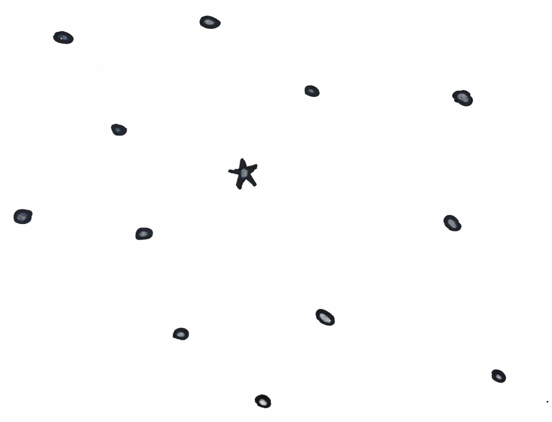
Lesson 9. Vehicle Routing Problem (VRP)

1 VRP: An extension of TSP

Vehicle Routing Problems are similar to TSP, except that instead of having a single "salesperson", we have multiple "salespeople" (vehicles), all *starting and ending at the same location.* We must decide the "cheapest" way to cover all of the locations via tours with our fleet of salespeople.

Problem 1. The pizza restaurant at $\star$ has 12 delivery orders ready and three drivers. Each driver can serve at most 5 customers. Draw a *feasible* solution to the VRP problem on the graph below.



Vehicle routing is a very practical real world application.

Basic VRP

- Fixed number of vehicles.
- All vehicles start and end a common location, or **depot**.
- All vehicles have the same capacity.
- Vehicles may serve many different customers, up to capacity.
- Each customer's demand must be served by a single vehicle.
- Goal: Find a minimum cost collection of *tours*, all starting and ending at the depot, that contain all customers and do not violate vehicle capacities.

Basic VRP: Mathematical Description
Notation:

- Let $G = (N, E)$ be an undirected graph.
  - One of the nodes is distinguished as the *depot*: usually vertex "0".
  - Each edge has a cost (or distance): c_{ij}, for $(i, j) \in E$ Remember that $(i, j) \in E$ if $i < j$.
  - Each node has a demand: d_i, for $i \in N$.
- Exactly k tours (vehicles) used, each with the same demand capacity (D).

Requirements:

- Each tour must include the depot.
- Each vertex (customer) must be be served.
- The demand of each tour must not exceed D.

Formulation assumptions:

- Every vehicle visits at least two customers.
- The direction an edge is traversed does not affect the cost.
- (Variations of the formulation can work around these assumptions.)

Goal:

- Minimize total cost of tours.

VRP can be modified to incorporate variations including:

- **Capacitated VRP:** Each vehicle has a capacity (problem we are doing)
- **VRP with time windows:** Each customer must be visited within a certain time window
- **Multidepot VRP:** Fixed number of vehicles starting at each of multiple depots.
- **Split delivery VRP:** Customer demand can be met by more than one vehicle

2 VRP Standard Formulation: Pizza Deliveries (Example 4.1 in Rader, page 127)

Problem 2. A local pizza shop received 10 orders for delivery last night with three delivery persons working. The shop uses a coordinate system to mark where houses are located (using the nearest intersection as locations). The 10 deliveries are to go to the following places:

	1	2	3	4	5	6	7	8	9	10
E/W	20	40	180	130	160	40	30	100	90	75
N/S	90	70	20	100	10	80	50	60	120	15

All streets in town go either north-south or east-west, so distance must be measured rectilinearly. Assuming that the pizza shop is located at position (0,0) and that each driver can deliver at most five orders, how should the delivery routes be determined to minimize the total travel distance?

1. Write an abbreviated concrete model to minimize the distance traveled by the pizza shop. Do **not** include subtour elimination constraints. *Hint: there are two types of constraints, one for each node other than the depot and one for the depot.*

Problem 3. Suppose a solution to the (incomplete) model above gives the following cycles: $\{0, 2, 4\}, \{0, 6, 9\}, \{0, 8, 10\}, \{1, 3, 5, 7\}$.

(a) Sketch the graph associated with this solution.

(b) What are the values of the edges corresponding to this solution?

(c) Why is this solution infeasible to the (complete) VRP formulation?

(d) Using the MST/TSP style, write a concrete constraint that would eliminate this solution from the feasible region.

Problem 4. Suppose a solution to the (incomplete) model above gives the following cycles: $\{0, 2, 4\}, \{0, 6, 9\}, \{0, 1, 3, 5, 7, 8, 10\}$.

