Proof of $v_p(1) = 0$.

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1 Definiton of a (tangent) vector.

A (tangent) vector at a point $p \in M$, where M is a smooth manifold, is an operator:

$$v_p : \mathcal{F}M \to \mathbb{R}$$

Our vector v_p, the operator, takes a smooth function (which itself maps our manifold to a scalar quantity) and provides a real number, given these conditions are satisfied:

i. $v_p(f + g) = v_p(f) + v_p(g)$

ii. $v_p(cf) = c(v_p(f)), \quad c \in \mathbb{R}$

iii. $v_p(fg) = v_p(f)g(p) + f(p)v_p(g)$

2 Proof of why $v_p(1) = 0$

From the definition of the vector, we see that it acts like a derivative. Both operate on functions and provide a scalar quantity in return. From derivatives, we know that:

$$\frac{d}{dx}(1) = 0$$

Can we prove then, that the (tangent) vector operating on unity is also nothing, i.e. $v_p(1) = 0$?

Proof. Let our tangent vector v_p act on the following smooth function from $\mathcal{F}M$:

$$f = x^0 + c^1x^1 + c^2x^2 + \cdots + c^nx^n, \quad c^i \in \mathbb{R}$$

From f, the first term $x^0 \in \mathbb{R}$. If we take $f + 1$, then we'd have:

$$f = 1 + x^0 + c^1x^1 + c^2x^2 + \cdots + c^nx^n, \quad c^i \in \mathbb{R}$$

But, $1 + x^0 \in \mathbb{R}$, so really, the constant can be absorbed into the 0^{th}-order term.

$$\implies f + 1 = f \tag{1}$$

So, given $v_p(f + 1)$, by (i):

$$v_p(f + 1) = v_p(f) + v_p(1) \tag{2}$$

Now, using Eq. 1 and Eq. 2:

$$\begin{aligned} v_p(f + 1) &= v_p(f) \\ \implies v_p(f) + v_p(1) &= v_p(f) \\ \implies v_p(1) &= v_p(f) - v_p(f) = 0 \end{aligned}$$

$\square$