Exercise 1: Consider a 2-dimensional manifold M with coordinate chart $\{\, x, y \,\}$. The following objects are given:

- $\mathbf{v} = 2\partial_x + \partial_y$; at point $p = (3, 1)$; $\mathbf{v} \in T_pM$
- $f = x^2 + xy + 2$; scalar function; $f \in \mathcal{F}M$
- $\mathbf{A} = 2x^2\partial_x + xy\partial_y$; vector field; $\mathbf{A} \in \mathcal{X}M$
- $\mathbf{B} = y\partial_x$; vector field; $\mathbf{B} \in \mathcal{X}M$
- $c(t) = (t^2 + t, 2\cos t)$; a curve; $\mathbf{c} \in \mathcal{C}M$

Calculate the following:

- $\mathbf{v}f$
- $\mathbf{A}f$
- $f \circ c$
- $\frac{\mathrm{d}}{\mathrm{d}t}(f \circ c)$ at $t = 0$
- $\dot{\mathbf{c}} \equiv \dot{c}(0)$
- $\dot{\mathbf{c}}f$
- $[\mathbf{A}, \mathbf{B}]$
- Draw the vector field $\mathbf{B}$ in the neighborhood of $(0, 0)$

Solution.

(a)

$$
\begin{aligned}
\mathbf{v}f &= (2\partial_x + \partial_y)(x^2 + xy + 2) \\
&= 2\partial_x(x^2 + xy + 2) + \partial_y(x^2 + xy + 2) \\
&= (2(2x + y) + x)\Big|_{(x,y)=(3,1)} \\
&= 2(2(3) + 1) + 3 \\
&= 17
\end{aligned}
$$

(b)

$$
\begin{aligned}
\mathbf{A}f &= (2x^2\partial_x + xy\partial_y)(x^2 + xy + 2) \\
&= 2x^2\partial_x(x^2 + xy + 2) + xy\partial_y(x^2 + xy + 2) \\
&= 2x^2(2x + y) + xy(x) \\
&= 4x^3 + 3x^2y
\end{aligned}
$$

(c)

$$
\begin{aligned}
f \circ c &= f(c(t)) \\
&= (t^2 + t)^2 + (t^2 + t)(2\cos t) + 2
\end{aligned}
$$

(d)

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}(f \circ c) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}[(t^2 + t)^2 + (t^2 + t)(2\cos t) + 2] \\
&= 2(t^2 + t)(2t + 1) + 2(2t + 1)\cos t - \sin t(t^2 + t)\Big|_{t=0} \\
&= 2(0)(1) + 2(1)(1) - 0(0) \\
&= 2
\end{aligned}
$$

(e) *Remember* $\dot{\mathbf{c}} = \dot{c}^i \partial_i = \dot{c}^i(x^i)\partial_i = \frac{\mathrm{d}(x^i \circ c)(t)}{\mathrm{d}t}\partial_i$. Operating on a function f,

$$\dot{\mathbf{c}}(f) = \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t_0} f \circ c(t)$$

Here, $t_0 = 0$,

$$\begin{aligned}\dot{\mathbf{c}} &= \frac{\mathrm{d}(t^2+t)}{\mathrm{d}t}\partial_x + \frac{\mathrm{d}(2\cos t)}{\mathrm{d}t}\partial_y \\ &= (2t+1)\partial_x - (2\sin t)\partial_y \bigg|_{t_0=0} \\ &= \partial_x\end{aligned}$$

(f)
From (c), we know that

$$(f \circ c)(t) = (t^2+t)^2 + (t^2+t)(2\cos t) + 2$$

So,

$$\begin{aligned}\dot{\mathbf{c}}f &= \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=t_0} (t^2+t)^2 + (t^2+t)(2\cos t) + 2 \\ &= 2(t^2+t)(2t+1) + (2t+1)(2\cos t) - (t^2+t)(2\sin t)\bigg|_{t=t_0} \\ &= 2(t_0^2+t_0)(2t_0+1) + (2t_0+1)(2\cos t_0) - (t_0^2+t_0)(2\sin t_0)\end{aligned}$$

(g) *Remember* the Lie bracket of two vector fields $\mathbf{A}$ and $\mathbf{B}$ is

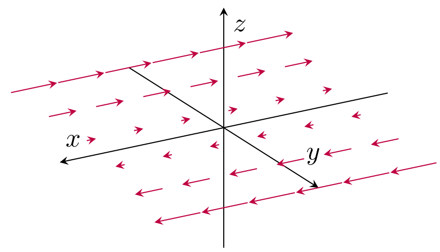
$$[\mathbf{A}, \mathbf{B}] = \left(A^j(\partial_j B^i) - B^j(\partial_j A^i)\right)\partial_i$$

Plugging in our vector fields,

$$\begin{aligned}[\mathbf{A}, \mathbf{B}] &= [A^j(\partial_j y) - B^j(\partial_j 2x^2)]\partial_x + [A^j(\partial_j 0) - B^j(\partial_j xy)]\partial_y \\ &= [A^j(\partial_j y)]\partial_x - [B^j(\partial_j 2x^2)]\partial_x - [B^j(\partial_j xy)]\partial_y \\ &= [2x^2(\partial_x y) + xy(\partial_y y)]\partial_x - [y(\partial_x 2x^2)]\partial_x - [y(\partial_x xy)]\partial_y \\ &= (0 + xy)\partial_x - 4xy\partial_x - y^2\partial_y \\ &= -3xy\partial_x - y^2\partial_y\end{aligned}$$

(h)
In the neighborhood of $(0, 0)$,

$$\mathbf{B} = y\partial_x$$



Exercise 2: Consider $M = \mathbb{R}^3$ with the chart of rectangular coordinates (x, y, z). Express each vector of the basis associated with the chart of spherical coordinates (r, φ, θ), namely $\{ \partial_r, \partial_\varphi, \partial_\theta \}$ in terms of the standard basis $\{ \partial_x, \partial_y, \partial_z \}$.

