ECE478

HW1

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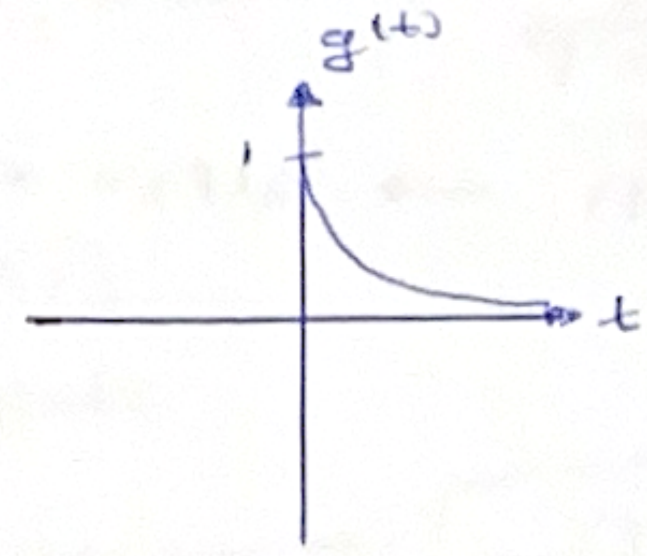
02/06/2025

(1) AN EXPONENTIALLY DECAYING PULSE IS DENOTED AS $g(t) = e^{-at}u(t)$, WHERE $u(t)$ IS THE UNIT STEP FUNCTION.



(A) USE F.T. DEFINITION TO DERIVE THE FREQUENCY DOMAIN REPRESENTATION OF THE EXPONENTIALLY DECAYING PULSE.

$$\mathcal{F}[g(t)] = G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi f t}\, dt$$

$\Rightarrow g(t) = e^{-at}u(t)$

$$\Rightarrow \mathcal{F}[g(t)] = G(f) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-j2\pi f t}\, dt$$

* THE INTEGRAND IS 0 FOR $(-\infty, 0)$, SO WE CAN CHANGE BOUNDS & DROP $u(t)$. (b/c of $u(t)$)

$$= \int_{0}^{\infty} e^{-at} e^{-j2\pi f t}\, dt$$

$$= \int_{0}^{\infty} e^{[-at + (-j2\pi f t)]}\, dt$$

$$= \int_{0}^{\infty} e^{-(j2\pi f + a)t}\, dt$$

$$= \frac{-1}{j2\pi f + a} e^{-(j2\pi f + a)t} \Big|_0^{\infty}$$

$$= \frac{-1}{j2\pi f + a} \left(\lim_{t\to\infty} \frac{1}{e^{(j2\pi f + a)t}} - e^0 \right)$$

$$= \frac{-1}{j2\pi f + a} (0 - 1)$$

$$\boxed{G(f) = \frac{1}{j2\pi f + a}}$$

(B) IS THE F.T. REAL OR COMPLEX VALUED?

$G(f)$ IS COMPLEX VALUED ($G: \mathbb{R} \to \mathbb{C}$) AS IT CONTAINS AN IMAGINARY COMPONENT.

THE FOURIER TRANSFORMS OF REAL AND EVEN SYMMETRIC SIGNALS ARE ALSO PURELY REAL, HOWEVER, THIS SIGNAL $g(t)$ IS ASYMMETRIC, THUS REQUIRING A COMPLEX FOURIER TRANSFORM PAIR.

ⓒ FIND THE AMPLITUDE SPECTRUM $|G(f)|$ AND PHASE SPECTRUM $\theta(f)$ OF $G(f)$.

$$G(f) = \frac{1}{a+j2\pi f}\cdot\frac{(a-j2\pi f)}{(a-j2\pi f)}$$

$$= \frac{a-j2\pi f}{a^2+4\pi^2 f^2}$$

$$= \underbrace{\frac{a}{a^2+4\pi^2 f^2}}_{X(f)} + j\underbrace{\frac{-2\pi f}{a^2+4\pi^2 f^2}}_{Y(f)} \longrightarrow G(f) = X(f) + jY(f)$$

$\boxed{|G(f)|}$

$$|G(f)| = \sqrt{X(f)^2+Y(f)^2} = \sqrt{\frac{a^2}{(a^2+4\pi^2 f^2)^2} + \frac{4\pi^2 f^2}{(a^2+4\pi^2 f^2)^2}}$$

$$= \sqrt{\frac{a^2+4\pi^2 f^2}{(a^2+4\pi^2 f^2)^2}}$$

$$\boxed{|G(f)| = \frac{1}{\sqrt{a^2+4\pi^2 f^2}}}$$

$\boxed{\theta(f)}$

$$\theta(f) = \arctan\left(\frac{\frac{-2\pi f}{a^2+4\pi^2 f^2}}{\frac{a}{a^2+4\pi^2 f^2}}\right) = \arctan\left(\frac{-2\pi f}{a}\right)$$

$$\Rightarrow \boxed{\theta(f) = \arctan\left(\frac{-2\pi f}{a}\right)}$$

ⓓ CODE & PLOT ATTACHED.

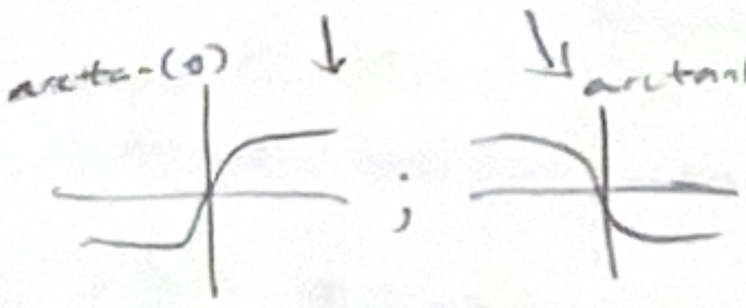
ⓔ SHOW F.T. OF $g(t)$ EXHIBITS CONJUGATE SYMMETRY; AMPLITUDE EVEN, PHASE ODD.

$$\rightarrow |G(f)| = \frac{1}{\sqrt{a^2+4\pi^2 f^2}} \Rightarrow |G(-f)| = \frac{1}{\sqrt{a^2+4\pi^2(-f)^2}} = \frac{1}{\sqrt{a^2+4\pi^2 f^2}} = |G(f)| \blacksquare.$$

$\therefore$ AMPLITUDE SPECTRUM IS EVEN-SYMMETRIC.

$$\rightarrow \theta(f) = \arctan\left(\frac{-2\pi f}{a}\right) \Rightarrow \theta(-f) = \arctan\left(\frac{-2\pi(-f)}{a}\right) = \arctan\left(-\left(\frac{-2\pi f}{a}\right)\right) = -\arctan\left(\frac{-2\pi f}{a}\right) = -\theta(f) \blacksquare.$$

$\arctan(\theta)$ ↓ ↓ $\arctan(-\theta) = -\arctan\theta$



$\therefore$ PHASE SPECTRUM IS ODD-SYMMETRIC.

$\Rightarrow$ BOTH CONDITIONS MET, SO THE F.T. OF $g(t)$ HAS CONJUGATE SYMMETRY.

