Lesson 10. Facility Location

1 Facility Location Introduction

The **facility location** problem is another famous OR problem which is very applied for industry/military applications. In this lesson, we will look at 3 different types of formulations of this problem.

In these problems, the **input data** is...

- a network of "customers" (demand nodes),
- a set of *possible* "facilities" (supply nodes),

facilities are subset of customers

- a set of edges between customers and facilities that *could* serve them
- distances on the edges (which could represent distance, time, cost, or some combination of these factors)

The **goal** is to choose a set of supply facilities to serve the customers' demand based on some metric.

- For example: minimize the number of supply facilities opened while requiring that all customers are served.
- The different problem types result from varied metrics and/or requirements.
- Real world problems of this type include locating
  - military installations,
  - fire/police stations,
  - cell phone towers,
  - retail distribution centers and stores,
  - schools,
  - vaccine clinics.

2 *General* Facility Location Problems

Goal: Choose a set of supply facilities to meet customer demand according to some metric.

Notation:

Sets:

C = set of customer nodes

S = set of possible supply nodes

E = edges (c, s) connecting a customer c with a supply facility s that *could* serve the customer

Parameters:

$d_{c,s}$ = distance (or cost or time) between customer c and supply location s, for $(c, s) \in E$

h_c = demand of customer c, for $c \in C$

Decision Variables:

$$x_s = \begin{cases} 1 \text{ if } \boxed{\text{facility open in location } s} \\ 0 \text{ otherwise} \end{cases}, \text{ for all } s \in S$$

Problem 1. We will use the network and data on the following page for all of our example problems.

(a) *All* vertices represent customers. *Boxed* vertices represent possible supply locations. Use set notation to list the elements of the sets C and S.

$C = \{1, 2, 3, \ldots, 12\}$ $S = \{1, 5, 6, 8, 10, 11\}$

(b) The distance between a customer c and a supplier s is the length of the shortest path between them. Find $d_{1,1}$, $d_{4,1}$, and $d_{8,5}$. (Note that the "edges" in the model do not correspond directly to the edges in the graph.)

$d_{11} = 0$ $d_{41} = 17$ $d_{85} = 9$

(c) How should the columns and rows in the distance matrix be labeled? Do your answers in part (a) agree with the corresponding values in the distance matrix?

DATA for FACILITY LOCATION EXAMPLES:

$d_{11} = 0$

$d_{41} = 17$

$d_{85} = 9$

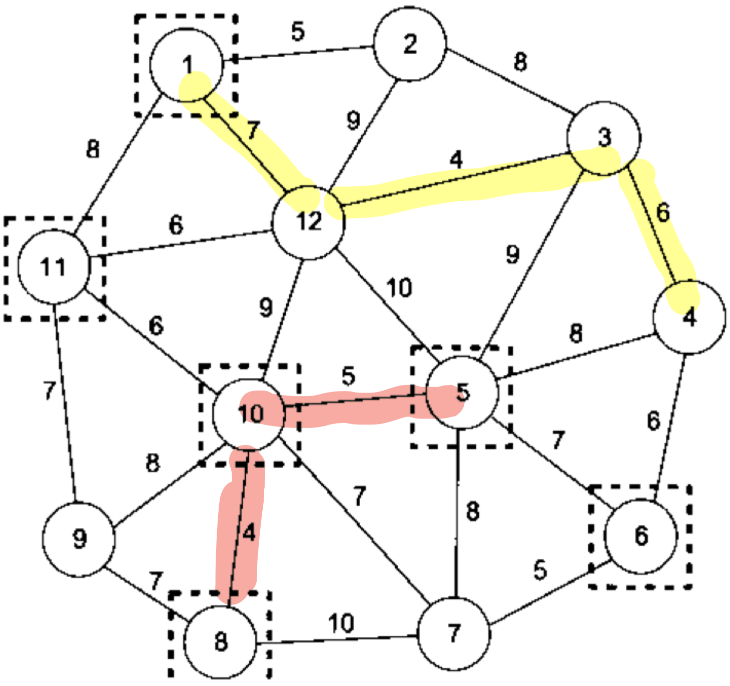
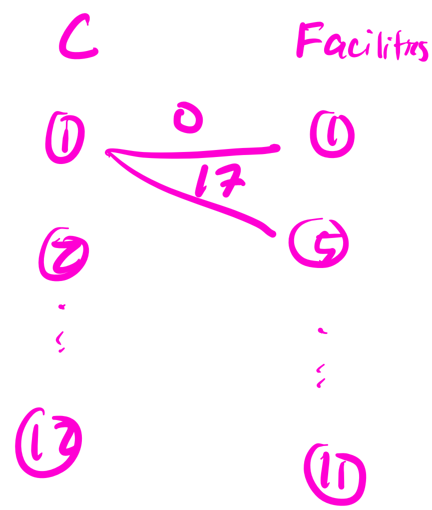


FIGURE 4.1 Example problem used for facility location models.



