# ECE 478/ECE 570 Principles of Communication Systems

Textbook:
Communication Systems, $5^{\text{th}}$ ed., Haykin & Moher, 2009, John Wiley & Sons, Inc.

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# Lecture 02 - Representation of Signals and Systems

- Fourier Transform –> provides the link between time domain and frequency domain description of signals
- Definition:

$$\boxed{G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)dt.}$$

- Inverse Fourier transform:

$$\boxed{g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi ft)df.}$$

- Notation:

$$\begin{aligned} \text{Fourier Transform (FT):} \quad G(f) &= \mathcal{F}[g(t)] \\ \text{Inverse Fourier Transform (IFT):} \quad g(t) &= \mathcal{F}^{-1}[G(f)] \\ \text{FT pair can be written as} \quad &\Rightarrow \quad g(t) \rightleftharpoons G(f) \end{aligned}$$

# Continuous Spectrum

- In general, FT is a complex function of frequency $(f)$.

$$G(f) = |G(f)| \exp(j\theta(f))$$

$$\begin{aligned} \text{Continuous Amplitude Spectrum} &= |G(f)| \\ \text{Continuous Phase Spectrum} &= \theta(f) \end{aligned}$$

- For the special case of a real-valued $g(t)$, FT has the Hermitian/conjugate symmetry property –> $G(-f) = G^*(f)$.
- Therefore, it follows that if $g(t)$ is a real-valued function of time $f$, then

$$\begin{aligned} |G(-f)| &= |G(f)| \rightarrow \text{amplitude spectrum is even-symmetric} \\ \theta(-f) &= -\theta(f) \rightarrow \text{phase spectrum is odd-symmetric} \\ &\Rightarrow \text{Spectrum of a real-valued signal exhibits conjugate symmetry} \end{aligned}$$

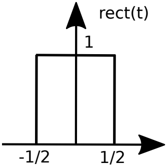
# Example 01 - Rectangular pulse

- Find the FT of a rectangular pulse $g(t)$

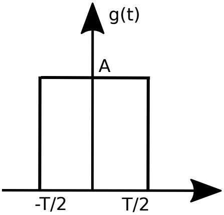
$$g(t) = \begin{cases} A, & -T/2 < t < T/2 \;\; \text{(or equivalently } \; |t| < T/2) \\ 0, & t \leq -T/2 \;\; \text{and} \;\; t \geq T/2 \;\; \text{(or equivalently } \; |t| \geq T/2). \end{cases}$$

- $g(t)$ can be represented by using the standard rectangular function $\text{rect}(t)$ as follows:

$$g(t) = A\,\text{rect}\left(\frac{t}{T}\right) \quad \text{where} \quad \text{rect}(t) = \begin{cases} 1, & |t| < 1/2 \\ 0, & |t| \geq 1/2. \end{cases}$$



Standard rectangular pulse



Rectangular pulse

# Example 01 - Rectangular pulse

- FT of g(t) can be derived as

$$G(f) \quad = \quad \int_{-T/2}^{T/2} A \exp(-j2\pi ft)dt$$

- Note that Euler's formula is given by

$$\exp(j\theta) = \cos(\theta) + j\sin(\theta)$$

- Similarly, we have

$$\exp(-j\theta) = \cos(-\theta) + j\sin(-\theta)$$

- Since $\cos(-\theta) = \cos(\theta)$ and $\sin(-\theta) = -\sin(\theta)$, we have

$$\exp(-j\theta) = \cos(\theta) - j\sin(\theta)$$

- Thus, we can substitute the following to evaluate $G(f)$ in above integral:

$$\exp(-j2\pi ft) = \cos(2\pi ft) - j\sin(2\pi ft)$$

# Example 01 - Rectangular pulse

- FT of g(t) can be derived as

$$
\begin{aligned}
G(f) &= \int_{-T/2}^{T/2} A\exp(-j2\pi ft)dt \\
&= \int_{-T/2}^{T/2} A\left[\cos(2\pi ft) - j\sin(2\pi ft)\right]dt \\
&= A\left[\frac{\sin(2\pi ft)}{2\pi f} + j\frac{\cos(2\pi ft)}{2\pi f}\right]_{-T/2}^{T/2} \\
&= \frac{A}{2\pi f}\left([\sin(\pi fT) + j\cos(\pi fT)] - [\sin(-\pi fT) + j\cos(-\pi fT)]\right) \\
&= \frac{A}{2\pi f}\left([\sin(\pi fT) + j\cos(\pi fT)] - [-\sin(\pi fT) + j\cos(\pi fT)]\right) \\
&= \frac{A}{2\pi f}\left(2\sin(\pi fT)\right) \\
&= A\left(\frac{\sin(\pi fT)}{\pi f}\right) = AT\left(\frac{\sin(\pi fT)}{\pi fT}\right)
\end{aligned}
$$