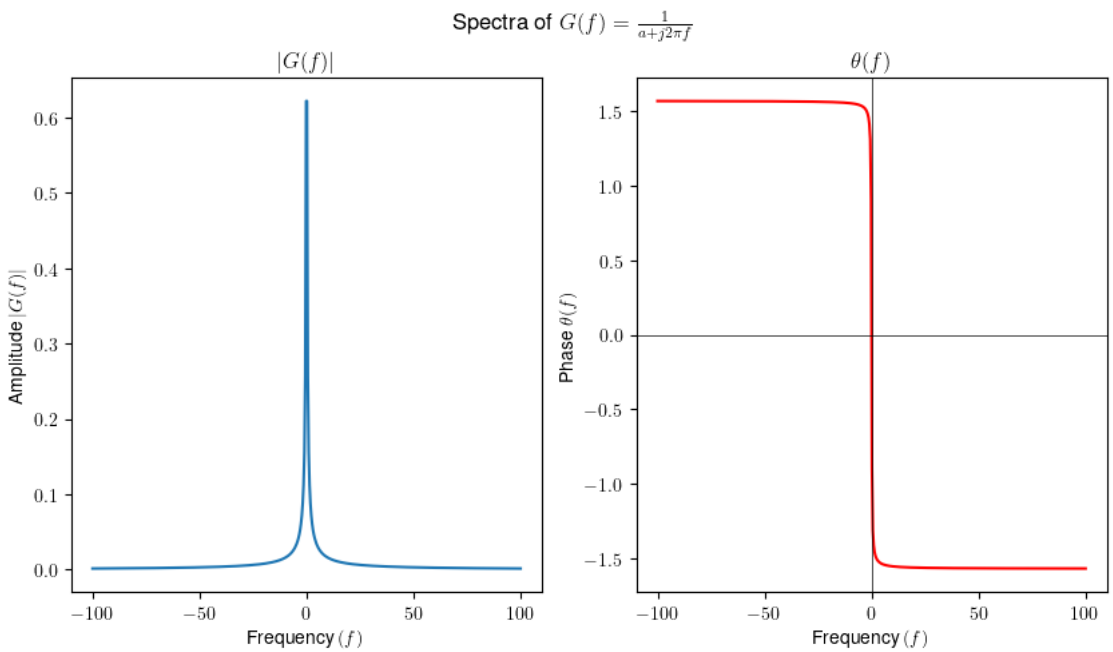
hw1-code

February 10, 2025

(3) Plot the amplitude spectrum and phase spectrum of $G(f)$

[26]:
```python
import matplotlib.pyplot as plt
import numpy as np

plt.rcParams.update({
    "text.usetex": True,
    "font.family": "Helvetica"
})

# DEFINE EACH AS FUNCTIONS:
# G(f) = 1/(a + j2pif)
# |G(f)| = 1/sqrt(a**2 + 4pi**2f**2)
# O(f) = arctan(-2pif/a)

def G(f, a):
    return 1 / (np.complex(a, 2*np.pi*f))

def amplitude_G(f, a):
    return 1 / np.sqrt(a**2 + (2*np.pi*f)**2)

def phase_G(f, a):
    return np.arctan(-2*np.pi*f / a)

# Create range of frequencies (-10kHz to 10kHz)
f = np.linspace(-100, 100, 500)

# Temp variable for a
a = 1

# Extract the spectra
amplitude = amplitude_G(f, a)
phase = phase_G(f, a)

# Plot the spectra
plt.figure(figsize=(10,5))

plt.subplot(1,2,1)
```

```
plt.plot(f, amplitude)
plt.title(r'$|G(f)|$')
plt.xlabel(r'Frequency $(f)$')
plt.ylabel(r'Amplitude $|G(f)|$')

plt.subplot(1,2,2)
plt.plot(f, phase, c='red')
plt.title(r'$\theta (f)$')
plt.xlabel(r'Frequency $(f)$')
plt.ylabel(r'Phase $\theta (f)$')
plt.axhline(y=0, color='black', linewidth=0.5)
plt.axvline(x=0, color='black', linewidth=0.5)

plt.suptitle(r'Spectra of $G(f)=\frac{1}{a + j 2 \pi f}$')
plt.show()
plt.close()
```



② CONSIDER THE TIME-DOMAIN REPRESENTATION OF A LOW-PASS FILTER:

$$g(t) = A\,\text{sinc}(2W(t-\tau))$$

Ⓐ FIND F.T. OF FILTER

*TIME SHIFTING: $g(t-\tau) \rightleftharpoons G(f)e^{-j2\pi f\tau}$

$\Rightarrow \mathcal{F}[g(t)] = \mathcal{F}[A\,\text{sinc}(2W(t-\tau))]$

$= \frac{A}{2W}\,\text{rect}(f/2W)\,e^{-j2\pi f\tau}$

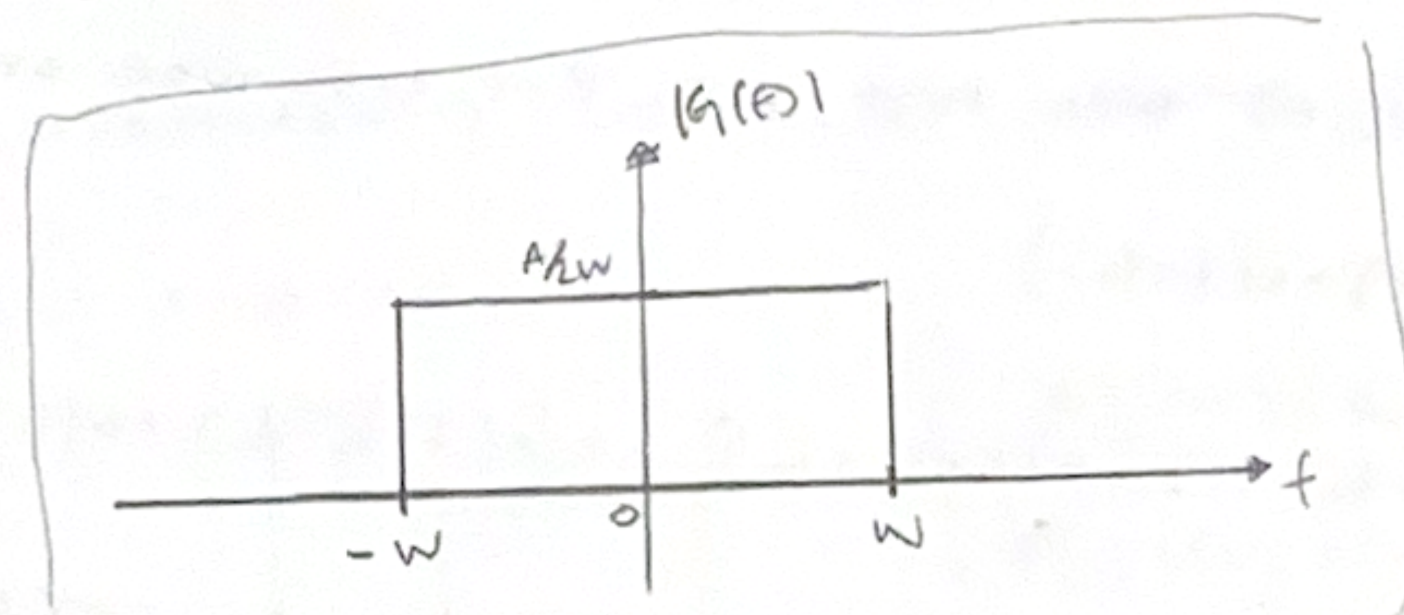
$$\Rightarrow \boxed{G(f) = \frac{A}{2W} e^{-j2\pi f\tau}\,\text{rect}(f/2W)}$$

Ⓑ PLOT THE AMPLITUDE RESPONSE

* $|e^{-j2\pi f t}| = |\cos(2\pi f t) - j\sin(2\pi f t)| = \sqrt{\cos^2(2\pi f t) + \sin^2(2\pi f t)} = \sqrt{1} = 1.$

