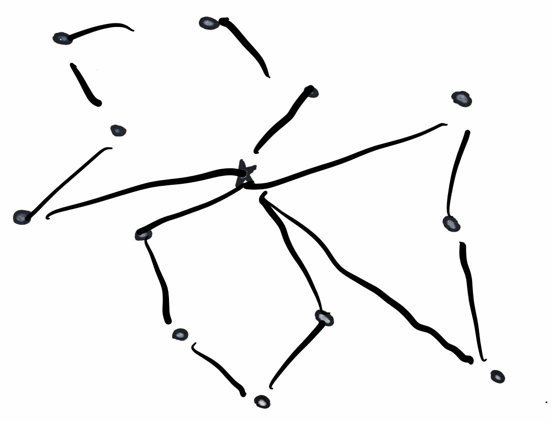
Lesson 9. Vehicle Routing Problem (VRP)

1 VRP: An extension of TSP

Vehicle Routing Problems are similar to TSP, except that instead of having a single "salesperson", we have multiple "salespeople" (vehicles), all *starting and ending at the same location*. We must decide the cheapest way to cover all of the locations via tours with our fleet of salespeople.

Problem 1. The pizza restaurant at ★ has 12 delivery orders ready and three drivers. Each driver can serve at most 5 customers. Draw a *feasible* solution to the VRP problem on the graph below.



Vehicle routing is a very practical real world application.

Basic VRP

- Fixed number of vehicles.] k: # of vehicles
- All vehicles start and end a common location, or **depot**.] node 0
- All vehicles have the same capacity.
- Vehicles may serve many different customers, up to capacity.
- Each customer's demand must be served by a single vehicle.
- Goal: Find a minimum cost collection of *tours*, all starting and ending at the depot, that contain all customers and do not violate vehicle capacities.

Basic VRP: Mathematical Description

Notation:

- Let $G = (N, E)$ be an undirected graph.
  - One of the nodes is distinguished as the *depot*: usually node "0".
  - Each edge has a cost (or distance): c_{ij}, for $(i, j) \in E$ Remember that $(i, j) \in E$ if $i < j$.
  - Each node has a demand: d_i, for $i \in N$.] demand = 1
- Exactly k tours (vehicles) used, each with the same demand capacity (D).

Requirements:

- Each tour must include the depot.] Routes start and end at 0
- Each vertex (customer) must be be served.] Every node has 2 edges
- The demand of each tour must not exceed D.] Hard constraints for VRP

Formulation assumptions:

- Every vehicle visits at least two customers.
- The direction an edge is traversed does not affect the cost.] undirected graph
- (Variations of the formulation can work around these assumptions.)

Goal:

- Minimize total cost of tours.

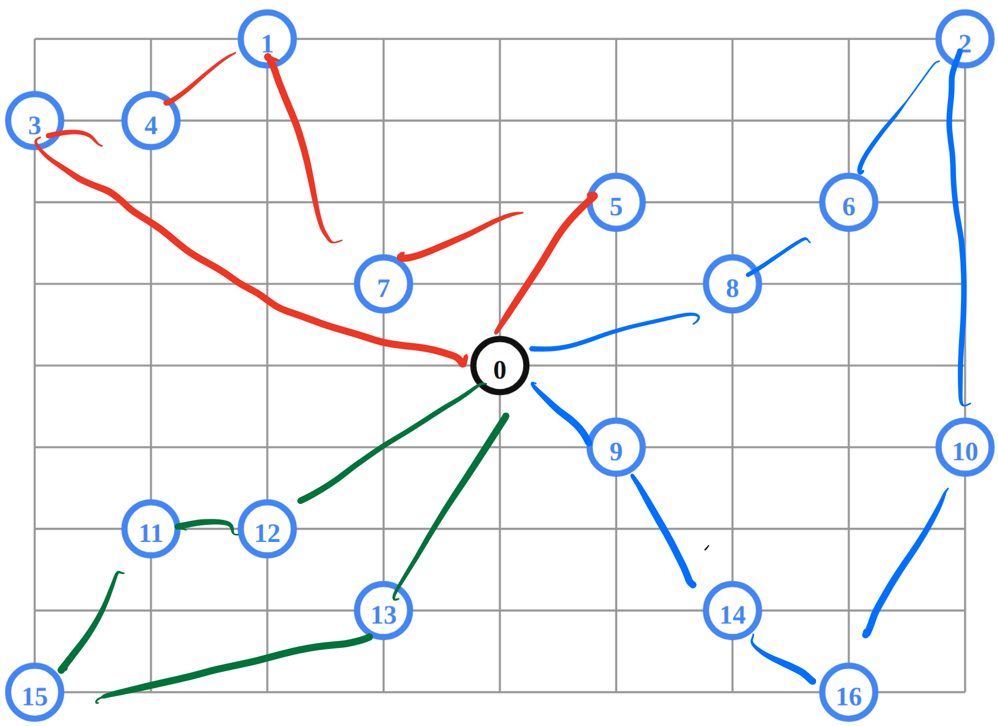
VRP can be modified to incorporate variations including:

