Impossibility of Collective Intelligence

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Abstract

Democratization of AI involves training and deploying machine learning models across heterogeneous and potentially massive environments. Diversity of data opens up a number of possibilities to advance AI systems, but also introduces pressing concerns such as privacy, security, and equity that require special attention. This work shows that it is theoretically impossible to design a rational learning algorithm that has the ability to successfully learn across heterogeneous environments, which we decoratively call collective intelligence (CI). By representing learning algorithms as choice correspondences over a hypothesis space, we are able to axiomatize them with essential properties. Unfortunately, the only feasible algorithm compatible with all of the axioms is the standard empirical risk minimization (ERM) which learns arbitrarily from a single environment. Our impossibility result reveals informational incomparability between environments as one of the foremost obstacles for researchers who design novel algorithms that learn from multiple environments, which sheds light on prerequisites for success in critical areas of machine learning such as out-of-distribution generalization, federated learning, algorithmic fairness, and multi-modal learning.

1 Introduction

Artificial intelligence (AI) systems are ubiquitous in every part of society and business. The main driving force of its success is a general-purpose method called machine learning (ML) that can turn gigantic amount of data into powerful predictive models. Some become recommendation engines (Konstan and Riedl 2012, Harper and Konstan 2015), some become facial recognition systems (Kamel et al. 1993, Zhu and Ramanan 2012, Schroff et al. 2015, Buolamwini and Gebru 2018), some become large language models (Aharoni et al. 2019, Brown et al. 2020), and so on. Traditionally, training data for ML algorithms are assumed identically and independently distributed (i.i.d) because they often come from a homogeneous environment. In this scenario, capability to train ML models with billion parameters

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at scale is the cornerstone of several AI-driven breakthroughs in science (Jumper et al. 2021) and engineering (Mirhoseini et al. 2021). However, democratization of AI requires these models to be trained and deployed across heterogeneous and potentially massive environments where the i.i.d. assumption is almost always violated. For example, multi-source data is an essential part in multi-task learning (Caruana 1997, Zhang and Yang 2021), domain generalization (DG) (Blanchard et al. 2011, Muandet et al. 2013, Mahajan et al. 2021, Wang et al. 2021, Zhou et al. 2021), and out-of-distribution (OOD) generalization (Arjovsky 2019, Wald et al. 2021) that improves model performance. Also, real-world data often arrive in different modalities ranging from visual data (e.g., images and videos) to language data (e.g., text and speech). Some believe that cross-modal learning could lead to artificial general intelligence (AGI) (Alayrac et al. 2022, Reed et al. 2022).

But with great power comes great responsibility. The exact same algorithm trains predictive models that aid high-stakes decision-making in health care (Tomašev et al. 2019, Wiens et al. 2019, Ghassemi and Mohamed 2022), employment (Deshpande et al. 2020), and criminal justice (Angwin et al. 2016). In human-centric domains, heterogeneity observed across data points, on the one hand, may represent an inherent diversity within a population that the learning algorithm must account for (Heckman 2001). On the other hand, it may reflect prejudice and societal biases against specific demographic groups that have historically influenced the collection and distribution of data. As a result, a growing concern on the disproportionate impact of algorithmic models has not only sparked a cross-disciplinary collaborations in algorithmic fairness to increase fairness and transparency in today's AI systems, but also created a sense of responsibility among legislatures to regulate them (Barocas et al. 2019, Zafar et al. 2019, Kilbertus et al. 2020, van Giffen et al. 2022). Additionally, when training data are scattered over potentially massive network of remote devices such as mobile phones or siloed data from hospitals, matters pertaining to privacy, security, and access rights may prevent data sharing across environments. Federated learning (FL) addresses these challenges by developing algorithms that rely only on privacy-aware aggregated information without direct access to local data sets (Konečný et al. 2016, McMahan et al. 2017, Li et al. 2020, Kairouz et al. 2021).

As we can see, heterogeneous data on the one hand offers opportunities to build more powerful AI systems. On the other hand, it also presents new challenges that might be in conflict with the principles of democracy.

To gain better understanding on this trade-off, this work aims to answer the following simple question: ***Given training data from n heterogeneous environments, is it possible to design a reasonable learning algorithm that can learn successfully across these environments?*** Inspired by the Arrow's Impossibility Theorem (Arrow 1950) and its descendants in social choice theory (Sen 1970; 2017, Patty and Penn 2019), we adopted an axiomatic method to systematically answer this question. In particular, we aim at deriving algorithms that are strictly better than the standard empirical risk minimization (ERM) commonly used in a homogeneous environment.[1] Our key insight is that we can view any learning algorithm as a choice correspondence, i.e., a function mapping a set of candidate models to a corresponding subset of models (cf. Definition 1), which allows us to subsequently specify the desirable properties of any conceivable learning algorithms, namely, Pareto Optimality (PO), Independence of Irrelevant Hypotheses (IIH), Invariance Restriction (IR), and Collective Intelligence (CI). Intuitively speaking, the PO property generalizes the notion of

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[1]In this paper, ERM loosely refers to the algorithm that learns by minimizing the empirical risk associated with a single homogeneous environment. When heterogeneity is disregarded, another popular baseline is ERM on data pooled from all environments as if it was a single i.i.d. data set.

"minimum risk" in homogeneous setting to heterogeneous environments. Next, the IIH property demands that learning algorithms do not rely on irrelevant information when deciding which hypotheses to choose from the hypothesis class. Similarly, the IR property ensures that the algorithm's outputs remain unchanged for the risk functionals that are informationally identical. Last but not least, we argue that the CI property, which demands that the algorithm leverages information across multiple environments, is indispensable for two reasons. First, an algorithm that lacks CI property fails to recognize the added diversity of data and thereby does not lead to any meaningful improvement over ERM. Second, the lack of CI means that one (or a few) of the heterogeneous environments, which may correspond to a specific individual, demographic group, or institution, over-proportionally determines the decisions of the ML algorithm. In-depth discussion on these axioms and their implications can be found in Section 2.2 and Section 6.

1.1 Our Contributions

The contributions of this work can be summarized as follows:

- We propose to represent any conceivable learning algorithms as choice correspondences that specify for any feasible set of models a nonempty subset of learned models (cf. Definition 1). Unlike conventional perspectives on machine learning, this characterization opens up new ways of analyzing learning algorithms. We also axiomatize the algorithms' behaviour (cf. Assumption 1) and provide connections to existing methods in machine learning (cf. Proposition 1 and Proposition 2).

- We formally show that when the number of environments is larger than two, there exists no learning algorithm that can satisfy PO, IIH, IR, and CI simultaneously (cf. Theorem 1). In other words, the unique algorithm that satisfies PO, IIH, and IR is ERM that simply learns from a single environment (cf. Corollary 1).

- We discuss direct implications of our general impossibility results on critical areas of modern machine learning including OOD generalization, federated learning, algorithmic fairness, and multi-modal learning (cf. Section 5). We also suggest ways to escaping this impossibility result (cf. Section 6).

- Most importantly, the lesson we, and hopefully the readers, can learn from our analyses is that, unlike learning problems in a homogeneous environment, learning problems in heterogeneous environments can become much more challenging because of the *informational incomparability*, i.e., lack of ability to make meaningful comparisons between environments) which limits information that can be shared across environments. *In order to build generalizable, fair, and democratic learning algorithms, it is therefore essential to first create a secure and trustworthy mechanism that incentivizes not only sharing of training data, but also dissemination of other relevant information.*