$\Rightarrow |G(f)| = A/2W\,\text{rect}(f/2W)$

*$T = 2W$



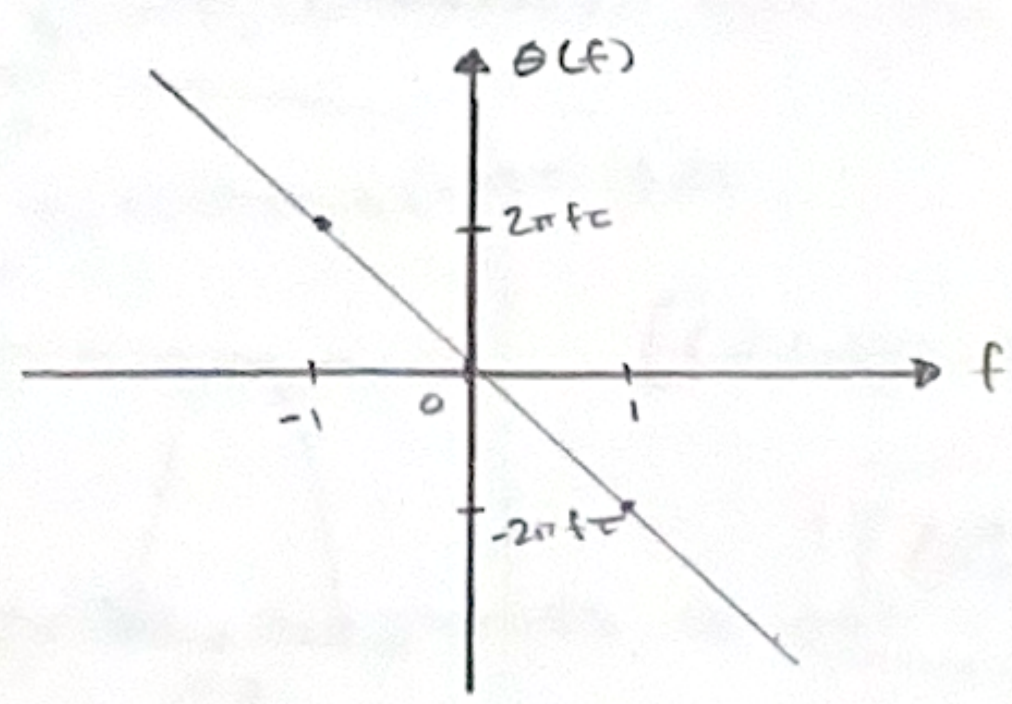
Ⓒ PLOT THE PHASE RESPONSE

FOR $G(f) = X(f) + jY(f) \Rightarrow \theta(f) = \arctan(Y(f)/X(f))$, SO WE CAN VIEW $G(f)$ AS:

$\rightarrow G(f) = A/2W\,\text{rect}(f/2W)(\cos(2\pi f\tau) - j\sin(2\pi f\tau))$

$\Rightarrow \theta(f) = \arctan\left(\frac{-\sin(2\pi f\tau)}{\cos(2\pi f\tau)}\right) = -\arctan(\tan(2\pi f\tau)) = -2\pi f\tau.$

$\rightarrow$



Ⓓ EFFECT OF TIME-SHIFT IN FREQ DOMAIN?

THE TIME-DELAY CREATES A CONSTANT CHANGE IN PHASE $\forall f \in \mathbb{R}$.

Ⓔ WHAT IS THE SIGNIFICANCE IN TIME SHIFTING IN DESIGNING FILTERS.

SINCE THE STANDARD SINC PULSE IS INFINITE IN DOMAIN, AND NONCAUSAL, THE PULSE NEEDS TO BE SHIFTED INTO POSITIVE TIME TO MAKE A PHYSICALLY REALIZABLE CASUAL SINC PULSE. TO DO SO, A TIME-SHIFT IS REQUIRED.

also needs truncation.

(3) CONSIDER: $y(t) = \beta g(t)\cos(2\pi f_c t)$, where $\beta \in \mathbb{R}$; $g(t) \rightleftharpoons G(f)$.

(A) FIND F.T OF $y(t)$

* $\mathcal{F}[\cos(2\pi f_c t)] = \frac{1}{2}(\delta(f-f_c)+\delta(f+f_c))$

* MULTIPLICATION IN TIME DOMAIN $\Rightarrow g_1(t)g_2(t) \rightleftharpoons G_1(f) \circledast G_2(f)$

$$\Rightarrow \mathcal{F}[y(t)] = \beta G(f) \circledast \tfrac{1}{2}(\delta(f-f_c)+\delta(f+f_c))$$

$$= \frac{\beta}{2}[G(f) * \delta(f-f_c) + G(f)\circledast\delta(f+f_c)]$$

* SIFTING PROPERTY $f(x)\circledast\delta(x-a) = f(x-a)$

$$= \beta/2\,[G(f-f_c)+G(f+f_c)]$$

$$\Rightarrow \boxed{Y(f) = \beta/2\,[G(f-f_c)+G(f+f_c)]}$$

(B) IF $g(t)=\cos(2\pi f_0 t)$, THEN FIND F.T. PLOT AMPLITUDE SPECTRUM FOR $f_c \gg f_0$.

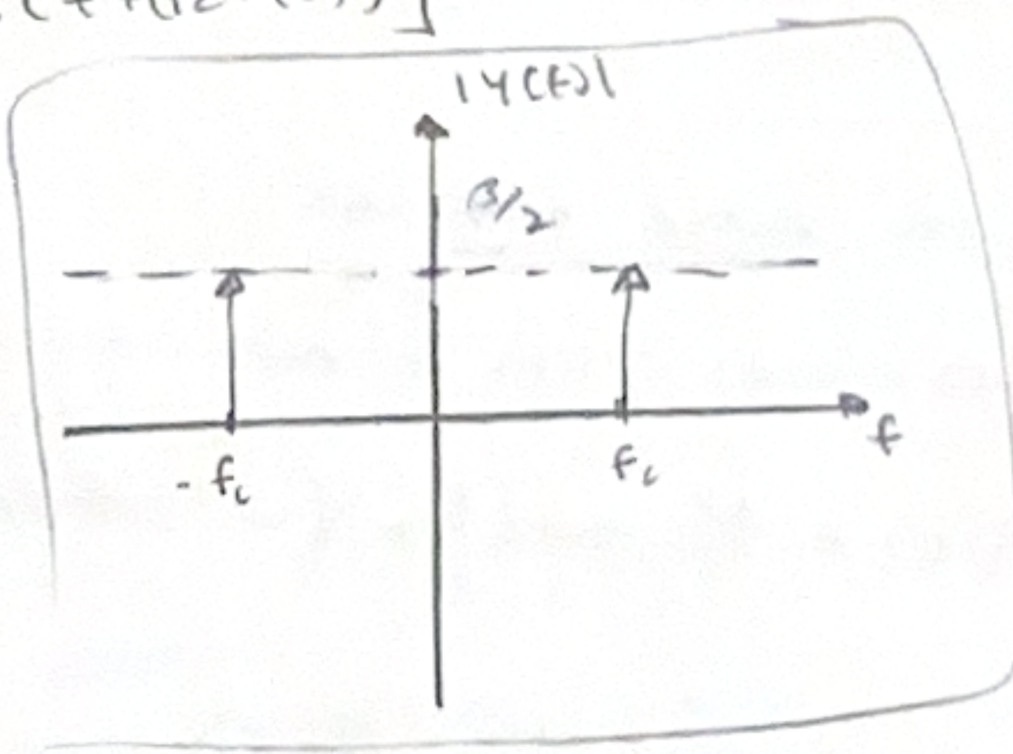
$$\Rightarrow \mathcal{F}[g(t)] = \tfrac{1}{2}[\delta(f-f_0)+\delta(f+f_0)]$$

$$Y(f) \Rightarrow \frac{\beta}{2}\left[\tfrac{1}{2}[\delta(f-f_c-f_0)+\delta(f-f_c+f_0)] + \tfrac{1}{2}[\delta(f+f_c-f_0)+\delta(f+f_c+f_0)]\right]$$

$$= \beta/4\,[\delta(f-(f_c+f_0))+\delta(f+(-f_c+f_0))+\delta(f+(f_c-f_0))+\delta(f+(f_c+f_0))]$$

