

User Guide for SLIP LU, A Sparse Left-Looking Integer Preserving LU Factorization

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1 Overview

SLIP LU is a software package designed to exactly solve unsymmetric sparse linear systems, $Ax = b$, where $A \in \mathbb{Q}^{n \times n}$, $b \in \mathbb{Q}^{n \times r}$, and $x \in \mathbb{Q}^{n \times r}$. This package performs a left-looking, roundoff-error-free (REF) LU factorization $PAQ = LDU$, where L and U are integer, D is diagonal, and P and Q are row and column permutations, respectively. Note that the matrix D is never explicitly computed nor needed; thus this package uses only the matrices L and U . The theory associated with this code is the Sparse Left-looking Integer-Preserving (SLIP) LU factorization [8]. Aside from solving sparse linear systems exactly, one of the key goals of this package is to provide a framework for other solvers to benchmark the reliability and stability of their linear solvers, as our final solution vector x is guaranteed to be exact. In addition, SLIP LU provides a wrapper class for the GNU Multiple Precision Arithmetic (GMP) [7] and GNU Multiple Precision Floating Point Reliable (MPFR) [6] libraries in order to prevent memory leaks and improve the overall stability of these external libraries. SLIP LU is written in ANSI C and is accompanied by a MATLAB interface.

For all primary computational routines in Section 8, the input argument A must be stored in a compressed sparse column (CSC) matrix with entries in `mpz_t` type (referred to as CSC `mpz_t` matrix henceforth), while b must be stored as a dense `mpz_t` matrix (i.e., a dense matrix with entries in `mpz_t` type). However, the original data type of entries in the user's input matrix A and right hand side (RHS) vectors b can be any one of: `double`, `int64_t`, `mpq_t`, `mpz_t`, or `mpfr_t`, and their format(s) are allowed to be CSC, sparse triplet or dense. A discussion of how to use these matrix formats and data types in the `SLIP_matrix`, and to perform conversions from the user's available format and type is given in Section 10.1.3.

The matrices L and U are computed using integer-preserving routines with the big integer (`mpz_t`) data types from the GMP Library [7]. The matrices L and U are computed one column at a time, where each column is computed via the sparse REF triangular solve detailed in [8]. All divisions performed in the algorithm are guaranteed to be exact (i.e., integer); therefore, no greatest common divisor algorithms are needed to reduce the size of entries.

The permutation matrices P and Q are either user specified or determined dynamically during the factorization. For the matrix P , the default option is to use a partial pivoting scheme in which the diagonal entry in column k is selected if it is the same magnitude as the smallest entry of k -th column, otherwise the smallest entry is selected as the k -th pivot. In addition to this approach, the code allows diagonal pivoting, partial pivoting which selects the largest pivot, or various tolerance based diagonal pivoting schemes. For the matrix Q , the default ordering is

the Column Approximate Minimum Degree (COLAMD) algorithm [4, 5]. Other approaches include using the Approximate Minimum Degree (AMD) ordering [1, 2], a user specified column ordering (i.e., the default column ordering applied to the input matrix). A discussion of how to select these permutations prior to factorization is given in Section 8.

Once the factorization $LDU = PAQ$ is computed, the solution vector x is computed via sparse REF forward and backward substitution. The forward substitution is a variant of the sparse REF triangular solve discussed above. The backward substitution is a typical column oriented sparse backward substitution. Both of these routines require b stored as a dense `mpz_t` matrix. At the conclusion of the forward and backward substitution routines, the final solution vector(s) x are guaranteed to be exact; thus are stored as a dense `mpq_t` matrix.

The final phase of SLIP LU comprises output routines. As mentioned above, the final solution x is by default a dense `mpq_t` matrix. Using the provided matrix copy functions, the solution vector(s) can be returned in either this format, or any of the 15 combinations of the set (CSC, triplet, dense) \times (`mpz_t`, `mpq_t`, `mpfr_t`, `int64_t`, or `double`). One key advantage of utilizing SLIP LU with floating-point output is that the solution is guaranteed to be exact until this final conversion; meaning that roundoff errors are only introduced in the final conversion from rational numbers. Thus, the solution x output in `double` precision are accurate to machine roundoff (approximately 10^{-16}) and SLIP LU utilizes higher precision for the MPFR output; thus it is also accurate to user specified precision.

Internally the vast majority of routines expect the input matrix A to be stored in CSC format. This data structure stores the matrix A as a sequence of three arrays:

- **A->p**: Column pointers; an array of size $n+1$. The row indices of column j are located in positions $A->p[j]$ to $A->p[j+1]-1$ of the array $A->i$. Data type: `int64_t`.
- **A->i**: Row indices; an array of size equal to the number of entries in the matrix. The entry $A->i[k]$ is the row index of the k th nonzero in the matrix. Data type: `int64_t`.
- **A->x**: Numeric entries. The entry $A->x[k]$ is the numeric value of the k th nonzero in the matrix. The array $A->x$ has a union type, and must be accessed via a suffix according to the type of A . For details, see Section 7.

An example matrix A with `mpz_t` type is stored as follows (notice that via C

convention, the indexing is zero based).

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 2 & 0 & 4 & 12 \\ 7 & 1 & 1 & 1 \\ 0 & 2 & 3 & 0 \end{bmatrix}$$

```
A->p      = [0, 3, 5, 8, 11]
A->i      = [0, 1, 2, 2, 3, 1, 2, 3, 0, 1, 2]
A->x.mpz = [1, 2, 7, 1, 2, 4, 1, 3, 1, 12, 1]
```

For example, the last column appears in positions 8 to 10 of `A->i` and `A->x.mpz`, with row indices 0, 1, and 2, and values $a_{03} = 1$, $a_{13} = 12$, and $a_{23} = 1$.

2 Availability

Copyright: This software is copyright by Christopher Lourenco, Jinhao Chen, Erick Moreno-Centeno, and Timothy Davis.

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License: This software package is dual licensed under the GNU General Public License version 2 or the GNU Lesser General Public License version 3. Details of this license can be seen in `SLIP_LU/License/license.txt`. For alternative licenses, please contact the authors.

Location: https://github.com/clouren/SLIP_LU and www.suitesparse.com

Required Packages: SLIP LU requires the installation of AMD [1, 2], COLAMD [5, 4], SuiteSparse_config [3], the GNU GMP [7] and GNU MPFR [6] libraries. AMD and COLAMD are available under a BSD 3-clause license, and no license restrictions apply to SuiteSparse_config. Notice that AMD, COLAMD, and SuiteSparse_config are included in this distribution for users' convenience. The GNU GMP and GNU MPFR library can be acquired and installed from <https://gmplib.org/> and <http://www.mpfr.org/> respectively.

If a user is running Unix that is Debian/Ubuntu based, a compatible version of GMP and MPFR can be installed with the following terminal commands:

```
sudo apt-get install libgmp3-dev
sudo apt-get install libmpfr-dev libmpfr-doc libmpfr4 libmpfr4-dbg
```

3 Installation

Installation of SLIP LU requires the `make` utility in Linux/MacOS, or Cygwin `make` in Windows. With the proper compiler, typing `make` under the main directory will compile AMD, COLAMD and SLIP LU to the respective `SLIP_LU/Lib` folder. To further install the libraries onto your computer, simply type `make install`. Thereafter, to use the code inside of your program, precede your code with `#include "SLIP_LU.h"`.

To run the statement coverage tests, go to the `Tcov` folder and type `make`. The last line of output should read:

```
statements not yet tested: 0
```

If you want to use SLIP LU within MATLAB, from your installation of MATLAB, `cd` to the folder `SLIP_LU/SLIP_LU/MATLAB` then type `SLIP_install`. This should compile the necessary code so that you can use SLIP LU within MATLAB. Note that this file does not add the correct directory to your path; therefore, if you want SLIP LU as a default function, type `pathtool` and save your path for future MATLAB sessions. If you cannot save your path because of file permissions, edit your `startup.m` by adding `addpath` commands (type `doc startup` and `doc addpath` for more information).

4 Managing the SLIP LU environment

4.1 SLIP_LU_VERSION: the software package version

SLIP LU defines the following strings with `#define`. Refer to the `SLIP_LU.h` file for details.

