Lesson 13. IP Formulations Part 2

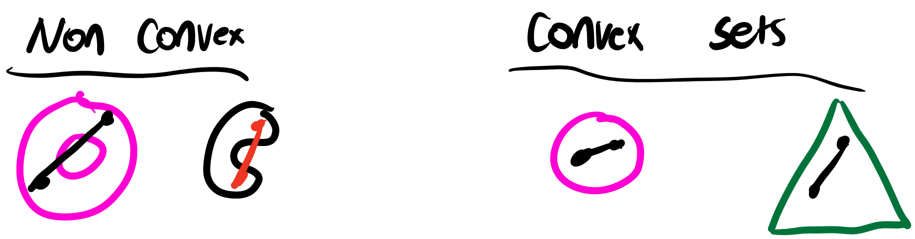
1 Convex Hull Formulation

1.1 Convex Set Review

Recall that a set is **convex** if, for any two points x and y in that set:

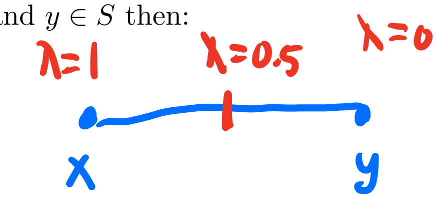
the line connecting them is also in that set.

Problem 1. Draw an example of a convex set and a set that is not convex.

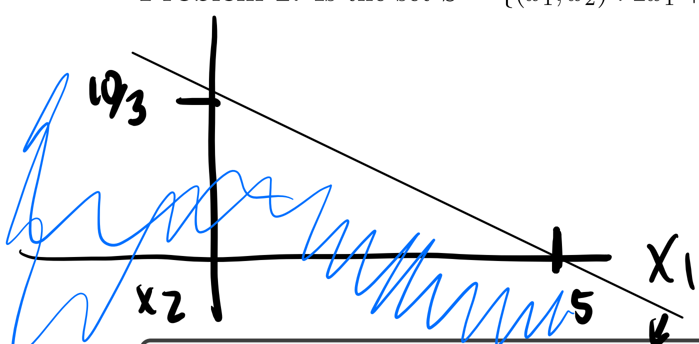


Mathematically, a set S is convex if, for any two points $x \in S$ and $y \in S$ then:

$\lambda x + (1-\lambda) y \in S \quad \forall \lambda \in [0,1]$



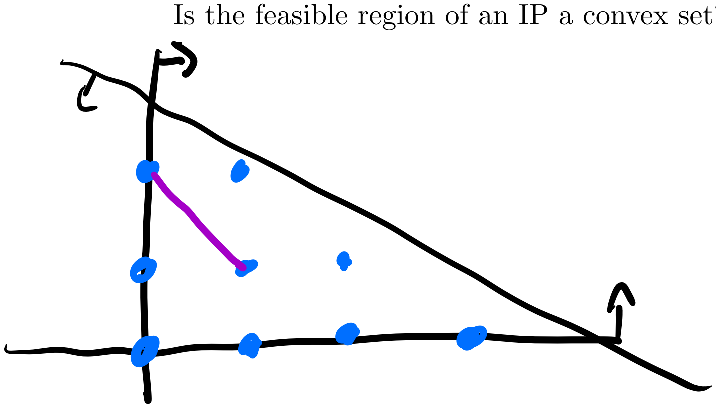
Problem 2. Is the set $S = \{(x_1, x_2) : 2x_1 + 3x_2 \leq 10\}$ convex?



Yes, any line connecting 2 points stays in shaded region.

Remember that the feasible region of an LP is a convex set.

Is the feasible region of an IP a convex set?



