HW10: Branch and Bound part 2

1. Solve the following problem using Branch and Bound

$$
\begin{aligned}
\max \quad & 20x_1 + 16x_2 + 25x_3 + 14x_4 + 9x_5 \\
\text{st} \quad & 3x_1 + 2x_2 + 5x_3 + 4x_4 + 2x_5 \leq 13 \\
& x_1, x_2, x_3, x_4, x_5 \in \{0, 1\}
\end{aligned}
$$

Solve each subproblem either using python or using the following algorithm:

- Compute the ratio r_i for each variable where

$$r_i = \frac{\text{coefficient of } x_i \text{ in objective function}}{\text{coefficient of } x_i \text{ in constraint}}$$

- Order r_i from largest to smallest
- Starting with the largest r_i, set x_i to its largest possible (continuous) value without violating the existing constraints.

For example, this algorithm to solve the initial problem would compute r_i as:

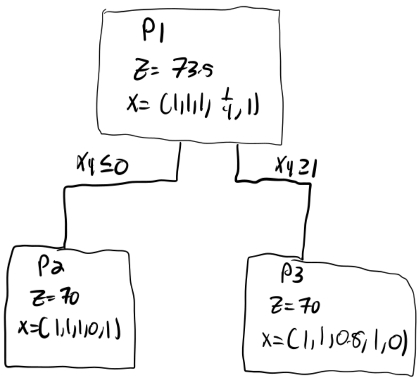
$$r_i = \{\frac{20}{3}, 8, 5, \frac{7}{2}, \frac{9}{2}\}$$

The largest of these is 8 so we would set $x_2 = 1$. The RHS of the constraint is now 11. The next largest is $\frac{20}{3}$, so we would set $x_1 = 1$ and the RHS of the constraint is now 8. The next largest is 5, so we would set $x_3 = 1$ and the RHS of the constraint becomes 3. The next largest is $\frac{9}{2}$, so we would set $x_5 = 1$ and the RHS of the constraint becomes 1. All that's left is x_4. We can't set it equal to 1, instead we maximize it and set $x_4 = \frac{1}{4}$. So the initial solution is $x = (1, 1, 1, 0.25, 1)$.

This one is very simple. Branch and bound tree is below.

- Step 1: Solve P1, initial solution is $z = 73.5$ and x_4 is fractional. I know $z_{IP} \leq 73$. Branch on x_4.
- Step 2: Solve P2 and P3. For P2, $z = 70$ and x is integral. For P3, $z = 70$ and x_3 is fractional. Since P2 is integral, I know $70 \leq z_I \leq 73$. Since P3 has a $z = 70$, there's no value in exploring this region. I stop.

Optimal solution is $x^* = (1, 1, 1, 0, 1)$ with optimal objective function value of $z^* = 70$.


P1
Z= 73.5
X= (1,1,1,1, 1/4, 1)
x4 ≤ 0
x4 ≥ 1
P2
Z= 70
X=(1,1,1,0,1)
P3
Z= 70
X=(1,1,0.8,1,0)