Macro	purpose
<code>SLIP_LU_VERSION</code>	current version of the code (as a string)
<code>SLIP_LU_VERSION_MAJOR</code>	major version of the code
<code>SLIP_LU_VERSION_MINOR</code>	minor version of the code
<code>SLIP_LU_VERSION_SUB</code>	sub version of the code

4.2 SLIP_info: status code returned by SLIP LU

Most SLIP LU functions return their status to the caller as their return value, an enumerated type called `SLIP_info`. All current possible values for `SLIP_info` are listed as follows:

0	SLIP_OK	The function was successfully executed.
-1	SLIP_OUT_OF_MEMORY	out of memory
-2	SLIP_SINGULAR	The input matrix A is exactly singular.
-3	SLIP_INCORRECT_INPUT	One or more input arguments are incorrect.
-4	SLIP_INCORRECT	The solution is incorrect.
-5	SLIP_PANIC	SLIP LU environment error

Either `SLIP_initialize` or `SLIP_initialize_expert` (but not both) must be called prior to using any other SLIP LU function. `SLIP_finalize` must be called as the last SLIP LU function.

Subsequent SLIP LU sessions can be restarted after a call to `SLIP_finalize`, by calling either `SLIP_initialize` or `SLIP_initialize_expert` (but not both), followed by a final call to `SLIP_finalize` when finished.

4.3 SLIP_initialize: initialize the working environment

```
SLIP_info SLIP_initialize
(
    void
);
```

`SLIP_initialize` initializes the working environment for SLIP LU functions. SLIP LU utilizes a specialized memory management scheme in order to prevent potential memory failures caused by GMP and MPFR libraries. Either this function or `SLIP_initialize_expert` must be called prior to using any other function in the library. Returns `SLIP_PANIC` if SLIP LU has already been initialized, or `SLIP_OK` if successful.

4.4 SLIP_initialize_expert: initialize environment (expert version)

```
SLIP_info SLIP_initialize_expert
(
    void* (*MyMalloc) (size_t),           // user-defined malloc
    void* (*MyCalloc) (size_t, size_t),   // user-defined calloc
    void* (*MyRealloc) (void *, size_t),   // user-defined realloc
    void (*MyFree) (void *)               // user-defined free
);
```


`SLIP_initialize_expert` is the same as `SLIP_initialize` except that it allows for a redefinition of custom memory functions that are used for SLIP LU and GM-P/MPFR. The four inputs to this function are pointers to four functions with the same signatures as the ANSI C `malloc`, `calloc`, `realloc`, and `free` functions. That is:

```
#include <stdlib.h>
void *malloc (size_t size) ;
void *calloc (size_t nmemb, size_t size) ;
void *realloc (void *ptr, size_t size) ;
void free (void *ptr) ;
```

Returns `SLIP_PANIC` if SLIP LU has already been initialized, or `SLIP_OK` if successful.

4.5 `SLIP_finalize`: free the working environment

```
SLIP_info SLIP_finalize
(
    void
) ;
```

`SLIP_finalize` finalizes the working environment for SLIP LU library, and frees any internal workspace created by SLIP LU. It must be called as the last `SLIP_*` function called, except that a subsequent call to `SLIP_initialize*` may be used to start another SLIP LU session. Returns `SLIP_PANIC` if SLIP LU has not been initialized, or `SLIP_OK` if successful.

5 Memory Management

The routines in this section are used to allocate and free memory for the data structures used in SLIP LU. By default, SLIP LU relies on the SuiteSparse memory management functions, `SuiteSparse_malloc`, `SuiteSparse_calloc`, `SuiteSparse_realloc`, and `SuiteSparse_free`. By default, those functions rely on the ANSI C `malloc`, `calloc`, `realloc`, and `free`, but this may be changed by initializing the SLIP LU environment with `SLIP_initialize_expert`.

5.1 SLIP_calloc: allocate initialized memory

```
void *SLIP_calloc
(
    size_t nitems,      // number of items to allocate
    size_t size         // size of each item
) ;
```

SLIP_calloc allocates a block of memory for an array of `nitems` elements, each of them `size` bytes long, and initializes all its bits to zero. If any input is less than 1, it is treated as if equal to 1. If the function failed to allocate the requested block of memory, then a NULL pointer is returned. Returns NULL if SLIP LU has not been initialized.

5.2 SLIP_malloc: allocate uninitialized memory

```
void *SLIP_malloc
(
    size_t size         // size of memory space to allocate
) ;
```

SLIP_malloc allocates a block of `size` bytes of memory, returning a pointer to the beginning of the block. The content of the newly allocated block of memory is not initialized, remaining with indeterminate values. If `size` is less than 1, it is treated as if equal to 1. If the function fails to allocate the requested block of memory, then a NULL pointer is returned. Returns NULL if SLIP LU has not been initialized.

5.3 SLIP_realloc: resize allocated memory

```
void *SLIP_realloc      // pointer to reallocated block, or original block
                        // if the realloc failed
(
    int64_t nitems_new,  // new number of items in the object
    int64_t nitems_old,  // old number of items in the object
    size_t size_of_item, // sizeof each item
    void *p,             // old object to reallocate
    bool *ok              // true if success, false on failure
) ;
```

SLIP_realloc is a wrapper for realloc. If `p` is non-NULL on input, it points to a previously allocated object of size `nitems_old` \times `size_of_item`. The object is reallocated to be of size `nitems_new` \times `size_of_item`. If `p` is NULL on input, then

a new object of that size is allocated. On success, a pointer to the new object is returned. Returns `ok` as `false` if SLIP LU has not been initialized.

If the reallocation fails, `p` is not modified, and `ok` is returned as `false` to indicate that the reallocation failed. If the size decreases or remains the same, then the method always succeeds (`ok` is returned as `true`), unless SLIP LU has not been initialized..

Typical usage: the following code fragment allocates an array of 10 `int`'s, and then increases the size of the array to 20 `int`'s. If the `SLIP_malloc` succeeds but the `SLIP_realloc` fails, then the array remains unmodified, of size 10.

```
int *p ;
p = SLIP_malloc (10 * sizeof (int)) ;
if (p == NULL) { error here ... }
printf ("p points to an array of size 10 * sizeof (int)\n") ;
bool ok ;
p = SLIP_realloc (20, 10, sizeof (int), p, &ok) ;
if (ok) printf ("p has size 20 * sizeof (int)\n") ;
else printf ("realloc failed; p still has size 10 * sizeof (int)\n") ;
SLIP_free (p) ;
```

5.4 SLIP_free: free allocated memory

```
void SLIP_free
(
    void *p          // Pointer to memory space to free
) ;
```

`SLIP_free` deallocates the memory previously allocated by a call to `SLIP_calloc`, `SLIP_malloc`, or `SLIP_realloc`. If `p` is `NULL` on input, then no action is taken (this is not an error condition). To guard against freeing the same memory space twice, the following macro `SLIP_FREE` is provided, which calls `SLIP_free` and then sets the freed pointer to `NULL`.

```
#define SLIP_FREE(p)          \
{                               \
    SLIP_free (p) ;           \
    (p) = NULL ;              \
}
```

Returns silently, with no action taken, if SLIP LU has not been initialized.

6 The SLIP_options object: parameter settings for SLIP LU

The `SLIP_options` object contains numerous parameters that may be modified by the user. Default values of these parameters will lead to good performance in most cases. Modifying this struct allows a user to control column orderings, pivoting schemes, and other components of the factorization.

6.1 SLIP_pivot: enum for pivoting schemes

There are six available pivoting schemes provided in SLIP LU. Users can set the pivoting method through the `SLIP_options` structure. If the matrix is non-singular (in an exact sense), then the pivot is always nonzero, and is chosen as the *smallest* nonzero entry, with the smallest magnitude. This may seem counter-intuitive, but selecting a small nonzero pivot leads to smaller growth in the number of digits in the entries of L and U. This choice does not lead to any kind of numerical inaccuracy, since SLIP LU is guaranteed to find an exact roundoff-error free factorization of a non-singular matrix (unless it runs out of memory), for any nonzero pivot choice.

The pivot tolerance for two of the pivoting schemes is specified by the `tol` component in `SLIP_options`. The pivoting schemes are as follows:

0	<code>SLIP_SMALLEST</code>	The k -th pivot is selected as the smallest entry in the k -th column.
1	<code>SLIP_DIAGONAL</code>	The k -th pivot is selected as the diagonal entry. If the diagonal entry is zero, this method instead selects the smallest pivot in the column.
2	<code>SLIP_FIRST_NONZERO</code>	The k -th pivot is selected as the first eligible nonzero in the column.
3	<code>SLIP_TOL_SMALLEST</code>	The k -th pivot is selected as the diagonal entry if the diagonal is within a specified tolerance of the smallest entry in the column. Otherwise, the smallest entry in the k -th column is selected. This is the default pivot selection strategy.
4	<code>SLIP_TOL_LARGEST</code>	The k -th pivot is selected as the diagonal entry if the diagonal is within a specified tolerance of the largest entry in the column. Otherwise, the largest entry in the k -th column is selected.
5	<code>SLIP_LARGEST</code>	The k -th pivot is selected as the largest entry in the k -th column.

