Lesson 1: Linear Programming Review, Intro to Integer Programming

1 Goals

- Formulate a concrete linear programming model.
- Introduce an integrality requirement on variables.
- Convert the integer program to parameterized form.

2 Review of SA305 Formulations

The five components of formulating an optimization model are:

1. Sets: Collection of indices for variables, parameters, and constraints
   - $T = \{1, 2, 3, 4, 5, 6, 7\}$

2. Decision Variables: representation of decision to be made
   - let x_1 be the amount of item 1 made

3. Parameters: letter that represents a CONSTANT number
   let c_1 be the cost of item 1
   → Always constant

4. Objective: Goal of problem
   max x_1
   min $\sum_{i=1}^{10} y_i$

5. Constraints: Math representation of real world restrictions
   $x_1 \geq 0$

What are the 3 assumptions/characteristics that make an optimization model a linear program?

Not in 405

1. All variables continuous → variables ≥ 0 and can be fractional

True in 405

2. Constraints and objective are linear → Never multiply variables

3. Proportionality: Add/ subtract /multiply variables and not change their meaning

3 Concrete Model

Chelsea is heading out on a camping trip, and she wants to carry only one pack that has 5.3 ft^3 of volumetric space. To keep from hurting her back, she needs to make sure that the contents of her backpack weighs no more than 12.5 lbs. You can assume the backpack weight is negligible. See the list of items that she is able to bring:

ID	Item	Volume (ft^3)	Usefulness Factor	Weight (lbs.)
1	Rope	2	1	3
2	Matches	0.01	5	0.1
3	Tent	3	7	10
4	Sleeping bag	2	6	4
5	Hammock	0.4	4.5	4
6	Granola bars	0.67	8	2

This problem is referred to as the **knapsack** problem and is a very widely used type of integer program. We'll see why we need IP shortly. We will first formulate it as an LP.

Problem 1. Write a concrete linear program whose solution maximizes the usefulness of the contents of Chelsea's bag given volume and weight requirements.

a) Define decision variables and then describe the objective function and the role of each constraint.

Decision: which items and how many of each one to bring

- let x_1 be amount of rope she brings

Goal: maximize usefulness of items

Constraints:
- weight ≤ 12.5
- volume ≤ 5.3
- non-negativity

b) Write the concrete model.

Decision Variables

let x_1 be the amount of item 1 she brings

⋮

let x_6 be the amount of item 6 she brings

Objective

max $1 \cdot x_1 + 5x_2 + 7x_3 + 6x_4 + 4.5x_5 + 8x_6$

Constraints

$3x_1 + 0.1x_2 + 10x_3 + 4x_4 + 4x_5 + 2x_6 \leq 125$

$2x_1 + 0.01x_2 + 3x_3 + 2x_4 + 0.4x_5 + 0.67x_6 \leq 5.3$

$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

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4 Integrality Restrictions

Suppose we solve this LP and get the following solution:

$$x_{granola} = x_{hammock} = x_{matches} = x_{sleeping} = 1$$

$$x_{tent} = 0.24$$

$$x_{rope} = 0$$

What is the objective function value of this solution? Denote this solution as z^*_{LP}.

$1.8 + 4.5 + 5 + 6 + 0.24(7) = 25.18$

Is this a reasonable solution?

No, can't have 0.24 of a tent

Suppose we change the problem so that x_i is **binary** instead of continuous. A binary variable is only allowed to take the value of 0 or 1. That is, the variables would be changed to:

$x_1 = 1$ if she takes rope and 0 if she doesn't

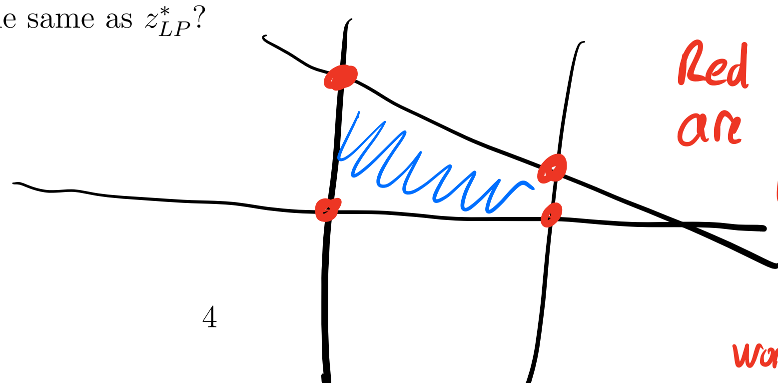
$$x_1 = \begin{cases} 1 & \text{if she takes rope} \\ 0 & \text{otherwise} \end{cases}$$

Our problem has now become a **integer program**. Why?

Variables can only be 0 or 1 so not continuous

Let z^*_{IP} be the optimal objective function value of the integer program. Would we expect z^*_{IP} to be smaller, larger, or the same as z^*_{LP}?

Smaller because no fractional variables



Red points are optimal LP. If not integer get a worse IP solution

5 Convert to Parameterized Models

Problem 2. Assuming integrality restrictions, convert your model to a parameterized model. Clearly define all sets, parameters, and decision variables.

Sets

I = set of items $\quad I = \{1,2,3,4,5,6\}$

Variables

$x_i = 1$ if she takes item i and 0 otherwise for all $i \in I$

Parameters

let u_i be the usefulness of item i for $i \in I$

let w_i be the weight of item i for $i \in I$

let v_i be the volume of item i for $i \in I$

Objective

$$\text{Max} \sum_{i \in I} u_i x_i = u_1 x_1 + u_2 x_2 + \dots + u_6 x_6$$

Constraints

$$\sum_{i \in I} w_i x_i \le 12.5$$

$$\sum_{i \in I} v_i x_i \le 5.3$$

$x_i \in \{0,1\}$ for $i \in I$

$$\begin{array}{c} \\ W \\ V \end{array}\begin{array}{c} \text{Items} \\ \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 & w_6 \\ v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{bmatrix} \end{array}\begin{array}{l} \\ \le 12.5 \\ \le 5.3 \end{array}$$

New set $J = \{W, V\}$

New parameter u_j is upperbound of j for $j \in J$

New parameter a_{ij} is amount of resource j used by item i for $i \in I$, $j \in J$

Constraints become

$$\sum_{i \in I} a_{ij} x_i \le u_j \quad \text{for } j \in J$$