SA405, Fall 2021 **Final Exam Practice**

Final exam guidance:

- The SA405 final exam is on Tues 14 Dec at 755 on the first deck of Chauvenet Hall. (Check mids for the exact classroom that your section is in.)
- You may bring in a single, two-sided, 8.5" x 11" sheet of notes that has been hand-written by you (no scanning or printing involved) to use during the exam.
- There will be no Python on the final exam. All other material is fair game.
- **The practice problems on this document do NOT represent an exhaustive list of topics that may appear on the final exam.**
- To prepare for the final exam, you should study all tests, quizzes, homework problems, and notes. I would recommend reworking any problems that you aren't comfortable with after reviewing the notes for those topics.
- Good luck!

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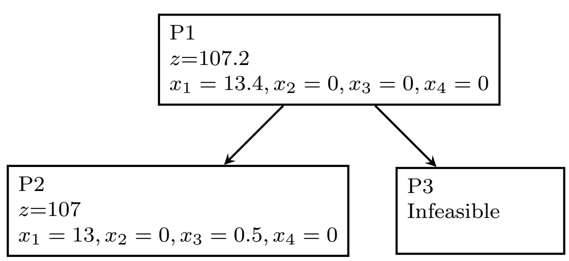
1. Consider the following: CDR Could B. Wright is attempting to solve a *maximizing* integer program using branch-and-bound. All four variables, x_1, x_2, x_3, and x_4, are required to be nonnegative integers.

   (a) He begins by solving the problem as an LP-relaxation. His solver returns the solution shown in the box labeled P1 below.

   At the initialization phase (based on P1 only), what are the upper and lower bounds on the optimal integer solution, which we will denote z^*?

   > **Solution:** $-\infty < z^* \leq 107 = \lfloor 107.2 \rfloor$. (The lower bound comes from an incumbent solution, and there is no incumbent solution yet.)

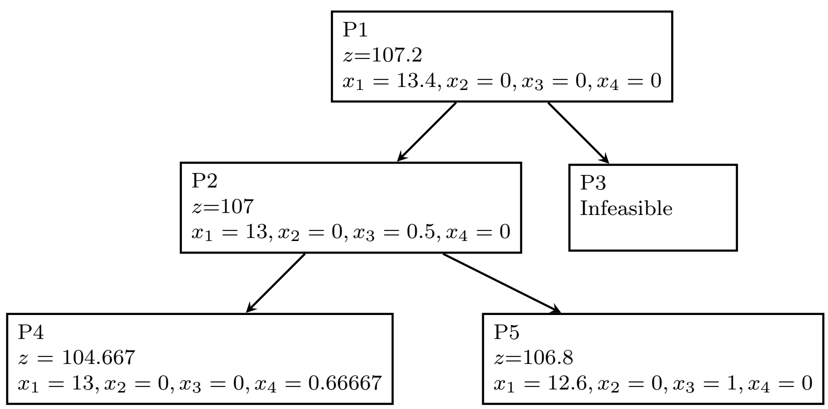
   (b) He then chooses a variable to branch on and adds the appropriate inequalities to the model (you may assume he does this correctly). Label both of the arcs from P1 with the branching variable and inequality that led to the solutions shown in the boxes labeled P2 and P3.

Solution: $x_1 \leq 13$, $x_1 \geq 14$

(c) CDR Could B. Wright initially claims that he can stop using branch-and-bound at this point because he has found an integer objective value with subproblem P2. Explain why this is not the case.

Solution: He should not terminate branch and bound at this point because he hasn't found a feasible integer SOLUTION (x_3 is not an integer), and P2 is still an active node that needs to be explored.

(d) CDR Could B. Wright takes your advice and continues using branch-and-bound and arrives at the tree shown below. Label both of the arcs from P2 with the branching variable and inequality that led to the solutions shown in the boxes labeled P4 and P5.



