Chapter 1

Numbers, vectors, tensors and all that

1.1 Sets

A **Cartesian product** of two sets A and B is the set of all ordered pairs of their elements, namely:

$$M = A \times B = \{\ (a,b) \mid a \in A \text{ and } b \in B\ \}$$

There are two natural maps called **projections**, defined:

$$\pi_A : A \times B \longrightarrow A \qquad \pi_B : A \times B \longrightarrow B$$

defined by

$$\pi_A(a,b) = a \qquad \pi_B(a,b) = b$$

Note that the Cartesian product may be equivalently defined through the pair of maps $\{\pi_A,\ \pi_B\}$. **Exercise:** What properties would have these maps to satisfy to make the definition equivalent to the above?

Relation in a set S is defined as a subset $R \subset S \times S$. We say that a and b are in relation R if $(a,b) \in R$. Most often we write aRb.

An **equivalence** relation in a set S, denoted by $\sim$, is a relation satisfying the following axioms: for any $a,b,c \in S$

$$1.\quad a \sim a \tag{1.1}$$

$$2.\quad \text{If} \quad a \sim b \quad \text{then} \quad b \sim a \tag{1.2}$$

$$3.\quad \text{If} \quad a \sim b \quad \text{and} \quad b \sim c \quad \text{then} \quad \sim c \tag{1.3}$$

The set of all elements in S that are in relation with a is denoted

$$[a] = \{x \in S \mid x \sim a\}$$

and called the **equivalence class** of the element a. If $b \in [a]$ then $[a] = [b]$. Two classes are either identical or disjoint. The set of equivalence classes is denoted as a quotient $S/\!\sim$. We have a natural map π called **projection**

$$\pi : S \ \longrightarrow\ S/\!\sim$$

which associates to any element a of S its class $[a]$.

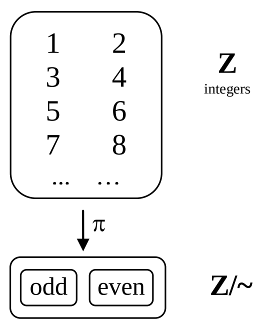
Examples:

1. The prototype of the notion of equivalence relation is equality "$=$" in a set (e.g., of numbers). In such a case every equivalence class contains only one element, $[a] = \{a\}$. On the other extreme end, if relation "$\sim$" is defined to hold for every pair of a set A, then the quotient $A/\sim$ consists of only one element.

2. $\{\mathbb{Z}, \sim\}$, where $\mathbb{Z}$ denotes the set of integers, and the relation $\sim$ is defined by

$$a \sim b \quad \text{if} \quad 2 \,\big|\, (a-b) \qquad \big(\text{2 divides the difference } (a-b)\big)$$

Thus, two numbers are in relation (belong to the same class), if they are both either even or odd.



The two-element set $\mathbb{Z}/\sim$ is denoted by $\mathbb{Z}_2$.

Problem: For any fix ed number $n \in \mathbb{Z}$, define relation of equivalence in $\mathbb{Z}$ by $a \sim b$ if $n \mid (a - b)$. Show that "$\sim$" is a relation of equivalence. Denote $\mathbb{Z}_n = \mathbb{Z}/\sim$. Define addition and multiplication of classes by

$$[a] + [b] = [a+b] \qquad [a] \cdot [b] = [a \cdot b]$$

Show that these operations are well defined in $\mathbb{Z}_n$, i.e. that the result does not depend on the particular choice of the representatives of the equivalence classes.

1.2 Numbers

Modeling physical reality starts numbers, the notion that starts with the art of counting. Natural numbers, denoted $\mathbb{N} = \{1.2.3....\}$ is the first step. But one soon arrives in the concept of a set $\mathbb{F}$ with two operations denoted "+" and "·" such that their 'inverses' denoted by "−" and ":", respectively, exist. The concept of *field* is an axiomatic formalization of such a concept.

Definition A *field* is a triple $\{\mathbb{F}, +, \cdot\}$ where $\mathbb{F}$ is a set, and "+" and "·" are binary operations satisfying the following conditions:

1.	$(a+b)+c = a+(b+c)$	$1'$.	$(a\cdot b)\cdot c = a\cdot(b\cdot c)$
2.	$a+b = b+a$	$2'$.	$a\cdot b = b\cdot a$
3.	$\exists 0 \in \mathbb{F}\ \forall\, a \in \mathbb{F}:\ \ a+0 = a$	$3'$.	$\exists 1 \in \mathbb{F}\ \forall\, a \in \mathbb{F}:\ \ 1\cdot a = a$
4.	$\forall a \in \mathbb{F}\ \ \exists\, a' \in \mathbb{F}:\ \ a+a' = 0$	$4'$.	$\forall a \in \mathbb{F}\ \ \exists\, a" \in \mathbb{F}:\ \ a\cdot a" = 1$
5.	$a\cdot(b+c) = a\cdot b + a\cdot c$		

The basic fields we shall deal with are the *rational* numbers $\mathbb{Q}$, *real* numbers $\mathbb{R}$, and the *complex* numbers $\mathbb{C}$. Real numbers are essential for classical physics. The structure of complex numbers is essential for quantum mechanics. If we disregard commutativity (Axiom 2'), we would include also **quaternions H** (see next section), which describe rotations of a three-dimensional space (and help to understand the curious property of the "double degeneration" of rotations). Because of the non-commutativity of quaternions they are called a "division algebra" rather than a field.

These number systems may be represented as a chain of generalization:

$$\begin{array}{ccccc} \mathbb{R} & \longrightarrow & \mathbb{C} & \longrightarrow & \mathbb{H} \\ \downarrow & & \downarrow & & \downarrow \\ \text{classical} & & \text{quantum} & & \text{inner} \\ \text{physics} & & \text{mechanics} & & \text{space} \end{array}$$

The dimensions of these fields are 1, 2, and 4, respectively. As to the further generalization, one may define so-called **octonions**, an eight-dimensional number system. The problem is that in the case of $\mathbb{H}$ one has to give up the property of commutativity of multiplication (2'), and the price of the generalization to octonions is associativity (1').

The above four cases exhaust the possibilities of real spaces that admit multiplication with an inverse (4') (see Frobenius Theorem).

Other examples of fields concern finite or not continuous sets. Rational numbers, $\mathbb{Q}$, form a field. The set $\mathbb{Z}_2 = \{0, 1\}$ with addition defined modulo 2, where $1+1 = 0$, is the smallest nontrivial field. Other examples are given as problems.

Note that the concept of a system of numbers is broader than that of a field. The process of generalization of numbers can be depicted so:

$$\begin{tikzcd}[column sep=small]
\mathbb{N} \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Q} \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{C} \arrow[r] & \mathbb{H} \arrow[r] & \mathbb{O} \\
\text{natural numbers} \arrow[u] & \text{ring of integers} \arrow[u] & \textit{field} \text{ of rationals} \arrow[u] & \text{field of reals} \arrow[u] & \text{field of complex numbers} \arrow[u] & \text{algebra of quaternions} \arrow[u] & \text{algebra of octonions} \arrow[u]
\end{tikzcd}$$

Problem 1: Prove from axioms that in a field $a \cdot 0 = 0$. Could the 0 and 1 coincide, $1 = 0$, in a field?

Problem 2: Consider set $\mathbb{Z}_n = \{1, 2, \dots, n\}$ with addition and multiplication understood 'modulo n. The basic fact is that $\mathbb{Z}$ is a field only if n is prime. Show that indeed $\mathbb{Z}_{12}$ is not a field.

Problem 3: Show that the set $\{a + b\sqrt{3} \mid a, b \in \mathbb{Q}\}$ with the usual addition '+' and multiplication '·' forms a field. Why the set $\{a + b\sqrt{6} \mid a, b \in \mathbb{Q}\}$ does not?

Problem 4: Consider *propositional calculus* $\{A, \wedge, \vee\}$ where A is a set ('sentences') and $\wedge$ and $\vee$ are the binary operations of *and* and *or* respectively. Which axioms fail to make it a field?

Problem 5: **[Duplex Numbers]** (Called also hyperbolic numbers, double numbers, or split complex numbers) Consider numbers of the form $a + bI$ where $a, b \in \mathbb{R}$ are real, and $I^2 = +1$ ("not quite the complex numbers"):

$$\mathbb{D} = \{a + bI \mid a, b \in \mathbb{R}\} \qquad I^2 = 1$$

Define, as one does for $\mathbb{C}$, the conjugation and the 'norm' for $z = a + bI$ by:

$$\bar{z} = a - bI \qquad |z|^2 = z\bar{z}$$

This suggests to try to get an inverse of z as $z^{-1} = \bar{z}/z\bar{z}$.

- Why $\mathbb{D}$ is not a field ?
- Draw a "unit circle" $|z|^2 = 1$ in the plane $\mathbb{D}$.
- What is an analog of the polar form of a complex number (recall: $z = r(\cos\varphi + i\sin\varphi)$) in the case of duplex numbers?
- What is $e^{\varphi I}$? Interpret geometrically multiplication of duplex numbers.
  Hint: Set a parameter $v = \tanh\varphi$ and express all terms in terms of v. Compare with the special theory of relativity if you are familiar with it.

Problem 6: Find a matrix representation for complex numbers and for duplex numbers.

1.2.1 Quaternions

The geometric interpretation of multiplication of complex numbers include rotations in the plane of $\mathbb{C}$. Taking this as the starting point, and taking into account the fact that the visual space in which we (seem to) live is three-dimensional, William Rowan Hamilton tried to introduce multiplication to a set of three-dimensional numbers of form $a+bi+cj$, with two 'imaginary units' i and j. It took him years to realize in 16 October 1843 that the three-dimensional space does not admit any invertible multiplication and to discover that one may turn the 4-dimensional linear space into a field.

Here we state only the basic definitions and properties; more is coming soon.

Definition: Quaternions $\mathbb{H}$ are the numbers of form

$$q = a + bi + cj + dk \qquad a,b,c,d \in \mathbb{R} \tag{1.4}$$

where the three *imaginary units* are "square roots of -1":

$$i^2 = j^2 = k^2 = -1$$

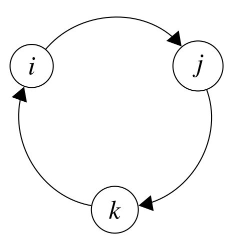
which anti-commute:

$$ij = -ji \qquad jk = -kj \qquad ki = -ik$$

and any two produce the third one:

$$ij = k \qquad jk = i \qquad ki = j$$

This "multiplication table" is often presented in the form of an oriented circle:



Geometrically, quaternions form a 4-dimensional space. For q as in (1.4), the first term, a, is called the *real* part of q, and $bi+cj+dk$ is called the *pure* part of q. One defines conjugation of q as

$$\bar{q} = a - bi - cj - dk$$

It is easy to show that the number defined by

$$|q|^2 = q\bar{q}$$

is real and equal to

$$|q|^2 = a^2 + b^2 + c^2 + d^2$$

and thus — for a non-zero quaternions — is always positive! Therefore, one may easily find an inverse of q (if $q \neq 0$), namely

$$q^{-1} = \bar{q} \;/\; |q|^2 \tag{1.5}$$

A quaternion is called **pure** if $\bar{q} = q$, i.e., it has vanishig real part. It is called **unit** if $|q|^2 = 1$.

Problem 1 (exercise):

(i) Show that (1.5) indeed defines an inverse. Find the inverse of $q = 1 + i - \sqrt{3}k$.

(ii) Let $q = 3 - 5i + 7j + k$. Find quaternion x such that $jq = xj$ (No extensive calculation! Learn how to pass j over q painlessly).

(iii) Show that $\overline{ab} = \overline{b}\,\overline{a}$, for any two quaternions.

Problem 2: Show that quaternions may be represented by the following 2×2 complex matrices:

$$1 \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad i \longrightarrow \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} \qquad j \longrightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad k \longrightarrow \begin{bmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{bmatrix}$$

so that addition and multiplication of quaternions correspond to addition and multiplication of matrices, respectively. (The bold **i** represents the imaginary unit in $\mathbb{C}$, whereas the plain i is from $\mathbb{H}$). Compare with *Pauli matrices*.

Problem 3: A quaternion $q = a + bi + cj + dk$ may be viewed as a pair

$$q = (a, \mathbf{v})$$

where a is a scalar, and $\mathbf{v}$ is a vector, $\mathbf{v} = (b, c, d)$. Show that the product of two quaternions

$$q = (a, \mathbf{v}) \qquad \text{and} \qquad p = (b, \mathbf{w})$$

results in

$$q \cdot p = (ab - \mathbf{v} \cdot \mathbf{w},\ \mathbf{v} \times \mathbf{w} + a\mathbf{w} + b\mathbf{v})$$

where $\mathbf{v} \cdot \mathbf{w}$ and $\mathbf{v} \times \mathbf{w}$ are the usual *scalar* and the *vector* product, respectively, in the three-dimensional Euclidean space.

Notice that for two pure quaternions $q = (0, \mathbf{v})$ and $p = (0, \mathbf{w})$, the above formula reduces to

$$q \cdot p = (-\mathbf{v} \cdot \mathbf{w},\ \mathbf{v} \times \mathbf{w})$$

Problem 4: Quaternions can describe rigid rotations in 3-dimensional space E. Identify E with the subspace of pure (imaginary) quaternions in $\mathbb{H}$:

$$v = (v_1, v_2, v_3) \longrightarrow v_1 i + v_2 j + v_3 k \tag{$\star$}$$

The Euclidean length may be now expressed as $|v|^2 = v\bar{v}$.

(a) Show that for any unit quaternion $q \in \mathbb{H}$, $|q|^2 = 1$, the map defined for v as in ($\star$) by:

$$v \longrightarrow v' = qv\bar{q}$$

is an orthogonal transformation of E. [Hint: It is enough to show that it preserves the length of v; see Problem (1,iii).]

(b) Show that, in particular, quaternion

$$q = \cos\varphi + i\sin\varphi$$

defines a rotation by angle 2φ around axis i (just compute the action of q on i, j, and k). How would rotations around the axis k be defined?

