1 PS

1.1 Induction

Problem 1 *Let n be a nonnegative integer. Prove that*

$$\underbrace{\frac{1}{1}-\frac{1}{2}+\frac{1}{3}...-\frac{1}{2n-1}+\frac{1}{2n}}_{L(n)}=\underbrace{\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}}_{R(n)}$$

Proof. We will apply induction on n. We

Base case: We must prove $L(0) = R(0)$. This is true, because $L(0)$ and $R(0)$ are empty sums and thus equal to 0.

Induction Step: Let m be a nonnegative integer. Assume that $L(m) = R(m)$(the induction hypothesis). We must prove that $L(m + 1) = R(m + 1)$. By definition,

$$\begin{aligned}
L(m) &= \frac{1}{1}-\frac{1}{2}+\frac{1}{3}+...+\frac{1}{2m-1}-\frac{1}{2m}\\
L(m+1) &= \frac{1}{1}-\frac{1}{2}+\frac{1}{3}+...+\frac{1}{2(m+1)-1}-\frac{1}{2(m+1)}\\
&= \frac{1}{1}-\frac{1}{2}+\frac{1}{3}+...+\frac{1}{2m+1}-\frac{1}{2m+2}\\
&= (\frac{1}{1}-\frac{1}{2}+\frac{1}{3}+...+\frac{1}{2m-1}-\frac{1}{2m})+\frac{1}{2m+1}-\frac{1}{2m+2}\\
&= L(m)+\frac{1}{2m+1}-\frac{1}{2m+2}
\end{aligned}$$

$$\begin{aligned}
R(m) &= \frac{1}{m+1}+\frac{1}{m+2}+...+\frac{1}{2m}\\
R(m+1) &= \frac{1}{m+2}+\frac{1}{m+3}+...+\frac{1}{2(m+1)}\\
&= (\frac{1}{m+1}+\frac{1}{m+2}+...+\frac{1}{2m})-\frac{1}{m+1}+\frac{1}{2m+1}+\frac{1}{2m+2}\\
&= R(m)-\frac{1}{m+1}+\frac{1}{2m+1}+\frac{1}{2m+2}
\end{aligned}$$

Thus we must show,

$$L(m+1)=L(m)+\frac{1}{2m+1}-\frac{1}{2m+2}=\underbrace{L(m)-\frac{1}{m+1}+\frac{1}{2m+1}+\frac{1}{2m+2}}_{\text{By IH},L(m)=R(m)}=R(m)-\frac{1}{m+1}+\frac{1}{2m+1}+\frac{1}{2m+2}$$

as desired. Hence the result follows.

This is an identity for finite sums. These types of questions can be proved in the same.

Problem 2 *Let n be a postive integer. A bit shall mean an element of M $\{0,1\}$. An n-bitstring shall mean an n-tuple of bits. For example, $\{1,0,1\}$ is a 3-bitstring. Prove that there exists a list $\{b_1, b_2, ..., b_{2^n}\}$ of all n-bitstrings exactly once and has the property that for each $i \in \{1,2,...,2^n\}$, the two n-bitstrings b_1 and b_{i-1} differ in exactly one entry. [Here b_0 means b_{2^n}]*

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Rough Sketch: The case for $n = 1$, we have the bitstrings $(0) \to (1)$. For $n = 2$, we have $(0,0)$, $(0,1)$,$(1,1)$ and $(1,0)$.

$$(0,0) \to (0,1) \to (1,1) \to (1,0)$$

For $n = 3$, we first copy the the two bit strings and add 0 infront of them, then do same with 1 in front of them). Then, we merge the two according to the conditions of the question.

We can use the following algorithm.

Algorithm 3 *1. Copy the previous $n-1$ bitstrings, append 0 infront of them and store them in a list L.*

2. Copy the previous $n-1$ bitstrings, append 1 infront of them and store them in a list R.

3. Reverse the order of elements in R.

4. Append R to L.

1. $(0,0,0) \to (0,0,1) \to (0,1,1) \to (0,1,0)$
2. $(1,0,0) \to (1,0,1) \to (1,1,1) \to (1,1,0)$
3. $(1,1,0) \to (1,1,1) \to (1,0,1) \to (1,0,0)$
4. $(0,0,0) \to (0,0,1) \to (0,1,1) \to (0,1,0) \to (1,1,0) \to (1,1,1) \to (1,0,1) \to (1,0,0)$

Proof. We will prove by induction on n.

Base case: $n = 1$, we have done this above.

Induction Step: Let m be a positive integer. Assume the claim holds for $n = m$ (the induction hypothesis). We must prove that the claim holds for $n = m+1$. The induction hypothesis tells us that there is a list $\{b_1, b_2, ..., b_{2^m}\}$ that satisfies the conditions in the problem.