Facilities Set S

$$\mathbf{d} = \begin{array}{cc} \begin{matrix} 1 & 5 & 6 & 8 & 10 & 11 \end{matrix} & \\ \begin{bmatrix} 0 & 17 & 23 & 18 & 14 & 8 \\ 5 & 17 & 20 & 22 & 18 & 13 \\ 11 & 9 & 12 & 17 & 13 & 10 \\ 17 & 8 & 6 & 17 & 13 & 16 \\ 17 & 0 & 7 & 9 & 5 & 11 \\ 23 & 7 & 0 & 15 & 12 & 18 \\ 21 & 8 & 5 & 10 & 7 & 13 \\ 18 & 9 & 15 & 0 & 4 & 10 \\ 15 & 13 & 20 & 7 & 8 & 7 \\ 14 & 5 & 12 & 4 & 0 & 6 \\ 8 & 11 & 18 & 10 & 6 & 0 \\ 7 & 10 & 16 & 13 & 9 & 6 \end{bmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \end{matrix} \end{array}$$

Customers Set C

$\mathbf{h} = (100,\ 90,\ 110,\ 120,\ 80,\ 100,\ 95,\ 75,\ 110,\ 90,\ 120,\ 85).$

3 Simplest Facility Location Model: *Set Covering*

Given a set of potential facilities to open, open the fewest number of facilities such that each customer is "covered" by at least one facility.

Problem 2. Find the minimum number of facilities required to serve all customers. A facility must be within $D = 9$ miles of a customer in order to serve the customer.

(a) We define a new set for each customer, referred to as the neighborhood of customer c, N_c. N_c is the set of facilities that can cover customer c:

$$N_c = \left\{ s \in S : \boxed{d_{cs} \le 9} \right\}, \text{ for all } c \in C.$$

(b) Complete the missing neighborhoods.

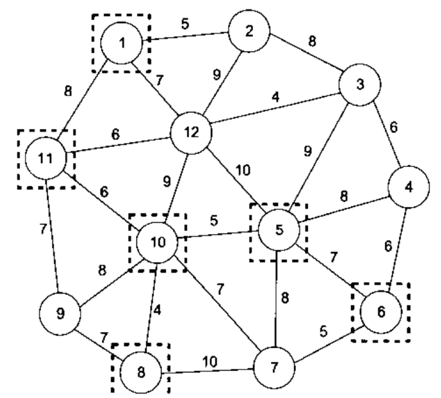


FIGURE 4.1 Example problem used for facility location models.

$S = \{1, 5, 6, 8, 10, 11\}$

$N_1 = \{1, 11\}$

$N_2 = \{1\}$

$N_3 = \boxed{\{5\}}$

$N_4 = \{5, 6\}$

$N_5 = \{5, 6, 8, 10\}$

$N_6 = \{5, 6\}$

$N_7 = \{5, 6, 10\}$

$N_8 = \boxed{\{5, 8, 10\}}$

$N_9 = \{8, 10, 11\}$

$N_{10} = \boxed{\{5, 8, 10, 11\}}$

$N_{11} = \{1, 10, 11\}$

$N_{12} = \boxed{\{1, 10, 11\}}$

(c) Write an abbreviated version of the concrete model using the x_s variables defined above.

$x_s = 1$ if facility s is open $\forall\ s \in S$

Objective

min: $x_1 + x_5 + x_6 + x_8 + x_{10} + x_{11}$] Total # of facilities open

Constraints

$x_1 + x_{11} \ge 1$] Node 1 is covered by a facility

$x_2 \ge 1$] Node 2 is covered

⋮

$x_1 + x_{10} + x_{11} \ge 1$] Node 12 is covered by a facility

$x_1, x_5, \ldots, x_{11} \in \{0, 1\}$

(d) Using the sets and variables defined below, complete the parameterized set covering facility location model.

