

**Exercise 1:** Consider a 2-dimensional manifold  $M$  with coordinate chart  $\{x, y\}$ . The following objects are given:

- $\mathbf{v} = 2\partial_x + \partial_y$ ; at point  $p = (3, 1)$ ;  $\mathbf{v} \in T_p M$
- $f = x^2 + xy + 2$ ; scalar function;  $f \in \mathcal{F}M$
- $\mathbf{A} = 2x^2\partial_x + xy\partial_y$ ; vector field;  $\mathbf{A} \in \mathcal{X}M$
- $\mathbf{B} = y\partial_x$ ; vector field;  $\mathbf{B} \in \mathcal{X}M$
- $c(t) = (t^2 + t, 2\cos t)$ ; a curve;  $\mathbf{c} \in \mathcal{C}M$

Calculate the following:

- $\mathbf{v}f$
- $\mathbf{A}f$
- $f \circ c$
- $\frac{d}{dt}(f \circ c)$  at  $t = 0$
- $\dot{\mathbf{c}} \equiv \dot{c}(0)$
- $\dot{\mathbf{c}}f$
- $[\mathbf{A}, \mathbf{B}]$
- Draw the vector field  $\mathbf{B}$  in the neighborhood of  $(0, 0)$

**Solution.**

(a)

$$\begin{aligned}
 \mathbf{v}f &= (2\partial_x + \partial_y)(x^2 + xy + 2) \\
 &= 2\partial_x(x^2 + xy + 2) + \partial_y(x^2 + xy + 2) \\
 &= (2(2x + y) + x) \Big|_{(x,y)=(3,1)} \\
 &= 2(2(3) + 1) + 3 \\
 &= 17
 \end{aligned}$$

(b)

$$\begin{aligned}
 \mathbf{A}f &= (2x^2\partial_x + xy\partial_y)(x^2 + xy + 2) \\
 &= 2x^2\partial_x(x^2 + xy + 2) + xy\partial_y(x^2 + xy + 2) \\
 &= 2x^2(2x + y) + xy(x) \\
 &= 4x^3 + 3x^2y
 \end{aligned}$$

(c)

$$\begin{aligned}
 f \circ c &= f(c(t)) \\
 &= (t^2 + t)^2 + (t^2 + t)(2\cos t) + 2
 \end{aligned}$$

(d)

$$\begin{aligned}
 \frac{d}{dt} \Big|_{t=0} (f \circ c) &= \frac{d}{dt} \Big|_{t=0} [(t^2 + t)^2 + (t^2 + t)(2\cos t) + 2] \\
 &= 2(t^2 + t)(2t + 1) + 2(2t + 1)\cos t - \sin t(t^2 + t) \Big|_{t=0} \\
 &= 2(0)(1) + 2(1)(1) - 0(0) \\
 &= 2
 \end{aligned}$$

(e) Remember  $\dot{\mathbf{c}} = \dot{c}^i \partial_i = \dot{c}^i(x^i) \partial_i = \frac{d(x^i \circ c)(t)}{dt} \partial_i$ . Operating on a function  $f$ ,

$$\dot{\mathbf{c}}(f) = \left. \frac{d}{dt} \right|_{t_0} f \circ c(t)$$

Here,  $t_0 = 0$ ,

$$\begin{aligned} \dot{\mathbf{c}} &= \frac{d(t^2 + t)}{dt} \partial_x + \frac{d(2 \cos t)}{dt} \partial_y \\ &= (2t + 1) \partial_x - (2 \sin t) \partial_y \Big|_{t_0=0} \\ &= \partial_x \end{aligned}$$

(f)

From (c), we know that

$$(f \circ c)(t) = (t^2 + t)^2 + (t^2 + t)(2 \cos t) + 2$$

So,

$$\begin{aligned} \dot{\mathbf{c}}f &= \left. \frac{d}{dt} \right|_{t=t_0} (t^2 + t)^2 + (t^2 + t)(2 \cos t) + 2 \\ &= 2(t^2 + t)(2t + 1) + (2t + 1)(2 \cos t) - (t^2 + t)(2 \sin t) \Big|_{t=t_0} \\ &= 2(t_0^2 + t_0)(2t_0 + 1) + (2t_0 + 1)(2 \cos t_0) - (t_0^2 + t_0)(2 \sin t_0) \end{aligned}$$

(g) Remember the Lie bracket of two vector fields  $\mathbf{A}$  and  $\mathbf{B}$  is

$$[\mathbf{A}, \mathbf{B}] = (A^j(\partial_j B^i) - B^j(\partial_j A^i)) \partial_i$$

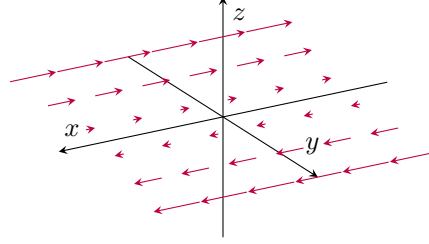
Plugging in our vector fields,

$$\begin{aligned} [\mathbf{A}, \mathbf{B}] &= [A^j(\partial_j y) - B^j(\partial_j 2x^2)] \partial_x + [A^j(\partial_j 0) - B^j(\partial_j xy)] \partial_y \\ &= [A^j(\partial_j y)] \partial_x - [B^j(\partial_j 2x^2)] \partial_x - [B^j(\partial_j xy)] \partial_y \\ &= [2x^2(\partial_x y) + xy(\partial_y y)] \partial_x - [y(\partial_x 2x^2)] \partial_x - [y(\partial_x xy)] \partial_y \\ &= (0 + xy) \partial_x - 4xy \partial_x - y^2 \partial_y \\ &= -3xy \partial_x - y^2 \partial_y \end{aligned}$$

(h)

In the neighborhood of  $(0, 0)$ ,

$$\mathbf{B} = y\partial_x$$



**Exercise 2:** Consider  $M = \mathbb{R}^3$  with the chart of rectangular coordinates  $(x, y, z)$ . Express each vector of the basis associated with the chart of spherical coordinates  $(r, \varphi, \theta)$ , namely  $\{\partial_r, \partial_\varphi, \partial_\theta\}$  in terms of the standard basis  $\{\partial_x, \partial_y, \partial_z\}$ .

