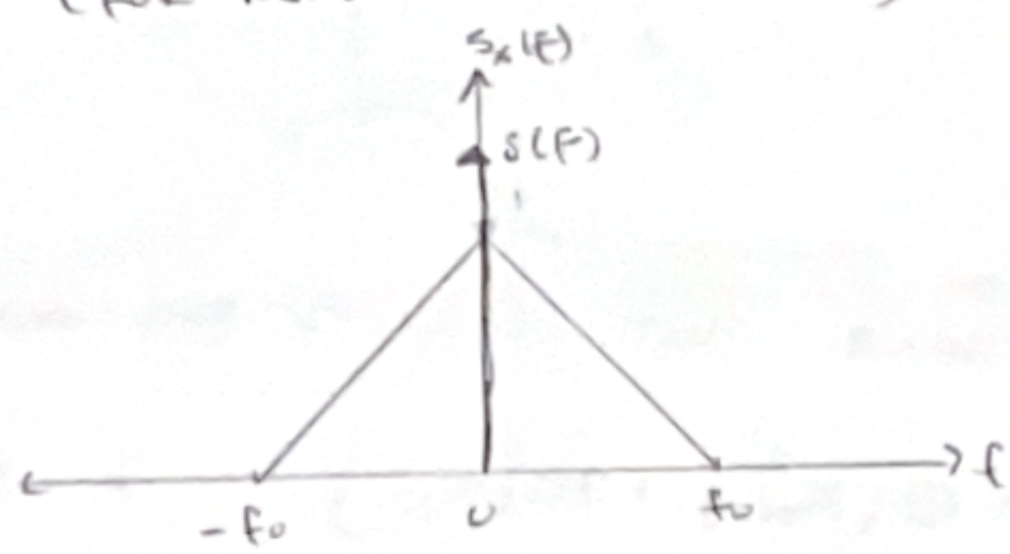
ELE478 | HW6 | C. LOTITO

① POWER SPECTRAL DENSITY (for random process $X(t)$)

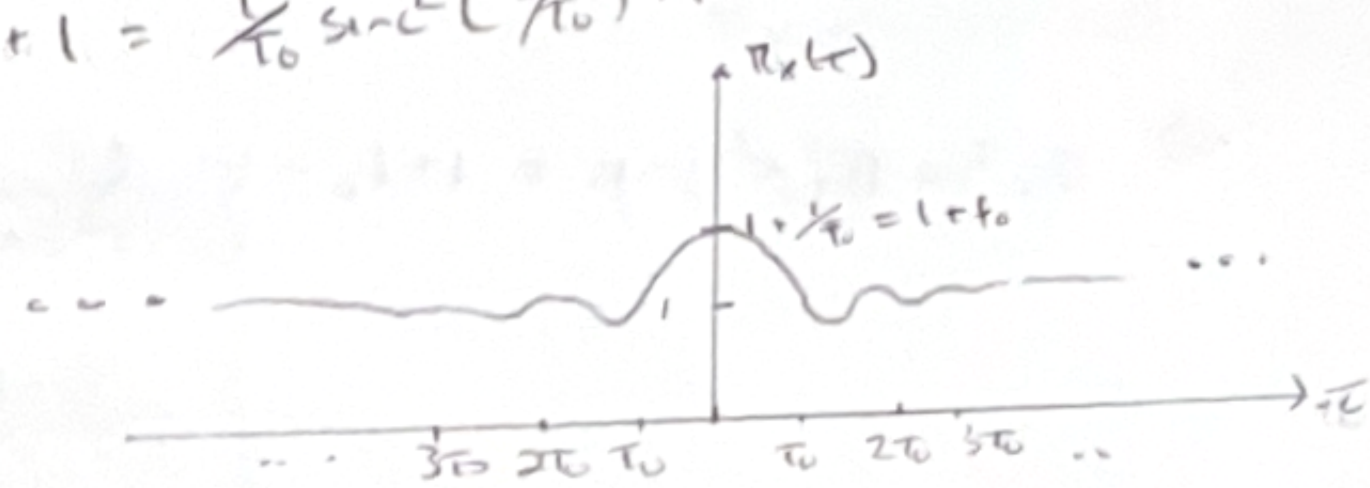


Ⓐ FIND & SKETCH THE AUTOCORRELATION FUNCTION $R_x(\tau)$ OF $X(t)$.

$$R \rightleftharpoons S$$

$$\rightarrow \mathcal{F}\{S_x(f)\} = \mathcal{F}\{\Delta(f/f_0) + \delta(f)\} = f_0 \operatorname{sinc}^2(f_0\tau) + 1 = \tfrac{1}{T_0}\operatorname{sinc}^2(\tau/T_0) + 1$$

$$\rightarrow \boxed{R_x(\tau) = \tfrac{1}{T_0}\operatorname{sinc}^2(\tau/T_0) + 1}$$



