MATH450

HW3

CHASE LOTTO

* I'm boxing some answers for my own mental clarity.

① Here are some differential forms on $M \cong \mathbb{R}^4$.

$$\left.\begin{aligned} \alpha &= x^2\,dx + z\,dy \\ \beta &= dy + 3dz + t\,dt \end{aligned}\right\} \text{1-forms}$$

$$\gamma = dx\wedge dy + 2y\,dx\wedge dt \quad \} \text{ 2-form}$$

$$\delta = dx\wedge dy\wedge dz + x\,dy\wedge dz\wedge dt \quad \} \text{ 3-form}$$

Calculate $\alpha\wedge\beta$, $\beta\wedge\gamma$, $\gamma\wedge\gamma$, $\alpha\wedge\delta$, $\beta\wedge\delta$.

(i) $\alpha\wedge\beta = (x^2dx + zdy)\wedge(dy + 3dz + tdt)$

$$= x^2\,dx\wedge dy + 3x^2\,dx\wedge dz + tx^2\,dx\wedge dt + \underbrace{z\,dy\wedge dy}_{0} + 3z\,dy\wedge dz + zt\,dy\wedge dt$$

$$= \boxed{x^2\,dx\wedge dy + 3x^2\,dx\wedge dz + tx^2\,dx\wedge dt + 3z\,dy\wedge dz + zt\,dy\wedge dt}$$

(ii) $\beta\wedge\gamma = (dy + 3dz + tdt)\wedge(dx\wedge dy + 2y\,dx\wedge dt)$

$$= \overset{0}{dy\wedge dx\wedge dy} + 2y\,dy\wedge dx\wedge dt + 3dz\wedge dx\wedge dy + 6y\,dz\wedge dx\wedge dt$$
$$+ t\,dt\wedge dx\wedge dy + \underbrace{2yt\,dt\wedge dx\wedge dt}_{0}$$

$$= -2y\,dx\wedge dy\wedge dt + t\,dx\wedge dy\wedge dt + 3dx\wedge dy\wedge dz - 6y\,dx\wedge dz\wedge dt$$

$$= \boxed{(t-2y)\,dx\wedge dy\wedge dt + 3\,dx\wedge dy\wedge dz - 6y\,dx\wedge dz\wedge dt}$$

(iii) $\gamma\wedge\gamma = (dx\wedge dy + 2y\,dx\wedge dt)\wedge(dx\wedge dy + 2y\,dx\wedge dt)$

$$= dx\wedge dy\wedge dx\wedge dy + 2y\,dx\wedge dy\wedge dx\wedge dt + 2y\,dx\wedge dt\wedge dx\wedge dy$$
$$+ 4y^2\,dx\wedge dt\wedge dx\wedge dt$$

$$= \boxed{0}$$

(iv) $\alpha \wedge \delta = (x^2 dx + z dy) \wedge (dx \wedge dy \wedge dz + x\, dy \wedge dz \wedge dt)$

$= \underbrace{x^2 dx \wedge dx \wedge dy \wedge dt}_{0} + x^3 dx \wedge dy \wedge dz \wedge dt + \underbrace{z dy \wedge dx \wedge dy \wedge dz}_{0}$

$+ \underbrace{xz\, dy \wedge dy \wedge dz \wedge dt}_{0}$

$= \boxed{x^3 dx \wedge dy \wedge dz \wedge dt}$

(v) $\beta \wedge \delta = (dy + 3dz + t dt)(dx \wedge dy \wedge dz + x dy \wedge dz \wedge dt)$

$= 0 + x(0) + 3(0) + 3x(0) + t\, dt \wedge dx \wedge dy \wedge dz + xt(0)$

$= \boxed{-t\, dx \wedge dy \wedge dz \wedge dt}$

② Calculate the exterior derivatives of the given exterior differential forms on a manifold w/ coordinates $\{x, y, z, t\}$.

