# Lesson 15. Lagrangian Relaxation

## 1 Motivation

When talking about branch and bound, we talked about how LP relaxations can be used to give upper bounds on integer programs. Recall that, if $z^*_{IP}$ is the optimal IP solution and $z^*_{LP}$ is the optimal LP solution then:

$$z^*_{IP} \boxed{\leq} z^*_{LP} \text{ if maximizing}$$

$$z^*_{IP} \boxed{\geq} z^*_{LP} \text{ if minimizing}$$

Many LP and IP problems have a special structure:

$$\begin{aligned} \max \quad & cx \\ \text{st} \quad & Ax = b \quad \text{] hard constraints} \\ & x \in \mathbb{X} \quad \text{] set of all easy constraints} \end{aligned}$$

The constraints $x \in \mathbb{X}$ are "easy" constraints such as:

flow balence, non-negativity, variable bounds

The constraints $Ax = b$ are "difficult" constraints such as:

fixed charge, subtour elimination

**Lagrangian Relaxation** is a method to obtain bounds on integer programs which have this structure. The basic idea of Lagrangian Relaxation is:

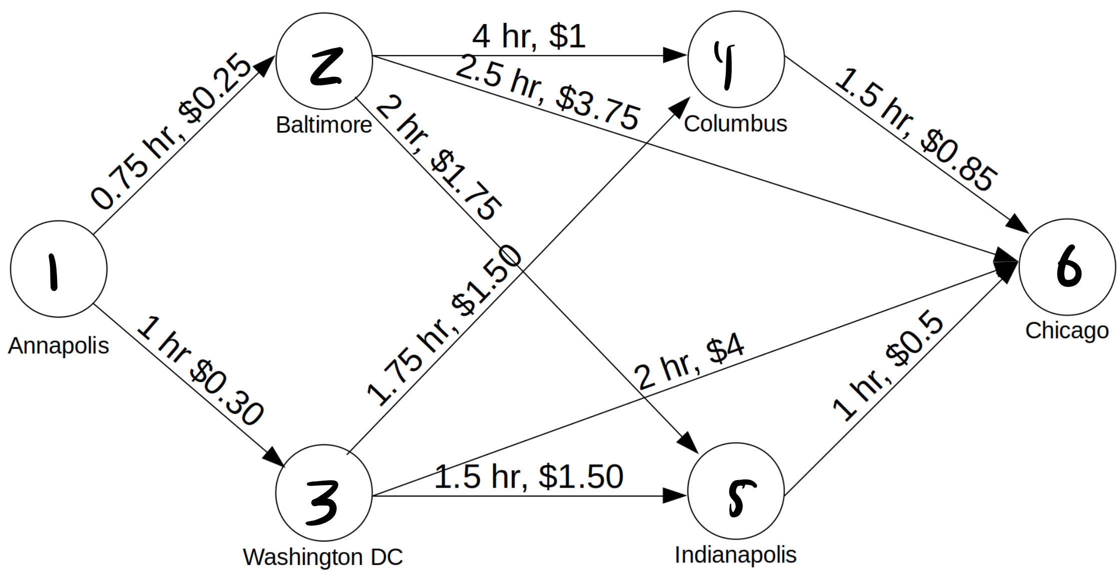
- Remove "hard" constraints from the problem and place them in the objective function (i.e., relax the IP) ↑ Forces you to pay a penalty if these are not satisfied
- Solve the relaxed problem that now consists of only "easy" constraints.
  ↗ Makes IP easy to solve

Doing so gives us a bound on the optimal solution.

In many cases, this bound is close to the optimal solution of the original problem.

## 2 Shortest Path Problem with Time Constraints

**Problem 1.** John is trying to travel from Annapolis to Chicago as cheaply as possible by taking public transportation. Unfortunately, there is no direct route between the two cities, so he will have to travel through some intermediate destinations. The network below shows the cost (in hundreds of dollars) and travel time required for each potential leg of his trip.




He would like to get to Chicago as quickly as possible on a $350 budget. He asks you for your help.

1. What is the shortest path from Annapolis to Chicago in terms of travel time? Is this path feasible?

$1 \to 3 \to 6$

1hr 2hr

$30 $400

Total time: 3hr

Total cost: $430

Not feasible

2. What is the shortest path from Annapolis to Chicago in terms of total cost? Is this the shortest path within his budget?

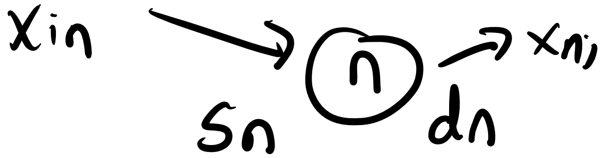
$1 \to 2 \to 4 \to 6$

$25 $100 $85

0.75 4 1.5

$210

6.25 hours




3. Using the variables $x_{i,j} = 1$ if edge $(i, j)$ is included in a path, formulate a concrete model that would allow John to minimize his total travel time within his \$350 budget.

