(f)), \quad c \in \mathbb{R}$

iii. $\mathbf{v}_p(fg) = \mathbf{v}_p(f)g(p) + f(p)\mathbf{v}_p(g)$

2 Proof of why $\mathbf{v}_p(1) = 0$

From the definition of the vector, we see that it acts like a derivative. Both derivatives and vectors operate on functions and provide a scalar quantity in return. From derivatives, we know that:

$$\frac{d}{dx}(1) = 0$$

Can we prove then, that the (tangent) vector operating on unity is also zero?

Proof. Given a (tangent) vector $\mathbf{v}_p$, let's apply the *Leibniz rule* as it operates on $1 \in \mathcal{F}M$:

$$\mathbf{v}_p(1) = \mathbf{v}_p(1 \cdot 1) \overset{\text{(iii)}}{=} \mathbf{v}_p(1) \cdot 1 + 1 \cdot \mathbf{v}_p(1) = \mathbf{v}_p(1) + \mathbf{v}_p(1) \tag{1}$$

Rearranging (1):

$$\begin{aligned} \mathbf{v}_p(1) - \mathbf{v}_p(1) &= \mathbf{v}_p(1) \\ \rightsquigarrow \mathbf{v}_p(1) &= 0 \end{aligned}$$

$\square$

Given Condition (ii), $\mathbf{v}_p(1) = 0$ generalizes $\forall c \in \mathbb{R}$:

$$\mathbf{v}_p(c) = \mathbf{v}_p(c \cdot 1) = c(\mathbf{v}_p(1)) = c \cdot 0 = 0$$