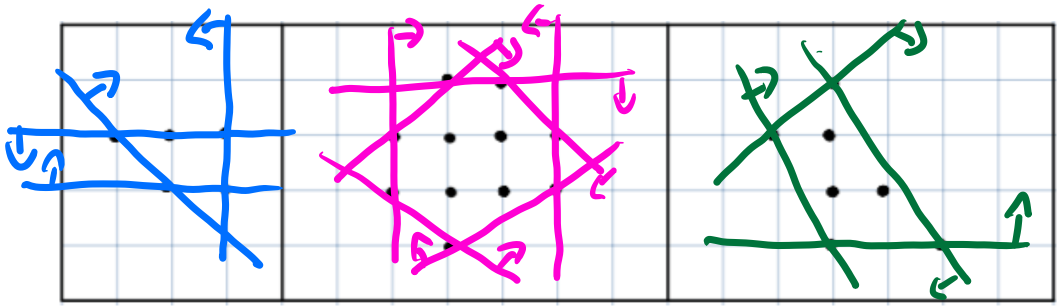
No, feasible region is integers only, any line connecting two points has fractions so not feasible.

Note: Non-convex optimization is hard

Convex hull is the smallest possible LP formulation

The **convex hull** of a set of integer feasible solutions is the **smallest convex set** that contains all of the points.

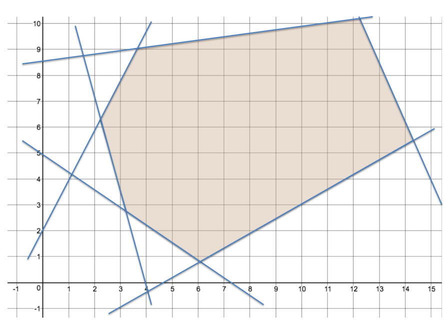
Problem 3. Given the following sets of integer points, sketch a convex hull formulation of these points.



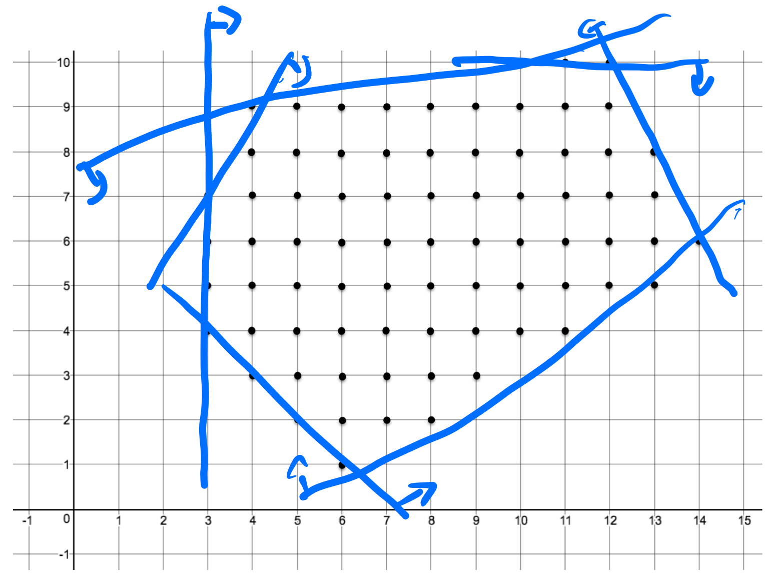
1) All corner points integer

2) Lots of constraints

Problem 4. A formulation for a set of feasible integer solutions is pictured on the left. The integer solutions are highlighted on the right. Sketch the **convex hull formulation** of this set of solutions.



6 constraints



8 constraints

The **convex hull formulation** of a finite set of integer feasible solutions is considered to be the **"ideal"** formulation.

Why? Formulating convex hull as LP gives integer corner points. Solving with simplex gives optimal IP solution.

have n variables takes about 2^n constraints

HOWEVER, for most problems <u>we can't use the ideal, convex hull formulation</u> because the number of $\boxed{\text{Constraints}}$ required to describe the convex hull is often very, very $\boxed{\text{large}}$, i.e., *exponential* in the number of variables.