Sets

Let S be the set of possible supply locations
Let C be the set of all customers
Let N_c be the neighborhood of customer c for all $c \in C$

Variables

Let $x_s = 1$ if a facility is placed at location s and 0 otherwise for all $s \in S$.

Objective

$$\min \sum_{s \in S} x_s$$] Total # of facilities open

Constraints

$$\sum_{i \in N_c} x_i \geq 1 \quad \forall \quad c \in C$$

$$x_s \in \{0,1\} \quad \forall \quad s \in S$$

$c=1$: $N_1 = \{1, 11\}$ $\quad \sum_{i \in N_1} x_i \geq 1 \rightarrow x_1 + x_{11} \geq 1$

$c=12$: $N_{12} = \{1, 10, 11\}$ $\quad \sum_{i \in N_{12}} x_i \geq 1 \rightarrow x_1 + x_{10} + x_{11} \geq 1$

What's the drawback of this model? In other words, what makes this problem unrealistic?

- Doesn't limit the # of facilities
- Doesn't account for demand

Still have $x_s = 1$ if facility s is open

4 *Maximal Covering* Location Problem

The **maximal covering location problem**: Given p facilities to open, maximize the customer demand that is covered. (A customer, c, can only be covered by a supply facility, s, in its neighborhood: $s \in N_c$.)

Maximal covering facility location model

***New* Parameters:**

$y_1 = 1 \rightarrow$ Customer 1 is covered
either $x_1 = 1$ or $x_{11} = 1$ or $x_1 = x_{11} = 1$

h_c = the demand at customer c, for all $c \in C$

p = the number of facilities to open

If $x_1 = 1 \rightarrow$ facility 1 is open
1,2,11,12 are covered
$y_1 = y_2 = y_{11} = y_{12} = 1$

***New* Decision Variables:**

$$y_c = \begin{cases} 1 \text{ if } \boxed{\text{Customer } c \text{ is covered}} \\ 0 \text{ otherwise} \end{cases}, \text{ for all } c \in C$$

Problem 3. Assuming that 2 facilities can be opened, write a concrete mode for the maximal covering facility location using the data from page 3.

Obj

Max covered demand: $h_1y_1 + h_2y_2 + \cdots + h_{12}y_{12}$] Total amount of demand that's covered

Constraints

$x_1 + x_5 + x_6 + x_8 + x_{10} + x_{11} = 2$] 2 facilities open

$y_1 \leq x_1 + x_{11}$] Forces $y_1 = 0$ if $x_1 = x_{11} = 0$

$y_2 \leq x_1$] Forces $y_2 = 0$ if $x_1 = 0$

$\vdots$

$y_{12} \leq x_1 + x_{10} + x_{11}$] Force $y_{12} = 0$ if $x_1 = x_{10} = x_{11} = 0$

$x_1, \ldots, x_{11}, y_1 \ldots, y_{12} \in \{0,1\}$

Customer 1: $y_1 = 1$ if either $x_1 = 1$ or $x_{11} = 1$
↳ If $x_1 = 1$ or $x_{11} = 1$ then $y_1 = 1$] Since max y_1 wants to be 1. Don't need to force $y_1 = 1$

Comment of constraint

$y_1 = 0$ if $x_1 = x_{11} = 0$
If $x_1 = 0$ and $x_{11} = 0$ then $y_1 = 0$] Need this constraint.

Problem 4. Write the parameterized version of the maximal covering facility location model.

Sets

C: set of customers

S: set of facilities

N_c: Neighborhood of customer c $\forall c \in C$

Variables

let $x_s = 1$ if facility s is open $\forall s \in S$

let $y_c = 1$ if customer c is covered $\forall c \in C$

Parameters

let h_c be the demand of customer c $\forall c \in C$

Objective

$$\max \sum_{c \in C} h_c y_c$$

Constraints

$$\sum_{s \in S} x_s = 2$$

$$y_c \leq \sum_{i \in N_c} x_i \quad \text{for all } c \in C$$

$$x_s \in \{0,1\} \quad \forall s \in S$$

$$y_c \in \{0,1\} \quad \forall c \in C$$

4.1 Example: Maximal Covering Facility Location Problem

Problem 5. Suppose that we can only afford to build and maintain $p = 2$ facilities, and the demand values for the customers (in order) are

$$\mathbf{h} = (100, 90, 110, 120, 80, 100, 95, 75, 110, 90, 120, 85).$$

(a) The optimal solution is to choose facilities 5 and 11. List the values of the decision variables x_s and y_c in the optimal solution. Illustrate the solution on the graph of the network.