The paper is organized as follows. Section 2 provides a formulation of learning problem in heterogeneous environments and presents our main results. Next, Section 3 gives characterizations of learning algorithms as choice correspondences. Proofs of our main results are subsequently presented in Section 4. Section 5 then elucidates the implications of our impossibility result on various applications of machine learning, followed by ways to escape this impossibility in Section 6. Finally, Section 7 concludes the paper.
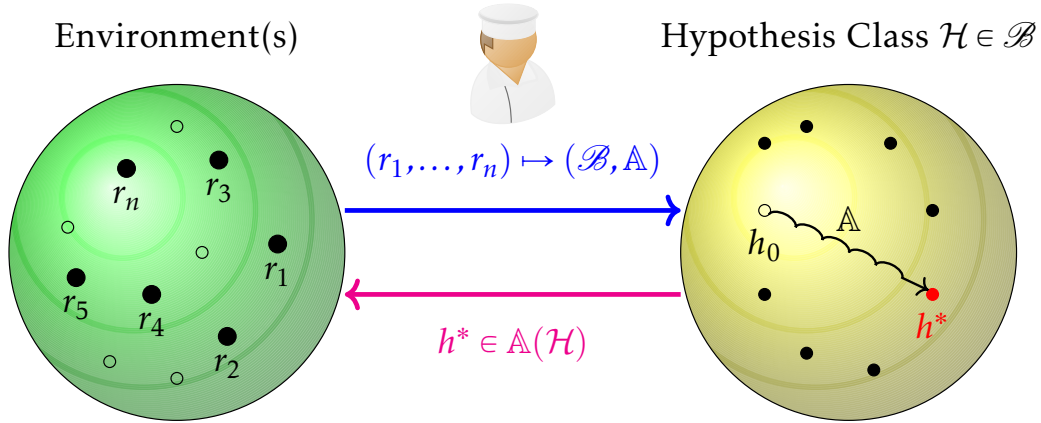
Figure 1: In heterogeneous environments, the task of ML researchers is to design an aggregation rule that takes a risk profile $(r_1, r_2, \ldots, r_n)$ representing the performance measures across n environments and produces a learning structure $(\mathscr{B}, \mathbb{A})$. For each hypothesis class $\mathcal{H} \in \mathscr{B}$, the algorithm $\mathbb{A}$ is implemented to choose the best hypotheses from $\mathcal{H}$. The lower arrow in the figure represents the deployment process.

2 Learning in Heterogeneous Environments

Notation Let $\mathscr{H}$ be a (possibly infinite) set of all conceivable hypotheses[2] and $\mathscr{B}$ a collection of nonempty subsets of $\mathscr{H}$. An algorithm $\mathbb{A}(\mathcal{H})$ specifies for any feasible nonempty subset $\mathcal{H} \in \mathscr{B}$ a nonempty subset $\mathbb{A}(\mathcal{H})$. Further restriction can be made so that any $\mathbb{A}(\mathcal{H})$ must be a unit set, with only one hypothesis chosen from $\mathcal{H}$, but we stick to the more general setting throughout. We call a tuple $(\mathscr{B}, \mathbb{A})$ a *learning structure* which specifies the learning algorithm $\mathbb{A}$ and situations $\mathscr{B}$ in which it is expected to work. Throughout this paper, we denote a set of feasible hypotheses associated with the learning structure $(\mathscr{B}, \mathbb{A})$ by $\mathcal{U} := \bigcup_{\mathcal{H} \in \mathscr{B}} \subseteq \mathscr{H}$. The set of natural numbers is denoted by $\mathbb{N}$, and for $n \in \mathbb{N}$, $[n] := \{1, 2, \ldots, n\}$. The set of real numbers and its positive part are denoted by $\mathbb{R}$ and $\mathbb{R}_+$, respectively.

Given data from n distinct environments[3] and a task-specific hypothesis class $\mathcal{H} \in \mathscr{B}$, the goal of machine learning (ML) is to choose the *best* hypothesis h^* from $\mathcal{H}$. In practice, we typically observe data via a risk profile $\mathbf{r} := (r_1, r_2, \ldots, r_n) : \mathcal{H} \to \mathbb{R}^n$. For each $h \in \mathcal{H}$, the profile $\mathbf{r}(h)$ corresponds to an n-tuple of risk functionals evaluated on h where each $r_i(h)$ is associated with the i-th environment. A learning algorithm $\mathbb{A}$ is defined generically as a map that takes the hypothesis class $\mathcal{H}$ and outputs a set of best hypotheses h^*, i.e., $h_* \in \mathbb{A}(\mathcal{H})$; cf. Section 3 for a formal definition. The task of ML researchers is to design an aggregation rule

$$F : (r_1, r_2, \ldots, r_n) \mapsto (\mathscr{B}, \mathbb{A}), \quad n \in \mathbb{N}, \tag{1}$$

that serves as a recipe of how $\mathbb{A}$ puts a risk profile $\mathbf{r} := (r_1, r_2, \ldots, r_n)$ together. In this paper, no restriction is made on the domain of F, i.e., for all $\mathbf{r}$, $F(\mathbf{r}) \neq \varnothing$. Figure 1 illustrates this perspective on machine learning.

The simplest example of a risk profile appears in supervised learning. Given a sample

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[2]We will write "hypothesis" and "model" interchangeably.

[3]The term "environments" refers to sources of data, e.g., individuals, demographic groups, tasks, mobile phones, modalities, etc, that gives rise to the observed heterogeneity.

$(x_i, y_i)_{i=1}^n$ of size n, we treat each (x_i, y_i) individually as a separate observation that forms the risk profile $\mathbf{r}(h) = (\ell(h(x_1), y_1), \ell(h(x_2), y_2), \ldots, \ell(h(x_m), y_m))$ where $r_i(h) = \ell(h(x_i), y_i)$ is a discrepancy between the true output y_i and the prediction $h(x_i)$. Common loss functionals for classification problem include a cross-entropy loss $\ell(h(x), y) = -\sum_{k=1}^K y_k \log(p_k)$ where K is the number of classes and p_k is the predicted probability that x is of class k, and a hinge loss $\ell(h(x), y) = \max(0, 1 - yh(x))$. For regression problems, the square loss $\ell(h(x), y) = (y - h(x))^2$ and absolute loss $\ell(h(x), y) = |y - h(x)|$ are among the most common choices. As we will discuss in Section 5, problems in multi-task learning, domain generalization, federated learning, algorithmic fairness, and multi-modal learning can also be formulated as (1).

Our main result shows that under some reasonable conditions there exists no aggregation rule F that is compatible with a learning structure $(\mathscr{B}, \mathbb{A})$ when the number of environments is at least three.[4]

Theorem 1. *For $n \geqslant 3$, there exists no learning structure $(\mathscr{B}, \mathbb{A})$ that is internally consistent and satisfies Pareto Optimality (PO), Independence of Irrelevant Hypotheses (IIH), Invariance Restriction (IR), and Collective Intelligence (CI) simultaneously.*

Internal consistency is a necessary property of any learning algorithms which we define in Section 3. The PO, IIH, IR, and CI are properties of learning structures defined with respect to the aggregation rule F. We define and discuss them further in Section 2.2. The following corollary follows immediately from Theorem 1.

Corollary 1. *A unique algorithm that satisfies PO, IIH, and IR simultaneously is of the form*

$$\mathbb{A}(\mathcal{H}) = \{h \in \mathcal{H} : r_i(h) \leqslant r_i(g) \text{ for all } g \in \mathcal{H}\}, \quad \mathcal{H} \in \mathscr{B}, \tag{2}$$

for some $i \in [n]$.

To summarize, Theorem 1 and Corollary 1 imply that if we treat PO, IIH, and IR as primitive properties that any learning algorithms in heterogeneous environments must satisfy, the only possibility that can come out of (1) is an algorithm that greedily minimizes an environment-specific risk functional. In other words, ***it is impossible for ML researchers to design learning algorithms that can leverage heterogeneity across environments***.

2.1 Homogeneous Environment

When $n = 1$, an undisputed recipe is risk minimization (RM): $\mathbb{A}(\mathcal{H}) = \{h \in \mathcal{H} : r(h) \leqslant r(g) \text{ for all } g \in \mathcal{H}\}$. A popular example is traditional supervised learning problem. Let $\mathcal{X}$ and $\mathcal{Y}$ be non-empty input and output spaces, respectively. Let X and Y be random variables taking values in $\mathcal{X}$ and $\mathcal{Y}$. We denote the realizations of X (resp. Y) by x (resp. y). Given a loss function $\ell : \mathcal{Y} \times \mathcal{Y} \to \mathbb{R}_+$, supervised learning aims to find $h \in \mathcal{H}$ that minimizes the expected loss

$$r(h) = \int \ell(y, h(x)) \, dP(x, y)$$

defined with respect to some unknown probability distribution $P(X, Y)$ on $\mathcal{X} \times \mathcal{Y}$. For example, in animal species categorization, the input x is a photo from a camera trap and the label y corresponds to one of the animal species (Beery et al. 2018; 2020). Accurate species

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[4]Interestingly, Stein (1956) proves that when the dimensionality is *at least three*, a usual maximum likelihood estimator (MLE) for the mean of the multivariate Gaussian distribution $\mathcal{N}(\theta, I)$ is inadmissible, i.e., there exists an estimator that always achieves smaller total mean squared error regardless of the true θ. This phenomenon is known to statisticians as Stein's paradox.

classification from camera traps can help ecologists better understand wildlife biodiversity and monitor endangered species.

