Homework 06

ECE 478/ECE 570 – Principles of Communication Systems

Posted date: 04/09/2025
Due by: 2.00 PM – 04/16/2025
Section: Random signal processing in Communication Systems
Number of problems: 05
Policy: Late submissions will not be accepted.
(Q.4) and (Q.5) are optional for undergraduate students.
(Q.5) is optional for graduate students.
Both student groups can earn bonus marks by solving optional problems.

Question (01): Power spectral density and autocorrelation function [20 marks]

Consider the power spectral density of a random process $X(t)$ shown in Fig. .

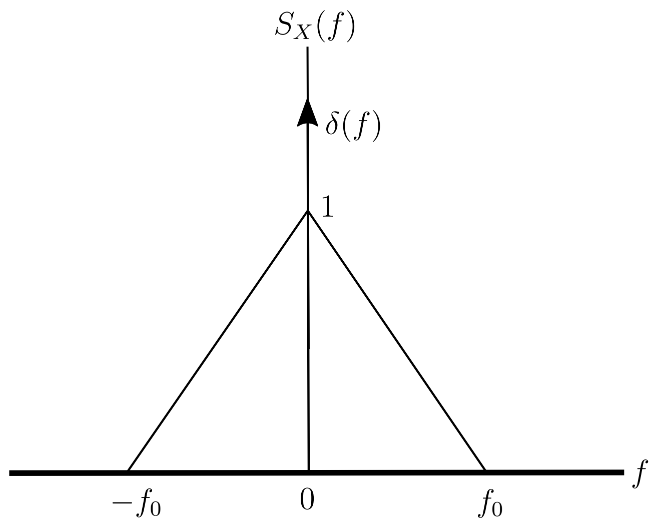

Figure 1: Power spectral density of X(t)

(a.) Determine an sketch the autocorrelation function $R_X(\tau)$ of $X(t)$.

(b.) What is the dc power contained in $X(t)$?

(c.) What is the ac power contained in $X(t)$?

(d.) What sampling rates will give uncorrelated samples of $X(t)$?

(e.) Are the samples statistically independent?

(f.) Find the mean and variance of these samples.

Question (02): RC low-pass filtered white noise [20 marks]

Consider the low-pass RC filter shown in Fig. 2. The transfer function of this low-pass RC filter is given by

$$H(f) = \frac{1}{1 + j2\pi f RC}.$$

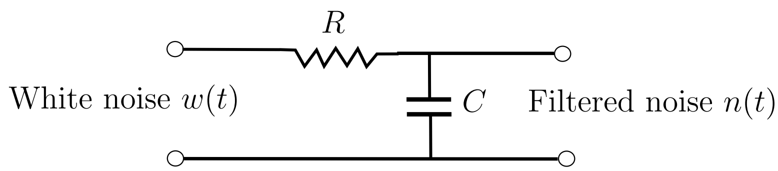

Figure 2: Low-pass RC filter

Suppose that a white Gaussian noise $w(t)$ of zero mean and power spectral density $N_0/2$ is applied to the low-pass RC filter given in Fig. 2. The filtered noise output is denoted by $n(t)$.

(a.) Show that the power spectral density of the noise $n(t)$ appearing at the low-pass RC filter output is given by

$$S_N(f) = \frac{N_0}{2\left[1 + (2\pi f RC)^2\right]}.$$

(b.) Sketch the output power spectral density $S_N(f)$ against frequency f.

(c.) Show that the autocorrelation function of the output noise $n(t)$ is given by

$$R_N(\tau) = \frac{N_0}{4RC} \exp\left(-\frac{|\tau|}{RC}\right).$$

(d.) Sketch the output autocorrelation function $R_N(\tau)$ against time-difference τ.

(e.) Decorrelation time τ_0 is defined as the τ for which the the autocorrelation function drops to 1 percent from its maximum value. Show that the output noise samples taken at a rate of $0.217/RC$ are uncorrelated. Are these output noise samples are Gaussian too?

(f.) What are the mean and variance of the output noise $n(t)$.

(g.) Show that the noise equivalent bandwidth of the low-pass RC filter in Fig. 2 is given by

$$B_N = \frac{1}{4RC}.$$

Question (03): Low-pass filtering of sinusoidal signals with random phases [20 marks]

(a). Consider a sinusoidal signal with random phase, defined by

$$X(t) = A\cos(2\pi f_c t + \Theta) \tag{1}$$

where A and f_c are constants. Further, Θ is a random variable that is uniformly distributed over the interval $(-\pi, \pi)$ and its probability density function is given by

$$f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi \leq \theta \leq \pi \\ 0, & \text{, elsewhere} \end{cases}$$

The autocorrelation function of $X(t)$ in (1) is defined as $R_X(\tau) = \mathbf{E}\left[X(t+\tau)X(t)\right]$.

