Branch-and-bound Graphical Example (Accompanying Lesson 16)

1 Branch-and-bound Example

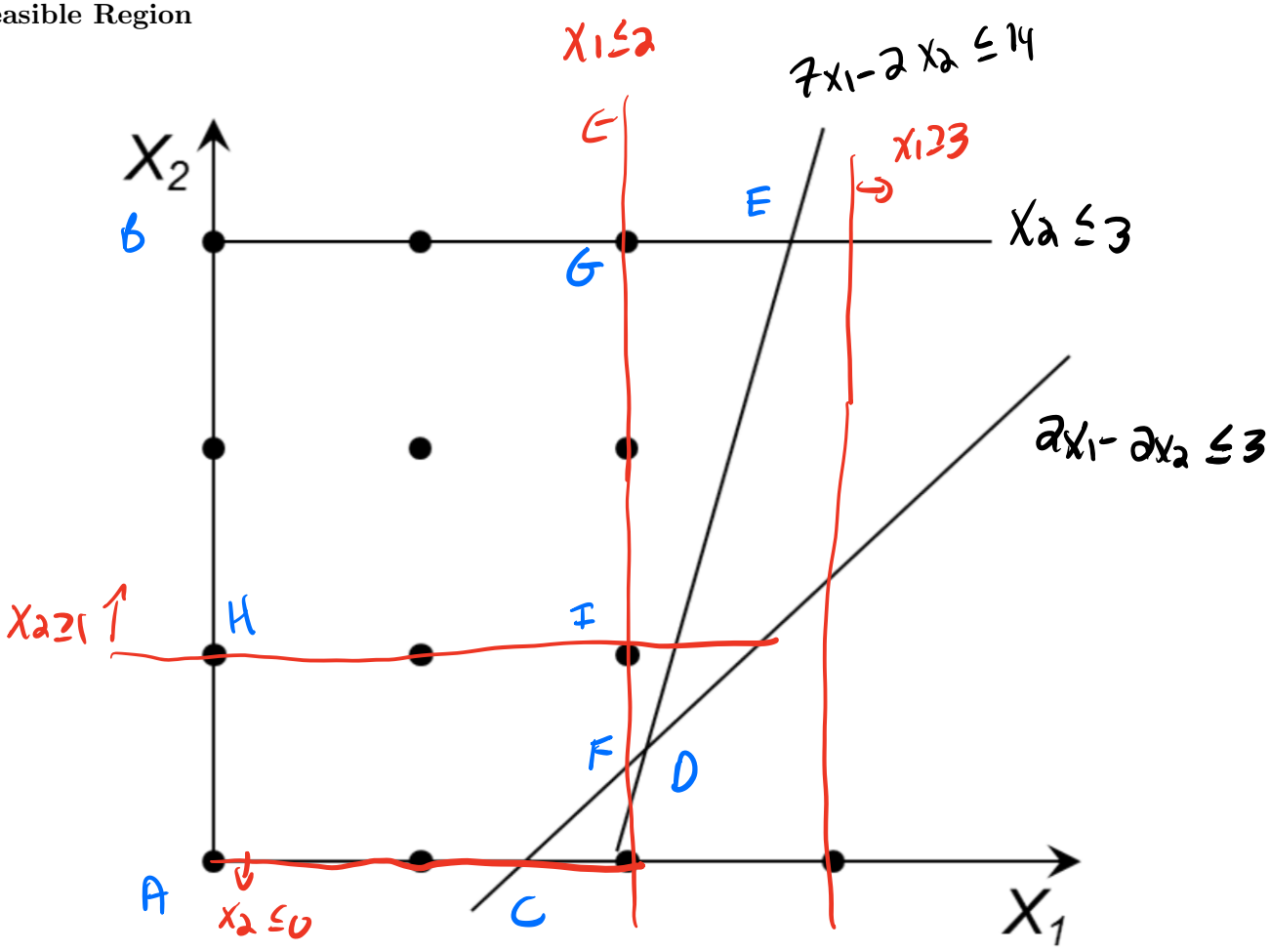
Solve the following IP using branch-and-bound. Solve each sub problem graphically.