In practice, we can only observe an independent and identically distributed (i.i.d.) sample $(x_1, y_1), \ldots, (x_m, y_m)$ of size m from $P(X, Y)$. The empirical risk minimization (ERM) framework, the backbone of supervised learning, approximates the expected loss $r(h)$ by a *regularized* empirical counterpart

$$\hat{r}(h) = \frac{1}{m} \sum_{i=1}^{m} \ell(y_i, h(x_i)) + \lambda \Omega(\|h\|_{\mathcal{H}}),$$

where $\Omega(\cdot)$ is a monotonically increasing function and $\lambda > 0$ is a regularization constant. If λ vanishes at an appropriate rate, it follows from the law of large number (LLN) that, for each $h \in \mathcal{H}$, $\hat{r}(h)$ converges in probability to $r(h)$. Hence, ERM chooses the best hypotheses by minimizing the empirical risk, i.e., $\mathbb{A}(\mathcal{H}) = \{h : h \in \mathcal{H}$ and $\hat{r}(h) \leqslant \hat{r}(g)$ for all $g \in \mathcal{H}\}$. The generalization capability of h to unseen data (e.g., unseen camera trap photos) has been studied extensively under the i.i.d. assumption and has been shown to be governed by the uniform consistency of $\hat{r}(h)$ over $\mathcal{H}$.

2.2 Heterogeneous Environments

On the other hand, when $n \geqslant 2$, there is an infinite number of possible recipes as well as additional challenges that arise from heterogeneity such as non-i.i.d. data, privacy, fairness, security, competition, and communication efficiency. To find the "right" recipes, we impose the following application-agnostic conditions.

1. **Pareto Optimality (PO):** For all $f, g \in \mathcal{U}$, $r_i(f) < r_i(g)$ for all $i \in [n]$ implies that $\{f\} = \mathbb{A}(\{f, g\})$.

2. **Independence of Irrelevant Hypotheses (IIH):** For any two profiles $\mathbf{r}$ and $\mathbf{r}'$ in the domain of F, and for any $\mathcal{H} \in \mathcal{B}$, with the learning algorithms $\mathbb{A}$ and $\mathbb{A}'$ corresponding to $\mathbf{r}$ and $\mathbf{r}'$, respectively:

$$[\forall i : (\forall f, g \in \mathcal{H} : r_i(f) \leqslant r_i(g) \Leftrightarrow r_i'(f) \leqslant r_i'(g))] \Rightarrow \mathbb{A}(\mathcal{H}) = \mathbb{A}'(\mathcal{H}).$$

3. **Invariance Restriction (IR):** For any pair of risk profiles $\mathbf{r}, \mathbf{r}'$ for which there exists $(a_1, \ldots, a_n) \in \mathbb{R}^n$ and $(b_1, \ldots, b_n) \in \mathbb{R}^n_+$ such that $r_i(h) = a_i + b_i r_i'(h)$ for all $i \in [n]$, $\mathbb{A}(\mathcal{H}) = \mathbb{A}'(\mathcal{H})$ for all $\mathcal{H} \in \mathcal{B}$.

4. **Collective Intelligence (CI):** There exists no $i \in [n]$ such that for all pair $f, g \in \mathcal{U}$ and for all $\mathbf{r}$ in the domain of F, $r_i(f) < r_i(g)$ implies that $\{f\} = \mathbb{A}(\{f, g\})$.

First of all, the PO condition requires that the algorithm simply chooses f over g if f is strictly better than g in *all* environments. It is hard to argue against PO as a desirable property for a learning algorithm. Surely, if f is unanimously superior to g, there is no reason for $\mathbb{A}$ to choose g over f. Secondly, the IIH requires that when choosing between two hypotheses, the algorithm cannot rely on any other information apart from the relative risks between these two hypotheses across all environments. Thirdly, the IR demands that the algorithm must be invariant to any transformation of the risk profile that is *informationally identical* to the original one. It generalizes the observation that most learning algorithms in the homogeneous environment are invariant to the strictly increasing transformation of the

risk functional, which is discussed later in Remark 1, to a set of n-tuples of risk functionals. Since r_i is informationally identical to r_i' for all $i \in [n]$, the behaviour of the algorithm must remain unchanged. Last but not least, the CI demands that the algorithm must exhibit a collective behaviour in the sense that it leverages information from multiple environments when learning from a hypothesis class $\mathcal{H}$.

It is instructive to understand intuitively how these assumptions respectively yield Theorem 1 and Corollary 1. First, we consider all conceivable algorithms defined on $\mathscr{H}$. The undesirable ones are then eliminated by demanding that they satisfy some essential properties. By successively imposing internal consistency, PO, IIH, and IR, (2) in Corollary 1 remains as the only possibility. Adding CI eliminates this only possibility, giving rise to the impossibility result in Theorem 1. The detailed proofs of these two results can be found in Section 4.

Three-Michelin-Starred algorithms   As a metaphor, let us think of $\mathscr{H}$ as all conceivable dishes and ML researcher as a chef who has the risk profile $(r_1,\dots,r_n)$ at their disposal as n different ingredients to develop a new recipe $\mathbb{A}$. From any feasible menu $\mathcal{H} \in \mathscr{B}$, $\mathbb{A}(\mathcal{H})$ consists of the most delicious dishes cooked by following the recipe $\mathbb{A}$; see Figure 1. To promote culinary excellence, a Michelin star is awarded to any recipe that adheres to PO, IIH, or IR, the criteria required by the Michelin guide. The recipes receive three Michelin stars if they meet all three criteria. Unfortunately, Theorem 1 and Corollary 1 show that despite these reasonable-looking criteria, *there cannot be a three-Michelin-starred recipe unless the chef is willing to use only one out of n ingredients*.

3   Learning Algorithm as Choice Correspondence

This section provides the details of learning structure $(\mathscr{B}, \mathbb{A})$ and the internal consistency condition which are the main building blocks of our contributions.

Definition 1. *Let $\mathscr{B}$ be a collection of nonempty subsets of a hypothesis space $\mathscr{H}$. A learning algorithm is a map $\mathbb{A} : \mathscr{B} \to 2^{\mathscr{H}} \setminus \varnothing$ such that $\mathbb{A}(\mathcal{H}) \subseteq \mathcal{H}$ for all $\mathcal{H} \in \mathscr{B}$.*

Note that we assume that $\mathbb{A}(\mathcal{H}) \neq \varnothing$ for all $\mathcal{H} \in \mathscr{B}$. That is, we require that the learning algorithm will always produce some outputs, which is a typical requirement in most applications. In the Probably Approximately Correct (PAC) learning model, for example, the setting in which the learning algorithms must output a hypothesis from $\mathcal{H}$ is known as a *proper* learning; see, e.g., Shalev-Shwartz and Ben-David (2014; Remark 3.2) for further discussion. Nevertheless, it is not impossible to deliberately construct a learning algorithm where $\mathbb{A}(\mathcal{H}) = \varnothing$ for some $\mathcal{H}$. While this requirement can be relaxed without altering our main results, it introduces several technical challenges. To help novice readers better understand our work, we thus adopt Definition 1 throughout. Expert readers who are interested in detailed exposition may consult, e.g., Kreps (2012; Sec. 1.6).

It is also reasonable in the context of our work to assume that $\{f\} \in \mathscr{B}$, $\{f,g\} \in \mathscr{B}$, and $\{f,g,h\} \in \mathscr{B}$ for any $f,g,h \in \mathcal{U}$ because these are the simplest settings in which we expect any learning algorithm to work. These conditions will also become crucial for Proposition 2.

