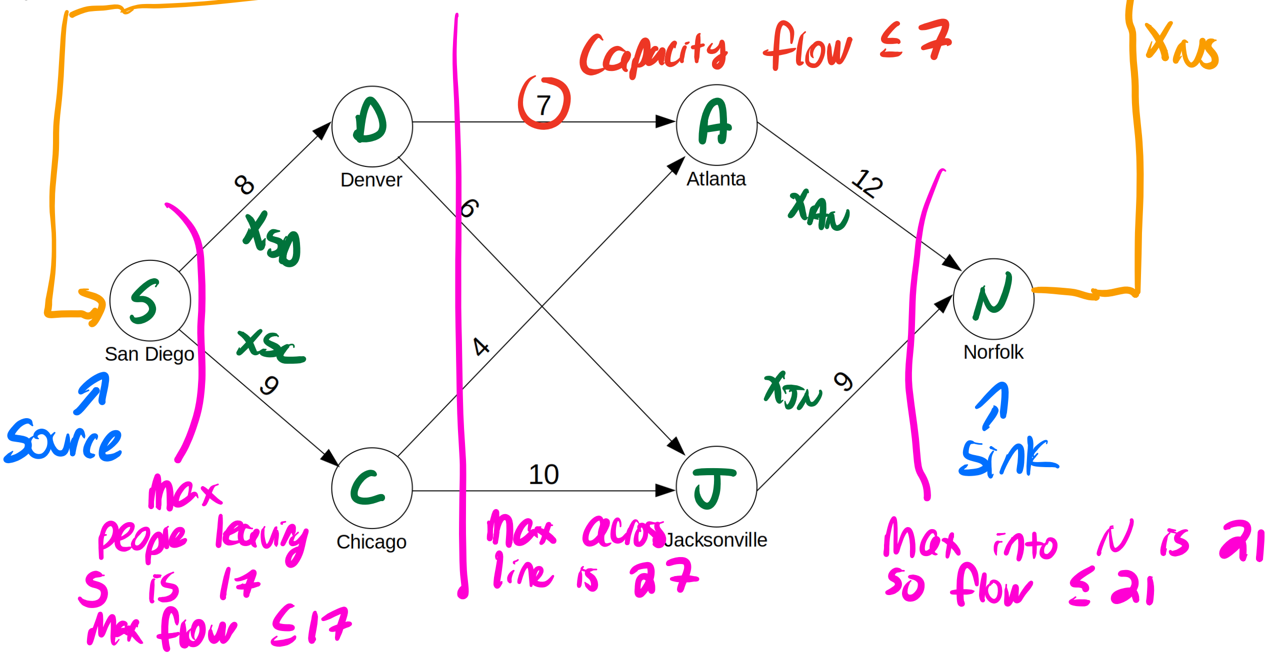
Lesson 4: Max Flow

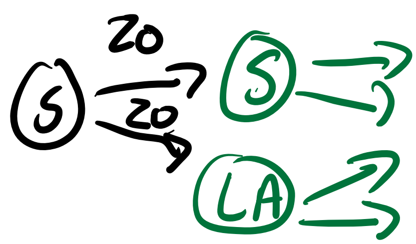
The Navy is trying to transport a large number of sailors from San Diego to Norfolk using airplanes. Unfortunately, they can not do direct flights, so they must make several stops along the way. The following diagram shows the flights available and the capacity (in hundreds) of each flight. Formulate a linear program that would allow the Navy to send as many sailors as possible from San Diego to Norfolk.



Notice that these problems have a special structure. Specifically, max flow problems generally have two special nodes called the **source** and **sink** nodes.

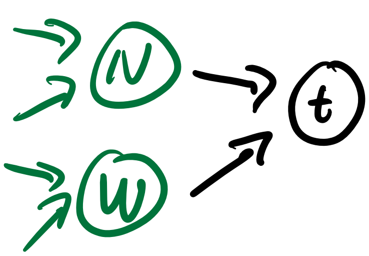
- **Source node:**

- origin of all flow
- Exactly 1 source node, if you have more than 1 use super source



- **Sink node:**

- Destination of all flow
- Exactly 1 sink node, if more than 1 make super sink.