# Example 01 - Rectangular pulse cont...

- The FT of $g(t)$ can alternatively be written by using sinc function as

$$G(f) = AT\left(\frac{\sin(\pi f T)}{\pi f T}\right) = AT\mathrm{sinc}(fT) \quad \text{where} \quad \mathrm{sinc}(\lambda) = \frac{\sin(\pi\lambda)}{\pi\lambda}.$$

- The sinc function plays an important role in communication theory. $\mathrm{sinc}(\lambda)$ has its maximum at $\lambda = 0$ and approaches 0 as $\lambda \to \infty$ oscillating through positive and negative values. It goes through 0 at $\lambda = \pm1, \pm2, \pm3, \ldots\ldots$
- The corresponding FT pair can be written as follows:

$$\boxed{A\,\mathrm{rect}\left(\frac{t}{T}\right) \rightleftharpoons AT\mathrm{sinc}(fT)}$$

- Amplitude spectrum is given by $|G(f)|$. The first zero-crossing of $|G(f)|$ occurs at $f = \pm 1/T$. Whenever the pulse duration $T$ decreases, the first zero-crossing moves up the frequency. Conversely, as $T$ increases, the first zero-crossing moves toward the origin.
- Thus, the relationship between the time-domain and frequency-domain is an inverse one. Therefore, a narrower pulse has a significant frequency description over a wide range of frequencies and vice versa.

# FT of a rectangular pulse

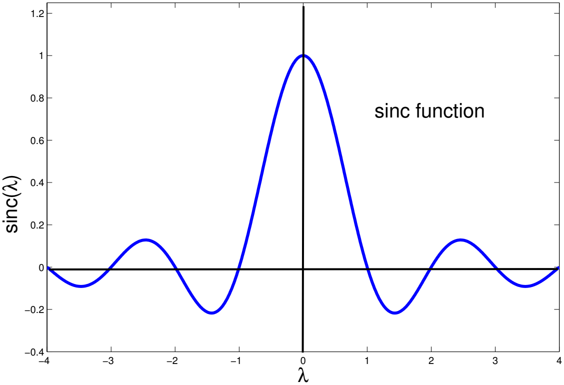
- The FT of a rectangular pulse $g(t) = A\text{rect}\left(\frac{t}{T}\right)$ can be summarized as follows:

$$\boxed{\mathcal{F}\left[A\text{rect}\left(\frac{t}{T}\right)\right] = AT\text{sinc}(fT)}$$

- Similarly, the FT of a standard rectangular $g(t) = \text{rect}\,(t)$ pulse is given by

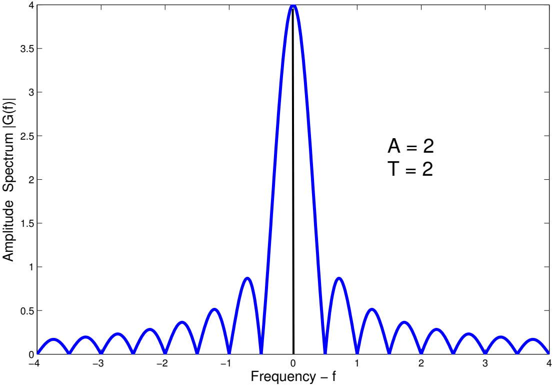
$$\boxed{\mathcal{F}\left[\text{rect}\,(t)\right] = \text{sinc}(f)}$$

# Figures for Example 01

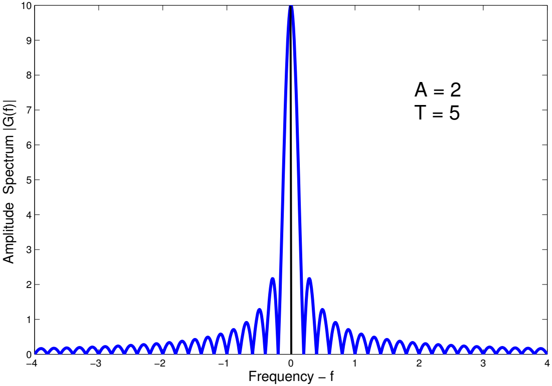



$$A = 1 \qquad \text{and} \qquad T = 1$$

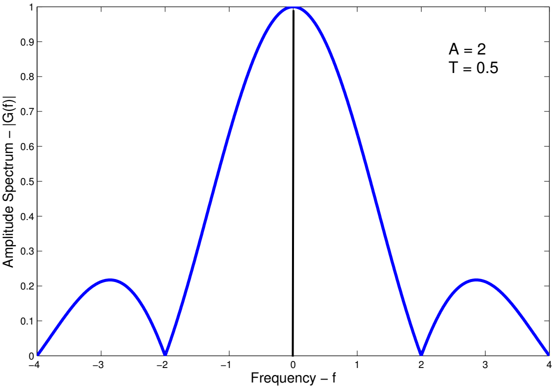
# Figures for Example 01



# Figures for Example 01



# Figures for Example 01



# Example 01 - Rectangular pulse cont...

- $g(t)$ is real-valued $\Rightarrow$ Amplitude spectrum has even-symmetry.
- The FT $G(f)$ is real-valued and even-symmetric function of frequency $f$.
- This is due to the fact that $g(t)$ is a real and even-symmetric function of time $t$.