Ⓑ WHAT IS THE DC POWER CONTAINED IN $X(t)$?

$$P_{DC} = \int_{-\infty}^{\infty} S_x(f)\Big|_{f=0} df = \int_{-\infty}^{\infty} \delta(f)\,df = 1 \Rightarrow \boxed{P_{DC} = 1}$$

Ⓒ WHAT IS THE AC POWER IN $X(t)$?

$$P_{AC} = \int_{-\infty}^{\infty} S_x(f)\Big|_{f\neq 0} df = \int_{-\infty}^{\infty} \Delta(f/f_0)\,df = \int_{-f_0}^{0}\left(\tfrac{f}{f_0}+1\right)df + \int_{0}^{f_0}\left(1 - f/f_0\right)df$$

$$= \left(\frac{f^2}{2f_0} + f\right)\Big|_{-f_0}^{0} + \left(f - \frac{f^2}{2f_0}\right)\Big|_{0}^{f_0}$$

$$= \frac{-f_0^2}{2f_0} + f_0 + f_0 - \frac{f_0^2}{2f_0} = 2f_0 - \frac{2f_0^2}{2f_0} = f_0 \rightarrow \boxed{P_{AC} = f_0}$$

Ⓓ WHAT SAMPLING RATES WILL GIVE UNCORRELATED SAMPLES OF $X(t)$?

THE AUTOCORRELATION FUNCTION HAS ZEROS AT... $\left(R_x(\tau) = 1 + \tfrac{1}{T_0}\operatorname{sinc}^2(\tau/T_0)\right)$

$$1 + \tfrac{1}{T_0}\operatorname{sinc}^2(\tau/T_0) = 0$$

$$\rightarrow \operatorname{sinc}^2(\tau/T_0) = -T_0$$

$$\rightarrow \operatorname{sinc}(\tau/T_0) = \sqrt{-T_0}$$

X

THE FUNCTION IS NONZERO $\forall \tau \in \mathbb{R}$, SO IT CAN NEVER BE UNCORRELATED

Ⓔ ARE THE SAMPLES STATISTICALLY INDEPENDENT?

$\boxed{\text{NO}}$ SINCE THE SAMPLES ARE CORRELATED.

(F) FIND THE MEAN μ_x AND VARIANCE σ_x^2.

$\mu_x = \mathbb{E}[X(t)]$, $\sigma_x^2 = \mathbb{E}[X^2(t)] - \mu_x^2$, $\mathbb{E}[X^2(t)] = R_x(0)$

$\rightsquigarrow \mathbb{E}[X^2(t)] = R_x(0) = 1 + f_0$

THE MEAN HAPPENS FOR $f = 0$ SINCE THAT ACCOUNTS FOR ALL TIME $T \to \infty$.

$\Rightarrow \mathbb{E}[X(t)] = \mu$ & $\mathbb{E}[X^2(t)] = \mathbb{E}[X_{DC}^2] + \mathbb{E}[X_{AC}^2]$ & $S(f) = S_{DC} + S_{AC}$

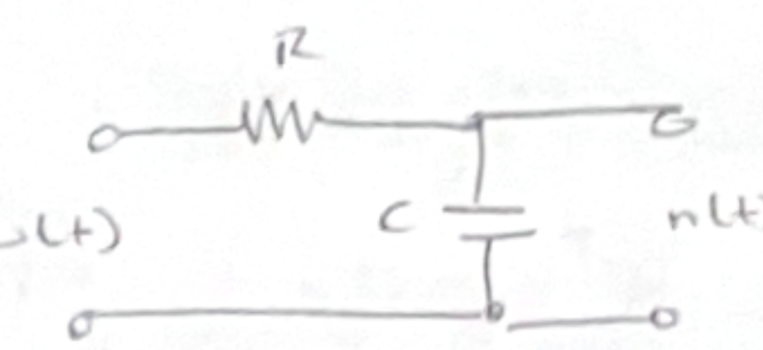
$= \underline{\mu^2} + \mathbb{E}[X_{AC}^2]$ $\qquad = \underset{\text{area}=1}{\underline{\delta(f)}} + \Lambda(f/f_0)$

$\Rightarrow \mu^2 = 1 \rightsquigarrow \boxed{\mu = 1}$

$\Rightarrow \sigma_x^2 = \mathbb{E}[X^2] - \mu = 1 + f_0 - 1 = f_0 \Rightarrow \boxed{\sigma_x^2 = f_0}$

(2) RC LPF WHITE NOISE

$$H(f) = \frac{1}{1 + j2\pi fRC}$$



w(t) IS GAUSSIAN WHITE NOISE w/ ZERO MEAN AND $S_w(f) = N_0/2$. n(t) IS THE FILTERED NOISE.