(a) Sketch the graph associated with this solution.

(b) What are the values of the edges corresponding to this solution?

(c) Why is this solution infeasible to the (complete) VRP formulation?

(d) Write a concrete constraint that would eliminate this solution from the feasible region *without including any edges connected to node 0 (we will see why soon).*

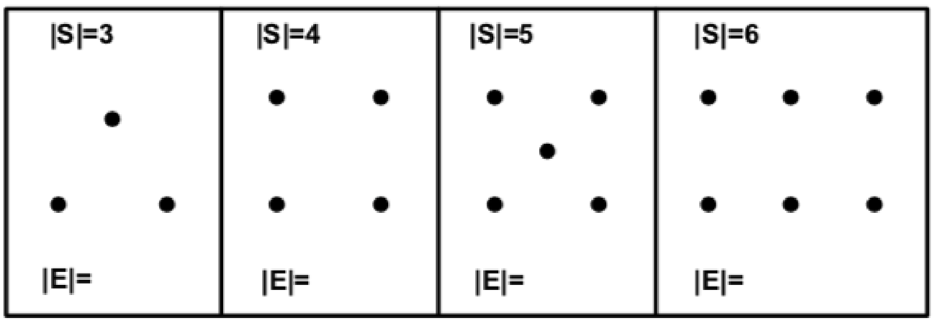
3 Route Splitting in VRP

The **subtour elimination constraints** that we learned about in MST and TSP would technically work for VRP but would make the solution process very difficult. Thus, prior to parameterizing the VRP problem; we introduce a new type of subtour elimination constraints called which work for the VRP (these are also the types of constraints you'll use for project 2!).

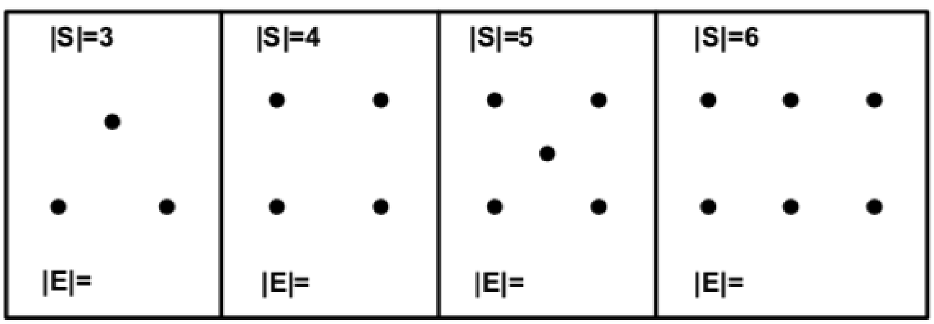
A **connected component** is a subset of nodes in a graph which are all connected to each other.

The question we ask in VRP is: How many edges must we include among $|S|$ vertices in order to have C connected components and NO CYCLES (importantly, this means you need C vehicles to visit each of these customers)? Let's try some test cases...

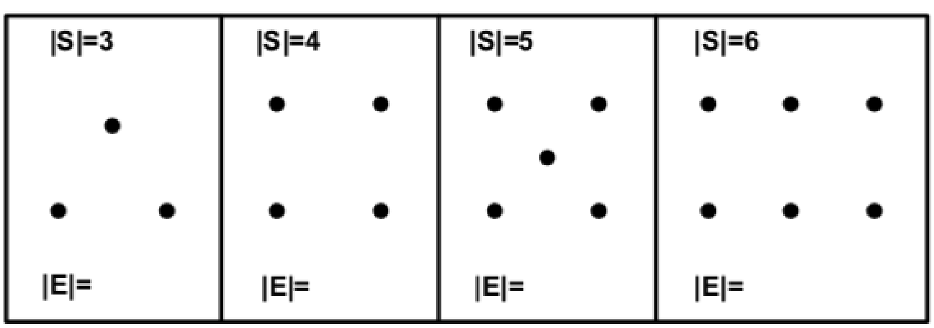
- $C = 1$: How many edges do we include to have 1 connected component and no cycles?



- $C = 2$: How many edges do we include to have 2 connected components and no cycles?



- $C = 3$: How many edges do we include to have 3 connected components and no cycles?



Given a general $C > 0$, how many edges must we include among $|S|$ nodes to have C connected components and no cycles? These are called **generalized subtour elimination** constraints.

4 Parameterized VRP Model

Problem 5. Write the full parameterized VRP model using the generalized subtour elimination constraints.