Solution: $x_3 \leq 0$, $x_3 \geq 1$

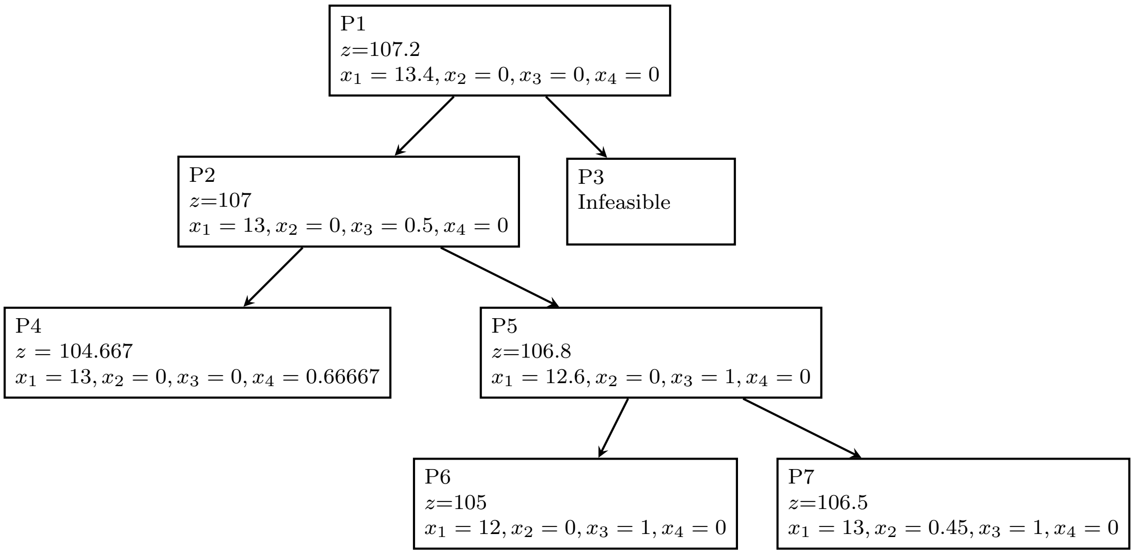
(e) What are the incumbent solution, global lower, and global upper bounds at this point in branch and bound?

Solution: There is no incumbent solution yet, so the current global lower bound is still $-\infty$. The global upper bound is 106, which is the largest upper bound among the active subproblems (P4 and P5).

(f) If CDR Could B. Wright wants to use a Best-First Search approach, which subproblem should he choose to branch on next? Briefly explain your answer for full credit.

Solution: He should branch on P5 because $106.8 > 104.667$. Best-First Search means we branch on the subproblem with the largest objective value when we have a maximization objective function.

(g) CDR Could B. Wright continues using branch-and-bound and arrives at the tree shown below.



What are the incumbent solution, global lower bound, global upper bound, and MIP gap at this point in branch and bound?

Solution: Incumbent solution: $(12, 0, 1, 0)$. $105 \leq z^* \leq 106$. MIP gap $= \frac{|106-105|}{|105|} = 0.009$.

(h) Label the status of each node as active, branched, fathomed, or incumbent.

Solution: P1: branched, P2: branched, P3: fathomed, P4: fathomed, P5: branched, P6: incumbent, P7: active

(i) What should happen next in the algorithm? If the algorithm should terminate, what is the optimal solution? If the algorithm should continue, explain how the next subproblems are produced.

Solution: The algorithm should continue because there is still an active node: P7. Branch at P7 on x_2. Add the constraint $x_2 \leq 0$ to form P8, and add constraint $x_2 \geq 1$ to form P9.

2. Draw two 2-dimensional shapes with straight edges, one convex and one nonconvex. Explain your examples.

Solution:

3. Consider the following integer linear program:

$$\max 3x_1 + 14x_2 + 18x_3$$

s.t.

$$\begin{array}{rrrcl} 3x_1 & +5x_2 & +6x_3 & \leq & 10 \\ x_1, & x_2, & x_3 & \in & \{0,1\} \end{array} \qquad (P)$$

whose linear programming relaxation optimal solution is $x_1 = 0$, $x_2 = \frac{4}{5}$, and $x_3 = 1$.

Recall the following definitions:

- A linear inequality is a **valid inequality** for a given discrete optimization model if it holds for all (integer) feasible solutions to the model.
- To **strengthen** a relaxation, a valid inequality must cut off (render infeasible) some feasible solutions to the current linear programming relaxation that are not feasible in the full integer linear programming model.

(a) The inequality $x_1 + x_2 + x_3 \leq 1$ is (circle one): **valid** or **not valid** for this model. Briefly explain your answer for full credit.

> **Solution:** The inequality is **not valid**. For example, the inequality does not hold for the feasible solution $(1, 1, 0)$ of the IP.

