Branch-and-bound Graphical Example (Accompanying Lesson 16)

1 Branch-and-bound Example

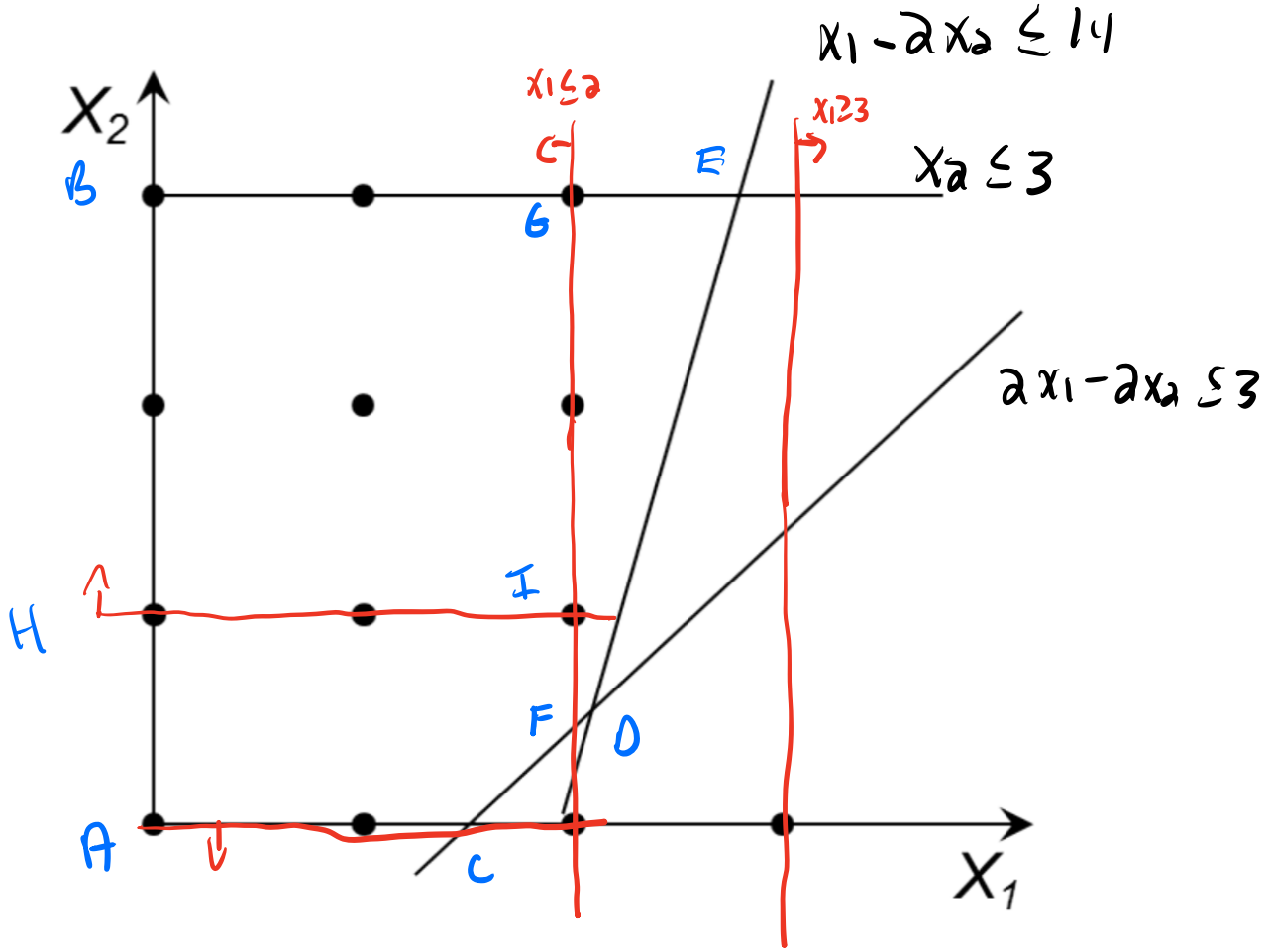
Solve the following IP using branch-and-bound. Solve each sub problem graphically.

$$\begin{aligned} \text{(P1)} \quad z^*_{IP} = \max\ & 4x_1 - x_2 \\ \text{s.t.}\quad & 7x_1 - 2x_2 \leq 14 \\ & 2x_1 - 2x_2 \leq 3 \\ & x_2 \leq 3 \\ & x_1, x_2 \in \mathbb{Z}^{\geq 0} \end{aligned}$$

- Solve each sub-problem graphically
- Branching Rules
  - Always select the active node with the largest upperbound for branching.
  - Branch on x_1 if it is fractional. Otherwise branch on x_2.
- Book-keeping
  - Keep track of the:
    - incumbent solution $\underline{x}$,
    - global lower bound $\underline{z}$, and
    - list of active nodes.
  - Draw the branch-and-bound tree:
    - Record the local upper bound (z) and relaxed optimal solution (x) for each subproblem.
    - Label each edge with the constraint that is added to form the child subproblem.
    - X-out fathomed nodes. Circle incumbent solution nodes.
  - Use the provided diagram to illustrate the (relaxed) feasible region of each subproblem.

incumbent solution $\underline{x}$ **global lower bound $\underline{z}$** **active nodes**