- **Capacitated VRP:** Each vehicle has a capacity (problem we are doing)
- **VRP with time windows:** Each customer must be visited within a certain time window
- **Multidepot VRP:** Fixed number of vehicles starting at each of multiple depots.
- **Split delivery VRP:** Customer demand can be met by more than one vehicle

2 VRP Standard Formulation: Coffee Deliveries

Problem 2. The MidStore coffee shop has decided to start delivering (iced) coffee around the yard. The diagram below shows the location of the coffee company (location 0) and the coordinates of each customer on the yard. Assume that each customer has the same demand. The MidStore coffee company has three (3) people who are available to deliver coffee and each of which can deliver to up to seven customers.

3 drivers
Capacity 7



1. Draw a feasible solution on the map using the three delivery people.
2. Suppose the distance between points is the straight line distance. What is the distance between nodes 0 and 1, 0 and 2, and 15 and 16?

$0 \to 1 : 2^2 + 4^2 = d^2 \to d = \sqrt{20}$

$0 \to 2 : 4^2 + 4^2 = d^2 \to d = \sqrt{32}$

$15 \to 16 : d = 7$

3. Write a concrete model to minimize the distance traveled by the MidStore coffee shop. Do **not** include subtour elimination constraints.

Vars

let $x_{ij} = 1$ if edge (i,j) is used for all $(i,j) \in E$

Objective

$$\min \quad \sqrt{20}\, x_{01} + \sqrt{32}\, x_{02} + \cdots + 7\, x_{15\,16}$$

Constraints

$$x_{01} + x_{02} + \cdots + x_{0\,16} = 2 \cdot 3$$ (node 0 has 6 edges)

$$x_{01} + x_{12} + \cdots + x_{1\,16} = 2$$

$$\vdots$$

$$x_{0\,16} + x_{1\,16} + \cdots + x_{15\,16} = 2$$

(All nodes 1–16 use 2 edges)

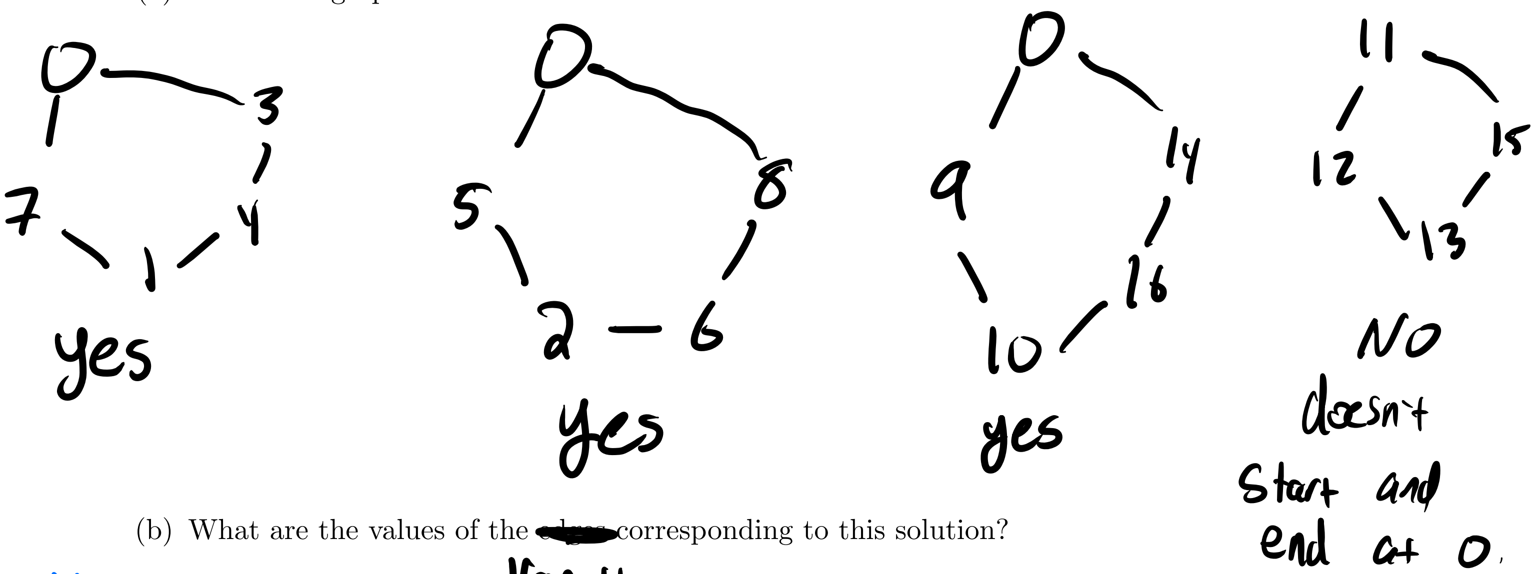
$$x_{01}, \ldots, x_{15\,16} \in \{0,1\}$$ (binary)