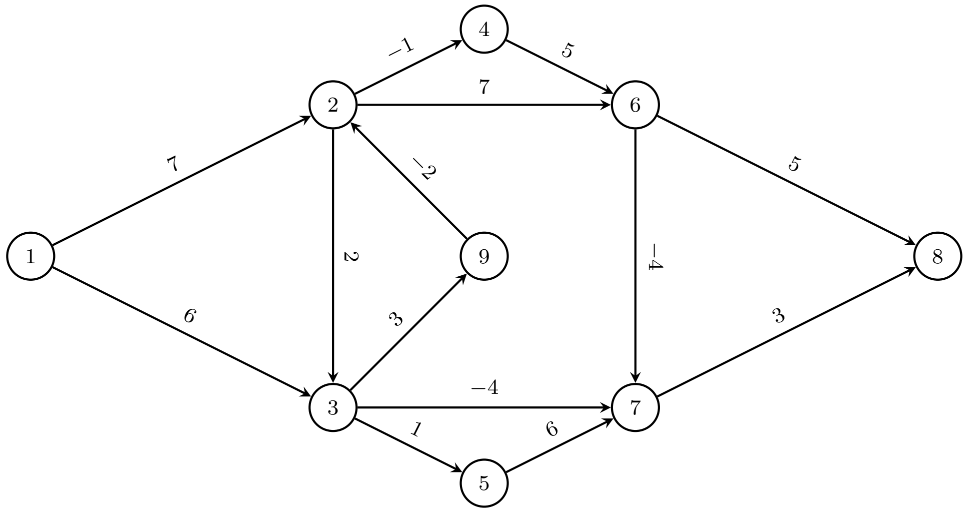
(b) Explain why the inequality $x_2 + x_3 \leq 1$ is valid for this model.

> **Solution:** The inequality is valid because it holds for all feasible solutions of the IP. There are six feasible solutions of the IP, $(0, 0, 0); (0, 0, 1); (0, 1, 0); (1, 0, 0); (1, 0, 1); (1, 1, 0)$ (the triplets $(0, 1, 1); (1, 1, 1)$ are not feasible) and it is easy to check that they all satisfy the inequality $x_2 + x_3 \leq 1$.

(c) Adding the constraint $x_2 + x_3 \leq 1$ would (circle one): **strengthen** or **not improve** the linear programming relaxation. Briefly explain your answer for full credit.

> **Solution:** Adding the constraint $x_2 + x_3 \leq 1$ would **strengthen** the original LP relaxation. The optimal solution of the original LP relaxation, $x_1 = 0$, $x_2 = \frac{4}{5}$, and $x_3 = 1$, does not satisfy the constraint $x_2 + x_3 \leq 1$, hence, part of the feasible region of the original LP relaxation is "cut off" and the new LP relaxation is "tighter".

4. Consider the following graph. The numbers above the arcs are costs.



(a) Formulate a concrete integer program whose solution provides the shortest paths from node 1 to each of the following nodes: 6, 8, and 9.

Solution:

Decision Variables

$x_{i,j}$ number of times arc (i,j) was used on shortest paths, for all $i, j \in \{1, \ldots, 9\}$ set of nodes

Formulation

$$
\begin{array}{lll}
\min & 7\ x_{1,2} + 6\ x_{1,3} + \ldots + (-2)\ x_{9,2} & \\
\text{s.t.} & x_{1,2} + x_{1,3} = 3 & \text{node 1, starting node} \\
& x_{1,2} + x_{9,2} = x_{2,3} + x_{2,4} + x_{2,6} & \text{node 2, transitional node} \\
& x_{1,3} + x_{2,3} = x_{3,5} + x_{3,7} + x_{3,9} & \text{node 3, transitional node} \\
& x_{2,4} = x_{4,6} & \text{node 4, transitional node} \\
& x_{3,5} = x_{5,7} & \text{node 5, transitional node} \\
& x_{2,6} + x_{4,6} - x_{6,7} - x_{6,8} = 1 & \text{node 6, destination node} \\
& x_{3,7} + x_{5,7} + x_{6,7} = x_{7,8} & \text{node 7, transitional node} \\
& x_{6,8} + x_{7,8} = 1 & \text{node 8, destination node} \\
& x_{3,9} - x_{9,2} = 1 & \text{node 9, destination node} \\
& x_{i,j} \geq 0 &
\end{array}
$$

(b) Convert your concrete model to abstract form. Define all notation used. Describe the purpose of the objective function and each constraint.

