Quantum Computing

"Master Vector Algebra before exploring quantum reality"

Day 2 – Vector Algebra

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Complex Numbers

- Imaginary unit: $i = \sqrt{-1}$
- Complex number: $z = a + bi$, where $a, b \in \mathbb{R}$.
- Euler's formula:

$$\boxed{e^{i\vartheta} = \cos\vartheta + i\sin\vartheta}$$

- Polar form: $z = re^{i\vartheta}$, where $r = |z|$, $\vartheta = \arg(z)$.
- Example: $e^{i\pi} + 1 = 0$ (Euler's Identity)

Vectors and Inner Product

- Vector in $\mathbb{C}^2$:

$$v = \begin{pmatrix} a \\ b \end{pmatrix}, \quad a, b \in \mathbb{C}.$$

- Hermitian conjugate: $v^\dagger = \begin{pmatrix} a^* & b^* \end{pmatrix}$.
- Inner product:

$$\langle u, v \rangle = u^\dagger v$$

- Properties:
  - $\langle u, v \rangle = \langle v, u \rangle^*$
  - $\langle u, u \rangle \geq 0$, equality if $u = 0$.
- Example: For $v = \begin{pmatrix} a \\ b \end{pmatrix}$, norm:

$$\boxed{\langle v, v \rangle = |a|^2 + |b|^2}$$

Explanation

1.Hermitian Conjugate:

$$\boxed{\left((u)^*\right)^T = \left((u)^T\right)^*}$$

Ex: $\begin{bmatrix}1\\0\end{bmatrix} = [1 \;\; 0]$

2.Inner Product:

$$\boxed{\langle u, v\rangle = u^\dagger v}$$

Ex: $U = \begin{bmatrix}1\\0\end{bmatrix}, v = \begin{bmatrix}0\\1\end{bmatrix} \Rightarrow U^+ = [1\ 0]\ ,\ v = \begin{bmatrix}0\\1\end{bmatrix}$

$$\langle u, v\rangle = [10]\begin{bmatrix}0\\1\end{bmatrix} = 0 + 0 = 0$$

Note : After inner product the output should always be scalar.

Matrix Multiplication

For A $(m \times n)$ and B $(n \times p)$, the product AB is $m \times p$:

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

then

$$AB = \begin{bmatrix} (1)(0) + (2)(1) & (1)(1) + (2)(0) \\ (3)(0) + (4)(1) & (3)(1) + (4)(0) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}.$$

Tensor Product

- Combines vector spaces.
- For $v = \begin{matrix} v_0 \\ v_1 \end{matrix}$, $w = \begin{matrix} w_0 \\ w_1 \end{matrix}$

$$\boxed{v \otimes w = \begin{matrix} v_o w_0 \\ v_0 w_1 \\ v_1 w_0 \\ v_1 w_1 \end{matrix}}$$

- Example:

$$\begin{matrix} 1 \\ 0 \end{matrix} \otimes \begin{matrix} 0 \\ 1 \end{matrix} = \begin{matrix} 0 \\ 1 \\ 0 \\ 0 \end{matrix}$$

Eigen values and Eigen vectors

For square matrix A, eigenvector v satisfies:

$$\boxed{Av = \lambda v.}$$

Solve: $\det(A - \lambda I) = 0$.

Example:

$$A = \begin{matrix} 2 & 0 \\ 0 & 3 \end{matrix}$$

Eigenvalues: $\lambda_1 = 2$, $\lambda_2 = 3$. Eigenvectors: $v_1 = \begin{matrix} 1 \\ 0 \end{matrix}$, $v_2 = \begin{matrix} 0 \\ 1 \end{matrix}$.

Trace of a Matrix

- Trace of square matrix A:

$$\boxed{\mathrm{Tr}(A) = \sum_{i=1}^{n} A_{ii}}$$

- Properties:
  - $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$
  - $\mathrm{Tr}(cA) = c\,\mathrm{Tr}(A)$
  - $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$
- Example:

$$\mathrm{Tr}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 1 + 4 = 5.$$

Types of Matrices

- **Hermitian**: $H^\dagger = H$. Real eigenvalues.
- **Unitary**: $U^\dagger U = I$. Preserves norms.
- **Pauli Matrices** (Note: $\sigma_x = X$, etc.):
  - $\sigma_x = X = \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}$
  - $\sigma_y = Y = \begin{matrix} 0 & -i \\ i & 0 \end{matrix}$
  - $\sigma_z = Z = \begin{matrix} 1 & 0 \\ 0 & -1 \end{matrix}$

  All are unitary and Hermitian.

Positive Definite Matrices

A Hermitian matrix A is positive definite if for every non-zero vector $x \in \mathbb{C}^n$:

$$\boxed{x^{\dagger} A x > 0.}$$

Equivalently, all eigenvalues of A are strictly positive.

Q.Find Postive Definite Or Not?

$$A = \begin{matrix} 2 & 0 \\ 0 & 3 \end{matrix}$$

Orthonormality

Vectors $\{u_1, \ldots, u_n\}$ are orthonormal if:

$$\langle u_i, u_j \rangle = \delta_{ij}.$$

Example: In $\mathbb{R}^2$, $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$\boxed{\langle e_1, e_1 \rangle = 1, \quad \langle e_2, e_2 \rangle = 1, \quad \langle e_1, e_2 \rangle = 0.}$$

Take Away

✓ Vectors in Hilbert space represent quantum states

✓ Matrices as operators: gates, observables, transformations

✓ Eigenvalues = measurement outcomes; eigenvectors = states

✓ Orthonormal bases = computational basis for qubits