

Lesson 15. Lagrangian Relaxation

1 Motivation

When talking about branch and bound, we talked about how LP relaxations can be used to give upper bounds on integer programs. Recall that, if z_{IP}^* is the optimal IP solution and z_{LP}^* is the optimal LP solution then:

$$\begin{array}{ccc} z_{IP}^* & \boxed{} & z_{LP}^* \text{ if maximizing} \\ z_{IP}^* & \boxed{} & z_{LP}^* \text{ if minimizing} \end{array}$$

Many LP and IP problems have a special structure:

$$\begin{array}{ll} \max & cx \\ \text{st} & Ax = b \\ & x \in \mathbb{X} \end{array}$$

The constraints $x \in \mathbb{X}$ are “easy” constraints such as:

The constraints $Ax = b$ are “difficult” constraints such as:

Lagrangian Relaxation is a method to obtain bounds on integer programs which have this structure. The basic idea of Lagrangian Relaxation is:

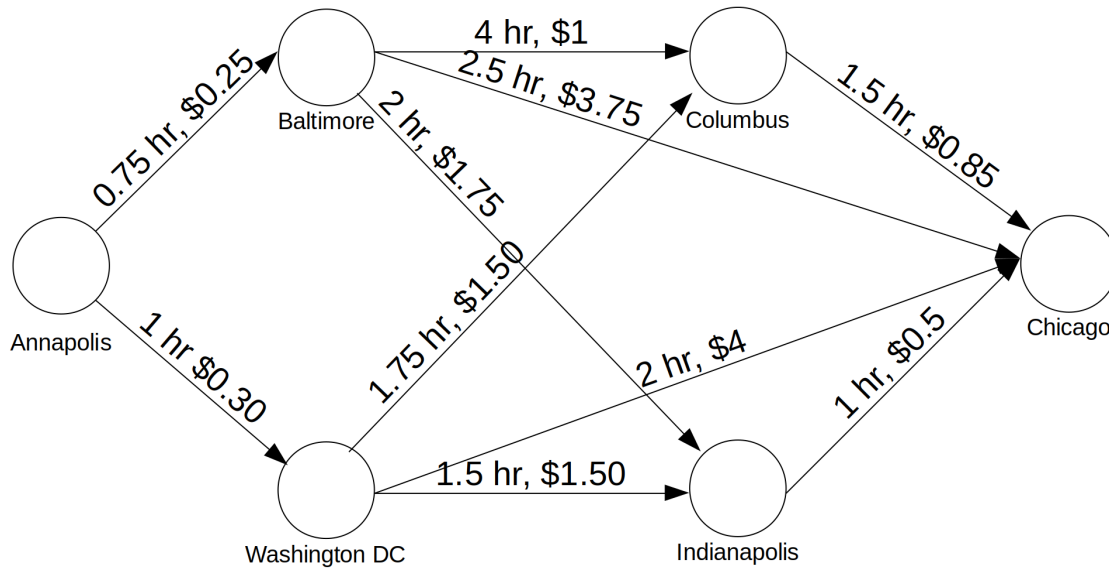
- Remove “hard” constraints from the problem and place them in the objective function (i.e., relax the IP)
- Solve the relaxed problem that now consists of only “easy” constraints.

Doing so gives us a bound on the optimal solution.

In many cases, this bound is close to the optimal solution of the original problem.

2 Shortest Path Problem with Time Constraints

Problem 1. John is trying to travel from Annapolis to Chicago as cheaply as possible by taking public transportation. Unfortunately, there is no direct route between the two cities, so he will have to travel through some intermediate destinations. The network below shows the cost (in hundreds of dollars) and travel time required for each potential leg of his trip.



He would like to get to Chicago as quickly as possible on a \$350 budget. He asks you for your help.

1. What is the shortest path from Annapolis to Chicago in terms of travel time? Is this path feasible?
2. What is the shortest path from Annapolis to Chicago in terms of total cost? Is this the shortest path within his budget?

3. Using the variables $x_{i,j} = 1$ if edge (i,j) is included in a path, formulate a concrete model that would allow John to minimize his total travel time within his \$350 budget.

4. Using the usual sets N as the nodes and E as the edges, convert your concrete model into a parameterized model.

5. In your model, which of your constraints are “easy” constraints? Why are they easy?

6. In your model, which of your constraints are “hard” constraints? Why are they hard?

3 Lagrangian Relaxation Idea

A Lagrangian Relaxation is a common method of finding bounds (and sometimes feasible solutions) to solve hard IP problems. They are similar to LP relaxations we use in Branch and Bound.

