FOURIER COEFFICIENTS OF THREE-DIMENSIONAL VECTOR-VALUED MODULAR FORMS

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Abstract. A thorough analysis is made of the Fourier coefficients for vector-valued modular forms associated to three-dimensional irreducible representations of the modular group. In particular, the following statement is verified for all but a finite number of equivalence classes: if a vector-valued modular form associated to such a representation has rational Fourier coefficients, then these coefficients have "unbounded denominators", i.e. there is a prime number p, depending on the representation, which occurs to an arbitrarily high power in the denominators of the coefficients. This provides a verification in the three-dimensional setting of a generalization of a long-standing conjecture about noncongruence modular forms.

Contents

1. Introduction

Since the seminal work of Atkin and Swinnerton-Dyer [1], evidence has been accumulating that modular forms for noncongruence subgroups of $\Gamma = SL_2(\mathbb{Z})$ have *unbounded denominators*. In other words, in stark contrast to the setting of congruence modular forms – where the theory of Hecke operators implies [22, Thm 3.52] that any space $\mathcal{M}_k(G)$ of (integral) weight k modular forms for a congruence subgroup G of Γ has a basis whose Fourier coefficients are rational integers – if G is a noncongruence subgroup then one expects that no such basis exists, and that instead the following holds:

Conjecture 1.1. *Suppose G is a finite index subgroup of Γ and $f \in \mathcal{M}_k(G)$ has integral Fourier coefficients. Then $f \in \mathcal{M}_k(G^c)$, where G^c denotes the*

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congruence closure of G (i.e. the intersection of all congruence subgroups containing G).

Thus if f is truly noncongruence and has rational Fourier coefficients, then conjecturally there is no sufficiently large integer N such that the Fourier coefficients of Nf are rational integers; this is what is meant[1] when one says the denominators of f are "unbounded". In keeping with common practice, we shall write f *is UBD* to denote this circumstance, and will say f *has bounded denominators* otherwise. More specifically, if there is a prime number p which occurs to an arbitrarily high power in the denominators of the Fourier coefficients of f, then we say that f *is p-UBD*, and otherwise that f *is p-bounded*.

In the years since the appearance of [1], a number of articles have appeared in the literature giving examples of noncongruence modular forms with unbounded denominators, or verifying Conjecture 1.1 for limited classes of modular forms; see e.g. the expository article [11], the more recent article [10], and references therein. Thus far, the deepest results concerning the Fourier coefficients of noncongruence modular forms are obtained via Scholl's generalization of Deligne's work on l-adic representations, to the setting of arbitrary finite index subgroups of the modular group. Among other things, Scholl has shown [19, Prop 5.2] (see also the expository article [20]) that for each finite index subgroup G of $PSL_2(\mathbb{Z}) = \Gamma/\{\pm 1\}$ there is an associated integer M such that for each integer k the following holds: if $\mathcal{M}_k(G)$ admits a basis with rational Fourier coefficients, then there exists a basis such that every prime factor occurring in the denominators of the basis coefficients is a factor of M. Consequently, if $f \in \mathcal{M}_k(G)$ is *UBD* then there is a prime factor p of M such that f is *p-UBD*.

The point of view taken in the present article is that the theory of modular forms for arbitrary finite index subgroups of Γ may be profitably viewed as a special case (corresponding to the representations of Γ having finite image) of the more general theory of vector-valued modular forms, and that working in this more general setting allows one to take advantage of a number of techniques not available to the classical theory. In particular, it will be seen that the bounded denominator question may be effectively probed, at least in low dimension, by utilizing the Fuchsian theory of differential equations as it pertains to vector-valued modular forms. This circle of ideas builds on a series of articles by M. Knopp and G. Mason [4, 5, 6, 7, 15] (see also the more recent contribution [2]), wherein a formal study of vector-valued modular forms was initiated. Mason himself has already undertaken the study of the bounded denominator question for vector-valued modular forms, obtaining in the two-dimensional setting [17] results quite similar to those of the present article. These results, and more general considerations, prompted him to posit the following generalization of Conjecture 1.1:

[1]This language is, of course, also used in the more general situation, where $\mathbb{Q}$ and $\mathbb{Z}$ are replaced by the number field defined by the Fourier coefficients of f and its ring of integers, respectively. In this article, however, we consider $\mathbb{Q}$-rational Fourier coefficients only.

Conjecture 1.2 (Mason). *Suppose F is a vector-valued modular form for a representation ρ of Γ, such that the components of F have Fourier expansions with bounded denominators. Then $\ker \rho$ is a congruence subgroup of Γ.*

This Conjecture is of course easily verified when the representation is one-dimensional, for here, as is well-known, there are only 12 possible representations to consider, each of whose kernel is congruence (of level dividing 12). And in the two-dimensional setting (loc. cit.), Mason has verified Conjecture 1.2 for all but a finite number (up to equivalence of representation) of open cases. In this article, we shall utilize similar techniques to verify Conjecture 1.2 in the three-dimensional setting, again up to a finite number of remaining open cases. Explicitly, we shall prove

Theorem 1.3. *Up to equivalence of representation, only a finite number irreducible $\rho : \Gamma \to GL_3(\mathbb{C})$, with $\rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of finite order, admit vector-valued modular forms with bounded denominators.*

Unlike a two-dimensional representation of Γ, which (cf. Prop 3.2, loc. cit.) has either infinite image or a congruence kernel, there exist infinitely many inequivalent $\rho : \Gamma \to GL_3(\mathbb{C})$ such that $\ker \rho$ is a finite index, noncongruence subgroup of Γ. Thus Theorem 1.3 provides, among other things, infinitely many new examples of modular forms for (infinitely many distinct) finite index, noncongruence subgroups, normal in the the full modular group, which are known to have unbounded denominators; see Section 5 for details. Furthermore, and perhaps even more interestingly, Theorem 1.3 provides infinitely many new examples of subgroups of infinite index in Γ, still of finite level (in the more general sense of Wohlfahrt [24], cf. Section 2 below), which admit holomorphic functions with "classical" q-expansions having unbounded denominators; see Section 6 for further discussion of this.

The layout of the remainder of this article is as follows. In the next Section, we review those aspects of the theory of vector-valued modular forms which are relevant to the task at hand. In Section 3 we utilize the Fuchsian theory of differential equations to establish a recursive formula for the Fourier coefficients of the minimal weight vector (3.6) associated to an arbitrary three-dimensional ρ, and from this formula the unboundedness of the denominators of (3.6) follows. In Section 4 we complete the proof of Theorem 1.3 by showing how the unboundedness of an arbitrary vector-valued modular form for ρ follows from that of the minimal weight vector (3.6). Section 5 contains explicit information regarding the three-dimensional ρ which have finite image; in particular, we discuss the infinite families of three-dimensional representations which have a noncongruence subgroup of finite index as kernel. As just mentioned, this provides (in light of Theorem 1.3) infinitely many new examples of noncongruence modular forms which are known to have unbounded denominators. We end the article with some final remarks in Section 6, regarding the arithmetic of the generalized modular forms (arising as components of vector-valued modular forms associated to finite level, infinite image representations of Γ) that appear in this setting, and also the prospects for extending the work presented here and in [17] to higher dimension.

2. Background

Let $\mathbb{H}$ denote the complex upper half-plane, $\mathcal{H}$ the complex linear space of holomorphic functions $f : \mathbb{H} \to \mathbb{C}$, and Γ the full modular group of 2×2 matrices with integer entries and determinant 1. We denote by $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ the well-known generators of Γ. For each integer k we write

$$\begin{aligned} |_k : \mathcal{H} \times \Gamma &\to \mathcal{H}, \\ (f, \gamma) &\mapsto f|_k\gamma \end{aligned}$$

to denote the k^{th} slash action of Γ on $\mathcal{H}$; thus for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $\tau \in \mathbb{H}$ we have

$$f|_k\gamma(\tau) = (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right).$$

A holomorphic function $F : \mathbb{H} \to \mathbb{C}^d$ is a d-dimensional *vector-valued modular form* of weight $k \in \mathbb{Z}$ if the component functions comprising F satisfy a moderate growth condition at the cusps of Γ, just as in the classical theory of modular forms, and if the span of these components is an invariant subspace of $\mathcal{H}$ under the $|_k$ action of Γ on $\mathcal{H}$. Explicitly, if one writes F as a column vector $F = (f_1, \cdots, f_d)^t$ then the above action of Γ on the span of the f_j takes the form of a matrix representation $\rho : \Gamma \to GL_d(\mathbb{C})$, and we say that F is a vector-valued modular form of weight k for the representation ρ if the following conditions are satisfied:

(1) The components f_j of F are of *moderate growth at infinity*, i.e. there is an integer N such that for each j we have $|f_j(x+iy)| < y^N$ for any fixed $x \in \mathbb{R}$ and $y \gg 0$.
(2) The functional equation $F|_k\gamma = \rho(\gamma)F$ is satisfied for each $\gamma \in \Gamma$ (here $|_k$ acts componentwise on F).

