Branch-and-bound Graphical Example (Accompanying Lesson 16)

1 Branch-and-bound Example

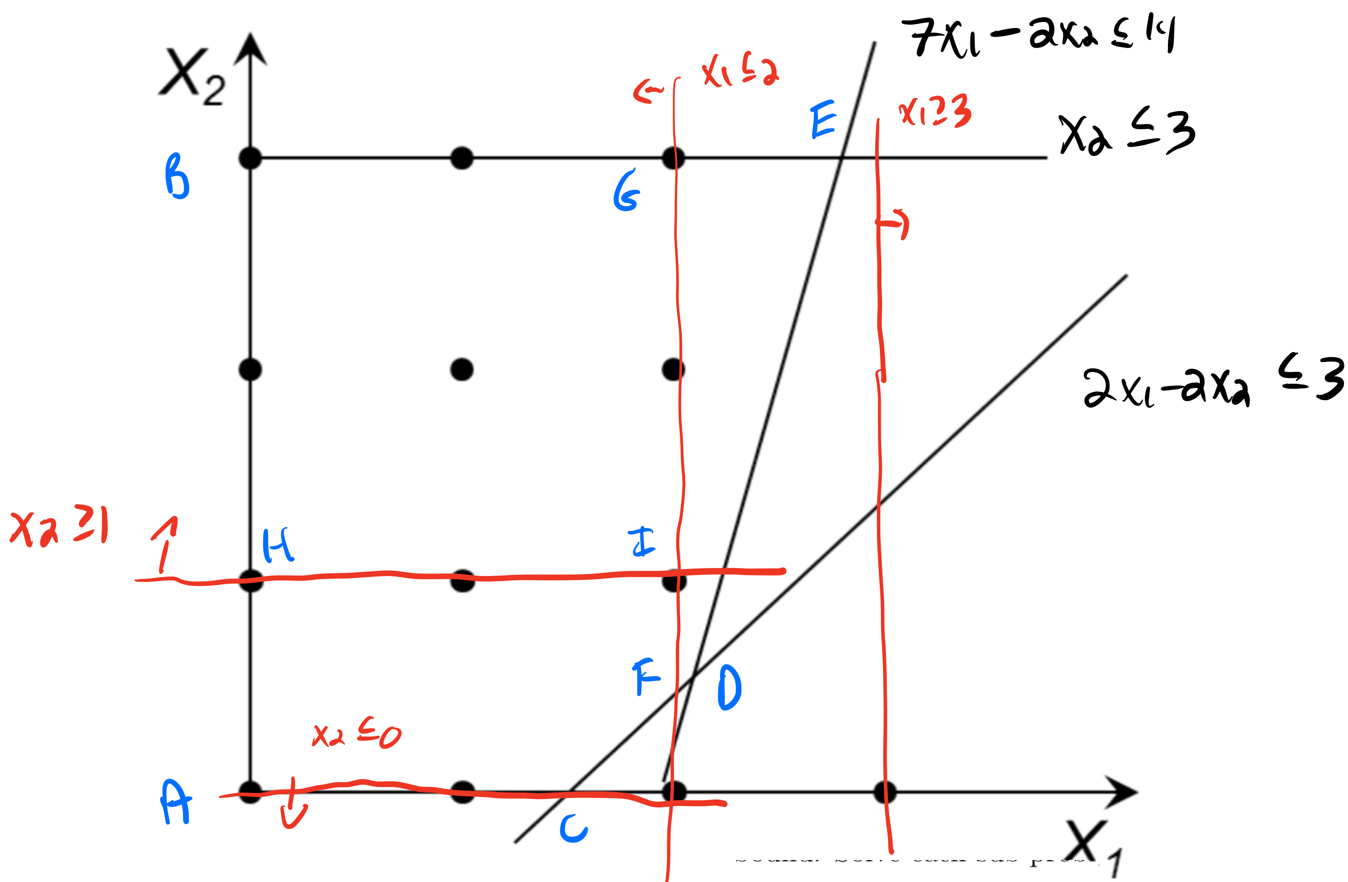
Solve the following IP using branch-and-bound. Solve each sub problem graphically.

$$
\begin{aligned}
\text{(P1)} \quad z^*_{IP} = \max\ & 4x_1 - x_2 \\
\text{s.t.}\quad & 7x_1 - 2x_2 \leq 14 \\
& 2x_1 - 2x_2 \leq 3 \\
& x_2 \leq 3 \\
& x_1, x_2 \in \mathbb{Z}^{\geq 0}
\end{aligned}
$$

- Solve each sub-problem graphically
- Branching Rules
  - Always select the active node with the largest upperbound for branching.
  - Branch on x_1 if it is fractional. Otherwise branch on x_2.
- Book-keeping
  - Keep track of the:
    - incumbent solution $\underline{x}$,
    - global lower bound $\underline{z}$, and
    - list of active nodes.
  - Draw the branch-and-bound tree:
    - Record the local upper bound (z) and relaxed optimal solution (x) for each subproblem.
    - Label each edge with the constraint that is added to form the child subproblem.
    - X-out fathomed nodes. Circle incumbent solution nodes.
  - Use the provided diagram to illustrate the (relaxed) feasible region of each subproblem.

incumbent solution $\underline{x}$	**global lower bound $\underline{z}$**	**active nodes**