(a.1) Show that the autocorrelation function of $X(t)$ is given by

$$R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau).$$

(a.2) Determine and sketch the power spectral density of $X(t)$.

(b). Suppose that the sinusoidal signal with random phase $X(t)$ in (1) is passed through the low-pass RC filter given in Fig. 2. The filtered output is denoted by $Y(t)$.

(b.1) Show that the power spectral density of the filtered output signal $Y(t)$ is given by

$$S_Y(f) = \frac{\delta(f - f_c) + \delta(f + f_c)}{2\left[1 + 2\pi f_c RC\right]}.$$

(b.2) Determine and sketch the autocorrelation function of the filtered output signal $Y(t)$.

(b.3) Determine the rate at which the output signal $Y(t)$ can be sampled to obtain uncorrelated samples.

(c). Suppose that $X(t)$ in (1) is passed through a ideal bandpass filter of bandwidth W, unit zero frequency response, and center frequency f_c. The filtered output is denoted by $Y(t)$.

(c.1) Determine and sketch the power spectral density of the filtered output signal $Y(t)$.

(c.2) Determine and sketch the autocorrelation function of the filtered output signal $Y(t)$.

(c.3) Determine the rate at which the output signal $Y(t)$ can be sampled to obtain uncorrelated samples.

Question (04): Mixing of random processes with sinusoidal processes [25 marks]

(a). Consider that a wise-sense stationary (WSS) random process $X(t)$ is applied to a mixer with a sinusoidal input with random noise. The output process of the mixer is thus written as

$$Y(t) = A_c X(t)\cos(2\pi f_c t + \Theta) \tag{2}$$

where the phase Θ is a random variable that is uniformly distributed over the interval $(0, 2\pi)$. Further, the sources of $X(t)$ and Θ are independent. The autocorrelation function of the output is defined as $R_Y(\tau) = \mathbf{E}\left[Y(t+\tau)Y(t)\right]$.

(a.1) Show that the autocorrelation function of $Y(t)$ is given by

$$R_Y(\tau) = \frac{A_c^2}{2} R_X(\tau)\cos(2\pi f_c \tau),$$

where $R_X(\tau)$ is the autocorrelation function of $X(t)$.

(a.2) Is $Y(t)$ a wide-sense stationary process?

(a.3) Show that the power spectral density of $Y(t)$ is given by

$$S_Y(f) = \frac{A_c^2}{4}\left[S_X(f - f_c) + S_X(f + f_c)\right],$$

where $S_X(f)$ is the power spectral density of $X(t)$.

(b). White Gaussian noise of zero mean and power spectral density $N_0/2$ is applied to the filtering scheme shown in Fig. 3.

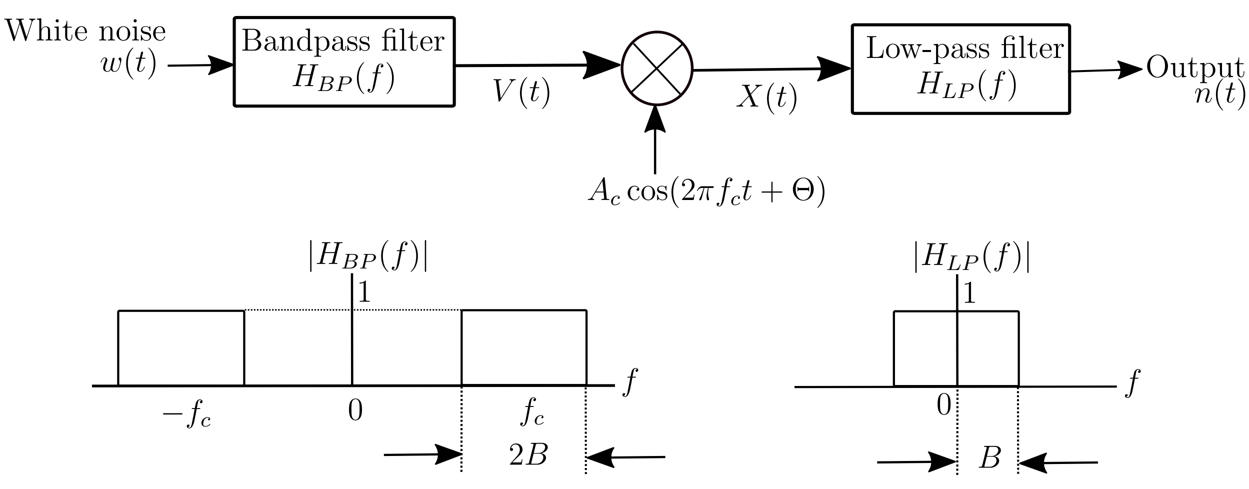

Figure 3: Filter-mixer cascaded system

(b.1) Determine and sketch the power spectral density $S_V(f)$ of $V(t)$.