Feasible Region



A: (0,0) $z=0$

B: (0,3) $z=-3$

C: $(\frac{3}{2}, 0)$ $z=6$

D: $(\frac{11}{5}, \frac{7}{10})$, $z=8.1$

E: $(\frac{20}{7}, 3)$, $z=8.4$ ←

F: $(2, \frac{1}{2})$, $z=7.5$

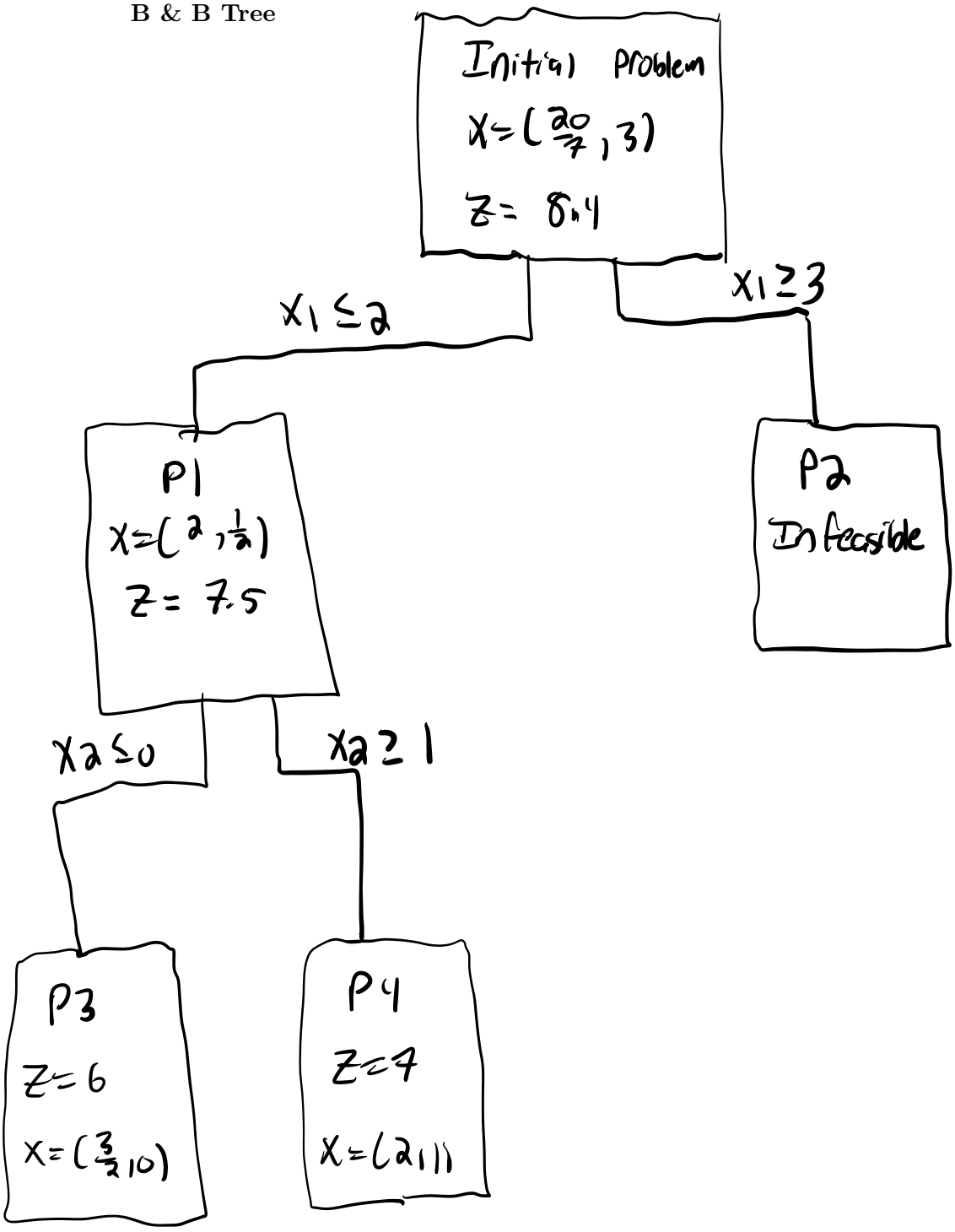
G: (2,3) $z=5$

H: (0,1) $z=-1$

I: (2,1) 7

$$
\begin{aligned}
z^*_{IP} = \max\ & 4x_1 - x_2 \\
\text{s.t.}\quad & 7x_1 - 2x_2 \leq 14 \\
& 2x_1 - 2x_2 \leq 3 \\
& x_2 \leq 3 \\
& x_1, x_2 \in \mathbb{Z}^{\geq 0}
\end{aligned}
$$

B & B Tree


Initial Problem
X=(20/7, 3)
Z= 8.4
x1 ≤ 2
x1 ≥ 3
P1
X=(2, 1/2)
Z= 7.5
P2
Infeasible
x2 ≤ 0
x2 ≥ 1
P3
Z=6
X=(3/2, 0)
P4
Z=7
X=(2, 1)

Branch and Bound logic

Step 1: Solve LP relaxation of original problem

Optimal solution is $(\frac{29}{7}, 3)$ with $Z^*_{LP} = 8.4$

$Z^*_{IP} \leq 8.4 \rightarrow Z^*_{IP} \leq 8$

x_1 is fractional so branch on x_1. P_1 is original problem and $x_1 \leq 2$

P_2 is original problem and $x_1 \geq 3$

Step 2: Solved problems P_1 and P_2

P_1: $X = (2, \frac{1}{2})$ $Z = 7.5$

P_2: Infeasible

old bound $Z_{IP} \leq 8$

new bound $Z_{IP} \leq 7$

x_2 is fractional so I want all $x_2 \leq 0$
$x_2 \geq 1$

STEP 3: Solve P_3 and P_4

P_3: $(\frac{3}{2}, 0)$ $Z = 6$

P_4: $(2, 1)$ $Z = 7$

Integer solution where $Z = 7$

$Z^*_{IP} \geq 7$ and $Z^*_{IP} \leq 7$

Stop P_4 is optimal

$Z^* = 7$ $X^* = (2, 1)$