Practice Questions for Exam 1

Name: Solution

- This practice exam consists of 6 problems. **Please note that clearly this is much longer than the actual exam 1 so that you can have more practice problems**
- No credit will be given to unjustified answers.
- Please read the entire exam carefully. Good Luck!

1. Walmart's shipping network in our area has 3 warehouses located in Baltimore, Laurel, and DC. Each warehouse has a capacity of goods it produces and each city it ships to has a demand. Walmart must ship goods throughout the area for the costs illustrated in the below table.

	Destinations				
Warehouse	Annapolis	Fort Meade	Ocean City	Arlington	Capacity
Baltimore	5	2	9	12	30000
Laurel	8	4	7	9	35000
DC	7	5	12	5	20000
Demand	10000	15000	25000	25000	

a. Draw the network diagram associated with this problem.

Skipped. You must add a dummy demand node!!!

b. What type of network model is this?

This is a transportation problem

c. Formulate a concrete model that will allow Walmart to ship its goods at minimum cost.

Decision Variables

Let $x_{B,A}$ be the flow from Baltimore to Annapolis

Let $x_{B,F}$ be the flow from Baltimore to Fort Meade

$\vdots$

Let $x_{D,Dum}$ be the flow from DC to Dummy Node

Parameters

None

Objective Function

$$\text{minimize cost: } 5x_{B,A} + 2x_{B,A} + \cdots + 5x_{D,A}$$

Constraints

$$\begin{aligned}
x_{B,A} + x_{B,F} + x_{B,O} + x_{B,Ar} + x_{B,Dum} &= 30000 && \text{(Baltimore Capacity} \\
x_{L,A} + x_{L,F} + x_{L,O} + x_{L,Ar} + x_{L,Dum} &= 35000 && \text{(Laurel Capacity} \\
x_{D,A} + x_{D,F} + x_{D,O} + x_{D,Ar} + x_{D,Dum} &= 20000 && \text{(DC Capacity} \\
x_{B,A} + x_{L,A} + x_{D,A} &= 10000 && \text{(Annapolis Demand)} \\
x_{B,F} + x_{L,F} + x_{D,F} &= 15000 && \text{(Fort Meade Demand)} \\
x_{B,O} + x_{L,O} + x_{D,O} &= 25000 && \text{(Ocean City Demand)} \\
x_{B,O} + x_{L,O} + x_{D,O} &= 25000 && \text{(Arlington Demand)} \\
x_{B,Dum} + x_{L,Dum} + x_{D,Dum} &= 10000 && \text{(Dummy Demand)} \\
x_{B,A}, x_{B,F}, \ldots, x_{D,Dum} &\geq 0
\end{aligned}$$

d. Formulate an parameterized model that will allow Walmart to ship its goods at minimum cost. Clearly label all sets, parameters and variables used.

Sets
Let N be the set of nodes
Let E be the set of edges

Decision Variables
Let $x_{i,j}$ be the flow along edge (i,j) for $(i,j) \in E$

Parameters
Let $c_{i,j}$ be the cost of shipping along edge (i,j) for all $(i,j) \in E$
Let b_n be the supply-demand at node n for all $n \in N$

Objective Function

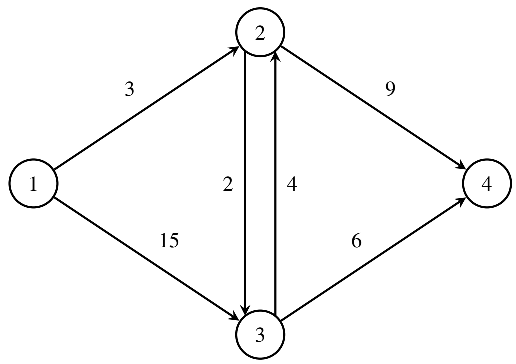
$$\text{minimize cost:} \sum_{(i,j)\in E} c_{i,j}x_{i,j}$$

Constraints

$$\begin{aligned} \sum_{(i,r)\in E} x_{i,r} - \sum_{(r,j)\in E} x_{r,j} &= b_n \quad \text{for all } r \in N \qquad \text{(Supply/demand)} \\ x_{i,j} &\geq 0 \quad \text{for all } (i,j) \in E \end{aligned}$$

Note: It's equivalent to model this problem with 3 sets: supply nodes, demand nodes, and edges.

