Lesson 6: Set Covering, Packing, and Partitioning

1 Covering Students

The USNA would like all plebes to hear a presentation about major selection. An OR student, Hannah, decides to visit some of the students in order to tell them about which major is the absolute best. She wants to make sure that each plebe sees the presentation, but would like to visit as few classes as possible. She develops the following mini-version of the problem in order to help write a model that will solve the large-scale optimization problem.

Let S be the set of students:

$S :=$ { Kyle, Aaron, Ryan, Jordan, Monika, Brandon, Sharon, Adam, Natalie, Joshua }

Let $\mathscr{C}$ be the set of classes:

$\mathscr{C} :=$ { Naval history, Fencing, Sailing, Boxing, Wrestling, Calculus }

Each element C of $\mathscr{C}$ is itself a set, a subset of S ($C \subseteq S$, for all $C \in \mathscr{C}$):

Naval history	:= { Kyle, Ryan, Monika, Brandon }
Fencing	:= { Kyle, Jordan, Sharon, Natalie }
Sailing	:= { Aaron, Monika, Adam }
Boxing	:= { Aaron, Ryan, Jordan, Sharon }
Wrestling	:= { Jordan, Brandon, Joshua }
Calculus	:= { Adam, Natalie, Joshua }

Hannah defines the following set of binary variables:

$$z_C := \begin{cases} 1 & \text{if she should visit class } C \\ 0 & \text{if she should not visit class } C \end{cases}, \text{ for } C \in \mathscr{C}$$

$z_N = 1$: Visits Naval History
Kyle, Ryan, Monika, and Brandon hear talk

	N	F	S	B	W	C
Kyle	1	1	0	0	0	0
Aaron	0	0	1	1	0	0
Ryan	1	0	0	1	0	0
⋮						
Joshua	0	0	0	0	1	1

2 Set Covering

1. Write two concrete constraints: one that ensures that Jordan will see the presentation, and one that ensures that Brandon will see the presentation.

Jordan: $z_F + z_B + z_W \geq 1$

she visits one of these, he sees the talk

Brandon: $z_N + z_W \geq 1$

2. Why are these called **set covering constraints**? (Think of the set of students.)

students are set S, each element in S is covered by a constraint

3. How many set covering constraints are needed?

10 students → 10 constraints

4. Using the same sets as above and the variable z_c, how would we write a general parameterized set covering constraint for the students?

C: set of classes

S: set of students

Every student is a member of at least 1 class

For each student, sum of classes they're in ≥ 1

$$\underbrace{\sum_{\substack{i \in C: \\ s \in i}}}_{} z_i \geq 1 \quad \text{for all } s \in S$$

sum all i in C such that student s is in class i

The parameterized constraint above works but is a bit messy. There's another way to parameterize it using what's called an **adjacency matrix**. The adjacency matrix is a matrix where the rows correspond to the classes and the columns correspond to the students.

5. Let the adjacency matrix be $a_{c,s}$ for all $c \in \mathscr{C}$ and all $s \in \mathscr{S}$ Illustrate this matrix.

$$a_{cs} = \begin{array}{c|ccc} & K & A & J \\ \hline N & 1 & 0 & 0 \\ F & 1 & 0 & 0 \\ S & 0 & 1 & 0 \\ B & 0 & 1 & 0 \\ W & 0 & 0 & 1 \\ C & 0 & 0 & 1 \end{array}$$

6. Write the parameterized set covering constraints using the adjacency matrix.

$$\sum_{c \in C} a_{c\,\text{kyle}} \cdot z_c = 1 \cdot z_N + 1 \cdot z_F + 0 \cdot z_S + 0 \cdot z_B + 0 \cdot z_W + 0 \cdot z_C$$

$$\downarrow$$

$$\sum_{c \in C} a_{cs} z_c \geq 1 \quad \text{for } s \in S$$

> Either approach works, it's really up to you when it comes to modeling.

7. Write a condensed ~~abstract~~ Parameterized model to find a set of classes that covers all students while requiring the fewest possible presentations using the sets, variables, and parameters defined above.

Parameter

let $a_{cs} = 1$ if student s is in class c for $s \in S$ and $c \in C$

Objective

$$\min \sum_{c \in C} z_c$$ (Total # of presentations)

Constraint

$$\sum_{c \in C} a_{cs} z_c \geq 1 \quad \text{for } s \in S$$ (covering every person as cheaply as possible)

$$z_c \in \{0,1\} \quad \forall c \in C$$

3 Set Packing

Eventually Hannah realizes that no student can stand to hear the presentation multiple times, but that she really wants lots of practice with public speaking. She wants to give the presentation as many times as possible without any student seeing it more than once.

1. Write two concrete constraints: one that ensures that Ryan will see the presentation *at most once*, and one that ensures that Brandon will see the presentation *at most once*.

Ryan: $z_N + z_B \leq 1$

Brandon: $z_N + z_W \leq 1$

2. Why are these called **set packing constraints**? (Think of the set of classes.)

Pack elements in as many sets as possible

3. Write a condensed abstract model to find a collection of classes that maximizes the number of classes Hannah visits, while not seeing any student more than once.

Obj

Max $\sum_{c \in C} z_c$

Constraints

(as many classes as possible with no overlap)

$\sum_{c \in C} a_{cs} z_c \leq 1$ for $s \in S$

$z_c \in \{0,1\} \ \forall c \in C$

4 Set Partitioning] Hardest one to solve

Hannah receives a message of encouragement from the Chief of Staff and is told to be sure to show the presentation to *every single student*. But she still knows that no student can possibly sit through it twice, so she must revise her model again.

1. Write two concrete constraints: one that ensures that Aaron will see the presentation *exactly once*, and one that ensures that Sharon will see the presentation *exactly once*.

Aaron: $z_S + z_B = 1$

Sharon: $z_F + z_B = 1$

2. Why are these called **set partitioning constraints**? (Think of the set of students.)

Each person broken into seperate disjoint sets

3. Write an abstract model to find a collection of classes that minimizes the number of classes Hannah visits, while seeing every student exactly once.

Obj

$$\min \sum_{c \in C} z_c$$

Constraints

$$\sum_{c \in C} a_{cs} z_c = 1 \quad \text{for } s \in S$$ (Every person attends exactly 1 talk)

$$z_c \in \{0,1\} \ \forall c \in C$$