FOR $f_c \gg f_0$: $L_0 = \beta/4\,[\delta(f-f_c)+\delta(f-f_c)+\delta(f+f_c)+\delta(f+f_c)]$

$$\boxed{Y(f) = \beta/2\,[\delta(f-f_c)+\delta(f+f_c)] = |Y(f)|}$$

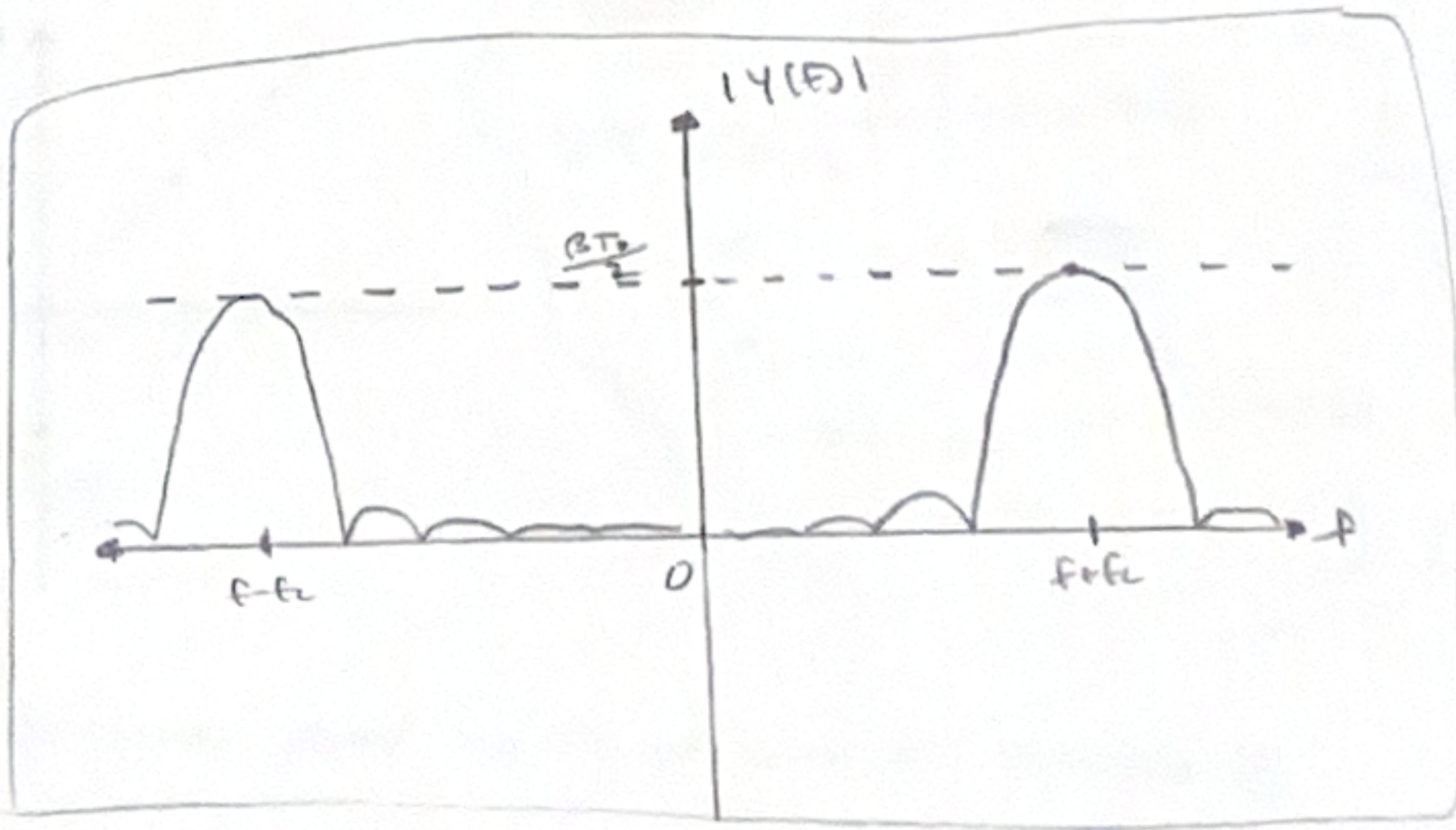


(C) IF $g(t) = \mathrm{rect}(t/T_0)$, FIND F.T. PLOT AMPLITUDE SPECTRUM FOR $f_c \gg f_0 = 1/T_0$.

$$\mathcal{F}[g(t)] = T_0\,\mathrm{sinc}(fT_0)$$

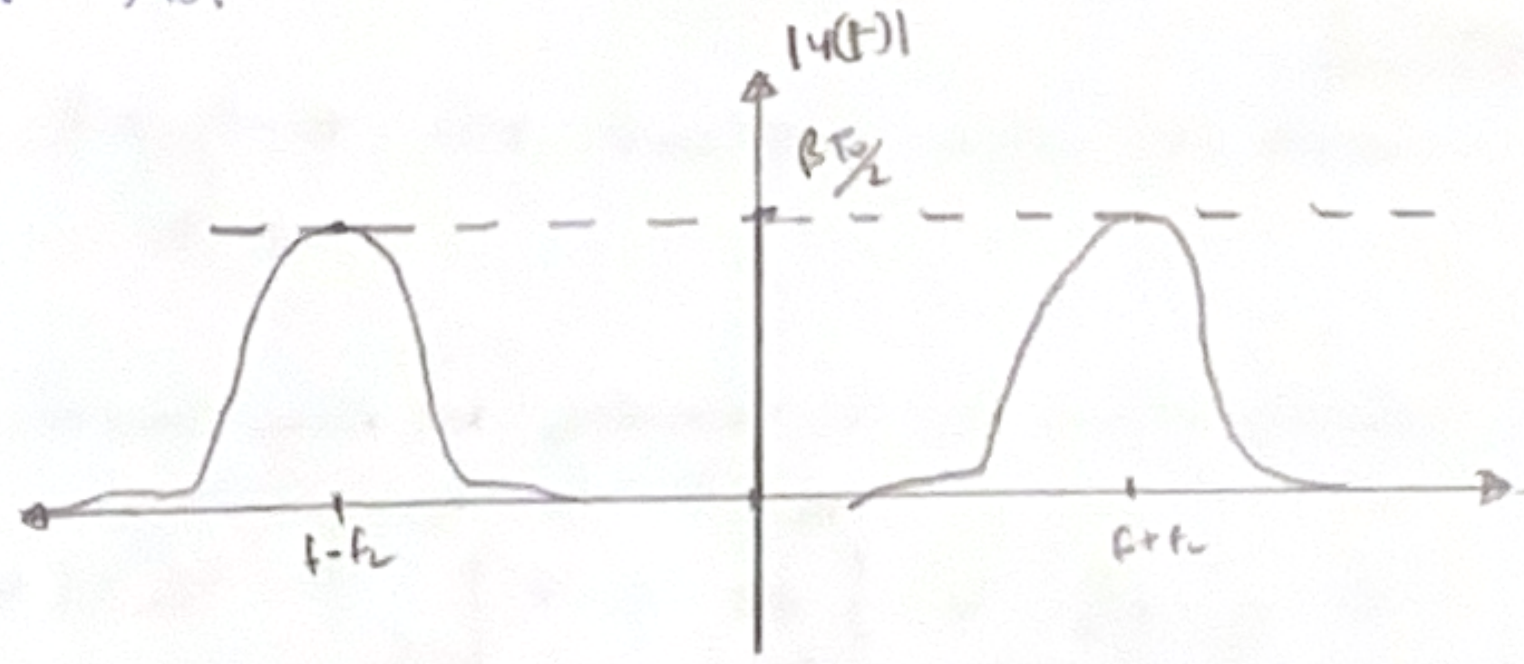
$$\Rightarrow Y(f) = \frac{\beta T_0}{2}[\mathrm{sinc}((f-f_c)T_0)+\mathrm{sinc}((f+f_c)T_0)]$$