Obj: $\min \; 0.75x_{12} + 1x_{13} + \cdots + 1 \cdot x_{56}$

Constraints: $0.25x_{12} + 0.3x_{13} + \cdots + 0.5x_{56} \leq 3.5$ (budget)

$1 = x_{12} + x_{13}$ (node 1)

$x_{12} = x_{24} + x_{25} + x_{26}$ (node 2)

$\vdots$

$x_{26} + x_{36} + x_{46} + x_{56} = 1$ (node 6)

$x_{12}, x_{13}, \ldots, x_{56} \in \{0,1\}$

4. Using the usual sets $N$ as the nodes and $E$ as the edges, convert your concrete model into a parameterized model.

Params: $t_{ij}$ is the time along edge $(i,j)$ $\forall$ $ij \in E$
$c_{ij}$ is the cost along edge $(i,j)$ $\forall$ $ij \in E$
$s_n$ is the supply of node $n$ $\forall$ $n \in N$
$d_n$ is the demand of node $n$ $\forall$ $n \in N$

Obj: $\min \sum_{(i,j) \in E} t_{ij} x_{ij}$

Constraints: $\sum_{(i,j) \in E} c_{ij} x_{ij} \leq 3.5$

$$\sum_{(i,n) \in E} x_{in} + s_n = \sum_{(n,j) \in E} x_{nj} + d_n \quad \forall \; n \in N$$

$x_{ij} \in \{0,1\} \quad \forall \; ij \in E$

5. In your model, which of your constraints are "easy" constraints? Why are they easy?

binary, flow balence → only these 2 have normal shortest path

6. In your model, which of your constraints are "hard" constraints? Why are they hard?

budget → doesn't allow you to solve with normal shortest path

## 3 Lagrangian Relaxation Idea

A Lagrangian Relaxation is a common method of finding bounds (and sometimes feasible solutions) to solve hard IP problems. They are similar to LP relaxations we use in Branch and Bound.

Recall that to **relax** a constraint we: remove it from feasible region

In an LP relaxation, the integrality constraints of the IP are relaxed.

In a Lagrangian relaxation, the hard constraints of the IP are relaxed.

Lagrangian relaxation: For an IP of the form:

$$\begin{aligned} \min \quad & cx \\ \text{st} \quad & Ax \leq b \\ & x \in \mathbb{X} \end{aligned}$$

Where $x \in \mathbb{X}$ are easy constraints and $Ax \leq b$ are hard constraints

1. Select a penalty $\lambda \geq 0$ to "pay" if the hard constraints are violated.
2. Relax the constraints from the feasible region and instead put them in the objective function multiplied by the penalty. That is, your objective function becomes:

$$\min cx + \lambda(Ax - b)$$

**Problem 2.** Write the Lagrangian Relaxation of your parameterized IP problem for any value of $\lambda$.

$$\min \sum_{(i,j)\in E} t_{ij} x_{ij} + \lambda\left(\sum_{(i,j)\in E} c_{ij} x_{ij} - 3.5\right)$$

$$\text{s.t.} \quad \sum_{(i,n)\in E} x_{in} + s_n = \sum_{(n,j)\in E} x_{nj} + d_n \quad \forall n \in N$$

$$x_{ij} \in \{0,1\} \quad \forall (i,j) \in E$$

$$\lambda \geq 0$$

## 4 Properties of Lagrangian Relaxation

Let $z^*(\lambda)$ be the optimal objective function value of the Lagrangian Relaxation. Let's think about what happens for various values of $\lambda$:

- If $\lambda = 0$ what solution is found?

min $\sum t_{ij} x_{ij} + \lambda ( \sum c_{ij} x_{ij} - 3.5)$

flow balance

binary

$\lambda = 0 \to$ normal shortest path

1) Shortest path is within budget so optimal for IP

$z^*(\lambda) = z^*_{IP}$

2) Shortest path not within budget

$z^*(\lambda) < z^*_{IP}$

Then, $z^*(\lambda) \boxed{\leq} z^*_{IP}$

- As $\lambda$ increases from 0, what happens?

min $\sum t_{ij} x_{ij} + \lambda ( \sum c_{ij} x_{ij} - 3.5)$

If $x$ is feasible to IP then its within the budget

so $\sum c_{ij} x_{ij} - 3.5 \leq 0$

since $\lambda > 0$

subtracting from obj

so its smaller.

Thus, $z^*(\lambda) \boxed{\leq} z^*_{IP}$ for **any value of** $\lambda$

This is the key property of Lagrangian relaxation!