Now for an m bitstring b let $0b$ be the $(m+1)$ bitstring obtained by inserting 0 to the left end of the bitstring.

$$b = c_1, c_2, ..., c_m \implies 0b = 0, c_1, c_2, ..., c_m$$

Similarly let $1b$ denote the $(m+1)$ bitstring obtained by inserting 1 to the left end of the bitstring.

$$b = c_1, c_2, ..., c_m \implies 1b = 1, c_1, c_2, ..., c_m$$

Claim 4

$$\{0b_1, 0b_2, ..., 0b_{2^m}, 1b_{2^m}, 1b_{2^m-1}, ..., 1b_1\}$$

forms an $(m+1)$ bitstring that satisfies all the conditions.

Proof. *For $\{0b_1, 0b_2, ..., 0b_{2^m}\}$ and $\{1b_1, 1b_2, ..., 1b_{2^m}\}$ it is clear that the given conditions are satisfied. For $\{0b_{2^m}, 1b_{2^m}\}$, it is clear that the conditions are satisfied because only one entry is changed. The same reasoning shows that $(1b_1 \to 0b_1)$ also satisfies the conditions. Since all the conditions are satisfied, our claim follows.* ■

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Problem 5 *Prove that any nonempty finite set of integers has a maximum*

Rough Sketch: We restate the problem: Let n be a positive integer, any set of n integers must have maximum.

Claim 6 *Let n be a positive integer, any set of n integers must have maximum.*

Proof. *We induct on n.*

Base Case: For $n = 1$, there is only one integer in the set and it is the maximum.

Induction Step: Let m be a postive integer. Assume that any m-element set of integers has a maximum(IH). We must show that any $(m+1)$-element set of integers has a maximum.

Let s be an arbitrary element of S, the $(m+1)$-element set of integers. Consider $S \backslash \{\dot{s}\}$. We can do this because S is nonempty by definition.

By the IH, $S \backslash \{\dot{s}\}$ has a maximum. Let t be that maximum.

Claim 7 *S has a maximum, namely:*

- *If $t \geq s$, then t is the maximum.*
- *otherwise, s is the maximum.*

Thus, our induction step is complete, and our result follows.

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Problem 8 *Let g and h be integers s.t. $g \leq h$. Let $b_g, b_{g+1}, ..., b_h$ be any $h-g+1$ nonzero integers. Assume that $b_g \geq 0$. Assume further that*

$$|b_{i+1} - b_i| \leq 1... \textit{ for every } i \in \{g, g+1, ..., h-1\}$$

Then, $b_n > 0$ for each n $\in \{g, g+1, ..., h\}$.

Rough Sketch: We induct on n.

Proof. We induct on n. For $n = g$, $b_g \geq 0$ and is nonzero thus $b_g > 0$.

Let $m \in \{g, g+1, ..., h-1\}$. Assume that the claim holds for $b_g, b_{g+1}, ..., bm$. We must show that the claim holds for $b_g, b_{g+1}, ..., b_{m+1}$.

Our IH, tells us that $b_m > 0(b_m \geq 1)$, and we have $|b_{m+1} - b_m| \leq 1$. This implies $b_{m+1} \geq b_m - 1 \geq 0$. Since b_{m+1} is nonnegative and nonzero, it must be positive. Hence, our result follows and our induction is complete. ■

Problem 9 *Prove that every integer $n \geq 0$ satisfies*

$$f_1 + f_2 + ... + f_n = f_{n+2} - 1$$

Proof. *We induct on n.*

Base Case: For $n = 1$, we have

$$f_1 = f_3 - 1 = f_2 + f_1 - 1 = 2f_1 - 1 \rightarrow f_1 = 1, \text{ as desired}$$

Inductive Step: Let m be a postive integer. Assume that $f_1 + f_2 + ... + f_m = f_{m+2} - 1$(IH). We must show that $f_1 + f_2 + ... + f_{m+1} = f_{m+3} - 1$.

$$f_1 + f_2 + ... f_m + f_{m+1} = (f_1 + f_2 + ... f_m) + f_{m+1} \underset{IH}{=} (f_{m+2} - 1) + f_{m+1} = f_{m+3} - 1$$

Hence, our induction step is complete and our result follows. ■

Problem 10 *Prove that every integer $n > 0$ satisfies*

$$f_{n+1}f_{n-1} - f_n^2 = (-1)^n$$

Trick: To rewrite

$$f_{n+m+2}$$

as

$$f_{n+(m+1)+1}$$

instead.

1.2 Pigeonhole Principle

For Injections

Let U and V be finite sets such that $|U| > |V|$. Let $f : U \rightarrow V$ be any map. Then f cannot be injective.

For "Multi Injections"

Let U and V be finite sets and $k \in N$. Let $f : U \rightarrow V$ be any map. Assume that for each $v \in V$ and $S = \{u \mid [u \in U] \wedge \ f(u) = v\}$.

$$|S| \ \leq k$$

Then $U \leq k|V|$.

The contrapositive is the pigeonhole principle. Of course, the case for equality is left for the reader to prove.

For Surjections

Let U and V be finite sets such that $|U| < |V|$. Let $f : U \to V$ be any map. Then f cannot be surjective.