- No cycles not connected to node 0 (TSP subtour elim)

- No routes larger than 7 nodes (Route splitting)

Problem 3. Suppose a solution to the (incomplete) model above gives the following cycles: $\{0, 7, 1, 4, 3\}, \{0, 5, 2, 6, 8\}, \{0, 9, 10, 16, 14\}, \{11, 12, 13, 15\}$.

(a) Sketch the graph associated with this solution.



(b) What are the values of the [redacted] corresponding to this solution?

Variables

$x_{03} = 1$

$x_{07} = 1$

$\vdots$

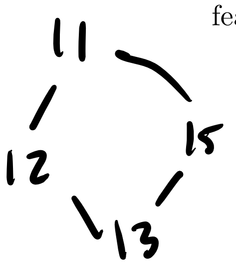
$x_{1315} = 1$

All other Variables = 0

(c) Why is this solution infeasible to the (complete) VRP formulation?

Route 11-12-13-15 doesn't include 0

(d) Using the TSP style, write a concrete constraint that would eliminate this solution from the feasible region.



$x_{1112} + x_{1213} + x_{1115} + x_{1315} \leq 3$

only stops cycle 11-12-13-15

Goal: stop all cycles on 11, 12, 13, 15

Recall: $\sum x_{ij} \leq |S| - 1 \quad \forall\ S \subset N$

$(i,j) \in E$:
$i \in S,\ j \in S$

$S = \{11, 12, 13, 15\}$: $x_{1112} + x_{1113} + x_{1115} + x_{1213} + x_{1215} + x_{1315} \leq 3$

Problem 4. Suppose a solution to the (incomplete) model above gives the following cycles: $\{0, 7, 1, 4, 3\}$, $\{0, 11, 15, [redacted], 12\}$, $\{0, 9, 14, 16, 10, 8, 6, 2, 5\}$.

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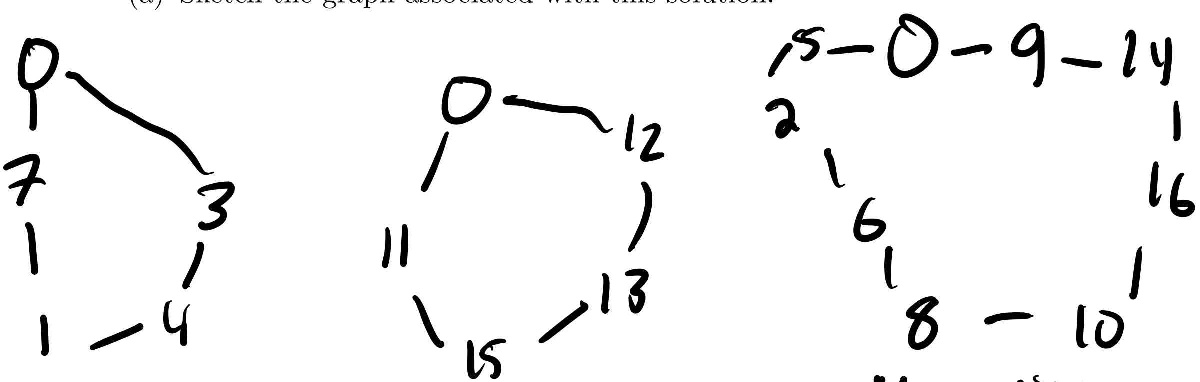
(a) Sketch the graph associated with this solution.



yes

yes

No, visits 8 people

Capacity = 7

(b) What are the values of the edges corresponding to this solution?

