Lesson 2: Network Models: Transportation & Minimum Cost Network Flow Models

1 A Transportation Problem: Bakeries

A local baked goods company has two bakeries where they bake their goods, which they then ship to three different area stores to sell. Each bakery can produce up to 50 truckloads of baked goods per week, and each bakery can supply any of the stores. The weekly demands (in truckloads) anticipated at each store along with the transportation costs (per truckload) are provided in the tables below. Note that partial truckloads cost just as much as full truckloads. How many truckloads should be sent from each bakery to each store in order to minimize total shipping cost?

	Demand
Store 1	35
Store 2	25
Store 3	40

	Store 1	Store 2	Store 3
Bakery a	\$20	\$45	\$35
Bakery b	\$35	\$35	\$50

Problem 1. Draw a directed graph to represent the problem. Bakeries and stores are represented by *nodes* (or *vertices*). Directed *arcs* (or *edges*) represent flow of goods. Label each node (a, b, 1, 2, 3), and beside each node indicate its corresponding supply or demand.

Every network has nodes and edges

Edges represent flow between 2 nodes

Node is a decision point

1

? node 1

Directed edge

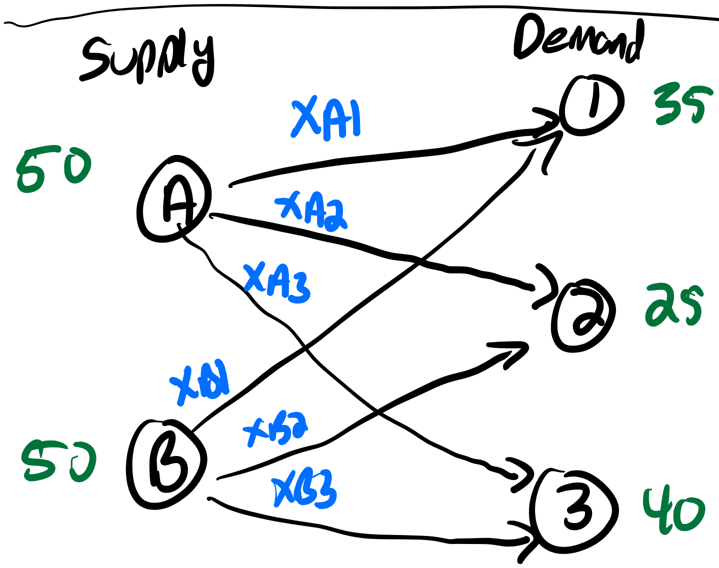
$1 \xrightarrow{x_{12}} 2$

flow from 1 to 2

Undirected edge

$1 — 2$

Flow from 1 to 2 or from 2 to 1



Decision: How many truckloads from each bakery to each store?

Objective: minimize shipping cost

Constraints:

- Every bakery up to 50 truckloads
- Meet demand at each store
- non-negativity and integers

$S = \{A, B\}$

$D = \{1, 2, 3\}$ → x_{ij} [illegible]

$E = \{(A,1), (A,2), \ldots, (B,3)\}$ → x_{ij} for $(i,j) \in E$

1.1 Concrete model

Problem 2. Using the usual format, write a concrete model to find a feasible transportation.

Variables

let x_{A1} be the truckloads from A to 1

⋮

let x_{B3} be the truckloads from B to 3

Objective

$$\min \; 20x_{A1} + 45x_{A2} + \cdots + 50x_{B3}$$

Constraints

$$x_{A1} + x_{A2} + x_{A3} \le 50 \quad \text{(supply A)}$$
$$x_{B1} + x_{B2} + x_{B3} \le 50 \quad \text{(supply B)}$$
$$x_{A1} + x_{B1} = 35 \quad \text{(demand 1)}$$
$$x_{A2} + x_{B2} = 25 \quad \text{(demand 2)}$$
$$x_{A3} + x_{B3} = 40 \quad \text{(demand 3)}$$
$$x_{A1}, x_{A2}, \ldots, x_{B3} \ge 0$$

In basic network flow models all variables are automatically integer

Supply nodes

$\sum_{j \in D} x_{ij}$

Demand node

$\sum_{i \in S} x_{ij}$

1.2 Parameterized model

Flow out of i

flow into j

Problem 3. Convert your model to a parameterized model.