6.2 SLIP_col_order: enum for column ordering schemes

The SLIP LU library provides three column ordering schemes: no pre-ordering, COLAMD, and AMD. Users can set the column ordering method through `order` component in the `SLIP_options` structure described in Section 6.3. In general, it is recommended that the user selects the COLAMD ordering, however, no preordering can be preferable if the user's matrix already has a good preordering.

0	SLIP_NO_ORDERING	No pre-ordering is performed on the matrix A , that is $Q = I$.
1	SLIP_COLAMD	The columns of A are permuted prior to factorization using the COLAMD [4] ordering. This is the default ordering.
2	SLIP_AMD	The nonzero pattern of $A + A^T$ is analyzed and the columns of A are permuted prior to factorization based on the AMD [2] ordering of $A + A^T$. This works well if A has a mostly symmetric pattern, but tends to be worse than COLAMD on matrices with unsymmetric pattern. [5].

6.3 SLIP_options structure

The `SLIP_options` struct stores key command parameters for various functions used in the SLIP LU package. The `SLIP_options* option` struct contains the following components:

- `option->pivot`: An enum `SLIP_pivot` type (discussed in Section 6.1) which controls the type of pivoting used. Default value: `SLIP_TOL_SMALLEST` (3).
- `option->order`: An enum `SLIP_col_order` type (discussed in Section 6.2) which controls what column ordering is used. Default value: `SLIP_COLAMD` (1).
- `option->tol`: A `double` which tells the tolerance used if the user selects a tolerance based pivoting scheme, i.e., `SLIP_TOL_SMALLEST` or `SLIP_TOL_LARGEST`. `option->tol` must be in the range of $(0, 1]$. Default value: 1 meaning that the diagonal entry will be selected if it has the same magnitude as the smallest entry in the k the column.
- `option->print_level`: An `int` which controls the amount of output: 0: print nothing, 1: just errors, 2: terse, with basic stats from COLAMD/AMD and SLIP, 3: all, with matrices and results. Default value: 0.
- `option->prec`: An `int32_t` which specifies the precision used if the user desires multiple precision floating point numbers, (i.e., MPFR). This can be any integer

larger than `MPFR_PREC_MIN` (value of 1 in MPFR 4.0.2 and 2 in some legacy versions) and smaller than `MPFR_PREC_MAX` (usually the largest possible integer available in your system). Default value: 128 (quad precision).

- `option->round`: A `mpfr_rnd_t` which determines the type of MPFR rounding to be used by SLIP LU. This is a parameter of the MPFR library. The options for this parameter are:
 - `MPFR_RNDN`: round to nearest (roundTiesToEven in IEEE 754-2008)
 - `MPFR_RNDZ`: round toward zero (roundTowardZero in IEEE 754-2008)
 - `MPFR_RNDU`: round toward plus infinity (roundTowardPositive in IEEE 754-2008)
 - `MPFR_RNDD`: round toward minus infinity (roundTowardNegative in IEEE 754-2008)
 - `MPFR_RNDA`: round away from zero
 - `MPFR_RNDF`: faithful rounding. This is not stable.

We refer the reader to the MPFR user guide available at <https://www.mpfr.org/mpfr-current/mpfr.pdf> for details on the MPFR rounding style and any other utilized MPFR convention. Default value: `MPFR_RNDN`.

- `option->check`: A `bool` which indicates whether the solution to the system should be checked. Intended for debugging only; the SLIP LU library is guaranteed to return the exact solution. Default value: `false`.

All SLIP LU routines except basic memory management routines in Sections 5.1-4.5 and `SLIP_options` allocation routine in 6.4 require `option` as an input argument. However, users can avoid creating one by passing `NULL` if the default settings are desired. Otherwise, users can use the following function/macro to create and destroy a `SLIP_options` object.

function/macro name	description	section
<code>SLIP_create_default_options</code>	create and return <code>SLIP_options</code> pointer with default parameters upon successful allocation	6.4
<code>SLIP_FREE</code>	destroy <code>SLIP_options</code> object	5.4

6.4 SLIP_create_default_options: create default SLIP_options object

```
SLIP_options* SLIP_create_default_options
(
    void
) ;
```

`SLIP_create_default_options` creates and returns a pointer to a `SLIP_options` struct with default parameters upon successful allocation, which are discussed in Section 6.3. To safely free the `SLIP_options* option` structure, simply use `SLIP_FREE(option)`. All functions that require `SLIP_options *option` as an input argument can have a `NULL` pointer passed instead. In this case, the default value of the corresponding command option is used. For example, if a `NULL` pointer is passed to the symbolic analysis routines, COLAMD is used. As a result, defaults are desired, the `SLIP_options` struct need not be allocated. Returns `NULL` if SLIP LU has not been initialized.

7 The SLIP_matrix object

All matrices for SLIP LU are stored as a `SLIP_matrix` object (a pointer to a `struct`). The matrix can be held in three formats: CSC, triplet or dense matrix (as discussed in Section 7.1) with entries stored as 5 different types: `mpz_t`, `mpq_t`, `mpfr_t`, `int64_t` and `double` (as discussed in Section 7.2). This gives a total of 15 different combinations of matrix format and entry type. Note that not all functions accept all 15 matrix types. Indeed, most functions expect the input matrix A to be a CSC `mpz_t` matrix while vectors (such as x and b) are in dense format.

7.1 SLIP_kind: enum for matrix formats

The SLIP LU library provides three available matrix formats: sparse CSC (compressed sparse column), sparse triplet and dense. Details for matrices in SLIP LU are discussed in Section 7.

0	SLIP_CSC	Matrix is in compressed sparse column format.
1	SLIP_TRIPLET	Matrix is in sparse triplet format.
2	SLIP_DENSE	Matrix is in dense format.

7.2 SLIP_type: enum for data types of matrix entry

The SLIP LU library provides five data types for matrix entries: `mpz_t`, `mpq_t`, `mpfr_t`, `int64_t` and `double`. Details for matrices in SLIP LU are discussed in Section 7.

0	SLIP_MPZ	Matrix entries are in <code>mpz_t</code> type: an integer of arbitrary size.
1	SLIP_MPQ	Matrix entries are in <code>mpq_t</code> type: a rational number with arbitrary-sized integer numerator and denominator.
2	SLIP_MPFR	Matrix entries are in <code>mpfr_t</code> type: a floating-point number of arbitrary precision.
3	SLIP_INT64	Matrix entries are in <code>int64_t</code> type.
4	SLIP_FP64	Matrix entries are in <code>double</code> type.

7.3 SLIP_matrix structure

A matrix `SLIP_matrix *A` has the following components:

- `A->m`: Number of rows in the matrix. Data Type: `int64_t`.
- `A->n`: Number of columns in the matrix. Data Type: `int64_t`.
- `A->nz`: The number of nonzeros in the matrix A , if A is a triplet matrix (ignored for matrices in CSC or dense formats). Data Type: `int64_t`.
- `A->nzmax`: The allocated size of the vectors `A->i`, `A->j` and `A->x`. Note that $A->nzmax \geq \text{nnz}(A)$, where $\text{nnz}(A)$ is the return value of `SLIP_matrix_nnz(A, option)`. Data Type: `int64_t`.
- `A->kind`: Indicating the kind of matrix A : CSC, triplet or dense. Data Type: `SLIP_kind`.
- `A->type`: Indicating the type of entries in matrix A : `mpz_t`, `mpq_t`, `mpfr_t`, `int64_t` or `double`. Data Type: `SLIP_type`.
- `A->p`: An array of size `A->n+1` which contains column pointers of A , if A is a CSC matrix (NULL for matrices in triplet or dense formats). Data Type: `int64_t*`.
- `A->p_shallow`: A boolean indicating whether `A->p` is shallow. Data Type: `bool`.