(A) SHOW THAT THE PSD OF n(t) IS:

$$S_N(f) = \frac{N_0}{2[1 + (2\pi fRC)^2]}$$

THE SYSTEM: $w(t) \longrightarrow \boxed{h(t)} \longrightarrow n(t)$

$$\Rightarrow H(f) = \frac{1 - j2\pi fRC}{(1 + j2\pi fRC)(1 - j2\pi fRC)} = \frac{1 - j2\pi fRC}{1 + (2\pi fRC)^2}$$

$$|H(f)|^2 = H(f)H^*(f) = \left(\frac{1 - j2\pi fRC}{1 + (2\pi fRC)^2}\right)\left(\frac{1 + j2\pi fRC}{1 + (2\pi fRC)^2}\right) = \frac{1 + (2\pi fRC)^2}{(1 + (2\pi fRC)^2)^2} = \frac{1}{1 + (2\pi fRC)^2}$$

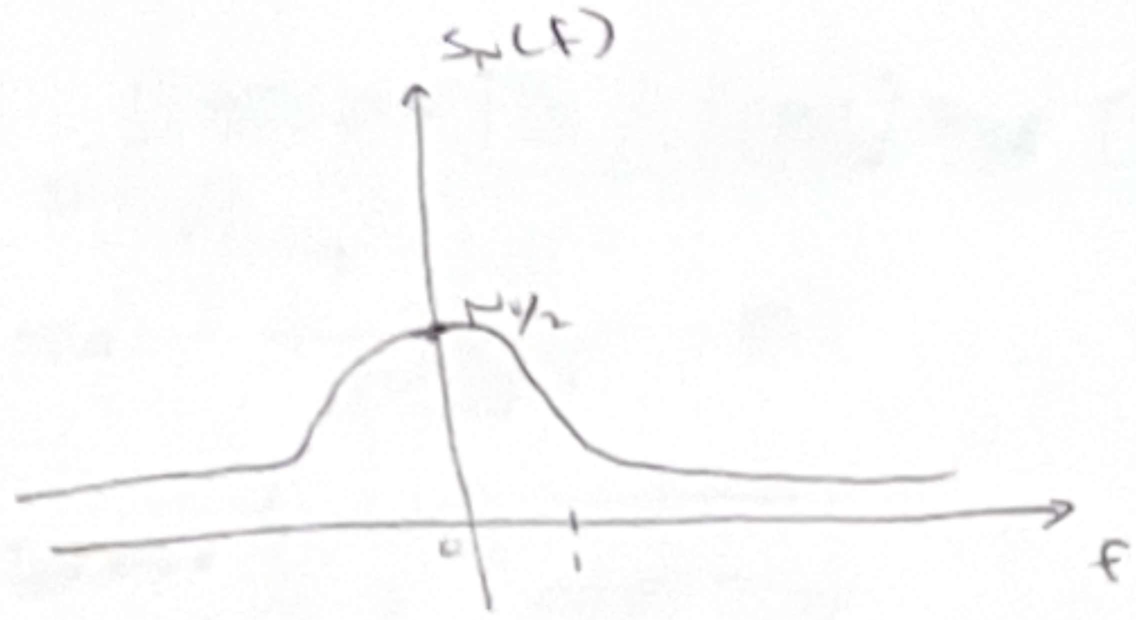
$$\rightarrow S_N(f) = |H(f)|^2 S_w(f) = \left(\frac{1}{1 + (2\pi fRC)^2}\right)\left(N_0/2\right)$$

$$\Rightarrow \boxed{S_N(f) = \frac{N_0}{2[1 + (2\pi fRC)^2]}}$$

(B) SKETCH $S_N(f)$.

$$S_N(f) = \frac{N_0/2}{1+(2\pi fRC)^2}$$

$$1+(2\pi fRC)^2 > 0$$



(C) SHOW AUTOCORRELATION OF $n(t)$ IS

$$R_N(\tau) = \frac{N_0}{4RC}\exp\left(\frac{-|\tau|}{RC}\right)$$

$$^{*}\exp(-a|t|) \rightleftharpoons \frac{2a}{a^2+(2\pi f)^2}$$

$$\Rightarrow \mathcal{F}\{S_N(f)\} = \mathcal{F}\left\{\frac{N_0/2}{1+(2\pi fRC)^2}\right\} = \frac{N_0}{2}\mathcal{F}\left\{\frac{1}{1+(2\pi fRC)^2}\right\}$$

$$= \frac{N_0}{2}\mathcal{F}\left\{\frac{1/(RC)^2}{(1/RC)^2+(2\pi f)^2}\right\}$$