We write $\mathcal{H}(k, \rho)$ for the complex linear space of all such vectors. Regardless of ρ, if k is large enough then $\mathcal{H}(k, \rho) \neq 0$ [7, Cor 3.12]. If ρ is furthermore assumed to be indecomposable – as shall be the case throughout this article – then it follows directly from the definition of vector-valued modular form that $\rho(S^2) = (-1)^k$ whenever $\mathcal{H}(k, \rho) \neq 0$, and there is an integer $k_0 \geq 1 - d$ such that the $\mathbb{Z}$-graded space

$$\mathcal{H}(\rho) = \bigoplus_{k \geq 0} \mathcal{H}(k_0 + 2k, \rho) \tag{2.1}$$

contains all the nonzero integral weight vector-valued modular forms for ρ. If $\rho = \mathbf{1}$ is the trivial one-dimensional representation of Γ, then[2] $k_0 = 0$ and we

[2]More generally, if ρ is unitary then the minimal weight k_0 is nonnegative.

write

$$\mathcal{H}(\mathbf{1}) = \mathcal{M} = \bigoplus_{k\geq 0} \mathcal{M}_{2k}$$

for the graded ring of holomorphic, integral weight modular forms for Γ. As is well-known, $\mathcal{M} = \mathbb{C}[E_4, E_6]$ is a graded polynomial algebra in $E_4 \in \mathcal{M}_4$ and $E_6 \in \mathcal{M}_6$, where for each even integer $k \geq 2$ we write

$$E_k(\tau) = 1 - \frac{2k}{B_k} \sum_{n\geq 0} \sigma_{k-1}(n) q^n \tag{2.2}$$

for the normalized Eisenstein series in weight k; here $q = q(\tau) = e^{2\pi i\tau}$, B_k denotes the k^{th} Bernoulli number, and $\sigma_k(n) = \sum_{1\leq d|n} d^k$. Each space (2.1) is a graded module over the ring $\mathcal{M}$ of scalar modular forms, via componentwise multiplication.

Note that if ρ' is equivalent to ρ – meaning there is a $U \in GL_d(\mathbb{C})$ such that $\rho'(\gamma) = U\rho(\gamma)U^{-1}$ for each $\gamma \in \Gamma$ – then multiplication by U defines an isomorphism of graded $\mathcal{M}$-modules $\mathcal{H}(\rho) \cong \mathcal{H}(\rho')$. This allows one to study vector-valued modular forms for representations which have desirable matrix forms within their particular equivalence class. In particular, in this article we shall be concerned only with those ρ which are *T-unitarizable*, meaning that $\rho(T)$ is similar to a unitary matrix. Since every unitary matrix may be diagonalized, the above isomorphism allows us to assume that

$$\rho(T) = \mathrm{diag}\left\{\mathbf{e}\left(r_1\right), \cdots, \mathbf{e}\left(r_d\right)\right\} \tag{2.3}$$

with $0 \leq r_j < 1$ for each j; here and throughout we write $\mathbf{e}\left(r\right) = e^{2\pi i r}$ for the exponential of $r \in \mathbb{R}$. In this case, it follows directly from the above definition of vector-valued modular form that each $F \in \mathcal{H}(\rho)$ has a holomorphic q-expansion

$$F(\tau) = \begin{pmatrix} f_1(\tau) \\ \vdots \\ f_d(\tau) \end{pmatrix} = \begin{pmatrix} \sum_{n\geq 0} a_1(n) q^{r_1+n} \\ \vdots \\ \sum_{n\geq 0} a_d(n) q^{r_d+n} \end{pmatrix}, \tag{2.4}$$

with $a_j(n) \in \mathbb{C}$ for each j, n. If the $a_j(n)$ are all rational numbers, then it makes sense to apply the terms *UBD*, *p-UBD*, and *p-bounded* to F. Note that for (2.4) to be *UBD*, *p-UBD* in this generalized sense, it is neccesary and sufficient for a single f_j to be *UBD*, *p-UBD* respectively, and in particular nothing further is implied by the use of this term in the vector-valued context.

A fundamental fact concerning the $\mathcal{M}$-module structure of (2.1) is the following

Theorem 2.1. *If ρ is indecomposable and T-unitarizable, then $\mathcal{H}(\rho)$ is a free $\mathcal{M}$-module of rank $d = \dim\rho$.* □

See [14, Thm 1] for a proof, or [6] for a more general result.

As in the scalar theory, the main objects of study in the theory of vector-valued modular forms are the Fourier coefficients $a_j(n)$ of the q-expansions in (2.4), and the present article is no exception. Here we will be concerned with vector-valued modular forms associated to T-unitarizable, three-dimensional representations of Γ which are irreducible. The structure of the corresponding space (2.1) is particularly well-understood in this setting, and this permits a thorough investigation of the associated Fourier coefficients. The main tools used in this regard come from the theory of modular differential equations, as we discuss next.

Recall that the modular derivative in weight $k \in \mathbb{R}$ is the operator

$$\begin{aligned} D_k &: \mathcal{H} \to \mathcal{H}, \\ D_k f &= \frac{1}{2\pi i}\frac{df}{d\tau} - \frac{k}{12}E_2 f, \end{aligned} \tag{2.5}$$

with E_2 as in (2.2). It is well-known in the scalar setting that D_k takes modular forms of weight k to those of weight $k+2$. This generalizes to higher dimension and yields a weight two operator D, which acts (componentwise) on each graded space (2.1) by acting as D_k on $\mathcal{H}(k,\rho)$. One defines for any $n \geq 2$ the composition

$$D_k^n = D_{k+2(n-1)} \circ \cdots \circ D_k \tag{2.6}$$

and in this way powers of D are well-defined operators on (2.1). This allows one to define a skew polynomial ring $\mathcal{R}$, which as an additive group is just the polynomial ring in one variable $\mathcal{M}[d]$, and whose multiplication is defined by the identity $dM = Md + DM$ for each $M \in \mathcal{M}$. Each space (2.1) of vector-valued modular forms is then a graded left $\mathcal{R}$-module (finitely generated thanks to Theorem 2.1), where $\mathcal{M}$ again acts by componentwise multiplication, and d^n acts as the n^{th} power of the modular derivative, i.e. d^n acts as (2.6) on $\mathcal{H}(k,\rho)$. A *modular differential equation* is simply an equation $L[f] = 0$ with $L \in \mathcal{R}$ homogeneous by weight. A very special case – which includes the setting of the present article – occurs when L is monic, say

$$L[f] = D_k^d f + M_4 D_k^{d-2} f + \cdots + M_{2d} f = 0 \tag{2.7}$$

for some $d \geq 1$, $k \in \mathbb{Z}$, and $M_j \in \mathcal{M}_j$ for each j. Because E_2 and the M_j are holomorphic in $\mathbb{H}$, the only singular point of (2.7) is $i\infty$, i.e. $q = 0$, and this is seen to be a regular singular point, in the sense of Fuchs [3, Chs 5,9]. Thus (2.7) defines a Fuchsian differential equation in the punctured disk $0 < |q| < 1$, and if the indicial roots $r_1, \cdots, r_d$ of (2.7) are real and distinct (mod $\mathbb{Z}$), then (cf. [16, Thm 4.1]) there is a vector-valued modular form $F \in \mathcal{H}(k, \rho_L)$ of the form (2.4), with $a_j(0) = 1$ for each j, such that the f_j form a fundamental system of solutions (i.e. a basis of the solution space) of (2.7). The representation ρ_L associated to F is indecomposable (though not necessarily irreducible, cf. [12, Cor 3.5.8]), and is given by the $|_k$ action of Γ on the solution space of (2.7). The matrix $\rho_L(T)$ may be assumed to have the form (2.3), and $\rho_L(T)$ is the monodromy of the fundamental system at the regular singular point $q = 0$.