3.1 Learning Structure

The hypothesis space $\mathscr{H}$ is predetermined by the learning problems, whereas $(\mathscr{B}, \mathbb{A})$ depends on the researchers who design the algorithm. The collection $\mathscr{B}$ provides the contexts under which the algorithm $\mathbb{A}$ is expected to be operational while $\mathbb{A}$ constitutes all instructions on how to choose the best hypotheses from any $\mathcal{H} \in \mathscr{B}$; see Figure 2a for an illustration. The Axiom of Choice (AC) allows us to reason about existence of $\mathbb{A}$ for any $\mathscr{B}$ (Zermelo 1904, Moore 1982).

More specifically, when faced with a learning problem, ML researchers develop either a parametric or nonparametric model and a learning algorithm that constitutes $(\mathscr{B}, \mathbb{A})$. The model consists of two important components, namely, parameters and hyperparameters. The former need to be fine-tuned by the algorithm, whereas the latter must be specified by the researchers or via model selection procedures such as cross validation (CV) and probabilistic measures like AIC, BIC, and MDL. For example, parameters of a deep neural network (DNN), one of the most popular models in machine learning, include weights and biases. Essential hyperparameters include number of hidden layers, learning rate, activation function, minibatch size, and the number of epoches among others. The associated algorithm $\mathbb{A}$ consists of initialization, optimization, and regularization strategies to fine-tune model parameters. In this case, elements of $\mathscr{B}$ are generated by the values of all hyperparameters, while each $\mathcal{H} \in \mathscr{B}$ is composed of all models that share the same hyperparameters.

Model selection Model selection is an integral part of the machine learning pipeline (Shalev-Shwartz and Ben-David 2014; Ch. 11). In addition to configuring model hyperparameters, ML practitioners may also have to choose among different types of existing models, e.g., logistic regression, SVM, KNN, and DNN. Although there is a growing interest in the deep learning, AutoML and meta-learning communities to automate model selection, the learning structure remains unchanged. That is, model selection is a learning problem where the learning structure is defined over $\mathscr{B}$, i.e., $(\mathscr{C}, \mathbb{M})$ where $\mathscr{C}$ is a collection of nonempty subsets of $\mathscr{B}$ and $\mathbb{M}$ is a model selection strategy. We will pursue this exciting direction in future work.

3.2 Internal Consistency

Since not all of the learning structures implied by Definition 1 exhibit desirable behaviours, we further impose restrictions known as *internal consistency* on them.

Assumption 1 (Internal consistency). *For $\mathcal{F}, \mathcal{G} \in \mathscr{B}$,*

  (1) if $h \in \mathbb{A}(\mathcal{F})$ and $h \in \mathcal{G} \subseteq \mathcal{F}$, then $h \in \mathbb{A}(\mathcal{G})$.

  (2) If $h \in \mathcal{G}$ and, for $\mathcal{G} \subseteq \mathcal{F}$, $h \in \mathbb{A}(\mathcal{F})$, then $\mathbb{A}(\mathcal{G}) \subseteq \mathbb{A}(\mathcal{F})$.

These two conditions, known as Property α (Chernoff 1954)(Sen 1970; Ch. 1*) and Property β (Sen 2017; pp. 320), are together one of the interpretations of rationality of choice in mainstream economic theory. In the context of our work, the former demands that any hypothesis h that is chosen from $\mathcal{F}$ must also be chosen from $\mathcal{G}$, if it is a contraction of $\mathcal{F}$ that also contains h. For example, suppose that $\mathcal{F}$ is composed of all polynomials of degree smaller than p and $\mathcal{G}$ consists of all polynomials of degree smaller than q where $q \leqslant p$, i.e., $\mathcal{G} \subseteq \mathcal{F}$. Then, if $\mathbb{A}$ chooses the polynomial of degree $t < q \leqslant p$ from $\mathcal{F}$ as a solution, it must also be chosen again from $\mathcal{G}$. Figure 2b illustrates a violation of the α Property. The latter, albeit less intuitive, can be interpreted as follows. If there is the hypothesis h that is chosen
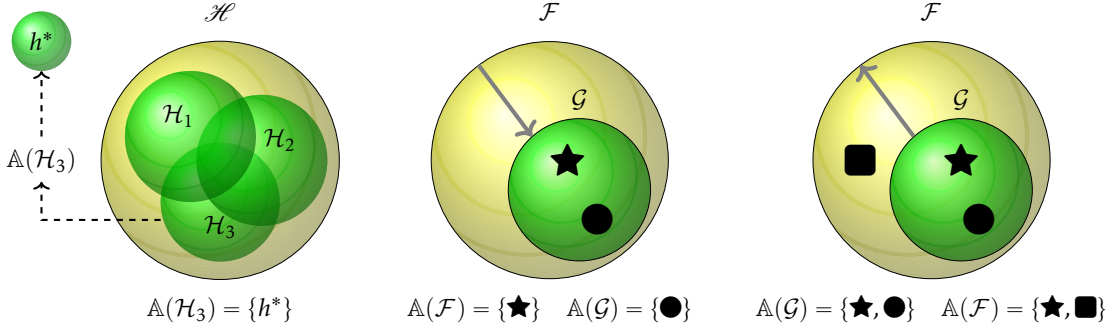
(a) Learning Structure $(\mathscr{B}, \mathbb{A})$ (b) Violation of α Property   (c) Violation of β Property

Figure 2: (a) Given a problem-dependent hypothesis space $\mathscr{H}$, ML researchers design a learning structure $(\mathscr{B}, \mathbb{A})$ composed of a collection of subsets $\mathscr{B}$ and learning algorithm $\mathbb{A}$. For each $\mathcal{H} \in \mathscr{B}$, the set $\mathbb{A}(\mathcal{H})$ consists of the hypotheses learned by the algorithm $\mathbb{A}$. (b) The α Property is violated when a contraction of hypothesis class, e.g., by fixing the values of some parameters, can change the behaviour of $\mathbb{A}$. (c) The β Property is violated when an expansion of the hypothesis class, e.g., by adding more parameters, can change how the optimal hypotheses are chosen.

from $\mathcal{G}$ and subsequently from $\mathcal{F}$, which is an expansion of $\mathcal{G}$, then all other hypotheses that are considered *equally good*[5] to h in $\mathcal{G}$ must also be chosen from $\mathcal{F}$. For instance, let $f, g \in \mathcal{G}$ be polynomials of degree t with different values of coefficients. If $\mathbb{A}$ chooses both f and g from $\mathcal{G}$, and f from $\mathcal{F}$, then it must also choose g from $\mathcal{F}$. Unlike the first condition, this condition characterizes the behaviour of the algorithm under the expansion of hypothesis space. Figure 2c illustrates a violation of the β property.

A violation of Assumption 1 can lead to an inconsistent behaviour. For example, it is possible that for three distinct hypotheses f, g, h from $\mathscr{H}$, $\mathbb{A}(\{f, g\}) = \{f\}$, but $\mathbb{A}(\{f, g, h\}) = \{g\}$.

While rationality of choice has been studied from different angles in economic theory (Sen 2017; Ch. A2) and other characterizations such as γ Property (Sen 2017; pp. 318), δ Property (Sen 2017; pp. 320), and choice coherence (Kreps 2012; Def. 1.1) also exist, we chose the α and β properties for two reasons. First, contraction and expansion of the hypothesis class are the most common scenarios encountered in machine learning. For example, adjusting the values of hyperparameters such as regularization constant and number of hidden nodes will either contract or expand the hypothesis class. Second, while there exist evidences of scenarios in which human decision may violate internal consistency (see, e.g., Sen (1993) and Sen (2017; Ch. A2) for counterexamples), it is hard to argue against internal consistency as a desirable property for learning algorithms. The lack thereof makes model selection cumbersome, as discussed further in Section 6. Moreover, α and β properties suffice to obtain Theorem 1 without additional rationality conditions.

One of the immediate result is that any risk minimization (RM) algorithm satisfies the internal consistency property.

Proposition 1. *Let $\mathbb{A}_r$ be a risk minimization (RM) learning algorithm, i.e.,*

$$\mathbb{A}_r(\mathcal{H}) = \{h \in \mathcal{H} : r(h) \leqslant r(g) \text{ for all } g \in \mathcal{H}\} \tag{3}$$

for some real-valued risk functional $r : \mathcal{U} \to \mathbb{R}$. Then, $\mathbb{A}_r$ satisfies internal consistency.

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[5]In this work, $h, g \in \mathcal{U}$ are considered by $\mathbb{A}$ as *equally good* if and only if $h, g \in \mathbb{A}(\mathcal{H})$.