Sets

let S be the supply nodes (Bakeries)

let D be the demand nodes (Stores)

let E be the set of edges

Variables

let x_{ij} be the units from $i \in S$ to $j \in D$

same as x_{ij} for $(i,j) \in E$

Parameters

let c_{ij} be the cost of shipping from $i \in S$ to $j \in D$

let s_i be the supply of node $i \in S$

let d_j be the demand of node $j \in D$

Objective

$$\min \sum_{i \in S} \sum_{j \in D} c_{ij} x_{ij}$$

Total Shipping cost

Constraints

$\sum_{i \in S} x_{ij} = d_j$ for $j \in D$ (flow into node j = demand)

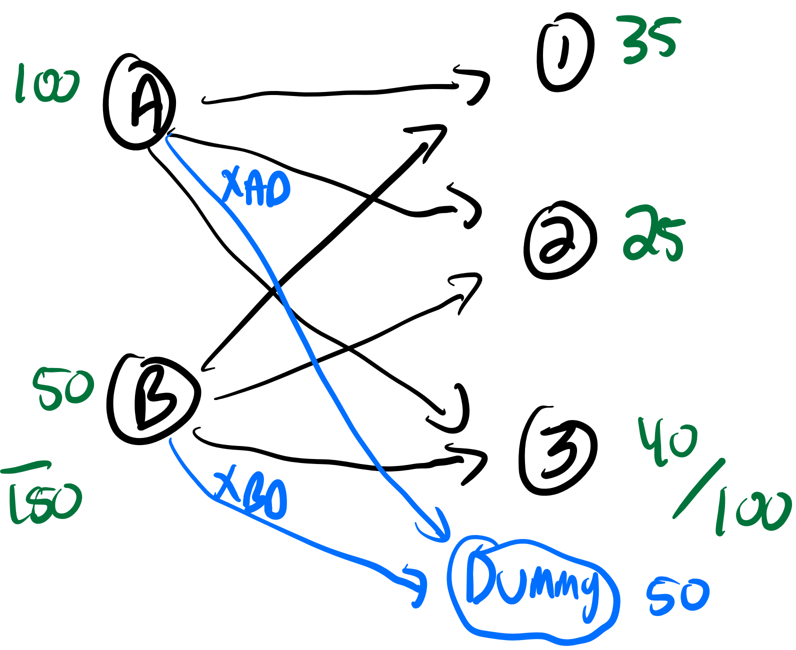
$\sum_{j \in D} x_{ij} \leq s_i$ for $i \in S$ (flow out of node $i \leq$ supply)

$x_{ij} \geq 0$ for $i \in S, j \in D$

1.3 Assumptions and Special Scenarios

Notice that this problem has a very important assumption: the total supply is equal to the total demand. If this isn't true, the basic model we formulated above does not work. This assumption is not true in general networks, but we can make it true.

1. What if bakery ~~B~~ A could produce 100 units instead of just 50? How would that modify our network diagram/formulation?



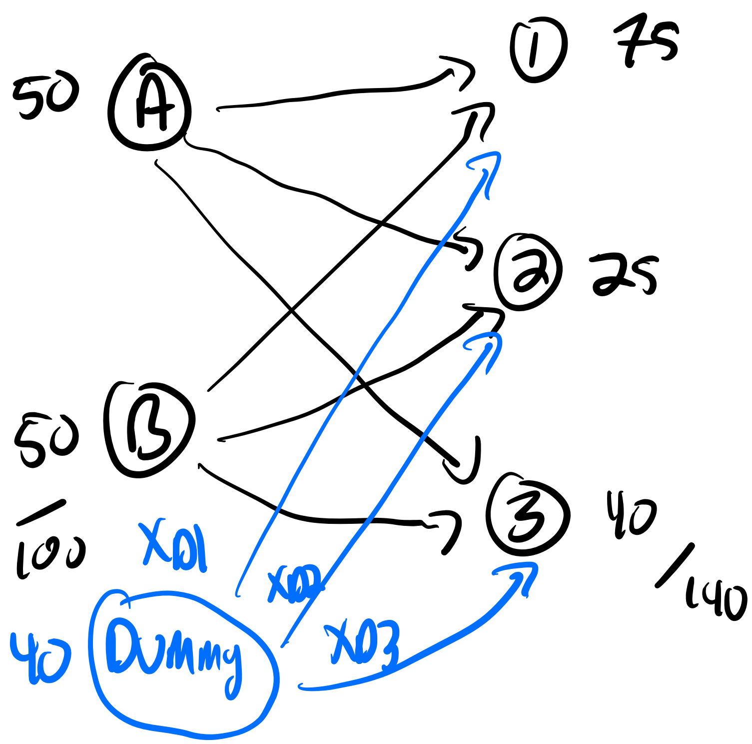
① If I solve the Problem on previous page for this network do I get an optimal solution?