- **A->i**: An array of size **A->nzmax** which contains the row indices of the nonzeros in *A*, if *A* is a CSC or triplet matrix (NULL for dense matrices). The matrix is zero based therefore indices are in the range of $[0, A \rightarrow n - 1]$. Data Type: **int64_t***.
- **A->i_shallow**: A boolean indicating whether **A->i** is shallow. Data Type: **bool**.
- **A->j**: An array of size **A->nzmax** which contains the column indices of the nonzeros in *A*, if *A* is a triplet matrix (NULL for matrices in CSC or dense formats). The matrix is zero based therefore indices are in the range of $[0, A \rightarrow n - 1]$. Data Type: **int64_t***.
- **A->j_shallow**: A boolean indicating whether **A->j** is shallow. Data Type: **bool**.
- **A->x**: An array of size **A->nzmax** which contains the numeric values of the matrix. This array is a union, and must be accessed via one of: **A->x.mpz**, **A->x.mpq**, **A->x.mpfr**, **A->x.int64**, or **A->x.fp64**, depending on the **A->type** parameter. Data Type: **union**.
- **A->x_shallow**: A boolean indicating whether **A->x** is shallow. Data Type: **bool**.
- **A->scale**: A scaling parameter for matrix of **mpz_t** type. For all matrices whose entries are stored in data type other than **mpz_t**, **A->scale = 1**. This is used to ensure that entry can be represented as an integer in an **mpz_t** matrix if these entries are converted from non-integer type data (such as double, variable precision floating point, or rational). Data Type: **mpq_t**.

Specifically, for different kinds of *A* of size $A \rightarrow m \times A \rightarrow n$ with **nz** nonzero entries, its components are defined as:

- (0) **SLIP_CSC**: A sparse matrix in CSC (compressed sparse column) format. **A->p** is an **int64_t** array of size $A \rightarrow n + 1$, **A->i** is an **int64_t** array of size **A->nzmax** (with $nz \leq A \rightarrow nzmax$), and **A->x.TYPE** is an array of size **A->nzmax** of matrix entries (**TYPE** is one of **mpz**, **mpq**, **mpfr**, **int64**, or **fp64**). The row indices of column *j* appear in **A->i** $[A \rightarrow p[j] \dots A \rightarrow p[j+1]-1]$, and the values appear in the same locations in **A->x.TYPE**. The **A->j** array is NULL. **A->nz** is ignored; the number of entries in *A* is given by **A->p** $[A \rightarrow n]$.

- (1) **SLIP_TRIPLET**: A sparse matrix in triplet format. **A->i** and **A->j** are both `int64_t` arrays of size **A->nzmax**, and **A->x.TYPE** is an array of values of the same size. The k th tuple has row index **A->i [k]**, column index **A->j [k]**, and value **A->x.TYPE [k]**, with $0 \leq k < \mathbf{A}\text{->nz}$. The **A->p** array is **NULL**.
- (2) **SLIP_DENSE**: A dense matrix. The integer arrays **A->p**, **A->i**, and **A->j** are all **NULL**. **A->x.TYPE** is a pointer to an array of size **A->m*A->n**, stored in column-oriented format. The value of $A(i, j)$ is **A->x.TYPE [p]** with $p = i + j * \mathbf{A}\text{->m}$. **A->nz** is ignored; the number of entries in **A** is **A->m** \times **A->n**.

A may contain shallow components, **A->p**, **A->i**, **A->j**, and **A->x**. For example, if **A->p_shallow** is true, then a non-NULL **A->p** is a pointer to a read-only array, and the **A->p** array is not freed by **SLIP_matrix_free**. If **A->p** is **NULL** (for a triplet or dense matrix), then **A->p_shallow** has no effect.

To simplify the access the entries in **A**, **SLIP LU** package provides the following macros (Note that the **TYPE** parameter in the macros is one of: **mpz**, **mpq**, **mpfr**, **int64** or **fp64**):

- **SLIP_1D(A,k,TYPE)**: used to access the k th entry in **SLIP_matrix *A** using 1D linear addressing for any matrix kind (CSC, triplet or dense), in any type with **TYPE** specified corresponding
- **SLIP_2D(A,i,j,TYPE)**: used to access the (i, j) th entry in a dense **SLIP_matrix *A**.

The **SLIP LU** package has a set of functions to allocate, copy(convert), query and destroy a **SLIP LU** matrix, **SLIP_matrix**, as shown in the following table.

function name	description	section
SLIP_matrix_allocate	allocate a m -by- n SLIP_matrix	7.4
SLIP_matrix_copy	make a copy of a matrix, into another kind and/or type	7.6
SLIP_matrix_nnz	get the number of entries in a matrix	7.7
SLIP_matrix_free	destroy a SLIP_matrix and free its allocated memory	7.5

7.4 SLIP_matrix_allocate: allocate a m -by- n SLIP_matrix

```
SLIP_info SLIP_matrix_allocate
(
    SLIP_matrix **A_handle, // matrix to allocate
    SLIP_kind kind,         // CSC, triplet, or dense
    SLIP_type type,         // mpz, mpq, mpfr, int64, or double (fp64)
    int64_t m,              // # of rows
    int64_t n,              // # of columns
    int64_t nzmax,          // max # of entries
    bool shallow,           // if true, matrix is shallow. A->p, A->i,
                           // A->j, A->x are all returned as NULL and must
                           // be set by the caller. All A->*_shallow are
                           // returned as true.
    bool init,              // If true, and the data types are mpz, mpq, or
                           // mpfr, the entries are initialized (using the
                           // appropriate SLIP_mp*_init function). If
                           // false, the mpz, mpq, and mpfr arrays are
                           // allocated but not initialized.
    const SLIP_options *option
);
```

SLIP_matrix_allocate allocate memory space for a m -by- n SLIP_matrix whose kind (CSC, triplet or dense) and data type (mpz, mpq, mpfr, int64 or fp64) is specified. If 'shallow' is true, all components (p, i, j, x) are NULL, and their shallow flags are all true. The user can then set A->p, A->i, A->j, and/or A->x accordingly, from their own arrays. If 'shallow' is false, the appropriate individual arrays are calloc'd. The second boolean parameter is used if the entries are mpz_t, mpq_t, or mpfr_t. Specifically, if init is true, the individual entries within A->x.TYPE are initialized using the appropriate SLIP_mp*_init) function. Otherwise, if init is false, the A->x.TYPE array is calloc'd but the individual values are not initialized. Attempting to access the values of these initialized entries will lead to undefined behavior. Returns SLIP_PANIC if SLIP LU has not been initialized.

7.5 SLIP_matrix_free: free a SLIP_matrix

```
SLIP_info SLIP_matrix_free
(
    SLIP_matrix **A_handle, // matrix to free
    const SLIP_options *option
);
```

SLIP_matrix_free frees the SLIP_matrix *A. Note that the input of the function is the pointer to the pointer of a SLIP_matrix structure. This is because this

function internally sets the pointer of a `SLIP_matrix` to be `NULL` to prevent potential segmentation fault that could be caused by double `free`. If default settings are desired, `option` can be input as `NULL`.

7.6 `SLIP_matrix_copy`: make a copy of a `SLIP_matrix`

```
SLIP_info SLIP_matrix_copy
(
    SLIP_matrix **C,          // matrix to create (never shallow)
    // inputs, not modified:
    SLIP_kind kind,          // CSC, triplet, or dense
    SLIP_type type,          // mpz_t, mpq_t, mpfr_t, int64_t, or fp64
    SLIP_matrix *A,          // matrix to make a copy of (may be shallow)
    const SLIP_options *option
) ;
```

`SLIP_matrix_copy` creates a `SLIP_matrix *C` which is a modified copy of a `SLIP_matrix *A`. This function can convert between any pair of the 15 kinds of matrices, so the new matrix `C` can be of any type or kind different than `A`. On input `C` must be non-`NULL`, and the value of `*C` is ignored; it is overwritten, output with the matrix `C`, which is a copy of `A` of kind `kind` and type `type`.

The input matrix is assumed to be valid. It can be checked first with `SLIP_matrix_check` (Section 7.8), if desired. Results are undefined for an invalid input matrix `A`. Returns `SLIP_PANIC` if `SLIP LU` has not been initialized.

7.7 `SLIP_matrix_nnz`: get the number of entries in a `SLIP_matrix`

```
int64_t SLIP_matrix_nnz      // return # of entries in A, or -1 on error
(
    const SLIP_matrix *A,      // matrix to query
    const SLIP_options *option
) ;
```

`SLIP_matrix_nnz` returns the number of entries in a `SLIP_matrix *A`. For details regarding how the number of entries is obtained for different kinds of matrices, users can refer to Section 7. For any matrix with invalid dimension(s), this function returns -1. If default settings are desired, `option` can be input as `NULL`. Returns -1 if `SLIP LU` has not been initialized.

7.8 SLIP_matrix_check: check and print a SLIP_matrix

```
SLIP_info SLIP_matrix_check    // returns a SLIP_LU status code
(
    const SLIP_matrix *A,      // matrix to check
    const SLIP_options* option // defines the print level
) ;
```

`SLIP_matrix_check` checks the validity of a `SLIP_matrix *A` in any of the 15 different matrix types (CSC, triplet, dense) \times (mpz, mpq, mpfr, int64, fp64). Users can adjust the print level by changing `option->print_level` (refer to Section 6 for more details). If default settings are desired, `option` can be input as `NULL`. Returns `SLIP_PANIC` if SLIP LU has not been initialized.