1. It is possible to get bounds on the optimal objective function value of this problem without formulating or solving it. Based on the network flow diagram, what are some upper bounds on the max flow of this network, z?

- max out of S = 17 → $z \le 17$
- max into N = 21 → $z \le 21$
- max across middle = 27 → $z \le 27$

Best bound is 17

2. Based on these bounds, can you think of an equivalent minimization problem? (This is the dual problem)

From duality can reformulate as min cut which is to find the minimum capacity cut that seperates S and N

3. This problem does not specify supply or demand. How can we write balance of flow constraints for this problem? (Hint: there are two correct ways to do this).

Option 1: Flow out of source = Flow into sink

$x_{SD} + x_{SC} = x_{AN} + x_{JN}$

Option 2: Use dummy edge for balance

$x_{NS} = x_{SC} + x_{SD}$

$x_{AN} + x_{JN} = x_{NS}$

Before we formulate this problem, there's a couple of Theorems that are important to conclude our study of basic network problems.

- Max Flow Integrality Theorem

  - Solving a max flow problem with integer capacities as an LP gives an integer solution.

- Min Cost Integrality Theorem

  - Solving MCNF problems (lesson 2) with LP gives integer solutions

The remainder of the network problems that we study do not have this nice property so they must be formulated as an IP.

4. Formulate the concrete LP associated with this max flow model.

Variables

let x_{SD} be the flow from San Diego to Denver

$\vdots$

let x_{JN} be the flow from Jacksonville to Norfolk

Objective

Max: option 1: $x_{SD} + x_{SC}$ (Flow out of source)

option 2: $x_{AN} + x_{JN}$ (Flow into sink)

Constraints

$x_{SD} = x_{DA} + x_{DJ}$ (Flow balance Denver)

$\vdots$

$x_{DJ} + x_{CJ} = x_{JN}$ (Flow balance Jacksonville)

Flow balance interior nodes

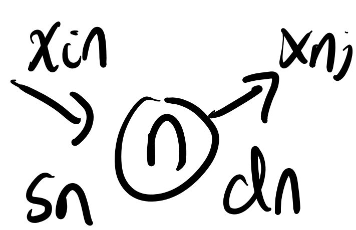
$x_{SD} + x_{SC} = x_{AN} + x_{JN}$ (Flow balance source/sink option 1)

$0 \le x_{SD} \le 8$

$\vdots$

$0 \le x_{JN} \le 9$

(Capacity and non-negativity)



5. Generalize your LP model to a parameterized model.

Sets

let E be the set of edges $\quad E = \{(S,D), (S,C), \ldots\}$

let V be the set of nodes

Variables

let x_{ij} be the flow on edge (i,j) $\forall (i,j) \in E$

Parameter

c_{ij} is the capacity of edge (i,j) $\forall (i,j) \in E$

Objective

Max option 1: $\sum_{(S,j) \in E} x_{Sj}$ sum of edges leaving S

option 2: $\sum_{(i,N) \in E} x_{iN}$ sum of edges entering N

Constraints

$\sum_{(i,n) \in E} x_{in} = \sum_{(n,j) \in E} x_{nj}$ for $n \in V \setminus \{S, N\}$ Flow balance for all nodes except S and N

$\sum_{(S,j) \in E} x_{Sj} = \sum_{(i,N) \in E} x_{iN}$ Flow out of Source = Flow into Sink

$0 \le x_{ij} \le c_{ij} \quad \forall (i,j) \in E$

Note if used option 2 on #3

Concrete model:

Source and sink are: $x_{SP} + x_{SC} = x_{NS}$

$x_{AN} + x_{JN} = x_{NS}$

Parameterized model:

$$\sum_{(i,n)\in E} x_{in} = \sum_{(n,j)\in E} x_{nj} \quad \forall \; n \in V$$