If the indicial roots also satisfy $0 \leq r_j < 1$ for each j, then by [14, Thm 3] the space $\mathcal{H}(\rho_L)$ of vector-valued modular forms for ρ_L is a cyclic $\mathcal{R}$-module, with generator F as in the previous paragraph; in light of Theorem 2.1, this implies that the set $\{F, D_k F, \cdots, D_k^{d-1} F\}$ is a basis of $\mathcal{H}(\rho_L)$ as $\mathcal{M}$-module. Furthermore, the minimal weight k_0 of $\mathcal{H}(\rho_L)$ satisfies the formula $k_0 = \frac{12r}{d} +$

$1-d$, where $d = \dim \rho$ and $r = \sum r_j$ is the sum of the indicial roots. And conversely, if (2.1) is a cyclic $\mathcal{R}$-module $\mathcal{R}F$ then there is a monic differential equation (2.7) such that ρ is equivalent to the representation ρ_L described above, and the components of F form a fundamental system of solutions of (2.7). Furthermore, in this case the indicial roots r_j of (2.7) are distinct and satisfy $0 \leq r_j < 1$ for each j, so (again up to equivalence) we may assume that $\rho(T)$ has the form (2.3) and F is as in (2.4), with $a_j(0) = 1$ for each j.

It is of fundamental importance for the analysis undertaken in this article that every irreducible, T-unitarizable $\rho : \Gamma \to GL_3(\mathbb{C})$ is equivalent to some ρ_L with L monic as in (2.7). This fact allows one to deduce the unboundedness of the denominators of an arbitrary vector-valued modular form in $\mathcal{H}(\rho)$ from that of the minimal weight vector F which generates $\mathcal{H}(\rho) = \mathcal{R}F$. We shall expound upon this more in the next Section.

Following Wohlfahrt [24], we refer to the order of $\rho(T)$ in $\rho(\Gamma)$ as the *level* of ρ. If ρ is of finite level N then $\ker \rho$ is a normal subgroup of Γ that contains T^N, thus it also contains the normal closure $\Delta(N)$ of the subgroup of Γ generated by T^N. It is proven in loc. cit. that if $N < 6$ then $\Delta(N)$ is the principal congruence subgroup $\Gamma(N)$, so $\ker \rho$ is necessarily a congruence subgroup of level N in this case; in particular, the components of any vector-valued modular form for a representation of level less than six are congruence modular forms, and consequently have bounded denominators. On the other hand, it is also proven in loc. cit. that $\Delta(N)$ is of infinite index in $\Gamma(N)$ when $N \geq 6$, so in this setting the image of ρ may, or may not, be finite.

If ρ has finite level and $\rho(T)$ is diagonal as in (2.3), then obviously the exponents r_j of the eigenvalues of $\rho(T)$ are rational numbers. In the next Section it will be shown that if the dimension of such a representation is three, then the associated space $\mathcal{H}(\rho)$ of vector-valued modular forms contains an $\mathcal{M}$-basis whose Fourier coefficients are rational numbers. The converse seems to be more subtle: if $\rho(T)$ has the form (2.3) and $\mathcal{H}(\rho)$ admits vector-valued modular forms with rational Fourier coefficients, does this force ρ to have finite level? The answer is currently not known, although a quick look at the recursive formula (3.9) leads one to believe that in the present setting of dimension three this converse must hold. In any event, we shall only be concerned with finite level representations in this article.

3. Three dimensional vector-valued modular forms

In this Section we analyze the Fourier coefficients of the minimal weight vector-valued modular form (3.6) associated to a generic three-dimensional representation of the modular group. This analysis forms the core of the proof of the Main Theorem 1.3, which will be completed in Section 4.

Suppose that

$$\rho : \Gamma \to GL_3(\mathbb{C}) \tag{3.1}$$

is irreducible and T-unitarizable. Up to equivalence of representation, we may and shall now assume that $\rho(T)$ is diagonal as in (2.3). It follows directly from [14, Thm 1] (or see [13, Thm 4.1] for a proof) that the space (2.1) of holomorphic vector-valued modular forms for ρ is a cyclic $\mathcal{R}$-module

$$\mathcal{H}(\rho) = \mathcal{R}F_0 = \mathcal{M}F_0 \oplus \mathcal{M}DF_0 \oplus \mathcal{M}D^2F_0, \tag{3.2}$$

and by [14, Thm 3] the components of the generator F_0 form a fundamental system of solutions of a modular differential equation

$$L[f] = D_{k_0}^3 f + \alpha_4 E_4 D_{k_0} f + \alpha_6 E_6 f = 0. \tag{3.3}$$

Here $k_0 = 4r - 2 \in \mathbb{Z}$, with $r = r_1 + r_2 + r_3$ the sum of the exponents in (2.3) (which are also the indicial roots of (3.3)), E_k is the Eisenstein series (2.2), and the complex numbers α_4, α_6 are uniquely determined by the r_j; cf. [13, Lemma 2.3], or see (3.12) for the explicit formulae. Note also that the r_j are distinct by [14, Thm 3]. Since we are interested in those ρ which admit vector-valued modular forms with rational Fourier coefficients, we assume henceforth that ρ is of finite level (cf. the discussion at the end of the previous Section). Denoting this level by N, we have

$$\rho(T) = \operatorname{diag}\left\{\mathbf{e}\left(\tfrac{A}{N}\right), \mathbf{e}\left(\tfrac{B}{N}\right), \mathbf{e}\left(\tfrac{C}{N}\right)\right\}, \tag{3.4}$$

where the integers N, A, B, C are distinct and satisfy

$$0 \le A, B, C \le N-1, \quad (A, B, C, N) = 1. \tag{3.5}$$

Knowing this, we may now assume that the minimal weight vector $F_0 \in \mathcal{H}(k_0, \rho)$ has a Fourier expansion of the form

$$F_0 = \begin{pmatrix} q^{\frac{A}{N}} + \sum_{n\ge1} a(n) q^{\frac{A}{N}+n} \\ q^{\frac{B}{N}} + \sum_{n\ge1} b(n) q^{\frac{B}{N}+n} \\ q^{\frac{C}{N}} + \sum_{n\ge1} c(n) q^{\frac{C}{N}+n} \end{pmatrix}. \tag{3.6}$$

Using (2.5) and (2.6), it follows after an elementary calculation that when (3.3) is rewritten in terms of the variable q, the equation

$$q^3 \frac{d^3 f}{dq^3} + g_2 q^2 \frac{d^2}{dq^2} + g_1 q \frac{df}{dq} + g_0 f = 0$$

obtains, where

$$
\begin{aligned}
g_2(q) &= \sum_{n\geq 0} G_2(n)q^n \\
&= 3+(3k_0+6)P,
\end{aligned}
$$

$$
\begin{aligned}
g_1(q) &= \sum_{n\geq 0} G_1(n)q^n \\
&= 1+(3k_0+6)P+(3k_0^2+9k_0+6)P^2 \\
&\quad +(3k_0+2+144\alpha_4)Q,
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
g_0(q) &= \sum_{n\geq 0} G_0(n)q^n \\
&= k_0(3k_0+2+144\alpha_4)PQ+k_0(k_0+1)(k_0+2)P^3 \\
&\quad +(k_0-432\alpha_6)R;
\end{aligned}
$$

here we set (with apologies to Ramanujan)

$$
P=-\frac{1}{12}E_2, \quad Q=\frac{1}{144}E_4, \quad R=-\frac{1}{432}E_6. \tag{3.8}
$$

Using this notation (and setting $a(0)=1$), we may write the Fuchsian recursive relation for the coefficients of the first component of (3.6) as

$$
a(n)=-\frac{1}{\phi\left(\frac{A}{N}+n\right)}\sum_{j=0}^{n-1}a(j)\phi_{n-j}\left(\frac{A}{N}+j\right), \tag{3.9}
$$

where

$$
\begin{aligned}
\phi_j(\lambda) &= G_2(j)\lambda(\lambda-1)+G_1(j)\lambda+G_0(j), \\
\phi(\lambda) &= \lambda(\lambda-1)(\lambda-2)+\phi_0(\lambda).
\end{aligned} \tag{3.10}
$$