$x_{07} = 1$

$x_{03} = 1$

$\vdots$

$x_{1416} = 1$

All variables not drawn = 0

(c) Why is this solution infeasible to the (complete) VRP formulation?

Route on right exceeds capacity

(d) Using the TSP style, write a concrete constraint that would eliminate this solution from the feasible region.

TSP style $\quad \sum x_{ij} \leq |S| - 1 \quad \forall\, S \subset N$

[illegible]

$S = \{0, 2, 5, 6, 8, 9, 10, 14, 16\}$ $\quad |S| = 9$

$x_{02} + x_{05} + \cdots + x_{1416} \leq 8$] Stops any cycle on those 9 nodes

(e) For this route, is there any drawback to the TSP style constraint? In other words, does the TSP style constraint eliminate all infeasible cycles on these nodes from this graph?

It works but doesn't enforce capacity.

Drawbacks 1) Delete 0 have route on 8 nodes

2) Doesn't stop these 8 customers from being part of a larger group

Goal: Write a constraint that splits these 8 people into 2 groups

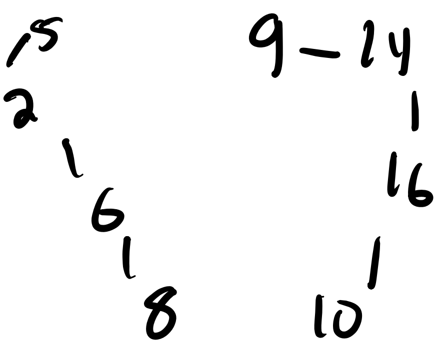
(f) How many delivery people would be needed to visit the nodes on this cycle?

2 → Capacity = 7 8 customers

(g) What is the maximum number of edges you could pick on these nodes in order to split it into two delivery people?

1 Route through 8 people → 7 edges

2 Routes through 8 people → 6 edges



(h) Write a constraint that enforces this requirement on the nodes involved in this cycle (not including node 0).

$S = \{2, 5, 6, 8, 9, 10, 14, 16\}$ $|S| = 8$

$x_{25} + x_{26} + \dots + x_{1416} \leq 6$

↗ Forces the 8 nodes to be split in at least 2 seperate groups

This type of constraint is referred to as **route-splitting** in the vehicle routing problem.

3 Route Splitting & Generalized Subtour Elimination Constraints

The **subtour elimination constraints** that we learned about in TSP would technically work for VRP but are not robust enough for VRP.

> A **connected component** is a subset of nodes in a graph which are all connected to each other. VRP : One Route

The question we ask in VRP is: How many edges must we include among $|S|$ vertices in order to have C connected components and NO CYCLES (importantly, this means you need C vehicles to visit each of these customers)? Let's try some test cases...

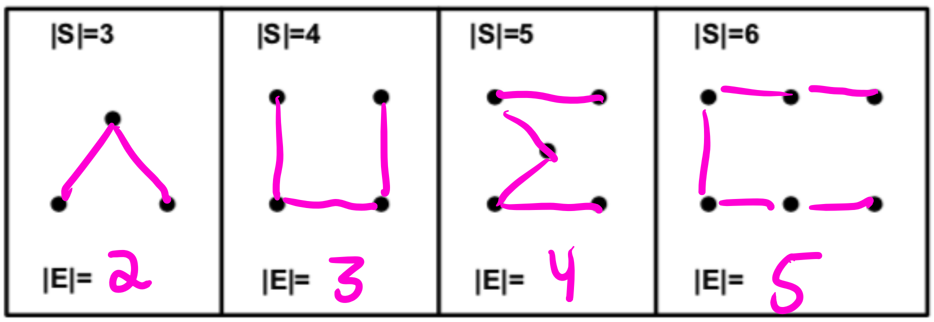
C: # of vehicles

- $C = 1$: How many edges do we include to have 1 connected component and no cycles?