2. Solve the following problem using Branch and Bound:

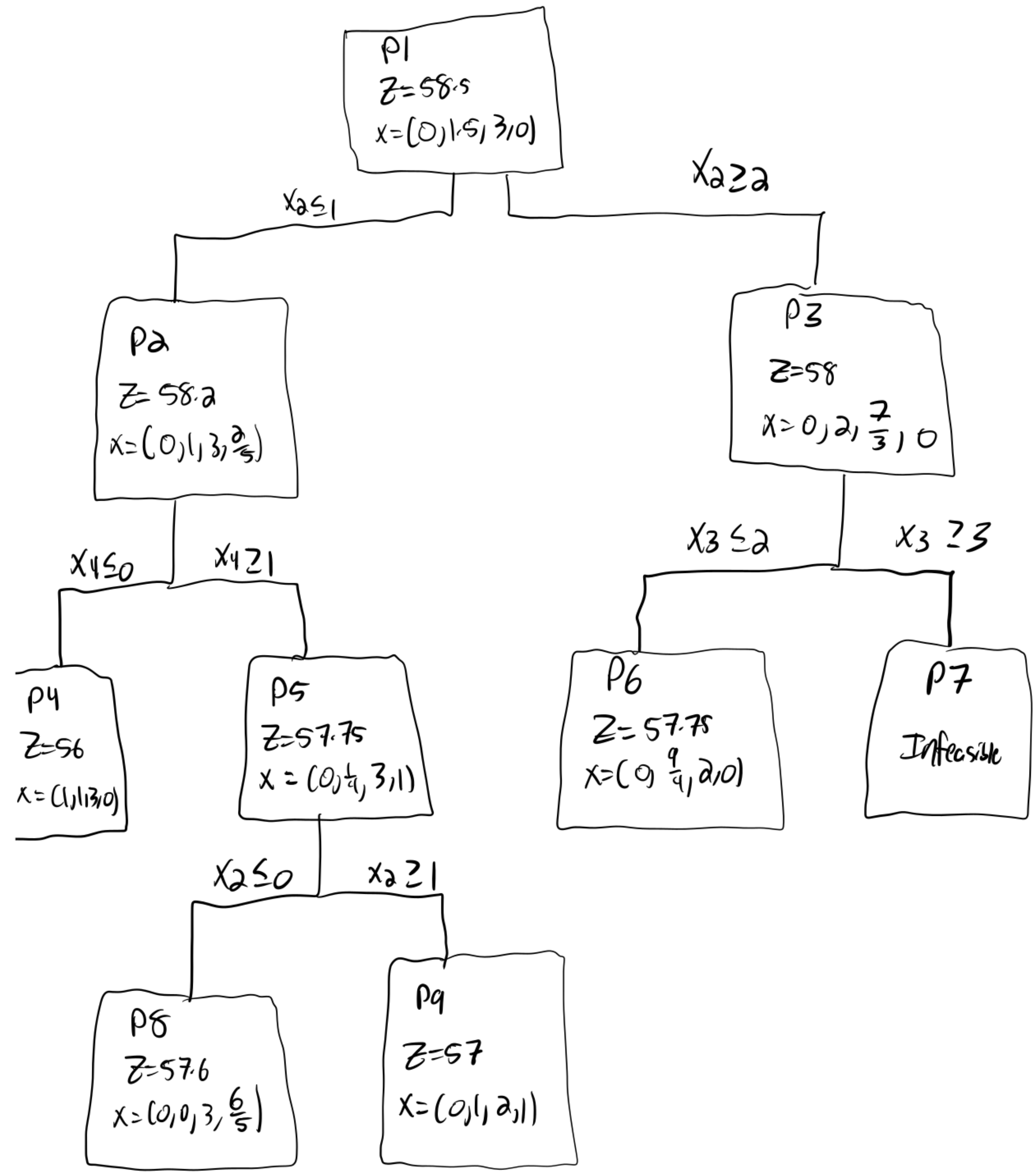
$$
\begin{aligned}
\max \quad & 5x_1 + 15x_2 + 12x_3 + 18x_4 \\
\text{st} \quad & 20x_1 + 40x_2 + 30x_3 + 50x_4 \leq 150 \\
& x_1 \leq 5 \\
& x_2 \leq 3 \\
& x_3 \leq 3 \\
& x_4 \leq 2 \\
& x_1, x_2, x_3, x_4 \in \mathbb{Z}^+
\end{aligned}
$$

Solve each subproblem either using python or using the same algorithm as problem 2 (notice that this time the variables are not binary so you'd maximize the value based on their upper bounds).

Branch and bound tree is on the next page. Steps of the algorithm are as follows.

- Step 1, solve P1. Initial solution $z = 58.5$, x_2 is fractional. At this point, I know $z_{IP} \leq 58$. I can branch on either x_2.
- Step 2, solve P2 and P3. For P2, $z = 58.2$ and x_4 is fractional. For P3, $z = 58$, x_3 is fractional. I can branch on either P2 or P3. I choose P2 because $58.2 > 58$.
- Step 3: Solve P4 and P5. For P4, $z = 56$ and solution is integer. For P5, $z = 57.75$ and x_2 is fractional. So at this point, I know $56 \leq z_{IP} \leq 58$. I can choose to branch on either P3 or P5. I choose P3
- Step 4: Solve P6 and P7. For P6, $z = 57.75$ and x_2 is fractional. P7 is infeasible. At this point, my active nodes are P5 and P6. Note that both have the same z value. So I can update my bounds to be $56 \leq z_{IP} \leq 57$. I choose to branch on P5.
- Step 5: Solve P8 and P9. For P8, $z = 57.6$ and x_4 is fractional. For P9, $z = 57$ and x is integral. Note that this means my bound is $57 \leq z_{IP} \leq 57$. This means my current solution is optimal, I can eliminate nodes P8 and P6.

Optimal solution is $x^* = (1, 1, 3, 0)$ with optimal objective function value of $z^* = 56$.


P1
Z=58.5
x=(0,1.5,3,0)
x2≤1
x2≥2
P2
Z=58.2
x=(0,1,3,2/5)
P3
Z=58
x=0,2,7/3,0
x4≤0
x4≥1
x3≤2
x3≥3
P4
Z=56
x=(1,1,3,0)
P5
Z=57.75
x=(0,1/4,3,1)
P6
Z=57.75
x=(0,9/4,2,0)
P7
Infeasible
x2≤0
x2≥1
P8
Z=57.6
x=(0,0,3,6/5)
P9
Z=57
x=(0,1,2,1)