2. Consider the directed network shown below, where the numbers on the arcs represent cost, $c_{i,j}$, to send one unit of flow along the arc. Use the start of a formulation given below to answer the following questions.



<u>**Sets**</u>

$N =$ the set of nodes i for all $i \in N$
$E =$ the set of edges (i, j) for all $(i, j) \in E$

<u>**Decision Variables**</u>

Let $x_{i,j}$ be the flow along edge (i, j) for all $(i, j) \in E$

<u>**Parameters**</u>

Let $c_{i,j}$ be the cost to send 1 unit of flow along edge (i, j) for all $(i, j) \in E$

a. We place one unit of supply at node 1, 1 unit of demand at node 4 and zero units of supply at all other nodes. Formulate an objective function to compute the minimum cost strategy for meeting the demand at node 4.

$$\text{minimize: } 3x_{1,2} + 15x_{1,3} + 2x_{2,3} + 4x_{3,2} + 6x_{3,4} + 9x_{2,4}$$

b. What type of network problem is this?

This is a shortest path problem.

c. Write the constraints for this problem using concrete form.

$$\begin{aligned} 1 &= x_{1,2} + x_{1,3} && \text{(node 1)} \\ x_{1,2} + x_{3,2} &= x_{2,3} + x_{2,4} && \text{(node 2)} \\ x_{1,3} + x_{2,3} &= x_{3,2} + x_{3,4} && \text{(node 3)} \\ x_{2,4} + x_{3,4} &= 1 && \text{(node 4)} \\ x_{1,2}, x_{1,3}, \ldots, x_{3,4} &\geq 0 \end{aligned}$$

d. Write the constraints for this problem using parameterized form.

Define an additional parameter b_n which is 1 for node 4, -1 for node 1, and 0 for all other nodes

$$
\begin{aligned}
\sum_{(i,r)\in E} x_{i,r} - \sum_{(r,j)\in E} x_{r,j} &= b_n \quad \text{for all } r \in N \qquad \text{(Flow in = flow out)} \\
x_{i,j} &\geq 0 \quad \text{for all } (i,j) \in E
\end{aligned}
$$

3. You have been asked to assign 1000 plebes to 50 different Calculus 1 sections. This problem can be modeled as a network. Specifically, the network has 1000 supply nodes and 50 demand nodes. Each student ranks their section choices from 1 to 50. For example, a plebe would be happiest with choice 1 and most unhappy with choice 50. Each section must have exactly 20 plebes assigned.

Sets

Let P be the set of plebes
Let S be the set of calculus sections

a. Using these sets, formulate the parameterized model associated with this problem.

Decision Variables

Let $x_{i,j} = 1$ if plebe i is assigned to class j for $i \in P$ and $j \in S$

Parameters

Let $h_{i,j}$ be the happiness score that plebe i assigns to section j (remember that a small happiness is good)
Let s_i be the supply of node i for all $i \in P$ ($s_i = 1 \forall i$)
Let d_j be the demand of node j for all $j \in S$ ($d_j = 20 \forall j$)

Objective Function

$$\text{minimize: } \sum_{i \in P, j \in S} h_{i,j} x_{i,j}$$

Constraints

$$\begin{aligned}
\sum_{j \in S} x_{i,j} &= s_i && \text{(for all } i \in P\text{)} && \text{(supply of plebe at each node)} \\
\sum_{i \in P} x_{i,j} &= d_j && \text{(for all } j \in S\text{)} && \text{(demand of each class)} \\
x_{i,j} &\in \{0,1\} && \text{for all } i \in P, j \in S
\end{aligned}$$

Note, another (better) formulation is to define the set E and keep $x_{i,j}$ to be continuous. This formulation is here to give you another idea.