Recall that to **relax** a constraint we:

In an LP relaxation, the constraints of the IP are relaxed.

In a Lagrangian relaxation, the constraints of the IP are relaxed.

Lagrangian relaxation: For an IP of the form:

$$\begin{array}{ll}\min & cx \\ \text{st} & Ax \leq b \\ & x \in \mathbb{X}\end{array}$$

Where $x \in \mathbb{X}$ are easy constraints and $Ax \leq b$ are hard constraints

1. Select a penalty $\lambda \geq 0$ to “pay” if the hard constraints are violated.
2. Relax the constraints from the feasible region and instead put them in the objective function multiplied by the penalty. That is, your objective function becomes:

$$\min cx + \lambda(Ax - b)$$

Problem 2. Write the Lagrangian Relaxation of your parameterized IP problem for any value of λ .

4 Properties of Lagrangian Relaxation

Let $z^*(\lambda)$ be the optimal objective function value of the Lagrangian Relaxation. Let's think about what happens for various values of λ :

- If $\lambda = 0$ what solution is found?

Then, $z^*(\lambda)$ z_{IP}^*

- As λ increases from 0, what happens?

Thus, $z^*(\lambda)$ z_{IP}^* for **any value of λ**

This is the key property of Lagrangian relaxation!

In general, for a maximization problem:

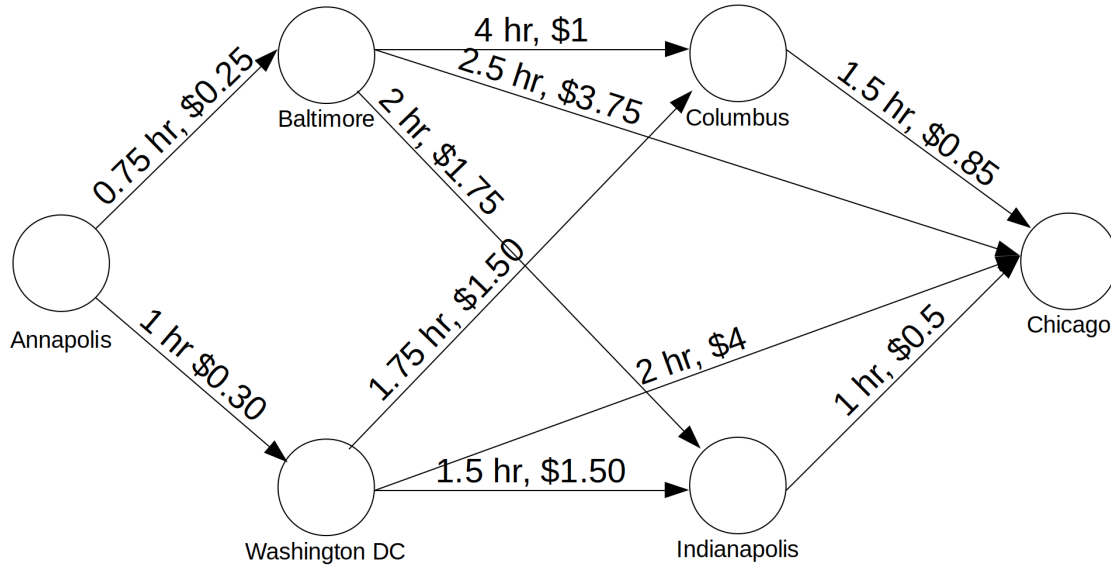
$$z^*(\lambda) \geq z_{IP}^*$$

For a minimization problem:

$$z^*(\lambda) \leq z_{IP}^*$$

For all values of $\lambda \geq 0$

Problem 3. Let's see an example, suppose $\lambda = 1$.



1. What is the shortest path in this network?

2. Is this solution feasible to the original problem?

3. What is the objective function value of this solution in the original problem?

4. What is the objective function value of this solution in the Lagrangian Relaxation?

Since the Lagrangian relaxation is an upper bound for any value of λ , a related problem is to find the optimal λ to find the best bound. In this case we should be trying to find as of a bound as possible.

Mathematically, this is called the **Lagrangian Dual** problem.

5 Other Considerations

- There are equivalent cases of Lagrangian relaxation for equality and \geq constraints; usually all that changes is a sign in the objective function.
- There are cases where the Lagrangian Relaxation provides an exact bound to the optimal solution of the IP. This makes it a very powerful tool.
- Some problems are so hard to solve; the best we can do is compute bounds on the optimal solution. Lagrangian Relaxation is a good tool for this.