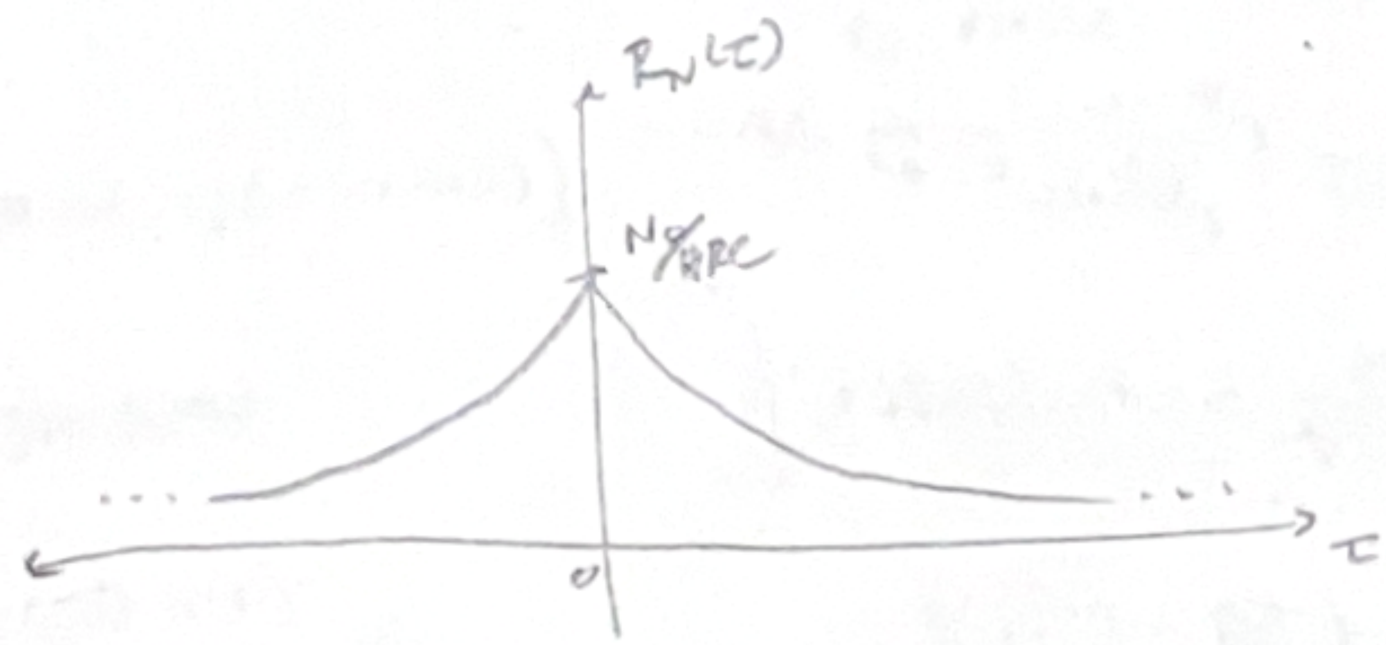
$$= \frac{N_0}{2RC}\mathcal{F}\left\{\frac{1/RC}{(1/RC)^2+(2\pi f)^2}\right\}$$

$$= \frac{N_0}{2RC}\mathcal{F}\left\{\frac{2}{2}\frac{1/RC}{(1/RC)^2+(2\pi f)^2}\right\}$$

$$= \frac{N_0}{4RC}\mathcal{F}\left\{\frac{2/RC}{(1/RC)^2+(2\pi f)^2}\right\}$$

$$\boxed{R_N(\tau) = \frac{N_0}{4RC}\exp\left(-\frac{|\tau|}{RC}\right)}$$

(D) SKETCH R_N AGAINST TIME DIFFERENCE τ.



(E) SHOW SAMPLES TAKEN AT A RATE OF $0.217/RC$ ARE UNCORRELATED. ARE THEY GAUSSIAN?

$$\Rightarrow f_s = \frac{0.217}{RC} \rightsquigarrow \tau_s = 4.608\,RC$$

$$\Rightarrow \cancel{\tfrac{N_0}{4RC}}\exp(-|\tau|/RC) = 0.01$$

$\tau > 0$

$$\Rightarrow \ln\left(\exp(-\tau/RC)\right) = \ln(0.01)$$

$$\Rightarrow -\tau/RC = \ln(0.01)$$

$$\tau = -\ln(0.01)RC \approx 4.605\,RC$$

$\tau_s > \tau$, SO SAMPLES ARE UNCORRELATED AS THEY'RE SAMPLED AT A PERIOD LONGER THAN THE DECORRELATION TIME.