$$\boxed{Y(f) = \frac{\beta T_0}{2}\left[\mathrm{sinc}\left(\frac{f-f_c}{f_0}\right)+\mathrm{sinc}\left(\frac{f+f_c}{f_0}\right)\right]}$$



② $g(t) = \Delta(t/T_0)$, FIND F.T, PLOT AMP. SPEC FOR $f_c \gg f_0 = 1/T_0$.

$\ast\, Y(f) = \frac{\beta}{2}\left[G(f-f_c) + G(f+f_c)\right]$



$\Rightarrow \mathcal{F}[g(t)] = T_0 \operatorname{sinc}^2(f/T_0)$

$$\Rightarrow \boxed{Y(f) = \frac{\beta T_0}{2}\left[\operatorname{sinc}^2\left(\frac{f-f_c}{T_0}\right) + \operatorname{sinc}^2\left(\frac{f+f_c}{T_0}\right)\right]}$$

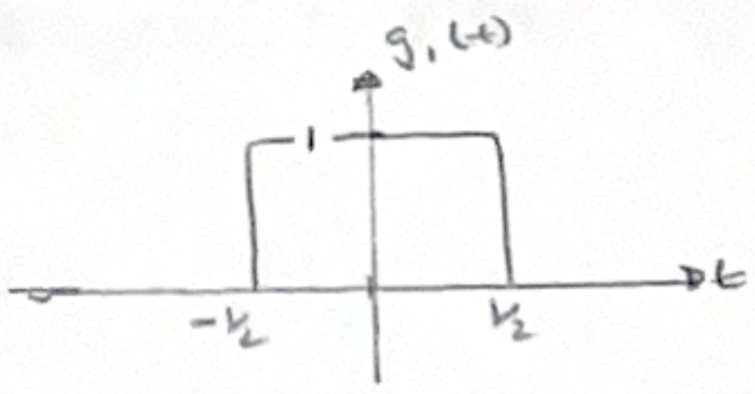
③ WHAT ARE SOME INSIGHTS OF THE FREQ. DOMAIN ANALYSIS OF $y(t) = \beta g(t)\cos(2\pi f_c t)$.

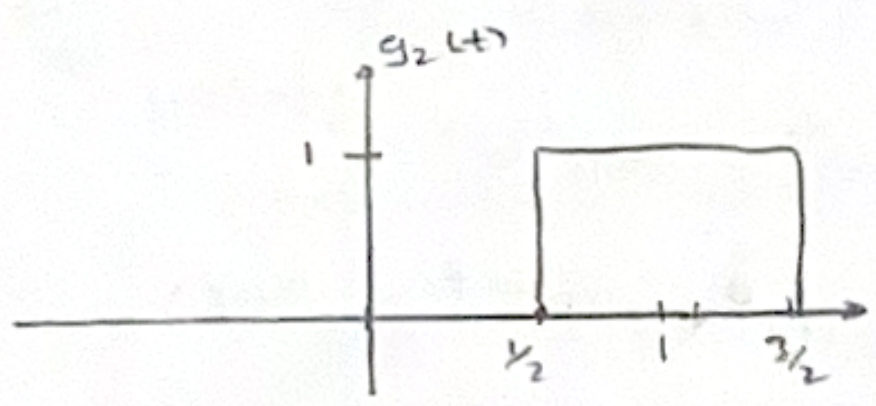
THIS $y(t)$ CAUSES A SCALING BY β (ATTENUATION OR BOOSTING), AND A CENTERING OF MAJOR FREQUENCY CONTENT AROUND $f = +/- f_c$.

THIS MODULATION COULD BE USEFUL IN EITHER CREATING A BANDPASS FILTER OR CHANGING THE CHANNEL INFORMATION IS TRANSMITTED.

④ CONSIDER $g_1(t) = \operatorname{rect}(t)$ AND $g_2(t) = \operatorname{rect}(t-1)$. LET $g_{12}(t) = g_1(t) * g_2(t) = \int_{-\infty}^{\infty} g_1(\tau)\, g_2(t-\tau)\, d\tau$.

ⓐ PLOT $g_1(t)$ & $g_2(t)$ IN TIME-DOMAIN. FIND F.T.S.





$$\boxed{\begin{aligned}\mathcal{F}[g_1(t)] &= G_1(f) = \operatorname{sinc}(f) \\ \mathcal{F}[g_2(t)] &= G_2(f) = e^{-i2\pi f}\operatorname{sinc}(f)\end{aligned}}$$

ⓑ FIND F.T. OF $g_{12}(t)$.

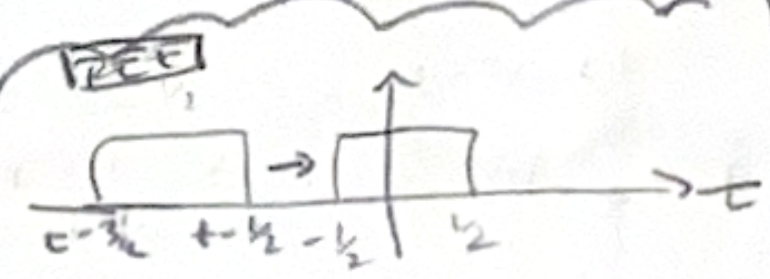
$\mathcal{F}[g_{12}(t)] = \mathcal{F}[(g_1 \circledast g_2)(t)] = G_1(f)\,G_2(f) = \boxed{e^{-i2\pi f}\operatorname{sinc}^2(f) = G_{12}(f)}$

ⓒ EVALUATING CONVOLUTION INTEGRAL, SHOW THAT $g_{12}(t)$ IS $\Delta(t-1) = \begin{cases} 0, & t<0 \\ t, & 0 \le t < 1 \\ -t+2, & 1 \le t < 2 \\ 0, & t \ge 2 \end{cases}$

$g_1(t) = \operatorname{rect}(t)$
$g_2(t) = \operatorname{rect}(t-1)$

$\rightarrow g_{12}(t) = \int_{-\infty}^{\infty} g_1(\tau)\, g_2(t-\tau)\, d\tau$

$= \int_{-\infty}^{\infty} \operatorname{rect}(\tau)\operatorname{rect}(t-(\tau-1))\, d\tau$



$\ast \quad -\frac{1}{2} \le \tau \le \frac{1}{2}$

$\ast \quad \frac{1}{2} \le t-\tau \le \frac{3}{2}$

$\Rightarrow \frac{1}{2} - t \le -\tau \le -t + \frac{3}{2}$

$\Rightarrow t - \frac{3}{2} \le \tau \le t - \frac{1}{2}$

CASE 1

HERE $t - \frac{1}{2} < -\frac{1}{2}$, OR, $t<0$, THE TWO PULSES DO NOT OVERLAP $\therefore\ g_{12}(t)\big|_{t<0} = 0$.

