SA405, Fall 2020 **Exam 2**
6 November 2020

Midshipmen are persons of integrity. **Name:** ____________________

- No books, notes, or any other outside help allowed.
- Show work clearly and neatly.
- Define all notation used.
- Please read each question carefully. If you are not sure what a question is asking, ask for clarification.
- If you start over on a problem, please CLEARLY indicate what your final answer is, along with its accompanying work.

Grade Table (for teacher use only)

Question	Points	Score
1	30	
2	35	
3	35	
Total:	100	

1. Answer the following

   (a) (14 points) You're modeling an integer program on continuous variables (x_1, x_2, and x_3). To complete the model, one of the following two constraints needs to be included:

$$2x_1 + 3x_2 - 3x_3 \geq 10 \quad (1)$$
$$x_1 - x_2 + 2x_3 \leq 30 \quad (2)$$

   Use binary variable y to modify above constraints in order to execute the following strategy; If $y = 0$, the first constraint (constraint 1) is included, and if $y = 1$, then second constraint (constraint 2) is included in the model.

   Solution:

$$2x_1 + 3x_2 - 3x_3 \geq 10 - My$$
$$x_1 - x_2 + 2x_3 \leq 30 + M(1 - y)$$

   where M represents a big number.

   (b) Consider the subtour elimination constraints in the traveling salesperson problem on **7 total nodes**. Next to each of the following set sizes, circle yes or no to indicate whether you would eliminate subtours of that size (e.g., if you circle yes next to 3, this means you would want to eliminate all subtours on sets of size 3).

      i. (2 points) Sets of size 2? YES or NO

      Solution: No

      ii. (2 points) Sets of size 3? YES or NO

      Solution: Yes

      iii. (2 points) Sets of size 7? YES or NO

      Solution: No

   (c) (10 points) Suppose you're modeling a vehicle routing problem with 2 vehicles on 8 nodes (specifically nodes 0, 1, 2, 3, 4, 5, 6, 7). Write the concrete constraints which tell the number of edges that must be connected to node 0 (the depot) and node 4.

   Solution:

For node 0, we would have:

$$x_{0,1} + x_{0,2} + x_{0,3} + x_{0,4} + x_{0,5} + x_{0,6} + x_{0,7} = 2 * 2 = 4$$

For node 4, we would have:

$$x_{0,4} + x_{1,4} + x_{2,4} + x_{3,4} + x_{4,5} + x_{4,6} + x_{4,7} = 2$$

2. In order to improve food variety, you've decided, along with one of your friends, to open up two taco stands on the yard. You're trying to determine where to open these two stands. You've identified 4 possible locations for your stands: Nimitz, Gate 1, Chauvenet courtyard, and Bancroft. You want to service demand at each of the following buildings: Nimitz (NI), Visitor center (VC), Chauvenet (CH), Michelson (MI), Alumni Hall (AH), Bancroft (BA), and Ward (WA). The table below gives the distances between each potential taco stand location and each customer as well as the demand of each customer:

	Destinations						
Potential Locations	NI	VC	CH	MI	AH	BA	WA
Nimitz (NI)	0	20	8	7	3	11	15
Gate 1 (G1)	15	1	12	12	15	10	6
Chauvenet Courtyard (CC)	6	15	1	1	8	5	12
Bancroft (BA)	11	10	5	5	12	0	7
Demand	50	12	25	22	7	80	15

A customer can only be serviced if its distance is at most 12 units away.

Consider the partial model given below:

Sets

Let S be the set of potential locations for your taco stand $S = \{NI, G1, CC, BA\}$
Let C be the set of customers of your taco stand $C = \{NI, VC, CH, MI, AH, BA, WA\}$

Parameters

Let h_c be the demand of customer c for all $c \in C$
Let $d_{s,c}$ be the distance between customer c and taco stand s for all $c \in C$ and for all $s \in S$
Let N_c be the neighborhood of customer c for all $c \in C$.

Decision Variables

Let $x_s = 1$ if a taco stand is placed at location s for all $s \in S$.

(a) (5 points) A taco stand is in the neighborhood of a customer if it is at most 12 units away. What is N_{NI} and N_{WA}?

Solution: The neighborhood of NI is all suppliers at most 12 units away. This means that:

$$N_{NI} = \{NI, CC, BA\}$$

Likewise, the neighborhood of WA is all suppliers at most 12 units away. Thus:

$$N_{WA} = \{G1, CC, BA\}$$

(b) Since you only have 2 taco stands, you decide to formulate an IP to optimally determine where to place your taco stands. The goal of this IP is to cover as much demand as possible with the limitation that only two taco stands can open.

i. (5 points) Introduce any new decision variables needed to formulate this IP. If none are needed write NONE NEEDED and continue to question ii

Solution: We need to define a new variable here y_c which equals 1 if customer c is covered and $y_c = 0$ if customer c is not covered for all $c \in C$

ii. (5 points) Write a concrete constraint which says that exactly 2 taco stands can be opened.