(b.2) Determine and sketch the autocorrelation function $R_V(\tau)$ of $V(t)$.

(b.3) Determine and sketch the autocorrelation function $R_X(\tau)$ of $X(t)$.

(b.4) Determine and sketch the power spectral density $S_X(f)$ of $X(t)$.

(b.5) Determine and sketch the power spectral density $S_N(f)$ of $n(t)$.

(b.6) Determine and sketch the autocorrelation function $R_N(\tau)$ of $n(t)$.

(b.7) Find the mean and variance of $n(t)$.

(b.8) What is the rate at which $n(t)$ can be sampled so that the resulting samples are essentially uncorrelated.

Question (05): Random binary signal [25 marks]

(a.) Consider the random binary signal $x(t)$ shown in Fig. 4. This signal is a sample function of a random process $X(t)$ consisting of a random sequence of binary symbols 1 and 0.

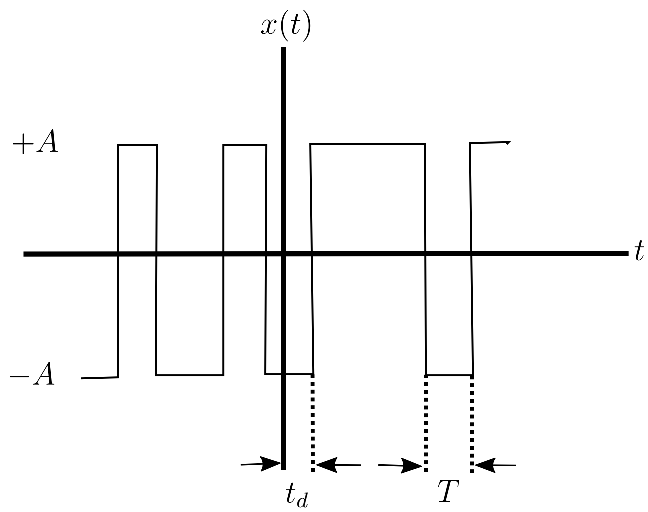

Figure 4: Sample function of a random binary signal

The symbols 1 and 0 are represented by pulses of amplitude $+A$ and $-A$. The occurrence of symbols 1 and 0 are independent and equally likely. The starting time t_d of the first complete pulse for positive time is equally likely to lie anywhere between 0 and T. Here T is the pulse duration. Therefore, t_d is random and is uniformly distributed and its probability density function is given by

$$f_{t_d}(t) = \begin{cases} \frac{1}{T}, & 0 \le t \le T \\ 0, & \text{elsewhere} \end{cases}$$

(a.1) Determine the mean of $X(t)$.

(a.2) The autocorrelation function of $X(t)$ is defined as $R_X(t_k, t_i) = \mathbf{E}[X(t_k)X(t_i)]$ where $X(t_k)$ and $X(t_i)$ are random variables obtained by observing the random process $X(t)$ at times t_k and t_i, respectively.

(a.2.1) Determine the autocorrelation function of $X(t)$ for the case of $|t_k - t_i| > T$.

(a.2.2) Determine the autocorrelation function of $X(t)$ for the case of $|t_k - t_i| < T$ with $t_k = 0$ and $t_i < t_k$.

(a.2.3) Show that the autocorrelation fucntion of $X(t)$ for all t is given by

$$R_X(\tau) = \begin{cases} A^2 \left(1 - \frac{|\tau|}{T}\right), & |\tau| < T \\ 0, & |\tau| \geq T \end{cases}$$

where $\tau = t_k - t_i$.

(a.3) Determine the power spectral density of $X(t)$.

(a.4) Determine the average power of the random binary wave.

(b.) A binary wave consists of a random sequence of symbols 1 and 0, similar to that described in part (a) with one difference: symbol 1 is now represented by a pulse of amplitude A and symbol zero is represented by zero volt. All other parameters are the same as in part (a).

(b.1) Show that the autocorrelation function of the above binary wave is given by

$$R_X(\tau) = \begin{cases} \frac{A^2}{4} + \frac{A^2}{4} \left(1 - \frac{|\tau|}{T}\right), & |\tau| < T \\ \frac{A^2}{4}, & |\tau| \geq T \end{cases}$$

(b.2) Show that its power spectral density is given by

$$S_X(f) = \frac{A^2}{4}\delta(f) + \frac{A^2 T}{4}\text{sinc}^2(fT)$$

(b.3) What percentage of power contained in the dc component of this binary wave.