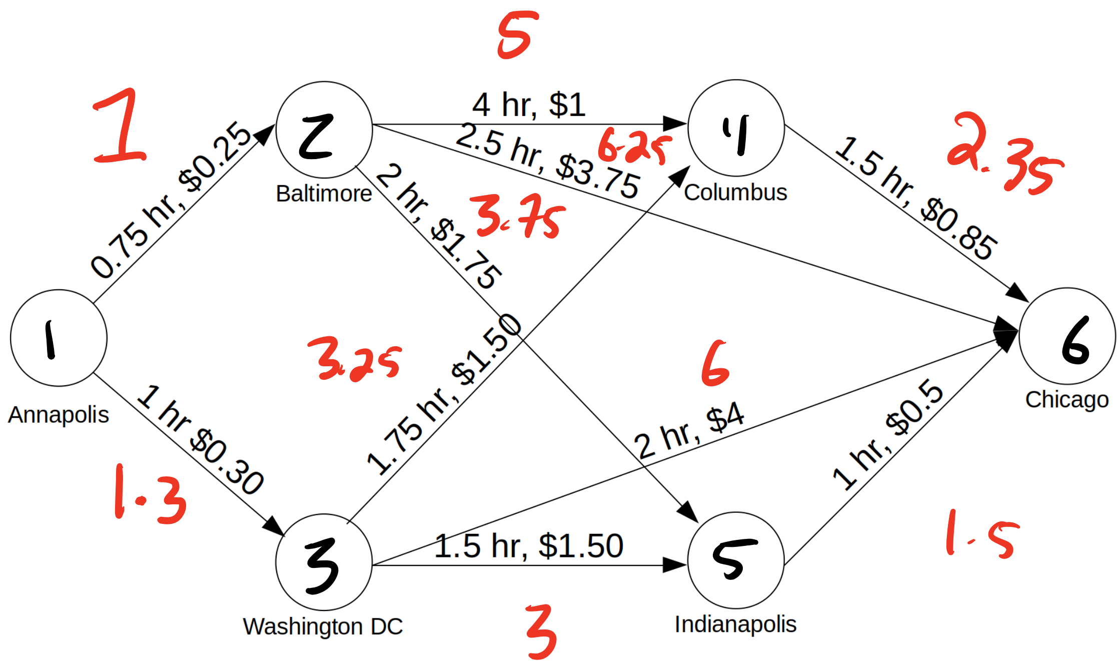
> In general, for a maximization problem:
>
> $$z^*(\lambda) \geq z^*_{IP}$$
>
> For a minimization problem:
>
> $$z^*(\lambda) \leq z^*_{IP}$$
>
> For all values of $\lambda \geq 0$

Set up objective so this is true

$Z_{LR} = \sum t_{ij} x_{ij} + \lambda(\sum c_{ij} x_{ij} - 3.5)$

$\lambda = 1: \sum (t_{ij} + c_{ij}) x_{ij} - 3.5$

**Problem 3.** Let's see an example, suppose $\lambda = 1$.




1. What is the shortest path in this network?

$1 \to 3 \to 5 \to 6$ Total cost: 5.8

1.3 3 1.5

2. Is this solution feasible to the original problem?

Yes cost is \$2.30

3. What is the objective function value of this solution in the original problem?

Time is $1 + 1.5 + 1 = 3.5$

$Z_{IP} = 3.5$

4. What is the objective function value of this solution in the Lagrangian Relaxation?

Total cost $-3.5 \Rightarrow 5.8 - 3.5 = 2.3$

$Z_{LR} = 2.3$

If $\min Z_{LR} \leq Z_{IP}$ for all $\lambda \geq 0$

Goal: Find best $\lambda$

Since the Lagrangian relaxation is an ~~upper~~ lower bound for any value of $\lambda$, a related problem is to find the optimal $\lambda$ to find the best bound. In this case we should be trying to find as $\boxed{\text{tight}}$ of a bound as possible.

Mathematically, this is called the **Lagrangian Dual** problem.

$$\begin{aligned} \min\ & cx \\ & Ax \leq b \\ & x \in X \end{aligned} \longrightarrow \begin{aligned} \min\ & cx + \lambda(Ax - b) \\ & x \in X \end{aligned}$$

$$\underbrace{cx + \lambda(Ax-b)}_{\text{large as possible}} \leq cx$$

$$LD: \quad \max\ cx + \lambda(Ax-b)$$
$$\lambda \geq 0$$

## 5 Other Considerations

- There are equivalent cases of Lagrangian relaxation for equality and $\geq$ constraints; usually all that changes is a sign in the objective function.
- There are cases where the Lagrangian Relaxation provides an exact bound to the optimal solution of the IP. This makes it a very powerful tool.
- Some problems are so hard to solve; the best we can do it compute bounds on the optimal solution. Lagrangian Relaxation is a good tool for this.

# LR for other types of IPs

$$\begin{aligned} &\max\ cx \\ &Ax \le b \\ &x \in X \end{aligned} \longrightarrow \begin{aligned} &\max\ cx + \lambda(b - Ax) \\ &\quad\text{OR} \\ &\max\ cx - \lambda(Ax - b) \end{aligned}$$

Goal: $Z_{LR} \geq Z_{IP}$