(i) $f = xy + e^{zt}$

$\rightsquigarrow df = d(xy + e^{zt}) = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial t} dt$

$= \boxed{y dx + x dy + t e^{zt} dz + z e^{zt} dt}$

(ii) $\alpha = x dx + y dy$

$\rightsquigarrow d\alpha = d(x dx + y dy)$

$= d(x) \wedge dx + x d(dx) + d(y) \wedge dy + y d(dy)$

$= dx \wedge dx + 0 + dy \wedge dy + 0$

$= \boxed{0}.$

(iii) $\beta = x dy + y dx$

$\rightsquigarrow d\beta = d(x dy + y dx)$

$= d(x) \wedge dy + x d(dy) + d(y) \wedge dx + y d(dx)$

$= dx \wedge dy + 0 + dy \wedge dx + 0$

$= dx \wedge dy - dx \wedge dy$

$= \boxed{0}.$

(iv) $\gamma = x\,dy - y\,dx$

$\rightsquigarrow d\gamma = d(x\,dy - y\,dx)$

$= d(x)\wedge dy + x\,d(dy) - \left[d(y)\wedge dx + y\,d(dx)\right]$

$= dx\wedge dy + 0 - dy\wedge dx - 0$

$= dx\wedge dy + dx\wedge dy$

$= \boxed{2\,dx\wedge dy}$

(v) $\sigma = xyzt\; dx\wedge dy$

$\rightsquigarrow d\sigma = d(xyzt\; dx\wedge dy)$

$= d(xyzt)\wedge dx\wedge dy + xyzt\; d(dx\wedge dy)$

$= \left[yzt\,dx + xzt\,dy + xyt\,dz + xyz\,dt\right]\wedge dx\wedge dy + 0$

$= 0 + 0 + xyt\,dz\wedge dx\wedge dy + xyz\,dt\wedge dx\wedge dy$

$= \boxed{xyt\,dx\wedge dy\wedge dz + xyz\,dx\wedge dy\wedge dt}$

(vi) $\mu = x\,dx\wedge dy\wedge dt + xyzt\,dy\wedge dz\wedge dt$

$\rightsquigarrow d\mu = \underbrace{d(x\,dx\wedge dy\wedge dt)}_{0} + d(xyzt\,dy\wedge dz\wedge dt)$

$= \left[yzt\,dx + xzt\,dy + xyt\,dz + xyz\,dt\right]\wedge dy\wedge dz\wedge dt + 0$

$= \boxed{yzt\,dx\wedge dy\wedge dz\wedge dt}$

SKIPPED SOME STEPS ON THESE...

(vii) $\sigma = (1 + yx + yzx)\,dx + (x + xz)\,dy + x^2y\,dz$

$\rightsquigarrow d\sigma = d\big((1+yx+yzx)dx\big) + d\big((x+xz)dy\big) + d(x^2y\,dz)$

$= \left[(y+yz)dx + (x+xz)dy + xy\,dz\right]\wedge dx + \left[(1+z)dx + x\,dz\right]\wedge dy + \left[2xy\,dx + x^2\,dy\right]\wedge dz$

$= -(x+xz)\,dx\wedge dy - xy\,dx\wedge dz + (1+z)\,dx\wedge dy - x\,dy\wedge dz + 2xy\,dx\wedge dz + x^2\,dy\wedge dz$

$= \boxed{(1+z-x-xz)\,dx\wedge dy + xy\,dx\wedge dz + (x^2-x)\,dy\wedge dz}$

(viii) $\omega = (x^2+y^2+z^2+t^2)\, dx\wedge dy\wedge dz\wedge dt$

$\rightsquigarrow d\omega = d(x^2+y^2+z^2+t^2)\wedge dx\wedge dy\wedge dz\wedge dt + 0.$

All of the components of $d(x^2+y^2+z^2+t^2)$ would collapse when wedged w/ $dx\wedge dy\wedge dz\wedge dt$.

$\Rightarrow \boxed{d\omega = 0.}$

(3) Consider the most general forms on a 3-dimensional manifold w/ coordinates $\{x^1, x^2, x^3\}$.

0-form: $f = f(x^1, x^2, x^3)$

1-form: $\alpha = a_1 dx^1 + a_2 dx^2 + a_3 dx^3$

2-form: $\beta = A_1\, dx^2\wedge dx^3 + A_2\, dx^3\wedge dx^1 + A_3\, dx^1\wedge dx^2$

3-form: $\omega = g\, dx^1\wedge dx^2\wedge dx^3$

where f, g, a_i and A_i are unspecified differentiable functions

Calculate: df, $d\alpha$, $d\beta$, $d\omega$, ddf.

