Branch-and-bound Python Example (Accompanying Lesson 16)

1 Today...

- This example is part of Lesson 16, Branch-and-bound.

2 Branch-and-bound Example

Solve the following IP using branch-and-bound.

$$(\text{P1}) \quad z_{IP}^* = \max 8x + 7y$$
$$\text{s.t.} \quad -18x + 38y \leq 133$$
$$13x + 11y \leq 125$$
$$10x - 8y \leq 55$$
$$x, y \in \mathbb{Z}^{\geq 0}$$

- Use Python to solve LP relaxations of subproblems

- Branching Rules

  ○ Always select the active node with the largest upperbound for branching.
  ○ Branch on x if it is fractional. Otherwise branch on y.

- Book-keeping

  ○ Keep track of the:
    ◇ incumbent solution $\underline{x}$,
    ◇ global lower bound $\underline{z}$, and
    ◇ list of active nodes.
  ○ Draw the branch-and-bound tree:
    ◇ Record the local upper bound (z) and relaxed optimal solution (x) for each subproblem.
    ◇ Label each edge with the constraint that is added to form the child subproblem.
    ◇ X-out fathomed nodes. Circle incumbent solution nodes.
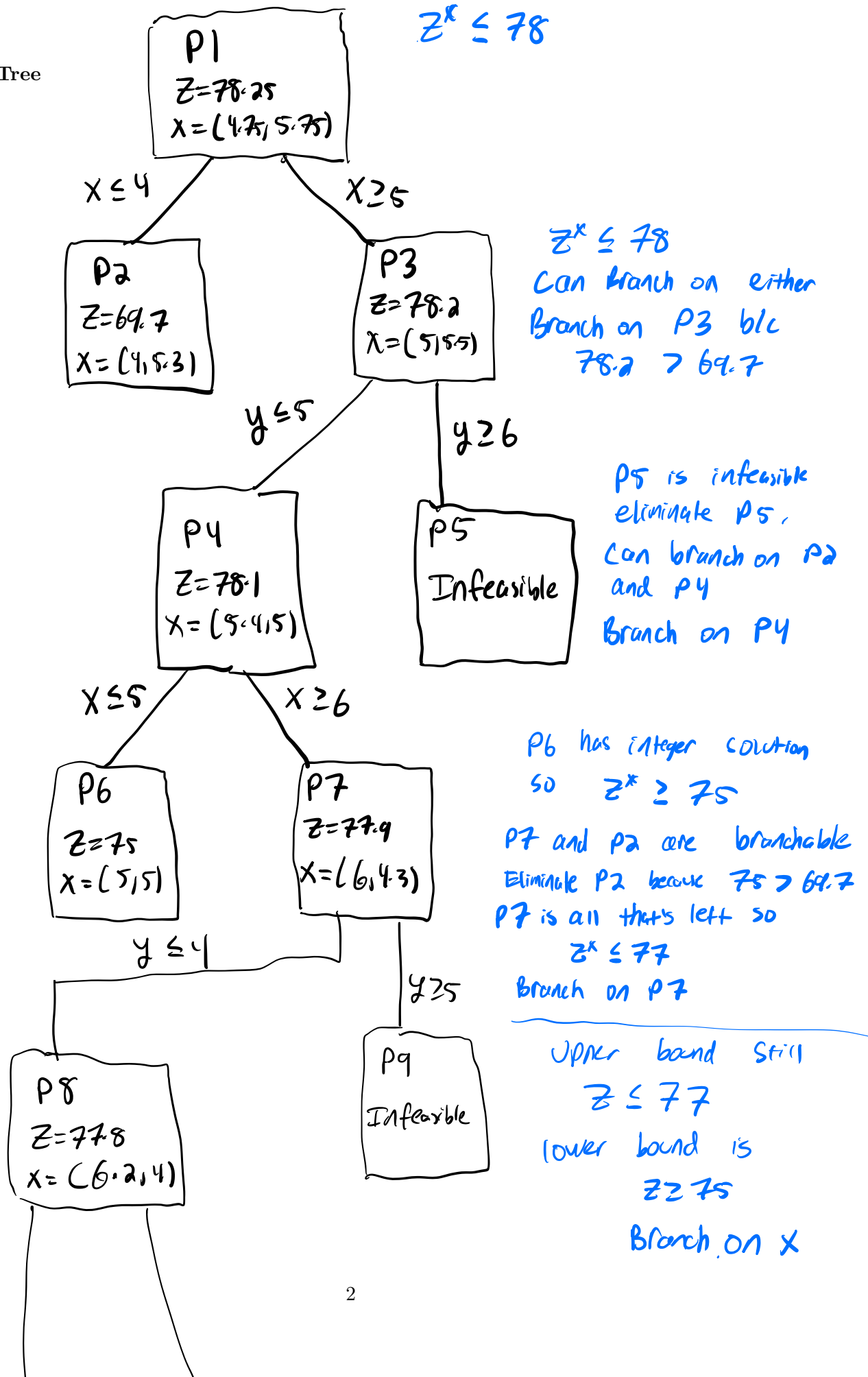 ○ Use the provided diagram to illustrate the (relaxed) feasible region of each subproblem.
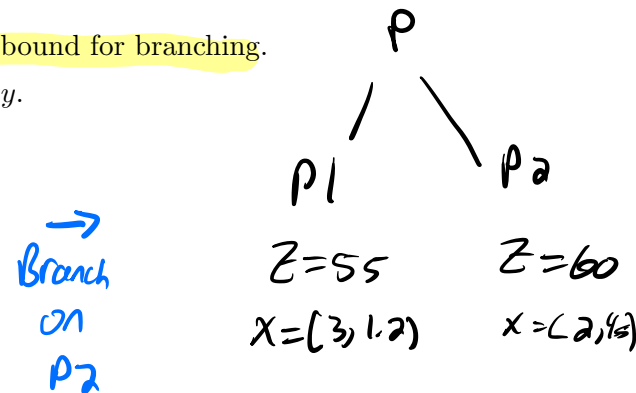
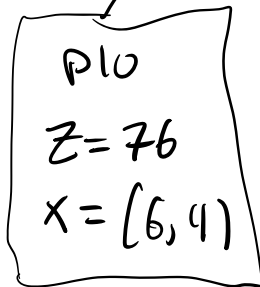**incumbent solution $\underline{x}$**   **global lower bound $\underline{z}$**   **active nodes**
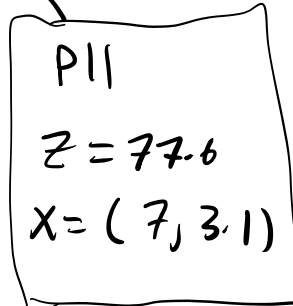
B & B Tree