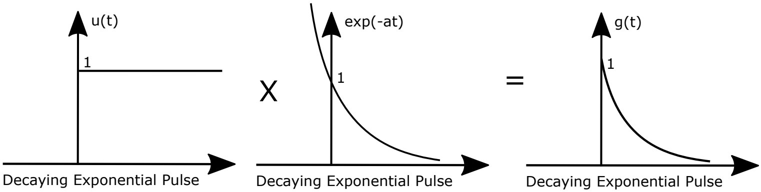
The FT of a real and even-symmetric function is real and exhibits even-symmetry.

# Example 02 - Decaying Exponential pulse

- The truncated decaying exponential pulse can be written as

$$g(t) = \exp(-at)u(t) \quad \text{where} \quad u(t) \text{ is the unit step funtion}$$

$$u(t) = \begin{cases} 1, & t > 0 \\ 1/2, & t = 1 \\ 0, & t < 0. \end{cases}$$



# Example 02 - Decaying Exponential pulse

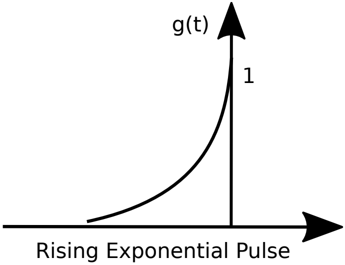
- FT of truncated decaying exponential pulse $g(t)$ is given by

$$
\begin{aligned}
G(f) &= \int_{-\infty}^{\infty} \exp(-at)u(t)\exp(-j2\pi ft)dt \\
G(f) &= \int_{0}^{\infty} \exp(-at)\exp(-j2\pi ft)dt \\
&= \int_{0}^{\infty} \exp(-t(a+j2\pi f)dt \\
&= \frac{1}{a+j2\pi f}
\end{aligned}
$$

# Example 03 - Rising Exponential pulse

- Similarly, the truncated rising exponential pulse is given by
- The truncated decaying exponential pulse can be written as

$$g(t) \quad = \quad \exp(at)u(-t)$$



# Example 03 - Rising Exponential pulse

- FT of the truncated rising exponential pulse can be derived as

$$G(f) = \frac{1}{a - j2\pi f}$$

- Both these pulses are asymmetric functions of time. Therefore, their FTs are complex-valued.
- Both decaying and rising exponential pulses have the same amplitude spectrum. However, the phase spectrum is the negative of that of the other.

The FT of a real and asymmetric function is complex-valued.

# Properties of the Fourier Transform

Table 1: Summary of Fourier Transform properties

| Property | Mathematical Description |
|---|---|
| Superposition | $\alpha g_1(t)+\beta g_2(t) \rightleftharpoons \alpha G_1(f)+\beta G_2(f)$, $\alpha$ & $\beta$ constants |
| Time scaling | $g(at) \rightleftharpoons \frac{1}{\lvert a\rvert} G\left(\frac{f}{a}\right)$ |
| Duality | If $g(t) \rightleftharpoons G(f)$, then $G(t) \rightleftharpoons g(-f)$ |
| Time shifting | $g(t-\tau) \rightleftharpoons G(f)\exp(-j2\pi f\tau)$, $\tau$ is constant |
| Frequency shifting | $g(t)\exp(j2\pi f_0 t) \rightleftharpoons G(f-f_0)$, $f_0$ is constant |
| Area under $g(t)$ | $\int_{-\infty}^{\infty} g(t)dt = G(0)$ |
| Area under $G(f)$ | $\int_{-\infty}^{\infty} G(f)df = g(0)$ |
| Differentiation | $\frac{d}{dt}g(t) \rightleftharpoons j2\pi f G(f)$ |
| Integration | $\int_{-\infty}^{t} g(\lambda)d\lambda \rightleftharpoons \frac{1}{j2\pi f}G(f) + \frac{G(0)}{2}\delta(f)$ |
| Conjugate | If $g(t) \rightleftharpoons G(f)$ then $g^*(t) \rightleftharpoons G^*(-f)$ |
| Multiplication | $g_1(t)g_2(t) \rightleftharpoons \int_{-\infty}^{\infty} G_1(f)G_2(f-\lambda)d\lambda$ |
| Convolution | $\int_{-\infty}^{\infty} g_1(t)g_2(t-\tau)d\tau \rightleftharpoons G_1(f)G_2(f)$ |
| Energy | $\int_{-\infty}^{\infty} \lvert g(t)\rvert^2 dt = \int_{-\infty}^{\infty} \lvert G(f)\rvert^2 df$ |