4. A clothing company manufactures shirts, shorts, and pants. Each shirt takes 3 hours of labor and 4 yards of cloth, each pair of shorts takes 2 hours of labor and 3 units of cloth, and each pair of pants takes 6 hours of labor and 4 units of cloth. Each week the company has 150 hours of labor and 160 units of cloth. The profit of selling a shirt is \$12, shorts is \$8, and pants are \$15. They formulate the following concrete model to maximize their profits:

Decision Variables

Let x_1 be the number of shirts made each week
Let x_2 be the number of shorts made each week
Let x_3 be the number of pants made each week

Parameters

None

Objective Function

$$\text{maximize profit: } 12x_1 + 8x_2 + 15x_3$$

Constraints

$$\begin{aligned} \text{subject to:} \quad 3x_1 + 2x_2 + 6x_3 &\leq 150 \qquad \text{(Labor)} \\ 4x_1 + 3x_2 + 4x_3 &\leq 160 \qquad \text{(Cloth)} \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

a. Convert their model to parameterized form. *Use only one parameter of the form $a_{i,j}$ for the left hand side of the constraints*

Sets

Let I be the types of item of clothing to be made $I = \{$ shirts, shorts, pants $\}$
Let R be the type of restriction $R = \{$ labor, cloth $\}$

Decision Variables

Let x_i be the number of clothing item i made each week for all $i \in I$

Parameters

Let $a_{i,r}$ be the amount of resource r required to produce clothing item i
Let u_r be the upper bound of resource r
Let p_i be the profit of clothing item i

Objective Function

$$\text{maximize profit: } \sum_{i \in I} p_i x_i$$

Constraints

$$\text{subject to:} \quad \sum_{i \in I} a_{i,r} x_i \leq u_r \quad \text{for } r \in R \quad \text{(Resources)}$$
$$x_i \geq 0 \quad \text{for } i \in I$$

b. The suppliers for the clothing company have changed their method of charging. Specifically, if the company wants shirts, they must order at least 10 shirts from the supplier and pay a fixed charge of \$50. If they want shorts, they must order at least 8 pairs of shorts from the supplier and pay a fixed charge of \$60. If they want pants, they must order at least 15 pairs from the supplier and pay a fixed charge of \$100.

(a) What new variable(s) would you need to introduce to model this new charging scheme? Let $y_i = 1$ if they decide to produce item i for all $i \in I$

(b) Modify your objective function so that it reflects these new fixed charges.
Define a parameter f_i as the fixed charge of producing item i. Objective function becomes:

$$\text{maximize profit: } \sum_{i \in I} (p_i x_i - f_i y_i)$$

(c) Add a constraint to the concrete model which says if they order shirts, they must order at least 10 of them from the supplier. We would add the following constraints to the model:

$$x_1 \geq 10 y_1$$

$$x_1 \leq M y_1$$

5. The Naval academy must purchase 1100 computers and St Johns must purchase 600 computers from three vendors. Vendor 1 charges \$500 per computer plus a delivery charge of \$5000. Vendor 2 charges \$350 per computer plus a delivery charge of \$4000. Vendor 3 charges \$250 per computer plus a delivery charge of \$6000. Vendor 1 can sell at most 800 computers, vendor 2 can sell at most 700 computers, and vendor 3 can sell at most 1000 computers.

a. Formulate a concrete model which allows both the Naval Academy and St Johns to purchase computers at minimum cost.

This problem is much trickier than I intended when I wrote it! If you got it right then that's great