(c) Conclude that $q = \cos\varphi/2 + I\sin\varphi/2$, where I is a unit quaternion, describes rotation by angle φ around I. Notice that $-q$ determines the *same* rotation as q. What does q^{-1} do?

1.2.2 Octonions

Octonions are numbers that form an 8-dimensional real space and can be written as

$$p = a_0 + a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k} + a_4\,\mathbf{E} + a_5\,\mathbf{I} + a_6\,\mathbf{J} + a_7\,\mathbf{K} \qquad a_i \in \mathbb{R}$$

or simply

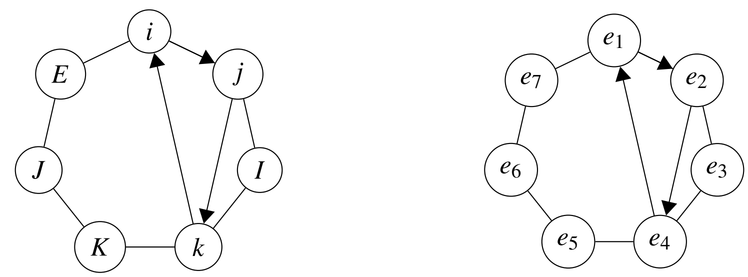
$$p = p_0 + p_1\,\mathbf{e}_1 + \ldots + p_7\,\mathbf{e_7} \qquad p_i \in \mathbb{R}$$

where each of the "imaginary units" is a "square root of minus one", that is $(\mathbf{e}_i)^2 = -1$. The product of distinct imaginary units is skew-symmetric, that is, for any i and j:

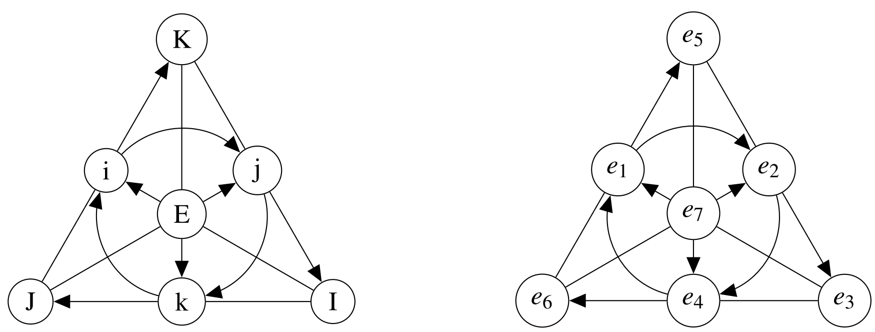
$$\begin{aligned} (\mathbf{e}_i)^2 &= -1 \\ \mathbf{e}_i \cdot \mathbf{e}_j &= -\mathbf{e}_j \cdot \mathbf{e}_i \qquad \text{if } i \neq i \end{aligned}$$

The choice of the letter symbols for the units is dictad by the fact that octonions generalize quaternions. In particular, $\{1, i, j, k\}$ form a subalgebra isomorphic with quaternions, and a new imaginary unit E produces yet another triple of imaginary units denoted: $I = Ei$, $J = Ej$, and $K = Ek$. The whole multiplication table may be presented in three equivalent ways:

1. Cyclic $\mathbb{Z}_7$ structure. The product of two vertices of the triangle made of arrows is the third one — the sign agrees with the triangle orientation, like in the case of quaternions. The triangle is free to rotate inside the circle. (In the right circle, the numbers indicate the indices, i.e., $i \to e_i$.)



2. Fano structure. The product table can be also organized in a form of the so-called Fano plane (a discrete projective space with 7 points and seven lines, each containing 3 points). The product of two elements, $x \cdot y$, lies in the same line as x and y. The orientation of the line determines the sign of the product.



3. Combinatorics. Notice that the multiplication in the first rule may be simply expressed in the form

$$e_i \circ e_{i+1} = e_{i+3}$$

where the indices are added modulo 7. The following triples generate quaternion subalgebras:

$$(1,2,4) \quad (2,3,5) \quad (3,4,6) \quad (4,5,7) \quad (5,6,1) \quad (6,7,2) \quad (7,1,3)$$

Remark: [to a combinatoritian] This structure corresponds to the fact that the numbers $(1,2,4)$ form a perfect difference set in $\mathbb{Z}_7$.

Exercise: Define for octonions the norm, $|p|^2$, and check that the inverse p^{-1} is well-defined (if $p \neq 0$).

Exercise: Show that multiplication of octonions is not associative. However this is true: $p(qq) = (pq)q$ for any $p, q \in \mathbb{O}$.

1.3 Linear space and its dual

Here is a review of the basic notions of linear algebra; the choice of these notes is dictated by their later use in differential geometry. In particular, we shall point the following:

1. The geometry of the dual space (how to imagine and draw covectors)
2. Interpret scalar product as a map $L \to L^*$
3. What is a matrix. Do not confuse with the matrix of an endomorphism with the matrix of a scalar product. (Not to mention the matrix of a change of basis). For instance, the meaning of determinant or trace is well defined for endomorphisms only. But the meaning of signature makes sense for the inner product, but not for endomorphism.

It is very important to distinguish between a *linear space*, L, and its *dual space*, L^*. They have different geometric and algebraic properties and lead to different pictures. While concepts such as *velocity*, *acceleration* or *force* are represented by the elements of a vector space, concepts such as *momentum*, *electro-magnetic field*, or *gauge potential* live in the realm of the dual space.

1.3.1 Vectors

Vectors are elements of a linear space. Here is a spectrum of notations most frequently used for a vector (for simplicity, we assumed 2 dimensional space):

Abstract symbol

Incorrect notation used in some physics literature

Correct – a sum of components

Einstein's summation convention

Matrix form

Dirac notation

Diagrammatic notation

$$\mathbf{v} = \underbrace{\mathrm{v}^{\mathrm{i}} = \sum_i v^i \mathbf{e}_i = v^i \mathbf{e}_i = \begin{bmatrix} v^1 \\ v^2 \end{bmatrix}}_{\text{in the presence of a basis}} = |\,v\,\rangle = \boxed{\mathrm{V}}$$

This diagram of symbols is sort of a summary of the next few sections.

1.3.2 What is a linear space (vector space)

It has been an important achievement in the history of mathematics to conceptualize linear space *without* Euclidean structure automatically associated with it. The essence of linearity is captured in the following definition:

Definition: A *linear space* over a field $\mathbb{F}$ is a set L equipped with a binary operation denoted $+$, and multiplication by a scalar (from $\mathbb{F}$) such that the following conditions are satisfied for every $\mathbf{v}, \mathbf{w}, \mathbf{z} \in L$ and $a, b \in \mathbb{F}$:

1.	$(\mathbf{v}+\mathbf{w})+\mathbf{z} = \mathbf{v}+(\mathbf{w}+\mathbf{z})$	5.	$a(\mathbf{v}+\mathbf{w}) = a\mathbf{v}+a\mathbf{w}$
2.	$\mathbf{v}+\mathbf{w} = \mathbf{w}+\mathbf{v}$	6.	$(a+b)\mathbf{v} = a\mathbf{v}+b\mathbf{v}$
3.	$\exists\, \mathbf{0} \in L \ \forall\, \mathbf{v} \in L: \ \mathbf{v}+\mathbf{0} = \mathbf{v}$	7.	$(ab)\mathbf{v} = a(b\mathbf{v})$
4.	$\forall\, \mathbf{v} \in L \ \exists\, \mathbf{v}' \in L: \ \mathbf{v}+\mathbf{v}' = \mathbf{0}$	8.	$1 \cdot \mathbf{v} = \mathbf{v}$

The elements of a linear space are called *vectors*.

Remark: A **vector space** is a linear space of finite dimension.

Exercise: Notice that Property 5 (distributivity) is the linear algebra reflection of the Thales Theorem on similarity. Why?

Exercise: Show that $0\mathbf{v} = \mathbf{0}$.

Examples:

1. The fields of the *real* and *complex* numbers, and *quaternions*, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, are linear spaces over real numbers.

2. The Cartesian product of a field,

$$L = \mathbb{F} \times \mathbb{F} \times \ldots \times \mathbb{F} = \mathbb{F}^n$$

   is a linear space over $\mathbb{F}$. The elements of $\mathbb{F}^n$ are often called *n-tuples*, and are written as $v = (v^1, v^2, \ldots, v^n)$. We shall often use a matrix notation, in which case typically $\mathbf{v} \in \mathbb{F}^n$ is represented by a column:

$$\mathbf{v} = \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^n \end{bmatrix} = \begin{bmatrix} v^1 & v^2 & \ldots & v^n \end{bmatrix}^T$$

   where $[\,]^T$ denotes a transposition of a matrix. It is a space-saving handy symbol when a column is to be written in a line in the body text.

3. Matrices of a fixed degree, say $n \times m$, over any field $\mathbb{F}$ form a linear space, which we will denote $\mathrm{Mat}_{n\times m}\mathbb{F}$.

4. Let A be any set (finite or infinite), and $\mathbb{F}$ be a field. Show that the set of functions from A to $\mathbb{F}$ is a linear space (denoted $\mathbb{F}^A$) where the addition and multiplication by a scalar is defined

$$(f+g)(a) = f(a) \qquad f,g \in \mathbb{F}^A, \quad a \in A \tag{1.6}$$
$$(cf)(a) = cf(a) \qquad c \in \mathbb{F} \tag{1.7}$$

Notice that the real n-dimensional space $\mathbb{R}^n$ may also be viewed as a set of functions from a finite set of n elements (indices) to $\mathbb{R}$; namely vector $\mathbf{v} = (v^1, v^2, \ldots, v^n)$ as a map $n \mapsto v^n$, $n \in \mathbb{N}$.

Definition: A set of k vectors $\{v_1, v_2, \ldots, v_k\}$ in a linear space L, are called **linearly independent**, if neither of them may be represented as a linear combination of the remaining ones, i.e., if any set of k numbers $\{a_1, a_2, ..., a_k\}$ it is

$$a_1 v_1 + a_2 v_2 + \ldots + a_k v_k = 0 \qquad \Rightarrow \qquad a_1 = a_2 = \cdots = a_k = 0$$

Space L is said to have **dimension** equal n, written $\dim L = n$, if it admits a set of at most n independent vectors. To indicate the field over which the space is considered, one may use notation $\dim_{\mathbb{F}} L$. In these notes, the field most often considered will be $\mathbb{R}$ or $\mathbb{C}$.

Examples: $\dim \mathbb{R}^n = n$, $\dim_{\mathbb{R}} \mathbb{C} = 2$, $\dim_{\mathbb{R}} \mathbb{H} = 4$, $\dim_{\mathbb{C}} \mathbb{C} = 1$, $\dim_{\mathbb{H}} \mathbb{H} = 1$, $\dim M_{k \times n} = k \cdot n$, $\dim_{\mathbb{F}} \mathbb{F}^S = \text{Card}\, S$.

A **basis** of a linear space L is any set of $n = \dim L$ linearly independent vectors of L, for instance $\mathbf{e} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$, where $\mathbf{e}$ is the name of the basis. Notice the use of *subscripts* for basis vectors.

An example of a basis of $\mathbb{F}^n$ is the set of n vectors represented by column matrices:

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Notation: If a particular basis $\mathbf{e}$ is fixed in a linear space, then each vector $\mathbf{v} \in L$ may be represented as a column matrix:

$$\mathbf{v} = \sum_i v^i\, \mathbf{e}_i = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}_{\mathbf{e}}$$

In other words, fixing a basis fixes a particular isomorphism between the linear space L and the "standard" linear space of the corresponding dimension, $L \cong \mathbb{R}^n$ (see Example 2 above).

Summation convention. From now on we shall use the "index summation convention" (also called "Einstein's convention"), by which a summation is implied along any index which appears in a term twice (on different levels). Thus the above formula may be written simply as

$$\mathbf{v} = v^i\, \mathbf{e}_i \equiv \sum_i v^i\, \mathbf{e}_i$$

Exercise: Check that the set of vectors

$$\mathbf{f}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{f}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{f}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

forms a basis in the space $\mathbb{R}^3$.

1.3.3 Dual space

Given a linear space L, the first natural thing to consider are linear **forms**. They also equivalently called **1-forms**, or **covectors**.

Definition: A **covector**, or a **1-form**, is a linear real function on L. That is to say, it is a map

$$\alpha : L \longrightarrow \mathbb{R}$$

such that for any $\mathbf{v}, \mathbf{w} \in L$ and $c \in \mathbb{R}$ it is

$$\begin{array}{ll} (i) & \alpha(\mathbf{v}+\mathbf{w}) = \alpha(\mathbf{v}) + \alpha(\mathbf{w}) \\ (ii) & \alpha(c \cdot \mathbf{v}) = c \cdot \alpha(\mathbf{v}) \end{array}$$

(Notice that in (i) the first "+" is in linear space, the second is in the field, $\mathbb{R}$).

Proposition: The set of all covectors is itself a linear space, which is called the *dual space* and denoted by L^*. Addition of two forms α and β and scaling by $c \in \mathbb{R}$ are defined as follows

$$(\alpha+\beta)(\mathbf{v}) = \alpha(\mathbf{v}) + \beta(\mathbf{v})$$
$$(c\,\alpha)(\mathbf{v}) = c\,(\alpha(\mathbf{v}))$$

Proof: Exercise (show that $\alpha+\beta \in L^*$ and $c\alpha \in L^*$). □

First naïve intuition: If a 3-dimensional space L has fixed basis, then a vector may be represented as a triple of numbers (x, y, z). Any linear function $\alpha : L \to \mathbb{R}$ must be of the form:

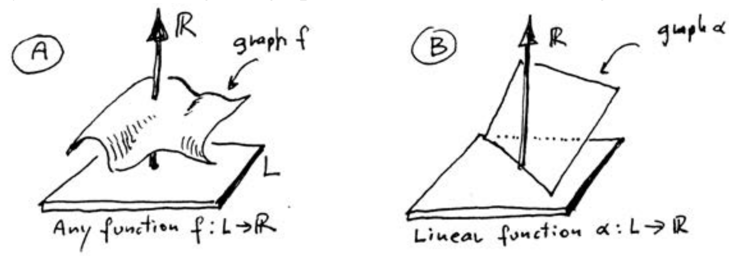
$$\alpha(\mathbf{v}) = ax + by + cz$$

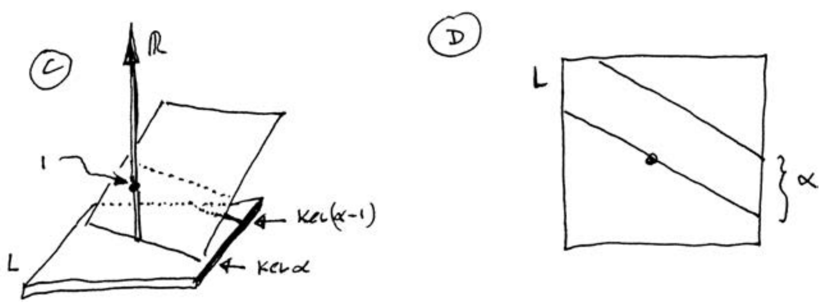
for some constants $a, b, c \in \mathbb{R}$. Thus, intuitively, L^* may be viewed as the space of all such possible coefficients (a, b, c), and is of dimension 3.