Solution:

$$x_{NI} + x_{G1} + x_{CC} + x_{BA} = 2$$

iii. (10 points) Write an objective function which maximizes the total demand covered.

Solution:

$$\text{maximize: } \sum_{c \in C} h_c y_c$$

iv. (10 points) In order to formulate this correctly, each customer node must either be covered or not covered. Write a concrete constraint for Alumni Hall that says if no taco stand is opened in its neighborhood, then it is not covered. Once you have written this constraint, explain its logic in words (it is suffice to say if BLEH then BLAH else MEH).

Solution: We need to write a constraint that says y_{AH} can only be covered if one of the nodes in its neighborhood is selected. First, we consider it's neighborhood which is $N_{AH} = \{NI, CC, BA\}$. So my constraint I want is:

$$x_{NI} + x_{CC} + x_{BA} \geq y_{AH}$$

The logic of this constraint is as follows. If we place the taco stand at either NI, CC, or BA, it covers AH. Thus, y_{AH} will become 1. Conversely, if none of these facilities are built, then $x_{NI} = x_{CC} = x_{BA} = 0$ thereby forcing y_{AH} to be 0 since it will not be covered.

3. The Military Command Center 'East' needs to install special secure communication lines between the Command Center (CC) and five of Military Installation posts (MI1,..., MI5) under its command. The communication line from a MI post need not be connected directly to the CC. It can be connected indirectly by being connected to another MI post that is connected (directly or indirectly) to the CC. The only requirement is that every MI post be connected by some route to the CC. The installation of a special communication line is costly; it costs $100,000$ per mile. Hence, the total cost of installation of special communication lines is $100,000$ times the number of miles involved, where the distances (in miles) between CC and MIs and between MIs are given in the table below:

	Distances in miles					
	CC	MI1	MI2	MI3	MI4	MI5
Command center (CC)	-	190	70	115	270	160
MI1		-	100	110	215	50
MI2			-	140	120	220
MI3				-	175	80
MI4					-	310
MI5						-

The CC wants to determine which pairs of MIs, and CC and MIs should be directly connected by special communication lines in order to connect every MI (directly or indirectly) to the CC at a minimum total cost.

Below we list the sets, parameters and variables of the model

Sets

Let S be the set the CC and MIs $S = \{CC, MI1, MI2, MI3, MI4, MI5\}$
Let E be the set of edges $E = \{(i,j) : i,j \in S, i < j\}$

Parameters

Let $d_{i,j}$ be the distance between nodes i and j for all $i,j \in S$ in miles.

Let P be the the price per mile of the special communication line.

Decision Variables

Let $x_{i,j} = 1$ if a communication line is laid between nodes i and j for all $i,j \in S$.

(a) (2 points) Which type of Network optimization model best describes this problem?

> **Solution:**
> Minimum Spanning Tree Model

(b) (5 points) Write the objective function for the concrete model. You can use ellipses, however, write at least 3 terms explicitly.

Solution:

$$\text{Minimize} \quad 190x_{CC,MI1} + 70x_{CC,MI1} + \cdots + 310x_{MI4,MI5}$$

(c) (8 points) Write a concrete constraint that ensures a given node is part of at least one edge for each of the following nodes: CC, MI1, and MI5.

Solution:

$$\begin{aligned}
\text{Node CC:} \quad & x_{CC,MI1} + x_{CC,MI2} + \cdots + x_{CC,MI5} \geq 1 \\
\text{Node MI1:} \quad & x_{CC,MI1} + x_{MI1,MI2} + \cdots + x_{MI1,MI5} \geq 1 \\
\text{Node MI 5:} \quad & x_{CC,MI5} + x_{MI1,MI5} + \cdots + x_{MI4,MI5} \geq 1
\end{aligned}$$

We have 6 such constraints, for each node one.

(d) (5 points) How many edges must the optimal solution have? Write the concrete constraint that ensures the solution has exactly that many edges.

Solution: Since we have 6 nodes, the MST will have $6 - 1 = 5$ edges.

$$\sum_{(i,j)\in E} x_{i,j} = 5$$

(e) Suppose that at one stage of solving the problem the following solution is obtained: $(CC, MI2), (CC, MI5), (MI1, MI5), (MI1, MI2), (MI2, MI3), (MI3, MI4)$

i. (3 points) Graph this solution.

Solution:

ii. (2 points) Is this solution feasible? Answer Yes or No.

Solution: NO

iii. (10 points) If Yes, explain. If Not, write the constraint that will prevent this solution from occurring again.

Solution: This solution has a cycle

$$(CC, MI2), (CC, MI5), (MI1, MI5), (MI1, MI2).$$

MST cannot have cycles, hence, they need to be eliminated. The set of nodes in the cycle is $S = \{CC, MI1, MI2, MI5\}$, therefore, the concrete cycle elimination constraint is

$$x_{CC,MI1} + x_{CC,MI2} + x_{CC,MI5} + x_{MI1,MI2} + x_{MI1,MI5} + x_{MI2,MI5} \leq 4 - 1 = 3$$