2 Comparing Formulations

When choosing which constraints to include in an IP formulation, there is a **tradeoff**:

- use **enough** constraints to make a reasonably tight "container" for the feasible points,
- but **few enough** constraints so the resulting problem is of manageable size.

Add all Constraints LP relaxation would be very hard to solve

One strategy is to iteratively add constraints as we need them, to $\boxed{\text{remove}}$ *fractional* solutions obtained by solving LP $\boxed{\text{relaxations}}$. We discussed this **separation** strategy in the context of both the

- $\boxed{\text{traveling}}$ $\boxed{\text{Salesperson}}$ problem, and
- $\boxed{\text{Vehicle}}$ $\boxed{\text{Routing}}$ problems.

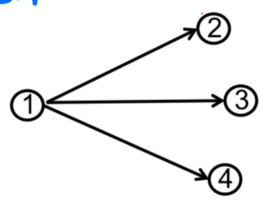
2.1 Example: Fixed-Charge Weak Vs. Strong Formulations

Many common IP problems have been studied extensively to determine effective modeling strategies. One such problem type is the **fixed-charge facility location problem** that we modeled earlier in the semester.

Problem 5. Suppose there is a possible warehouse at location 1 with maximum capacity C_1, and customers at locations 2, 3, and 4. The binary variable z_1 indicates whether or not facility 1 is used. Integer variables x_{12}, x_{13}, and x_{14} represent the amount of flow on the edges leaving facility 1.

$z_1 = 1$ if facility 1 is open

x_{12}, x_{13}, x_{14} flow from 1 to 2,3,4.



We saw two different ways to enforce the requirement that if facility 1 is closed, there is no flow out of facility 1.

Weak formulation:

$x_{12} + x_{13} + x_{14} \le C \cdot z_1$

1 constraint per facility

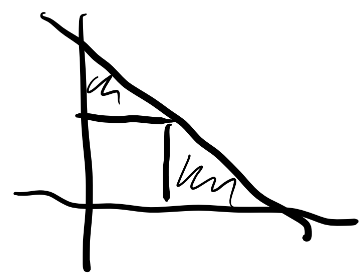
Strong formulation:

$x_{12} \le C \cdot z_1$

$x_{13} \le C \cdot z_1$

$x_{14} \le C \cdot z_1$

1 constraint per edge



Why is the formulation on the left referred to as weak while the one on the right is strong?

Feasible region of weak constraints is larger than strong constraints. Using strong constraints gives easier to solve IP. But need more constraints for strong formulation.

To summarize, finding the convex hull of an integer program is the gold standard of IP formulations. That said, there are several issues with this:

1. Exponential number of constraints] n variables $\approx 2^n$ constraints
2. Potential numerical issues with tons of constraints] larger problems are harder to solve

In general, we do not look for the convex hull. We do, however, use this idea to generate **cuts** when solving IPs.

Lesson 14: Add constraints as necessary to remove "bad solutions"

3 Bounds for IPs

In the next few classes, we will look at "branch-and-bound", the algorithmic framework that most MIP (mixed-integer *linear* programming) solvers use. A critical component of this algorithm is producing bounds on the integer optimal solution.

x_1, x_2 integer

3.1 Upper and lower bounds for IPs

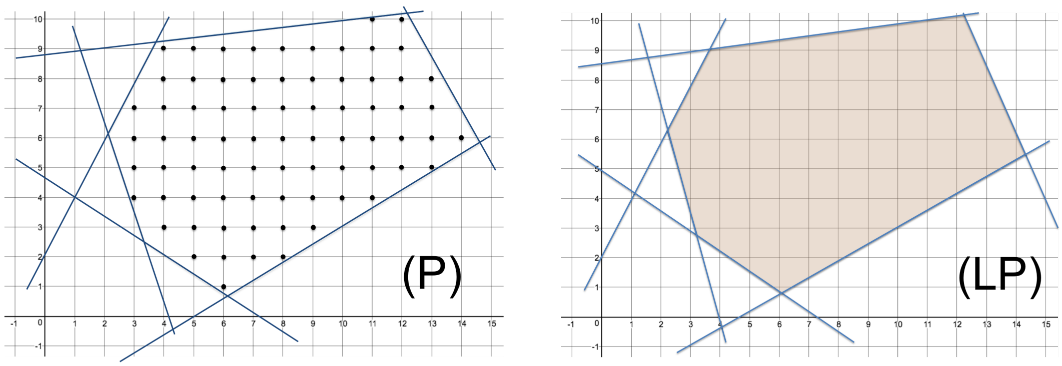
Problem 6. Suppose (P) is an IP with a maximizing objective function,