Solution:

Sets

N	set of nodes
Node 1	starting node for any shortest path
$D \subseteq N - \{1\}$	destination nodes
A	set of arcs (i,j) for $i,j \in N$

Parameters

$c_{i,j}$	the cost on the arc $(i,j) \in A$

Decision Variables

$x_{i,j}$	number of times arc $(i,j) \in A$ was used on shortest paths

Formulation

$$\min \sum_{(i,j)\in A} c_{i,j} x_{i,j} \tag{1}$$

$$\text{s.t.} \sum_{j \in N:(1,j)\in A} x_{1,j} - \sum_{i \in N:(i,1)\in A} x_{i,1} = |D| \qquad |D| \text{ cardinality of set} D \tag{2}$$

$$\sum_{j \in N:(n,j)\in A} x_{n,j} - \sum_{i \in N:(i,n)\in A} x_{i,n} = 1 \qquad \forall n \in D \tag{3}$$

$$\sum_{j \in N:(n,j)\in A} x_{n,j} - \sum_{i \in N:(i,n)\in A} x_{i,n} = 0 \qquad \forall n \in N - D - \{1\} \tag{4}$$

$$x_{i,j} \geq 0 \qquad \forall (i,j) \in A \tag{5}$$

Objective and constraint descriptions

(1) The objective is to minimize the cost of traveling from the starting node to each of the destination nodes.

(2) Determining the min cost path from the starting node to the destination node is facilitated as delivering one unit from starting node (source) to destination node (sink). Since we have $|D|$ destination nodes "flow" from the staring node is exactly $|D|$.

(3) "Flow in" - "flow out" of the destination node must be one which is a consequence of what was discussed above.

(4) "Flow in" - "flow out" of the transitional nodes must be zero.

(5) No integer negativity requirement. It is sufficient to require that values of the variables are no-negative real numbers, model will automatically produce values that are no-negative integers.

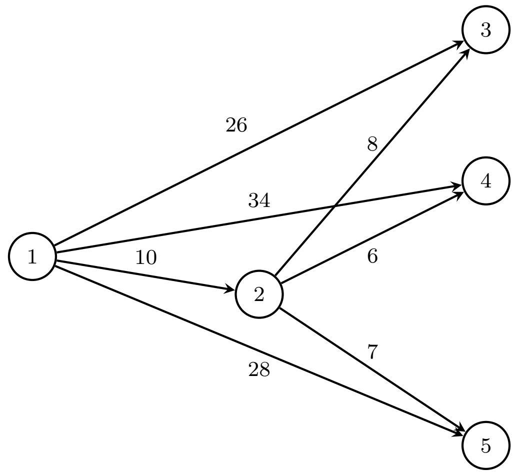
(c) Suppose you ran the solver and discovered the optimal solution results in the following shortest paths: 1-2-4-6, 1-3-7-8, and 1-3-9. What were the values of the decision variables in the optimal solution?

Solution:

$x_{1,3} = 2, x_{1,2} = x_{2,4} = x_{4,6} = x_{3,7} = x_{3,9} = x_{7,8} = 1$

All other variables have value zero.

5. The directed graph below shows possible routes for fiber optic lines from backbone node 1 to customers at nodes 3, 4, and 5. Node 2 is an optical repeater that may or may not be included in the ultimate system. Numbers on arcs are the fixed cost f_{ij} of the corresponding fiber optic line.



The net supplies $b_i = s_i - d_i$, where s_i is the supply and d_i is the demand at node i, for the nodes are given in the following table:

node	1	2	3	4	5
b_i	3	0	-1	-1	-1

The carrying capacity of the arcs u_{ij} is given in the following table:

arc	(1,2)	(1,3)	(1,4)	(1,5)	(2,3)	(2,4)	(2,5)
u_{ij}	3	1	1	1	1	1	1

For the questions that follow, use the decision variables x_{ij} to indicate the amount of flow on arc (i, j) and binary variables y_{ij}, where $y_{ij} = 1$ if arc (i, j) is to be used, and 0 otherwise, for all arcs (i, j).

(a) Write an objective function that seeks to minimize the cost of providing connectivity to all three customers.