CASE 2

HERE $t - \frac{1}{2} > -\frac{1}{2} \Rightarrow t > 0$ BUT $t - \frac{3}{2} \le -\frac{1}{2} \Rightarrow t \le 1$ SO IN TOTAL $\Rightarrow 0 < t \le 1$

WE START TO SEE OVERLAP BTWN $g_1(\tau)$ AND $g_2(t-\tau)$.

TO INTEGRATE THIS OVERLAP IN THE τ-DOMAIN, WE WILL ACCOUNT $\forall \tau$ s.t. $-\frac{1}{2} \le \tau \le t - \frac{1}{2}$.

KNOWING HERE $\operatorname{rect}(\tau)\operatorname{rect}(t-\tau) = 1 \cdot 1 = 1$.

$$\Rightarrow g_{12}(t)\Big|_{0<t\le 1} = \int_{-1/2}^{t-1/2} d\tau = \tau\Big|_{-1/2}^{t-1/2} = (t-\tfrac{1}{2}) - (-\tfrac{1}{2}) = t \quad \therefore\ g_{12}(t)\Big|_{0<t\le 1} = t$$

cont.

CASE 3

THERE IS STILL OVERLAP FOR $t-\frac{1}{2} > \frac{1}{2}$ AND $t - \frac{3}{2} \le \frac{1}{2}$ $\xrightarrow{\text{COMB.}}$ $\underline{1 < t \le 2}$

$\quad\quad t > 1 \quad\quad\quad\quad t \le 2$

INTEGRATING IN τ-DOMAIN, WE WILL SWIPE FROM $t - \frac{3}{2} \le \tau \le \frac{1}{2}$

$$\Rightarrow g_{12}(t)\Big|_{1<t\le 2} = \int_{t-3/2}^{1/2} d\tau = \tau\Big|_{t-3/2}^{1/2} = \tfrac{1}{2} - (t - \tfrac{3}{2}) = \underline{2-t}$$

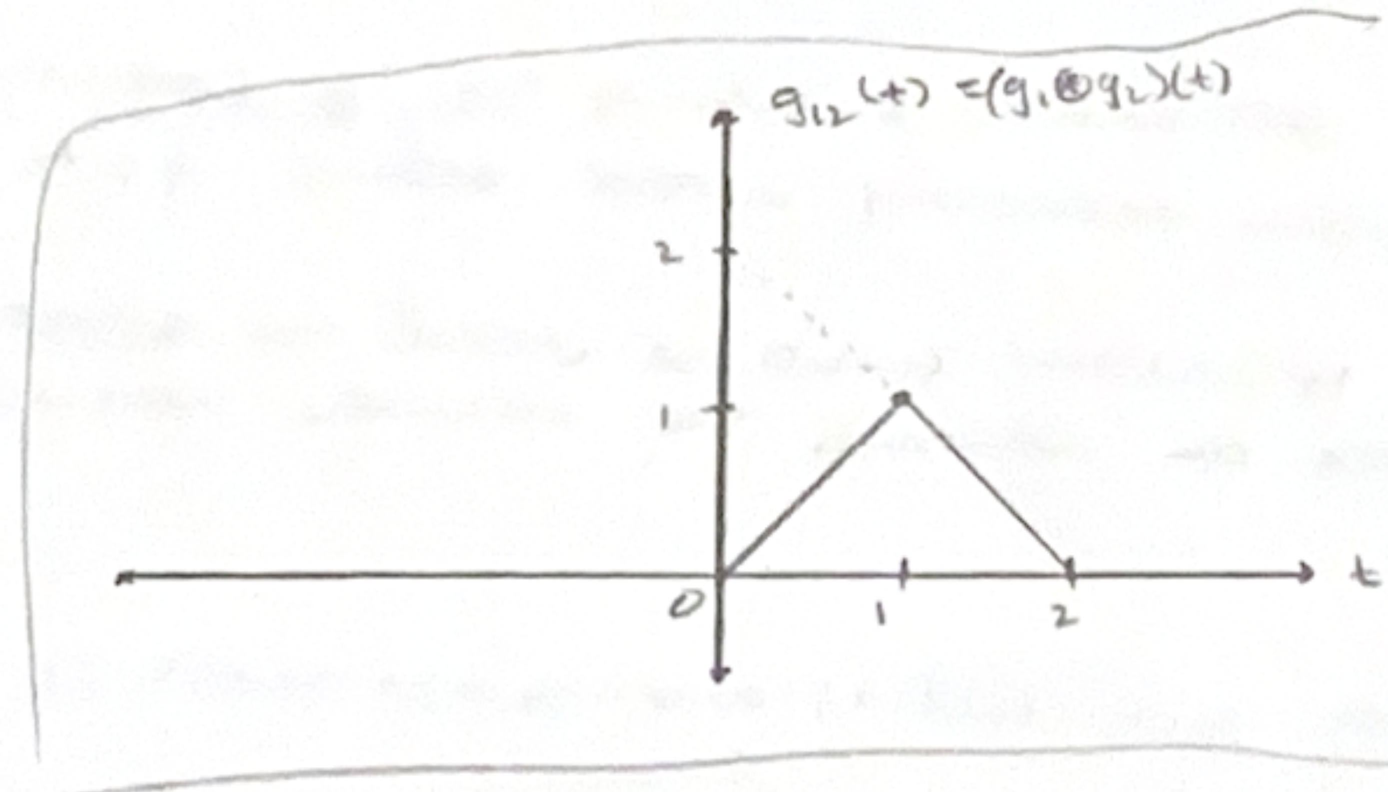
$$\therefore g_{12}(t)\Big|_{1<t\le 2} = 2-t.$$

CASE 4

WHEN $t - \frac{3}{2} > \frac{1}{2} \Rightarrow t > 2$, THEN WE REACH NO MORE OVERLAP $\therefore g_{12}(t)\Big|_{t>2} = 0.$

COMBINED:

$$\Rightarrow \boxed{g_{12}(t) = \begin{cases} 0, & t < 0 \\ t, & 0 < t \le 1 \\ 2-t, & 1 < t \le 2 \\ 0, & t > 2 \end{cases}}$$



(D) FIND F.T. OF $g_{12}(t)$ USING TIME-SHIFTING PROPERTY.

$* \ \mathcal{F}[\Delta(t)] = \text{sinc}^2(f) \quad \Rightarrow \quad \mathcal{F}[\Delta(t-1)] = \text{sinc}^2(f) \cdot e^{-j2\pi f(1)}$

