Lesson 5: Fixed-Charge Facility Location

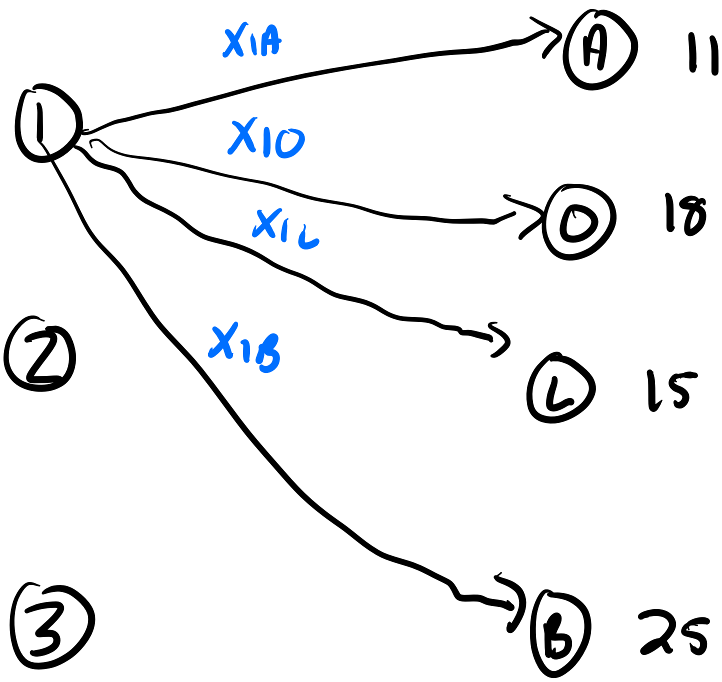
1 Gotit Grocery

Gotit Grocery Company is considering 3 locations for new distribution centers to serve its customers in Maryland. The following table shows the fixed cost (in millions of dollars) of opening each potential center, the number (in thousands) of truckloads forecasted to be demanded at each city over the next 5 years, and the transporation cost (in millions of dollars) per thousand truckloads moved from each center location to each city.

– If used pay fixed cost

	Fixed	Transport Costs			
	Cost	Annapolis	Ocean City	Laurel	Baltimore
Distribution Center 1	200	6	5	9	3
Distribution Center 2	400	4	3	5	6
Distribution Center 3	225	5	8	2	4
Demand	—	11	18	15	25

According to management, if Gotit does open a new distribution center, that center must send at least 10 thousand truckloads in order to be a worthwhile investment. Gotit seeks a minimum cost distribution system assuming any distribution center can meet any or all demands.



2 Concrete Model

Let $\mathbf{x_{i,j}}$ be the flow along edge (i, j). *Ignoring the facility opening costs/variables* write a concrete model for this transportation problem.

obj

min $6x_{1A} + \dots + 4x_{3B}$

Constraints

$x_{1A} + x_{2A} + x_{3A} = 11$
$x_{1D} + x_{2D} + x_{3D} = 18$
$x_{1L} + x_{2L} + x_{3L} = 15$
$x_{1B} + x_{2B} + x_{3B} = 25$

→ Demand

$x_{1A}, \dots, x_{3B} \geq 0$ → Non-negativity

Missing:

1. Fixed cost for opening DC
   - DC 1 is open pay \$200
2. DC logic constraints
   - DC 1 is open flow out of DC 1
   - DC 1 is closed no flow
3. DC lower bounds
   - DC 1 is open send at least 10 units

3 Binary Decision Variables to Represent Decisions

The use of binary $\{0, 1\}$ variables to model yes/no decisions is very common. In this model, we use a binary decision variable for each potential distribution center that indicates whether or not it is opened:

a value of $\boxed{1}$ indicates that the facility is used, and we must pay the "fixed-cost" to open it;

a value of $\boxed{0}$ indicates that the facility is not used, and therefore we don't have to pay for it.

