

Exercise 1: Consider a 2-dimensional manifold M with coordinate chart $\{x, y\}$. The following objects are given:

- $\mathbf{v} = 2\partial_x + \partial_y$; at point $p = (3, 1)$; $\mathbf{v} \in T_p M$
- $f = x^2 + xy + 2$; scalar function; $f \in \mathcal{F}M$
- $\mathbf{A} = 2x^2\partial_x + xy\partial_y$; vector field; $\mathbf{A} \in \mathcal{X}M$
- $\mathbf{B} = y\partial_x$; vector field; $\mathbf{B} \in \mathcal{X}M$
- $c(t) = (t^2 + t, 2\cos t)$; a curve; $\mathbf{c} \in \mathcal{C}M$

Calculate the following:

- $\mathbf{v}f$
- $\mathbf{A}f$
- $f \circ c$
- $\frac{d}{dt}(f \circ c)$ at $t = 0$
- $\dot{\mathbf{c}} \equiv \dot{c}(0)$
- $\dot{\mathbf{c}}f$
- $[\mathbf{A}, \mathbf{B}]$
- Draw the vector field \mathbf{B} in the neighborhood of $(0, 0)$

Solution.

(a)

$$\begin{aligned}\mathbf{v}f &= (2\partial_x + \partial_y)(x^2 + xy + 2) \\ &= 2\partial_x(x^2 + xy + 2) + \partial_y(x^2 + xy + 2) \\ &= (2(2x + y) + x) \Big|_{(x,y)=(3,1)} \\ &= 2(2(3) + 1) + 3 \\ &= 17\end{aligned}$$

(b)

$$\begin{aligned}\mathbf{A}f &= (2x^2\partial_x + xy\partial_y)(x^2 + xy + 2) \\ &= 2x^2\partial_x(x^2 + xy + 2) + xy\partial_y(x^2 + xy + 2) \\ &= 2x^2(2x + y) + xy(x) \\ &= 4x^3 + 3x^2y\end{aligned}$$

(c)

$$\begin{aligned}f \circ c &= f(c(t)) \\ &= (t^2 + t)^2 + (t^2 + t)(2\cos t) + 2\end{aligned}$$

(d)

$$\begin{aligned}\frac{d}{dt} \Big|_{t=0} (f \circ c) &= \frac{d}{dt} \Big|_{t=0} [(t^2 + t)^2 + (t^2 + t)(2\cos t) + 2] \\ &= 2(t^2 + t)(2t + 1) + 2(2t + 1)\cos t - \sin t(t^2 + t) \Big|_{t=0} \\ &= 2(0)(1) + 2(1)(1) - 0(0) \\ &= 2\end{aligned}$$

(e) Remember $\dot{\mathbf{c}} = \dot{c}^i \partial_i = \dot{c}^i(x^i) \partial_i = \frac{d(x^i \circ c)(t)}{dt} \partial_i$. Operating on a function f ,

$$\dot{\mathbf{c}}(f) = \left. \frac{d}{dt} \right|_{t_0} f \circ c(t)$$

Here, $t_0 = 0$,

$$\begin{aligned} \dot{\mathbf{c}} &= \frac{d(t^2 + t)}{dt} \partial_x + \frac{d(2 \cos t)}{dt} \partial_y \\ &= (2t + 1) \partial_x - (2 \sin t) \partial_y \Big|_{t_0=0} \\ &= \partial_x \end{aligned}$$

(f)

From (c), we know that

$$(f \circ c)(t) = (t^2 + t)^2 + (t^2 + t)(2 \cos t) + 2$$

So,

$$\begin{aligned} \dot{\mathbf{c}}f &= \left. \frac{d}{dt} \right|_{t=t_0} (t^2 + t)^2 + (t^2 + t)(2 \cos t) + 2 \\ &= 2(t^2 + t)(2t + 1) + (2t + 1)(2 \cos t) - (t^2 + t)(2 \sin t) \Big|_{t=t_0} \\ &= 2(t_0^2 + t_0)(2t_0 + 1) + (2t_0 + 1)(2 \cos t_0) - (t_0^2 + t_0)(2 \sin t_0) \end{aligned}$$

(g) Remember the Lie bracket of two vector fields \mathbf{A} and \mathbf{B} is

$$[\mathbf{A}, \mathbf{B}] = (A^j(\partial_j B^i) - B^j(\partial_j A^i)) \partial_i$$