1.3.4 Geometry of covectors

Clearly, a *vector* is a vector of L, and a *covector* is a vector in the space L^*.

There is no natural identification between the two spaces. Yet one would like to "see" a covector directly in the space L. Here is how such a visualization can be done. First, we shall consider a 2-dimensional space L (plane). Any form α —being a linear function on L— may be viewed through its graph in $L \times \mathbb{R}$, as shown in (Fig (B) below.





A graph of any function $f : \mathbb{R}^2 \to \mathbb{R}$ requires 3 dimensions for its drawing (see Figure A below). A way to reduce this drawing to only 2-dimensional image of f one typically draws the level curves directly on $\mathbb{R}^2$ with omission of the third dimension. We shall do the same trick with a covector.

Notice that covector α may be characterized by two factors (Fig C):

1. The intersection of the plane L with the graph of α. Here it is a line P_0 through the origin. The orientation of this line represents the "direction" of α (clearly, we observe here a "2-dimensional freedom" of such orientation).
2. The slope of the plane of graph(α); this corresponds to the "length" of α. To represent it within L, draw the line of vectors which are anti-image of the unit

$1 \in \mathbb{R}$.

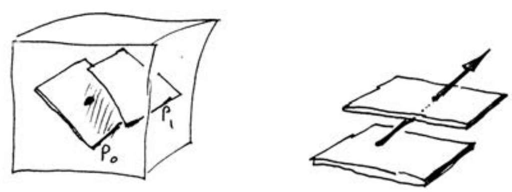
As a result, we get two parallel lines in L which entirely determine α: one goes through 0, the other is parallel to it and goes in some distance from 0 (Fig. D). They are just the level curves for α understood as a real function.

This idea can be generalized to linear spaces of any dimension. We arrive at the following **visualization of forms**. A form (covector) $\alpha \in L^*$ may be represented by a pair of two parallel **hyper-planes** in L (spaces of dimension $= \dim L - 1$), namely:

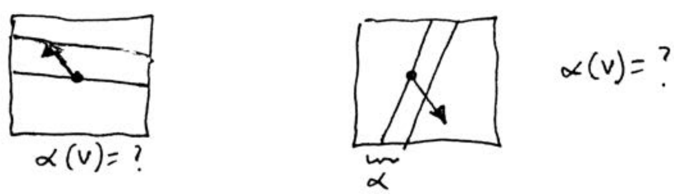
$$P_0 = \{\, \mathbf{v} \in L \mid \alpha(\mathbf{v}) = 0 \,\} \cong \operatorname{Ker}\alpha \tag{1.8}$$

$$P_1 = \{\, \mathbf{v} \in L \mid \alpha(\mathbf{v}) = 1 \,\} \cong \operatorname{Ker}(\alpha - 1) \tag{1.9}$$

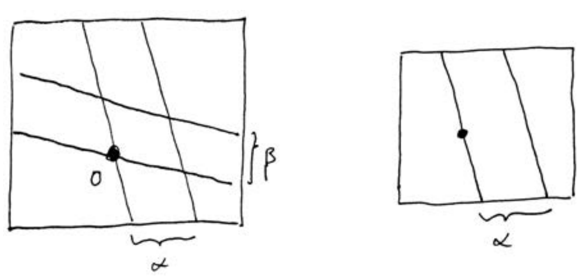
The figure below shows a covector (1-form) in a 3-dimensional space (left side). What is the approximate value of the covector on the vector shown on the right side?



Exercise: What are the values of the form on the indicated vectors?



Exercise: Add two arbitrary forms $\alpha + \beta$ using only their visual representation. Similarly, multiply a form by 2.



1.3.5 Dual basis of the dual space

We already know that the set L^* is a vector space but we do not know how "big" it is.

Proposition The dual space of a vector space is of the same dimension as the initial space:

$$\dim L^* = \dim L \qquad (\text{if } \dim L < \infty)$$

PROOF: We need to show that one can build a basis of L^* of exactly $n = \dim L$ elements. Suppose a basis $\{\mathbf{e}_i\}$ in L is chosen, $i = 1, 2, \ldots, n$. Let us deffine a set of n covectors $\{\varepsilon^i\}$ by this formula:

$$\varepsilon^i(\mathbf{e}_j) = \delta^i_j \equiv \begin{cases} 1 & \text{if } i = j; \\ 0 & \text{otherwise} \end{cases} \tag{1.10}$$

The function denoted by δ is called **Kronecker delta**.

Claim: This set forms a basis in L^*.

To show it we need to illustrate that any covector α may be written as a linear combination of this basis covectors. Indeed, $\alpha \in L^*$, define a new covector:

$$\widetilde{\alpha} := \alpha(\mathbf{e}_i)\, \varepsilon^i$$

Now one can show that $\alpha \equiv \widetilde{\alpha}$. This is left as an exercise. [Hint: Show that for any vector $\mathbf{v} \in L$ one has $\widetilde{\alpha}(\mathbf{v}) = \alpha(\mathbf{v})$. Write the vector as $\mathbf{v} = v^k \mathbf{e}_k$]. $\square$

Exercise: Show that the set of covector defined in 1.10 is linearly independent.

Thus, one may express any covector $\alpha \in L^*$ as a sum $\alpha = \alpha_i\, \varepsilon^i$ where the coefficients are restorable via $\alpha_i = \alpha(\mathbf{e}_i)$. The basis (1.10) is called the **dual basis** of L^*. Note the opposite positions of the indices for coefficients of covectors and those of vectors (low versus high).

Equivalently, one may represent α as a **row matrix**. The dual basis of covectors is

$$\varepsilon_1 = [1,0,0,...,0], \quad \varepsilon_2 = [0,1,0,...,0], \quad \ldots\,, \quad \varepsilon_n = [0,0,0,...,1]\,.$$

and a general covector becomes

$$\alpha = \alpha_i\, \varepsilon^i = \begin{bmatrix} \alpha_1 & \alpha_2 & \ldots & \alpha_n \end{bmatrix}$$

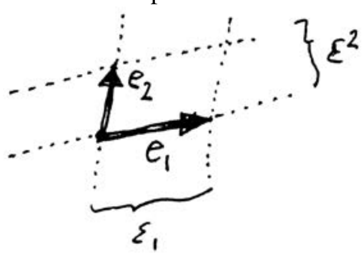
The evaluations of forms of vectors boils down to matrix multiplication. As an example see these two ways. Foe example, consider a form and a vector:

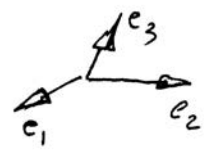
$$\alpha = 3\varepsilon^1 + 7\varepsilon^2 = [3,\ 7] \qquad \text{and} \qquad \mathbf{v} = 5\mathbf{e}_1 + 9\mathbf{e}_2 = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

The value of $\alpha(\mathbf{v})$ may be calculated in two ways: directly or via matrices:

$$\alpha(\mathbf{v}) = (3\varepsilon^1 + 7\varepsilon^2)(5\mathbf{e}_1 + 9\mathbf{e}_2) = 71 \qquad \text{or} \qquad [3,\ 7] \begin{bmatrix} 5 \\ 9 \end{bmatrix} = [3 \cdot 5 + 7 \cdot 8] = [71]$$

Exercise: The left-hand side figure shows geometric interpretation of the dual basis in a 2-dimensional space. Draw dual basis in the 3-dimensional case.





Exercise: In a 2-dimensional space L with a basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, draw the following forms: ε_1, ε_2, $\varepsilon_1 + \varepsilon_2$, $2\varepsilon_1$, $3\varepsilon_1$, $2\varepsilon_1 + \varepsilon_2$, $2\varepsilon_1 - \varepsilon_2$ in both L and L^*.

Question: If there is **no natural identification** of L with L^*, how was it possible to induce a basis in L^*? Doesn't it provide such a unique identification via $\mathbf{e}_i \to \varepsilon^i$?

Various notations for a form are summarized below:

Abstract symbol

Incorrect notation used in some physics literature

Correct – a sum of components

Einstein's summation convention

Matrix form

Dirac notation

Diagrammatic notation

$$\boldsymbol{\alpha} = \underbrace{\alpha_i = \sum_i \alpha_i\, \varepsilon^i = \alpha_i\, \varepsilon^i = [\alpha_1 \quad \alpha_2]}_{\text{in the presence of a basis}} = \langle\, \alpha \,| = \boxed{\alpha}$$

1.3.6 Change of basis in L and L^*

The same vector $\mathbf{v} \in L$ has different matrix representations in different bases. If $\mathbf{e} = \{\mathbf{e}_i\}$ and $\mathbf{f} = \{\mathbf{f}_i\}$ are two bases, vector $\mathbf{v}$ may be represented in two ways:

$$\mathbf{v} = v^i\, \mathbf{e}_i = v'^i\, \mathbf{f}_i\ . \tag{1.11}$$

Every element of one basis may be expressed in terms of the other basis, say

$$\mathbf{e}_i = B_i^j\, \mathbf{f}_j \tag{1.12}$$

This easily leads to the formula expressing the i-th component of $\mathbf{v}$ in basis $\mathbf{e}$ in terms of its components in basis $\mathbf{f}$

$$v'^i = B_j^i\, v^j \tag{1.13}$$

Indeed, using (1.12): $\mathbf{v} = v^i\, \mathbf{e}_i = v^i\, (B_i^j\, \mathbf{f}_j) = (B_i^j\, v^i)\, \mathbf{f}_j$. Comparing this with (1.11), one gets (1.13).

The summary below shows how change of bases affects vectors' and covectors' coefficients in a matrix notation. Proofs are left as exercises.

Clearly, the two spaces, L and its dual L^*, may be equipped with bases that are unrelated. However, if we insist on using bases that are dual to each other, then changing a basis in L requires appropriate changing of the dual basis in L^*. Let $\mathbf{e}$ and $\mathbf{f}$ be two bases in L, and ε and φ their corresponding dual bases. Represent them collectively by a row and a column matrix, respectively note that the entries are not numbers!):

initial basis	new basis
$\mathbf{e} = \begin{bmatrix} \mathbf{e}_1, & \dots, & \mathbf{e}_n \end{bmatrix}$	$\mathbf{f} = \begin{bmatrix} \mathbf{f}_1, & \dots, & \mathbf{f}_n \end{bmatrix}$
$\varepsilon = \begin{bmatrix} \varepsilon^1 \\ \vdots \\ \varepsilon^n \end{bmatrix}$	$\varphi = \begin{bmatrix} \varphi^1 \\ \vdots \\ \varphi^n \end{bmatrix}$

Then any vector $\mathbf{v}$ can be written in basis $\mathbf{e}$ as a matrix product:

$$\mathbf{v} \;=\; [\mathbf{e}]\,[v]_{\mathbf{e}} \;=\; \begin{bmatrix} \mathbf{e}_1, \dots, \mathbf{e}_n \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$$

and similarly for a form α:

$$\varepsilon \;=\; [\alpha]_\varepsilon\,[\varepsilon] \;=\; \begin{bmatrix} \alpha_1, \dots, \alpha_n \end{bmatrix} \begin{bmatrix} \varepsilon^1 \\ \vdots \\ \varepsilon^n \end{bmatrix}$$

The connection between the bases may be described in matrix notations follows:

$$[\mathbf{f}^1, \dots, \mathbf{f}^n] = [\mathbf{e}^1, \dots, \mathbf{e}^n] \begin{bmatrix} & & \\ & B & \\ & & \end{bmatrix}$$

Result: If **e** and ε are the "old" bases, and **f** and φ the "new" ones, then coefficients of vector **v** and of form α transforms (in the matrix notation):

$$\begin{bmatrix} \\ \mathbf{v} \\ \\ \end{bmatrix}_{\mathbf{f}} = \begin{bmatrix} & & \\ & B^{-1} & \\ & & \end{bmatrix} \begin{bmatrix} \\ \mathbf{v} \\ \\ \end{bmatrix}_{\mathbf{e}} \qquad \begin{bmatrix} & \alpha & \end{bmatrix}_{\mathbf{e}} = \begin{bmatrix} & \alpha & \end{bmatrix}_{\mathbf{f}} \begin{bmatrix} & & \\ & B & \\ & & \end{bmatrix}$$

For this reason, the following terminology was established:

vector $\longleftarrow$ a **contravariant** geometric object
form $\longleftarrow$ a **covariant** geometric object

These terms are inaccurate and come from the times when the geometric objects were viewed through their coefficients rather than as legitimate coordinate-free entities. (One of an annoying leftover of this is the incorrect expression "given a vector v^i ".) As we shall see later, the terms should be switched. This exercise however would be a too expensive...