Yes, meet demand, ≤ supply

② If I wanted to keep track of leftover cakes without changing the model what can I do?

Add dummy demand node

2. What if store 1 needed 75 units instead of 35? How would that modify our network diagram/formulation?



① Same question?

No, this problem is infeasible. Impossible to meet demand.

② Fix this?

Dummy supply.

Cost of each edge is problem specific.

2 Minimum Cost Network Flow Models

Consider the bakery again, only this time there are two warehouses that baked goods must be delivered to before they are taken to their final destinations. The warehouses ($w1$ and $w2$) have no supply nor demand. These are the transportation costs:

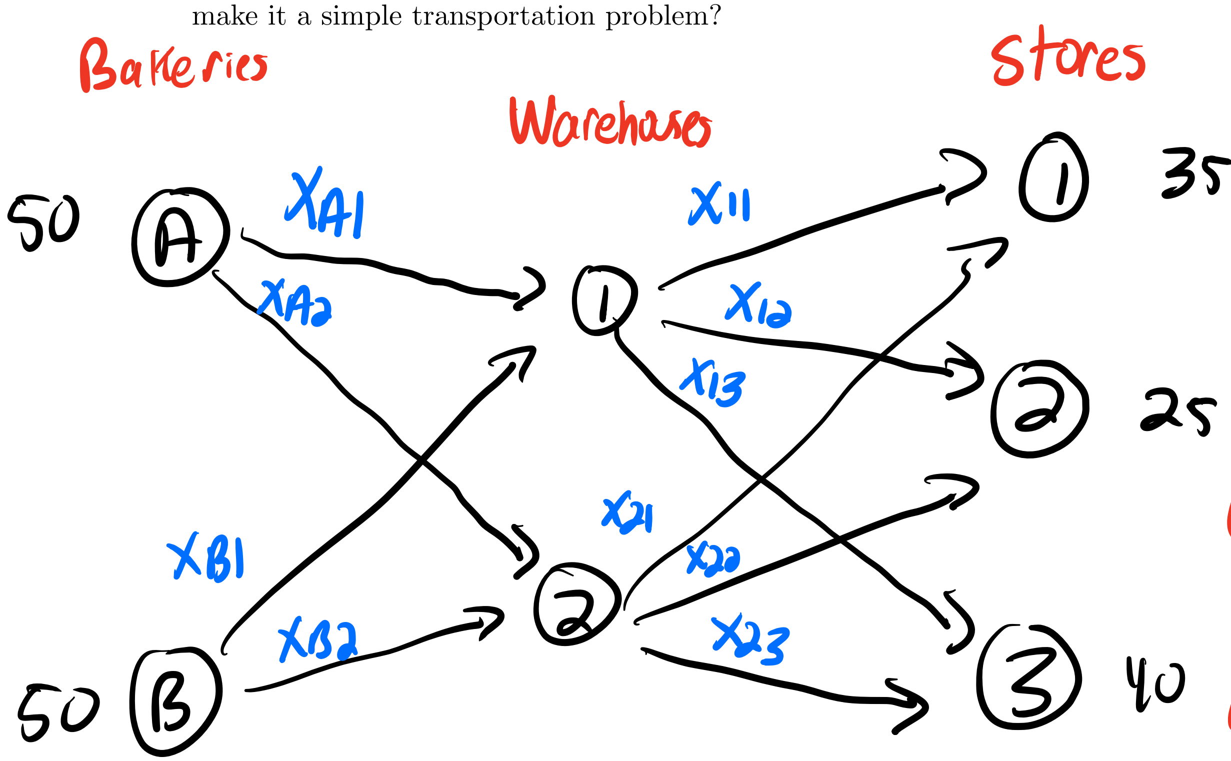
	Warehouse 1	Warehouse 2
Bakery 1	\$10	\$15
Bakery 2	\$20	\$25