(i) df

$$\rightsquigarrow df = \frac{\partial f}{\partial x^1}dx^1 + \frac{\partial f}{\partial x^2}dx^2 + \frac{\partial f}{\partial x^3}dx^3 = \frac{\partial f}{\partial x^i}dx^i$$

(ii) $d\alpha$

$$\rightsquigarrow d\alpha = \left(\frac{\partial a_1}{\partial x^1}dx^1 + \frac{\partial a_1}{\partial x^2}dx^2 + \frac{\partial a_1}{\partial x^3}dx^3\right)\wedge dx^1 + 0 + \left(\frac{\partial a_2}{\partial x^1}dx^1 + \frac{\partial a_2}{\partial x^2}dx^2 + \frac{\partial a_2}{\partial x^3}dx^3\right)\wedge dx^2 + 0$$

$$+ \left(\frac{\partial a_3}{\partial x^1}dx^1 + \frac{\partial a_3}{\partial x^2}dx^2 + \frac{\partial a_3}{\partial x^3}dx^3\right)\wedge dx^3 + 0$$

$$= \frac{\partial a_1}{\partial x^2}dx^2\wedge dx^1 + \frac{\partial a_1}{\partial x^3}dx^3\wedge dx^1 + \frac{\partial a_2}{\partial x^1}dx^1\wedge dx^2 + \frac{\partial a_2}{\partial x^3}dx^3\wedge dx^2$$

$$+ \frac{\partial a_3}{\partial x^1}dx^1\wedge dx^3 + \frac{\partial a_3}{\partial x^2}dx^2\wedge dx^3$$

$$= \boxed{\left(\frac{\partial a_3}{\partial x^2} - \frac{\partial a_2}{\partial x^3}\right)dx^2\wedge dx^3 + \left(\frac{\partial a_1}{\partial x^3} - \frac{\partial a_3}{\partial x^1}\right)dx^3\wedge dx^1 + \left(\frac{\partial a_2}{\partial x^1} - \frac{\partial a_1}{\partial x^2}\right)dx^2\wedge dx^1}$$

(iii) $d\beta$

$$\rightsquigarrow d\beta = d(A_1)\wedge dx^2\wedge dx^3 + 0 + d(A_2)\wedge dx^3\wedge dx^1 + 0 + d(A_3)\wedge dx^1\wedge dx^2$$

$$= \left(\frac{\partial A_1}{\partial x^1}dx^1 + \frac{\partial A_1}{\partial x^2}dx^2 + \frac{\partial A_1}{\partial x^3}dx^3\right)\wedge dx^2\wedge dx^3$$

$$+ \left(\frac{\partial A_2}{\partial x^1}dx^1 + \frac{\partial A_2}{\partial x^2}dx^2 + \frac{\partial A_2}{\partial x^3}dx^3\right)\wedge dx^3\wedge dx^1$$

$$+ \left(\frac{\partial A_3}{\partial x^1}dx^1 + \frac{\partial A_3}{\partial x^2}dx^2 + \frac{\partial A_3}{\partial x^3}dx^3\right)\wedge dx^1\wedge dx^2$$

$$= \frac{\partial A_1}{\partial x^1}dx^1\wedge dx^2\wedge dx^3 + \frac{\partial A_2}{\partial x^2}dx^2\wedge dx^3\wedge dx^1 + \frac{\partial A_3}{\partial x^3}dx^3\wedge dx^1\wedge dx^2$$

$$= \boxed{\left(\frac{\partial A_1}{\partial x^1} + \frac{\partial A_2}{\partial x^2} + \frac{\partial A_3}{\partial x^3}\right) dx^1\wedge dx^2\wedge dx^3}$$