The exponents $\frac{A}{N},\frac{B}{N},\frac{C}{N}$ in (3.4) are the solutions of the indicial equation $\phi(\lambda)=0$ associated to (3.3), and from this one obtains directly

$$
\begin{aligned}
G_2(0) &= 3-\frac{\sigma}{N}, \\
G_1(0) &= G_2(0)+\frac{\omega}{N^2}-2, \\
G_0(0) &= -\frac{\pi}{N^3},
\end{aligned}
$$

where we set

$$
\sigma=A+B+C, \quad \omega=AB+AC+BC, \quad \pi=ABC.
$$

Using this information, it is now straightforward to compute and find

$$
\phi\left(\frac{A}{N}+n\right)=\frac{n\lambda(n)}{N^2},
$$

where for each $n \geq 1$ we set

$$\lambda(n) = Nn[Nn + (A - B) + (A - C)] + (A - B)(A - C). \tag{3.11}$$

Furthermore, comparing the above formulae for the $G_j(0)$ with those obtained directly from (3.7) yields

$$\begin{aligned} k_0 &= \frac{x_0}{N}, \\ \alpha_4 &= \frac{x_4}{(12N)^2}, \\ \alpha_6 &= \frac{x_6}{(12N)^3}, \end{aligned} \tag{3.12}$$

where we define the integers

$$\begin{aligned} x_0 &= 4\sigma - 2N, \\ x_4 &= 144\omega + x_0(12N - 3x_0) + 8N^2, \\ x_6 &= x_0x_4 + x_0(x_0 + 2N)(x_0 + 4N) - 1728\pi. \end{aligned} \tag{3.13}$$

A final round of elementary computations yields

$$\begin{aligned} G_2(1) &= \frac{24\sigma}{N}, \\ G_1(1) &= \frac{240\omega - 48\sigma(2\sigma - N)}{N^2}, \\ G_0(1) &= \frac{504\pi + \big(2\sigma - N\big)\big(8\sigma(4\sigma - N) - 120\omega\big)}{N^3}, \end{aligned}$$

and from this it is trivial to verify the following

Lemma 3.1. *For each $n \geq 0$ we have $\phi_1\left(\frac{A}{N} + n\right) = \frac{z_n}{N^3}$ with*

$$\begin{aligned} z_n = {} & 24[10\omega Nn + \sigma(A + Nn)(A + N(n-1))] + \\ & 8[2\sigma - N][\sigma(4\sigma - N) - 15\omega - 6\sigma(A + Nn)] + \\ & 240A\omega + 504\pi. \end{aligned} \tag{3.14}$$

□

It is easy to see that the x_j in (3.12) satisfy $2|x_0$, $4|x_4$, $8|x_6$, and this will be used to prove

Lemma 3.2. *For $j = 2, 3$ set*

$$\delta_j = \begin{cases} 0 & j \mid N, \\ 1 & j \nmid N. \end{cases}$$

Then for each $n \geq 2$ the following hold:

(1) $G_2(n) \in \mathbb{Z}$.

(2) $G_1(n) \in \frac{1}{3^{\delta_3} N^2}\mathbb{Z}$.

(3) $G_0(n) \in \frac{1}{2^{\delta_2} 3^{\delta_3} N^3}\mathbb{Z}$.

Proof. Since the minimal weight k_0 is an integer, one sees from (3.12), (3.13) that $N|4\sigma$. Furthermore, from (3.7) and (2.2) we obtain

$$G_2(n) = \frac{24\sigma}{N}\sigma_1(n)$$

for each $n \geq 1$. This implies statement (1). For statements (2) and (3), it is sufficient to verify the analogous statement for the coefficients of each summand of g_1 and g_2, respectively. This amounts to a routine verification and we omit the proof. We do note, however, that in addition to the 2-adic properties of the x_j mentioned above, we also have that $3|x_4$ iff $3|N$, and $3|N$ implies $3|x_0$, $9|x_6$; these observations are all that is required to fill in the remaining details. □

For each prime p we write ν_p to denote the p-adic valuation of $\mathbb{Q}$, thus if $x = p^k v$ is an integer with $(v, p) = 1$ then $\nu_p(x) = k$, and if $\frac{x}{y} \in \mathbb{Q}$ then $\nu_p\left(\frac{x}{y}\right) = \nu_p(x) - \nu_p(y)$. The most important step in the proof of Theorem 1.3 is the determination of $\nu_p(z_n)$ for various primes p, and we turn now to this task:

Proposition 3.3. *Let p be a prime dividing the level N of (3.1). After relabeling (if needed) the indicial roots $\frac{A}{N}, \frac{B}{N}, \frac{C}{N}$ of (3.3), the following statements hold for all $n \geq 0$:*

(1) *If $p > 7$, then $\nu_p(z_n) = 0$.*
(2) *If $p = 7$ and $7 \mid \pi$, then $\nu_7(z_n) = 0$.*
(3) *If $p = 7$ and $7 \nmid \pi$, $7^2|N$, then one of the following holds:*
    (a) *$7 \mid \omega$ and $\nu_7(z_n) = 1$.*
    (b) *$7 \nmid \omega$ and $\nu_7(z_n) = 0$.*
(4) *If $p = 5$ and $5 \nmid \pi$, then $\nu_5(z_n) = 0$.*
(5) *If $p = 5$ and $5|\pi$, $5^2|N$, then $\nu_5(z_n) = 1$.*
(6) *If $p = 3$ and $3^2|N$, $3 \nmid \omega$, then $\nu_3(z_n) = 1$.*
(7) *If $p = 3$ and 3^3N, $3|\omega$, then $\nu_3(z_n) = 2$.*
(8) *If $p = 2$ and $2^5|N$, then $\nu_2(z_n) = 4$.*

Proof. From the formula (3.12) for k_0, it follows that if $p \geq 3$ is a prime dividing N then $\nu_p(\sigma) \geq \nu_p(N) > 0$, and if $p = 2$ then $\nu_2(\sigma) \geq \nu_p(N) - 2$. On the other hand, it is seen from (3.14) that if p divides both N and σ, then $\nu_p(z_n) > 0$ iff p divides

$$240A\omega + 504\pi = 24A[10A(B + C) + 31BC]. \tag{3.15}$$

By assuming, as we may, that $p \nmid AB$, it is clear that if $p|C$ and $p > 5$, then p does not divide (3.15); this implies statement (2) and part of statement (1) of the Proposition.

Assume now that $\nu_p(N) = k \geq 1$ with $5 \leq p \nmid \pi$. Then (3.15) shows that $\nu_p(z_n) > 0$ iff $p|10A(B+C)+31BC$. Transposing A and B throughout the calculations which led to (3.14) will yield the analogue y_n of (3.14) for the numerator of $\phi_1\left(\frac{B}{N}+n\right)$, and one finds similarly that $\nu_p(y_n) > 0$ iff $p|10B(A+C)+31AC$. Noting the fact we also have $\nu_p(\sigma) \geq k$ in this context, a trivial calculation shows that for any $1 \leq m \leq k$ we have

$$\begin{aligned} 10A(B+C)+31BC &\equiv -(10A^2+31AB+31B^2) \pmod{p^m}, \\ 10B(A+C)+31AC &\equiv -(10B^2+31AB+31A^2) \pmod{p^m}, \end{aligned}$$

so a necessary condition for $\nu_p(z_n) \geq m$ and $\nu_p(y_n) \geq m$ to *both* hold is that p^m divides the difference

$$10A^2+31AB+31B^2-(10B^2+31AB+31A^2) = 21(B^2-A^2). \tag{3.16}$$

Note that $p \nmid (B+A)$, since $p|\sigma$ and $p \nmid C$, so $p^m|(B^2-A^2)$ iff $A \equiv B \pmod{p^m}$. Furthermore, if this holds then it follows immediately that

$$10A(B+C)+31BC \equiv -72A^2 \pmod{p^m}, \tag{3.17}$$

and because we are assuming $5 \leq p \nmid A$, this cannot be. In particular, this implies that p does not divide (3.17) if $p = 5$ or $p > 7$. Taking $m = 1$ and relabeling (if needed) then completes the proof of statement (1) of the Proposition and yields statement (4) as well. On the other hand, assuming $p = 7$, $k \geq 2$, $m = 2$ makes it clear that $\nu_7(z_n) \leq 1$ (after relabeling if needed), and from this and (3.15) statement (3) follows immediately.

