HW3: Logical Constraints Part 1-Solution

Problem 1: Charlene is considering opening a food stand to sell refreshments to tourists who visit Annapolis. She can sell each of the following: hamburgers, hotdogs, dark beer and IPA. Each hamburger requires two pieces of bread and 0.5 pound of meat. Likewise, each hot dog requires 1 piece of bread and 0.75 pounds of meat. She has 200 pieces of bread, 100 pounds of meat, 200 ounces of dark beer, and 160 ounces of IPA. Lastly, she can sell each hamburger for \$5, each hotdog for \$4, 12 ounces of dark beer for \$5, and 12 ounces of IPA for \$4.

1. Formulate an integer program that would allow Charlene to maximize her revenue.

   Decision Variables

   Let x_h be the number of hamburgers sold
   Let x_o be the number of hotdogs sold
   Let x_d be the number of 12 ounce dark beers sold
   Let x_i be the number of 12 ounce IPAs sold

   Objective Function

   $$\text{max revenue: } 5x_h + 4x_o + 5x_d + 4x_i$$

   Constraints

   $$\begin{aligned} \text{st} \quad 2x_h + x_o &\leq 200 && \text{(Bread available)} \\ 0.5x_h + 0.75x_o &\leq 100 && \text{(Meat available)} \\ x_d &\leq 200/12 && \text{(Dark beer available)} \\ x_i &\leq 160/12 && \text{(IPA available)} \\ x_h, x_o, x_d, x_i &\geq 0 && \text{(non-negativity)} \end{aligned}$$

2. Charlene is now considering adding fried chicken sandwiches to her menu. For \$150, she can obtain 50 pounds of fried chicken tenders. She can then make a fried chicken sandwich with 2 piece of bread and 0.75 pounds of fried chicken and sell it for \$6. If she does buy the chicken, she wants to sell at least 25 chicken sandwiches. Modify your model from part 1 to account for this new requirement.

To incorporate this, we introduce the following new variables:

Let x_c be the number of chicken sandwiches sold
Let $y = 1$ if she buys the chicken and 0 otherwise

Then, we change the objective function to be:

$$\text{max revenue: } 5x_h + 4x_o + 5x_d + 4x_i + 6x_c - 150y$$

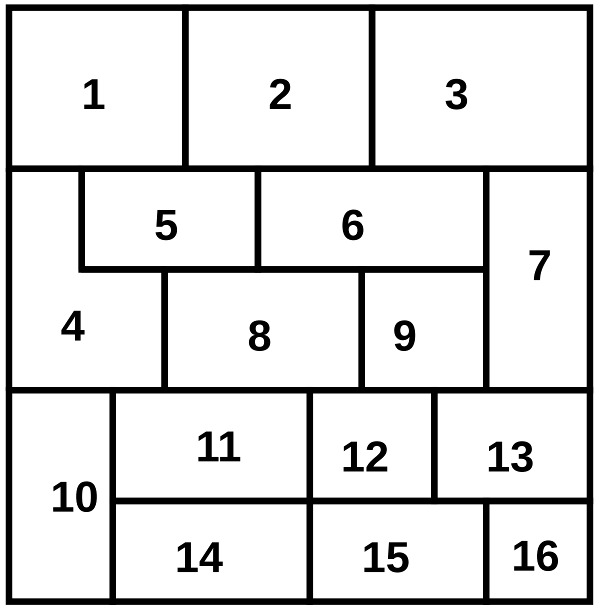
We also modify the bread constraint to be:

$$2x_h + x_o + x_c \leq 200$$

Lastly, we add the following two constraints which are the fixed charge style constraints:

$$\begin{aligned} x_c &\leq 50/0.75 * y && \text{(upper bound)} \\ x_c &\geq 25y && \text{(minimum number made)} \end{aligned}$$

Problem 2: The city of Annapolis is rezoning the city into 16 separate districts (see the figure below). Based on this rezoning, answer the following questions:



1. The city wants to build hospitals to service Annapolis. They want to make sure every district either contains a hospital, or is adjacent to another district containing a hospital. Formulate a concrete model that will allow them to make sure every district is serviced by a hospital while building the minimum number of hospitals.

   Decision Variables

   Let $x_1 = 1$ if a hospital is built in zone 1 and 0 otherwise

   $\vdots$

   Let $x_{16} = 1$ if a hospital is build in zone 16 and 0 otherwise

   Objective Function

   $$\text{minimize } x_1 + x_2 + \cdots + x_{16}$$

   Constraints

   $$\begin{aligned}
   x_1 + x_2 + x_4 + x_5 &\geq 1 && \text{(Zone 1 is covered)} \\
   x_1 + x_2 + x_3 + x_5 + x_6 &\geq 1 && \text{(Zone 2 is covered)} \\
   &\vdots \\
   x_{13} + x_{15} + x_{16} &\geq 1 && \text{(Zone 16 is covered)} \\
   x_1, x_2, \ldots, x_{16} &\in \{0, 1\} && \text{(binary)}
   \end{aligned}$$

2. Convert your concrete model from part 1 into a parameterized model *Hint: It may be helpful to define some extra sets that aren't directly in the problem or use the adjacency matrix idea from class.*

   There are two ways to solve this problem. The first uses an extra parameter. The second uses extra sets. Both options should make sense to you. I'll give the first option below in green and the second option in red.

   OPTION 1

   Sets

   Let Z be the set of zones

   Parameters

   Let $a_{i,j} = 1$ if zone i is adjacent to zone j for all $i \in Z$ and all $j \in Z$. (Note that this is a parameter not a variable).

   Decision Variables

   Let $x_i = 1$ if a hospital is build in zone i for all $i \in Z$

   Objective Function

$$\min \sum_{i \in Z} x_i$$

   Constraints

$$\begin{aligned} \sum_{i \in Z} a_{i,j} x_i &\geq 1 \qquad \text{for all } j \in Z \\ x_i &\in \{0,1\} \quad \text{for all } i \in Z \end{aligned}$$

OPTION 2

Sets

Let Z be the set of zones
Let N_i be the set of zones that can cover node i for all $i \in Z$ (thus $N_1 = \{1, 2, 4, 5\}$, $N_2 = \{1, 2, 3, 5, 6\}$, etc)

Parameters

None

Decision Variables

Let $x_i = 1$ if a hospital is build in zone i for all $i \in Z$

Objective Function

$$\min \sum_{i \in Z} x_i$$

Constraints

$$\begin{aligned} \sum_{j \in N_i} x_i &\geq 1 & \text{for all } i \in Z \\ x_i &\in \{0, 1\} & \text{for all } i \in Z \end{aligned}$$