Decision Variables

Let $x_{N,1}$ be the number of computers USNA purchases from Vendor 1
Let $x_{N,2}$ be the number of computers USNA purchases from Vendor 2
Let $x_{N,3}$ be the number of computers USNA purchases from Vendor 3
Let $x_{S,1}$ be the number of computers St Johns purchases from Vendor 1
Let $x_{S,2}$ be the number of computers St Johns purchases from Vendor 2
Let $x_{S,3}$ be the number of computers St Johns purchases from Vendor 3
Let $y_{N,1} = 1$ if USNA purchases from Vendor 1 and 0 otherwise
Let $y_{N,2} = 1$ if USNA purchases from Vendor 2 and 0 otherwise
Let $y_{N,3} = 1$ if USNA purchases from Vendor 3 and 0 otherwise
Let $y_{S,1} = 1$ if St Johns purchases from Vendor 1 and 0 otherwise
Let $y_{S,2} = 1$ if St Johns purchases from Vendor 2 and 0 otherwise
Let $y_{S,3} = 1$ if St Johns purchases from Vendor 3 and 0 otherwise

Parameters

None

Objective Function

minimize cost: $500x_{N,1}+500x_{S,1}+350x_{N,2}+350x_{S,2}+250x_{N,3}+250x_{S,3}+5000y_{N,1}+5000y_{S,1}+4000y_{N,2}$

$$+4000y_{S,2}+6000y_{N,3}+6000y_{S,3}$$

Constraints

$$
\begin{aligned}
\text{subject to:}\quad x_{N,1}+x_{N,2}+x_{N,3} &= 1100 && \text{(USNA Demand)}\\
x_{S,1}+x_{S,2}+x_{S,3} &= 600 && \text{(St Johns Demand)}\\
x_{N,1} &\leq 800y_{N,1} && \text{(Strong forcing)}\\
x_{N,2} &\leq 700y_{N,2} && \text{(Strong forcing)}\\
x_{N,3} &\leq 1000y_{N,3} && \text{(Strong forcing)}\\
x_{S,1} &\leq 800y_{S,1} && \text{(Strong forcing)}\\
x_{S,2} &\leq 700y_{S,2} && \text{(Strong forcing)}\\
x_{S,3} &\leq 1000y_{S,3} && \text{(Strong forcing)}\\
x_{N,1}+x_{S,1} &\leq 800 && \text{(Supply of 1)}\\
x_{N,2}+x_{S,2} &\leq 700 && \text{(Supply of 2)}\\
x_{N,3}+x_{S,3} &\leq 1000 && \text{(Supply of 3)}\\
x_{N,1}\dots x_{S,3} &\geq 0\\
y_{N,1}\dots y_{S,3} &\in \{0,1\}
\end{aligned}
$$

b. Formulate an parameterized model which allows both the Naval Academy and St Johns to purchase computers at minimum cost.

I will formulate it as a variant of min cost network flow. There will be 5 total nodes, one for each USNA and St Johns and one for each vendor. Again, this is harder than I expected when I first wrote it.

Sets

Let N be the set of nodes
Let E be the set of edges
Let I be the set of customers
Let J be the set of vendors

Decision Variables

Let $x_{i,j}$ be the number of computers shipped along edge (i,j) for all $(i,j)\in E$
Let $y_{i,j}=1$ if edge (i,j) is used and 0 otherwise

Parameters

Let s_j be the supply of vendor j
Let d_i be the demand of customer i
Let $c_{i,j}$ be the cost of shipping along edge (i,j) for all $(i,j)\in E$
Let f_j be the fixed cost of purchasing from vendor j

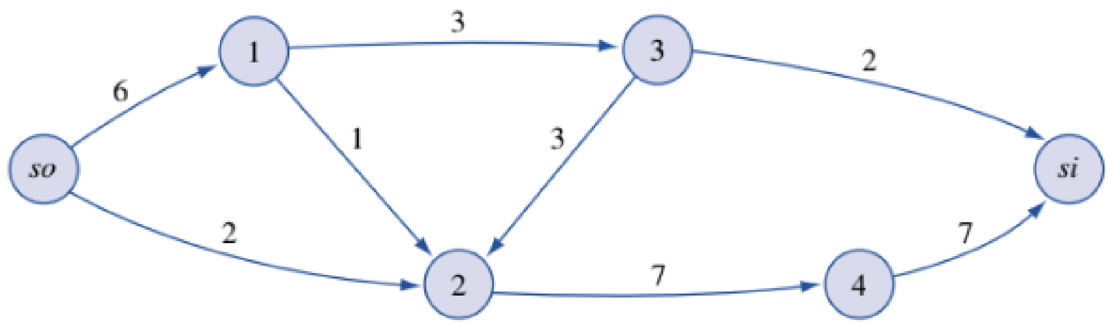
Objective Function

$$\text{minimize cost:}\quad \sum_{(i,j)\in E} c_{i,j}x_{i,j} + \sum_{(i,j)\in E:\,j\in J} f_j y_{i,j}$$