$\uparrow$ $(\mathrm{div} A)\cdot\eta$

(iv) $d\omega$

$$\rightsquigarrow d\omega = d(g)\wedge dx^1\wedge dx^2\wedge dx^3 + 0$$

$$= \left(\frac{\partial g}{\partial x^1}dx^1 + \frac{\partial g}{\partial x^2}dx^2 + \frac{\partial g}{\partial x^3}dx^3\right)\wedge dx^1\wedge dx^2\wedge dx^3$$

$$= \boxed{0}.$$

(v) ddf

$$\rightsquigarrow ddf = d\left(\frac{\partial f}{\partial x^1}dx^1 + \frac{\partial f}{\partial x^2}dx^2 + \frac{\partial f}{\partial x^3}dx^3\right)$$

$$= d\left(\frac{\partial f}{\partial x^1}\right)\wedge dx^1 + 0 + d\left(\frac{\partial f}{\partial x^2}\right)\wedge dx^2 + 0 + d\left(\frac{\partial f}{\partial x^3}\right)\wedge dx^3 + 0$$

$$= \left(\frac{\partial^2 f}{\partial (x^1)^2}dx^1 + \frac{\partial^2 f}{\partial x^2\partial x^1}dx^2 + \frac{\partial^2 f}{\partial x^3\partial x^1}dx^3\right)\wedge dx^1 + \left(\frac{\partial^2 f}{\partial x^1\partial x^2}dx^1 + \frac{\partial^2 f}{\partial (x^2)^2}dx^2 + \frac{\partial^2 f}{\partial x^3\partial x^2}dx^3\right)\wedge dx^2$$

$$+ \left(\frac{\partial^2 f}{\partial x^1\partial x^3}dx^1 + \frac{\partial^2 f}{\partial x^2\partial x^3}dx^2 + \frac{\partial^2 f}{\partial (x^3)^2}dx^3\right)\wedge dx^3$$

CLAIRUT'S THM: If f_{xy} & f_{yx} are continuous, $f_{xy} = f_{yx}$

$\uparrow$ The functions are differentiable $\therefore$ continuous $\therefore$ Clairaut's Thm applies.

$$= \frac{\partial^2 f}{\partial x^2\partial x^1}dx^2\wedge dx^1 + \frac{\partial^2 f}{\partial x^3\partial x^1}dx^3\wedge dx^1 + \frac{\partial^2 f}{\partial x^1\partial x^2}dx^1\wedge dx^2 + \frac{\partial^2 f}{\partial x^3\partial x^2}dx^3\wedge dx^2$$

$$+ \frac{\partial^2 f}{\partial x^1\partial x^3}dx^1\wedge dx^3 + \frac{\partial^2 f}{\partial x^2\partial x^3}dx^2\wedge dx^3$$

$$= \left(\frac{\partial^2 f}{\partial x^1\partial x^2} - \frac{\partial^2 f}{\partial x^2\partial x^1}\right)dx^1\wedge dx^2 + \left(\frac{\partial^2 f}{\partial x^3\partial x^1} - \frac{\partial^2 f}{\partial x^1\partial x^3}\right)dx^3\wedge dx^1 + \left(\frac{\partial^2 f}{\partial x^3\partial x^2} - \frac{\partial^2 f}{\partial x^2\partial x^3}\right)dx^3\wedge dx^2$$

$$= \boxed{0}.$$

(4) Let $\alpha = x\,dy + y\,dx$ and $\omega = dx \wedge dy + dz \wedge dt$.

Calculate: $\alpha \wedge \alpha$, $\alpha \wedge \omega$, and $\omega \wedge \omega$.

(i) $\alpha \wedge \alpha$

$$= (x\,dy + y\,dx) \wedge (x\,dy + y\,dx)$$

$$= x^2\,dy \wedge dy + xy\,dy \wedge dx + xy\,dx \wedge dy + y^2\,dx \wedge dx$$

$$= (xy - xy)\,dx \wedge dy$$

$$= \boxed{0}.$$

(ii) $\alpha \wedge \omega$

$$= (x\,dy + y\,dx) \wedge (dx \wedge dy + dz \wedge dt)$$

$$= 0 + 0 + x\,dy \wedge dz \wedge dt + y\,dx \wedge dz \wedge dt$$

$$= \boxed{x\,dy \wedge dz \wedge dt + y\,dx \wedge dz \wedge dt}$$

(iii) $\omega \wedge \omega$

$$= (dx \wedge dy + dz \wedge dt) \wedge (dx \wedge dy + dz \wedge dt)$$

$$= 0 + dx \wedge dy \wedge dz \wedge dt + dz \wedge dt \wedge dx \wedge dy + 0$$

$$= \boxed{2\,dx \wedge dy \wedge dz \wedge dt}$$