Next we assume $5^2|N$, $5|C$, say $C = 5X$ for some integer X. Then it follows directly from (3.15) that $\nu_5(z_n) \geq 1$, and $\nu_5(z_n) \geq 2$ iff $5|(2A+X)$. As in the previous paragraph, we pursue an identical analysis for the integer y_n which is the numerator of $\phi_1\left(\frac{B}{N}+n\right)$, and find this time that $5^2|y_n$ iff $5|(2B+X)$. Thus 5^2 necessarily divides the difference $2(A-B)$ of these terms if 5^2 divides the numerators of both $a(1)$ and $b(1)$, which is to say $5|(A-B)$. But $5|C$, $5|\sigma$ imply that $5|(A+B)$, thus $5 \nmid (A-B)$ since $(A,B,C) = 1$ and $5|C$. This proves Statement (5) of the Proposition.

Now assume that $\nu_3(N) = k \geq 1$, $3 \nmid A$. Statement (6) of the Proposition follows immediately from (3.15) by assuming $k \geq 2$. On the other hand, if $k \geq 3$ and $3|\omega$, then (3.15) makes it clear that $\nu_3(z_n) \geq 2$. But examining the calculations which led to (3.17), one sees that this logic remains valid for the prime 3, and taking $m = 3$ shows that, up to relabeling, we have $\nu_3(z_n) \leq 2$, and this implies statement (7) of the Proposition.

Finally, assume that $\nu_2(N) \geq 4$. Then $\nu_2(\sigma) \geq 2$, and this implies $\nu_2(\pi) \geq 1$. Note that $2 \nmid \omega$ since $2|\pi$. If $\nu_2(\pi) \geq 2$, then we may assume that $2 \nmid A$, $2^2|BC$, and this makes it clear that the first two terms of (3.14) are divisible by 2^5, whereas the last term is divisible only by 2^4. On the other hand, if $\nu_2(\pi) = 1$, then we may assume that $\nu_2(A) = 1$, $2 \nmid BC$, and in this case we find that 2^6 divides the first two terms of (3.14), but the last term is divisible by only 2^4. Thus statement (7) of the Proposition holds, and this completes the proof of the Proposition. $\square$

It follows immediately that if p is a prime satisfying one of conditions $(1)-(8)$ in Proposition 3.3 then the difference

$$\delta = \delta(p) = \nu_p(z_n) - \nu_p(N) < 0 \tag{3.18}$$

is well-defined, independently of the integer $n \geq 0$. With this notation, we may now prove

Proposition 3.4. *Suppose p is a prime satisfying one of conditions $(1)-(8)$ of Proposition 3.3, and assume furthermore that that $\nu_p(N) > 2\nu_p(z_0)$. Then for all $n \geq 1$ we have*

$$\nu_p(a(n)) = n\delta - \nu_p\left(\prod_{k=1}^{n} k\lambda(k)\right). \tag{3.19}$$

In particular, $\nu_p(a(n))$ is a strictly decreasing, negative function of n, and (3.6) is p-UBD.

Proof. The proof will be made by induction on n. Since

$$a(1) = -\frac{N^2}{\lambda(1)} \cdot \phi_1\left(\frac{A}{N}\right) = -\frac{z_0}{N\lambda(1)},$$

it is clear from Proposition 3.3 that if p is a prime satisfying the hypothesis of the current Proposition, then (3.19) holds for $n = 1$. Now assume that $n \geq 2$ and (3.19) holds for all $1 \leq j \leq n-1$. Examining (3.9), we find that

$$\begin{aligned}
\nu_p(a(n)) &= \nu_p(N^2) - \nu_p(n\lambda(n)) + \nu_p\left(a(n-1)\phi_1\left(\frac{A}{N} + (n-1)\right)\right) \\
&= n\delta - \nu_p\left(\prod_{k=1}^{n} k\lambda(k)\right),
\end{aligned}$$

and the Proposition is proved, so long as we have

$$\nu_p\left(a(n-1)\phi_1\left(\frac{A}{N} + (n-1)\right)\right) < \nu_p\left(a(j)\phi_{n-j}\left(\frac{A}{N} + j\right)\right) \tag{3.20}$$

for all $0 \leq j \leq n-2$. Using the inductive hypothesis, one sees immediately that a sufficient condition for (3.20) to hold is that $2\nu_p(z_0) - \nu_p(N) < 0$, which is exactly what is assumed. $\square$

Corollary 3.5. *Suppose there is a prime p which divides $\frac{N}{(N, 2^8\cdot 3^4\cdot 5^2\cdot 7^2)}$. Then (3.6) is p-UBD.* $\square$

With this Corollary in hand, we are now well-situated to complete the proof of Theorem 1.3.

4. Proof of Main Theorem

We now complete the proof of Theorem 1.3 by establishing some general facts about cyclic $\mathcal{R}$-modules of vector-valued modular forms.

Assume throughout this Section that $\rho : \Gamma \to GL_d(\mathbb{C})$ is an irreducible representation of arbitrary dimension d, with $\rho(T)$ as in (2.3), such that

$$\mathcal{H}(\rho) = \bigoplus_{j=0}^{d-1} \mathcal{M} D^j F_0$$

is a cyclic $\mathcal{R}$-module with generator F_0. We set

$$\begin{aligned} \mathcal{H}(\rho)_{\mathbb{Q}} &= \{F \in \mathcal{H}(\rho) \mid F \text{ has rational Fourier coefficients}\}, \\ \mathcal{M}_{\mathbb{Q}} &= \{f \in \mathcal{M} \mid f \text{ has rational Fourier coefficients}\}. \end{aligned}$$

Then $\mathcal{H}(\rho)_{\mathbb{Q}}$ is clearly an $\mathcal{M}_{\mathbb{Q}}$-module, and we have

Lemma 4.1. *Assume that $F_0 \in \mathcal{H}(\rho)_{\mathbb{Q}}$. Then*

$$\mathcal{H}(\rho)_{\mathbb{Q}} = \bigoplus_{j=0}^{d-1} \mathcal{M}_{\mathbb{Q}} D^j F_0$$

is a free $\mathcal{M}_{\mathbb{Q}}$-module of rank d.

Proof. It follows directly from (2.5) and (2.2) that $D^j F_0 \in \mathcal{H}(\rho)_{\mathbb{Q}}$ for any integer $j \geq 0$, so clearly the free $\mathcal{M}_{\mathbb{Q}}$-module $\bigoplus_{j=0}^{d-1} \mathcal{M}_{\mathbb{Q}} D^j F_0$ is contained in $\mathcal{H}(\rho)_{\mathbb{Q}}$. On the other hand, suppose

$$F = \begin{pmatrix} f_1 \\ \vdots \\ f_d \end{pmatrix} = \begin{pmatrix} \sum_{n\geq 0} c_1(n) q^{r_1+n} \\ \vdots \\ \sum_{n\geq 0} c_d(n) q^{r_d+n} \end{pmatrix} \in \mathcal{H}(\rho)$$

has rational Fourier coefficients. Then there are unique $M_j \in \mathcal{M}$ such that $F = \sum_{j=1}^{d} M_j D^{j-1} F_0$, and we need to show that in fact $M_j \in \mathcal{M}_{\mathbb{Q}}$ for each j. A simple inductive argument shows that

$$D^j f_i = \sum_{n\geq 0} \beta_{ij}(n) q^{r_i+n},$$

where

$$\beta_{ij}(0) = \prod_{k=0}^{j-1} \left(r_i - \frac{k_0 + 2k}{12} \right),$$

with $k_0 \in \mathbb{Z}$ the weight of F_0. Writing $M_j = \sum_{n\geq 0} \alpha_j(n) q^n$ for each j, we obtain the formula

$$\begin{pmatrix} c_1(0) \\ \vdots \\ c_d(0) \end{pmatrix} = B \begin{pmatrix} \alpha_1(0) \\ \vdots \\ \alpha_d(0) \end{pmatrix}$$

for the leading Fourier coefficients of F; here B denotes the $d \times d$ matrix whose (i, j) entry is $\beta_{ij}(0)$. Noting that each of these entries is rational, as are the $c_j(0)$, one observes that the invertibility of B would yield $\alpha_j(0) \in \mathbb{Q}$ for each