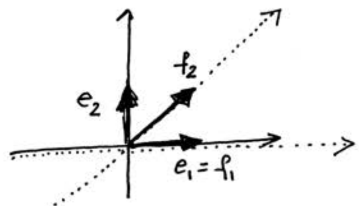
VECTORS & COVECTORS UNDER CHANGE OF BASIS

EXAMPLE

1ST BASIS $e = [e_1, e_2]$

2ND BASIS $f = [f_1, f_2]$

ENTRIES OF THESE ROWS ARE VECTORS (NOT "COEFFICIENTS")



MATRIX B

$$\begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 \end{bmatrix}$$

That is: $\begin{cases} f_1 = e_1 \\ f_2 = e_1 + e_2 \end{cases}$

Expectations: vector $v = e_1 + e_2 = f_2$

covector $\alpha = \varepsilon_2 = \ldots\ldots$ (guess)

GENERAL RULES

VECTOR

$$\begin{bmatrix} v \end{bmatrix}_f = \begin{bmatrix} B^{-1} \end{bmatrix} \begin{bmatrix} v \end{bmatrix}_e$$

COVECTOR

$$\begin{bmatrix} \alpha \end{bmatrix}_f = \begin{bmatrix} \alpha \end{bmatrix}_e \begin{bmatrix} B \end{bmatrix}$$

CHECK GRAPHICALLY THAT

$v \rightarrow$

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}_e = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_f$$

$$\begin{bmatrix} 1 & -1 \\ & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}_e = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_f$$

etc

$\alpha \rightarrow$

$$\begin{bmatrix} 0 & 1 \end{bmatrix}_e \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix}_f$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix}_e \begin{bmatrix} 1 & 1 \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}_f$$

etc

1.3.7 Duality summarized

Situation: we have two vector spaces, L and its dual L^*. We observed that there is a *natural* pairing of the elements of these spaces:

$$\begin{aligned} L^* \times L &\longrightarrow \mathbb{R}: \\ \alpha, \mathbf{v} &\longrightarrow \alpha(\mathbf{v}) \end{aligned}$$

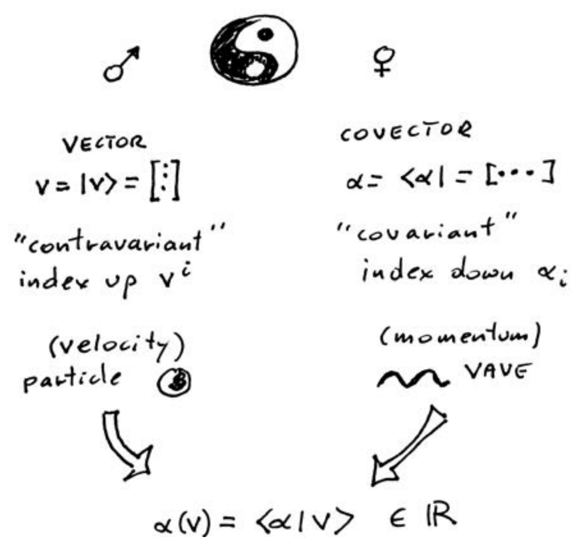
which does not depend on basis or any extra-structure of L. In the Dirac's notation, vectors are denoted as $|\mathbf{v}\rangle$, and covectors as $\langle\alpha|$. Their product is then

$$\alpha(\mathbf{v}) \equiv \langle \alpha \mid \mathbf{v} \rangle \tag{1.14}$$

In the folklore of physics, covector is called 'bra', and vector is called 'ket', their product is then 'bracket'. Although pairing (1.14) resembles scalar product, it is important to keep in mind that one may have *many* different scalar products in a given space, whereas pairing (1.14) is unique!

A vector space L and its dual space L^* are isomorphic as linear spaces, but **no** particular isomorphism is fixed: there is no natural map $L \to L^*$ distinguished.

The distinction L versus L^* is as fundamental as *Yin – Yang* duality in the Far East philosophy of *Tao*.



The dual space L^*, being a vector space itself, has its own dual space, $(L^*)^*$. However, for the finite-dimensional spaces, the dual to the dual space may be identified with the original space:

$$L \cong (L^*)^* \tag{1.15}$$

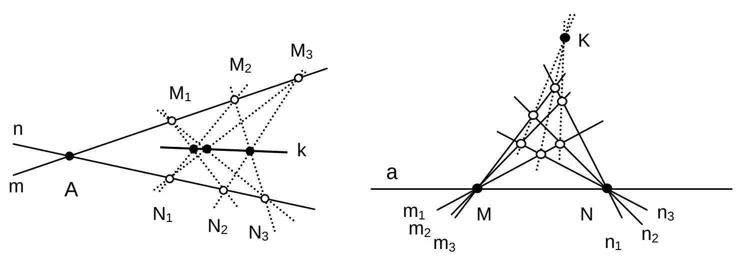
where the identification $L \to (L^*)^* : \mathbf{v} \to \bar{\mathbf{v}}$ is realized by defining the value of $\bar{\mathbf{v}}$ on any $\alpha \in L^*$ as

$$\bar{\mathbf{v}}(\alpha) = \alpha(\mathbf{v}) \tag{1.16}$$

This stresses again that the evaluation of a form on a vector is as well an evaluation of a vector for a form. The duality is fulfilled.

Remark: In the case if the infinite-dimensional spaces, Eq.(1.16) implies only $L \subset (L^*)^*$. However, in the case of Hilbert space (the fundamental concept in Quantum Mechanics) the isomorphism (1.15) *is* implied by the Hermitian structure — this is the essence of Riesz' theorem.

Remark: Duality is deeply rooted already in classical geometry. The duality of polyhedra, like that of Platonic solids, was well-known to the ancient mathematician. In projective geometry, one has a duality principle that states that any proposition remains true if every instance of 'point' is replaced by 'line' and *vice versa*. For example: "two point determine a line" has its dual "two lines determine a point". Theorems concerning various configurations come in pairs: Brianchon's theorem is dual to Pascal's theorem, theorem on 20 Cayley lines is dual to that on 20 Steiner points, while Desargues' theorem is self-dual.



The figure shows two propositions that are dual to each other.

On the left side: Start with a point A and two lines through it, m and n. Chose three points on each line, say m_1, m_2, m_3 and n_1, n_2, n_3. Their pairs (with of different indices) determine six lines, and these intersect at three point (describe the process). Result: the three points are collinear (line k).

On the right side: Start with a line a and two points on it, M and N. Chose three lines through each point, say M_1, M_2, M_3 and N_1, N_2, N_3. Their pairs (with of different indices) determine six points, and these determine three lines (describe the process). Result: the three lines are concurrent (intersect at one point, K).

1.4 Remarks on inner products

A scalar product can be viewed as an identification between a space and its dual space. Below we explain why.

1.4.1 Euclidean space — review

A **Euclidean space** is usually defined as a pair (E, g) where E is a linear space equipped with a map

$$g : E \times E \longrightarrow \mathbb{R}$$

called a **scalar product** (or **dot product**). The product of two vectors denoted alternatively in a number of ways:

$$g(\mathbf{v}, \mathbf{w}) \equiv \mathbf{v} \cdot \mathbf{w} \equiv \langle \mathbf{v}, \mathbf{w} \rangle,$$

which is (1) bilinear, (2) symmetric, (3) non-degenerate, and (4)positive-definite:

1. $\mathbf{v} \cdot (a\mathbf{w} + b\mathbf{y}) = a\mathbf{v} \cdot \mathbf{w} + b\mathbf{v} \cdot \mathbf{y}$
2. $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
3. If $\mathbf{v} \cdot \mathbf{w} = 0$ for each $\mathbf{w}$ then $\mathbf{v} = 0$
4. $\mathbf{v} \cdot \mathbf{v} \geq 0$

for any $\mathbf{v}, \mathbf{w}, \mathbf{y} \in E$, and $a, b \in \mathbb{R}$.

Remark on notation. Note the difference:

$$\begin{array}{ll} \langle \alpha \mid \mathbf{w} \rangle & \leftarrow \text{ a vector evaluated by a form} \\ \langle \mathbf{v}, \mathbf{w} \rangle & \leftarrow \text{ a “dot” product of two vectors} \end{array}$$

Euclidean structure conceals the everyday experience of "rigidness" and "orthogonality" in a mathematical form. It is the algebraic form of the device of compass.

Definition: Two vectors $\mathbf{v}, \mathbf{w} \in L$ are called **orthogonal**, $\mathbf{v} \perp \mathbf{w}$, if $g(\mathbf{v}, \mathbf{w}) = 0$. A norm (squared) is defined $|\mathbf{v}|^2 = g(\mathbf{v}, \mathbf{v})$. Vector $\mathbf{v}$ is called **unit**, if $|\mathbf{v}| = 1$.

A basis $\{\mathbf{e}_i\}$ is called **orthogonal** if each pair of distinct basis vectors is orthogonal to each other, $\mathbf{e}_i \perp \mathbf{e}_j$. The basis is **orthonormal** if it is orthogonal, and the basis vectors are unit, $|\mathbf{e}_i| = 1 \quad \forall\, i$.

Problem 1: Show that if $\{\mathbf{e}_i\}_{i=1,\ldots,n}$ is a basis in a Euclidean space, then Axiom 1 implies that g is —as a function— a second degree polynomial in coefficients of the vectors:

$$g(\mathbf{v}, \mathbf{w}) = g_{ij}\, v^i w^j$$

for some $n \times n$ numbers $\{g_{ij}\}$ defined $g_{ij} = g(\mathbf{e}_i, \mathbf{e}_j)$.

Matrix description. We shall denote the matrix formed by entries g_{ij} these numbers as $[g]$. Axiom 2 implies that the matrix $[g]$ is symmetric, $[g] = [g]^T$. Axiom 3 is equivalent to $\det[g] \neq 0$. Thus, in the presence of a basis, the dot products becomes:

$$g(\mathbf{v}, \mathbf{v}) = [\mathbf{v}]^T\, [g]\, [\mathbf{w}]$$

where $[\mathbf{v}]$ and $[\mathbf{w}]$ denote the matrix representations of the vectors (columns) and T denotes transposition. (If the context of a particular basis $\mathbf{e}$ needs to be stressed, one can use we shall use symbol $[g]_{\mathbf{e}}$.)

Problem 2: Show that under a change of basis the coefficients g_{ij} must be replaced by g'_{ij} so that

$$g'_{ij} = g_{kl}\, B^k_i\, B^l_j$$

We say that g_{ij} is covariant in both indices i and j.

Theorem: For any Euclidean space $\{L, g\}$ there exists a basis in which the matrix of g is the unit matrix, $g_{ij} = \delta_{ij}$.

PROOF: Perform the Gram-Schmidt orthonormalization of any basis. $\square$

It is important to realize that the same space may be equipped with different Euclidean structures; this fact is useful in physics, where Euclidean structure represents a *physical object*. In physicists' slang the structure g is called a *metric* in L.

Exercise 1: [Geometric visualization of Euclidean structure] The set of points $S = \{\, \mathbf{v} \in L \mid \mathbf{v} \cdot \mathbf{v} = 1 \,\}$ will be called the **unit sphere** (with respect to a particular Euclidean structure). Show that if two scalar products g and g' are given in the same vector space, then in an orthonormal basis of g, sphere of metric g' is an ellipsoid. (This implies that in an arbitrary basis the unit sphere of a Euclidean structure is an ellipsoid).

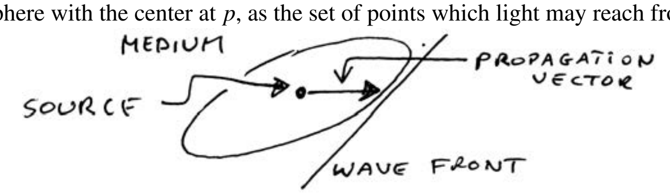
Exercise 2: Consider three Euclidean structures in the same 2-dimensional vector space: g, g', and g", which in a certain basis are represented by the following matrices:

$$g = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad g' = \begin{bmatrix} 2 & 0 \\ 0 & 1/3 \end{bmatrix} \qquad g" = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

In the orthogonal basis of g, (i) draw the unit sphere of g', and (ii) of g".

Exercise 3: Let g and g' are Euclidean structures. Show that $g + g'$ is a Euclidean metric, and $g - g'$ in general is not.

Example: Consider a medium in which the speed of light depends on the direction, $|\mathbf{v}| = (\,\gamma_{ij} v^i v^j\,)^{1/2}$. The visual metric and the physical metric will be different (think of visual unit sphere with the center at p, as the set of points which light may reach from p in 1 second).



Problem 3: Show that the set $\mathcal{F}$ of all continuous real functions on $[0,1]$ forms a space.

1. Show that the map $\alpha : \mathcal{F} \to \mathbb{R}$ defined $\alpha(f) = f(1)$ is a covector (1-form). Actually, for any $x \in [0,1]$, a map $\widehat{x} : \mathcal{F} \to \mathbb{R}$ defined as $\widehat{x}(f) = f(x)$ is a 1-form.
2. Show that the product defined for any two functions $f, g \in \mathcal{F}$ as

$$f \cdot g = \int_0^1 f(x)\, g(x)\, dx$$

is a candidate for a positive definite scalar product. What additional condition must the set $\mathcal{F}$ satisfy?

1.4.2 Euclidean structure reconsidered

Note that the Euclidean structure is a **bilinear** map

$$g:\ L\times L \longrightarrow \mathbb{R}:\ \ \mathbf{v},\mathbf{w}\ \to g(\mathbf{v},\mathbf{w}),$$

fixes an isomorphism between the space and its dual space:

$$g' : L \longrightarrow L^* \tag{1.17}$$

which maps a vector $\mathbf{v}\in L$ to a covector $\widetilde{\mathbf{v}}\in L^*$ defined as

$$\widetilde{\mathbf{v}} = g'(\mathbf{v}) := g\,(\mathbf{v},\,\cdot\,)$$

where the dot "$\cdot$" stands for a place waiting to consume a vector. The value of the form $\widetilde{\mathbf{v}}$ on any vector $\mathbf{w}\in L$ is

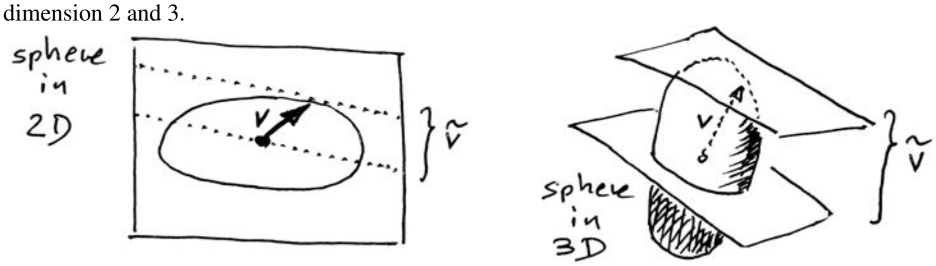
$$\langle\widetilde{\mathbf{v}}\mid\mathbf{w}\rangle\ =\ g(\mathbf{v},\mathbf{w})\equiv\mathbf{v}\cdot\mathbf{w}$$

Exercise: Suppose whave a basis in L and the dual basis in L^*. Denote $\mathbf{v}=v^i\mathbf{e}_i$ and $\widetilde{\mathbf{v}}=v_i\varepsilon^i$. We may use the same letter v for the components of the covector, for the position of the index indicates which is the case. Show that

$$v_j = g_{ij}\,v^i$$

and $\widetilde{\mathbf{v}} = g_{ij}\,v^i\,\varepsilon^j$.

