Lesson 12. IP Formulations Part 1

LP Relaxation

1 Solving Integer Programs can be *Really* Hard!

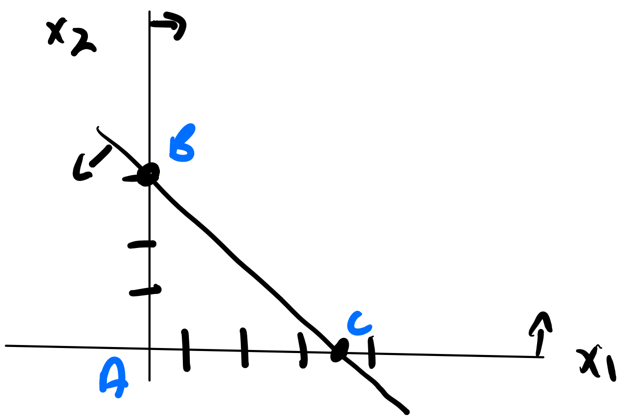
Suppose we are solving the following integer program:

$$\begin{aligned} \max \quad & 12x_1 + 13x_2 \\ \text{st} \quad & 6x_1 + 7x_2 \leq 21 \\ & x_1, x_2 \in \mathbb{Z}^+ \end{aligned}$$

$x_1, x_2 \geq 0$ integer

$$\begin{aligned} \text{Max} \quad & 12x_1 + 13x_2 \\ & 6x_1 + 7x_2 \leq 21 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Usually, everyone's first thought for solving IPs is to solve the LP and then round to the nearest integer. Let's try that here:



A: $(0,0)$ $z=0$

B: $(0,3)$ $z=39$

C: $(3.5,0)$ $z=42$

If we solve the LP we get the solution:

$(3.5, 0)$ $z = 42$

Rounding this solution, we get an IP solution of:

$(3, 0)$ $z = 36$

Is this the optimal solution to the IP?

No because $(0,3)$ is optimal to IP

In general, IPs are **significantly harder** to solve than LPs.

- In the next two lessons, we will discuss why IPs are harder than LPs and why the way we model IP problems can impact solver performance.
- In lesson 14 we will learn about the **branch and bound algorithm** which is a method to solve IPs.

2 Review Linear Programming Solution Techniques

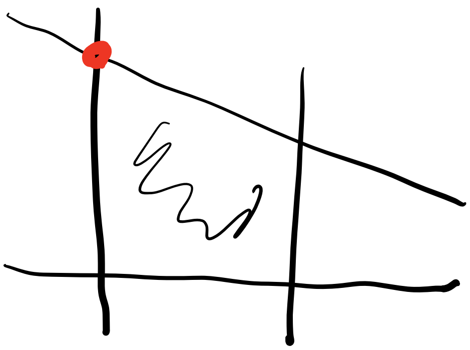
2.1 Types of LP solutions

Theorem: Every LP's solution is EXACTLY one of the following:

1. Unique optimal solution
2. Multiple optimal solutions → In LP infinite, IP 2 or more
3. Unbounded LP
4. Infeasible LP

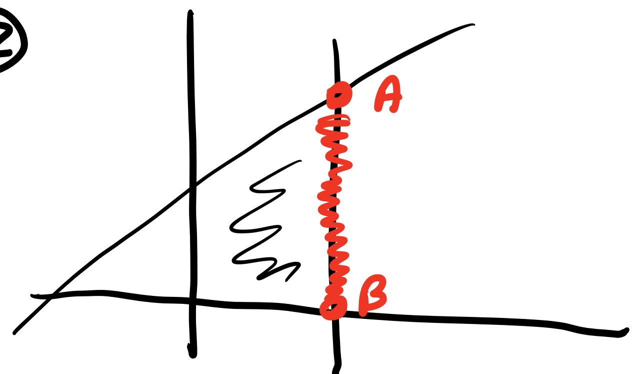
1

Problem 1. Sketch graphs which illustrate each of the types of LP solutions.