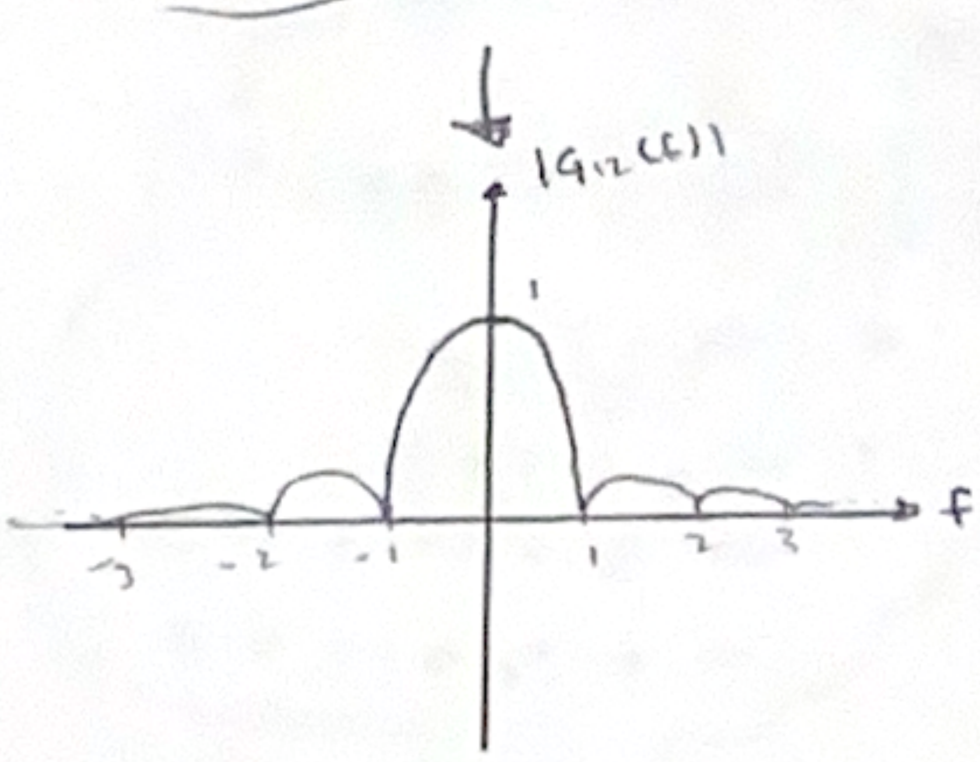
$$\Rightarrow \boxed{G_{12}(f) = e^{-j2\pi f}\,\text{sinc}^2(f)}$$

(E) VERIFY (B) & (D) EQUIV.

THEY'RE THE SAME. ✓

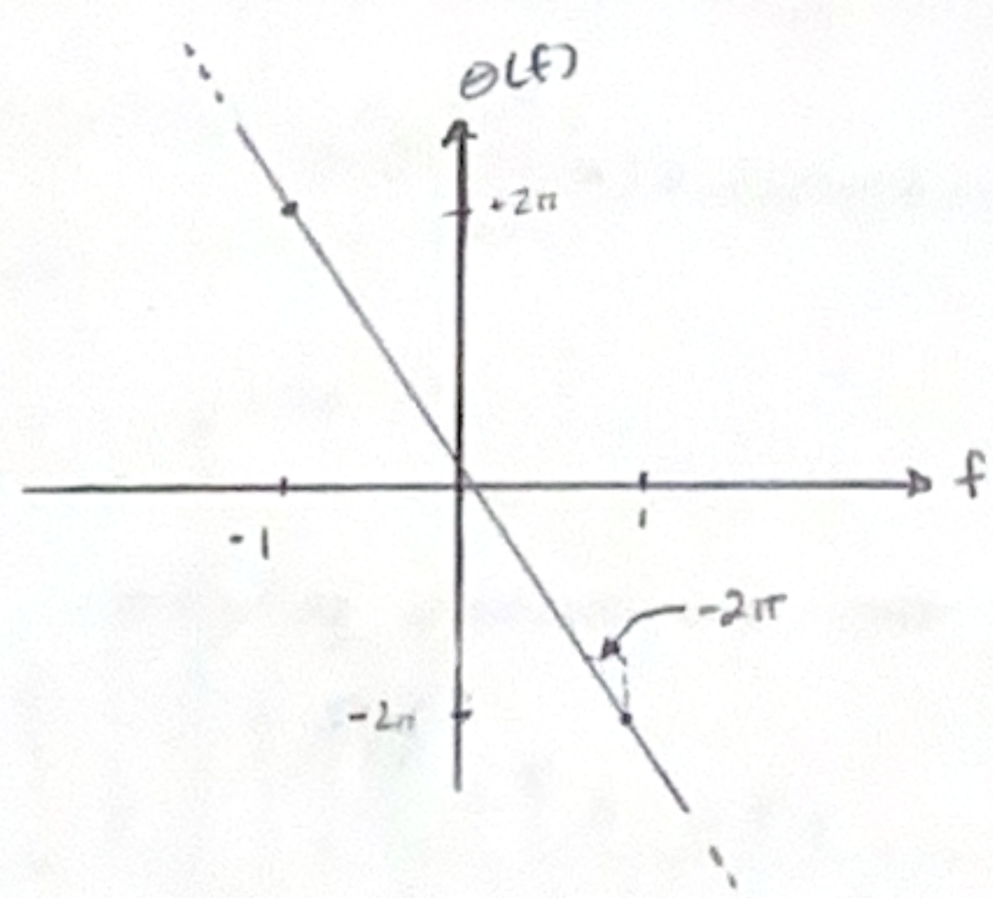
(F) PLOT AMPLITUDE AND PHASE SPECTRUM.

$\underline{|G_{12}(f)| = \text{sinc}^2(f)}$

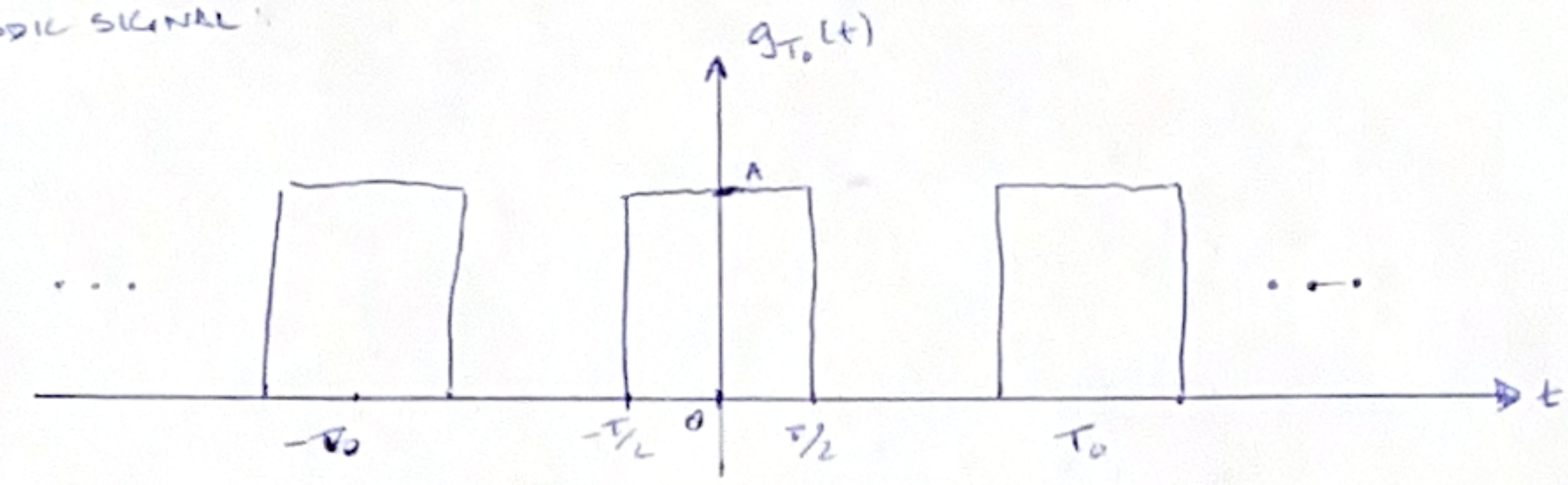


$\theta(f) = \arctan(\tan(-2\pi f))$

$= -2\pi f$



(5) CONSIDER THE PERIODIC SIGNAL:



(A) FIND THE GENERATING FUNCTION $g(t)$ OF $g_{T_0}(t)$.

