Branch-and-bound Graphical Example (Accompanying Lesson 14)

1 Branch-and-bound Example

Solve the following IP using branch-and-bound. Solve each sub problem graphically.

$$
\begin{aligned}
\text{(P1)} \quad z^*_{IP} = \max\ & 4x_1 - x_2 \\
\text{s.t.}\ \ & 7x_1 - 2x_2 \le 14 \\
& 2x_1 - 2x_2 \le 3 \\
& x_2 \le 3 \\
& x_1, x_2 \in \mathbb{Z}^{\ge 0}
\end{aligned}
$$

P1

P2 P3

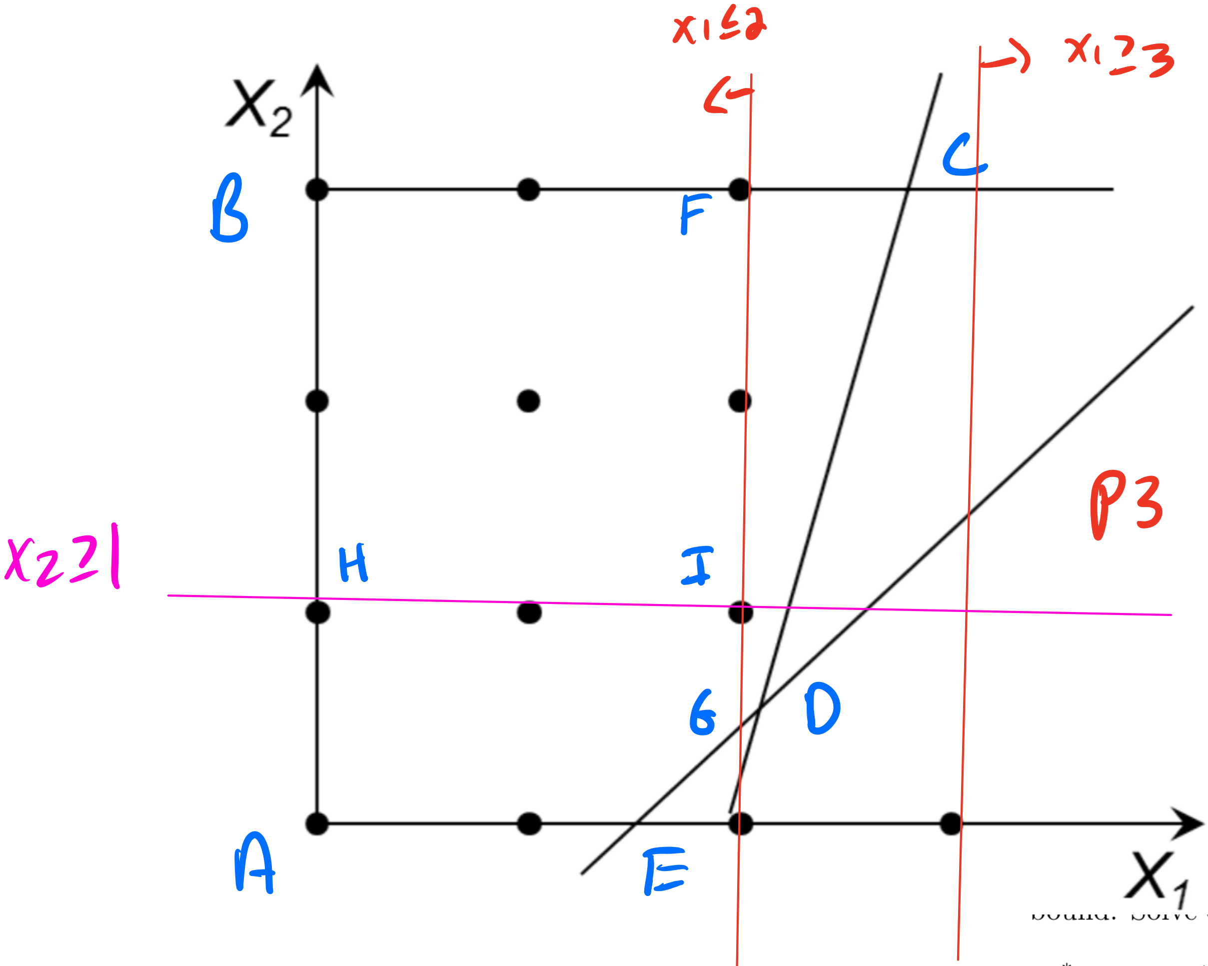
Z= 77.1 Z=61.2

pick

- Solve each sub-problem graphically
- Branching Rules
  - Always select the active node with the largest upperbound for branching.
  - Branch on x_1 if it is fractional. Otherwise branch on x_2.
- Book-keeping
  - Keep track of the:
    - incumbent solution $\underline{x}$,
    - global lower bound $\underline{z}$, and
    - list of active nodes.
  - Draw the branch-and-bound tree:
    - Record the local upper bound (z) and relaxed optimal solution (x) for each subproblem.
    - Label each edge with the constraint that is added to form the child subproblem.
    - X-out fathomed nodes. Circle incumbent solution nodes.
  - Use the provided diagram to illustrate the (relaxed) feasible region of each subproblem.

incumbent solution $\underline{x}$ **global lower bound $\underline{z}$** **active nodes**

Feasible Region



bound. Solve each sub proble

$$
\begin{aligned}
z_{IP}^* = \max\ & 4x_1 - x_2 \\
\text{s.t.}\quad & 7x_1 - 2x_2 \le 14 \\
& 2x_1 - 2x_2 \le 3 \\
& x_2 \le 3 \\
& x_1, x_2 \in \mathbb{Z}^{\ge 0}
\end{aligned}
$$