Whenever we use binary decision variables, we must include $\boxed{}$ that enforce the correct behavior of the variable in the context of the model. **This can require some thought, careful logic, and even creativity.**

DC 1: $z_1 = 1$
① Pay \$200
② $x_{1A}, x_{1C}, x_{1B}, x_{1O}$ can be > 0

$z_1 = 0$ Closed
① Pay \$0
② $x_{1A} = x_{1B} = x_{1C} = x_{1O} = 0$
No flow from 1

1. Define three new decision variables, z_1, z_2, z_3, which encapsulate the facility opening logic.

let $z_1 = 1$ if DC 1 is used and 0 otherwise

let $z_2 = 1$ if DC 2 is used and 0 otherwise

let $z_3 = 1$ if DC 3 is used and 0 otherwise

2. Using these new decision variables, modify the objective function of your formulation to incorporate the fixed costs of opening the distribution centers.

Old obj: Min flow cost: $6x_{1A} + \cdots + 4x_{3B}$

If $z_1 = 1$ pay 200
$z_1 = 0$ pay 0

New obj: Min $\underbrace{\text{Flow cost}}$ + $\underbrace{\text{Facility cost}}$

$6x_{1A} + \cdots + 4x_{3B} + 200z_1 + 400z_2 + 225z_3$

4 Fixed-Charge Forcing Constraints

Explicitly, using the binary variables above, if $z_1 = 1$ then we are opening distribution center 1. However, implicitly, we must force the logic that if $z_1 = 0$ then:

No flow from 1 so $x_{1A} = x_{1B} = x_{1L} = x_{1O} = 0$

Constraints that enforce this logic are called **forcing constraints**. There are two options for this, single variable OR multiple variable.

3. Write a constraint that enforces the logic that if $z_1 = 0$ then $x_{1,A}$ must also be zero.

Goal: If $z_1 = 1$ x_{1A} free

$z_1 = 0$ $x_{1A} = 0$

Replace x_{1A} with

$z_1 \cdot x_{1A}$

$z_1 = 1$: x_{1A}

$z_1 = 0$: $x_{1A} = 0$

X Nonlinear

$x_{1A} = z_1$

$z_1 = 0 \Rightarrow x_{1A} = 0$ ✓

$z_1 = 1 \Rightarrow x_{1A} = 1$

X Fixing value of x_{1A}

Already have

$x_{1A} \geq 0$

Force $x_{1A} = 0$

$x_{1A} \leq 0 \cdot \underbrace{1}$ $z_1 = 0$

$x_{1A} \leq 11 \cdot \underbrace{1}$ $z_1 = 1$

Max value of x_{1A}

$\boxed{x_{1A} \leq 11 \cdot z_1}$

This type of constraint is often referred to as a **strong forcing constraint**.

$z_1 = 1$: $x_{1A} \leq 11$

$z_1 = 0$: $x_{1A} \leq 0$

4. Write all of the strong forcing constraints needed for this model.

DC 1

$x_{1A} \leq 11 z_1$

$x_{1O} \leq 18 \cdot z_1$

$x_{1L} \leq 15 \cdot z_1$

$x_{1B} \leq 25 \cdot z_1$

DC 2

$x_{2A} \leq 11 z_2$

$x_{2O} \leq 18 z_2$

$x_{2L} \leq 15 z_2$

$x_{2B} \leq 25 z_2$

DC 3

$x_{3A} \leq 11 z_3$

$x_{3O} \leq 18 z_3$

$x_{3L} \leq 15 z_3$

$x_{3B} \leq 25 z_3$

$z_1 = 1$ DC open

$x_{1A} \leq 11$ $x_{1C} \leq 15$

$x_{1O} \leq 18$ $x_{1B} \leq 25$

$z_1 = 0$ DC 1 closed

$x_{1A} \leq 0$ $x_{1L} \leq 0$

$x_{1O} \leq 0$ $x_{1B} \leq 0$

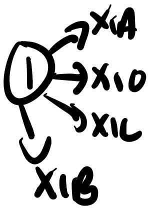
12 total constraints

1 for each edge

Idea: only have 1 constraint per DC

The next type of constraint is generally referred to as **weak forcing constraints**.