	Store 1	Store 2	Store 3
Warehouse 1	\$20	\$45	\$35
Warehouse 2	\$35	\$35	\$50

Also, recall that the store demands are $d_{s1} = 35$, $d_{s2} = 25$, and $d_{s3} = 40$, so that *total supply* $=$ *total demand* $= 100$.

- The warehouses are called **transshipment nodes**.
- This is called a **transshipment problem** or a **minimum cost network flow problem**.

Problem 4. Draw the new network diagram. What's the major difference this time that doesn't make it a simple transportation problem?



Objective: min Shipping cost

Constraints
- Supply
- demand
- non-negativity

node 1:

$$X_{A1} + X_{B1} = X_{11} + X_{12} + X_{13}$$

2.1 Concrete model

Problem 5. Using the usual format, write a concrete model to find a feasible transportation.

Variables

let x_{A1} be the flow from A to W1

⋮

let x_{23} be the flow from W2 to S3

One variable for every edge

Obj function

min $10x_{A1} + \dots + 50x_{23}$

Constraints

$x_{A1} + x_{A2} \leq 50$ (supply A)

$x_{B1} + x_{B2} \leq 50$ (supply B)

$x_{11} + x_{21} = 35$ (demand 1)

$x_{12} + x_{22} = 25$ (demand 2)

$x_{13} + x_{23} = 40$ (demand 3)

$x_{A1}, x_{A2}, \dots x_{23} \geq 0$

$x_{A1} + x_{B1} = x_{11} + x_{12} + x_{13}$ (flow balance W1)

$x_{A2} + x_{B2} = x_{21} + x_{22} + x_{23}$ (flow balance W2)

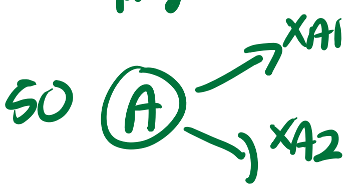
2.2 Balance of Flow

Before writing the parameterized model, it will be helpful to rearrange the constraints so that they all have the same form. First, verify that you have one constraint per node, and that all of your constraints have one of the following forms:

$$\textit{flow out} \leq \textit{supply} \quad \text{or} \quad \textit{demand} \leq \textit{flow in}.$$

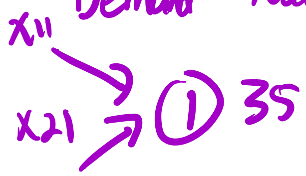
We can write all of our constraints using the same form by adding these two forms and we obtain the general flow balance constraint:

Supply node



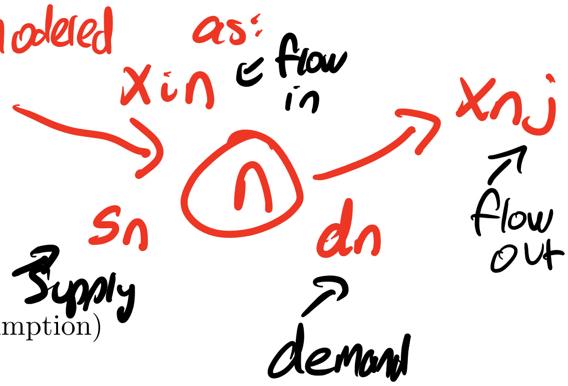
$X_{A1} + X_{A2} \leq 50$

Demand Node



$x_{11} + x_{21} = 35$

Every node can be modeled as:



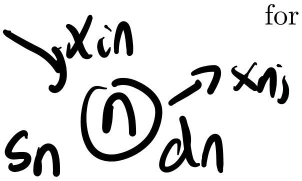
These are called **flow balance** constraints. (Recall this is an assumption)

- Why does it make sense for *supply* at a node to be combinded with *flow in* and *demand* at a node to be combined with flow out?

- Supply is like flow in because it is taken into the node and sent elsewhere

- demand is like flow out because satisfying demand is like sending flow elsewhere.

