Lesson 11: Scheduling Problems

1 Overview

Scheduling models are a type of optimization problem that determine how jobs/people/etc should be assigned given scheduling requirements while optimizing a given objective.

Just like in other models that we've seen, in many cases we also need to include **logical constraints** to account for complicating factors in the models.

2 Problem Description

The USS George H.W. Bush (CVN-77) is the tenth and final Nimitz-class supercarrier of the United States Navy. You've been tasked with scheduling the following test flights on the USS Bush:

Flights	Total Flight Time (hours)	Flight Importance
1	2	4
2	5	3
3	1	2
4	4	5 ← most important
5	3	1 ← least important
6	7	1.5

max

min
$1/4$
$1/3$
$1/2$
$1/5$
1
$1/1.5$

For this problem, you can to assume that each flight must take off and land before the next flight can leave. In other words, the test flights cannot commence simultaneously. Your ship captain has tasked you with scheduling the in order in which these test flights should commence order to minimizing the total weighted (by importance) cumulative flight completion time.

3 Concrete Model

To model this problem, it is convenient to define a set T which is the time horizon. To solve this problem, we need to plan for $\boxed{22}$ hours. Our variables can then be like the typical assignment variable. Specifically, we'll define:

$$x_{f,t} = \begin{cases} 1 & \text{if flight } f \text{ is scheduled to begin at time } t \\ 0 & \text{otherwise} \end{cases}$$

$f \in \{1,2,3,\ldots,6\}$

$t \in \{1,2,\ldots,22\}$

Objective Function: We want to minimize the weighted flight completion time. Suppose $x_{2,1} = 1$. What is the completion time of flight 2?

$x_{21}=1 \Rightarrow$ flight 2 starts at time 1, takes 5 hours, done at $1+5=6$

Likewise, suppose $x_{4,6} = 1$. What is the completion time of flight 4?

$x_{46}=1 \Rightarrow$ flight 4 starts at time 6, takes 4 hours, done at $6+4=10$

So what is the completion time of flight 4 if it can start at any time t?

- Flight 4 starts at time $t \Rightarrow$ done at $t+4$
- only starts at one time $\Rightarrow$ only 1 $x_{4t}=1$ all others are 0

Completion time: $(4+1)x_{41} + (4+2)x_{42} + \cdots + (4+22)x_{4\,22} = \sum_{t=1}^{22}(4+t)\,x_{4t}$

If we weight this by importance, we obtain the following objective function:

Importance of flight 4: 5 $\Rightarrow \frac{1}{5}\sum_{t=1}^{22}(4+t)\,x_{4t}$ ← prioritizes flight 4 in a min

$$\Rightarrow \min \ \frac{1}{4}\sum_{t=1}^{22}(2+t)\,x_{1t} + \frac{1}{3}\sum_{t=1}^{22}(5+t)\,x_{2t} + \cdots + \frac{1}{15}\sum_{t=1}^{22}(7+t)\,x_{6t}$$

Constraints

- Each flight must be completed.
  - Write a constraint that ensures flight 1 is completed in some time slot.

$x_{11} + x_{12} + x_{13} + \cdots + x_{1\,22} = 1$

↗ flight 1 starts at exactly 1 time

  - Likewise, write a constraint which ensures that flight 6 is scheduled in some time slot.

$x_{61} + x_{62} + \cdots + x_{6\,22} = 1$

↗ Flight 6 starts in one slot

- No two flights can occupy the carrier at the same time.
  - Write a constraint which ensures that no two flights can be scheduled at time 1.

$x_{11} + x_{21} + x_{31} + x_{41} + x_{51} + x_{61} = 1$

Ensures one flight starts at time 1

  - Using the same style, write a constraint that ensures no two flights can be scheduled at time 2.

$x_{12} + x_{22} + x_{32} + x_{42} + x_{52} + x_{62} = 1$

one flight starts at time 2

For ex: If $x_{11} = 1$ then all flights at time 2 are 0

  - Why does the constraint above not work for time 2?

Doesn't account for what happens at time 1.

  - Correct the constraint for time 2. We would need a similar constraint like this for each other time slot.

One idea: Change to $\leq 1 \Rightarrow$ Needs tons of logical constraints.

Better idea: Change constraint to mean carrier is occupied at time 2

$x_{11} + x_{12} + x_{21} + x_{22} + x_{32} + x_{41} + x_{42} + x_{51} + x_{52} + x_{61} + x_{62} = 1$

LHS: every possible flight / time combination that occupies time 2

- Which constraint is remaining?

$x_{ft} \in \{0, 1\}$ for $f \in \{1, 2, 3, 4, 5, 6\}$, $t \in \{1, 2, \ldots, 22\}$

4 Parameterized Model

Before we parameterize the model, let's clearly define the sets, parameters, and variables and explain what they do.

Sets

F: set of flights $F = \{1, 2, \ldots, 16\}$

T: set of times $T = \{1, 2, \ldots, 223\}$

F_K: set of all flight times that occupy time k $\forall k \in T$

Parameters

p_f: time of flight f $\forall f \in F$

I_f: importance of flight f $\forall f \in F$

Variables

$x_{ft} = 1$ if flight f starts at time t

$\forall f \in F \quad t \in T$

Objective Function

- What is the completion time of each flight?

for flight f: $\sum_{t \in T} (p_f + t) x_{ft}$

- What is the objective function?