Feasible Region



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$$

A: $(0,0)$, $z=0$

B: $(0,3)$, $z=-3$

C: $(\frac{3}{2}, 0)$, $z=6$

D: $(\frac{11}{5}, \frac{7}{10})$, $z=8.1$

E: $(\frac{20}{7}, 3)$, $z=8.4$

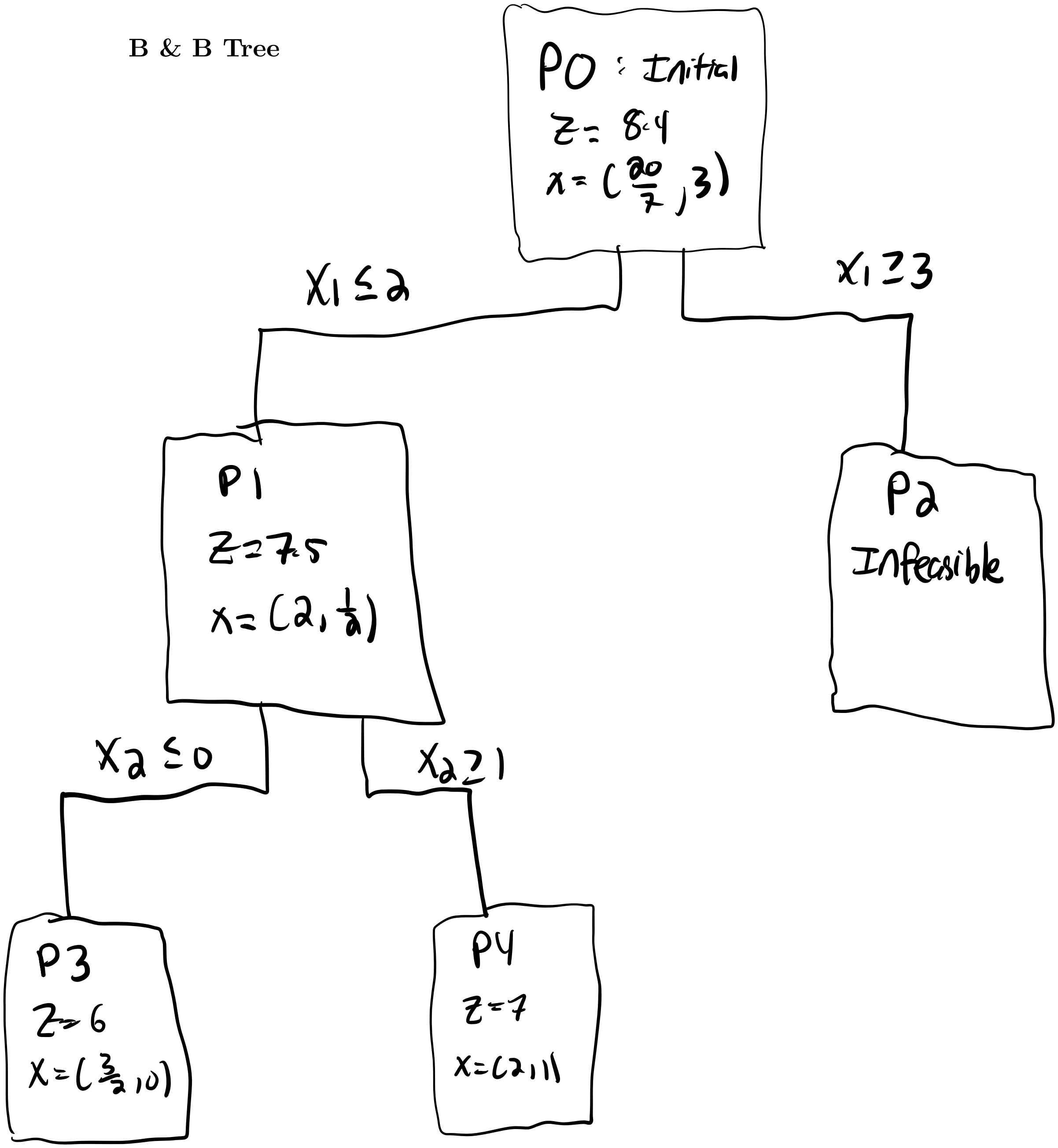
F: $(2, \frac{1}{2})$, $z=7.5$

G: $(2,3)$, $z=5$

H: $(0,1)$, $z=-1$

I: $(2,1)$, $z=7$

B & B Tree


P0 : Initial
Z = 8.4
x = (20/7, 3)
x1 ≤ 2
x1 ≥ 3
P1
Z = 7.5
x = (2, 1/2)
P2
Infeasible
x2 ≤ 0
x2 ≥ 1
P3
Z = 6
x = (3/2, 0)
P4
Z = 7
x = (2, 1)

Steps taken

Step 1: Solve the LP relaxation of original problem

Got $Z = 8.4$ $x = (\frac{20}{7}, 3)$ Not optimal for IP

$$Z^*_{IP} \le 8.4 \rightarrow Z^*_{IP} \le 8$$

x_1 is fractional. We branch on x_1 and create 2 problems P_1 and P_2.

$P1$: Original problem and $x_1 \le 2$

$P2$: Original problem and $x_1 \ge 3$

Step 2: Solve LP relaxation of P1 and P2

$P1 \Rightarrow Z = 7.5$ $x = (2, \frac{1}{2})$ $Z^* \le 7$

$P2 \Rightarrow$ Infeasible

Eliminate P2.

Branch P1 on x_2. Create P3 and P4

P3: Original Problem and $x_1 \le 2$, $x_2 \le 0$

P4: Original problem and $x_1 \le 2$, $x_2 \ge 1$

Step 3: solve LP relaxation of P3 and P4

$P3 \rightarrow Z = 6$ $X = (\frac{3}{2}, 0)$

$P4 \rightarrow Z = 7$ $X = (2, 1)$

From P4 we know $Z^* \ge 7$

All that's left is P3. But $6 < 7$, so there's no value in exploring P3. Eliminate P3.

Optimal solution $Z^* = 7$

$x^* = (2, 1)$