Solution:

$$\min 10y_{12} + 26y_{13} + 34y_{14} + 28y_{15} + 8y_{23} + 6y_{24} + 7y_{25}$$

or

$$\min \sum_{(i,j)\in A} f_{ij}y_{ij}$$

where A is the set of arcs given in the table above

(b) Write the balance of flow constraint for node 2.

Solution:

$$x_{23} + x_{24} + x_{25} - x_{12} = 0$$

or

$$\sum_{j|(2,j)\in A} x_{2j} - \sum_{j|(j,2)\in A} x_{j2} = 0$$

where A is the set of arcs given in the table above

(c) Write a constraint that ensures the flow from the backbone node (node 1) to the optical repeater (node 2) meets the following criteria: is only nonzero if we pay to use arc $(1,2)$ in the objective function, and does not exceed the carrying capacity of arc $(1,2)$.

Solution:

$$x_{12} \leq 3y_{12}$$

(d) A new cost is introduced into the problem: a fixed charge of 5 for using node 2, the optical repeater. Modify the model to incorporate this new cost, including any changes to the objective function and/or constraints. Define any new notation that you use.

Solution: Let z_2 denotes a binary variable that has value 1 if node 2 is used and 0 otherwise.
The objective function changes to the following for

$$\min 10y_{12} + 26y_{13} + 34y_{14} + 28y_{15} + 8y_{23} + 6y_{24} + 7y_{25} + 5z_2$$

We also need to add either a weak constraint or strong constraints for node 2.
Weak constraint:

$$x_{23} + x_{24} + x_{25} \leq 3z_2$$

Strong constraints:

$$x_{23} \leq 3z_2$$

$$x_{24} \leq 3z_2$$

$$x_{25} \leq 3z_2$$

Note: Alternatively, we could have used incoming arcs for node 2 to formulate weak and strong constraints. Balance of flow constraint at node 2 will ensure that using incoming arcs will produce the same results as using outgoing arcs. Since we have only one incoming arc, weak and strong constraints are the same.
Weak and strong constraint:

$$x_{12} \leq 3z_2$$

Hence, it is more beneficial to use the second option with incoming arc $(1,2)$.

6. You are trying to decide where you want to attend graduate school. You plan to leave from and return to USNA after visiting each school one time. You can visit the schools in any order, but you would like to determine a route of shortest total length. A table of the corresponding distances, d_{ij}, from city i to city j is given in the table below, with city 1 representing Annapolis.

	1	2	3	4	5	6	7
1	0	8	9	11	17	12	3
2	8	0	11	5	6	10	7
3	9	11	0	2	7	8	13
4	11	5	2	0	3	4	4
5	17	6	7	3	0	9	1
6	12	10	8	4	9	0	6
7	3	7	13	4	1	6	0

(a) Write an abstract model whose solution provides a minimal cost way to begin at city 1, visit each of the other cities exactly one time each, and return to city 1. Be sure to define any notation used in the model.

Solution: This is a traveling salesperson problem. We define the following:

Sets

Let V be the set of nodes

Let E be the set of edges

Decision Variables

Let $x_{i,j} = 1$ if edge (i, j) is included in the tour and 0 otherwise for all $(i, j) \in E$

Parameters

Let $d_{i,j}$ be the distance between cities i and j for all $(i, j) \in E$

Objective function

$$\text{min distance:} \sum_{(i,j)\in E} d_{i,j} x_{i,j}$$

Constraints

$$\begin{aligned}
\sum_{(r,j)\in E} x_{r,j} + \sum_{(i,r)\in E} x_{i,r} &= 2 && \text{for all } r \in V \\
\sum_{(i,j)\in S} x_{i,j} &\le |S| - 1 && \text{for all } S \subset V,\ |S| \ge 3 \\
x_{i,j} &\in \{0, 1\} && \text{for all } (i, j) \in E
\end{aligned}$$

(b) Provide a concrete example of each type of constraint that appears in the abstract model. Explain the purpose and logic of each of the example constraints.

Solution: For the first constraint consider node 1. We have:

$$x_{1,2} + x_{1,3} + x_{1,4} + x_{1,5} + x_{1,6} + x_{1,7} = 2$$

The logic of this constraint is that every node is touched exactly twice by an edge. Essentially this means that each node is visited and left in the tour. Additionally, no node is visited more than once.