(F) WHAT ARE THE MEAN + VARIANCE OF $n(t)$?

$*\ \mathbb{E}[W(f)] = 0$!!!

$$\Rightarrow \mathbb{E}[n(t)] = \mathbb{E}[N(f)] = \mathbb{E}[W(f)H(f)]$$
$$= \mathbb{E}[0 \cdot H(f)]$$
$$= \mathbb{E}[0]$$
$$= 0.$$

$$\Rightarrow \boxed{\mu_N = 0}$$

$$\Rightarrow \sigma_N^2 = \mathbb{E}[n^2(t)] - (\mathbb{E}[n(t)])^2$$
$$= \mathbb{E}[n^2(t)] - 0$$
$$= R_N(0)$$
$$\boxed{\sigma_N^2 = \frac{N_0}{4RC}}$$

(G) SHOW NOISE EQUIVALENT BANDWIDTH OF LOWPASS RC FILTER

$$B_N = \frac{1}{4RC}$$

FROM ONLINE SOURCE ↳ $B_N = \dfrac{\int_{-\infty}^{\infty} |H(f)|^2 df}{|H(f_0)|^2}$, where f_0 maximizes $H(f)$.

$H(f)$ is maximized for $f_0 = 0$. → $H(0) = 1$ ∴ $|H(0)|^2 = 1$.

$* \ \dfrac{1}{a^2+x^2} = \dfrac{1}{a}\arctan(x/a) + C$

$$\Rightarrow B_N = \int_{-\infty}^{\infty} \frac{1}{1+(2\pi f RC)^2} df$$
$$= \int_{-\infty}^{\infty} \frac{\left(\frac{1}{2\pi RC}\right)^2}{\left(\frac{1}{2\pi RC}\right)^2 + f^2} df$$
$$= \frac{1}{(2\pi RC)^2} \int_{-\infty}^{\infty} \frac{1}{\left(\frac{1}{2\pi RC}\right)^2 + f^2} df$$
$$= \frac{2\pi RC}{(2\pi RC)^2} \arctan(2\pi RCf)\Big|_{-\infty}^{\infty}$$
$$= \frac{1}{2\pi RC}\left(\pi/2 - (-\pi/2)\right)$$
$$= \pi / 2\pi RC$$
$$= \frac{1}{2RC} \boxtimes .$$

↑ is 1/4RC a typo?

3) LPF OF SIGNALS W/ RANDOM PHASES

A) CONSIDER A SINUSOIDAL SIGNAL W/ RANDOM PHASE Θ

$\hookrightarrow$ $X(t) = A\cos(2\pi f_c t + \Theta)$, $A, f_c \in \mathbb{R}$, $\Theta = \{(\theta) : -\pi < \theta < \pi\}$

PDF $\hookrightarrow$ $f_\Theta(\theta) = \begin{cases} 1/2\pi & ; -\pi < \theta < \pi \\ 0 & ; \text{else} \end{cases}$

THE AUTOCORRELATION OF $X(t)$ IS $R_X(\tau) = \mathbb{E}[X(t+\tau)X(t)]$

A.1) SHOW $R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau)$

$$\rightarrow R_X(\tau) = \mathbb{E}[X(t+\tau)X(t)]$$

$$= \mathbb{E}[(A\cos(2\pi f_c(t+\tau)+\Theta))(A\cos(2\pi f_c t+\Theta))]$$

$$= \mathbb{E}[A^2\cos(2\pi f_c t + 2\pi f_c \tau + \Theta)\cos(2\pi f_c t + \Theta)]$$

* $\cos\theta\cos\phi = \frac{1}{2}(\cos(\theta-\phi) + \cos(\theta+\phi))$

$$= \mathbb{E}\left[\frac{A^2}{2}\left(\cos(2\pi f_c \tau) + \cos(4\pi f_c t + 2\pi f_c \tau + 2\Theta)\right)\right]$$

* $\mathbb{E}[g(Y)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$

$$= \mathbb{E}\Big[\underbrace{\frac{A^2}{2}\cos(2\pi f_c \tau) + \frac{A^2}{2}\cos(4\pi f_c t + 2\pi f_c \tau + 2\Theta)}_{g(\theta) = \frac{A^2}{2}\cos(2\pi f_c\tau) + \frac{A^2}{2}\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)}\Big]$$

$$= \frac{A^2}{2}\int_{-\pi}^{\pi} g(\theta) f_\Theta(\theta)\,d\theta$$

$$= \frac{A^2}{2}\int_{-\pi}^{\pi}\left(\cos(2\pi f_c \tau) + \cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)\right)\left(\frac{1}{2\pi}\right)d\theta$$

$$= \frac{A^2}{4\pi}\int_{-\pi}^{\pi}\cos(2\pi f_c \tau) + \cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)\,d\theta$$

$\rightarrow$ $\cos(2\pi k + 2x)$ IS EVEN?