8 Primary Computational Routines

These routines perform symbolic analysis, compute the LU factorization of the matrix A , and solve $Ax = b$ using the factorization of A .

8.1 SLIP_LU_analysis structure

The `SLIP_LU_analysis` data structure is used for storing the column permutation for LU and the guess on the number of nonzeros that may appear in L and U . Users do not need to modify this struct, just pass it into the functions. A `SLIP_LU_analysis` structure has the following components:

- `S->q`: The column permutation stored as a dense `int64_t` vector of size $n + 1$, where n is the number of columns of the analyzed matrix. Currently this vector is obtained via COLAMD, AMD, or is set to no ordering (i.e., $[0, 1, \dots, n - 1]$).
- `S->lnz`: An `int64_t` which is a guess for the number of nonzeros in L . `S->lnz` must be in the range of $[n, n^2]$. If `S->lnz` is too small, the program may waste time performing extra memory reallocations. This is set during the symbolic analysis.
- `S->unz`: An `int64_t` which is a guess for the number of nonzeros in U . `S->unz` must be in the range of $[n, n^2]$. If `S->unz` is too small, the program may waste time performing extra memory reallocations. This is set during the symbolic analysis.

The SLIP LU package provides the following functions to create and destroy a `SLIP_LU_analysis` object:

function/macro name	description	section
<code>SLIP_LU_analyze</code>	create <code>SLIP_LU_analysis</code> object	8.2
<code>SLIP_LU_analysis_free</code>	destroy <code>SLIP_LU_analysis</code> object	8.3

8.2 SLIP_LU_analyze: perform symbolic analysis

```
SLIP_info SLIP_LU_analyze
(
    SLIP_LU_analysis **S, // symbolic analysis (column permutation
                          // and nnz L,U)
    const SLIP_matrix *A, // Input matrix
    const SLIP_options *option // Control parameters
);
```

`SLIP_LU_analyze` performs the symbolic ordering for SLIP LU. Currently, there are three options: no ordering, COLAMD, or AMD, which are passed in by `SLIP_options *option`. For more details, users can refer to [Section 6](#).

The `SLIP_LU_analysis *S` is created by calling `SLIP_LU_analyze(&S, A, option)` with `SLIP_matrix *A` properly initialized as CSC matrix and `option` be NULL if default ordering (COLAMD) is desired. The value of `S` is ignored on input. On output, `S` is a pointer to the newly created symbolic analysis object and `SLIP_OK` is returned upon successful completion, or `S = NULL` with error status returned if a failure occurred. Returns `SLIP_PANIC` if SLIP LU has not been initialized.

The analysis `S` is freed by `SLIP_LU_analysis_free`.

8.3 SLIP_LU_analysis_free: free SLIP_LU_analysis structure

```
SLIP_info SLIP_LU_analysis_free
(
    SLIP_LU_analysis **S, // Structure to be deleted
    const SLIP_options *option
);
```

`SLIP_LU_analysis_free` frees a `SLIP_LU_analysis` structure. Note that the input of the function is the pointer to the pointer of a `SLIP_LU_analysis` structure. This is because this function internally sets the pointer of a `SLIP_LU_analysis` to be NULL to prevent potential segmentation fault that could be caused by double `free`. If default settings are desired, `option` can be input as NULL. Returns `SLIP_PANIC` if SLIP LU has not been initialized.

8.4 SLIP_LU_factorize: perform LU factorization

```
SLIP_info SLIP_LU_factorize
(
    // output:
    SLIP_matrix **L_handle,    // lower triangular matrix
    SLIP_matrix **U_handle,    // upper triangular matrix
    SLIP_matrix **rhos_handle, // sequence of pivots
    int64_t **pinv_handle,     // inverse row permutation
    // input:
    const SLIP_matrix *A,      // matrix to be factored
    const SLIP_LU_analysis *S, // stores guess on nnz
                                // and column permutation
    const SLIP_options* option
) ;
```

SLIP_LU_factorize performs the SLIP LU factorization. This factorization is done via n (number of rows or columns of the square matrix A) iterations of the sparse REF triangular solve function. The overall factorization is $PAQ = LDU$. This routine allows the user to separate factorization and solve. For example codes, please refer to either SLIP_LU/Demos/SLIPLU.c or Section 10.3.

On input, L , U , $rhos$, and $pinv$ are undefined and ignored. A must be a CSC mpz matrix. Default settings are used if $option$ is input as NULL.

Upon successful completion, the function returns SLIP_OK, and L and U are lower and upper triangular matrices, respectively, which are CSC matrices of type mpz. $rhos$ contains the sequence of pivots as an n -by-1 dense vector of type mpz.

After factorizing the matrix, the determinant of A can be obtained from $rhos[n-1]$ and $A->scale$ as follows:

```
mpq_t determinant ;
SLIP_mpq_init (determinant) ;
SLIP_mpq_set_z (determinant, rhos->x.mpz[rhos->n-1]) ;
SLIP_mpq_div (determinant, determinant, A->scale) ;
```

The output array $pinv$ contains the inverse row permutation (that is, the row index in the permuted matrix PA . For the i th row in A , $pinv[i]$ gives the row index in PA).

Returns SLIP_PANIC if SLIP LU has not been initialized. Otherwise, if another error occurs, L , U , $rhos$, and $pinv$ are all returned as NULL, and an error code will be returned correspondingly.

8.5 SLIP_LU_solve: solve the linear system $Ax = b$

```
SLIP_info SLIP_LU_solve          // solves the linear system  $LD^{(-1)}U x = b$ 
(
    // Output
    SLIP_matrix **X_handle,      // rational solution to the system
    // input:
    const SLIP_matrix *b,        // right hand side vector
    const SLIP_matrix *A,        // Input matrix
    const SLIP_matrix *L,        // lower triangular matrix
    const SLIP_matrix *U,        // upper triangular matrix
    const SLIP_matrix *rhos,     // sequence of pivots
    const SLIP_LU_analysis *S,   // symbolic analysis struct
    const int64_t *pinv,         // inverse row permutation
    const SLIP_options* option
);
```

SLIP_LU_solve obtains the solution of `mpq_t` type to the linear system $Ax = b$ upon a successful factorization. This function may be called after a successful return from SLIP_LU_factorize, which computes L, U, rhos, and pinv.

On input, `SLIP_matrix *x` is undefined. A, L and U must be CSC `mpz_t` matrices while `b` and `rhos` must be dense `mpz_t` matrices. All matrices must have matched dimensions: the matrices L and U must be square lower and upper triangular matrices the same size as A, and `rhos` must be a dense n-by-1 vector. The input matrix `b` must have same number of rows as A. Default settings are used if `option` is input as NULL.

Upon successful completion, the function returns SLIP_OK, and `x` contains the solution of `mpq_t` type with dense format to the linear system $Ax = b$. If desired, `option->check` can be set to `true` to enable a post-check of the solution of this function. However, this is intended for debugging only; the SLIP LU library is guaranteed to return the exact solution. Otherwise (in case of error occurred), the function returns corresponding error code.

Like some other routines discussed in this section, this function is primarily for advanced users who might want intermediate calculation results; thus for usage information please refer to either SLIP_LU/Demos/SLIPLU.c or Section 10.3. Returns SLIP_PANIC if SLIP LU has not been initialized.

8.6 SLIP_backslash: solve $Ax = b$ and return x in user desired type

```
SLIP_info SLIP_backslash
(
    // Output
    SLIP_matrix **X_handle,      // Final solution vector
    // Input
    SLIP_type type,              // Type of output desired:
                                // Must be SLIP_MPQ, SLIP_MPFR,
                                // or SLIP_FP64
    const SLIP_matrix *A,        // Input matrix
    const SLIP_matrix *b,        // Right hand side vector(s)
    const SLIP_options* option
);
```

`SLIP_backslash` solves the linear system $Ax = b$ and returns the solution as a dense matrix of `mpq_t`, `mpfr_t` or `double` numbers. This function performs symbolic analysis, factorization, and solving all in one line. It can be thought of as an exact version of MATLAB sparse backslash.

On input, `SLIP_matrix *x` is undefined. `type` must be one of: `SLIP_MPQ`, `SLIP_MPFR` or `SLIP_FP64` to specify the data type of the solution entries. `A` should be a square CSC `mpz_t` matrix while `b` should be a dense `mpz_t` matrix. In addition, `A->m` should be equal to `b->m`. Default settings are used if `option` is input as `NULL`.