$$\begin{aligned} \text{(P1)} \quad z^*_{IP} = \max\ & 4x_1 - x_2 \\ \text{s.t.}\quad & 7x_1 - 2x_2 \leq 14 \\ & 2x_1 - 2x_2 \leq 3 \\ & x_2 \leq 3 \\ & x_1, x_2 \in \mathbb{Z}^{\geq 0} \end{aligned}$$

- Solve each sub-problem graphically
- Branching Rules
  - Always select the active node with the largest upperbound for branching.
  - Branch on x_1 if it is fractional. Otherwise branch on x_2.
- Book-keeping
  - Keep track of the:
    - incumbent solution $\underline{x}$,
    - global lower bound $\underline{z}$, and
    - list of active nodes.
  - Draw the branch-and-bound tree:
    - Record the local upper bound (z) and relaxed optimal solution (x) for each subproblem.
    - Label each edge with the constraint that is added to form the child subproblem.
    - X-out fathomed nodes. Circle incumbent solution nodes.
  - Use the provided diagram to illustrate the (relaxed) feasible region of each subproblem.

incumbent solution $\underline{x}$ **global lower bound $\underline{z}$** **active nodes**

Feasible Region



A: (0,0), Z=0

B: (0,3), Z=-3

C: $(\frac{3}{2}, 0)$, Z=6

D: $(\frac{11}{5}, \frac{7}{10})$, Z=8.1

E: $(\frac{20}{7}, 3)$, Z=8.4

F: $(2, \frac{1}{2})$, Z=7.5

G: (2,3), Z=5