$x_5 = 1 \quad x_{11} = 1$

$x_1 = x_6 = x_8 = x_{10} = 0$

$y_2 = 0$

$y_1 = y_3 = \cdots = y_{12} = 1$

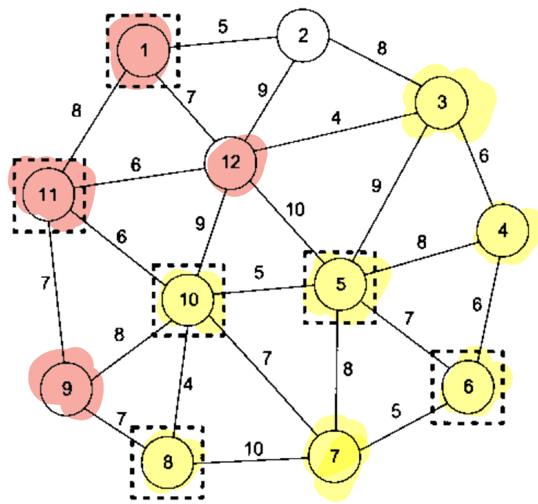


FIGURE 4.1 Example problem used for facility location models.

(b) Find the optimal objective function value. What does it mean?

$\sum_{c \in C} h_c y_c \rightarrow 100 \cdot 1 + 90 \cdot 0 + 110 \cdot 1 + \cdots + 85 \cdot 1 = 1085$

1085: Amount of satisfied demand

(c) Suppose we allow $p = 3$

i Which new facility would we open to cover the most demand?

Open 1 because it can satisfy the demand of 2

ii What would be our new objective function value?

$1085 + 90 = 1175$

Keep $x_s = 1$ if facility s is open

5 *p-Center* Facility Location Problem: Minimize Maximum Distance

Goal of the p-**center problem**: Choose p supply facilities to open in order to minimize the maximum distance between any customer and the supply facility that serves it.

Problem 6. Complete the p-center facility location formulation below by filling in the missing constraints and descriptions.

p-center facility location model

***New* Decision Variables:**

$$z_{c,s} = \begin{cases} 1 \text{ if } \boxed{\text{customer } c \text{ is assigned to facility } s} \\ 0 \text{ otherwise} \end{cases}, \text{ for all } c \in C,\ s \in S$$

W = the maximum distance between a customer and the facility chosen to serve it

Notice that by the way $z_{c,s}$ is defined, we assume that each customer could be served by any supplier.

Objective and constraint descriptions:

(1) and (2) C1): W is defined as max distance, minimizing this max distance.
C2): setting a bound on W for each customer.

(3) Exactly p facilities are opened

(4) Every customer is assigned to exactly 1 facility.

(5) Customer c cannot be assigned to a facility that is not open

$$\text{minimize} \quad W \tag{1}$$

$$\text{subject to} \quad \sum_{s \in S} d_{c,s} z_{c,s} \le W, \ \text{for } c \in C \tag{2}$$

$$\boxed{\sum_{s \in S} x_s = p} \tag{3}$$

$$\sum_{s \in S} z_{c,s} = 1, \ \text{for } c \in C \tag{4}$$

$$\boxed{z_{cs} \le x_s \ \forall s \in S, c \in C} \tag{5}$$

$$x_s \in \{0,1\}, \ \text{for } s \in S$$

$$z_{c,s} \in \{0,1\}, \ \text{for } c \in C,\ s \in S$$

Alt version of (2)

$z_{cs} d_{cs} \le W \quad \forall c \in C, s \in S$

$z_{cs} = 1$ customer c is assigned to facility s.

$W \ge d_{cs}$ so max distance can't be smaller than d_{cs}.

$z_{cs} = 0 \rightarrow W \ge 0$ no bound

If $x_s = 0$ then $z_{cs} = 0$

If $(1 - x_s) = 1$ then $(1 - z_{cs}) = 1$

$1 - x_s \le 1 - z_{cs}$

5.1 Example: p-Center Facility Location Problem

Problem 7. Assume the same network, distances, and demand values that we have already been using. We wish to find the $p = 2$ facilities that can serve all of the customers so that the maximum distance between a customer and the facility it is served by is minimized.

