SA405, Fall 2021

Exam 1
29 Sept 2021

Name: ______________________________

- **One sided** 8.5 by 11 inch formula/note sheet is allowed.
- Show work clearly and neatly.
- Define all notation used.
- Please read each question carefully. If you are not sure what a question is asking, ask for clarification.
- If you start over on a problem, please CLEARLY indicate what your final answer is, along with its accompanying work.

Grade Table

Question	Points	Score
1	50	
2	50	
Total:	100	

Please write the following statement and sign it.
The Naval Service I am a part of is bound by honor and integrity. I will not compromise our values by giving or receiving unauthorized help on this exam.

1. The U.S. Marine Corps (USMC) has placed a series of wireless sensors within a hostile environment. There three types of sensors: ***gathering***, ***relaying***, and ***analyzing***.

   - Gathering sensors strictly gather s bits of information within the environment.
   - Relaying sensors can only relay information from a gathering sensor to an analyzing sensor.
   - Analyzing sensors can only receive and analyze bits information from a relaying sensor.

   The USMC must pay a per unit cost to transmit a single bit of information from one sensor to another. The cost per unit is equal to the distance (in feet) between each pair of sensors.

   Additionally, there exists a bandwidth restriction that limits the number of bits that can be transmitted from one sensor to another.

(a) (20 points) Using only the following sets and parameters, formulate the **parameterized** mathematical programming model that minimizes the total cost to gather, transmit, and analyze all information gathered by the wireless sensor network. Be sure to clearly define your decision variables.

$\mathcal{V} :=$ Set of all sensors

$\mathcal{A} :=$ Set of all arcs on which information can be transmitted

$b_i = s_i - d_i,\ \forall i \in \mathcal{V}$, the net supply/demand value for each node i.

$c_{ij}, \forall (i,j) \in \mathcal{A}$, bandwidth restriction on arc (i,j)

$d_{ij}, \forall (i,j) \in \mathcal{A}$, the distance between sensors i and j

Solution: Decision Variables: Let $x_{i,j}$ be the bits transmitted along edge (i,j) for all $(i,j) \in A$

Objective:

$$\text{min distance: } \sum_{(i,j)\in E} d_{i,j} x_{i,j}$$

Constraints:

$$\begin{aligned}
x_{i,j} &\leq c_{i,j} && \text{for all } (i,j) \in A \\
\sum_{(i,n)\in E} x_{i,n} - \sum_{(n,j)\in E} x_{n,j} &= -bn && \text{for all } n \in N \\
x_{i,j} &\geq 0 && \text{for all } (i,j) \in E
\end{aligned}$$

(b) (5 points) Assuming that sensors 1 and 2 receive a total of 150 bits of information, write out the concrete flow-balance constraint to ensure that sensor 5 receives and analyzes all information.

Solution:

$$x_{3,5} + x_{4,5} = 150$$

(c) (5 points) Write a concrete constraint to ensure that the amount of information transmitted from sensor 1 to sensor 3 does not exceed its bandwidth restriction of 50 bits of information.

Solution:

$$x_{1,3} \leq 50$$

The USMC is now considering placing a new **relaying** sensor (6) that can relay information from gathering sensors along to analyzing sensor 5. The table below details the relevant information related to this sensor (placement cost and distance to other sensors). You can assume that sensor 6 does not have any bandwidth restrictions on information transmitted to or from it.

Values	Sensor 6
Placement Cost	75
Distance to 1	45
Distance to 2	55
Distance to 3	60
Distance to 4	35
Distance to 5	30

(d) (7 points) Clearly define any new decision variable(s) you may need to model the possible presence of sensor 6.

Solution:
Let $z = 1$ if sensor is placed
Let $x_{1,6}, x_{2,6}\ldots$ be the flow from node i to node 6

(e) (6 points) In the **concrete** form, write any changes/additions the objective function resulting in the consideration of the addition of sensor 6.. *There's no need to re-write what you provided in part (a). You just need to provide any changes or new terms.*

Solution:

$$z_{new} = z_{old} + 45x_{1,6} + 55x_{2,6} + 60x_{3,6} + 35x_{4,6} + 30x_{5,6} + 75z$$

(f) (7 points) Write *concrete* constraint(s) to enforce that no information is transmitted to or from sensor 6 if it is not placed. They should also allow sensor 6 to receive and transmit information if placed.