A majority of learning algorithms including empirical risk minimization (ERM) (Vapnik 1991), structural risk minimization (SRM) (Shawe-Taylor et al. 1996), and invariant risk minimization (IRM) (Arjovsky et al. 2019, Ahuja et al. 2020) fall into this category. We say that $\mathbb{A}$ can be represented by r if it minimizes r.

Remark 1. *A few remarks on RM algorithms follow.*

1. *Note that a risk functional that represents $\mathbb{A}_r$ is not unique. For any strictly increasing function $c : \mathbb{R} \to \mathbb{R}$, $r'(h) = c(r(h))$ is a new risk functional that also represents $\mathbb{A}_r$. From learning perspective, they carry the same information, i.e., $\mathbb{A}_r$ and $\mathbb{A}_{r'}$ must behave identically. Any two algorithms $\mathbb{A}_r$ and $\mathbb{A}_{r'}$ that can be represented by r and r' respectively are said to be equivalent if such a strictly increasing function exists.*

2. *This property also makes learning algorithms immune to trivial manipulations (e.g., strategic behaviours and adversarial attacks) and perturbations (e.g., unreliable communication over massive networks) of the risk functional. The IR condition generalizes this property to a set of n-tuples of risk functionals.*

3. *When $\mathscr{H}$ is finite, it has been shown that any $\mathbb{A}$ that satisfies Assumption 1 must be a risk minimizer (3) (Kreps 2012; Prop. 1.2c). However, this is not always the case for an infinite $\mathscr{H}$, at least not without further assumptions on the learning structure; see, e.g., Kreps (2012; Sec. 1.5). Thus, a learning algorithm that satisfies internal consistency is not necessarily a risk minimizer.*

Proposition 2. *Suppose that $\mathbb{A}$ satisfies the internal consistency property. For every pair $f, g \in \mathcal{U}$, let $f \succeq_\mathbb{A} g$ if $f \in \mathbb{A}(\{f, g\})$. Then, the binary relation $\succeq_\mathbb{A}$ is complete and transitive.[6] Moreover, for every $\mathcal{H} \in \mathscr{B}$,*

$$\mathbb{A}(\mathcal{H}) = \mathbb{A}_{\succeq_\mathbb{A}}(\mathcal{H}) := \{h \in \mathcal{H} : h \succeq_\mathbb{A} g \text{ for all } g \in \mathcal{H}\}.$$

The binary relation $\succeq_\mathbb{A}$ is known as a *revealed preference* of $\mathbb{A}$ and this proposition implies that as long as $\mathbb{A}$ satisfies internal consistency, $\mathbb{A}(\mathcal{H})$ for any $\mathcal{H} \in \mathscr{B}$ coincides with those obtained from the learning algorith $\mathbb{A}_{\succeq_\mathbb{A}}$ defined in terms of preferences that are revealed by $\mathbb{A}$ operated on one- and two-element subsets of hypotheses; see, e.g., Sen (1971) for details on revealed preference. By virtue of Proposition 2, we can work with $\mathbb{A}$ by restricting it over pairs of hypotheses only.

To close this section, we elucidate the advantages and disadvantages of representing learning algorithms as choice correspondences. On the one hand, choice correspondences abstract away the more complicated aspects of algorithms, allowing us to focus at a conceptual level. As Proposition 1 shows, any conceivable algorithms, including not only those that behave as a risk minimizer, can be captured by our formulation. As a result, it broadens the scope of our analyses to include scenarios that might involve moral philosophy and value judgements, and those in which the objective function cannot be represented numerically. On the other hand, our analyses completely ignore the *optimization* strategies. In practice, the contraction and expansion of hypothesis class may have an influence on the results optimization methods, leading to a violation of internal consistency. However, this does not invalidate the internal consistency as a desirable property.

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[6]The binary relation $\succeq$ on $\mathcal{U}$ is *complete* if for every pair f and g from $\mathcal{U}$, either $f \succeq g$ or $g \succeq f$ (or both). It is *transitive* if $f \succeq g$ and $g \succeq h$ implies $f \succeq h$.

4 Proofs

This section provides detailed proofs of our results, especially Theorem 1, which relies heavily on the insights from the original proof of Arrow's General Possibility Theorem (Arrow 1950) and its simplification in Sen (2017; pp. 286).

4.1 Theorem 1 and Corollary 1

Let $\mathcal{E}$ be a set of environments. Crucial to the proof is the idea of a set $\mathcal{E}$ being "decisive".

Definition 2 (Decisiveness). *A set of environments $\mathcal{E}$ is said to be* locally decisive *over a pair of hypotheses f, g if $r_e(f) < r_e(g)$ for all $e \in \mathcal{E}$ implies that $\{f\} = \mathbb{A}(\{f, g\})$. It is said to be* globally decisive *if it is locally decisive over every pairs of hypotheses.*

The following two intermediate results provide basic properties of decisive set of environments $\mathcal{E}$.

Lemma 1. *If a set of environments $\mathcal{E}$ is decisive over any pair $\{f, g\}$, then $\mathcal{E}$ is globally decisive.*

Proof. Let $\{p, q\}$ be any other pair of hypotheses that is different from $\{f, g\}$. Assume that in every environment e in $\mathcal{E}$, $r_e(p) < r_e(f)$, $r_e(f) < r_e(g)$, and $r_e(g) < r_e(q)$. For all other environments e' not in $\mathcal{E}$, we assume that $r_{e'}(p) < r_{e'}(f)$ and $r_{e'}(g) < r_{e'}(q)$ and leave the remaining relations unspecified. By PO condition, $\{p\} = \mathbb{A}(\{p, f\})$ and $\{g\} = \mathbb{A}(\{q, g\})$. By the decisiveness of $\mathcal{E}$ over $\{f, g\}$, we have $\{f\} = \mathbb{A}(\{f, g\})$. Then, it follows from the transitivity implied by Proposition 2 that $\{p\} = \mathbb{A}(\{p, q\})$. By IIH condition, this must be related only to the relation between p and q. Since we have only specified information in $\mathcal{E}$, $\mathcal{E}$ must be decisive over $\{p, q\}$ and for all other pairs. Hence, $\mathcal{E}$ is globally decisive. ∎

Lemma 2. *If a set of environments $\mathcal{E}$ consists of more than one element and is decisive, then some proper subset of $\mathcal{E}$ is also decisive.*

Proof. Since $n \geqslant 3$, we can partition $\mathcal{E}$ into two subsets $\mathcal{E}_1$ and $\mathcal{E}_2$. Assume that $r_e(f) < r_e(g)$ and $r_e(f) < r_e(h)$ in every environment $e \in \mathcal{E}_1$ with the relation between g and h unspecified. Let $r_{e'}(f) < r_{e'}(g)$ and $r_{e'}(h) < r_{e'}(g)$ in every environment $e' \in \mathcal{E}_2$. By the decisiveness of $\mathcal{E}$, we have $\{f\} = \mathbb{A}(\{f, g\})$. Now, if h is at least as good as f for some environments over $\{h, f\}$, then we must have $\{h\} = \mathbb{A}(\{h, g\})$ for that configuration. Since we do not specify relation over $\{g, h\}$ other than those in $\mathcal{E}_2$, and $r_{e'}(h) < r_{e'}(g)$ in $\mathcal{E}_2$, $\mathcal{E}_2$ is decisive over $\{g, h\}$. By Lemma 1, $\mathcal{E}_2$ must be globally decisive. That is, some proper subset of $\mathcal{E}$ is indeed decisive for that particular case. To avoid this possibility, we must remove the assumption that h is at least as good as f. But then f must be better than h. However, no environment has this relation over $\{f, h\}$ other than those in $\mathcal{E}_1$ where f is better than h. Clearly, $\mathcal{E}_1$ is decisive over $\{f, h\}$. Thus, by Lemma 1, $\mathcal{E}_1$ is globally decisive. So either E_1 or E_2 must be decisive. This completes the proof. ∎

An important observation from the proofs of Lemma 1 and Lemma 2 is that we rely only on the relative rankings of hypotheses. The justification of this restriction will be made in Section 6. For now, we are in a position to prove Theorem 1.