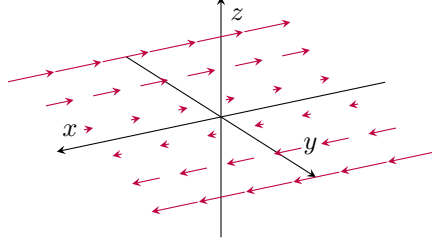
Plugging in our vector fields, $\mathbf{A} = 2x^2 \partial_x + xy \partial_y$, $\mathbf{B} = y \partial_x$,

$$\begin{aligned} [\mathbf{A}, \mathbf{B}] &= [A^j(\partial_j y) - B^j(\partial_j 2x^2)] \partial_x + [A^j(\partial_j 0) - B^j(\partial_j xy)] \partial_y \\ &= [A^j(\partial_j y)] \partial_x - [B^j(\partial_j 2x^2)] \partial_x - [B^j(\partial_j xy)] \partial_y \\ &= [2x^2(\partial_x y) + xy(\partial_y y)] \partial_x - [y(\partial_x 2x^2)] \partial_x - [y(\partial_x xy)] \partial_y \\ &= (0 + xy) \partial_x - 4xy \partial_x - y^2 \partial_y \\ &= -3xy \partial_x - y^2 \partial_y \end{aligned}$$

(h)

In the neighborhood of $(0, 0)$,

$$\mathbf{B} = y\partial_x$$



Exercise 2: Consider $M \cong \mathbb{R}^3$ with the chart of rectangular coordinates (x, y, z) . Express each vector of the basis associated with the chart of spherical coordinates (r, φ, θ) , namely $\{\partial_r, \partial_\varphi, \partial_\theta\}$ in terms of the standard basis $\{\partial_x, \partial_y, \partial_z\}$.

Solution.

The link between Cartesian coordinates and Spherical coordinates is:

$$x = r \sin \varphi \cos \theta$$

$$y = r \sin \varphi \sin \theta$$

$$z = r \cos \varphi$$

A position vector on this manifold can be described in the Cartesian basis as:

$$x^i \partial_i = x \partial_x + y \partial_y + z \partial_z$$

Substituting x^i with the Spherical coordinate equivalents:

$$x^i \partial_i = r \sin \varphi \cos \theta \partial_x + r \sin \varphi \sin \theta \partial_y + r \cos \varphi \partial_z$$

From this we can find equations for the Spherical basis vectors via this change of coordinates:

$$\frac{\partial}{\partial y^k} = \frac{\partial x^i}{\partial y^k} \frac{\partial}{\partial x^i}$$

Where here, $\frac{\partial}{\partial y^1} = \frac{\partial}{\partial r} = \partial_r$, and so on:

For ∂_r :

$$\begin{aligned} \frac{\partial}{\partial y^1} &= \frac{\partial}{\partial r} = \frac{\partial x^i}{\partial r} \partial_i = \frac{\partial(r \sin \varphi \cos \theta)}{\partial r} \partial_x + \frac{\partial(r \sin \varphi \sin \theta)}{\partial r} \partial_y + \frac{\partial(r \cos \varphi)}{\partial r} \partial_z \\ &= \sin \varphi \cos \theta \partial_x + \sin \varphi \sin \theta \partial_y + \cos \varphi \partial_z \end{aligned}$$

For ∂_φ :

$$\begin{aligned} \frac{\partial}{\partial y^2} &= \frac{\partial}{\partial \varphi} = \frac{\partial x^i}{\partial \varphi} \partial_i = \frac{\partial(r \sin \varphi \cos \theta)}{\partial \varphi} \partial_x + \frac{\partial(r \sin \varphi \sin \theta)}{\partial \varphi} \partial_y + \frac{\partial(r \cos \varphi)}{\partial \varphi} \partial_z \\ &= r \cos \varphi \cos \theta \partial_x + r \cos \varphi \sin \theta \partial_y - r \sin \varphi \partial_z \\ &= r(\cos \varphi \cos \theta \partial_x + \cos \varphi \sin \theta \partial_y - \sin \varphi \partial_z) \end{aligned}$$