Upon successful completion, the function returns `SLIP_OK`, and `x` contains the solution of data type specified by `type` to the linear system $Ax = b$. If desired, `option->check` can be set to `true` to enable solution checking process in this function. However, this is intended for debugging only; SLIP LU library is guaranteed to return the exact solution. Otherwise (in case of error occurred), the function returns corresponding error code.

Returns `SLIP_PANIC` if SLIP LU has not been initialized.

For a complete example, users can refer to `SLIP_LU/Demos/example.c`, `SLIP_LU/Demos/example2.c`, or Section [10.2](#).

9 SLIP LU wrapper functions for GMP and MPFR

SLIP LU provides a wrapper class for all GMP and MPFR functions used by SLIP LU. The wrapper class provides error-handling for out-of-memory conditions that are not handled by the GMP and MPFR libraries. These wrapper functions are

used inside all SLIP LU functions, wherever any GMP or MPFR functions are used. These functions may also be called by the end-user application.

Each wrapped function has the same name as its corresponding GMP/MPFR function with the added prefix `SLIP_`. For example, the default GMP function `mpz_mul` is changed to `SLIP_mpz_mul`. Each SLIP GMP/MPFR function returns `SLIP_OK` if successful or the correct error code if not. The following table gives a brief list of each currently covered SLIP GMP/MPFR function. For a detailed description of each function, please refer to `SLIP_LU/Source/SLIP_gmp.c`.

If additional GMP and MPFR functions are needed in the end-user application, this wrapper mechanism can be extended to those functions. Below, we give instructions on how to do this.

Given a GMP function `void gmpfunc(TYPEa a, TYPEb b, ...)`, where `TYPEa` and `TYPEb` can be GMP type data (`mpz_t`, `mpq_t` and `mpfr_t`, for example) or non-GMP type data (`int`, `double`, for example), and they need not to be the same. In order to apply our wrapper to a new function, one can create it as follows:

```
SLIP_info SLIP_gmpfunc
(
    TYPEa a,
    TYPEb b,
    ...
)
{
    // Start the GMP Wrapper
    // uncomment one of the followings that meets the needs
    // If this function is not modifying any GMP/MPFR type variable, then use
    //SLIP_GMP_WRAPPER_START;
    // If this function is modifying mpz_t type (say TYPEa = mpz_t), then use
    //SLIP_GMPZ_WRAPPER_START(a) ;
    // If this function is modifying mpq_t type (say TYPEa = mpq_t), then use
    //SLIP_GMPQ_WRAPPER_START(a) ;
    // If this function is modifying mpfr_t type (say TYPEa = mpfr_t), then use
    //SLIP_GMPFR_WRAPPER_START(a) ;

    // Call the GMP function
    gmpfunc(a,b,...) ;

    //Finish the wrapper and return ok if successful.
    SLIP_GMP_WRAPPER_FINISH;
    return SLIP_OK;
}
```

Note that, other than `SLIP_mpfr_fprintf`, `SLIP_gmp_fprintf`, `SLIP_gmp_printf`

and `SLIP_gmp_fscanf`, all of the wrapped GMP/MPFR functions always return `SLIP_info` to the caller. Therefore, for some GMP/MPFR functions that have their own return value. For example, for `int mpq_cmp(const mpq_t a, const mpq_t b)`, the return value becomes a parameter of the wrapped function. In general, a GMP/MPFR function in the form of `TYPEr gmpfunc(TYPEa a, TYPEb b, ...)`, users can create the wrapped function as follows:

```
SLIP_info SLIP_gmpfunc
(
    TYPEr *r,          // return value of the GMP/MPFR function
    TYPEa a,
    TYPEb b,
    ...
)
{
    // Start the GMP Wrapper
    //SLIP_GMP_WRAPPER_START;

    // Call the GMP function
    *r = gmpfunc(a,b,...) ;

    //Finish the wrapper and return ok if successful.
    SLIP_GMP_WRAPPER_FINISH;
    return SLIP_OK;
}
```

MPFR Function	SLIP_MPFR Function	Description
<code>n = mpfr_asprintf(&buff, fmt, ...)</code>	<code>n = SLIP_mpfr_asprintf(&buff, fmt, ...)</code>	Print format to allocated string
<code>mpfr_free_str(buff)</code>	<code>SLIP_mpfr_free_str(buff)</code>	Free string allocated by MPFR
<code>mpfr_init2(x, size)</code>	<code>SLIP_mpfr_init2(x, size)</code>	Initialize x with size bits
<code>mpfr_set(x, y, rnd)</code>	<code>SLIP_mpfr_set(x, y, rnd)</code>	$x = y$
<code>mpfr_set_d(x, y, rnd)</code>	<code>SLIP_mpfr_set_d(x, y, rnd)</code>	$x = y$ (double)
<code>mpfr_set_q(x, y, rnd)</code>	<code>SLIP_mpfr_set_q(x, y, rnd)</code>	$x = y$ (mpq_t)
<code>mpfr_set_z(x, y, rnd)</code>	<code>SLIP_mpfr_set_z(x, y, rnd)</code>	$x = y$ (mpz_t)
<code>mpfr_get_z(x, y, rnd)</code>	<code>SLIP_mpfr_get_z(x, y, rnd)</code>	(mpz_t) $x = y$
<code>x = mpfr_get_d(y, rnd)</code>	<code>SLIP_mpfr_get_d(x, y, rnd)</code>	(double) $x = y$
<code>mpfr_mul(x, y, z, rnd)</code>	<code>SLIP_mpfr_mul(x, y, z, rnd)</code>	$x = y * z$
<code>mpfr_mul_d(x, y, z, rnd)</code>	<code>SLIP_mpfr_mul_d(x, y, z, rnd)</code>	$x = y * z$
<code>mpfr_div_d(x, y, z, rnd)</code>	<code>SLIP_mpfr_div_d(x, y, z, rnd)</code>	$x = y / z$
<code>mpfr_ui_pow_ui(x, y, z, rnd)</code>	<code>SLIP_mpfr_ui_pow_ui(x, y, z, rnd)</code>	$x = y^z$
<code>mpfr_log2(x, y, rnd)</code>	<code>SLIP_mpfr_log2(x, y, rnd)</code>	$x = \log_2(y)$
<code>mpfr_free_cache()</code>	<code>SLIP_mpfr_free_cache()</code>	Free cache after log2
GMP Function	SLIP_GMP Function	Description
<code>n = gmp_fscanf(fp, fmt, ...)</code>	<code>n = SLIP_gmp_fscanf(fp, fmt, ...)</code>	Read from file fp
<code>mpz_init(x)</code>	<code>SLIP_mpz_init(x)</code>	Initialize x
<code>mpz_init2(x, size)</code>	<code>SLIP_mpz_init2(x, size)</code>	Initialize x to size bits
<code>mpz_set(x, y)</code>	<code>SLIP_mpz_set(x, y)</code>	$x = y$ (mpz_t)
<code>mpz_set_ui(x, y)</code>	<code>SLIP_mpz_set_ui(x, y)</code>	$x = y$ (signed int)
<code>mpz_set_si(x, y)</code>	<code>SLIP_mpz_set_si(x, y)</code>	$x = y$ (unsigned int)
<code>mpz_set_d(x, y)</code>	<code>SLIP_mpz_set_d(x, y)</code>	$x = y$ (double)
<code>x = mpz_get_d(y)</code>	<code>SLIP_mpz_get_d(x, y)</code>	$x = y$ (double out)
<code>mpz_set_q(x, y)</code>	<code>SLIP_mpz_set_q(x, y)</code>	$x = y$ (mpq_t)
<code>mpz_mul(x, y, z)</code>	<code>SLIP_mpz_mul(x, y, z)</code>	$x = y * z$
<code>mpz_add(x, y, z)</code>	<code>SLIP_mpz_add(x, y, z)</code>	$x = y + z$
<code>mpz_addmul(x, y, z)</code>	<code>SLIP_mpz_addmul(x, y, z)</code>	$x = x + y * z$
<code>mpz_submul(x, y, z)</code>	<code>SLIP_mpz_submul(x, y, z)</code>	$x = x - y * z$
<code>mpz_divexact(x, y, z)</code>	<code>SLIP_mpz_divexact(x, y, z)</code>	$x = y / z$
<code>gcd = mpz_gcd(x, y)</code>	<code>SLIP_mpz_gcd(gcd, x, y)</code>	$gcd = gcd(x, y)$
<code>lcm = mpz_lcm(x, y)</code>	<code>SLIP_mpz_lcm(lcm, x, y)</code>	$lcm = lcm(x, y)$
<code>mpz_abs(x, y)</code>	<code>SLIP_mpz_abs(x, y)</code>	$x = y $
<code>r = mpz_cmp(x, y)</code>	<code>SLIP_mpz_cmp(r, x, y)</code>	$r = 0$ if $x = y$ $r \neq 0$ if $x \neq y$
<code>r = mpz_cmpabs(x, y)</code>	<code>SLIP_mpz_cmpabs(r, x, y)</code>	$r = 0$ if $ x = y $ $r \neq 0$ if $ x \neq y $
<code>r = mpz_cmp_ui(x, y)</code>	<code>SLIP_mpz_cmp_ui(r, x, y)</code>	$r = 0$ if $x = y$ $r \neq 0$ if $x \neq y$
<code>sgn = mpz_sgn(x)</code>	<code>SLIP_mpz_sgn(sgn, x)</code>	$sgn = 0$ if $x = 0$
<code>size = mpz_sizeinbase(x, base)</code>	<code>SLIP_mpz_sizeinbase(size, x, base)</code>	size of x in base
<code>mpq_init(x)</code>	<code>SLIP_mpq_init(x)</code>	Initialize x
<code>mpq_set(x, y)</code>	<code>SLIP_mpq_set(x, y)</code>	$x = y$
<code>mpq_set_z(x, y)</code>	<code>SLIP_mpq_set_z(x, y)</code>	$x = y$ (mpz)
<code>mpq_set_d(x, y)</code>	<code>SLIP_mpq_set_d(x, y)</code>	$x = y$ (double)
<code>mpq_set_ui(x, y, z)</code>	<code>SLIP_mpq_set_ui(x, y, z)</code>	$x = y / z$ (unsigned int)
<code>mpq_set_num(x, y)</code>	<code>SLIP_mpq_set_num(x, y)</code>	$num(x) = y$
<code>mpq_set_den(x, y)</code>	<code>SLIP_mpq_set_den(x, y)</code>	$den(x) = y$
<code>mpq_get_den(x, y)</code>	<code>SLIP_mpq_get_den(x, y)</code>	$x = den(y)$
<code>x = mpq_get_d(y)</code>	<code>SLIP_mpq_get_d(x, y)</code>	(double) $x = y$
<code>mpq_abs(x, y)</code>	<code>SLIP_mpq_abs(x, y)</code>	$x = y $
<code>mpq_add(x, y, z)</code>	<code>SLIP_mpq_add(x, y, z)</code>	$x = y + z$
<code>mpq_mul(x, y, z)</code>	<code>SLIP_mpq_mul(x, y, z)</code>	$x = y * z$
<code>mpq_div(x, y, z)</code>	<code>SLIP_mpq_div(x, y, z)</code>	$x = y / z$
<code>r = mpq_cmp(x, y)</code>	<code>SLIP_mpq_cmp(r, x, y)</code>	$r = 0$ if $x = y$ $r \neq 0$ if $x \neq y$
<code>r = mpq_cmp_ui(x, n, d)</code>	<code>SLIP_mpq_cmp_ui(r, x, n, d)</code>	$r = 0$ if $x = n/d$ $r \neq 0$ if $x \neq n/d$
<code>r = mpq_equal(x, y)</code>	<code>SLIP_mpq_equal(r, x, y)</code>	$r = 0$ if $x = y$ $r \neq 0$ if $x \neq y$