$\hookrightarrow$ $\cos(2\pi k - 2x)$

$\uparrow$ must equal $\cos(2\pi k - 2x)$ AS IT HAS Σ INTEGER SHIFT OF PERIOD 2π, .. SO

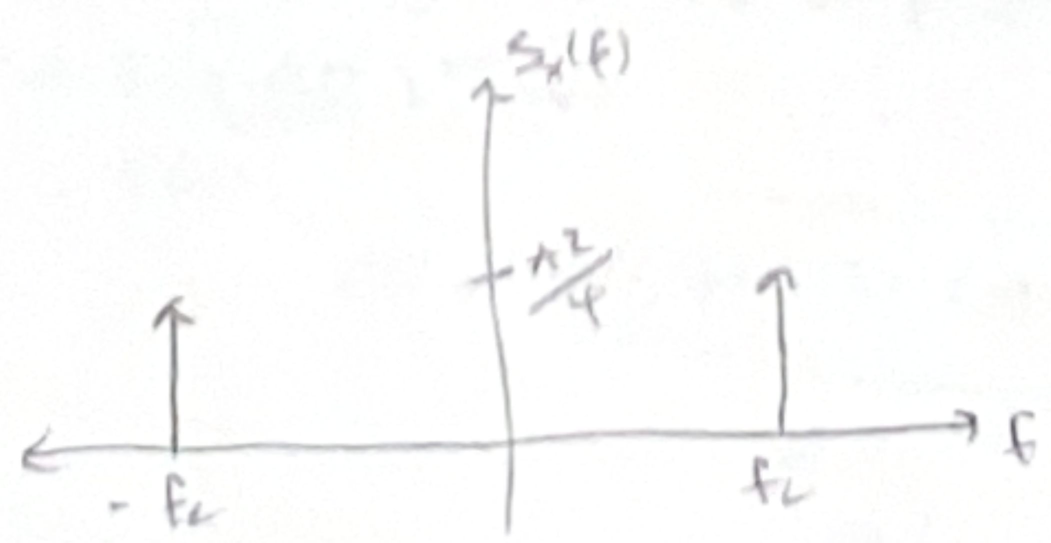
$$= \frac{A^2}{4\pi}\left(\theta\cos(2\pi f_c \tau)\right)\Big|_{-\pi}^{\pi} + \left(\frac{A^2}{4\pi}\cdot\frac{1}{2}\underbrace{\cos(4\pi f_c t + 2\pi f_c \tau + 2\theta)\Big|_{-\pi}^{\pi}}_{\rightarrow 0 \text{ B/C EVEN}}\right)$$

$$= \frac{A^2}{4\pi}\left[\cos(2\pi f_c \tau)(\pi - (-\pi))\right]$$

$$= \frac{2\pi A^2}{4\pi}\cos(2\pi f_c \tau)$$

$$= \frac{A^2}{2}\cos(2\pi f_c \tau) \quad \blacksquare$$

A.2) SKETCH POWER SPECTRAL DENSITY.



(B) SUPPOSE $x(t)$ IS SENT THROUGH RC LPF. OUTPUT IS $y(t)$.

(B.1) SHOW PSD $y(t)$ IS $S_Y(f) = \dfrac{\delta(f-f_c)+\delta(f+f_c)}{2(1+2\pi f_c RC)}$