For ∂_θ :

$$\begin{aligned}\frac{\partial}{\partial y^3} &= \frac{\partial}{\partial \theta} = \frac{\partial x^i}{\partial \theta} = \frac{\partial(r \sin \varphi \cos \theta)}{\partial \theta} \partial_x + \frac{\partial(r \sin \varphi \sin \theta)}{\partial \theta} \partial_y + \frac{\partial(r \cos \varphi)}{\partial \theta} \partial_z \\ &= -r \sin \varphi \sin \theta \partial_x + r \sin \varphi \cos \theta \partial_y \\ &= r \sin \varphi (-\sin \theta \partial_x + \cos \theta \partial_y)\end{aligned}$$

Removing the greatest common divisor from each, making them unit vectors, we are able to express the Spherical basis vectors as follows:

$$\begin{aligned}\partial_r &= \sin \varphi \cos \theta \partial_x + \sin \varphi \sin \theta \partial_y + \cos \varphi \partial_z \\ \partial_\varphi &= \cos \varphi \cos \theta \partial_x + \cos \varphi \sin \theta \partial_y - \sin \varphi \partial_z \\ \partial_\theta &= -\sin \theta \partial_x + \cos \theta \partial_y\end{aligned}$$

Exercise 3: Show that the Lie bracket of vector fields satisfies the Jacobi identity:

$$[\mathbf{A}, [\mathbf{B}, \mathbf{C}]] + [\mathbf{B}, [\mathbf{C}, \mathbf{A}]] + [\mathbf{C}, [\mathbf{A}, \mathbf{B}]] = 0$$

Solution.

Remembering that $\mathbf{A} = A^i \partial_i$, $\mathbf{B} = B^i \partial_i$ & $\mathbf{C} = C^i \partial_i$, let's expand the LHS of the Jacobi identity:

$$\begin{aligned}&= (\mathbf{A}(\mathbf{B}\mathbf{C} - \mathbf{C}\mathbf{B}) - (\mathbf{B}\mathbf{C} - \mathbf{C}\mathbf{B})\mathbf{A}) + (\mathbf{B}(\mathbf{C}\mathbf{A} - \mathbf{A}\mathbf{C}) - (\mathbf{C}\mathbf{A} - \mathbf{A}\mathbf{C})\mathbf{B}) \\ &\quad + (\mathbf{C}(\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A}) - (\mathbf{A}\mathbf{B} - \mathbf{B}\mathbf{A})\mathbf{C}) \\ &= \mathbf{A}\mathbf{B}\mathbf{C} - \mathbf{A}\mathbf{C}\mathbf{B} - \mathbf{B}\mathbf{C}\mathbf{A} + \mathbf{C}\mathbf{B}\mathbf{A} + \mathbf{B}\mathbf{C}\mathbf{A} - \mathbf{B}\mathbf{A}\mathbf{C} \\ &\quad - \mathbf{C}\mathbf{A}\mathbf{B} + \mathbf{A}\mathbf{C}\mathbf{B} + \mathbf{C}\mathbf{A}\mathbf{B} - \mathbf{C}\mathbf{B}\mathbf{A} - \mathbf{A}\mathbf{B}\mathbf{C} + \mathbf{B}\mathbf{A}\mathbf{C} \\ &= 0\end{aligned}$$

Beautifully, every term cancels.

Exercise 4: Let $\mathbf{C} = [\mathbf{A}, \mathbf{B}]$ be the Lie bracket of two vector fields. In a chart, the vector fields are given as $\mathbf{A} = A^i \partial_i$, $\mathbf{B} = B^i \partial_i$ and $\mathbf{C} = C^i \partial_i$. Express the coefficients C^i in terms of the coefficients of the other two vector fields.

Solution.

Calculating \mathbf{C} , the Lie bracket of \mathbf{A} and \mathbf{B} :

$$\begin{aligned}\mathbf{C} &= [\mathbf{A}, \mathbf{B}] \\ &= A^j \partial_j B^i \partial_i - B^j \partial_j A^i \partial_i \\ &= (A^j \partial_j B^i - B^j \partial_j A^i) \partial_i\end{aligned}$$

From this, $C^i = A^j \partial_j B^i - B^j \partial_j A^i$, which can also be expressed:

$$C^i = A^j \frac{\partial B^i}{\partial x^j} - B^j \frac{\partial A^i}{\partial x^j}$$