SA405, Fall 2022

Exam 1
30 September 2022

Name: ____________________________

- **One sided** 8.5 by 11 inch formula/note sheet is allowed.
- Show work clearly and neatly.
- Define all notation used.
- Please read each question carefully. If you are not sure what a question is asking, ask for clarification.
- If you start over on a problem, please CLEARLY indicate what your final answer is, along with its accompanying work.

Grade Table (for teacher use only)

Question	Points	Score
1	32	
2	35	
3	33	
Total:	100	

Please write the following statement and sign it.

The Naval Service I am a part of is bound by honor and integrity. I will not compromise our values by giving or receiving unauthorized help on this exam.

1. Gandalf has decided to transport the 19 rings of power from various parts of Middle-Earth to the Southlands in order to combat evil Sauron's power. Five of the rings originate in Numenor, ten rings originate in Lothloren, and the remaining four rings originate in Rivendell. To transport the rings from these cities to the Southlands, they can be transported through the following cities/towns: Mithlond, the Shire, and Helm's Deep. To send rings through various cities is quite risky. The following table depicts the level of risk for transporting a single ring from one city to another:

	To			
From	(M) Mithlond	(T) The Shire	(H) Helm's Deep	(S) The Southlands
(N) Numenor	5	3	4	–
(L) Lothloren	7	2	5	–
(R) Rivendell	2	6	7	–
(M) Mithlond	–	–	–	17
(T) The Shire	–	–	–	12
(H) Helm's Deep	–	–	–	18

Assuming that you cannot transport the rings directly from their origination points, Gandalf wants to minimize the total risk of moving the 19 rings from the three cities all to the Southlands. All 19 rings must be transported through Mithlond, The Shire, or Helm's Deep.

We define the following variables:

Let $x_{i,j}$ be the number of rings shipped from city i to city j (that is $x_{N,M}$, $x_{N,T}$,. . ., $x_{H,S}$)

(a) (8 points) In the *abbreviated* concrete form, write the objective function set forth from Gandalf.

Solution:

$$\min\ 5x_{N,M} + 3x_{N,T} + \cdots + 18x_{H,S}$$

(b) (6 points) In the concrete from, write the flow-balance constraint for the island city of Numenor.

Solution:

$$5 = x_{N,M} + x_{N,T} + x_{N,H}$$

(c) (6 points) In the concrete from, write the flow-balance constraint for the castle of Helm's Deep.

Solution:

$$x_{N,H} + x_{L,H} + x_{R,H} = x_{H,S}$$

(d) (6 points) In the concrete from, write the flow-balance constraint for The Southlands.

Solution:

$$x_{M,S} + x_{T,S} + x_{H,S} = 19$$

(e) (6 points) Finally, write constraint(s) to ensure that no more than 9 rings can travel through each of the three relay towns (Mithlond, The Shire, and Helm's Deep).

Solution: There's two solutions, either restrict flow in to the cities or flow out of the cities. I'll do the flow in below:

$$\begin{aligned} x_{N,M} + x_{L,M} + x_{R,M} &\le 9 \\ x_{N,T} + x_{L,T} + x_{R,T} &\le 9 \\ x_{N,H} + x_{L,H} + x_{R,H} &\le 9 \end{aligned}$$

2. Last Spring, Professor Alameda decided to leave USNA to open up his own muffin bakery. Against all odds, business has been great! He is considering opening up 2 new bakeries that would serve 3 stores in the nearby area. The following table shows the cost of opening each bakery (in thousands), the amount of muffins truckloads each store demands, and the transportation cost (in thousands) per truckload of muffins from each bakery to each store. Assume each bakery can supply 44 truckloads of muffins. Professor Alameda needs your help to decide which bakeries to open in order to minimize his costs!

	Bakery Cost	Transport Costs Store 1	Store 2	Store 3
Bakery 1	200	6	5	9
Bakery 2	400	4	3	5
Demand	—	11	18	15

(a) (10 points) Ignoring the bakery opening cost, write out a concrete model that minimizes the total transportation cost (make sure to define all variables for full credit).