j. But $\beta_{i1}(0) = 1$ for each i, and for $1 < k \leq d-1$ there are polynomials $p_k \in \mathbb{Q}[k_0]$ such that

$$\beta_{ij}(0) = r_i^j + \sum_{k=1}^{j-1} p_k r^{j-k},$$

thus B reduces to the $d \times d$ Vandermonde matrix V_d, whose (i,j) entry is r_i^{j-1}. It is well-known that

$$\det(V_d) = \prod_{1 \leq i < j \leq d} (r_j - r_i),$$

and by [14, Thm 3] the r_j are distinct, so $\det(V_d) = \det(B) \neq 0$ and we have $\alpha_j(0) \in \mathbb{Q}$ for each j. Continuing in this way, one arrives at the formula

$$\begin{pmatrix} c_1(n) \\ \vdots \\ c_d(n) \end{pmatrix} = B \begin{pmatrix} \alpha_1(n) \\ \vdots \\ \alpha_d(n) \end{pmatrix} + \vec{v},$$

for the n^{th} Fourier coefficients of F, where the i^{th} entry of $\vec{v}$ is a $\mathbb{Q}$-linear combination of the $\alpha_j(k)$, $1 \leq j \leq d-1$, $0 \leq k \leq n-1$. Assuming inductively that these entries are rational shows that the $\alpha_j(n)$ are also, and the Lemma is proved. □

For a prime number p, we write $\mathcal{B}_p \leq \mathcal{H}(\rho)_{\mathbb{Q}}$ for the $\mathcal{M}_{\mathbb{Q}}$-submodule of *p-bounded* vectors in $\mathcal{H}(\rho)_{\mathbb{Q}}$, i.e. the vectors which are not *p-UBD*. Note that if $F_0 \in \mathcal{B}_p$, then clearly $D^j F_0 \in \mathcal{B}_p$ for any $j \geq 0$, so by the previous Lemma we have

Corollary 4.2. *If $F_0 \in \mathcal{B}_p$, then $\mathcal{B}_p = \mathcal{H}(\rho)_{\mathbb{Q}}$ is a free $\mathcal{M}_{\mathbb{Q}}$-module of rank d.* □

On the other hand, we have

Proposition 4.3. *Suppose there is a prime p such that the Fourier coefficients of the first component of (3.6) satisfy (3.19). Then $\mathcal{B}_p = \{0\}$.*

Proof. Let $0 \neq g = \sum_{n \geq 0} \alpha(n) q^n \in \mathcal{M}_{\mathbb{Q}}$, so that

$$g f_1 = \sum_{n \geq 0} \beta(n) q^{r_1 + n}, \quad \beta(n) = \sum_{j=0}^{n} \alpha(j) a(n-j).$$

For any given $n \geq 0$, we have by (3.19) that

$$\nu_p(\beta(n)) = \nu_p(\alpha(0)) + n\delta - \nu_p\left(\prod_{k=1}^{n} k\lambda(k)\right) \tag{4.1}$$

so long as $\nu_p(\alpha(0)a(n)) < \nu_p(\alpha(j)a(n-j))$ for all $1 \leq j \leq n$. Again using (3.19), it is seen that this inequality will hold if

$$(n-j)\delta - \nu_p\left(\prod_{k=n-j+1}^{n} k\lambda(k)\right) < \nu_p(\alpha(j)) - \nu_p(\alpha(0))$$

for each $1 \leq j \leq n$. Since $g \in \mathcal{M}_{\mathbb{Q}}$, there is an integer M such that $\nu_p(\alpha(k)) \geq M$ for all $k \geq 0$, so for any integer $m \geq 0$ satisfying $m > \nu_p(\alpha(0)) - M$, by setting $n = p^m$ in (4.1) we obtain

$$(n-j)\delta - \nu_p\left(\prod_{k=n-j+1}^{n} k\lambda(k)\right) \leq -m < M - \nu_p(\alpha(0)) \leq \nu_p(\alpha(j)) - \nu_p(\alpha(0)),$$

for any $1 \leq j \leq p^m$. Thus for any such m we have $\nu_p(\beta(n)) < -m$, so $\lim_{m\to\infty} \nu_p(\beta(p^m)) = -\infty$. Since g was arbitrary, we have $\mathcal{M}_{\mathbb{Q}}F_0 \cap \mathcal{B}_p = \{0\}$.

Now suppose $F \in \mathcal{B}_p$. Since $D^jF \in \mathcal{B}_p$ for any $j \geq 0$, we obtain from Lemma 4.1 a relation

$$\begin{pmatrix} F \\ \vdots \\ D^{d-1}F \end{pmatrix} = A \begin{pmatrix} F_0 \\ \vdots \\ D^{d-1}F_0 \end{pmatrix},$$

with $A = (\alpha_{ij}) \in Mat_d(\mathcal{M}_{\mathbb{Q}})$. Now, if A were invertible, then we could write $A^{-1} = \det(A)^{-1}C$, where $\det(A)$ and the entries of the cofactor matrix C lie in $\mathcal{M}_{\mathbb{Q}}$. But this would yield a relation

$$C\begin{pmatrix} F \\ \vdots \\ D^{d-1}F \end{pmatrix} = \det(A) \begin{pmatrix} F_0 \\ \vdots \\ D^{d-1}F_0 \end{pmatrix},$$

whose left hand side lies in $\mathcal{B}_p$, and whose right hand side, by the work of the previous paragraph, does not. This contradiction implies that A is therefore *not* invertible, and consequently there is a relation

$$M_1D^{d-1}F + \cdots + M_dF = 0,$$

where at least one of the $M_j \in \mathcal{M}_{\mathbb{Q}}$ is nonzero. Thus each of the d components of F satisfies the same Fuchsian differential equation of order less than d. In particular, the d components must be linearly dependent, so the irreducibility of ρ forces $F = 0$, and the proof is complete. □

We are now able to complete the proof of Theorem 1.3. Consider once again an irreducible, three-dimensional representation (3.1) with $\rho(T)$ as in (3.4). By [23, Thm 2.9], the eigenvalues $\mathbf{e}\left(\frac{A}{N}\right), \mathbf{e}\left(\frac{B}{N}\right), \mathbf{e}\left(\frac{C}{N}\right)$ of $\rho(T)$ uniquely determine the equivalence class of irreducible representations to which ρ belongs. If the level N of ρ satisfies the hypothesis of Corollary 3.5, then the minimal weight vector (3.6) of $\mathcal{H}(\rho)$ is *p-UBD* for some prime p dividing N, thus by Proposition 4.3 every nonzero $F \in \mathcal{H}(\rho)_{\mathbb{Q}}$ is *p-UBD*. On the other hand, it is clear that the hypothesis of Corollary 3.5 will be satisfied by all but a finite number of positive integers N, and for each such N there are only a finite number of triples (A, B, C) satisfying the conditions (3.5). Thus there are only a finite number of equivalence classes of finite level, irreducible representations $\rho : \Gamma \to GL_3(\mathbb{C})$ that admit vector-valued modular forms with bounded denominators, and Theorem 1.3 is proved. □

5. Finite image representations

In this Section we examine the implications of Theorem 1.3 for the classical theory of modular forms. As discussed in the Introduction, if f is a component of a vector-valued modular form for a finite image representation ρ of the modular group Γ, then by definition f is a modular form for the finite index subgroup $\ker \rho \leq \Gamma$. Conversely (cf. [21, pgs 6-7]), if one begins with a modular form $f \in \mathcal{M}_k(G)$ for a subgroup G of index μ in Γ then f is, say, the first component of a vector-valued modular form

$$F = \begin{pmatrix} f|_k\gamma_1 \\ \vdots \\ f|_k\gamma_\mu \end{pmatrix} \in \mathcal{H}(k, \rho),$$

where $\{1 = \gamma_1, \cdots, \gamma_\mu\}$ is a complete set of coset representatives for Γ/G and ρ is the monomial representation arising from the natural action of Γ on the set Γ/G; thus $\rho = Ind_G^\Gamma(\mathbf{1})$ is realized by inducing the trivial one-dimensional representation of G up to the full modular group. More generally, if f is a modular form for G which transforms according to a character (i.e. one-dimensional representation) $\chi : G \to \mathbb{C}^\times$, then by the same procedure one obtains a vector-valued modular form with f as component, for the induced representation $Ind_G^\Gamma(\chi)$.