A: $(0,0)$, $Z=0$

B: $(0,3)$, $Z=-3$

C: $(\frac{20}{7}, 3)$, $Z = \frac{59}{7} \approx 8.4$

D: $(\frac{11}{5}, \frac{7}{10})$, $Z = \frac{81}{10} = 8.1$

E: $(\frac{3}{2}, 0)$, $Z = 6$

F: $(2,3)$, $Z = 5$

G: $(2, \frac{1}{2})$, $Z = 7.5$

H: $(0,1)$, $Z = -1$

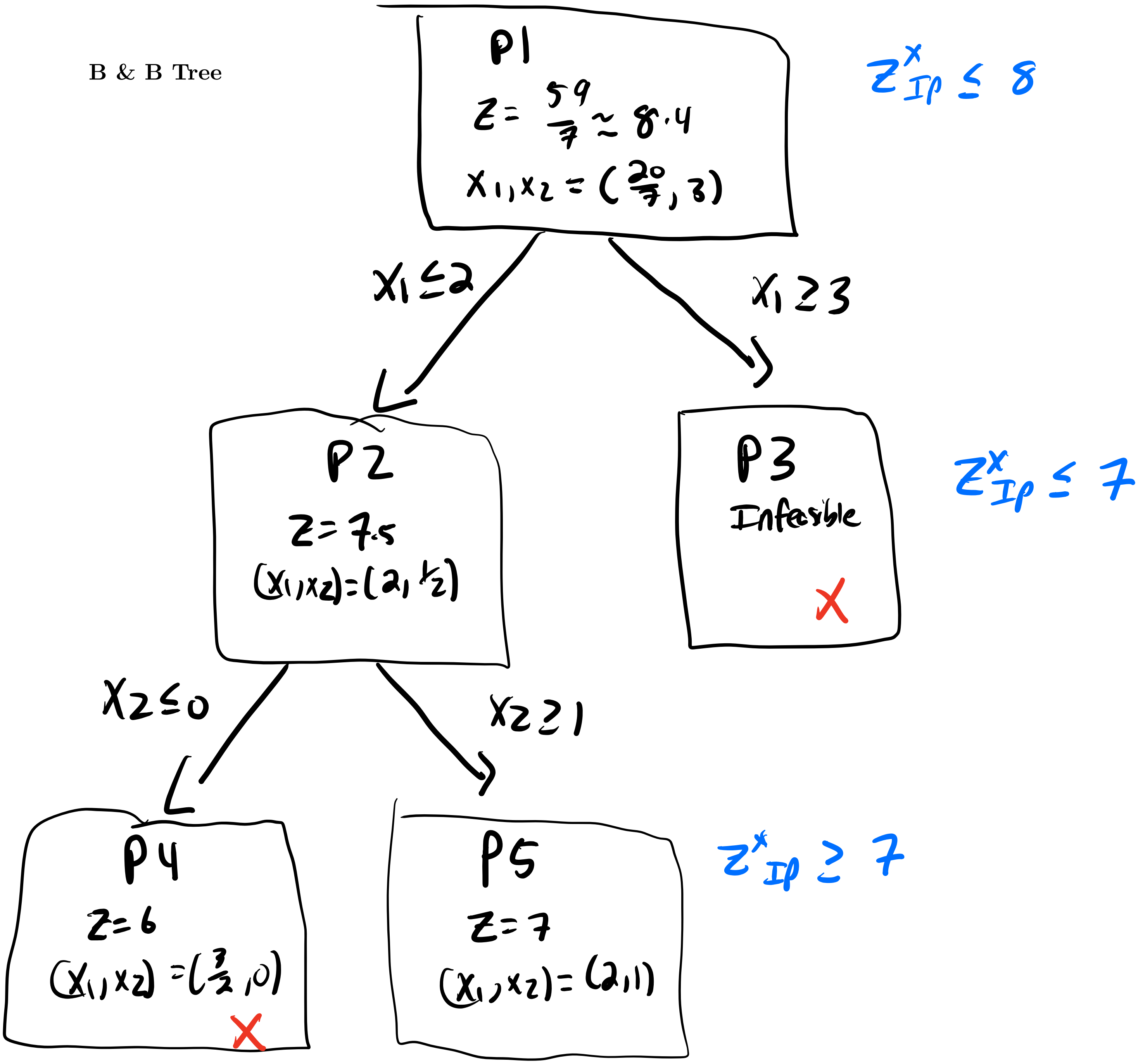
I: $(2,1)$, $Z = 7$

Optimal solution

$(x_1, x_2) = (2,1)$

$Z = 7$

B & B Tree



P1
$Z = \frac{59}{7} \approx 8.4$
$x_1, x_2 = (\frac{20}{7}, 3)$
$z^*_{IP} \leq 8$
$x_1 \leq 2$
$x_1 \geq 3$
P2
$Z = 7.5$
$(x_1, x_2) = (2, \frac{1}{2})$
P3
Infeasible
$z^*_{IP} \leq 7$
$x_2 \leq 0$
$x_2 \geq 1$
P4
$z = 6$
$(x_1, x_2) = (\frac{3}{2}, 0)$
P5
$z = 7$
$(x_1, x_2) = (2, 1)$
$z^*_{IP} \geq 7$

① Solve LP relaxation of original problem

We get $Z = 8.4$ $(x_1, x_2) = (\frac{20}{7}, 3)$

- Upper bound $Z_{IP} \leq 8.4 \rightarrow Z_{IP} \leq 8$
- Branch P1 on x_1 because x_1 can't be a fraction.

x_1 is any int $< 20/7$ $x_1 \leq 2$ P2

x_1 is any int $> 20/7$ $x_1 \geq 3$ P3

② Solve P2 and P3

- IP Region of P2 and P3 = P1 but LP region is smaller.

P2 $\Rightarrow$ $Z = 7.5$ $x_1, x_2 = (2, \frac{1}{2})$

P3 $\Rightarrow$ infeasible

New upper bound $Z \leq 7$

Branch P2 on x_2

$x_2 \leq 0 \rightarrow$ P4

$x_2 \geq 1 \rightarrow$ P5

③ Solve P4 and P5

P4 $\rightarrow$ $(\frac{3}{2}, 0)$ $Z = 6$

P5 $\Rightarrow$ $(2, 1)$ $Z = 7$

- Because P5 is integer, we get $Z^*_{IP} \geq 7$

STOP

① Upper bound = lower bound = 7

② 6 < lower bound so don't want P4