The second constraint, consider a set of size 3 on nodes 1, 2, and 3. We would have:

$$x_{1,2} + x_{1,3} + x_{2,3} \leq 2$$

This constraint ensures that no cycle is formed on the three nodes 1, 2, and 3 because it is impossible to form a cycle on 3 nodes with only two edges.

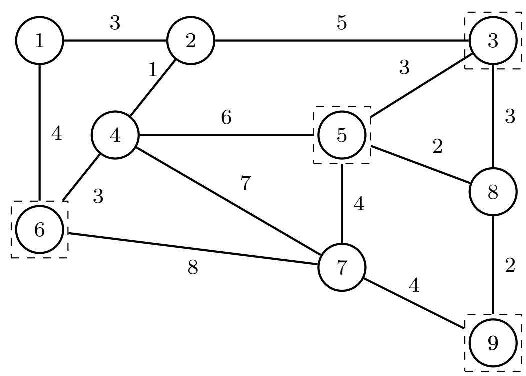
(c) Write the concrete constraint(s) from the abstract model from part (a) that would prevent a solution that includes a tour among cities 1, 4, and 5, and a separate tour among the rest of the cities.

Solution:

$$x_{1,4} + x_{1,5} + x_{4,5} \leq 2$$

This ensures a separate cycle is not formed just on nodes 1, 4, and 5.

7. The city of Examopolis has hired you to help them decide where to place some new emergency response facilities. The figure below will be used to help make the decision, where each of the $n = 9$ nodes $I = \{1, 2, ..., 9\}$ correspond to the boroughs who need service and the nodes $J = \{3, 5, 6, 9\}$ in dashed boxes are the possible emergency response facility locations. The weight on each edge (i, j) corresponds to direct distance between i and j. Let the distance d_{ij} be the length of the shortest path in this graph between nodes i and j.



The city council has asked that you use the following notation when formulating your responses to their questions.

Indices and Sets

$i \in I$	boroughs, $I = \{1, 2, ..., 9\}$
$j \in J$	possible emergency response facility locations, $J = \{3, 5, 6, 9\}$

Data

h_i	population at node i in thousands
D	maximum allowable distance between a borough and its servicing facility
N_i	neighborhood of i, where $N_i = \{j \in J : d_{ij} \leq D\}$ is the set of all facilities j that can serve node i
d_{ij}	the length of the shortest path between node i and node j

Decision Variables

x_j	1 if node j is the location of an emergency response facility, 0 otherwise
y_{ij}	1 if borough i has its demand satisfied by emergency response facility j, 0 otherwise

(a) Write an objective function that minimizes the number of emergency response facilities needed.

Solution:

$$\min \sum_{j \in J} x_j$$

(b) If $D = 7$, write out all the elements of N_4.

Solution: $N_4 = \{3, 5, 6\}$

(c) Write a set of constraints which ensures each borough must be associated with an emergency response facility in its neighborhood, N_i.

Solution:

$$\sum_{j \in N_i} x_j \geq 1, \quad i \in I$$

The city council now wants a model that attempts to minimize the maximum distance between a borough and its closest emergency response facility.

(d) Let W be the maximum distance between a borough i and its assigned facility j. Write an objective function and corresponding set of constraints that minimizes this maximum distance.

Solution:

$$\begin{array}{ll} \min & W \\ \text{s.t.} & \sum_{j \in J} d_{ij} y_{ij} \leq W, \quad i \in I \end{array}$$

or

$$\begin{array}{ll} \min & W \\ \text{s.t.} & d_{ij} y_{ij} \leq W, \quad i \in I, \ j \in J \end{array}$$

(e) Write a constraint which ensures exactly 2 emergency response facilities are opened.

Solution:

$$\sum_{j \in J} x_j = 2$$

(f) Write a set of constraints which ensure that borough i's demand is met by emergency response facility j only if facility j is open.

Solution:

$$y_{ij} \leq x_j, \quad i \in I, \ j \in J$$

(g) Write one or more constraints to ensure that if a facility is opened at either 3 or 9, then a facility must not be opened at 5. The constraint(s) should not impose any restriction on the facility at 5 if neither 3 nor 9 are opened.

Solution: $x_5 \leq 1 - x_3$, $x_5 \leq 1 - x_9$