$* \ |H(f)|^2 = \dfrac{1}{1+(2\pi f RC)^2}$

$$S_Y(f) = |H(f)|^2 S_X(f)$$

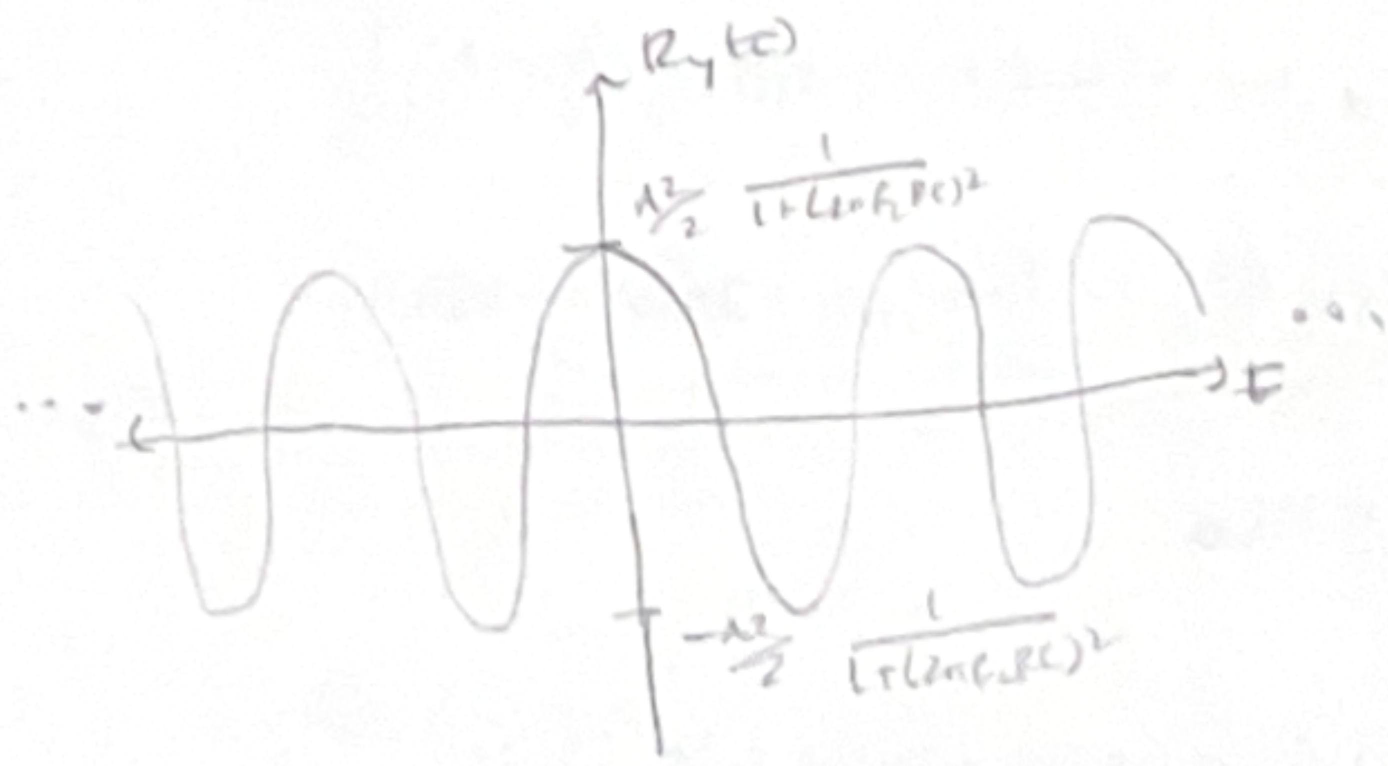
$$= \frac{1}{1+(2\pi f RC)^2}\left(\frac{A^2}{4}\left(\delta(f-f_c)+\delta(f+f_c)\right)\right)$$

$$= \frac{A^2}{4}\cdot\frac{1}{1+(2\pi f_c RC)^2}\,\delta(f-f_c) + \frac{A^2}{4}\,\frac{1}{1+(2\pi(-f_c)RC)^2}\,\delta(f+f_c)$$

$$\boxed{S_Y(f) = \frac{A^2}{4}\,\frac{\delta(f-f_c)+\delta(f+f_c)}{1+(2\pi f_c RC)^2}}$$

(B.2) SKETCH $R_Y(\tau)$

$$\Rightarrow \mathcal{F}\{S_Y(f)\} = R_Y(\tau) = \frac{A^2}{2}\,\frac{1}{1+(2\pi f_c RC)^2}\cos(2\pi f_c \tau)$$



(B.3) FIND RATE OF UNCORRELATED SAMPLES.

$\hookrightarrow \cos(2\pi f_c \tau) = 0$ $\qquad \cos\theta = 0$ for $\theta = k\pi/2$

$$2\pi f_c \tau = k\frac{\pi}{2}$$

$$\tau = \frac{k}{4f_c} \Rightarrow \boxed{\tau_0 = \frac{1}{4f_c}} \to \boxed{f_0 = 4f_c}$$

(C) SUPPOSE $x(t)$ IS PASSED THROUGH IDEAL LPF OF BANDWIDTH W, UNIT FREQ RESPONSE, AND CENTER FREQ f_c. OUTPUT IS $y(t)$.

(C.1) DETERMINE & SKETCH PSD OF $y(t)$.