Constraints

$$
\begin{aligned}
\text{subject to:} \quad & \sum_{(i,j)\in E} x_{i,j} = d_i && \text{for all } i \in I \\
& \sum_{(i,j)\in E} x_{i,j} \le s_j && \text{for all } j \in J \\
& x_{i,j} \le s_j y_{i,j} && \text{for all } i \in I \text{ and } j \in J \\
& x_{i,j} \ge 0 && \text{for all } (i,j) \in E \\
& y_{i,j} \in \{0,1\} && \text{for all } (i,j) \in E
\end{aligned}
$$

6. Consider the following network. Note that so is the source node and si is the sink node:



An OR student writes the following model:

Sets

Let N be the set of nodes, $n \in N$
Let E be the set of edges, $(i, j) \in E$

Decision Variables

Let $x_{i,j}$ be the flow along edge (i, j) for all $(i, j) \in E$

Parameters

Let $c_{i,j}$ be the cost of edge (i, j)
Let b_i be the supply/demand at node i. Specifically, at node so, $b_{so} = -1$ at nodes 1, 2, 3, and 4, $b_i = 0$ and at node si, $b_{si} = 1$.

Objective Function

$$\text{minimize cost: } \sum_{(i,j)\in E} c_{i,j} x_{i,j}$$

Constraints

$$\begin{aligned} \text{subject to: } \quad \sum_{(i,n)\in E} x_{i,n} - \sum_{(n,j)\in E} x_{n,j} &= b_n && \text{for } n \in N \\ x_{i,j} &\geq 0 && \text{for all } (i,j) \in E \end{aligned}$$

a. What type of network problem is this?

This is a shortest path problem

b. Why is the supply at node so 1? The supply at node so is 1 because we are looking for a shortest path from source to sink. By putting a supply of 1 at node so and a demand of 1 and node si, we are sending 1 unit at minimum cost from so to si. Since this unit is sent at minimum cost from so to si it must be the shortest path from so to si.

c. Using the given network, transform this parameterized model into a concrete model.

Decision Variables

Let $x_{so,1}$ be the flow along edge $(so,1)$
Let $x_{so,2}$ be the flow along edge $(so,2)$
Let $x_{1,2}$ be the flow along edge $(1,2)$
Let $x_{1,3}$ be the flow along edge $(1,3)$
Let $x_{2,4}$ be the flow along edge $(2,4)$
Let $x_{3,2}$ be the flow along edge $(3,2)$
Let $x_{3,si}$ be the flow along edge $(3,si)$
Let $x_{4,si}$ be the flow along edge $(4,si)$

Parameters

None

Objective Function

$$\text{minimize cost: } 6x_{so,1} + 2x_{so,2} + \ldots + 7x_{4,si}$$

Constraints

$$\begin{aligned}
\text{subject to:} \qquad 1 &= x_{so,1} + x_{so,2} && \text{(Flow in = flow out source)} \\
x_{so,1} &= x_{1,2} + x_{1,3} && \text{(Flow in = flow out node 1)} \\
x_{so,2} + x_{1,2} + x_{3,2} &= x_{2,4} && \text{(Flow in = flow out node 2)} \\
x_{1,3} &= x_{3,2} + x_{3,si} && \text{(Flow in = flow out node 3)} \\
x_{2,4} &= x_{4,si} && \text{(Flow in = flow out node 4)} \\
x_{3,si} + x_{4,si} &= 1 && \text{(Flow in = flow out sink)} \\
x_{so,1} \ldots x_{4,si} &\geq 0
\end{aligned}$$