Solution: Decision Variables: Let $x_{1,1}$ be the flow from bakery 1 to store 1, $x_{1,2}$ be the flow from bakery 1 to store 2, etc.

Objective

$$\min\ 6x_{1,1} + 5x_{1,2} + \cdots + 5x_{2,3}$$

Constraints

$$\begin{aligned}
x_{1,1} + x_{1,2} + x_{1,3} &\leq 44 && \text{(supply 1)}\\
x_{2,1} + x_{2,2} + x_{2,3} &\leq 44 && \text{(supply 2)}\\
x_{1,1} + x_{2,1} &= 11 && \text{(demand 1)}\\
x_{1,2} + x_{2,2} &= 18 && \text{(demand 2)}\\
x_{1,3} + x_{2,3} &= 15 && \text{(demand 3)}\\
x_{1,1}, \ldots, x_{2,3} &\geq 0 && \text{(non-negativity)}
\end{aligned}$$

(b) (10 points) Modify your formulation from part (a) to incorporate the bakery opening costs and any constraints associated with opening the bakeries. Be sure to clearly define any new variable(s) used.

Solution: Define two new binary variables: $z_1 = 1$ if bakery 1 is open and $z_2 = 1$ if bakery 2 is open.

Modify objective function to be:

$$\min\ 6x_{1,1} + 5x_{1,2} + \cdots + 5x_{2,3} + 200z_1 + 400z_2$$

Add forcing constraints. One way to do this are the following two constraints:

$$\begin{aligned}
x_{1,1} + x_{1,2} + x_{1,3} &\leq 44z_1\\
x_{2,1} + x_{2,2} + x_{2,3} &\leq 44z_2
\end{aligned}$$

Also add binary constraints:

$$z_1, z_2 \in \{0, 1\}$$

(c) (15 points) Parameterize your completed model. Be sure to clearly define all sets, variables, and parameters for full credit.

Solution: Sets

Let B be the set of bakeries
Let S be the set of stores

Parameters

Let $c_{i,j}$ be the cost of shipping from bakery i to store j for all $i \in B$ and $j \in S$
Let s_i be the supply of bakery i for all $i \in B$
Let d_j be the demand of store j for all $j \in S$
Let f_i be the fixed cost of opening store i for all $i \in B$

Variables

Let $x_{i,j}$ be the number of muffins sent from bakery i to store j for all $i \in B$ and $j \in S$
Let $z_i = 1$ if bakery i is open and 0 otherwise for all $i \in B$