$Z^* \leq 78$

P1
$Z = 78.25$
$X = (4.75, 5.75)$

$X \leq 4$

$X \geq 5$

P2
$Z = 69.7$
$X = (4, 5.3)$

P3
$Z = 78.2$
$X = (5, 5.5)$

$y \leq 5$

$y \geq 6$

P4
$Z = 78.1$
$X = (5.4, 5)$

P5
Infeasible

$X \leq 5$

$X \geq 6$

P6
$Z = 75$
$X = (5, 5)$

P7
$Z = 77.9$
$X = (6, 4.3)$

$y \leq 4$

$y \geq 5$

P8
$Z = 77.8$
$X = (6.2, 4)$

P9
Infeasible

$Z^* \leq 78$
Can branch on either
Branch on P3 b/c
$78.2 > 69.7$

P5 is infeasible
eliminate P5.
Can branch on P2
and P4
Branch on P4

P6 has integer solution
so $Z^* \geq 75$
P7 and P2 are branchable
Eliminate P2 because $75 > 69.7$
P7 is all that's left so
$Z^* \leq 77$
Branch on P7

Upper bound still
$Z \leq 77$
lower bound is
$Z \geq 75$
Branch on X

2

$X \leq 6$

$X \geq 7$

```
┌─────────────┐
│    P10      │
│             │
│  Z = 76     │
│             │
│  X = (6, 4) │
└─────────────┘
```

```
┌─────────────┐
│     P11     │
│             │
│   Z = 77.6  │
│             │
│ X = (7, 3.1)│
└─────────────┘
```

upper bound is

$Z \leq 77$
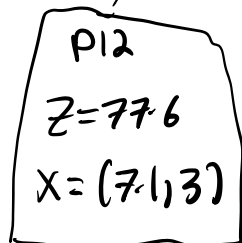
p10 is integer

and $76 > 75$.

So P10 becomes the current
solution, lower bound is

$Z \geq 76$

$y \leq 3$

$y \geq 4$
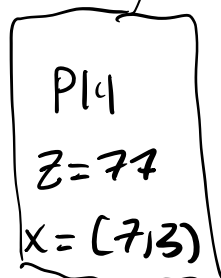
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┌─────────────┐
│     P12     │
│             │
│   Z = 77.6  │
│             │
│  X = (7.1, 3)│
└─────────────┘
```

```
┌─────────────┐
│    P13      │
│             │
│  Infeasible │
└─────────────┘
```
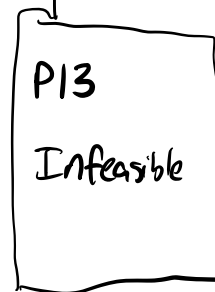
upper bound

$Z \leq 77$

So branch on
X

$X \leq 7$

$X \geq 8$

```
┌─────────────┐
│    P14      │
│             │
│   Z = 77    │
│             │
│  X = (7, 3) │
└─────────────┘
```

```
┌──────────┐
│   P15    │
│          │
│ Infeasible│
└──────────┘
```

lower bound is

$Z \geq 77$

No nodes left to
branch, so done.

Also can stop because
lower bound = upper bound.

Optimal solution $Z^k = 77$

$X^k = (7, 3)$

P1

P2 $x \leq 4$ $x \geq 5$ P3

P4 $y \leq 5$ $y \geq 6$ P5

$x \leq 5$ $x \geq 6$

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