C=1: One delivery person to visit these nodes.

|S|-1 edges

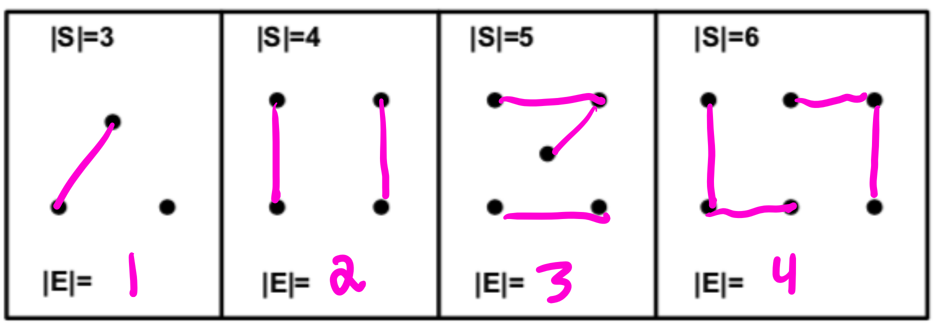
TSP



- $C = 2$: How many edges do we include to have 2 connected components and no cycles?

C=2: 2 delivery people to visit these nodes.

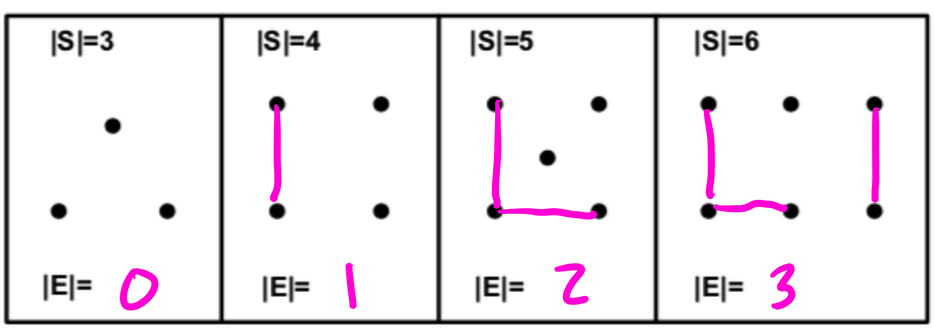
|S|-2 edges can be picked



- $C = 3$: How many edges do we include to have 3 connected components and no cycles?

C=3: 3 delivery people

|S|-3 edges Can be picked



C : # of delivery people for S nodes

D: Capacity

$C = \left\lceil \frac{|S|}{D} \right\rceil$ ← Round up

Given a general $C > 0$, how many edges must we include among $|S|$ nodes to have C connected components and no cycles? These are called **generalized subtour elimination** constraints.

$$\sum_{\substack{(i,j)\in E:\\ i\in S,\, j\in S}} x_{ij} \le |S| - C \quad \forall\ S \subset N,\ |S| \ge 3,\ 0 \notin S$$

4 Parameterized VRP Model

Problem 5. Write the full parameterized VRP model using the generalized subtour elimination constraints.

Sets

N: set of nodes

E: set of edges

Parameters

d_{ij}: distance on edge $(i,j)\in E$

K: # of delivery people

D: capacity of each route

Vars

let $x_{ij} = 1$ if edge (i,j) is used $\forall\ (i,j)\in E$

Objective

$$\min \sum_{(i,j)\in E} d_{ij} x_{ij}$$

(Total distance traveled)

Constraints

$$\sum_{(0,j)\in E} X_{0j} = 2\cdot K$$

$$\sum_{(i,n)\in E} x_{in} + \sum_{(n,j)\in E} x_{nj} = 2 \quad \forall\ n \in N \setminus \{0\}$$

$$x_{ij} \in \{0,1\} \quad \forall\ (i,j)\in E$$

$$\sum_{\substack{(i,j)\in E:\\ i\in S,\, j\in S}} x_{ij} \le |S| - C \quad \forall\ S \subset N,\ |S| \ge 3,\ 0 \notin S$$

Illustrate subtour elim

$$\sum_{\substack{(i,j)\in E:\\ i\in S,\ j\in S}} x_{ij} \le |S| - c \quad \forall\, S \subset N,\ |S| \ge 3,\ 0 \notin S$$

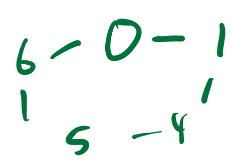
- TSP: Always 1 route $c = 1$, no node 0

$S = \{1, 3, 4\}$

$$x_{13} + x_{14} + x_{34} \le 3 - 1$$

- VRP. Capacity is 3

Route: $\{0, 1, 4, 5, 6\}$



$$c = \#\text{ of delivery people} = \left\lceil \frac{4}{3} \right\rceil = 2$$

$S = \{1, 4, 5, 6\}$ ← Don't include 0

$$x_{14} + x_{15} + x_{16} + x_{45} + x_{46} + x_{56} \le 4 - 2$$