Completion time of all flights: $\sum_{f \in F} \sum_{t \in T} (p_f + t) x_{ft}$

$$\min \sum_{f \in F} \sum_{t \in T} \frac{1}{I_f} (p_f + t) x_{ft}$$

Constraints

a. To ensure that every flight must be scheduled on the Bush, write a constraint that every flight must be scheduled to begin in some time slot:

Each flight is scheduled to start at some time

$$\sum_{t \in T} x_{ft} = 1 \quad \forall \; f \in F$$

F_k: Set of flights and times occupying time K $\forall K \in T$

b. Write a constraint to ensure that no two flights simultaneously occupy the carrier in any time slot. There are several ways to do so, but we can define a set F_k, for each $k \in \boldsymbol{T}$, that represents all combinations of flights f and time slots t that would occupy the carrier at time k if flight f is scheduled at time t. Formally:

$$F_k = \{(f,t) : f \in F \text{ and } t \in \{\max\{0, k - p_f + 1\}, \ldots, \min\{k, T - p_f\}\}\} \qquad \forall k \in \boldsymbol{T}.$$

What are the elements of F_k for $k = 3$?

$$F_3 = \left\{ \begin{array}{l} (1,3), (2,3), (3,3), (4,3) (5,3) (6,3), \\ (1,2) (2,2), (4,2), (5,2), (6,2), \\ (2,1), (4,1), (5,1), (6,1) \end{array} \right\}$$

Using the set F_k above, write a constraint to ensure that every time slot must be assigned to one flight:

$$\sum_{(f,t) \in F_K} x_{ft} = 1 \qquad \forall \; K \in T$$

All time periods are occupied

c. Write the final binary constraints.

$$x_{ft} \in \{0,1\} \quad \forall \; \begin{array}{l} f \in F \\ t \in T \end{array}$$

5 Multiple Carriers

Now, it turns out that we can use any of the 11 carriers in the fleet to complete our flights. We are now going to convert the parameterized version of this model for the single carrier case to a parameterized version of the model with multiple carriers.

New Parameters and Sets:

- Suppose now that there exists a set of m carriers. $M = \{1, 2, 3, \ldots, 11\}$

New Variables

- Variables $x_{ftc} \in \{0, 1\}$ equal 1 if flight $f \in F$ is scheduled at time $t \in T$ on carrier $c \in \{1, \ldots, m\}$.

Notes:

In this model, we want to allow idleness (especially at the end of the time horizon) on the carriers. This is necessary because when multiple carriers are used, we typically do not use each carrier over the entire time horizon, nor do we know how long each carrier will be utilized before solving the scheduling problem.

How would you revise your constraints from the single carrier example?

1) Old: $\sum_{t \in T} x_{ft} = 1 \quad \forall f \in F$

New $\sum_{c \in M} \sum_{t \in T} x_{ftc} = 1 \quad \forall f \in F$

2) Old: $\sum_{f,t \in F_k} x_{ft} = 1 \quad \forall k \in T$

$\sum_{f,t \in F_k} x_{ftc} \leq 1 \quad \forall k \in T \quad \forall c \in M$

The new objective weighted completion time objective becomes:

Old: $\min \sum_{f \in F} \sum_{t \in T} \frac{1}{I_f} (t + p_f) x_{ft}$

New: $\min \sum_{f \in F} \sum_{t \in T} \sum_{c \in M} \frac{1}{I_f} (t + p_f) x_{ftc}$

d_f : due time of flight f

6 Alternative Objectives:

Minimize weighted tardiness

For *weighted tardiness*, the completion time of flight f is $(t + p_f)$ if $x_{ftc} = 1$ for some carrier c. In that case, the tardiness of flight f would be given by $\max\{0, t + p_f - d_f\}$. Before solving the model, we can simply compute tardiness parameters $\tau_{ft} = \max\{0, t + p_f - d_f\}$ for each $f \in F$ and $t \in \{0, \ldots, T - p_f\}$. The objective would then be:

Flight completes at $t+p_f$

$t+p_f \leq d_f \to$ Not tardy tardiness $=0$

$t+p_f > d_f \to$ tardy tardiness $(t+p_f)-d_f$

τ_{ft} : how tardy flight f is if it starts at time t

$$\min \sum_{f \in F} \sum_{t \in T} \sum_{c \in M} \tau_{ft} x_{ftc} \cdot \frac{1}{I_f}$$

Minimize the number of late flights

A similar approach also handles the case in which we *minimize the number of late flights*. Define binary parameters $\ell_{ft} = 1$ if $(t + p_f) > d_f$ (and thus scheduling flight f at time t would cause it to be late), and $\ell_{ft} = 0$ otherwise, for all $f \in F$ and $t \in \{0, \ldots, T - p_f\}$. To minimize the number of late flights, we would use the objective function:

$t+p_f \leq d_f \to$ Not late lateness $=0$

$t+p_f > d_f \to$ late lateness $=1$

$\ell_{ft}=1$ if flight f is late if it starts at time t.

$$\min \sum_{f \in F} \sum_{t \in T} \sum_{c \in M} \ell_{ft} x_{ftc}$$

Minimize makespan

Minimizing makespan requires a slightly different approach. Define a variable, z, that represents the maximum completion time of any flight. Because the completion time of flight f is given by $\sum_{t=0}^{T-p_f} \sum_{c=1}^{m} (t + p_f) x_{ftc}$, the following set of constraints guarantees that z is *at least* as large as the makespan:

$$z \geq \sum_{t=0}^{T-p_f} \sum_{c=1}^{m} (t + p_f) x_{jtc} \qquad \forall f \in F. \tag{1}$$

Minimizing the makespan requires the following simple objective:

$$\text{Min } z. \tag{2}$$

Because z is being minimized, an optimal solution will exist in which z equals the completion time of the latest flight to finish on a carrier. This is very similar idea to the p-center facility location.

Makespan $\to$ when last flight finishes