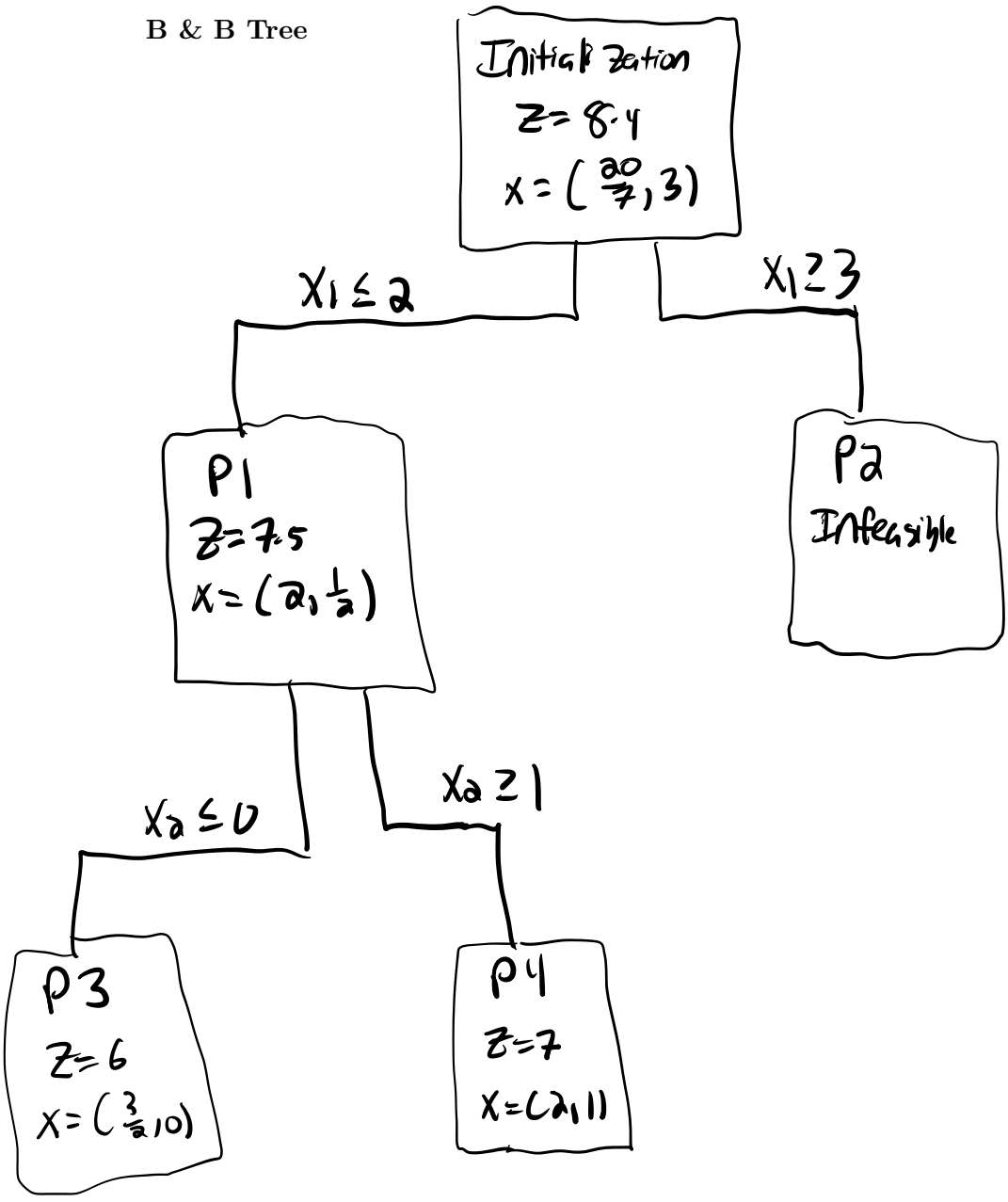
H: (0,1), Z=-1

I: (2,1), Z=7

$$
\begin{aligned}
z^*_{IP} = \max\ & 4x_1 - x_2 \\
\text{s.t.}\quad & 7x_1 - 2x_2 \le 14 \\
& 2x_1 - 2x_2 \le 3 \\
& x_2 \le 3 \\
& x_1, x_2 \in \mathbb{Z}^{\ge 0}
\end{aligned}
$$

y

B & B Tree


Initialization
$z = 8.4$
$x = (\frac{20}{7}, 3)$
$x_1 \leq 2$
$x_1 \geq 3$
P1
$z = 7.5$
$x = (2, \frac{1}{2})$
P2
Infeasible
$x_2 \leq 0$
$x_2 \geq 1$
P3
$z = 6$
$x = (\frac{3}{2}, 0)$
P4
$z = 7$
$x = (2, 1)$

Step 1: Solve the LP relaxation of original problem

Initial solution as $Z = 8.4$ $x = (\frac{20}{7}, 3)$ so

$$Z^*_{IP} \leq 8$$

x_1 is fractional so I want to eliminate $x_1 = \frac{20}{7}$. Branch on x_1 and create problems P1 and P2

P1: Original problem with $x_1 \leq 2$

P2: original problem with $x_1 \geq 3$

Step 2: solve P1 and P2

P1: $Z = 7.5$ $x = (2, \frac{1}{2})$

P2: Infeasible

$$Z^*_{IP} \leq 7 \quad \text{New bound}$$

Branch P1 on x_2 to create P3 and P4

P3: Original problem and $x_1 \leq 2$, $x_2 \leq 0$

P4: Original problem and $x_1 \leq 2$, $x_2 \geq 1$

Step 3: solve P3 and P4

P3: $(\frac{3}{2}, 0)$ $Z = 6$

P4: $(2, 1)$ $Z = 7$

P4 $\Rightarrow$ $Z^* \geq 7$

Eliminate P3 b/c $6 < 7$

$x^* = (2, 1)$, $Z^* = 7$