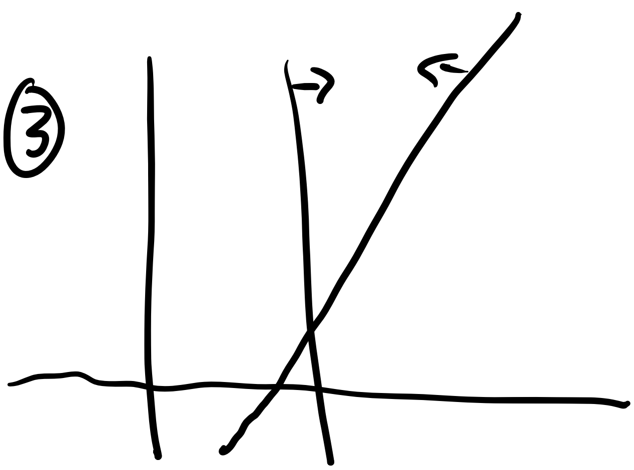
Red point has best Objective value, so only optimal solution

2



A and B both have best value entire line connecting them is optimal.

3



If max, objective can infinitely increase

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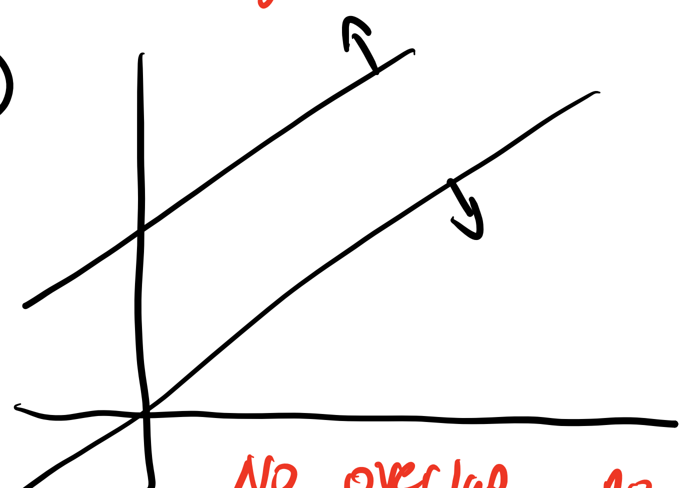
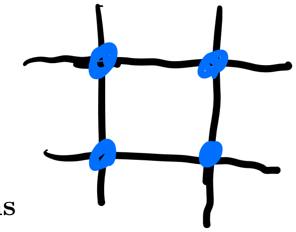


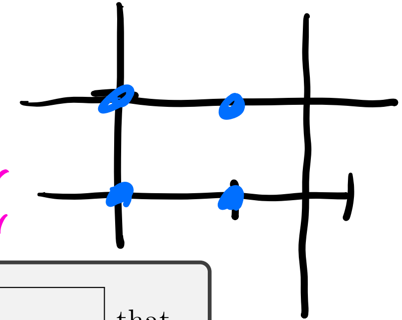
No overlap, no points satisfy constraints

> These types of solutions are also true for integer programs. Recall that if an LP has an optimal solution, it always occurs at a corner point.

Max x_1
$x_1 \leq 1$ $x_1 \geq 0$ integer
$x_2 \leq 1$ $x_2 \geq 0$ integer



Max x_1
$x_1 \leq 1.7$
$x_2 \leq 1$
$x_1 \geq 0$ integer
$x_2 \geq 0$ integer



3 Integer Program Formulations

A **formulation** of an integer (linear) program is a set of linear $\boxed{\text{constraints}}$ that capture ALL of the $\boxed{\text{feasible}}$ integer points, and NO OTHER integer points.

IP Max cx
$Ax=b$
$x \geq 0$ integer

LP Relaxation Max cx
$Ax=b$
$x \geq 0$

The **LP relaxation** of an IP is the LP that is formed by relaxing (i.e., removing) the integer requirement on the variables.

Problem 2. Below are two integer programs, along with the diagrams of their constraints.

Integer Program A

$$\begin{aligned} \text{maximize} \quad & 8x + 7y \\ \text{subject to} \quad & -18x + 38y \leq 133 \\ & 13x + 11y \leq 125 \\ & 10x - 8y \leq 55 \\ & x,\ y \geq 0, \text{ integer} \end{aligned}$$

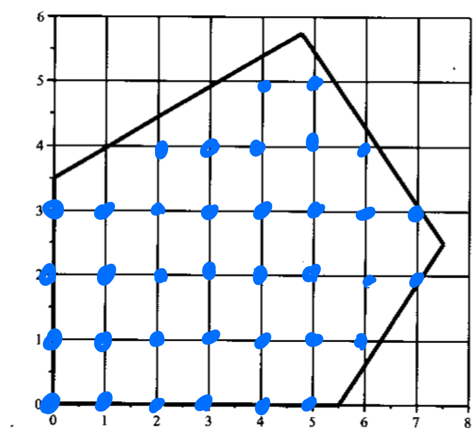


FIGURE 13.1 Feasible region for integer program (13.3).