(a) Again, the optimal solution is to choose facilities 5 and 11. Facility 5 serves the customers 3, 4, 5, 6, 7, and 8. Facility 11 serves the rest. Write the values of the decision variables for this optimal solution and draw the solution on the network.

$$\mathbf{d} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccccc|c} 1 & 5 & 6 & 8 & 10 & 11 & \\ \hline 0 & 17 & 23 & 18 & 14 & 8 & 1 \\ 5 & 17 & 20 & 22 & 18 & 13 & 2 \\ 11 & 9 & 12 & 17 & 13 & 10 & 3 \\ 17 & 8 & 6 & 17 & 13 & 16 & 4 \\ 17 & 0 & 7 & 9 & 5 & 11 & 5 \\ 23 & 7 & 0 & 15 & 12 & 18 & 6 \\ 21 & 8 & 5 & 10 & 7 & 13 & 7 \\ 18 & 9 & 15 & 0 & 4 & 10 & 8 \\ 15 & 13 & 20 & 7 & 8 & 7 & 9 \\ 14 & 5 & 12 & 4 & 0 & 6 & 10 \\ 8 & 11 & 18 & 10 & 6 & 0 & 11 \\ 7 & 10 & 16 & 13 & 9 & 6 & 12 \end{array}$$

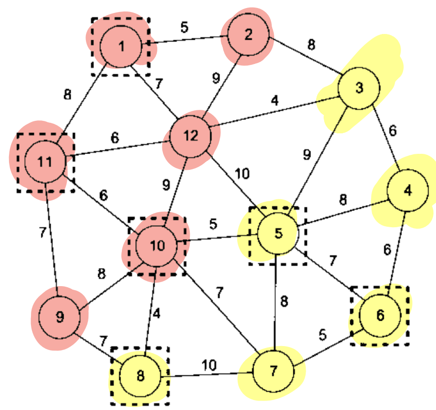


FIGURE 4.1 Example problem used for facility location models.

$x_5 \to$ facility open or not

$x_5 = 1,\ x_{11} = 1,\ x_1 = x_6 = x_8 = x_{10} = 0$

$z_{cs} = 1$ if c assigned to s

$z_{35} = z_{45} = \cdots = z_{85} = 1$

$z_{1\,11} = z_{2\,11} = z_{9\,11} = \cdots z_{12\,11} = 1$

All other $z_{cs} = 0$

(b) Find the optimal objective function value. What does it mean?

$W = 13$ Max distance traveled is 13

(c) Write concrete versions of the following:

i constraint (2) for customers 2 and 3:

$\sum d_{cs} z_{cs} \le W$

$c=2:\ 5z_{21} + 17z_{25} + 20z_{26} + 22z_{28} + 18z_{2\,10} + 13z_{2\,11} \le W$

$c=3:\ 11z_{31} + 9z_{35} + \cdots + 10z_{3\,11} \le W \to$ plug in z

$W \ge 13$

$W \ge 9$

ii constraint (4) for customer 2:

$\sum z_{cs} = 1$

$c=2:\ z_{21} + z_{25} + \cdots + z_{2\,11} = 1$

iii constraints (5) for supplier 5 and all of its potential customers:

$z_{cs} \le x_s$

$s = 5$

$z_{15} \le x_5$

$z_{25} \le x_5$

$\vdots$

$z_{12\,5} \le x_5$

Always have $x_s = 1$ if facility s is open $\forall s \in S$

6 Summary

We discuss 3 common types of facility location models. These models, and their goals were:

- Set covering
  - Ensure every customer is covered with fewest # of facilities
  - Doesn't acount for demand or # of facility
- Maximal covering
  - Maximize covered demand with limited facilities
  - Make sure you understand logical constraints
- p-center
  - min max distance travel
  - Make sure understand (and replicate) all constraints except W.

This is **not** an all inclusive list of models. Typically, one could use one of these as a starting point and then expand from there. Other things to model include:

- A fixed charge associated with each facility (think back to Lesson 5)
- Incorporating logical constraints like in Lesson 7
- Multiperioud facility location (can be quite difficult)
- Different objective functions such as:
  - Minimizing the average distance traveled
  - Minimizing a weighted distance (i.e., prioritize customers with high demand)
  - Opening facilities based on need (i.e., some areas have a priority)
  - etc...