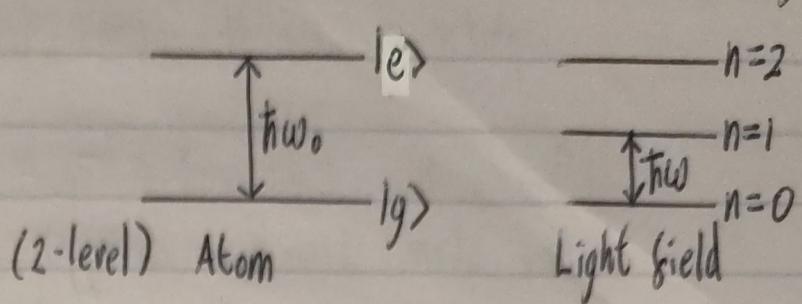


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JM Model

interaction Hamiltonian:

$$H = \lambda \hbar (\alpha \sigma_+ + \alpha^\dagger \sigma_-) \quad (1)$$



$|g\rangle$: ground state $|e\rangle$: excited state.

- Can be viewed like a spin- $\frac{1}{2}$ system, which is why we use the notation; $\sigma_{\pm} = \sigma_x \pm i \sigma_y$. The operator σ_+ raises the state from ground to excited; σ_- lowers the state from excited to ground.

Note: As we only need 2 levels, the atomic state is described by a 2-D complex vector. We say it lives in a 2D Hilbert space.

- In Quantum optics, light is described not just as a wave but also in terms of photons - the quanta of the EM field. A single mode of the EM field can be labeled by the number of photons $n=0, 1, 2, \dots$. These are called Fock states or number states:

$|0\rangle$ = vacuum (no photons)

$|1\rangle$ = one photon

$|2\rangle$ = 2 photons, etc.

a^\dagger = creation operator (creates one photon)

a = annihilation operator (annihilates one photon).

These act on the number states as:

$$a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle, \quad a |n\rangle = \sqrt{n} |n-1\rangle \quad (2)$$

Note: The field (laser mode) and the atom together live in a joint Hilbert space; the tensor product of the atom's 2D space and the photonic Fock space. (initial state: $|n\rangle \otimes |g\rangle \iff |e\rangle \otimes |n-1\rangle$)

Looking at the Hamiltonian:

- $a\sigma_+$: If the atom is in $|g\rangle$, σ_+ slips it to $|e\rangle$. At the same time, a must remove a photon, because that photon's energy has been absorbed by the atom.

$\cdot a^\dagger \sigma_-$: If the atom is in $|e\rangle$, σ_- slips it down to $|g\rangle$. Simultaneously, a^\dagger creates a photon, because the atom's lost energy goes into the field as a new photon.

(This is how light and a TLA exchange energy when the laser frequency matches the atom's transition energy)

- In QM, the time evolution of a state $|\psi(t)\rangle$ is governed by:

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle, \text{ where } U(t) = e^{-iHt} \quad (3)$$

Here, H is our interaction Hamiltonian. So the goal is to exponentiate the operator $(a\sigma_+ + a^\dagger\sigma_-)$ (potentially I think)

- The Hamiltonian couples states only in specific pairs:

$$|n, g\rangle \leftrightarrow |n+1, e\rangle$$

This means the Hamiltonian forms two-dimensional blocks in the bigger space. Each 2-D subspace is spanned by:

$$\{|n+1, g\rangle, |n, e\rangle\}, \quad (n=0, 1, 2, \dots)$$

is this helpful?

Yes - because we can handle the exponential of H in these small 2×2 chunks.

Mathematically, that is much simpler than directly exponentiating an infinite-dimensional matrix.

$$H = \lambda \hbar (a\sigma_+ + a^\dagger\sigma_-)$$

Let's introduce the basis order:

$$|1\rangle \equiv |n+1, g\rangle, \quad |2\rangle \equiv |n, e\rangle \quad (4)$$

We want the matrix representation:

$$[H]_{\alpha\beta} = \langle \alpha | H | \beta \rangle, \quad \alpha, \beta \in \{1, 2\} \quad (5)$$

Off-diagonal elements:

$$\langle 1 | H | 2 \rangle = \langle n+1, g | \hbar \lambda (a\sigma_+ + a^\dagger\sigma_-) | n, e \rangle$$