Solution:

$$x_{1,6} + x_{2,6} + x_{3,6} + x_{4,6} + x_{5,6} \leq Mz$$

2. The Navy is transporting six types of cargo on a plane. The table below shows the weight and volume requirements of each piece of cargo:

Cargo	Weight (pounds)	Volume (ft^3)
Type 1	200	100
Type 2	270	75
Type 3	150	125
Type 4	400	235
Type 5	335	150
Type 6	100	115

The plane itself is split into two separate sections. The first section has a weight and volume capacity of 700 pounds and 300 ft^3, respectively. The second section has a weight and volume capacity of 800 pounds and 250 ft^3, respectively. They want to transport as much of the cargo as possible on a single flight. LCDR Thompson, who was an OR major as USNA, is tasked to develop an integer programming formulation to model this problem. She looks at her favorite textbook and writes the following formulation:

Sets

Let C be the types of cargo
Let S be the sections of the plane

Decision Variables

Let

$$x_{c,s} = \begin{cases} 1 & \text{if cargo type } c \text{ is placed in section } s \\ 0 & \text{otherwise} \end{cases} \quad \text{for all } c \in C \text{ and } s \in S$$

Parameters

Let w_c be the weight of each cargo of type c for all $c \in C$
Let v_c be the volume of each cargo of type c for all $c \in C$
Let m_s be the maximum weight capacity of each section s for all $s \in S$
Let u_s be the maximum volume capacity of each section s for all $s \in S$

Objective Function

$$\text{maximize: } \sum_{c \in C} \sum_{s \in S} x_{c,s}$$

Constraints

$$\begin{aligned} \text{st} \quad \sum_{c \in C} w_c x_{c,s} &\leq m_s && \text{for all } s \in S && \text{(weight requirement)} \\ \sum_{c \in C} v_c x_{c,s} &\leq u_s && \text{for all } s \in S && \text{(volume requirement)} \\ x_{c,s} &\in \{1, 0\} && \text{for all } c \in C, s \in S && \text{(binary)} \end{aligned}$$

(a) (15 points) Using the variables $x_{c,s}$ defined above (i.e., $x_{1,1}, x_{2,1}, \ldots, x_{6,2}$), write the objective function and the constraints of this model in concrete form. You can use ellipses once the summation/constraints are clear.

Solution:

Objective

$$\text{maximize: } x_{1,1} + x_{2,1} + x_{3,1} + \cdots + x_{5,2}$$

Constraints

$$\begin{aligned}
200x_{1,1} + 270x_{2,1} + 150x_{3,1} + 400x_{4,1} + 335x_{5,1} + 100x_{6,1} &\leq 700 \\
200x_{1,2} + 270x_{2,2} + 150x_{3,2} + 400x_{4,2} + 335x_{5,2} + 100x_{6,2} &\leq 800 \\
100x_{1,1} + 75x_{2,1} + 125x_{3,1} + 235x_{4,1} + 150x_{5,1} + 115x_{6,1} &\leq 300 \\
100x_{1,2} + 75x_{2,2} + 125x_{3,2} + 235x_{4,2} + 150x_{5,2} + 115x_{6,2} &\leq 250 \\
x_{1,1}, x_{2,1}, \ldots, x_{5,2} &\in \{0,1\}
\end{aligned}$$

(b) After looking at your concrete version of the model, LCDR Thompson realizes that she forgot to enforce some constraints that belong in this model. Specifically, since there is one of each type of cargo, constraints need to be added to this formulation to restrict the same piece of cargo being assigned to two sections of the plane.

i. (5 points) Write a concrete constraint which enforces the rule that cargo type 3 can be placed in at most one section.

Solution:

$$x_{3,1} + x_{3,2} \leq 1$$

ii. (5 points) Based on your constraint for cargo 3, write a parameterized set of constraints that enforce that each type of cargo is placed in at most one section.

Solution:

$$\sum_{c \in C} x_{t,c} \leq 1 \text{ for each } t \in T$$

(c) LCDR Thompson decides that one section needs to be tightly packed; that is, she wants one section of the plane to hold at least 3 types of cargo. She defines a new binary variable z which equals 1 if section 1 is tightly packed and 0 if section 2 is tightly packed.

i. (5 points) Write a concrete constraint that enforces that section 1 has at least 3 pieces of cargo in it if $z = 1$ and is relaxed if $z = 0$.

Solution:

$$x_{1,1} + x_{2,1} + x_{3,1} + x_{4,1} + x_{5,1} + x_{6,1} \geq 3 - M(1 - z)$$

ii. (5 points) Write a concrete constraint that enforces that section 2 has at least 3 pieces of cargo in it if $z = 0$ and is relaxed if $z = 1$.

Solution:

$$x_{1,2} + x_{2,2} + x_{3,2} + x_{4,2} + x_{5,2} + x_{6,2} \geq 3 - Mz$$

iii. (3 points) Does the objective function need to change for the binary variable z? If yes what is the new objective function? If no why not?

Solution: No the objective function doesn't need to change as z has no impact on the objective.

(d) Once the model is complete, LCDR Thompson again realizes that new constraints need to be added to the model to account for some other requirements introduced to her.

i. (6 points) Cargo type 1 and cargo type 4 are made of volatile components; thus should not be packed together. Write a (concrete) constraint which enforces the logic that if cargo type 1 is placed in section 1, then cargo type 4 can not be placed in section 1.

Solution:

$$x_{1,1} \leq 1 - x_{4,1}$$

ii. (6 points) Cargo types 1, 2, and 5 are also incompatible. Write a logical constraint which enforces the logic that if cargo type 1 and 2 are placed in section 1, then cargo type 5 can not be placed in section 1.

Solution:

$$x_{1,1} + x_{2,1} \leq (1 - x_{5,1}) + 1$$