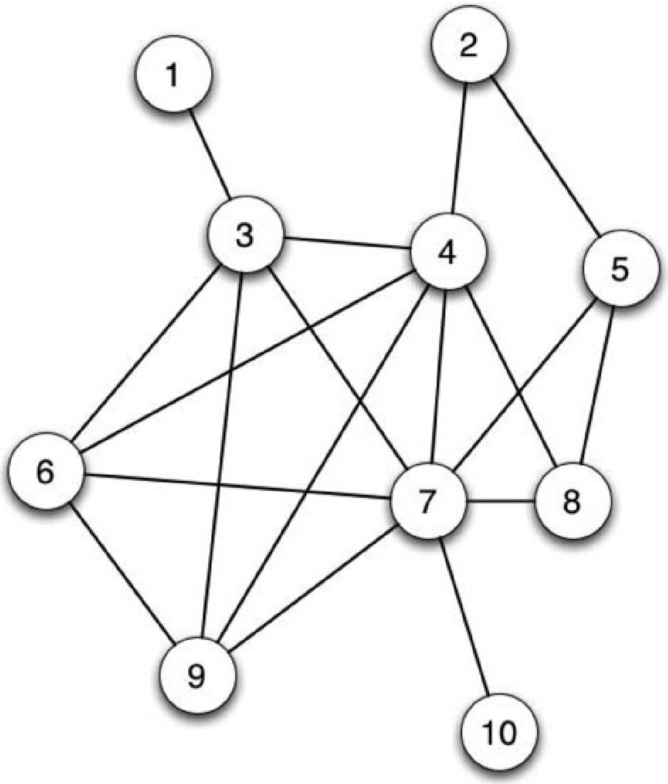
Objective

$$\min \sum_{i \in B} \sum_{j \in S} c_{i,j} x_{i,j} + \sum_{i \in B} f_i z_i$$

Constraints

$$\begin{aligned}
\sum_{j \in S} x_{i,j} &\le s_i z_i && \text{for } i \in B && \text{(supply and forcing)} \\
\sum_{i \in B} x_{i,j} &= d_j && \text{for } j \in S && \text{(demand)} \\
x_{i,j} &\ge 0 && \text{for } i \in B \text{ and } j \in S && \text{(non-negative)} \\
z_i &\in \{0,1\} && \text{for } i \in B && \text{(binary)}
\end{aligned}$$

3. Consider the following network with nodes N and edges E:



A **node covering** is a collection of nodes in the graph such that every edge in E has at least one endpoint in the collection. For example, for edge $(1, 3)$ selecting either node 1 or node 3 covers this edge. The goal is to minimize the number of nodes selected for the covering.

Let

$$x_i = \begin{cases} 1 & \text{if node } i \text{ is selected for the covering} \\ 0 & \text{if node } i \text{ is not selected for the covering} \end{cases}$$

for all $i \in N$.

(a) (5 points) Write a concrete objective function which could be used to select a covering consisting of the fewest number of nodes.

Solution:

$$\min\ x_1 + x_2 + \cdots + x_{10}$$

(b) (6 points) Write two constraints, one that would ensure edge $(2, 5)$ is covered and another which would ensure edge $(6, 7)$ is covered.

Solution:

$$x_2 + x_5 \geq 1$$
$$x_6 + x_7 \geq 1$$

(c) (5 points) Parameterize your constraints from part (b) such that we have one parameterized constraint that covers all edges in the network.

Solution:

$$x_i + x_j \geq 1 \text{ for all } (i, j) \in E$$

(d) Suppose that there is a cost associated with placing a sensor at each node. Two companies offer bids for the sensors. The first company offers a flat rate of \$100 for each sensor but at least 5 sensors much be purchased. The second company offers a rate of \$150 for each sensor with no minimum number requirement. There's a total budger of \$600. You write the following constraints:

$$\begin{array}{lrcl} \text{(A)} & 100x_1 + 100x_2 + \cdots + 100x_{10} & \leq & 600 \\ \text{(B)} & x_1 + x_2 + \cdots + x_{10} & \geq & 5 \\ \text{(C)} & 150x_1 + 150x_2 + \cdots + 150x_{10} & \leq & 600 \end{array}$$

You define a binary variable z. $z = 1$ if sensors are purchased from company 1 and $z = 0$ if sensors are purchased from company 2. Modify these constraints such that:

i. (6 points) Constraints (A) and (B) are enforced if $z = 1$ and relaxed if $z = 0$.

Solution:

$$\begin{array}{lrcl} \text{(A)} & 100x_1 + 100x_2 + \cdots + 100x_{10} & \leq & 600 + M(1 - z) \\ \text{(B)} & x_1 + x_2 + \cdots + x_{10} & \geq & 5 - M(1 - z) \end{array}$$

ii. (4 points) Constraint (C) is relaxed if $z = 1$ and enforced if $z = 0$.

Solution:

$$150x_1 + 150x_2 + \cdots + 150x_{10} \leq 600 + Mz$$

(e) (3 points) Suppose that if node 3 is not selected than node 4 must be selected. Write a constraint to enforce this requirement.

Solution:

$$1 - x_3 \leq x_4$$

(f) (4 points) Likewise, suppose that if node 6 is selected then node 3 must also be selected and node 4 can not be selected. Write constraint(s) to enforce this requirement.

Solution:

$$2x_6 \leq x_3 + (1 - x_4)$$