- Some like to combine supply and demand into a single parameter, $b_i = supply_i - demand_i$, for each node. Write a general form for the **balance of flow constraint** using b_i:



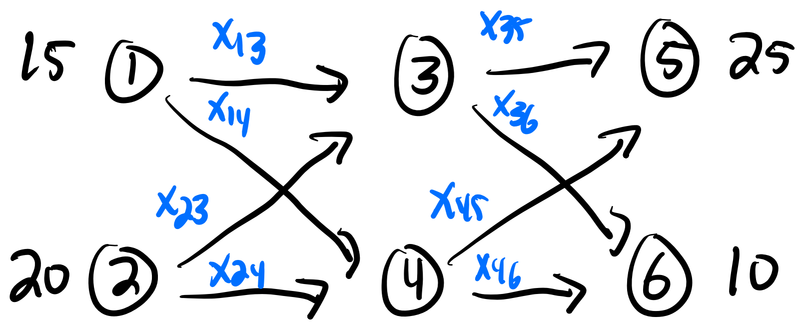
Flow in + supply = Flow out + demand

For ALL Nodes in network

$$\sum_{(i,n)\in E} x_{in} + s_n = \sum_{(n,j)\in E} x_{nj} + d_n \quad \text{for all } n \in N$$

Book: $\sum x_{in} - \sum x_{nj} = \boxed{d_n - s_n}$ let $b_n = d_n - s_n$

$\hookrightarrow \sum x_{in} + \sum x_{nj} = b_n$



Supply: $S_n = [15, 20, 0, 0, 0, 0]$

Flow in: $X_{in} = [0, 0, X_{13}+X_{23}, X_{14}+X_{24}, X_{35}+X_{45}, X_{36}+X_{46}]$

Demand: $d_n = [0, 0, 0, 0, 25, 10]$

Flow out: $X_{nj} = [X_{13}+X_{14}, X_{23}+X_{24}, X_{35}+X_{36}, X_{45}+X_{46}, 0, 0]$

Supply node: node 1: $0+15 = X_{13}+X_{14} + 0$ ✓

Transshipment node: node 4: $X_{14}+X_{24}+0 = X_{45}+X_{46}+0$ ✓

Demand: node 6: $X_{36}+X_{46}+0 = 0+10$ ✓

2.3 Parameterized model

Problem 6. Convert your model to a parameterized model.

Sets

let N be the set of nodes $N = \{A, B, W_1, W_2, \ldots\}$

let E be the set of edges $E = \{(A,1), (A,2), \ldots\}$

Variables

let x_{ij} be the flow along edge (i,j) for $(i,j) \in E$

Parameters

let c_{ij} be the cost of edge (i,j) $\forall (i,j) \in E$

let s_n be the supply of node n $\forall\ n \in N$

let d_n be the demand of node n $\forall\ n \in N$

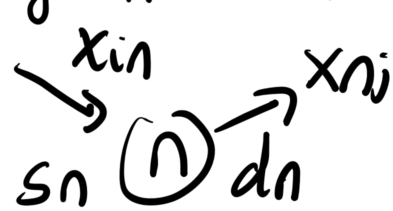
Every node in graph

Obj

$$\min \sum_{(i,j) \in E} c_{ij} x_{ij}$$

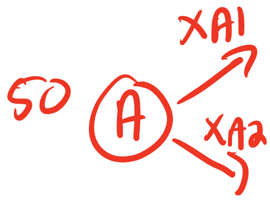
Constraints

For any node $n \in N$



$$s_n + \sum_{(i,n) \in E} x_{in} = d_n + \sum_{(n,j) \in E} x_{nj} \quad \text{for } n \in N$$

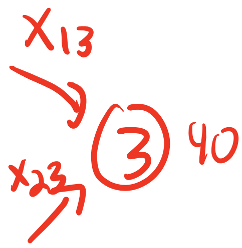
$$x_{ij} \geq 0 \quad \forall (i,j) \in E$$



$n = A$

$s_A + \Sigma x_{iA} = d_A + \Sigma x_{Aj}$

$50 + 0 = 0 + x_{A1} + x_{A2}$



$n = 3$

$0 + x_{13} + x_{23} = 40 + 0$