c_1, c_2 integer

$$\text{maximize } f(\mathbf{x}) = c_1x_1 + c_2x_2,$$

where c_1 and c_2 are integers. The feasible regions of (P) and (LP), the LP relaxation of (P), are pictured below.

IP feasible Region LP Relaxation



Let z^* be the optimal objective value of (P), which we want to find upper and lower bounds for as part of the branch and bound algorithm.

z^*_{IP}

(a) Suppose we solve the LP relaxation (LP) and get an optimal objective value of 83.9. What can we say about z^* relative to 83.9? Explain.

Lesson 12: $z^*_{IP} \le z^*_{LP}$ $\rightarrow$ $z^*_{IP} \le 83.9$

$z^*_{LP} = 83.9$ $\nearrow$

(b) We already stated that the c_1 and c_2 are integers. What does that tell us about z^*? Explain. *Hint: Suppose (x_1^*, x_2^*) is an optimal solution to (P). What do we know about x_1^* and x_2^*?*

$$z^*_{IP} = \underbrace{c_1x_1^* + c_2x_2^*}_{\text{Integers} \rightarrow \text{can't add to get } 83.9} \le 83.9$$

(c) Combining parts (a) and (b), find a better bound for z^*. Explain.

$z^*_{IP} \le 83$ → Get rid of decimals when integer objective functions

(d) Now suppose that $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)$, is some *feasible* solution to (P) (not necessarily optimal). What can we say about $f(\hat{x}) = c_1\hat{x}_1 + c_2\hat{x}_2$ relative to z^*? Explain.

$\hat{x}$ is feasible to IP with objective value $\hat{z}$

1) $\hat{x}$ optimal → $\hat{z} = z^*_{IP}$

2) $\hat{x}$ not optimal → $\hat{z} < z^*_{IP}$

→ $\hat{z} \le z^*_{IP}$

3.2 Better formulation leads to better (LP) bounds

The quality of the bound obtained by solving the LP relaxation depends on the formulation:

A **tighter** formulation provides a $\boxed{\text{better}}$ bound via its LP relaxation.

3.3 Summary of IP bounds

If (P) is a **maximizing** IP with integer objective coefficients and optimal objective value z^*,

- If z^*_{LP} is the optimal objective value to the LP relaxation of (P), then $\boxed{z^*_{LP}}$ is a/an $\boxed{\text{upper}}$ bound on z^*. z^*_{IP}
- The objective value for *any feasible* solution to (P) provides a/an $\boxed{\text{lower}}$ bound on z^*.

If (P) is a **minimizing** IP with integer objective coefficients and optimal objective value z^*,

- If z^*_{LP} is the optimal objective value to the LP relaxation of (P), then $\boxed{z^*_{LP}}$ is a/an $\boxed{\text{lower}}$ bound on z^*.
- The objective value for *any feasible* solution to (P) provides a/an $\boxed{\text{upper}}$ bound on z^*.

IP optimal Solution z^*_{IP}

LP Relaxation z^*_{LP}

Any Feasible Point $\hat{z}$

Max: $\hat{z} \le z^*_{IP} \le z^*_{LP}$

Min $\hat{z} \ge z^*_{IP} \ge z^*_{LP}$