

5. Explain how measurement affects the quantum state and probability outcomes. Also explain the measurement basis conversion.

* Measurement collapses the state into $|i\rangle$ with the probability $|\langle i|\psi\rangle|^2$.

* After measurement, state becomes $|i\rangle$.

* Measurement destroys the superposition.

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

Let $i=0$, $\Rightarrow |i\rangle = |0\rangle$ then the probability of getting $|0\rangle$ is

$$\Rightarrow |\langle 0|\psi\rangle|^2 = |\langle 0|\alpha|0\rangle + \beta|1\rangle|^2 = |\alpha\langle 0|0\rangle + \beta\langle 0|1\rangle|^2$$

$$\langle 0|0\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \quad \Rightarrow \quad \langle 0|1\rangle = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0$$

$$\Rightarrow |\langle 0|\psi\rangle|^2 = |\alpha|1\rangle + \beta|0\rangle|^2 = |\alpha|^2$$

Let $i=1$, $|i\rangle = |1\rangle$ then the probability of getting $|1\rangle$ is $|\beta|^2$

$$\Rightarrow |\langle 1|\psi\rangle|^2 = |\langle 1|\alpha|0\rangle + \beta|1\rangle|^2 = |\alpha\langle 1|0\rangle + \beta\langle 1|1\rangle|^2$$

$$\Rightarrow \langle 1|0\rangle = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0; \quad \langle 1|1\rangle = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$$

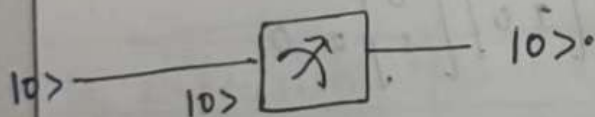
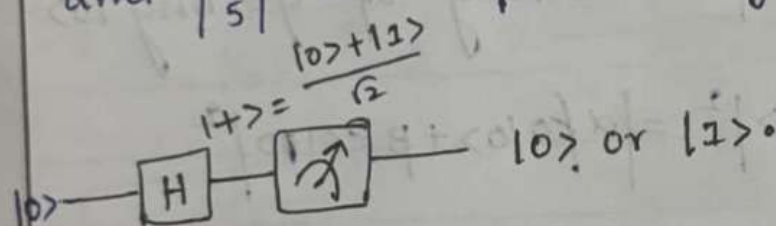
$$\Rightarrow |\alpha|0\rangle + \beta|1\rangle|^2 = |\beta|^2$$

Example:-

$\frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle \rightarrow \boxed{\nearrow} \rightarrow \left(\frac{3}{5}\right)^2$ is the probability of getting $|0\rangle$.

$\frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle \rightarrow \boxed{\nearrow} \rightarrow \left|\frac{4}{5}\right|^2$ is the probability of getting $|1\rangle$.

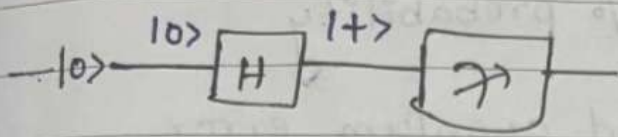
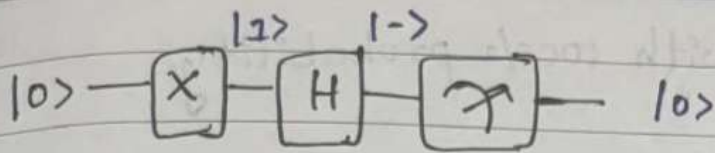
$\Rightarrow \frac{3}{5}|0\rangle + \frac{4}{5}|1\rangle$ is destroyed to $|0\rangle$ or $|1\rangle$ after measurement with $\left(\frac{3}{5}\right)^2$ probability of getting $|0\rangle$ and $\left|\frac{4}{5}\right|^2$ is the probability of getting $|1\rangle$.



Possible states are $|0\rangle \otimes |0\rangle$ or $|1\rangle \otimes |0\rangle$.

$$\begin{aligned} &\Rightarrow \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \otimes |0\rangle && \Rightarrow \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right] \otimes |0\rangle \\ &= \left(\frac{|00\rangle + |10\rangle}{\sqrt{2}} \right) && \Rightarrow \left[\frac{|00\rangle - |10\rangle}{\sqrt{2}} \right] \end{aligned}$$

Anyway after measurement, post measurement state must be $|00\rangle$ or $|10\rangle$.



$$\Rightarrow |0\rangle \otimes |0\rangle$$

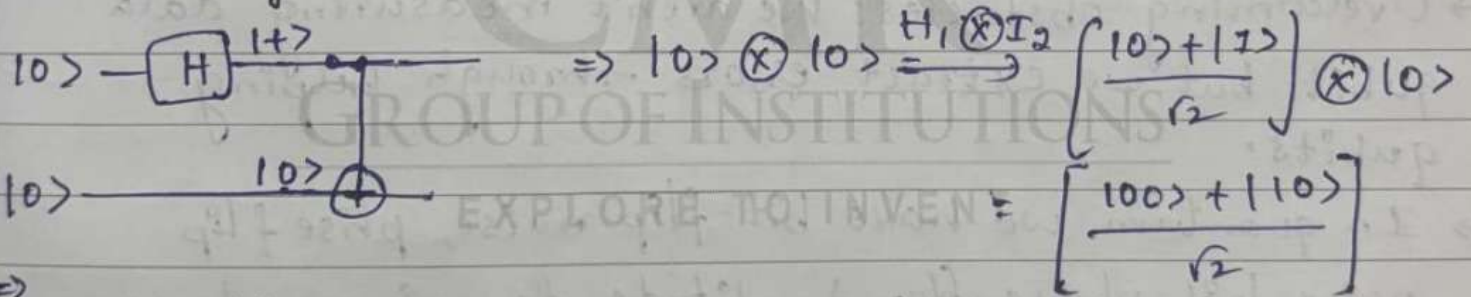
$$\downarrow X_1 \otimes I_2$$

$$\Rightarrow |1\rangle \otimes |0\rangle$$

$$\downarrow H_1 \otimes H_2$$

$$\Rightarrow \left[\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right] \otimes \left[\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right] = \left[\frac{|00\rangle - |11\rangle + |01\rangle - |10\rangle}{2} \right]$$

After measurement, we can get $|00\rangle$ or $|01\rangle$ or $|10\rangle$ or $|11\rangle$.
Bell State generator circuit:



$$\Rightarrow \frac{|00\rangle + |10\rangle}{\sqrt{2}} \xrightarrow{\text{CNOT}_{12}} \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

↓ This is called bell state/entangled pairs.