**Solution.**

The link between Cartesian coordinates and Spherical coordinates is:

$$x = r \sin \varphi \cos \theta$$

$$y = r \sin \varphi \sin \theta$$

$$z = r \cos \varphi$$

In  $\mathbb{R}^3$ , any position on this manifold can be described in the Cartesian basis as:

$$x^i \partial_i = x \partial_x + y \partial_y + z \partial_z$$

Substituting  $x^i$  with the Spherical coordinate equivalents:

$$x^i \partial_i = r \sin \varphi \cos \theta \partial_x + r \sin \varphi \sin \theta \partial_y + r \cos \varphi \partial_z$$

From this we can find equations for the Spherical basis vectors via this change of coordinates:

$$\frac{\partial}{\partial y^k} = \frac{\partial x^i}{\partial y^k} \frac{\partial}{\partial x^i}$$

Where here,  $\frac{\partial}{\partial y^1} = \frac{\partial}{\partial r} = \partial_r$ , and so on:

For  $\partial_r$ :

$$\begin{aligned} \frac{\partial}{\partial y^1} &= \frac{\partial}{\partial r} = \frac{\partial x^i}{\partial r} \partial_i = \frac{\partial(r \sin \varphi \cos \theta)}{\partial r} \partial_x + \frac{\partial(r \sin \varphi \sin \theta)}{\partial r} \partial_y + \frac{\partial(r \cos \varphi)}{\partial r} \partial_z \\ &= \sin \varphi \cos \theta \partial_x + \sin \varphi \sin \theta \partial_y + \cos \varphi \partial_z \end{aligned}$$

For  $\partial_\varphi$ :

$$\begin{aligned} \frac{\partial}{\partial y^2} &= \frac{\partial}{\partial \varphi} = \frac{\partial x^i}{\partial \varphi} \partial_i = \frac{\partial(r \sin \varphi \cos \theta)}{\partial \varphi} \partial_x + \frac{\partial(r \sin \varphi \sin \theta)}{\partial \varphi} \partial_y + \frac{\partial(r \cos \varphi)}{\partial \varphi} \partial_z \\ &= r \cos \varphi \cos \theta \partial_x + r \cos \varphi \sin \theta \partial_y - r \sin \varphi \partial_z \\ &= r(\cos \varphi \cos \theta \partial_x + \cos \varphi \sin \theta \partial_y - \sin \varphi \partial_z) \end{aligned}$$

For  $\partial_\theta$ :

$$\begin{aligned}\frac{\partial}{\partial y^3} &= \frac{\partial}{\partial \theta} = \frac{\partial x^i}{\partial \theta} = \frac{\partial(r \sin \varphi \cos \theta)}{\partial \theta} \partial_x + \frac{\partial(r \sin \varphi \sin \theta)}{\partial \theta} \partial_y + \frac{\partial(r \cos \varphi)}{\partial \theta} \partial_z \\ &= -r \sin \varphi \sin \theta \partial_x + r \sin \varphi \cos \theta \partial_y \\ &= r \sin \varphi (-\sin \theta \partial_x + \cos \theta \partial_y)\end{aligned}$$

Removing the greatest common divisor from each, making them unit vectors, we are able to express the Spherical basis vectors as follows:

$$\begin{aligned}\partial_r &= \sin \varphi \cos \theta \partial_x + \sin \varphi \sin \theta \partial_y + \cos \varphi \partial_z \\ \partial_\varphi &= \cos \varphi \cos \theta \partial_x + \cos \varphi \sin \theta \partial_y - \sin \varphi \partial_z \\ \partial_\theta &= -\sin \theta \partial_x + \cos \theta \partial_y\end{aligned}$$

**Exercise 3:** Show that the Lie bracket of vector fields satisfies the Jacobi identity:

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$$

**Solution.**

Remembering that  $A = A^i \partial_i$ ,  $B = B^i \partial_i$  &  $C = C^i \partial_i$ , let's expand the LHS of the Jacobi identity:

$$\begin{aligned}&= (A(BC - CB) - (BC - CB)A) + (B(CA - AC) - (CA - AC)B) \\ &\quad + (C(AB - BA) - (AB - BA)C) \\ &= ABC - ACB - BCA + CBA + BCA - BAC \\ &\quad - CAB + ACB + CAB - CBA - ABC + BAC \\ &= 0\end{aligned}$$

Beautifully, every term cancels.

**Exercise 4:** Let  $C = [A, B]$  be the Lie bracket of two vector fields. In a chart, the vector fields are given as  $A = A^i \partial_i$ ,  $B = B^i \partial_i$  and  $C = C^i \partial_i$ . Express the coefficients  $C^i$  in terms of the coefficients of the other two vector fields.

**Solution.**

Calculating  $C$ , the Lie bracket of  $A$  and  $B$ :

$$\begin{aligned}C &= [A, B] \\ &= A^j \partial_j B^i \partial_i - B^j \partial_j A^i \partial_i \\ &= (A^j \partial_j B^i - B^j \partial_j A^i) \partial_i\end{aligned}$$

From this,  $C^i = A^j \partial_j B^i - B^j \partial_j A^i$ , which can also be expressed:

$$C^i = A^j \frac{\partial B^i}{\partial x^j} - B^j \frac{\partial A^i}{\partial x^j}$$