Proof of Theorem 1. Consider any two risk profiles $\mathbf{r}$ and $\mathbf{r}^*$ such that for any f, g and for all $i \in [n]$, $r_i(f) < r_i(g) \Leftrightarrow r_i^*(f) < r_i^*(g)$. For every pair $\{f, g\}$, there exists a positive affine

transformation $\{\varphi_i\}$ applied to $\mathbf{r}^*$ such that

$$\mathbf{r}'_i(f) = \varphi_i(r^*_i(f)) = r_i(f) \quad \text{and} \quad \mathbf{r}'_i(g) = \varphi_i(r^*_i(g)) = r_i(g) \text{ for all } i \in [n].$$

By IIH condition, $\{f\} = \mathbb{A}(\{f,g\})$ if and only if $\{f\} = \mathbb{A}'(\{f,g\})$ and by IR condition, $\{f\} = \mathbb{A}'(\{f,g\})$ if and only if $\{f\} = \mathbb{A}^*(\{f,g\})$. Since this holds pair by pair, clearly $\mathbb{A}(\mathcal{H}) = \mathbb{A}'(\mathcal{H})$ for all $\mathcal{H} \in \mathscr{B}$. As a result, we can rely on the relative ranking of any two hypotheses when comparing them. Next, by PO condition, the set of all environments $\mathcal{E}$ is decisive. By Lemma 2, some proper subset of $\mathcal{E}$ must also be decisive. Given that smaller subset of environments, some proper subset of it must also be decisive, and so on. Since the number of environments is finite, the set will eventually contain just a single environment that is decisive. However, this violates CI condition, resulting in the impossibility. ∎

Corollary 1 follows by omitting the last step in the proof of Theorem 1.

4.2 Proposition 1

Proof of Proposition 1. (α Property): For any $\mathcal{F}, \mathcal{G} \in \mathscr{B}$ such that $\mathcal{G} \subseteq \mathcal{F}$, let $h \in \mathbb{A}_r(\mathcal{F})$. Thus, $r(h) \leqslant r(g)$ for all $g \in \mathcal{F}$. Assume that $h \in \mathcal{G}$ but $h \notin \mathbb{A}_r(\mathcal{G})$. This implies that there exists another hypothesis $f \in \mathcal{G}$ for which $r(f) < r(h)$. However, since f is also in $\mathcal{F}$, it contradicts with $r(h) \leqslant r(g)$ for all $g \in \mathcal{F}$. Hence, h must also be in $\mathbb{A}_r(\mathcal{G})$. ($\beta$ Property): For any $h,g \in \mathcal{G}$, let $h,g \in \mathbb{A}_r(\mathcal{G})$. Assume that $h \in \mathbb{A}_r(\mathcal{F})$ but $g \notin \mathbb{A}_r(\mathcal{F})$. This implies that $r(h) \leqslant r(f)$ for all $f \in \mathcal{F}$ and $r(g) > r(h)$, which contradicts the fact that $g \in \mathbb{A}_r(\mathcal{G})$. Hence, g must also be in $\mathbb{A}_r(\mathcal{F})$. This implies that $\mathbb{A}_r(\mathcal{G}) \subseteq \mathbb{A}_r(\mathcal{F})$. ∎

4.3 Proposition 2

Proof of Proposition 2. Since $\mathbb{A}(\{f,g\}) \neq \varnothing$, $\succeq_\mathbb{A}$ is complete. If $f \succeq_\mathbb{A} g$ and $g \succeq_\mathbb{A} h$, then $f \in \mathbb{A}(\{f,g\})$ and $g \in \mathbb{A}(\{g,h\})$. By β Property, if $g \in \mathbb{A}(\{f,g,h\})$, then $f \in \mathbb{A}(\{f,g,h\})$. Also, if $h \in \mathbb{A}(\{f,g,h\})$, then $g \in \mathbb{A}(\{f,g,h\})$. Hence, we have $f \in \mathbb{A}(\{f,g,h\})$ in any case. By α Property, $f \in \mathbb{A}(\{f,h\})$ and $f \succeq_\mathbb{A} h$, which shows that $\succeq_\mathbb{A}$ is transitive. Next, we show that $\mathbb{A}(\mathcal{H}) = \mathbb{A}_{\succeq_\mathbb{A}}(\mathcal{H})$ for every $\mathcal{H} \in \mathscr{B}$. Assume that $f \in \mathbb{A}(\mathcal{H})$. By α Property, we have for every $g \in \mathcal{H}$ that $f \in \mathbb{A}(\{f,g\})$. This implies that $f \succeq_\mathbb{A} g$ and thus $f \in \mathbb{A}_{\succeq_\mathbb{A}}(\mathcal{H})$. Now, let us assume that $f \neq g$, $f \in \mathbb{A}_{\succeq_\mathbb{A}}(\mathcal{H})$, and $g \in \mathbb{A}(\mathcal{H})$. Then, $f \in \mathbb{A}(\{f,g\})$ and by β Property, $f \in \mathbb{A}(\mathcal{H})$, which completes the proof. ∎

5 Implications on Related Work

This section elucidates some connections to social choice theory and discusses direct implications of our main result on several sub-fields of machine learning.

5.1 Social Choice Theory

Our main result is a reincarnation of the Arrow's Impossibility Theorem (Arrow 1950, Sen 2017) which forms the basis of modern social choice theory; see, e.g., Patty and Penn (2019) and references therein. To understand this, suppose that $\mathscr{H}$ consists of a finite number of at least three hypotheses representing a set of alternatives. In Arrow's setting, he is interested in the social welfare function (SWF):

$$F : (\succeq_1, \succeq_2, \ldots, \succeq_n) \mapsto \succeq$$

that aggregates preferences of n individuals $\geq_1, \ldots, \geq_n$ over the set of alternatives to obtain the *social* preference $\geq$. He also demands $\geq$ to be rational, i.e., complete and transitive. The impossibility result is established under similar set of axioms, namely, Universal Domain (no restriction on the domain of F), Pareto Principle (our PO), Independence of Irrelevant Alternatives (our IIH), and Non-dictatorship (our CI). As is apparent in the proof of Theorem 1 (cf. Section 4.1), our IR condition further restricts usable information of the risk profile to relative rankings of hypotheses. Moreover, as shown in Proposition 2, the internal consistency ensures that the revealed preference of $\mathbb{A}$ will be complete and transitive.

The most closely related to ours is Sen (1970) which considers the social welfare functional (SWFL)

$$F : (u_1, u_2, \ldots, u_n) \mapsto C(\cdot)$$

that aggregates cardinal utility functions of n individuals to obtain the social choice function (SCF) $C(\cdot)$ over the set of alternatives. By weakening some restrictions on $C(\cdot)$, possibility results start to emerge; see, e.g., Sen (2017; T.A2*.1, pp. 316). In fact, these connections are natural once one realizes that as soon as algorithmic models have societal impact, machine learning becomes a social choice problem, as succinctly put by the best-selling author Brian Christian in *The Alignment Problem: Machine Learning and Human Values*: "[...] every machine-learning system *is* a kind of parliament, in which the training data represent some larger electorate–and, as in any democracy, it's crucial to ensure that everyone gets a vote." (Christian 2020; pp. 33).

A few distinctions deserve further discussion though. First of all, the fact that the impossibility still persists even when the size of $\mathcal{H}$ becomes infinite suggests that neither adding more data nor scaling up the models alone will get us out of this roadblock. On the contrary, the increase in the number of alternatives may even lead to the impossibility results; see, e.g., Theorem 4*2 and 4*3 in Sen (2017). Secondly, choice correspondence is a basis of microeconomic theory in that it is a model of human choice behaviour (Kreps 2012; Chapter One). In this work, we instead use it to model the behaviour of artificial agents (i.e., learning algorithms). Lastly, our work differs from the textbook machine learning in that we start from all conceivable learning algorithms and then rule out those that are incompatible with the desirable properties until we arrive at the unique algorithm (or none at all).