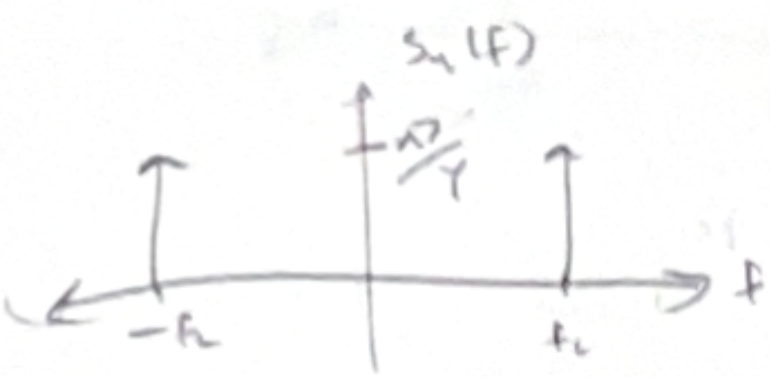
$$\Rightarrow H(f) = \mathrm{rect}\left(\frac{f-f_c}{W}\right) + \mathrm{rect}\left(\frac{f+f_c}{W}\right) \rightsquigarrow |H(f)|^2 = \mathrm{rect}^2\left(\frac{f-f_c}{W}\right) + \underbrace{2\,\mathrm{rect}\left(\frac{f-f_c}{W}\right)\mathrm{rect}\left(\frac{f+f_c}{W}\right)}_{0} + \mathrm{rect}^2\left(\frac{f+f_c}{W}\right)$$

b/c

$$\Rightarrow S_Y(f) = |H(f)|^2 S_X(f) = \frac{A^2}{4}\Big(\mathrm{rect}^2\left(\tfrac{f-f_c}{W}\right)\delta(f-f_c) + \underbrace{\mathrm{rect}^2\left(\tfrac{f-f_c}{W}\right)\delta(f+f_c)}_{0}$$
$$+ \underbrace{\mathrm{rect}^2\left(\tfrac{f+f_c}{W}\right)\delta(f-f_c)}_{0} + \mathrm{rect}^2\left(\tfrac{f+f_c}{W}\right)\delta(f+f_c)\Big)$$

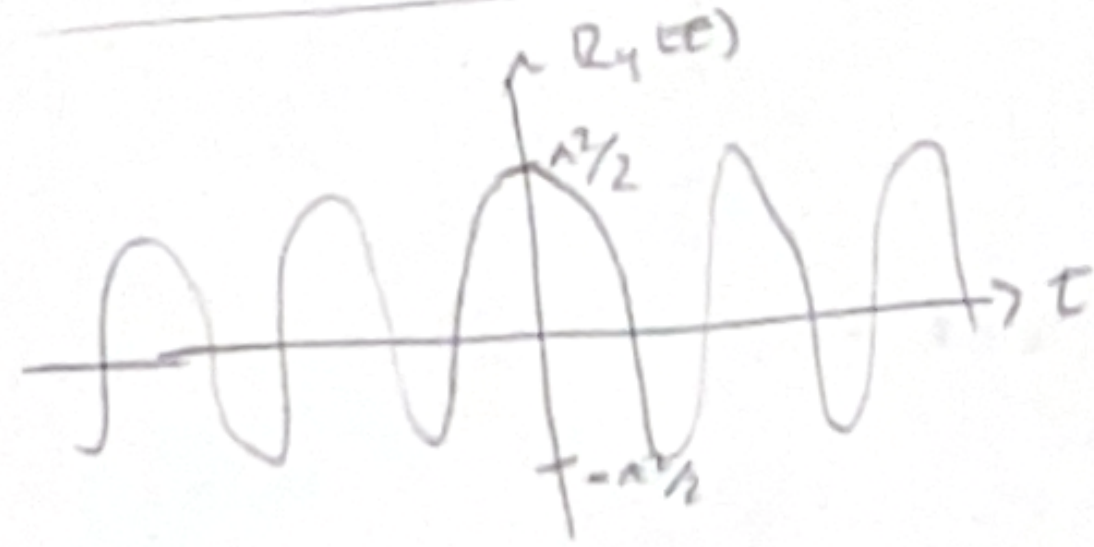
$$\boxed{S_Y(f) = \frac{A^2}{4}\left(\delta(f-f_c)+\delta(f+f_c)\right)} = S_X(f)$$

$\Rightarrow$



C-2 ~~[illegible]~~ FIND & SKETCH $R_y(\tau)$.

$$\mathcal{F}\{S_y(f)\} = \boxed{R_y(\tau) = \frac{A^2}{2}\cos(2\pi f_c \tau)}$$



C-3 DETERMINE RATE SAMPLES OF $y(t)$ ARE UNCORRELATED.

$R_y(\tau) = 0?$

$\cos(2\pi f_c \tau) = 0$

$\hookrightarrow 2\pi f_c \tau = \pi/2$

$\tau = \frac{\pi}{4\pi f_c}$

$\tau = \frac{1}{4f_c}$

$\Rightarrow \boxed{f_s = 4f_c}$