- $a\sigma_+ |n, e\rangle$ gives 0, because $\sigma_+ |e\rangle = 0$ (by defⁿ)

$$\begin{aligned} - a^\dagger\sigma_- |n, e\rangle : \quad & \sigma_- |e\rangle = |g\rangle, \quad a^\dagger |n\rangle = \sqrt{n+1} |n+1\rangle \\ & \Rightarrow a^\dagger\sigma_- |n, e\rangle = \sqrt{n+1} |n+1, g\rangle \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \langle 1 | H | 2 \rangle &= \hbar \lambda \langle n+1, g | a^\dagger \sigma_- | n, e \rangle \\
 &= \hbar \lambda \langle n+1, g | \sqrt{n+1} | n+1, g \rangle \\
 &= \hbar \lambda \sqrt{n+1} \langle n+1, g | n+1, g \rangle^{\frac{1}{2}} \\
 &= \hbar \lambda \sqrt{n+1}
 \end{aligned}$$

Similarly: $\langle 2 | H | 1 \rangle = \langle n, e | \hbar \lambda (a\sigma_+ + a^\dagger \sigma_-) | n+1, g \rangle$

$$\begin{aligned}
 &= \hbar \lambda \langle n, e | a\sigma_+ | n+1, g \rangle \\
 &= \hbar \lambda \langle n, e | \sqrt{n+1} | n, e \rangle \\
 &= \hbar \lambda \sqrt{n+1}
 \end{aligned}$$

Diagonal elements:

$$\begin{aligned}
 \langle 1 | H | 1 \rangle &= \langle n+1, g | \hbar \lambda (a\sigma_+ + a^\dagger \sigma_-) | n+1, g \rangle \\
 - a^\dagger \sigma_- | n+1, g \rangle &= 0 \quad \text{as } \sigma_- | g \rangle = 0 \\
 - a\sigma_+ | n+1, g \rangle &= \sqrt{n+1} | n, e \rangle \\
 \text{However } \langle n+1, g | n, e \rangle &= 0 \quad (\text{orthogonal}) \\
 \Rightarrow \langle 1 | H | 1 \rangle &= 0
 \end{aligned}$$

Similarly $\langle 2 | H | 2 \rangle = 0$

Putting all this together, in the basis $\{|n+1, g\rangle, |n, e\rangle\}$ the Hamiltonian is represented by:

$$H^{(n)} = \hbar \lambda \begin{pmatrix} 0 & \sqrt{n+1} \\ \sqrt{n+1} & 0 \end{pmatrix}$$

or: $H^{(n)} = \hbar \lambda \sqrt{n+1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \hbar \lambda \sqrt{n+1} \sigma_x$

subst. into $U(t) = e^{-iHt/\hbar}$

$$\Rightarrow U(t) = e^{-i\lambda\sqrt{n+1}\sigma_x t}$$

Pauli matrix.

There is a standard identity for exponentiating any Pauli matrix ($\sigma_x, \sigma_y, \sigma_z$), in particular:

$$\exp(-i\theta \sigma_x) = \cos \theta \mathbb{I} - i \sin(\theta) \sigma_x$$

where \mathbb{I} is the 2×2 identity matrix.

\Rightarrow for $\theta = \lambda \sqrt{n+1} t$, we have:

$$e^{-i\lambda\sqrt{n+1}\sigma_x t} = \mathbb{I} \cos(\lambda\sqrt{n+1}t) - i\sigma_x \sin(\lambda\sqrt{n+1}t)$$

renaming $\lambda\sqrt{n+1} = \Omega_n$:

$$\Rightarrow \boxed{\exp(-i\Omega_n \sigma_x t) = \mathbb{I} \cos(\Omega_n t) - i\sigma_x \sin(\Omega_n t)}$$

since $\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$:

$$\Rightarrow \exp(-i\Omega_n \sigma_x t) = \begin{pmatrix} \cos(\Omega_n t) & -i\sin(\Omega_n t) \\ -i\sin(\Omega_n t) & \cos(\Omega_n t) \end{pmatrix}$$

Subst. $\Omega_n = \lambda \sqrt{n+1}$:

$$\Rightarrow \boxed{U_n(t) = \begin{pmatrix} \cos(\lambda\sqrt{n+1}t) & -i\sin(\lambda\sqrt{n+1}t) \\ -i\sin(\lambda\sqrt{n+1}t) & \cos(\lambda\sqrt{n+1}t) \end{pmatrix}}$$

This is the precise matrix form for the time-evolution operator in the Jaynes-Cummings subspace spanned by $|n+1, g\rangle$ and $|n, e\rangle$

Physical interpretation:

- The matrix $U_n(t)$ describes oscillations (with frequency $\Omega_n = \lambda \sqrt{n+1}$) between $|n+1, g\rangle \leftrightarrow |n, e\rangle$.
- This $\sqrt{n+1}$ factor comes from the ladder operators a and a^\dagger , reflecting the fact that the transition amplitude depends on the number of photons present.
- These oscillations are known as "Rabi oscillations"
- Since each 2-D subspace $\{|g, n+1\rangle, |e, n\rangle\}$ evolves independently of the others, the full time-evolution operator is the direct sum of these 2×2 evolution blocks:

$$U(t) = \bigoplus_{n=0}^{\infty} U_n(t)$$

(for each n)