Integer Program B

$$\begin{aligned} \text{maximize} \quad & 8x + 7y \\ \text{subject to} \quad & -x + 2y \leq 6 \\ & x + y \leq 10 \\ & x - y \leq 5 \\ & x \leq 7 \\ & y \leq 5 \\ & x,\ y \geq 0, \text{ integer} \end{aligned}$$

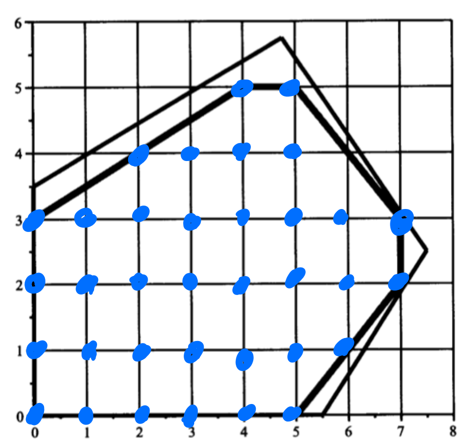


FIGURE 13.2 Feasible region for integer program (13.4).

(a) On the diagrams, identify all feasible solutions to both IPs.

Grid of blue points

(b) Are the integer feasible regions for IP A and IP B different or the same?

A and B are the same feasible region of the IP.

(c) Are the feasible regions of the LP relaxations of IP A and IP B different or the same?

LP Relaxation of A is larger than LP Relaxation of B.

(d) What does this mean about problems A and B?

A and B have same IP feasible region but different LP relaxations

(e) Based on these graphs, will the optimal solution of an IP always occur at a corner point?

No look at graph of A

(f) Which of these formulations is easier to solve? Why?

B is easier to solve because corner points are integer.

Lesson 13: IP B is the convex Hull formulation

IP LP Relaxation

Objective value z_{IP} z_{LP}

max $z_{LP} \ge z_{IP}$

min $z_{LP} \le z_{IP}$

3.1 Better Formulation ⇒ Better Bound

Now let's consider the relationship between an IP and its LP relaxation.

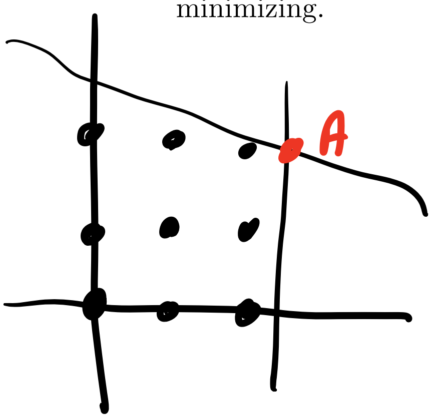
In general:

- If we are solving a **maximization** IP, the solution of its LP relaxation provides a $\boxed{\text{upper}}$ bound on the solution of the IP problem.
- If we are solving a **minimization** IP, the solution of its LP relaxation provides a $\boxed{\text{lower}}$ bound on the solution of the IP problem.

The **tighter** a formulation, the $\boxed{\text{better}}$ bound you obtain via the LP relaxation.

> This idea is key for solving IPs!

Problem 3. Sketch a problem which proves if we're maximizing, $z_{LP} \geq z_{IP}$ and vice versa if minimizing.



A is optimal to LP. 2 options

1) A is integer so $z_{LP} = z_{IP}$

2) A is not integer. No integer point in grid has higher objective value than A

$z_{LP} > z_{IP}$

Combining this $z_{LP} \ge z_{IP}$

Looking ahead to L13

Often the decision of how to formulate an IP comes down to a tradeoff between the formulation quality and number of constraints.

- More constraints can lead to a better (tighter) formulation, but:

  Solving LP relaxation is harder

- Fewer constraints lead to a weaker formulation but:

  Solving LP relaxation is easier