5.2 Multi-source Learning

Learning from multi-source data has a long history in machine lerning (Cortes et al. 2021, Hoffman et al. 2018, Zhao et al. 2018, Blanchard et al. 2011, Muandet et al. 2013, Mahajan et al. 2021, Wang et al. 2021, Zhou et al. 2021, Zhang and Yang 2021, Sener and Koltun 2018). In critical areas like health care, we typically have access to data from n distinct environments, which can be represented by n probability distributions $P_1(X, Y), \ldots, P_n(X, Y)$. In multiple-source adaptation (MSA) and domain generalization (DG), the risk profile can be expressed in terms of the empirical losses:

$$\mathbf{r}(h) = (\hat{r}_1(h), \ldots, \hat{r}_n(h)) = \left(\frac{1}{m_1} \sum_{k=1}^{m_1} \boldsymbol{\ell}(h(x_k^1), y_k^1), \ldots, \frac{1}{m_n} \sum_{k=1}^{m_n} \boldsymbol{\ell}(h(x_k^n), y_k^n) \right)$$

where $(x_k^i, y_k^i)_{k=1}^{m_i}$ denotes a sample of size m_i from $P_i(X, Y)$. The empirical losses $(\hat{r}_1(h), \ldots, \hat{r}_n(h))$ measures average performances of the hypothesis h across n environments. When there are different tasks across these environments, i.e., multi-task learning (MTL), we can rewrite

the risk profile as $\mathbf{r}(\mathbf{h}) = (\hat{r}_1(\mathbf{h}), \ldots, \hat{r}_n(\mathbf{h}))$ where

$$\hat{r}_i(\mathbf{h}) := \sum_{j=1}^{n} \frac{\mathbb{1}[j=i]}{m_j} \sum_{k=1}^{m_j} \ell(h_j(x_k^j), y_k^j)$$

where $\mathbb{1}[i = j] = 1$ if $i = j$ and zero otherwise, and $\mathcal{H}^n := \times_{i=1}^{n} \mathcal{H}_i$ such that for each $\mathbf{h}$ in $\mathcal{H}^n$, $\mathbf{h} = (h_1, \ldots, h_n)$ where $h_i \in \mathcal{H}_i$ for $i \in [n]$. In this case, Theorem 1 implies that under the PO, IIH, and IR conditions, multi-source adaptation and multi-task learning is impossible as the algorithm cannot leverage information across multiple sources or tasks. Similarly, DG algorithms cannot improve upon the standard ERM under the same conditions. Interestingly, this has previously been observed empirically in Koh et al. (2021) and Gulrajani and Lopez-Paz (2021). David et al. (2010) provides impossibility theorems for domain adaptation (DA) problems ($n = 2$) where our impossibility result does not hold.

5.3 Algorithmic Fairness

As AI systems become increasingly ubiquitous, societal impact of these systems also become more visible. To ensure that decisions guided by algorithmic models are equitable, researchers have started paying careful attention to algorithmic bias and unfairness that arise from deploying them in the real world.

In the field of fair machine learning, myriad formal definitions fairness have been proposed and studied by both computer science and economics communities (Verma and Rubin 2018, Hutchinson and Mitchell 2019, Mitchell et al. 2021). Dwork et al. (2012) calls for the idea that similar individuals should be treated similarly, which requires an appropriate measure of similarity. Group-based fairness requires that algorithms have equal errors rate across groups defined by protected attributes such as race and gender (Hardt et al. 2016, Kleinberg et al. 2018, Zafar et al. 2019, Rambachan et al. 2020, Mitchell et al. 2021). Popular fairness criteria include demographic parity, equal of opportunity, and equalized odds, to name a few. To promote these fairness criteria, the learning problem is often formulated as a constrained optimization problem and solved using relaxations of the fairness constraints. However, Lohaus et al. (2020) demonstrates that relaxations sometimes fail to produce fair solutions. A number of recent works also explore interventional and counterfactual approaches to mitigating unfairness (Kilbertus et al. 2017, Kusner et al. 2017, Nabi and Shpitser 2018, Chiappa 2019)

In this context, the risk profile $(r_1(h), r_2(h), \ldots, r_n(h))$ may encode the error rates of algorithmic model h across groups defined by protected attributes. When the number of groups is larger than two, Corollary 1 implies that there will be a single group that is indiscriminately favored by the learning algorithm, which hardly seems fair by any standard. Interestingly, this form of unfairness arises even before we start imposing any of the aforementioned fairness constraints. In other words, if we consider PO, IIH, and IR as primitive properties, then there is no room left for fairness (and anything else). Note that our impossibility result differs from that of Corbett-Davies et al. (2017), Chouldechova (2017), and Kleinberg et al. which shows the mathematical incompatibility between different fairness criteria.

Similar to our work, recent works have also advocated for preference-based notion of fairness (Zafar et al. 2017, Dwork et al. 2018, Ustun et al. 2019) as well as its welfare-economics interpretation (Hu and Chen 2020, Mullainathan 2018).

5.4 Federated Learning

As deep learning (DL) models keep growing in complexity, we are in need of huge amount of carefully curated data and substantial amount of computational energy for training them (Strubell et al. 2019; 2020). Unfortunately, amalgamating, curating, and maintaining a high-quality data set can take considerable time, effort, and expense. For example, health data is highly sensitive and its usage is tightly regulated (Rieke et al. 2020). Federated learning (FL) is a decentralized form of machine learning that has emerged as a promising alternative approach for overcoming these challenges (Konečný et al. 2016, McMahan et al. 2017, Li et al. 2020, Kairouz et al. 2021). While FL is designed to overcome data governance and privacy concerns by training ML models *collaboratively* without exchanging the data itself, it also paves the way for democratization of AI, more energy-efficient approaches for training DL models, and positive environmental impact of training large AI models. For instance, it has been shown that FL can lead to lower carbon emission than traditional learning (Qiu et al. 2020).

The main assumption of FL is that there exist private data sets $Z_1, Z_2, \ldots, Z_n$ residing at n local nodes (e.g., mobile phones, hospitals, planets, or galaxies[7]). The goal is then to train DL models on the entire data set $Z = \{Z_1, \ldots, Z_n\}$ while ensuring that each of them never leaves its local node. Formally, let r denotes a global loss functional obtained via a weighted combination of n local losses $r_1, \ldots, r_n$ computed from local data sets $Z_1, \ldots, Z_n$:

$$\min_{h \in \mathcal{H}} r(Z; h) \quad \text{with} \quad r(Z; h) := \sum_{i=1}^{n} w_i r_i(Z_i; h),$$

where $w_i > 0$ denote the respective weight coefficients; see, also, Li et al. (2021a) and Li et al. (2021b) for alternative loss functionals inspired by fair resource allocation (Moulin 2003). One of the most popular FL algorithms, FedAvg (McMahan et al. 2017), typically works by first initializing a global model and broadcasting it to local nodes. The local nodes update the model by executing the training on local data. The model updates, e.g., parameters and gradients, are subsequently sent back to the server where they are aggregated to update the global model. The process is repeated until convergence. It is not difficult to see that this training process is an instance of the aggregation rule (1).

FL is strikingly similar to a voting system, which is one of the most studied scenarios in social choice theory. To understand this connection, let $\mathcal{H}$ be a set of candidates and local nodes represent n voters. In each round of voting, voters cast the votes by submitting their preferences in the form of locally best parameters or gradient updates. The server then aggregates these preferences to obtain the globally best candidate. The common limiting factor of both FL and voting system is that information about voters cannot be revealed beyond their preferences. From this perspective, it is unsurprising that the same patterns of inconsistencies that have previously been observed in the voting systems would also arise in the FL setting.

Our impossibility result seems to suggest that any FL algorithms must violate at least one of the PO, IIH, and IR conditions. Otherwise, the founding principle of FL cannot be fulfilled.

5.5 Multi-modal Learning and Heterogeneous Data

Multi-modal machine learning has seen much progress in the past few years (Ngiam et al. 2011, Ramachandram and Taylor 2017, Baltrusaitis et al. 2019). Its goal is to build mod-

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[7]We envision an intergalactic learning in the near future.

els that can process and relate information from multiple modalities such as images, texts, and audios. Furthermore, heterogeneous data are also increasingly common (Nazabal et al. 2020, Valera et al. 2020). For example, human-centric data like the Electronic Health Record (EHR) are composed of attributes that have different formats including discrete (e.g., gender and race), continuous (e.g., salary), and positive count data (e.g., blood counts) among others.

