HW 5: Unsymmetric Traveling Salesperson problem

The TSP problem we did in class operated on an undirected graph; thus, an edge like $(1,2)$ can be from node 1 to node 2 or node 2 to node 1. In this homework problem, we will consider the unsymmetric traveling salesperson problem which operates on a directed graph (thus edge $(1,2)$ and $(2,1)$ are distinct).

Problem 1. Cameron is taking a vacation to New York City and wants to drive around to see some of the famous landmarks there. There are 8 total landmarks he wants to see. Since many of the roads in NYC are one way only, the path between two landmarks is not the same distance. The matrix below gives the distances (in miles) between each of the 8 landmarks he wants to visit.

$$D = \begin{bmatrix} 0 & 2 & 3 & 1 & 7 & 5 & 4 & 4 \\ 1 & 0 & 3 & 2 & 4 & 2 & 6 & 3 \\ 2 & 1 & 0 & 4 & 2 & 4 & 7 & 1 \\ 3 & 2 & 1 & 0 & 4 & 3 & 2 & 2 \\ 6 & 8 & 1 & 2 & 0 & 4 & 6 & 5 \\ 3 & 2 & 4 & 2 & 7 & 0 & 3 & 4 \\ 2 & 3 & 1 & 4 & 5 & 6 & 0 & 2 \\ 1 & 4 & 3 & 2 & 7 & 4 & 3 & 0 \end{bmatrix}$$

He wants to visit all of the landmarks while driving as little as possible and he asks you for help.

1. Formulate a concrete model whose solution will give Cameron a tour that visits all 8 landmarks exactly once. *Hint: Do not write all the subtour elimination constraints, you can write one or two then move on. Also, if you're having trouble getting started draw out the network for the problem.*

   Decision Variables

   Let $x_{1,2} = 1$ if edge $(1,2)$ is selected as part of the tour and 0 otherwise

   $\vdots$

   Let $x_{8,7} = 1$ if edge $(8,7)$ is selected as part of the tour and 0 otherwise

   Objective Function

   $$\text{minimize: } 2x_{1,2} + 3x_{1,3} + \cdots + 6x_{8,7}$$

   Constraints

   We have the typical constraints here, but notice that the big difference is that in a directed graph, you can have a cycle on only 2 nodes.

$$
\begin{aligned}
\text{st} \qquad & x_{1,2} + x_{1,3} + \cdots + x_{8,7} &= 8 \qquad & \text{(can only take 8 edges for a tour)} \\
& x_{1,2} + x_{1,3} + \cdots + x_{8,1} &= 2 \qquad & \text{(node 1 is visited once)} \\
& \vdots & \vdots & \\
& x_{1,8} + x_{2,8} + \cdots + x_{8,7} &= 2 \qquad & \text{(node 8 is visited once)} \\
& x_{1,2} + x_{2,1} &\leq 1 \qquad & \text{(Subtour elim for nodes 1,2)} \\
& x_{1,3} + x_{3,1} &\leq 1 \qquad & \text{(Subtour elim for nodes 1,3)} \\
& \text{and so on for all subtour elim} & & \\
& x_{1,2}, x_{1,3}, \cdots, x_{8,7} &\in \{0,1\} \qquad & \text{(binary)}
\end{aligned}
$$

2. Convert your concrete model above to a parameterized model.

Sets

Let N be the set of nodes
Let E be the set of edges

Decision Variables

Let $x_{i,j} = 1$ if edge (i,j) is included in the tour and 0 otherwise for all $(i,j) \in E$

Parameters

Let $d_{i,j}$ be the length/distance of edge (i,j) for all $(i,j) \in E$

Objective Function

$$\text{minimize:} \sum_{(i,j)\in E} d_{i,j} x_{i,j}$$

Constraints

We have the typical constraints here, but notice that the big difference is that in a directed graph, you can have a cycle on only 2 nodes.

$$
\begin{aligned}
\text{st} \qquad & \sum_{(i,j)\in E} x_{i,j} &= |N| \qquad & & (|N| \text{ edges in a tour}) \\
& \sum_{(i,n)\in E} x_{i,n} + \sum_{(n,j)\in E} x_{n,j} &= 2 \qquad & \text{for all } n \in N & \text{(each node visited once)} \\
& \sum_{(i,j)\in E: i\in S, j\in S} x_{i,j} &\leq |S| - 1 \qquad & \text{for all } S \subset N \ |S| \geq 2 & \text{(subtour elim)} \\
& x_{i,j} &\in \{0,1\} \qquad & \text{for all } (i,j) \in E & \text{(binary)}
\end{aligned}
$$

3. Suppose after solving your model, the solver returns the following solution.

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The following edges should be selected: (1,6), (2,1), (3,7), (4,3),
(5,2), (6,5), (7,8), (8,4)
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(a) What are the values of your variables associated with this solution?

$$x_{1,6} = x_{2,1} = x_{3,7} = x_{4,3} = x_{5,2} = x_{6,5} = x_{7,8} = x_{8,4} = 1$$

All other variables are 0.

(b) What is the total distance traveled by this solution?

Total distance traveled is:

$$5 + 1 + 7 + 1 + 8 + 7 + 2 + 2 = 33$$

(c) Is this solution optimal for your TSP problem?

No this is not the optimal solution. It consists of two cycles $1 - 2 - 5 - 6 - 1$ and $3 - 7 - 8 - 4 - 3$

(d) If the solution is not optimal, write a constraint you could add to your model to remove this solution from your feasible region.

I will write a constraint to eliminate the first cycle:

$$x_{1,2} + x_{1,5} + x_{1,6} + x_{2,1} + x_{2,5} + x_{2,6} + x_{5,1} + x_{5,2} + x_{5,6} + x_{6,1} + x_{6,2} + x_{6,5} \leq 3$$