Geometric interpretation: A metric in a vector space given determines a sphere — vectors of unit length. The map (1.17) is based on tangentiality. Here are images of unit vectors in L^* for dimension 2 and 3.



Notice that in optics $\mathbf{v}$ corresponds to a *ray*, and $\widetilde{\mathbf{v}}$ represents the *wave front*.

Exercise 4: Consider a metric in a 2-dimensional vector space given by g' of Exercise 2. Find the covector $\widetilde{\mathbf{v}}$ corresponding to a (unit) vector $\mathbf{v}=[1,\ \sqrt{2}/2]^T$, and draw it. Show that for any unit vector $\mathbf{v}$ its dual $\widetilde{\mathbf{v}}$ is represented by a hyper-plane tangent to the unit sphere of g at the end-point of $\mathbf{v}$. What about not unit vectors?

1.4.3 Inner product generalized

In mathematical and physical reality we often encounter a situation were the Euclidean structure is to restrictive. Hence we define a (generalized) **inner product** keeping only the bi-linearity and relaxing all other (like symmetry and non-degeneracy, etc). For such a general case we will not use the dot to denote it and will not call it "dot product". We may make use of the previous subsection:

Definition. An inner product in a linear space L is a linear map (homomorphism)

$$\gamma : L \longrightarrow L^*$$

This map defines a binary map (denoted by the same letter)

$$\gamma(\mathbf{v},\mathbf{w}) \;=\; \langle\, \gamma(\mathbf{v}) \mid \mathbf{w}\,\rangle$$

Clearly, in the presence of a basis the inner product is determined by a matrix $[\gamma]$.

Additional specification of the properties of γ leads to particular "geometries" beyond the Euclidean case. Here are some examples; the structure is called:

- pre-Euclidean — if γ is symmetric, $\gamma(\mathbf{v},\mathbf{w}) = \gamma(\mathbf{w},\mathbf{v})$.
- pseudo-Euclidean — if γ is symmetric, and nondegenerate ($\mathrm{Ker}\,\gamma = 0$).
- Euclidean — if γ is symmetric, nondegenerate, and positive definite (the last means: $\gamma(\mathbf{v},\mathbf{v}) = 0$ if and only if $\mathbf{v} = 0$).
- pre-symplectic — if γ is skew-symmetric, $\gamma(\mathbf{v},\mathbf{w}) = -\gamma(\mathbf{w},\mathbf{v})$.
- symplectic — if γ is skew-symmetric and nondegenerate, $\mathrm{Ker}\,\gamma = 0$).

A. Pseudo-Euclidean structure is obtained, if in the definition of Euclidean structure we drop the requirement of positive-definiteness.

Problem 1: Show that there exists a basis $\{e_i\}$, $i = 1,\ldots,n$ such that its vectors are orthogonal, and

$$\mathbf{e}_i\cdot\mathbf{e}_i = \begin{cases} +1 & \text{for } i = 1,\ldots,p;\\ -1 & \text{for } i = p+1,\ldots,n. \end{cases}$$

This means that the matrix of g is diagonal in this basis:

$$[\,g\,] = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & & \ddots \\ & & & & & -1 \end{bmatrix} = \begin{bmatrix} I_p & \\ & -I_q \end{bmatrix} \quad = \quad \mathrm{diag}\,(\underbrace{1,...,1}_{p}\,,\,\underbrace{-1,...,-1}_{q})$$

with the number p of "+1" and the number q of "−1". The **signature** of g, denoted sgn g, is the pair sgn (p,q). Clearly $p+q = n = \dim L$. (I_p denotes the $p\times p$ unit matrix. Yet another notation, "diag", lists the diagonal elements and indicates that all other entries are zero.)

Theorem (Sylvester): The signature of g does not depend of a basis.

If $p = 1$ or $q = 1$, the space $\{L,g\}$ is called **Minkowski space**. Such a space of signature sgn $g = (1,3)$ (or $(3,1)$) appears in the theory of relativity theory as the model of **space-time**. This non-positive definite structure is the principal cause of the strange non-intuitive properties of time relations. (See Chapter X.)

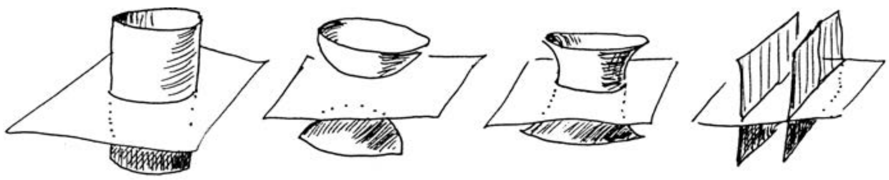
B. Pre-Euclidean space is (1) bilinear, and (2) symmetric, but may be degenerate, and is not necessarily positive definite.

Exercise: Show that there is a basis such that the matrix of g is diagonal in this basis:

$$[g] = \begin{bmatrix} I_k & & \\ & I'_k & \\ & & \mathbf{0}_p \end{bmatrix}$$

where $\mathbf{0}_p$ denotes $p \times p$ matrix of zeros. The signature (k, k', p) does not depend on the choice of the basis.

Examples of generalized spheres:



$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 0 \end{bmatrix} \quad \begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & & \\ & 1 & \\ & & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}$$

Exercise: Draw a sphere for $[g] = \text{diag }(1, -1, 0)$. Draw a "comix" illustrating what happens with the unit sphere when the metric

$$[g] = \begin{bmatrix} 1 & & \\ & \varepsilon & \\ & & 0 \end{bmatrix}$$

slowly changes; say ε grows from $\varepsilon = -1$ to $\varepsilon = 1$.

C. Symplectic space is a true departure from the standard dot product. Symplectic space is defined as $\{L, \omega\}$ in which, denoting $\omega(\mathbf{v}, \mathbf{w}) = \mathbf{v} \cdot \mathbf{w}$, we require:

1. $\mathbf{v} \cdot \mathbf{w} = -\mathbf{w} \cdot \mathbf{v}$
2. $\mathbf{v} \cdot (a\,\mathbf{w} + b\,\mathbf{y}) = a\,\mathbf{v} \cdot \mathbf{w} + b\,\mathbf{v} \cdot \mathbf{y}$
3. If $\mathbf{v} \cdot \mathbf{w} = 0$ for each $\mathbf{w}$, then $\mathbf{v} = 0$

Thus, a symplectic scalar product is (1) skew-symmetric, (2) bi-linear, and (3) non-degenerate.

Problem: (i) Show that the matrix of symplectic scalar product is skew-symmetric, i.e. $\omega_{ij} = -\omega_{ji}$ (that is, $[\omega] = -[\omega]^T$). (ii) Prove that the symplectic space must be of even dimension. Hint: use the fact that for any matrix M, it is $\det M = \det M^T$.

Problem: Show that there is a basis such that the matrix of ω is off-diagonal:

$$[\omega] = \begin{bmatrix} & -I_k \\ I_k & \end{bmatrix}$$

where $\dim L = 2k$, and I_k is a unit $k \times k$ matrix.

Notice that in a symplectic space each vector is 'orthogonal' to itself, $v \cdot v = 0$.

Although not popular in the popular expositions of geometry, the symplectic geometry is the one that underlines Hamiltonian and Lagrangian mechanics — a fact that was discovered long after the Lagrangian/Hamiltonian mechanics was formulated for the first time. It turns out that Gaussian optics as well as thermodynamics conceal symplectic geometry.

1.5 Note on linear maps

Let L_1 and L_2 be two vector spaces (of possibly different dimensions). A **transformation** is a map

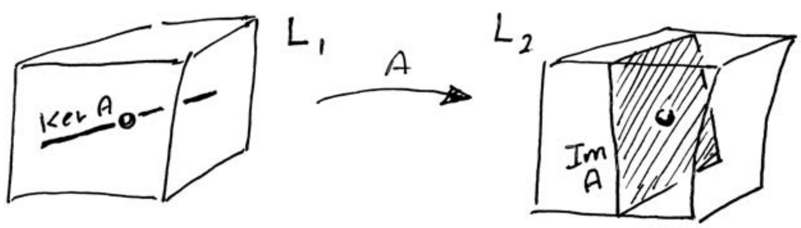
$$A : L_1 \longrightarrow L_2$$

that respects the linear structure of the spaces, i.e. if for any $\mathbf{v}, \mathbf{w} \in L$ and $a, b \in \mathbb{F}$, it satisfies:

$$A(a\,\mathbf{v} + b\,\mathbf{w}) = a\,A\,(\mathbf{v}) + b\,A(\mathbf{w})$$

The **kernel** of A, denoted $\mathrm{Ker}A$, is the set of all vectors in L_1, the value of which is the zero vector in L_2. The **image** of A, denoted $\mathrm{Im}A$, is the set of values of map A.

$$\begin{array}{lll} \mathrm{Ker}A & = \{\mathbf{v} \in L \mid A(\mathbf{v}) = \mathbf{0}\} & \subset L_1 \\ \mathrm{Im}A & = \{\mathbf{w} \in E \mid \mathbf{w} = A(\mathbf{v}) \ \text{ for some } \ \mathbf{v} \in L\} & \subset L_2 \end{array}$$



Problem: (a) Show that both sets are actually subspaces of the corresponding spaces.
(b) Show that $\dim \mathrm{Ker}A + \dim \mathrm{Im}A = \dim L$.

Other terms:

- **Isomorphism** is a linear map such that $\mathrm{Ker}A = 0$ and $\mathrm{Im}A = L_2$.
- **Endomorphism** is a linear map of a space into itself ($L_1 = L_2$).
- **Automorphism** is an endomorphism which is one-to-one, $\mathrm{Ker}A = 0$.

The set of all endomorphisms of L is denoted by $\mathrm{End}L$. The set of all automorphisms of space L is denoted by $\mathrm{Aut}L$.

Examples: Show that the following maps are endomorphisms. Which are automorphisms?

(i)	Identity	$\mathrm{id}: \mathbf{v} \to \mathbf{v}$	$\mathrm{Ker\,id} = 0$	$\mathrm{Im\,id} = L$
(ii)	Nul – operator	$\widehat{0}: \mathbf{v} \to 0$	$\mathrm{Ker}\,\widehat{0} = L$	$\mathrm{Im}\,\widehat{0} = 0$
(iii)	Point inversion	$I: \mathbf{v} \to -\mathbf{v}$	$\mathrm{Ker}\,I = 0$	$\mathrm{Im}\,I = L$
(iv)	Dilation$(c \in \mathbb{R})$	$\widehat{c}: \mathbf{v} \to c\mathbf{v}$		
(v)	$\mathbf{w}$ – projection $(\lvert\mathbf{w}\rvert = 1)$	$\widehat{\mathbf{w}}: \mathbf{v} \to (\mathbf{w}\cdot\mathbf{v})\mathbf{w}$	$\mathrm{Ker}\,\widehat{\mathbf{w}} = \mathbf{w}^{\perp}$	$\mathrm{Im}\,\widehat{\mathbf{w}} = \mathrm{span}\,\{\mathbf{w}\}$
(vi)	Projection onto L_1 along L_2	$P: \mathbf{v} \to \mathbf{v}'$	$\mathrm{Ker}\,P = L_2$	$\mathrm{Im}\,P = L_2$

Explanation of (vi): Let L_1 and L_2 be two subspaces of a vector space L such that their intersection is the zero vector and that they span the whole space, i.e.:

$$L_1 \cap L_2 = 0 \qquad\qquad L_1 \oplus L_2 = L$$

This determines a unique decomposition of each vector of $\mathbf{v} \in L$ into $\mathbf{v} = \mathbf{v}' + \mathbf{v}''$ such that $\mathbf{v}' \in L_1$ and $\mathbf{v}'' \in L_2$.

Exercise: Show that:
(i) A composition of two endomorphisms (automorphisms) as maps is an endomorphism (automorphism).
(ii) The set $\mathrm{End}\,L$ is a linear space, but $\mathrm{Aut}\,L$ is not.

1.5.1 Transformations and matrices

An endomorphism of space L is often called a *linear transformation*. Consider $A \in \mathrm{End}\,L$:

$$A : \mathbf{v} \longrightarrow \mathbf{v}' = A(\mathbf{v})$$

Let $\mathbf{e} = \{\mathbf{e}_i\}$ be a basis in L. Each basis vector $\mathbf{e}_i$ has its image in L and may be represented as a sum in the basis $\mathbf{e}$. Denote:

$$A(\mathbf{e}_i) = A_i^j\, \mathbf{e}_j$$

Since A is linear, the information in Eq (2) is sufficient to reconstruct action of A on any vector; indeed:

$$\mathbf{v}' = v'^i \mathbf{e}_i = A(\mathbf{v}) = A(v^j \mathbf{e}_j) = v^j\, A(\mathbf{e}_j) = v^j\, A_j^i\, \mathbf{e}_i$$

Thus the transformation is

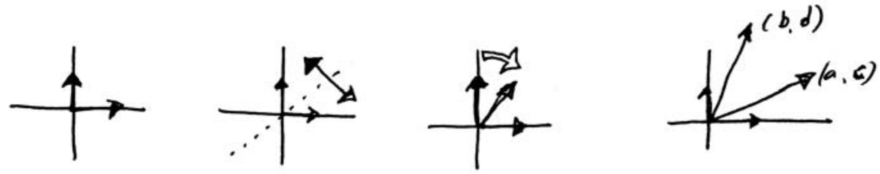
$$v'^i = A_j^i\, v^j$$

In the matrix notation, map A may be described as follows:

$$\begin{bmatrix} \\ \mathbf{v}' \\ \\ \end{bmatrix}_{\mathbf{e}} = \begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix}_{\mathbf{e}} \begin{bmatrix} \\ \mathbf{v} \\ \\ \end{bmatrix}_{\mathbf{e}}$$

Warning: Compare it —and never confuse— with a change of basis!