# Superposition (Linearity)

## Superposition/Linearity

$\alpha g_1(t)+\beta g_2(t) \rightleftharpoons \alpha G_1(f)+\beta G_2(f)$, where $\alpha$ and $\beta$ are constants

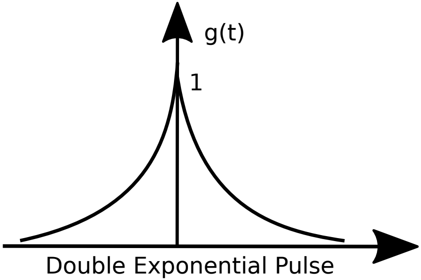
- Let $g_1(t) \rightleftharpoons G_1(f)$ and $g_2(t) \rightleftharpoons G_2(f)$.

$$\mathcal{F}[\alpha g_1(t) + \beta g_2(t)] = \int_{-\infty}^{\infty} (\alpha g_1(t) + \beta g_2(f)) \exp(-j2\pi f t) dt$$
$$= \alpha \int_{-\infty}^{\infty} g_1(t) \exp(-j2\pi f t) dt + \beta \int_{-\infty}^{\infty} g_2(t) \exp(-j2\pi f t) dt$$
$$= \alpha G_1(f) + \beta G_2(f)$$

# Example 04: Superposition (Linearity)

- Linearity property - Combination of exponential pulses
- The symmetric double exponential pulse is given by $g(t) = \exp(-a|t|)$ and can be written as a summation of decaying and rising exponential pulses.

$$g(t) = \exp(-a|t|) = \begin{cases} \exp(-at), & t > 0 \\ 1, & t = 0 \\ \exp(at), & t < 0 \end{cases} = \exp(-at)u(t) + \exp(at)u(-t)$$



# Example 04: Superposition (Linearity) cont...

- By using superposition property, the FT of $g(t)$ can be written as

$$\begin{aligned} G(f) &= \mathcal{F}\left[\exp(-at)u(t)\right] + \mathcal{F}\left[\exp(at)u(-t)\right] \\ &= \frac{1}{a + j2\pi f} + \frac{1}{a - j2\pi f} = \frac{2a}{a^2 + (2\pi f)^2} \end{aligned}$$

- Note that $g(t)$ is real and symmetric in time domain. Therefore, FT is also real and symmetric in frequency domain.

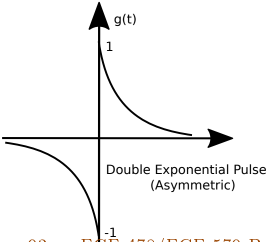
# Example 04: Superposition (Linearity) cont...

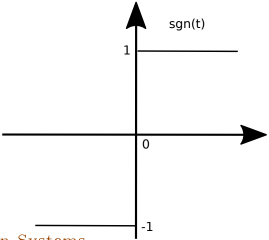
- The difference between a decaying and rising exponential pulses can be written as

$$g'(t) = \begin{cases} \exp(-at), & t > 0 \\ 1, & t = 0 \\ -\exp(at), & t < 0 \end{cases} \quad = \quad \exp(-a|t|)\mathrm{sgn}(t)$$

- Here $\mathrm{sgn}(t)$ is the signum function and given by

$$\mathrm{sgn}(t) = \begin{cases} 1, & t > 0 \\ 0, & t = 0 \\ -1, & t < 0 \end{cases}$$



# Example 04: Superposition (Linearity) cont...

- The function $g'(t)$

$$g(t) = \exp(-a|t|)\text{sgn}(t) = \exp(-at)u(t) - \exp(at)u(-t)$$

- Thus, the FT of $g'(t)$ is given by

$$\begin{aligned} G'(f) &= \mathcal{F}\left[\exp(-at)u(t)\right] - \mathcal{F}\left[\exp(at)u(-t)\right] \\ &= \frac{1}{a + j2\pi f} - \frac{1}{a - j2\pi f} = \frac{-j4\pi f}{a^2 + (2\pi f)^2} \end{aligned}$$

- Note that $g'(t)$ is real and odd-symmetric. Hence, the FT of $g'(t)$ is odd and purely imaginary.

The FT of a real and odd-symmetric function is purely imaginary.