5. Write an inequality that enforces the logic that, if $z_1 = 0$ then $x_{1,A} = x_{1,O} = x_{1,L} = x_{1,B} = 0$.



Goal: If $Z_1 = 1$ $X_{1A}, X_{1B}, X_{1L}, X_{1O}$ free
If $Z_1 = 0$ $X_{1A} = X_{1B} = X_{1L} = X_{1O} = 0$
1 constraint

$Z_1 = 1$: $X_{1A} + X_{1O} + X_{1L} + X_{1B} \leq 69$
$Z_1 = 0$ $X_{1A} + X_{1O} + X_{1L} + X_{1B} \leq 0$

DC 1: Add strong constraints

$$X_{1A} + X_{1O} + X_{1L} + X_{1B} \leq \underbrace{69}_{\text{max possible flow from DC 1}} Z_1$$

6. Write all of the weak forcing constraints for this model.

DC 1: $X_{1A} + X_{1O} + X_{1L} + X_{1B} \leq 69 Z_1$
DC 2: $X_{2A} + X_{2O} + X_{2L} + X_{2B} \leq 69 Z_2$
DC 3: $X_{3A} + X_{3O} + X_{3L} + X_{3B} \leq 69 Z_3$

only include this OR strong forcing.

5 Lower Bound Constraints

The last type of constraint in this model also often comes up in fixed charge models. It forces a minimum amount of production to occur in order to use a facility. Recall that Gotit will only open a distribution center if it sends at least 10 thousand truck loads.

7. Write an inequality that enforces the logic that, if ~~$z_1 = 0$~~ $Z_1 = 1$ then at least 10 thousand truck loads must come from DC 1.

$Z_1 = 1$ $X_{1A} + X_{1O} + X_{1L} + X_{1B} \geq 10 \cdot 1$
$Z_1 = 0$ No lower bound $X_{1A} + X_{1O} + X_{1L} + X_{1B} \geq 0 \cdot 10$

$$X_{1A} + X_{1O} + X_{1L} + X_{1B} \geq 10 \cdot Z_1$$

8. Write all of the lower bound constraints for this model.

DC 1: $X_{1A} + X_{1O} + X_{1L} + X_{1B} \geq 10 Z_1$
DC 2: $X_{2A} + X_{2O} + X_{2L} + X_{2B} \geq 10 Z_2$
DC 3: $X_{3A} + X_{3O} + X_{3L} + X_{3B} \geq 10 Z_3$

6 Parameterized Model: Fixed-Charge Facility Location

Minimize ~~Maximize~~ $\sum_{i \in I} \sum_{j \in J} c_{ij} x_{ij} + \boxed{\sum_{i \in I} f_i \cdot z_i}$] Flow + fixed cost

subject to $\sum_{i \in I} x_{ij} \geq d_j, \text{ for } j \in J$

weak forcing constraints:

$\boxed{\sum_{j \in J} x_{ij} \leq 69 z_i \quad \text{for}}$ $i \in I$

OR

strong forcing constraints:

$\boxed{x_{ij} \leq d_j \cdot z_i \quad \text{for } i \in I}$ and $j \in J$

lower bound constraints:

$\boxed{\sum_{j \in J} x_{ij} \geq 10 \cdot z_i \quad \text{for}}$ $i \in I$

$x_{ij} \geq 0, \quad \text{integer}, \forall i \in I, j \in J$

$\boxed{z_i \in \{0, 1\} \quad \text{for}}$ $i \in I$

Sets

I: set of Distribution centers $I = \{1, 2, 3\}$

J: set of demand Cities $J = \{A, O, L, B\}$

New Parameters

f_i is the fixed cost of DC i for $i \in I$

New Variables

$z_i = 1$ if DC i is open for all $i \in I$