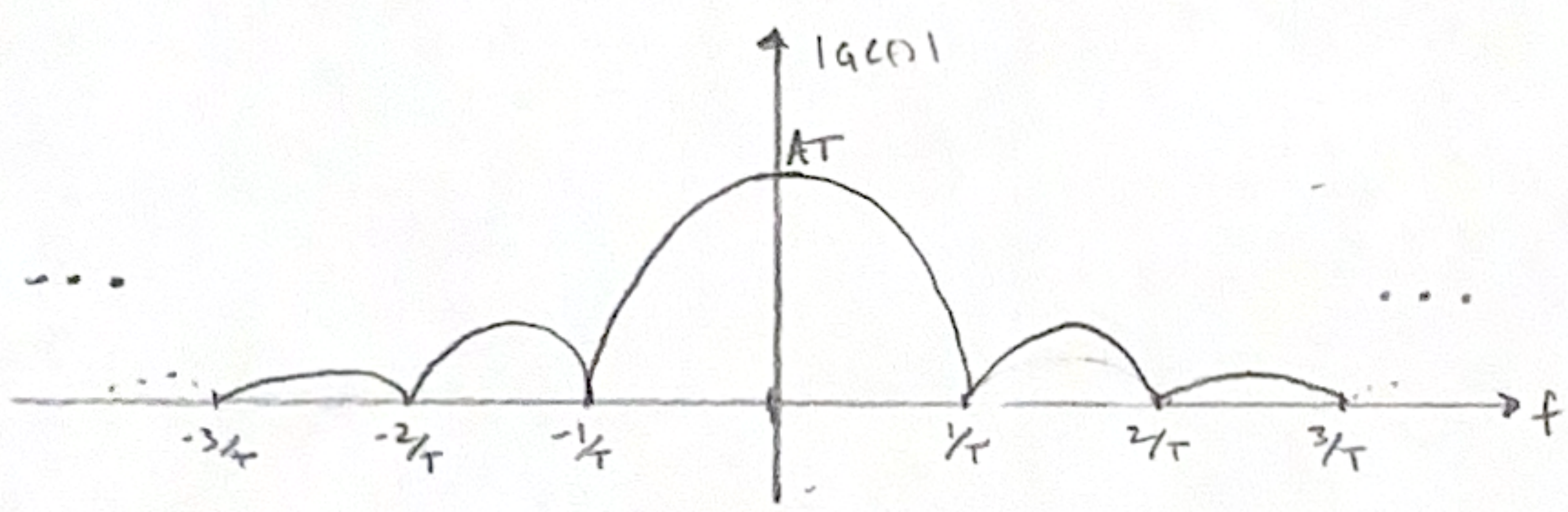
THE GENERATING FUNCTION $g(t)$ BY INSPECTION IS $\boxed{g(t) = A\,\text{rect}(t/T)}$

(B) EXPRESS $g_{T_0}(t)$ IN TERMS OF ~~THE~~ ITS GENERATING FUNCTION $g(t)$.

$$\Rightarrow \boxed{g_{T_0}(t) = \sum_{m=-\infty}^{\infty} A\,\text{rect}\left(\frac{t - mT_0}{T}\right)}\ , \text{ where } m \in \mathbb{Z}$$

(C) FIND THE F.T. OF THE GENERATING FUNCTION AND PLOT ITS AMPLITUDE SPECTRUM.

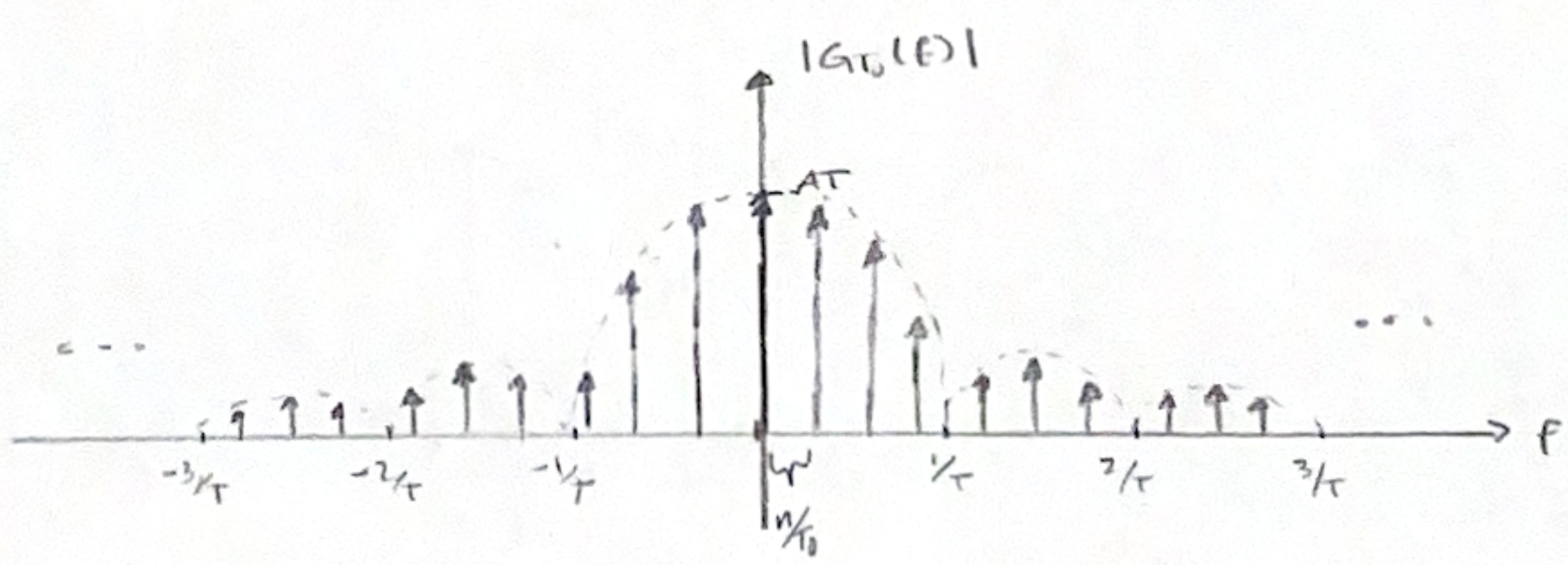
$$\Rightarrow \mathcal{F}[g(t)] = \mathcal{F}[A\,\text{rect}(t/T)] = \frac{A}{1/T}\,\text{sinc}^2\left(\frac{f}{1/T}\right) = \boxed{AT\,\text{sinc}^2(Tf) = G(f)}$$



(D) FIND THE F.T. OF THE PERIODIC FUNCTION $g_{T_0}(t)$ AND PLOT ITS AMPLITUDE SPECTRUM.

$$* \quad \mathcal{F}[g_{T_0}(t)] = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} G(n/T_0)\,\delta(f - n/T_0)$$

$$\Rightarrow \mathcal{F}[g_{T_0}(t)] = \boxed{\frac{1}{T_0} \sum_{n=-\infty}^{\infty} AT\,\text{sinc}^2(T \cdot n/T_0)\,\delta(f - n/T_0) = G_{T_0}(f)}$$



(E) INSIGHTS?

PART C'S PLOT SHOWS THE ENVELOPE OUR $G_{T_0}(f)$ FOLLOWS.

THE FINAL F.T. TO GIVE $G_{T_0}(f)$ SHOWS THE TRAIN OF RECTANGULAR PULSES IN TIME-DOMAIN WITH PERIOD T_0 PRODUCES IN FREQ. DOMAIN A TRAIN OF IMPULSES WHICH FOLLOW THE $AT\text{sinc}^2(fT)$ ENVELOPE w/ $n/T_0 = nf_0$ SPACING.

Fundamental Freq.