Formally, let $\mathcal{X}$ be a data space that can be partitioned into n different modalities as $\mathcal{X} = \times_{i=1}^{n} \mathcal{X}_i$ and $\mathcal{H}^n = \times_{i=1}^{n} \mathcal{H}_i$ denotes the corresponding hypothesis class. The sub-class $\mathcal{H}_i$ is the hypothesis class associated with the input space $\mathcal{X}_i$. In this case, we can express the risk profile as $\mathbf{r}(\mathbf{h}) = (\hat{r}_1(\mathbf{h}), \dots, \hat{r}_n(\mathbf{h}))$ where

$$\hat{r}_i(\mathbf{h}) := \sum_{j=1}^{n} \mathbb{1}[j = i] c_j(x^j \,|\, h_j)$$

and $\mathbf{h} = (h_1, \dots, h_n) \in \mathcal{H}^n$. Here, for each $\mathbf{x} = (x^1, \dots, x^n) \in \mathcal{X}$, $x^j \in \mathcal{X}_j$ for $j \in [n]$ and $c_j(x^j \,|\, h_j)$ denotes a score function, e.g., negative log-likelihood function, associated with the j-th modality. Hence, multi-modal learning can be viewed as an aggregation rule F in (1) where the learning structure is defined on the compound hypothesis class $\mathcal{H}^n$. Hence, the heterogeneity of multi-modal data makes it particularly challenging for coordinated and joint representation learning, especially when the PO, IIH, and IR must be imposed on the learning algorithms.

6 Escaping the Impossibility

To achieve collective intelligence, we must therefore give up at least one of the conditions in Theorem 1. The first way out is to remove the requirement that the learning algorithm must be internally consistent. In other words, the algorithm is allowed to behave differently depending on which hypothesis class it is learning from. This can happen even with the internal consistency, but as an artifact of optimization strategies. Unfortunately, model selection will become extremely hard without this property. The second way out is to restrict the domain of F. In fact, existing assumptions such as task relatedness in multi-task learning and (causal) invariance in domain generalization are domain restriction in disguise. That is, it somehow amounts to assuming that there is an invariant structure that is shared across all environments. The drawback of domain restriction however is that it is normally non-trivial to test whether this condition holds or not in practice. Pareto optimality (Pareto 1897) is a very simple and highly appealing criterion of comparison of hypotheses in the multi-objective setting which generalizes the notion of "minimum risk". Therefore, the consequence of dropping PO as a necessary criterion for machine learning in general must be immense. It also implies that the information contained in the risk profile is not sufficient for learning and some "irrelevant" information must be used. Hence, a violation of PO requires some caution. Dropping IIH opens up a number of possibilities, but also poses similar concern on the use of irrelevant information.

How about the IR condition? As apparent in the proof of Theorem 1, this condition restricts information that can be shared across environments to relative ranking between any two hypotheses. Is this too restrictive?

6.1 Informational Incomparability

To answer this question, consider two hypotheses h and h' from $\mathscr{H}$ and risk functionals r_i and r_j from the same risk profile. Suppose that $r_i(h') - r_i(h) = r_j(h') - r_j(h) < 0$, i.e., h' is better than h in both environments i and j and by the same margin. Then, our ability to relax the IR condition will depend on whether or not we can say "*h' leads to the same improvement over h in environment i as it does in environment j*". For instance, will the COVID-19 AI diagnosis system h' lead to the same improvement over the old system h for Johns Hopkins Hospital in Baltimore as it does for Siriraj Hospital in Thailand? Will the new autocorrection system h' lead to the same improvement over the existing one h in terms of satisfaction for users in Japan as it does for users in South Africa? Will the updated face recognition system h' lead to the same improvement over the existing one h for white people as it does for black people? and so on. If the answer to these questions is yes, then information beyond relative rankings is usable by the algorithm, which opens up a number of possibilities. On the other hand, if we cannot answer these questions with an affirmative yes, then the IR condition must still be in place. The reason is that we may not possess sufficient information to make a meaningful comparison between environments beyond the relative rankings of hypotheses, a shortcoming that we decoratively call *informational incomparability*.[8]

Towards informational comparability   Thus, it seems that before we can build generalizable, fair, and democratic learning algorithms in heterogeneous environments, *the first question we must ask is whether we know enough to make meaningful comparisons between them*. Some immediate challenges are already in sight. The first challenge is a physical one. In federated learning, for example, it is physically impossible to share all the data across a massive networks of mobile devices. Matters pertaining to privacy, security, and access rights will also limit data sharing across environments. The second challenge is a cultural one. An algorithmic model might have varied effects across different demographic groups simply because of the culture differences. Pretending there are means for us to tell all the differences between any two cultures is a sign of ignorance. In algorithmic fairness, for example, there can be a mismatch between measurement modelling and operationalization of social constructs, i.e., abstractions that describe phenomena of theoretical interest such as socioeconomic status and risk of recidivism, which makes it difficult to meaningfully compare different operationalizations (Jacobs and Wallach 2021). The third challenge is of subjective matter. In multi-modal learning, the relationship between modalities is often open-ended or subjective (Ramachandram and Taylor 2017, Baltrusaitis et al. 2019). Language is often seen as symbolic, but audio and visual data are represented as signals. Moreover, likelihood functions defined on different data types are generally imcomparable (Javaloy 2022). Last but not least, the obstacle can simply be a legal one. To protect its people, a government might regulate what kind of and to what extent information can be shared. Well-known examples of this attempt are the EU's General Data Protection Regulation (GDPR) and its upcoming AI Act.[9]

Limitations and future directions   Our work made a number of simplifying assumptions. First of all, by modelling learning algorithms as choice correspondences, some interesting aspects such as initialization strategies, model architectures, data augmentation, optimization methods, regularization strategies, and model selection are abstracted away.

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[8] In case of cardinal utility functions, this problem is known in economics as an interpersonal incomparability of utility; see, e.g., Sen (2017; Ch. 7). There has been a long debate on whether one can make a meaningful comparison of welfare of different individuals.

[9] https://digital-strategy.ec.europa.eu/en/policies/european-approach-artificial-intelligence

With an increasing number of new algorithms proposed every year, finding the optimal model that captures the right kind of behaviours while disregarding negligible details is one of the important future directions. In particular, generalizations of the internal consistency property might shed light into more sophisticated learning behaviours in CNN, RNN, and transformer, for example. Second, generalization to unseen data is another aspect that we omit in this work. Nevertheless, we have somehow demonstrate the generalization ability of ERM as a learning algorithm in the sense that it remains invariant under the same set of axioms regardless of the nature of the environments in which it operates. Last but not least, it remains to explore whether other impossibility results can be established. Notably, given a growing interest in learning under strategic behaviours and adversarial examples, one of the future directions is to generalize the Gibbard-Satterthwaite theorem (Gibbard 1973, Satterthwaite 1975) which shows that there exists no aggregation rule that is strategy-proof.

7 Conclusion

To conclude, we prove that it is impossible to design a learning algorithm that has the ability to successfully learn across heterogeneous environments whether they represent individuals, demographic groups, mobile phones, siloed data from hospitals, or data modalities. By representing any conceivable algorithm as an internally consistent choice correspondence over hypothesis space, we provide reasonable-looking axioms that are deemed necessary, namely, Pareto Optimality (PO), Independence of Irrelevant Hypotheses (IIH), and Invariance Restriction (IR). The unique algorithm compatible with all of the axioms is the standard empirical risk minimization (ERM) which simply learns from a single arbitrary environment. Unfortunately, this possibility result implies the impossibility of *Collective Intelligence* (CI), the algorithm's ability to successfully learn across heterogeneous environments. Our general impossibility theorem elucidates the fundamental trade-off in emerging areas of machine learning such as OOD generalization, federated learning, algorithmic fairness, and multi-modal learning.

More importantly, this result reveals a subtle challenge, which we decoratively call *informational incomparability*, that ML researchers may not be able to overcome. The true challenge of learning in heterogeneous environments is a heterogeneity itself (no pun intended). Unlike in the homogeneous environment, relative impacts of algorithmic models in heterogeneous environments could vary in ways that cannot be measured precisely by the risk functionals due to physical constraints, culture differences, or ethical and legal concerns. As a result, comparative information beyond the relative rankings of any two models cannot be leveraged by the algorithm. To make progress, it is thus imparative not only to strengthen privacy and information security such that information can be disseminated securely, but also to better understand the real impact of alogrithmic models in deployment.

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