10 Using SLIP LU in C

Using SLIP LU in C has three steps:

1. initialize and populate data structures,
2. perform symbolic analysis, factorize the matrix A and solve the linear system for each b vector, and
3. free all used memory and finalize.

Step 1 is discussed in Section 10.1. For Step 2, performing symbolic analysis and factorizing A and solving the linear $Ax = b$ can be done in one of two ways. If the user is only interested in obtaining the solution vector x , SLIP LU provides a simple interface for this purpose which is discussed in Section 10.2. Alternatively, if the user wants the actual L and U factors, please refer to Section 10.3. Finally, step 3 is discussed in Section 10.4. For the remainder of this section, `n` will indicate the dimension of A (that is, $A \in \mathbb{Z}^{n \times n}$) and `numRHS` will indicate the number of right hand side vectors being solved (that is, if `numRHS` = r , then $b \in \mathbb{Z}^{n \times r}$).

10.1 SLIP LU initialization and population of data structures

This section discusses how to initialize and populate the global data structures required for SLIP LU.

10.1.1 Initializing the environment

SLIP LU is built upon the GNU Multiple Precision Arithmetic (GMP) [7] and GNU Multiple Precision Floating Point Reliable (MPFR) [6] libraries and provides wrappers to all GMP/MPFR functions it uses. This allows SLIP LU to properly handle memory management failures, which GMP/MPFR does not handle. It may also allow the user to not need any direct access to the GMP/MPFR libraries. To enable this mechanism, SLIP LU requires initialization. The following must be done before using any other SLIP LU function:

```
SLIP_initialize ( ) ;  
// or SLIP_initialize_expert (...); if custom memory functions are desired
```

10.1.2 Initializing data structures

SLIP LU assumes three specific input options for all functions. These are:

- **SLIP_matrix* A** and **SLIP_matrix *b**: **A** contains the user's input matrix, while **b** contains the user's right hand side vector(s). If the input matrix was already an integer matrix, **A** is the user's input and **A->scale=1**. Otherwise, the input matrix is not integer and **A** contains the user's scaled input matrix. **b** is handled in the same way.
- **SLIP_LU_analysis* S**: **S** contains the column permutation used for **A** as well as guesses for the number of nonzeros in **L** and **U**.
- **SLIP_options* option**: **option** contains various control options for the factorization including column ordering used, pivot selection scheme, and others. For a full list of the contents of the **SLIP_options** structure, please refer to Section 6. If default settings are desired, **option** can be set to **NULL**.

10.1.3 Populating data structures

Of the three data structures discussed in Section 10.1.2, **S** is constructed during symbolic analysis (Section 8.2), and **option** can be provided as **NULL** unless user desires setting different from default. Please refer to Section 6 for the contents of **option**.

SLIP LU allows the input numerical data for **A** and **b** to come in one of 5 types: **int64_t**, **double**, **mpfr_t**, **mpq_t**, and **mpz_t**. Moreover, both **A** and **b** can be stored in CSC form, sparse triplet form or dense form. CSC form is discussed in Section 1. The triplet form stores the contents of the matrix **A** in three arrays **i**, **j**, and **x** where the k th nonzero entry is stored as $A(i[k], j[k]) = x[k]$. SLIP LU stores its dense matrices in in column-oriented format, that is, the (i, j) th entry in **A** is **A->x.TYPE[p]** with $p = i + j * \text{A->m}$.

If the data for matrices are in file format to be read, user can refer to **SLIP_LU/Demo/example2.c** on how to read in data and construct **A** and **b**. If the data for matrices are already stored in vectors corresponding to CSC form, sparse triplet form or dense form, the user can allocate a shallow **SLIP_matrix** and assign vectors accordingly, then use **SLIP_matrix_copy** to get a **SLIP_matrix** in the desired kind and type. For more details, refer to **SLIP_LU/Demo/example.c**. In a case when **A** is available in format other than CSC **mpz**, and/or **b** is available in format other than dense **mpz**, the following code snippet shows how to get **A** and **b** in a required format.

```

/* Get the matrix A. Assume that A1 is stored in CSC form
   with mpfr_t entries, while b1 is stored in triplet form
   with mpq_t entries. (for A1 and b1 in any other form,
   the exact same code will work) */

SLIP_matrix *A, *b;
// A is a copy of the A1. A is a CSC matrix with mpz_t entries
SLIP_matrix_copy(&A, SLIP_CSC, SLIP_MPZ, A1, option);
// b is a copy of the b1. b is a dense matrix with mpz_t entries.
SLIP_matrix_copy(&b, SLIP_DENSE, SLIP_MPZ, b1, option);

```

10.2 Simple SLIP LU routines for solving linear systems

After initializing the necessary data structures, SLIP LU obtains the solution to $Ax = b$ using the “simple” interface of SLIP LU, which requires only that user decides what data type that he/she wants for the entries of `SLIP_matrix *x`. SLIP LU allows entries of `x` to be `double`, `mpq_t`, or `mpfr_t` with an associated precision. This is done by using `SLIP_backslash` (Section 8.6).