Solution.

The link between Cartesian coordinates and Spherical coordinates is:

$$\begin{aligned} x &= r \sin\varphi \cos\theta \\ y &= r \sin\varphi \sin\theta \\ z &= r \cos\varphi \end{aligned}$$

In $\mathbb{R}^3$, any position on this manifold can be described in the Cartesian basis as:

$$x^i \partial_i = x\partial_x + y\partial_y + z\partial_z$$

Substituting x^i with the Spherical coordinate equivalents:

$$x^i \partial_i = r \sin\varphi \cos\theta \partial_x + r \sin\varphi \sin\theta \partial_y + r \cos\varphi \partial_z$$

From this we can find equations for the Spherical basis vectors via this change of coordinates:

$$\frac{\partial}{\partial y^k} = \frac{\partial x^i}{\partial y^k} \frac{\partial}{\partial x^i}$$

Where here, $\frac{\partial}{\partial y^1} = \frac{\partial}{\partial r} = \partial_r$, and so on:

For ∂_r:

$$\begin{aligned} \frac{\partial}{\partial y^1} = \frac{\partial}{\partial r} = \frac{\partial x^i}{\partial r} &= \frac{\partial(r \sin\varphi \cos\theta)}{\partial r} \partial_x + \frac{\partial(r \sin\varphi \sin\theta)}{\partial r} \partial_y + \frac{\partial(r \cos\varphi)}{\partial r} \partial_z \\ &= \sin\varphi \cos\theta \partial_x + \sin\varphi \sin\theta \partial_y + \cos\varphi \partial_z \end{aligned}$$

For ∂_φ:

$$\begin{aligned} \frac{\partial}{\partial y^2} = \frac{\partial}{\partial \varphi} = \frac{\partial x^i}{\partial \varphi} &= \frac{\partial(r \sin\varphi \cos\theta)}{\partial \varphi} \partial_x + \frac{\partial(r \sin\varphi \sin\theta)}{\partial \varphi} \partial_y + \frac{\partial(r \cos\varphi)}{\partial \varphi} \partial_z \\ &= r \cos\varphi \cos\theta \partial_x + r \cos\varphi \sin\theta \partial_y - r \sin\varphi \partial_z \\ &= r(\cos\varphi \cos\theta \partial_x + \cos\varphi \sin\theta \partial_y - \sin\varphi \partial_z) \end{aligned}$$

For ∂_θ:

$$\begin{aligned}\frac{\partial}{\partial y^3} = \frac{\partial}{\partial \theta} = \frac{\partial x^i}{\partial \theta} &= \frac{\partial(r\sin\varphi\cos\theta)}{\partial\theta}\partial_x + \frac{\partial(r\sin\varphi\sin\theta)}{\partial\theta}\partial_y + \frac{\partial(r\cos\varphi)}{\partial\theta}\partial_z \\ &= -r\sin\varphi\sin\theta\partial_x + r\sin\varphi\cos\theta\partial_y \\ &= r\sin\varphi(-\sin\theta\partial_x + \cos\theta\partial_y)\end{aligned}$$

Removing the greatest common divisor from each, making them unit vectors, we are able to express the Spherical basis vectors as follows:

$$\begin{aligned}\partial_r &= \sin\varphi\cos\theta\partial_x + \sin\varphi\sin\theta\partial_y + \cos\varphi\partial_z \\ \partial_\varphi &= \cos\varphi\cos\theta\partial_x + \cos\varphi\sin\theta\partial_y - \sin\varphi\partial_z \\ \partial_\theta &= -\sin\theta\partial_x + \cos\theta\partial_y\end{aligned}$$

Exercise 3: Show that the Lie bracket of vector fields satisfies the Jacobi identity:

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0$$

Solution.

Remembering that $A = A^i\partial_i$, $B = B^i\partial_i$ & $C = C^i\partial_i$, let's expand the LHS of the Jacobi identity:

$$\begin{aligned}&= (A(BC-CB) - (BC-CB)A) + (B(CA-AC) - (CA-AC)B) \\ &\qquad + (C(AB-BA) - (AB-BA)C) \\ &= ABC - ACB - BCA + CBA + BCA - BAC \\ &\qquad - CAB + ACB + CAB - CBA - ABC + BAC \\ &\qquad = 0\end{aligned}$$

Beautifully, every term cancels.

Exercise 4: Let $C = [A,B]$ be the Lie bracket of two vector fields. In a chart, the vector fields are given as $A = A^i\partial_i$, $B = B^i\partial_i$ and $C = C^i\partial_i$. Express the coefficients C^i in terms of the coefficients of the other two vector fields.

Solution.

Calculating C, the Lie bracket of A and B:

$$\begin{aligned}C &= [A,B] \\ &= A^j\partial_j B^i\partial_i - B^j\partial_j A^i\partial_i \\ &= (A^j\partial_j B^i - B^j\partial_j A^i)\partial_i\end{aligned}$$

From this, $C^i = A^j\partial_j B^i - B^j\partial_j A^i$, which can also be expressed:

$$C^i = A^j\frac{\partial B^i}{\partial x^j} - B^j\frac{\partial A^i}{\partial x^j}$$