Exercise — How to read a matrix: Show that the i^{th} column of matrix $[A]$ representing an endomorphism describes, if interpreted as a vector, the image of the i^{th} basis vector in the map A. As an illustration consider the following cases:



$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Using this trick show that rotation in $\mathbb{R}^2$ by angle α is described by $\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$.

1.5.2 Induced endomorphism in L^*

Each endomorphism $A \in \mathrm{End}\,L$ induces an endomorphism A^* of the dual space, $A^* \in \mathrm{End}\,L^*$, which can be defined as:

$$\langle A^*\alpha \mid \mathbf{v}\rangle \;=\; \langle \alpha \mid A\mathbf{v}\rangle$$

for any $\alpha \in L^*$ and $\mathbf{v} \in L$.

Notice that in matrix notation, both sides of the above equation have the same form, namely:

$$[\alpha]\;[A]\;[\mathbf{v}]$$

Matrix $[A]$ can be understood as acting in the “usual" way on $\mathbf{v}$, or in a “transposed" way on α.

Exercise: In the standard 2-dimensional space $\mathbb{R}^2$, consider a linear transformation described by matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Describe geometrically, what happens (in the transformation) to the basis vectors, and to the vectors of the dual basis. Draw all in L and compare.

1.6 Exterior forms

Exterior forms generalize the concept of a linear form (covector). We start friendly with the idea of exterior 2-form and later move to forms higher degrees.

1.6.1 Bi-forms

An **exterior form of the second degree** (or just “**2-form**," or “**bi-form**") is a map

$$\omega : L \times L \longrightarrow \mathbb{R}$$

which is (i) linear in both entries, and (ii) skew-symmetric, that is:

$$\begin{array}{rl} (i) & \omega(\mathbf{v}+\mathbf{v}', \mathbf{w}) = \omega(\mathbf{v}, \mathbf{w}) + \omega(\mathbf{v}', \mathbf{w}) \\ & \omega(c\mathbf{v}, \mathbf{w}) = c\,\omega(\mathbf{v}, \mathbf{w}) \\ (ii) & \omega(\mathbf{v}, \mathbf{w}) = -\omega(\mathbf{w}, \mathbf{v}) \end{array}$$

for all $\mathbf{v}, \mathbf{w} \in L,\ c \in \mathbb{R}$.

Clearly, (ii) implies linearity in the second entry, $\mathbf{w}$. It also implies that $\omega(\mathbf{v}, \mathbf{v}) = 0$.

The set of bi-forms in L is denoted in two ways: $\wedge^2 L^*$ or $L^* \wedge L^*$.

Proposition 1.6.1. *The set $\wedge^2 L^*$ is a linear space with addition and scaling defined for any $\omega_1, \omega_2, \omega \in \wedge^2 L$ and $c \in \mathbb{R}$ as follows:*

$$\begin{array}{rl} (i) & (\omega_1 + \omega_2)\,(\mathbf{v}, \mathbf{w}) := \omega_1\,(\mathbf{v}, \mathbf{w}) + \omega_2\,(\mathbf{v}, \mathbf{w}) \\ (ii) & (c \cdot \omega)\,(\mathbf{v}, \mathbf{w}) := c \cdot (\omega\,(\mathbf{v}, \mathbf{w})) \end{array}$$

for any $\mathbf{v}, \mathbf{w} \in L$.

PROOF: Simple exercise. □

There is a natural way to build a 2-form out of two covectors.

Definition: An **exterior product** (or a wedge product) is a map

$$L^* \times L^* \longrightarrow \wedge^2 L^* : \quad \alpha, \beta \rightsquigarrow \alpha \wedge \beta$$

where the 2-form $\alpha \wedge \beta$ evaluated on a pair of vectors $\mathbf{v}$ and $\mathbf{w}$ is

$$\alpha \wedge \beta\ (\mathbf{v}, \mathbf{w}) \quad = \quad \alpha\ (\mathbf{v})\ \beta\ (\mathbf{w}) - \alpha(\mathbf{w})\ \beta\ (\mathbf{v}) \tag{1.18}$$

The following simple properties for any covectors $\alpha, \beta, \gamma \in L^*$ are immediate:

$$\begin{array}{rll} (i) & (\alpha+\beta) \wedge \gamma = \alpha \wedge \gamma + \beta \wedge \gamma & \text{distributivity} \\ (ii) & (c\alpha) \wedge \beta = \alpha \wedge (c\beta) = c(\alpha \wedge \beta) & \\ (iii) & \alpha \wedge \beta = -\beta \wedge \alpha & \text{skew-symmetry} \\ (iv) & \alpha \wedge \alpha = 0 & \text{(implied by the above)} \end{array}$$

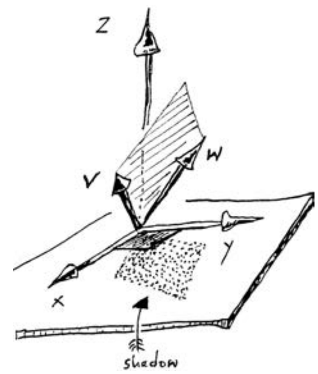
Example 1: Consider a 3-dimensional standard space $L \cong \mathbb{R}^3$ with basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Denote the dual basis in L^* as usual by $\{\varepsilon^1, \varepsilon^2, \varepsilon^3\}$. One can easily show that the following three 2-forms make a basis in the space of 2-forms $\wedge^2 L$:

$$\varepsilon^1\wedge\varepsilon^2 \quad \varepsilon^2\wedge\varepsilon^3 \quad \varepsilon^3\wedge\varepsilon^1$$

(left as an exercise). Thus any bi-form in $\mathbb{R}^3$ may be expressed as a linear combination

$$\omega = a\,\varepsilon^1\wedge\varepsilon^2 + b\,\varepsilon^2\wedge\varepsilon^2 + c\,\varepsilon^3\wedge\varepsilon^1 \tag{1.19}$$

for some numbers $a, b, c \in \mathbb{R}$. Note that this implies that the dimension of the space of bi-forms to be $\dim \wedge^2\mathbb{R}^3 = 3$. Geometric interpretation of $\varepsilon_1\wedge\varepsilon_2$ emerges when we evaluate it on two vectors, say $\mathbf{v} = v^1\mathbf{e}_i$, $\mathbf{w} = w^i\mathbf{e}_i \in L$:



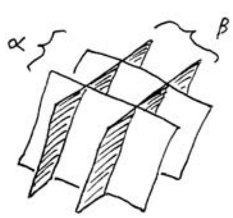
$$\varepsilon^1\wedge\varepsilon^2\,(\mathbf{v},\mathbf{w}) = v^1w^2 - v^2w^1$$

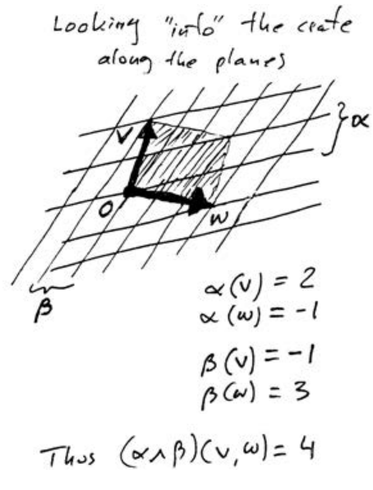
(Check with direct calculations). Thus

$$(\varepsilon^1\wedge\varepsilon^2)\,(\mathbf{v},\mathbf{w}) = \left\{\begin{array}{c}\text{`area' of the shadow}\\ \text{of the parallelogram spanned by } \mathbf{v} \text{ and } \mathbf{w}\\ \text{projected onto plane span}\,(\mathbf{e}_1,\mathbf{e}_2) \text{ along } \mathbf{e}_3\\ \text{with the `square' } \mathbf{e}_1\times\mathbf{e}_2 \text{ taken as area unit}\end{array}\right\}$$

Consequently, the general biform ω of (1.19) may be interpreted as a combination of such shadows with weights.

A geometric interpretation of $\omega = \alpha\wedge\beta$ for some one-forms α and β emerges similarly. Since each can be visualized as a pair of planes, or as a congruence of infinite number of parallel planes, together they make a “crate".





End of Example 1.

Now, we continue with general 2 forms.

Proposition 1.6.2. $\dim \wedge^2 L^* = \binom{n}{2}$.

PROOF: First, notice that if $\{\mathbf{e}_i\}$ and $\{\varepsilon^i\}$ is a basis and a dual basis of L and L^*, respectively, then

$$\{\ \varepsilon^i \wedge \varepsilon^j\ \} \qquad 0 \le i < j \le n = \dim L$$

forms a basis in $\wedge^2 L$. Indeed, denote

$$\omega_{ij} := \omega(\mathbf{e}_i, \mathbf{e}_j) \in \mathbb{R}$$

and check that a biform $\widetilde{\omega}$ defined as

$$\widetilde{\omega} := \sum_{i<j} \omega_{ij}\, \varepsilon^i \wedge \varepsilon^j$$

acts on any two vectors $\mathbf{v}$ and $\mathbf{w}$ the same way as the original ω, hence $\widetilde{\omega} \equiv \omega$. □

Definition. A biform ω is **simple** if it can be expressed as a product $\omega = \alpha \wedge \beta$ for some 1-forms $\alpha,\ \beta \in L^*$.

Example: Exterior two-form $\omega = \varepsilon^1 \wedge \varepsilon^2 - \varepsilon^2 \wedge \varepsilon^3 + 2\,\varepsilon^3 \wedge \varepsilon^1$ is simple. Indeed,

$$\begin{array}{lcll} \omega & = & \varepsilon^1 \wedge \varepsilon^2 - \varepsilon^2 \wedge \varepsilon^3 + 2\,\varepsilon^3 \wedge \varepsilon^1 & \\ & = & (\varepsilon^1 + \varepsilon^3) \wedge \varepsilon^2 + 2\,\varepsilon^3 \wedge \varepsilon^1 & \text{combine the first two terms} \\ & = & (\varepsilon^1 + \varepsilon^3) \wedge \varepsilon^2 + 2\,\varepsilon^3 \wedge (\varepsilon^1 + \varepsilon^3) & \text{adding } \varepsilon^3 \text{ doesn't change anything, since } \varepsilon^1 \wedge \varepsilon^3 = 0 \\ & = & (\varepsilon^1 + \varepsilon^3) \wedge (\varepsilon^2 - 2\,\varepsilon^3) & \text{factoring out} \end{array}$$

thus $\omega = \alpha \wedge \beta$ for $\alpha = \varepsilon^1 + \varepsilon^3$ and $\beta = \varepsilon^2 - 2\,\varepsilon^3$.

Exercise 1: Show that every 2-form in $L \cong \mathbb{R}^3$ is simple.

Exercise 2: Show that $\varepsilon^1 \wedge \varepsilon^2 + \varepsilon^3 \wedge \varepsilon^4 \in L^* \cong \mathbb{R}^4$ is not simple.

Remark: Notice that Definition 1.18 of the wedge product of two 1-forms may be expressed by a determinant:

$$\alpha \wedge \beta\ (\mathbf{v}, \mathbf{w}) \quad = \quad \alpha\ (\mathbf{v})\ \beta\ (\mathbf{w}) - \alpha(\mathbf{w})\ \beta\ (\mathbf{v}) \quad \equiv \quad \begin{vmatrix} \alpha\ (\mathbf{v}) & \alpha(\mathbf{w}) \\ \beta\ (\mathbf{v}) & \beta\ (\mathbf{w}) \end{vmatrix}$$

This gives us the window to generalization beyond 1-forms.

1.6.2 Exterior forms in general

Now, using the intuition just gained, let us generalize.

Definition. An **exterior form of degree** k, or just k-form, is a map

$$\omega : \underbrace{L \times \ldots \times L}_{k \text{ times}} \longrightarrow \mathbb{R}$$

which is linear in every entry and skew-symmetric in the sense that for any permutation σ of k elements, we have

$$\omega(\mathbf{v}_{\sigma(1)}, \mathbf{v}_{\sigma(2)}, \ldots, \mathbf{v}_{\sigma(k)}) = (-1)^{|\sigma|}\, \omega(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$$

We shall write $\deg \alpha = k$. In this new context, scalars will be called 0-forms and covectors will be called exterior 1-forms. An exterior form of degree $n = \dim L$ will also be called **volume form**. Notice that exterior forms of degree greater than $\dim L$ must vanish.

$0-$form	$c \in \mathbb{R}$	a scalar, by definition
$1-$form	$\alpha : L \longrightarrow \mathbb{R}$	linear map, a covector
$2-$form	$\beta : L \times L \longrightarrow \mathbb{R}$	bi-linear map, skew-symmetric: $\gamma(\mathbf{v},\mathbf{w}) = -\gamma(\mathbf{w},\mathbf{v})$
$3-$form	$\gamma : L \times L \times L \longrightarrow \mathbb{R}$	tri-linear map, skew-symmetric: $\gamma(\mathbf{v},\mathbf{w},\mathbf{z}) = \gamma(\mathbf{z},\mathbf{v},\mathbf{w}) = \gamma(\mathbf{w},\mathbf{z},\mathbf{v})$ $= -\gamma(\mathbf{v},\mathbf{z},\mathbf{w}) = -\gamma(\mathbf{w},\mathbf{v},\mathbf{z}) = -\gamma(\mathbf{z},\mathbf{w},\mathbf{v})$
$k-$form	$\omega : \overbrace{L \times \cdots \times L}^{k \text{ times}} \longrightarrow \mathbb{R}$	k-linear skew-symmetric map

Exercise: Show that the condition of skew-symmetricity of exterior form can be written without invoking the symbol of permutation σ, namely

$$\omega(\ldots, \mathbf{v}_i, \ldots, \mathbf{v}_k, \ldots) \;=\; -\omega(\ldots, \mathbf{v}_k, \ldots, \mathbf{v}_i, \ldots)$$

at any position i and k.