The following code snippet shows how to get solution as a dense `mpq_t` matrix. User can modify accordingly to meet one’s need.

```

SLIP_matrix *x;
SLIP_type my_type = SLIP_MPQ; // SLIP_MPQ, SLIP_MPFR, SLIP_FP64
SLIP_backslash(&x, my_type, A, b, option) ;

```

On successful return, this function returns `SLIP_OK` (see Section 4.2).

10.3 Expert SLIP LU routines

If a user wishes to perform the SLIP LU factorization of the matrix A while capturing information about the factorization itself and solving the linear system, extra steps must be performed that are all done internally in the `SLIP_backslash` method described in the previous subsection.

To split this work into its component steps, the following steps must be performed:

1. declare `L`, `U`, the solution matrix `x`, and others,
2. perform symbolic analysis,

3. compute the factorization $PAQ = LDU$,
4. solve the linear system $Ax = b$, and
5. convert the final solution into the final desired form.

Below, we discuss each of these steps followed by an example of putting it all together.

10.3.1 Declare workspace

Using SLIP LU in this form requires the intermediate variables be declared, such as L , U , etc. The following code snippet shows the detailed list.

```
// A and b are in required type and ready to use
SLIP_matrix *L = NULL;
SLIP_matrix *U = NULL;
SLIP_matrix *x = NULL;
SLIP_matrix *rhos = NULL;
int64_t* pinv = NULL;
SLIP_LU_analysis* S = NULL;

// option needs no declaration if default setting is desired
// only declare option for further modification on default setting
SLIP_options *option = SLIP_create_default_options();
```

10.3.2 SLIP LU symbolic analysis

The symbolic analysis phase of SLIP LU computes the column permutation and guesses for the number of nonzeros in L and U . This function is called as:

```
SLIP_LU_analyze (&S, A, option) ;
```

10.3.3 Computing the factorization

The matrices L and U , the pivot sequence $rhos$, and the row permutation $pinv$ are computed via the `SLIP_LU_factorize` function (Section 8.4). Upon successful completion, this function returns `SLIP_OK`.

10.3.4 Solving the linear system

After factorization, the next step is to solve the linear system and store the solution as a dense matrix `x` with entries of rational number `mpq_t`. This solution is done via the `SLIP_LU_solve` function (Section 8.5). Upon successful completion, this function returns `SLIP_OK`.

In this step, user can set `option->check` to `true` to enable the solution check process as discussed in Section 8.5. The process can verify that the solution vector `x` satisfies $Ax = b$ in perfect precision intended for debugging. This step is not needed, since the solution returned is guaranteed to be exact. It appears here simply as debugging tool, and as a verification that SLIP LU is computing its expected result. This test can fail only if it runs out of memory, or if there is a bug in the code (in which case, please notify the authors). Also, note that this process can be quite time consuming; thus it is not recommended to be used in general.

10.3.5 Converting the solution vector to the final desired form

Upon completion of the above routines, the solution to the linear system is in a dense `mpq_t` matrix. SLIP LU allows this to be converted into any form of matrix in the set of (CSC, sparse triplet, dense) \times (`mpfr_t`, `mpq_t`, `double`) using `SLIP_matrix_copy`. The following code snippet shows how to get solution as a dense `double` matrix; since this involves a floating-point representation, the solution `my_x` will no longer be exact, even though `x` is the exact solution.

```
SLIP_kind my_kind = SLIP_DENSE; // SLIP_CSC, SLIP_TRIPLET or SLIP_DENSE
SLIP_type my_type = SLIP_FP64;  // SLIP_MPQ, SLIP_MPFR, or SLIP_FP64

SLIP_matrix* my_x = NULL;        // New output
Create copy which is stored as my_kind and my_type:
SLIP_matrix_copy( &my_x, my_kind, my_type, x, option);
```

10.4 Freeing memory

As described in Section 5, SLIP LU provides a number of functions/macros to handle this for the user. Below, we briefly summarize which memory freeing routine should be used for specific data types:

- `SLIP_matrix*`: A `SLIP_matrix*` A data structure can be freed with a call to `SLIP_matrix_free(&A, NULL)` ;

- **SLIP_LU_analysis***: A **SLIP_LU_analysis*** S data structure can be freed with a call to **SLIP_LU_analysis_free(&S, NULL)** ;
- All others including **SLIP_options***: These data structures can be freed with a call to the macro **SLIP_FREE()**, for example, **SLIP_FREE(option)** for **SLIP_options* option**.

After all usage of the SLIP LU routines is finished, one must call **SLIP_finalize()** (Section 4.5) to finalize usage of the library.

10.5 Examples of using SLIP LU in a C program

The **SLIP_LU/Demo** folder contains three sample C codes which utilize SLIP LU. These files demonstrate the usage of SLIP LU as follows:

- **example.c**: This example generates a random dense 50×50 matrix and a random dense 50×1 right hand side vector b and solves the linear system. In this function, the **SLIP_backslash** function is used; and the output is given as a double matrix.
- **example2.c**: This example reads in a matrix stored in triplet format from the **ExampleMats** folder. Additionally, it reads in a right hand side vector from this folder and solves the associated linear system via the **SLIP_backslash** function, and, the solution is given as a matrix of rational numbers.
- **SLIPLU.c**: This example reads in a matrix and right hand side vector from a file and solves the linear system $Ax = b$ using the techniques discussed in Section 10.3. This file also allows command line arguments (discussed in **README.md**) and can be used to replicate the results from [8].

11 Using SLIP LU in MATLAB

After following the installation steps discussed in Section 3, using the SLIP LU factorization within MATLAB can be done via the **SLIP_LU.m** function. First, this section describes the **options** struct in Section 11.1 then we describe how to use the factorization in Section 11.2. Again, recall that by default the SLIP LU MATLAB routines are not natively installed into your MATLAB installation; thus if you want to use them in a different directory please add the **SLIP_LU/MATLAB** folder to your path.

11.1 Optional parameter settings

Much like the C routines described throughout, the SLIP LU MATLAB interface has various parameters that the user can modify to control the factorization. In MATLAB, these are stored in the `option` struct. Notice that this struct is optional can be avoided if default options are used. The elements of the `option` struct are listed below. Any fields not present in the struct are treated as their default values.

- `option.pivot`: This parameter is a string that controls the pivoting scheme used. When selecting a pivot entry in a given column, the factorization method uses one of the following pivoting strategies:
 - `'smallest'`: smallest pivot,
 - `'diagonal'`: diagonal pivot if possible, otherwise smallest pivot,
 - `'first'`: first nonzero pivot in each column,
 - `'tol smallest'`: (default) diagonal pivot with a tolerance (`option.tol`) for the smallest pivot,
 - `'tol largest'`: diagonal pivot with a tolerance (`option.tol`) for the largest pivot,
 - `'largest'`: largest pivot.

It is recommended that the user always selects either `'tol smallest'` or `'diagonal'` for this parameter, unless they are trying to extract the Doolittle factors, then `'largest'` may be appropriate (due to the size of numbers in Doolittle).

- `option.order`: This parameter is a string controls the fill-reducing column preordering used.
 - `'none'`: no column ordering; factorize **A** as-is.
 - `'colamd'`: COLAMD ordering (default)
 - `'amd'`: AMD ordering

It is usually recommended that the user keep this at `'colamd'` unless they already have a good column permutation. The `'AMD'` ordering is suitable if the nonzero pattern of **A** is mostly symmetric. In this case, `option.pivot = 'diagonal'` is a useful option.

- `option.tol`: This parameter determines the tolerance used if one of the threshold pivoting schemes is chosen. The default value is 0.1 and this parameter can take any value in the range $(0, 1]$.

11.2 SLIP_LU.m

The `SLIP_LU.m` function solves the linear system $Ax = b$ where $A \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^{n \times m}$. The final solution vector(s) obtained via this function are exact prior to their conversion to double precision.

The SLIP LU function expects as input a sparse matrix A and dense set of right hand side vectors b . Optionally, the user can also pass in the options struct. Currently, there are 2 ways to use this function outlined below:

- `x = SLIP_LU(A,b)` returns the solution to $Ax = b$ using default settings. The solution vectors are more accurate than the solution obtained via `x = A \ b`.
- `x = SLIP_LU(A,b,option)` returns the solution to $Ax = b$ using user specified settings from the options struct.

Note that the result `x` is held as a MATLAB dense vector, in conventional floating-point representation (`double`), having been converted from the `mpq` type to return to MATLAB. As a result, the solution `x` is guaranteed to be exact only if the exact solution can be held in `double` without modification.

A future version of the `SLIP_LU` MATLAB function could return `x` in the MATLAB `vpa` format, which would preserve the exact solution. This feature may be added in the future.

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