It transpires that the irreducible representations $\rho : \Gamma \to GL_3(\mathbb{C})$ with finite image may be identified via the eigenvalues of $\rho(T)$, as was shown in [18, Thm 2.1]. As was discussed in Section 2, every representation of level less than six has a kernel which is a congruence subgroup of Γ, so by [22, Thm 3.52] the vector-valued modular forms for these representations have bounded denominators. The classification of [18] splits the representations considered there into *primitive* and *imprimitive* types; in the present setting of dimension three a representation is imprimitive iff it is monomial, and otherwise is primitive.

By Theorem 2.1 (d), loc. cit., the only remaining equivalence classes of primitive, irreducible representations of finite image in dimension three are of level 7, and from the discussion in Section 3 it is seen immediately that only two equivalence classes appear here, with (A, B, C, N) in (3.5) equal to either $(1, 2, 4, 7)$ or $(3, 5, 6, 7)$. Although we shall not prove it here, the kernel of each of these representations is a congruence subgroup of Γ, so the associated vector-valued modular forms have bounded denominators.

On the other hand, in the imprimitive setting one finds – as we shall now demonstrate and as is also shown in [18] – that there are an infinite number of inequivalent, irreducible $\rho : \Gamma \to GL_3(\mathbb{C})$ which are monomial. Any such ρ is induced from a character $\chi : G \to \mathbb{C}^\times$ on a subgroup G of index three in Γ, and there are exactly four such subgroups: Γ^3, which is the normal subgroup of Γ generated by $\{\gamma^3 \mid \gamma \in \Gamma\}$, and the conjugate subgroups $(ST)^j\Gamma_0(2)(ST)^{-j}$, $j = 0, 1, 2$; here $\Gamma_0(2)$ denotes the subgroup of matrices in Γ which are upper-triangular (mod 2).

We consider first the subgroup $G = \Gamma^3$. This group has the single cusp ∞, with stabilizer generated by $\pm T^3$, and three elliptic points $i + j$, $j = 0, 1, 2$, with stabilizer generated by $T^j S T^{-j}$ respectively. This yields the presentation

$$G \cong \langle E_0, E_1, E_2, P \mid E_0^2 = E_1^2 = E_2^2,\ E_0^4 = E_1^4 = E_2^4 = E_0 E_1 E_2 P = 1\rangle,$$

where, e.g., one identifies P with $-T^3$ and E_j with $T^j S T^{-j}$ to obtain the isomorphism. From this one sees that a character χ of G is determined by specifying the fourth roots of unity $\chi(E_j) = \mathbf{e}\left(\frac{x_j}{4}\right)$, $j = 0, 1, 2$, subject to the constraint $x_i \equiv x_j \pmod 2$ for any i, j. Setting $\chi(P) = \mathbf{e}\left(\frac{x}{4}\right)$, with $x := -\sum x_j$, and using the coset decomposition $\Gamma = \cup_{j=0}^2 G T^j$, we obtain the induced representation $\rho = Ind_G^\Gamma(\chi)$, with[3] $\rho(\gamma) = (\chi(T^{i-1}\gamma T^{1-j}))$ for each $\gamma \in \Gamma$. In particular, we have that $\rho(S) = \mathrm{diag}\{\chi(E_0), \chi(E_1), \chi(E_2)\}$ is diagonal, and

$$\rho(T) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \chi(P) & 0 & 0 \end{pmatrix},$$

so the eigenvalues of $\rho(T)$ are the three cube roots of $\chi(P)$, i.e. $\mathbf{e}\left(\frac{x+4j}{12}\right)$ for $j = 0, 1, 2$. If ρ is irreducible, then by [23, Thm 2.9] these eigenvalues uniquely determine the equivalence class of ρ, thus inducing characters of Γ^3 up to Γ yields only a finite number of inequivalent representations of Γ, each of which has level dividing 12. Each of these representations is one of the open cases remaining after Theorem 1.3, possibly with a congruence subgroup as kernel, and as such we shall say no more about them.

We next consider the three conjugates of $\Gamma_0(2)$. Since inducing characters from conjugate subgroups yields equivalent representations, it is sufficient to consider only the characters of $G = \Gamma_0(2)$. This group has the two cusps ∞ and 0, with stabilizers generated by $\pm T$, $\pm S T^2 S^{-1}$ respectively, and a single elliptic point $\frac{i-1}{2}$, with stabilizer generated by $(ST)S(ST)^{-1}$. This yields the presentation

$$G \cong \langle E, P_1, P_2 \mid E^4 = E P_1 P_2 = 1\rangle,$$

with the identifications

$$\begin{aligned} E &\leftrightarrow (ST)S(ST)^{-1}, \\ P_1 &\leftrightarrow ST^2S^{-1}, \\ P_2 &\leftrightarrow T \end{aligned}$$

giving the isomorphism. This shows that a character χ of G is determined by any choice of integer $0 \leq x \leq 3$ and $c \in \mathbb{C}^\times$, so that $\chi(E) = \mathbf{e}\left(\frac{x}{4}\right)$, $\chi(P_2) = c$. Using the coset decomposition $\Gamma = \cup_{j=0}^2 G(ST)^j$, we obtain the induced representation $\rho = Ind_G^\Gamma(\chi)$, with $\rho(\gamma) = (\chi((ST)^{i-1}\gamma(ST)^{1-j}))$ for

[3]Here and in the next paragraph we abuse notation slightly by identifying G with its abstract presentation.

each $\gamma \in \Gamma$. In particular we have

$$\rho(T) = \begin{pmatrix} \chi(T) & 0 & 0 \\ 0 & 0 & \chi((ST)T(ST)^{-2}) \\ 0 & \chi((ST)^2T(ST)^{-1}) & 0 \end{pmatrix},$$

and from this it follows that the eigenvalues of $\rho(T)$ are $\chi(P_2) = c$ and the two square roots of $\chi(P_1) = c^{-1}\mathbf{e}\left(-\frac{x}{4}\right)$.

Assume χ has finite image, so that the first eigenvalue is $\lambda_1 = c = \mathbf{e}\left(\frac{A}{M}\right)$, with $0 \leq A < M$ and $(A, M) = 1$. Then the other eigenvalues are

$$\lambda_2 = \mathbf{e}\left(-\tfrac{4A+Mx}{8M}\right) = -\lambda_3,$$

and the level of ρ is the even integer $N := \frac{8M}{(4,Mx)}$. Since χ has finite image, so does ρ, and we have $\ker\rho \leq \ker\chi \leq G$. As the integer $M \geq 1$ is arbitrary, the corresponding level N of ρ may be any positive even integer. If the associated representation ρ is irreducible, then (again by [23, Thm 2.9]) we may assume up to equivalence that ρ is of the form analyzed in Section 3, and from this and Corollary 3.5 we obtain

Proposition 5.1. *Suppose $\rho : \Gamma \to GL_3(\mathbb{C})$ is as in (3.1), such that the integers A, B, C, N in (3.5) satisfy the additional constraints $N = 2M$, $C = B + M$ for some $M \geq 4$. Then ρ has finite image. If, furthermore, there is a prime p satisfying the hypothesis of Corollary 3.5, then* $\ker\rho$ *is noncongruence and every $F \in \mathcal{H}(\rho)_{\mathbb{Q}}$ is p-*UBD. □

The components of each such $F \in \mathcal{H}(\rho)_{\mathbb{Q}}$ are modular forms for the noncongruence subgroup $\ker\rho$, and at least one component is p-*UBD* whenever p satisfies the hypothesis of Corollary 3.5. This provides infinitely many new examples of noncongruence modular forms, for even level subgroups of $\Gamma_0(2)$, with unbounded denominators.

6. Concluding remarks

Although the theory of vector-valued modular forms can be a valuable tool in the study of classical modular forms, a perhaps more intriguing aspect of the theory (and one which was the original motivation for the writing of [4]) is found in the generic setting of infinite image representations of Γ. Here one observes, for the T-unitarizable representations, vector-valued modular forms whose components have classical q-expansions[4], yet nonetheless exhibit novel arithmetic behavior.