Exercise: Show that if vectors $\mathbf{v}_1, \ldots \mathbf{v}_k$ are linearly dependent, then for any k-form $\omega \in \wedge^k L^*$ it is $\omega(\mathbf{v}_1, \ldots \mathbf{v}_k) = 0$.

Exercise: Let $L \cong \mathbb{R}^3$ be an Euclidean space with the standard dot-product. Fix a vector $\mathbf{e} \in L$. Show that

$$\omega : L \longrightarrow \mathbb{R} : \quad \mathbf{v} \mapsto \omega(\mathbf{v}) := \mathbf{e} \cdot \mathbf{v}$$

is a 1-form. Show also that

$$\beta : L \times L \longrightarrow \mathbb{R} : \quad \mathbf{v}, \mathbf{w} \mapsto \beta(\mathbf{v}, \mathbf{w}) := \mathbf{e} \cdot (\mathbf{v} \times \mathbf{w})$$

is an example of a 2-form. Moreover, show that

$$\eta : L \times L \times L \longrightarrow \mathbb{R} : \quad \mathbf{v}, \mathbf{w}, \mathbf{z} \mapsto \beta(\mathbf{v}, \mathbf{w}, \mathbf{z}) := (\mathbf{v} \times \mathbf{w}) \cdot \mathbf{z}$$

is a 3-form. Ponder upon the “special dimension 3."

Proposition 1.6.3. *The set of k-forms on an n dimensional space L is a linear space, denoted $\wedge^k L$, of dimension* $\dim \wedge^k L = \binom{n}{k}$.

PROOF: Analogous to the case of bi-forms. □

$0-$forms	$\wedge^0 L^*$	$\cong \mathbb{R}$	scalars (by definition)
$1-$forms	$\wedge^1 L^*$	$\cong L^*$	linear forms = 1-forms = covectors
$2-$forms	$\wedge^2 L^*$	$\cong L^* \wedge L^*$	bi-firms
$k-$forms	$\wedge^k L^*$		k-forms
$n-$forms	$\wedge^n L^*$	$\cong \mathbb{R}$	volume form $(n = \dim L)$

Exterior forms can be combined into a new exterior form via a natural product — the **exterior product**:

$$\wedge^p L^* \times \wedge^q L^* \longrightarrow \wedge^{p+q} L^* :$$

We shall define this product in a few steps. First, note that any set of k 1-forms $\{\alpha_i\}$ defines an exterior k-form denoted by $\alpha_i \wedge \alpha_2 \wedge ... \wedge \alpha_k$ and defined through its value on k vectors $\{v_i\}$, $i = 1, ..., k$, as

$$\left(\alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^k\right)(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k) = \begin{vmatrix} \alpha_1(\mathbf{v}_1) & \dots & \alpha_1(\mathbf{v}_k) \\ \vdots & & \vdots \\ \alpha_k(\mathbf{v}_1) & \dots & \alpha_k(\mathbf{v}_k) \end{vmatrix}$$

(Note that for two 1-forms, $k = 2$, the above coincides with the definition of the wedge product given in (1.18).) The properties of determinant assure that the object defined is indeed an exterior form. It is easy to show that if $\{\varepsilon^i\}$ is a basis in L^*, then monomials

$$\varepsilon^{i_1} \wedge \varepsilon^{i_2} \wedge \cdots \wedge \varepsilon^{i_k} \qquad i_1 < i_2 < ... < i_k$$

form a basis in $\wedge^k L^*$. Therefore the dimension of the space of exterior k-forms is over an n-dimensional vector space L is

$$\dim \wedge^k L^* = \binom{n}{k}$$

Now, we can define an exterior product of monomials of different degree, say p and q simply as

$$\left(\alpha^{i_1} \wedge \alpha^{i_2} \wedge ... \wedge \alpha^{i_p}\right) \wedge \left(\beta^{i_1} \wedge \beta^{i_2} \wedge ... \wedge \beta^{i_q}\right) = \alpha^{i_1} \wedge ... \wedge \alpha^{i_p} \wedge \beta^{i_1} \wedge ... \wedge \beta^{i_q}$$

Now, extend it by linearity to any p- and q-form to get a linear product $\wedge$ in the space of all exterior forms.

Definition The **Grassmann** algebra over a vector space L^* is the pair $\{\wedge L^*, \wedge\}$, where $\wedge L^*$ is a 2^n-dimensional linear space

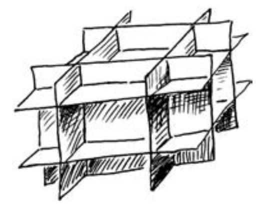
$$\wedge L^* = \mathbb{R} \oplus L^* \oplus \wedge^2 L^* \oplus \wedge^3 L^* \oplus \cdots \oplus \wedge^n L^*$$

and $\wedge$ is the exterior product extended onto all elements of $\wedge L$.

Let us introduce a function of degree, namely we shall write $\deg \omega = k$ if $\omega \in \wedge^k L^*$. The above Grassmann algebra $\wedge L^*$ is **graded** in the sense that

$$\deg \alpha = p \quad \text{and} \quad \deg \beta = q \quad \Rightarrow \quad \deg(\alpha \wedge \beta) = p + q$$

Other topics to think about:



If T is an endomorphism in a vector space L, then one defines a **determinant** of T as a number $\det T$ such that for any $\eta \in \wedge^n L$ and a set of $\mathbf{v}_i \in L$, it is

$$\eta(T(\mathbf{v}_1, T(\mathbf{v}_2), \ldots, T(\mathbf{v}_n)) \; = \; \det T \cdot \eta(\mathbf{v}_1), \mathbf{v}_2, \ldots, \mathbf{v}_n)$$

1.6.3 A symbol for contraction

Let *omega* $\in \wedge 2L*$ be a bi-form and $\mathbf{v} \in L$ be a vector. We define $\mathbf{v} \lrcorner \omega$ as a 1-form which evaluated with vector $\mathbf{w}$ is

$$\langle \mathbf{v} \lrcorner \omega \mid \mathbf{w} \rangle \; = \; \omega(\mathbf{v}, \mathbf{w}) \qquad \forall \mathbf{w} \in L$$

We may simply write it as $\mathbf{v} \lrcorner \omega = \omega(\mathbf{v}, \cdot)$. This generalizes k-forms:

$$\mathbf{v} \lrcorner \omega = \; \omega(\mathbf{v}, \cdot, ..., \cdot)$$

Exercise: Let α, β, γ be 1-forms (covectors). Show / explain that

$$\begin{array}{ll} \mathbf{v} \lrcorner \alpha & = \; \alpha(\mathbf{v}) \\ \mathbf{v} \lrcorner (\alpha \wedge \beta) & = \; \alpha(\mathbf{v})\, \beta - \beta(\mathbf{v})\, \alpha \\ \mathbf{v} \lrcorner (\alpha \wedge \beta \wedge \gamma) & = \; \alpha(\mathbf{v}) \wedge \beta \wedge \gamma - \alpha \wedge \beta(\mathbf{v}) \wedge \gamma + \alpha \wedge \beta \wedge \gamma\,(\mathbf{v}) \end{array}$$

The symbols may be also used for the map $L \to L^*$ defined by an inner product g, namely

$$\mathbf{v} \quad \mapsto \quad \tilde{\mathbf{v}} \; = \; \mathbf{v} \lrcorner g \; = \; g(\mathbf{v}, \cdot\,)\,.$$

1.7 What are tensors and what is $\otimes$

Tensor is a term that encompasses and generalizes the notions of scalar, vector, covector, k-forms, endomorphism, scalar product, and more. To make the definition of tensor clearer, let us start with the observation that both vector and covector can be defined as linear maps:

$$\begin{array}{rll} \text{COVECTOR} & L \longrightarrow \mathbb{R}: & \mathbf{v} \rightsquigarrow \langle \alpha | \mathbf{v} \rangle \\ \text{VECTOR} & L^* \longrightarrow \mathbb{R}: & \alpha \rightsquigarrow \langle \alpha | \mathbf{v} \rangle \end{array}$$

Tensors can be viewed as multi-linear version of these.

1.7.1 Covariant tensors

A k-**covariant tensor** T is a map

$$T: \quad \underbrace{L \times L \times \ldots \times L}_{k \text{ times}} \longrightarrow \mathbb{R}$$

that is **linear** in every entry. That is, for any set of k vectors, $\mathbf{v}_1$, $\mathbf{v}_2$,..., $\mathbf{v}_k$, $\mathbf{w}_i$, and two constants $a, b \in \mathbb{R}$, we have for every position i:

$$T(\mathbf{v}_1, , ..., a\mathbf{v}_i + b\mathbf{w}_i, ..., \mathbf{v}_k) = a\,T(\mathbf{v}_1, ..., \mathbf{v}_i, ..., \mathbf{v}_k) + b\,T(\mathbf{v}_1, ..., \mathbf{w}_i, ..., \mathbf{v}_k)$$

Notice that in particular

$$c \cdot T(\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k) = T(c\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k) = T(\mathbf{v}_1, c\mathbf{v}_2, ..., \mathbf{v}_k) = \ldots = T(\mathbf{v}_1, \mathbf{v}_2, ..., c\mathbf{v}_k)$$

for any $c \in \mathbb{R}$.

The set of such maps forms a linear space and will be denoted in any of these ways:

$$(L^*)^{\otimes k} \;\equiv\; \otimes^k L^* \;\equiv\; L^{(0,k)}$$

Its elements are also called $(0,k)$-variant tensors.

Examples:

(0) The usual **covector:** $\alpha : L \longrightarrow \mathbb{R}$, thus $L^* = L^{(0,1)}$

(1) **exterior k-forms:** are *anti-symmetric* $(0,k)$-variant tensors, $\wedge^k L \subset L^{(0,k)}$.

(2) **Scalar product**: An Euclidean structure is tantamount to a 2-covariant *symmetric* tensor $(0,k)$-variant tensors

$$g \in L^{(0,2)}$$

Tensor product of two covariant tensors T_1 and T_2 of variance k and l, respectively, is a $(k+l)$-covariant tensor $T_1 \otimes T_2$

$$T_1 \otimes T_2: \quad \underbrace{L \times \ldots \times L}_{k \text{ times}} \times \underbrace{L \times \ldots \times L}_{l \text{ times}} \longrightarrow \mathbb{R}$$

defined via the usual product of the scalar values:

$$(T_1 \otimes T_2)\ (v_1, v_2, \ldots, v_{k+l}) = T_1\ (v_1, v_2, \ldots, v_k) \cdot T_2\ (v_{k+1}, v_{k+2}, \ldots, v_{k+l})$$

It is easy to check that tensor product is associative, and distributive with respect to the sum of tensors.

$$\begin{array}{rll} (i) & T_1 \otimes (T_2 \otimes T_3) & = (T_1 \otimes T_2) \otimes T_3 \\ (ii) & (T_1 + T_2) \otimes T_3 & = T_1 \otimes T_3 + T_2 \otimes T_3 \end{array}$$

(No commutativity or anticommutativity is assumed.)

The tensor product allows one to introduce a basis of the space of k-covariant tensors on space L, built from the basis $\{\varepsilon^i\}$ of L^*:

Proposition 1.7.1. *The set of tensors*

$$\{\ \varepsilon^{i_1} \otimes \varepsilon^{i_2} \otimes \ldots \otimes \varepsilon^{i_k}\ \} \qquad i_j = 1, \ldots, n = \dim L \tag{1.20}$$

where the elements vary over all n^k *choices of the indices, forms a* **basis** *of tensor space* $L^{(0,k)}$*. Consequently,* $\dim L^{(0,k)} = n^k$.

PROOF: A simple exercise □

Example: A general 2-covariant tensor in a 2-dimensional space L is of the form:

$$t = t_{11}\ \varepsilon_1 \otimes \varepsilon_1 + t_{12}\ \varepsilon_1 \otimes \varepsilon_2 + t_{21}\ \varepsilon_2 \otimes \varepsilon_1 + t_{22}\ \varepsilon_2 \otimes \varepsilon_2$$

where t_{ij} are real numbers. Note that tensor t can be conveniently represented as a matrix

$$t = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}.$$

Similarly, a 3-covariant tensor $[t] = t_{ijk}\ \varepsilon_1 \otimes \varepsilon_1 \otimes \varepsilon_k$ in $\mathbb{R}^2$ would require rather a $2 \times 2 \times 2$ numerical "cube." Generalize.

Exercise: Show that (1.20) is indeed a basis of $L^{(0,k)}$ and therefore $\dim L^{(0,k)} = (\dim L)^k$.

1.7.2 Contravariant tensors

The prototype for a **contravariant** tensor is a vector. A **contra-variant tensor** T of type $(k,0)$ at point $p \in M$ is a linear map

$$T: \quad \underbrace{L^* \times \ldots \times L^*}_{k \text{ times}} \longrightarrow \mathbb{R}$$

The set of such maps forms a linear space under the obvious scaling and addition, denoted usually in one of these ways:

$$(L)^{\otimes k}\ \equiv\ \otimes^k L\ \equiv\ L^{(k,0)}$$

The tensor **product** is defined similarly to the covariant case: two contravariant tensors T_1 and T_2 of variance k and l, respectively, define a $(k+l)$-contravariant tensor $T_1 \otimes T_2$

$$T_1 \otimes T_2: \quad \underbrace{L^* \times \ldots \times L^*}_{k \text{ times}} \times \underbrace{L^* \times \ldots \times L^*}_{l \text{ times}} \longrightarrow \mathbb{R}:$$

defined simply by a product of their values:

$$T_1 \otimes T_2\ (\alpha^1, \ldots, \alpha^{k+l}) = T_1\ (\alpha^1, \ldots, \alpha^k)\ \cdot\ T_2\ (\alpha^{k+1}, \ldots, \alpha^{k+l})$$

The tensor product allows us to introduce a basis of k-tensors:

$$\{\ \mathbf{e}_{i_1} \otimes \mathbf{e}_{i_2} \otimes \ldots \otimes \mathbf{e}_{i_k}\ \}$$

where the set varies over all $(\dim L)^k$ choices of the indices. Hence, as before, $\dim L^{(k,0)} = (\dim L)^k$.