By way of example, let us return to the motivating problem for this article, and point out that in fact there are two fundamentally different ways that an infinite set of rational numbers may have unbounded denominators. It may be the case that there is a prime number p which occurs to an arbitrarily high power as a factor of the denominators, and when this occurs for the set

[4]In the non-T-unitarizable case one finds a more exotic mixture of components, which includes classical q-expansions multiplied by powers of $\log q$; this mimics (and as seen in this article is intimately tied up with) what is observed in the Fuchsian theory of linear differential equations; cf. [7] for further details.

of Fourier coefficients of a component of a vector-valued modular form F we say that F is *p-UBD*. Alternatively, a set of rational numbers has unbounded denominators if there are infinitely many distinct primes appearing as factors in the denominators, even if each of these primes is universally bounded as a factor. By the work of Scholl mentioned in the Introduction, this phenomenon does not occur for the set of Fourier coefficients of any modular form, on any finite index subgroup $G \leq PSL_2(\mathbb{Z})$ (or even for the set of Fourier coefficients of a basis of some $\mathcal{M}_k(G)$).

Nonetheless, there do exist q-expansions belonging to finite level, infinite image representations of the modular group which admit infinitely many primes in their denominators. Indeed, it has been observed experimentally by both Geoffrey Mason (in the two-dimensional setting) and the present author (using Sage while preparing for this article) that it is often the case – at least for low-dimensional representations – that one or more primes dividing the level of an infinite image representation will occur to arbitrarily high powers in the denominators of an associated vector-valued modular form, and at the same time infinitely many other primes will occur in the denominators. Furthermore, these primes often occur in arithmetic progressions determined by the eigenvalues for the action of T on the span of the components of the vector-valued form. The first concrete results along these lines are due to Kohnen and Mason [8, 9], where it is proven that certain *parabolic generalized modular forms*, transforming according to (nonunitary) characters $\chi : G \to \mathbb{C}^\times$ on congruence subgroups $G \leq \Gamma$, have q-expansions admitting infinitely many primes in the denominators. Mason has also shown (in unpublished work, cf. also [14, Lemma 4.3] et seq.) that some vector-valued modular forms associated to two-dimensional indecomposable, reducible representations of Γ have the property that all but finitely many prime numbers occur in the denominators of their Fourier expansion. In any event, the emerging theory of these generalized modular forms is quite intricate, and perhaps even richer arithmetically and geometrically than that of the classical (finite index subgroup) setting.

We also would like to comment here on the prospects for generalizing the results of this article to higher dimension. On the one hand, Section 4 shows that the analogue of Theorem 1.3 will follow immediately for cyclic $\mathcal{R}$-modules of vector-valued modular forms in arbitrary dimension, so long as an analogue of Proposition 3.4 can be found for the generating vector of the $\mathcal{R}$-module. Although the various computations are already somewhat tedious and intricate in dimension three, it does seem possible, in theory, to establish more general results along these lines.

Despite this, it must be noted that the case under consideration in this article – where the space of vector-valued modular forms is a cyclic $\mathcal{R}$-module – is far from generic. While it is true that $\mathcal{H}(\rho)$ is a cyclic $\mathcal{R}$-module whenever the dimension of ρ is less than four, in arbitrary dimension this will be a very special case indeed (cf. [12] for examples in dimensions four and five). Consequently, one would need to establish an analogue of Proposition 3.4 for each of the generators of $\mathcal{H}(\rho)$ as $\mathcal{R}$-module, and somehow deduce from this the analogue of Theorem 1.3. This will be further complicated by the fact that the modular differential equations which annihilate the generators will generally not be monic, thus the coefficients (3.7) of the differential equations will generally lie not in the polynomial ring $\mathbb{C}[E_2, E_4, E_6]$, but in its field of fractions. Consequently, the computations involved will be substantially more complicated than those of the present article.

Unsurprisingly, a natural conclusion is that the results of furthest reach in this area will be obtained via an understanding of the algebro-geometric aspects of the theory. Indeed, the work of Scholl mentioned in the Introduction has thus far been the basis for all investigations of the bounded denominator question, apart from the above-referenced contributions of Mason and the present article. The author is currently in collaboration with T. Gannon and G. Mason, developing a cohomology theory for the modular group, which will generalize the well-known work of Eichler, Shimura, et al., and it is hoped that an interpretation of the l-adic representations of Deligne and Scholl will emerge from this work. This provides the brightest prospect for generalizing the results and techniques of this article to higher dimension.

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References

1. A. O. L. Atkin and H. P. F. Swinnerton-Dyer, *Modular forms on noncongruence subgroups*, Combinatorics (Proc. Sympos. Pure Math., Vol. XIX, Univ. California, Los Angeles, Calif., 1968), Amer. Math. Soc., Providence, R.I., 1971, pp. 1–25.
2. Peter Bantay and Terry Gannon, *Vector-valued modular functions for the modular group and the hypergeometric equation*, Commun. Number Theory Phys. **1** (2007), no. 4, 651–680.
3. Einar Hille, *Ordinary differential equations in the complex domain*, Dover Publications Inc., Mineola, NY, 1997, Reprint of the 1976 original.
4. Marvin Knopp and Geoffrey Mason, *Generalized modular forms*, J. Number Theory **99** (2003), no. 1, 1–28.
5. ______, *On vector-valued modular forms and their Fourier coefficients*, Acta Arith. **110** (2003), no. 2, 117–124.
6. ______, *Vector-valued modular forms and Poincaré series*, Illinois J. Math. **48** (2004), no. 4, 1345–1366.
7. ______, *Logarithmic vector-valued modular forms*, Acta Arith. **147** (2011), no. 3, 261–282.
8. Winfried Kohnen and Geoffrey Mason, *On generalized modular forms and their applications*, Nagoya Math. J. **192** (2008), 119–136.

9. ______, *On the canonical decomposition of generalized modular functions*, arXiv:1003.2407v1 (2010).
10. Wen-Ching Winnie Li and Ling Long, *Fourier coefficients of noncongruence cusp forms*, arXiv:1012.3062 (2010).
11. Ling Long, *Finite index subgroups of the modular group and their modular forms*, Modular forms and string duality, Fields Inst. Commun., vol. 54, Amer. Math. Soc., Providence, RI, 2008, pp. 83–102.
12. Christopher Marks, *Classification of vector-valued modular forms of dimension less than six*, Ph.D. dissertation, arXiv:1003.4111 (2009).
13. ______, *Irreducible vector-valued modular forms of dimension less than six*, arXiv:1003.4111 (to appear in Ill. J. Math) (2010).
14. Christopher Marks and Geoffrey Mason, *Structure of the module of vector-valued modular forms*, J. Lond. Math. Soc. (2) **82** (2010), no. 1, 32–48.
15. Geoffrey Mason, *Vector-valued modular forms and linear differential operators*, Int. J. Number Theory **3** (2007), no. 3, 377–390.
16. ______, *Vector-valued modular forms and linear differential operators*, Int. J. Number Theory **3** (2007), no. 3, 377–390.
17. ______, *On the Fourier coefficients of two-dimensional vector-valued modular forms*, arXiv:1009.0781 (2010).
18. Eric C. Rowell and Imre Tuba, *Finite linear quotients of $\mathcal{B}_3$ of low dimension*, J. Knot Theory Ramifications **19** (2010), no. 5, 587–600.
19. A. J. Scholl, *Modular forms and de Rham cohomology; Atkin-Swinnerton-Dyer congruences*, Invent. Math. **79** (1985), no. 1, 49–77.
20. ______, *Modular forms on noncongruence subgroups*, Séminaire de Théorie des Nombres, Paris 1985–86, Progr. Math., vol. 71, Birkhäuser Boston, Boston, MA, 1987, pp. 199–206.
21. Atle Selberg, *On the estimation of Fourier coefficients of modular forms*, Proc. Sympos. Pure Math., Vol. VIII, Amer. Math. Soc., Providence, R.I., 1965, pp. 1–15.
22. Goro Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publications of the Mathematical Society of Japan, No. 11. Iwanami Shoten, Publishers, Tokyo, 1971, Kanô Memorial Lectures, No. 1.
23. Imre Tuba and Hans Wenzl, *Representations of the braid group $\mathcal{B}_3$ and of* $\mathrm{SL}(2, \mathbf{Z})$, Pacific J. Math. **197** (2001), no. 2, 491–510.
24. Klaus Wohlfahrt, *An extension of F. Klein's level concept*, Illinois J. Math. **8** (1964), 529–535.

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