Example: A 2-contravariant tensor in 2-dimensional space is in general

$$t = t^{11}\ \mathbf{e}_1 \otimes \mathbf{e}_1 + t^{12}\ \mathbf{e}_1 \otimes \mathbf{e}_2 + t^{21}\ \mathbf{e}_2 \otimes \mathbf{e}_1 + t^{22}\ \mathbf{e}_2 \otimes \mathbf{e}_2$$

and can be represented as a 2×2 matrix, which should not be confused with a similar matrix for a 2-covariant tensor.

1.7.3 General tensors over a vector space

Finally, wse are ready to define a (k,l)-**variant** tensor over a vector space L as a $(k+l)$-linear map

$$T: \quad \underbrace{L^* \times \ldots \times L^*}_{k \text{ times}} \times \underbrace{L \times \ldots \times L}_{l \text{ times}} \longrightarrow \mathbb{R}$$

Basis of (k,l)-variant tensors consists of tensors

$$\{\ \mathbf{e}_{i_1} \otimes \ldots \otimes \mathbf{e}_{i_k} \otimes \varepsilon^{i_{k+1}} \otimes \ldots \otimes \varepsilon^{i_{k+l}}\ \}$$

Example 1: Any **endomorphism** in L should be viewed as a tensor

$$A = A_i^j\ \mathbf{e}_j \otimes \varepsilon^i$$

The map of endomorphism is thus

$$\mathbf{v} \quad \longrightarrow \quad A\mathbf{v} = A(\,\cdot\,, \mathbf{v})$$

Check that in a basis notation we have the familiar:

$$A\mathbf{v} = A_i^j v^i\, \mathbf{e}_j$$

We represent typically this (1,1)-variant tensor as a matrix. Note also that the same tensor may be viewed as a linear map in the *dual* space; namely if $\alpha = \alpha_i \varepsilon^i$ is a form, then

$$\alpha \quad \longrightarrow \quad A\alpha = A(\alpha, \,\cdot\,) = A_i^j \alpha_j\ \varepsilon^i$$

Example 2: Any **inner product** in L should be viewed as a tensor

$$g = g_{ij}\ \varepsilon^j \otimes \varepsilon^i$$

The associated map $\mathring{g} : L \to L^*$ (which physicists call “lowering of an index" is

$$\mathbf{v} \quad \longrightarrow \quad A\mathbf{v} = A(\,\cdot\,, \mathbf{v})$$

which in a basis notation is

$$g(\mathbf{v}) = g_{ij}\, v^i \varepsilon^j$$

We represent typically this (0,2)-variant tensor as a matrix, and this matrix shold not be confused with the matrix of endomorphism.

1.7.4 Wedge product versus tensor product

We can easily verify that for any two 1-forms α and β we have

$$\alpha \wedge \beta \ = \ \alpha \otimes \beta - \beta \otimes \alpha$$

In general we have for any set of k 1-forms ω^i

$$\omega^1 \wedge \ldots \wedge \omega^k = \sum_\sigma \operatorname{sgn}(\sigma)\, \omega^{\sigma(1)} \otimes \omega^{\sigma(2)} \otimes \ldots \otimes \omega^{\sigma(k)}$$

where the sum runs over all permutations of k elements $\{1, 2, \ldots, k\}$.

Any skew-symmetric (0,2)-variant tensor $\omega = \omega_{ij} \varepsilon^i \otimes \varepsilon^j$ with $\omega_{ij} = -\omega_{ji}$ is a 2-form:

$$\omega = \omega_{ij}\, \varepsilon^i \otimes \varepsilon^j = \frac{1}{2}\ \omega_{ij}\ (\varepsilon^i \otimes \varepsilon^j - \varepsilon^j \otimes \varepsilon^i) = \frac{1}{2}\ \omega_{ij}\ \varepsilon^i \wedge \varepsilon^j$$

We can now similarly define exterior k-vectors in a vector space L as a set $\wedge^k L$ of all k-contravariant tensors in L.

Problem: What are the properties of k-vectors? Basis? What can be viewed as space dual to the space of k-vectors? What about the dual basis?

1.7.5 Tensor product of spaces

Let L and K be two vector spaces. The set of maps

$$L^* \times K^* \longrightarrow \mathbb{R}$$

that are linear in both arguments forms a linear space denoted $L \otimes K$ and called a **tensor product of spaces** L and K. Clearly, $\dim L \otimes K = \dim L \cdot \dim K$. Tensor product of spaces is associative:

$$(L_1 \otimes L_2) \otimes L_3 \cong L_1 \otimes (L_2 \otimes L_3)$$

for any three vector spaces L_1, L_2, L_3.

In particular, if we restrict ourselves to a vector space L and its dual L^*, this justifies our notation

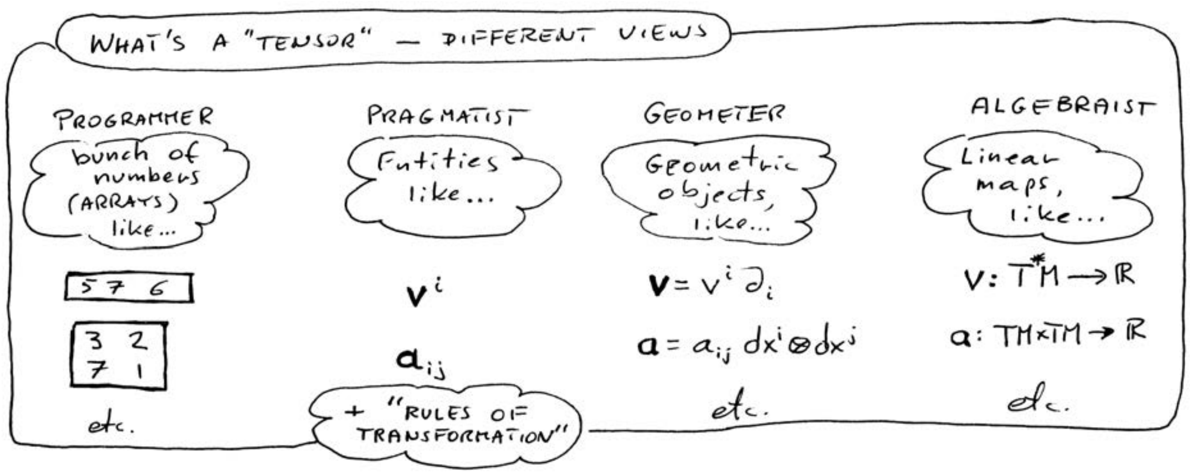
$$L^{(k,l)} = L^{\otimes k} \otimes L^{* \otimes l} = \underbrace{L \otimes \ldots \otimes L}_{k} \otimes \underbrace{L^* \otimes \ldots \otimes L^*}_{l}$$

We may define $L^* \otimes K^*$ as the space of linear maps

$$L \times K \longrightarrow \mathbb{R}$$

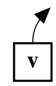
It may be shown that

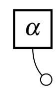
$$(L \otimes K)^* = L^* \otimes K^*$$

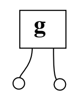


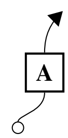
1.7.6 Graphical representation of tensors

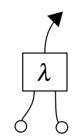
A (k,l)-variant tensor may be represented by a rectangular with k “exits" (arrows) and l “entries" (tails). For instance:

vector $\quad v = v^i\, \mathbf{e}_i$ 

co-vector (1-form) $\quad \alpha = \alpha_i\, \varepsilon^i$ 

biform $\quad \omega = \omega_{ij}\, \varepsilon^i \wedge\, \varepsilon^j$ 

endomorphism $\quad A = A^i_j\, \mathbf{e}_i \otimes\, \varepsilon^j$ 

vector-valued biform $\quad \lambda = \lambda_i^{jk}\, \varepsilon^j \wedge \varepsilon^k \otimes \mathbf{e}_i$ 

(One may also think of the attached strings as the representatives of the indices in the coordinate description of tensors). Any contraction may be now represented by connecting corresponding arrows with tails; for instance, for the tensors listed above we have

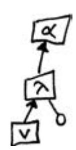
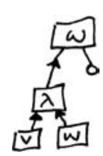
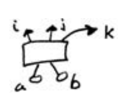
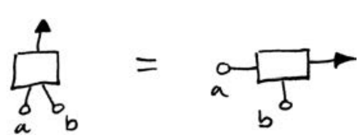
$v \lrcorner \omega =$  $\quad v \lrcorner \lambda \lrcorner \alpha =$  $\quad (v \wedge w) \lrcorner \lambda \lrcorner \omega =$ 

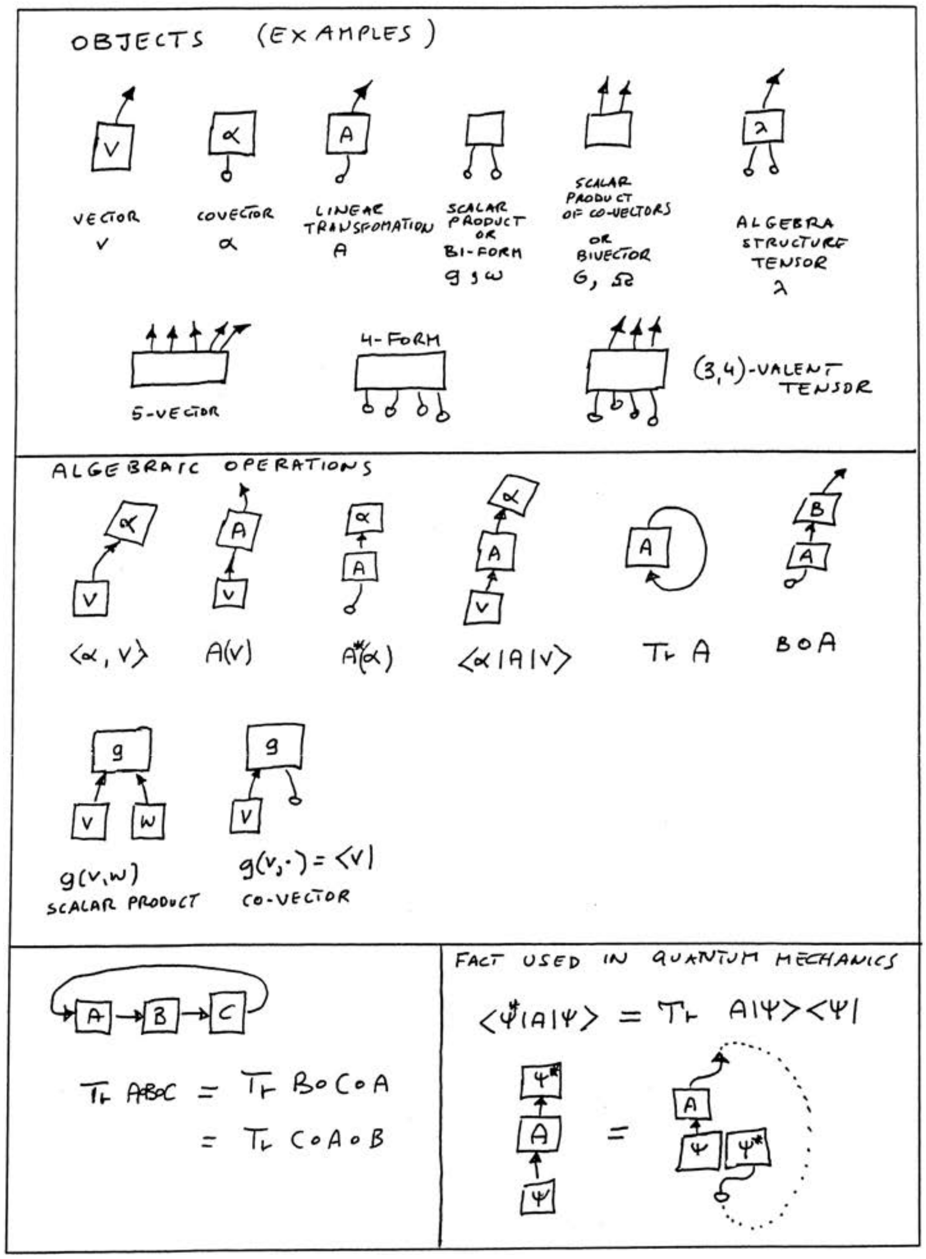
Figure A-1 is a brief presentations of this technique. The order of the contravariant indices corresponds to a clockwise arrangement of the arrows around the box, while the order of the covariant indices corresponds to a counterclockwise arrangement of the tails around the box. Thus, for a (3,2)-variant tensor T:

$$T^{ijk}_{ab} \quad \longrightarrow$$



This makes the shape of the box or the exact places of attachment irrelevant:




OBJECTS (EXAMPLES)
VECTOR
v
COVECTOR
α
LINEAR TRANSFOMATION
A
SCALAR PRODUCT OR BI-FORM
g, ω
SCALAR PRODUCT OF CO-VECTORS OR BIVECTOR
G, Ω
ALGEBRA STRUCTURE TENSOR
λ
5-VECTOR
4-FORM
(3,4)-VALENT TENSOR
ALGEBRAIC OPERATIONS
⟨α, v⟩
A(v)
A*(α)
⟨α|A|v⟩
Tr A
B∘A
g(v,w)
SCALAR PRODUCT
g(v,·) = ⟨v|
CO-VECTOR
Tr A∘B∘C = Tr B∘C∘A
= Tr C∘A∘B
FACT USED IN QUANTUM MECHANICS
⟨ψ|A|ψ⟩ = Tr A|ψ⟩⟨ψ|