All Together:

- Start: We have the interaction Hamiltonian $H = \lambda \hbar (a\sigma_+ + a^\dagger \sigma_-)$
- Observation: The total number of excitations (photons + atomic excitation) is conserved.
- Result: The Hilbert space splits into independent "excitation subspaces" of dimension 2.
- In each subspace: The Hamiltonian is a simple 2×2 matrix, which is straightforward to exponentiate.
- Full $U(t)$: Put all these 2×2 blocks on the diagonal in a big direct-sum sense, yielding an operator that evolves each subspace independently.

i.e. for each time independent operator (i.e. 2×2 matrix), $U_1, U_2, U_3 \dots$ acting on different subspaces (subspace of the full Hilbert space), the direct sum $\bigoplus_i A$ is defined as the block-diagonal operator that acts on the direct sum of those vector spaces $H_1 \oplus H_2 \oplus \dots$. Concretely, each U_i is a 2×2 matrix:

$$\bigoplus_i U_i = \begin{pmatrix} U_1 & 0 & 0 & \dots \\ 0 & U_2 & 0 & \dots \\ 0 & 0 & U_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Hence;

$$U(t) = \exp(-iHt/\hbar) = \bigoplus_{n=0}^{\infty} \exp(-i\lambda\sqrt{n+1}\sigma_z t) \quad (\text{acting on the relevant } 2D \text{ subspace})$$

$$\text{where: } \exp(-i\lambda\sqrt{n+1}\sigma_z t) = U_n(t) = \begin{pmatrix} \cos(\lambda\sqrt{n+1}t) & -i\sin(\lambda\sqrt{n+1}t) \\ -i\sin(\lambda\sqrt{n+1}t) & \cos(\lambda\sqrt{n+1}t) \end{pmatrix}$$

- Question: what happens if we do not assume that the laser light's frequency is in resonance with the transition energy between the levels?

laser freq. = ω , atomic transition freq. = ω_0

detuning term:

$$> \Delta = \omega_0 - \omega.$$

> Hamiltonian will acquire this detuning term, the resulting dynamics Field energy will differ from the purely resonant Rabi oscillations.

$$H = \hbar \omega a^\dagger a + \frac{1}{2} \hbar \omega_0 \sigma_2 + \hbar \lambda (a\sigma_+ + a^\dagger \sigma_-) - \text{interaction}$$

- In the rotating frame, the Hamiltonian can be written as:

$$H = \frac{\hbar\Delta}{2} \sigma_z + \hbar\lambda(a\sigma_+ + a^*\sigma_-)$$

$$\Delta = \omega_0 - \omega$$

if $\omega_0 = \omega \Rightarrow \Delta = 0$ we get original Hamiltonian.

- In a 2-level system context, it is typical to choose $|g\rangle$ and $|e\rangle$ to be eigenstates of the Pauli- z operator σ_z . By defn:

$$\sigma_z = |e\rangle\langle e| - |g\rangle\langle g|$$

$$\Rightarrow \sigma_z |e\rangle = +1 \cdot |e\rangle$$

$$\sigma_z |g\rangle = -1 \cdot |g\rangle$$

Applying the σ_z term to our matrix representation of our Hamiltonian:

$$\langle 1 | H | 1 \rangle = -\frac{1}{2}\hbar\Delta$$

$$\langle 2 | H | 2 \rangle = \frac{1}{2}\hbar\Delta$$

They have no contribution to the off-diagonal elements.

$$\Rightarrow H^{(n)} = \begin{pmatrix} -\frac{1}{2}\hbar\Delta & \hbar\lambda\sqrt{n+1} \\ \hbar\lambda\sqrt{n+1} & +\frac{1}{2}\hbar\Delta \end{pmatrix} \quad \text{where } \Delta = \omega_0 - \omega.$$

$$U_n(t) = e^{-\frac{i}{\hbar}H^{(n)}t}$$

Defining a dimensionless matrix:

$$M = \frac{1}{\hbar} H^{(n)} = \begin{pmatrix} -\frac{1}{2}\Delta & \lambda\sqrt{n+1} \\ \lambda\sqrt{n+1} & +\frac{1}{2}\Delta \end{pmatrix}$$

$$\Rightarrow U_n(t) = e^{-iMt}$$

Notice, the diagonal entries of M are $-\frac{1}{2}\Delta$ and $+\frac{1}{2}\Delta$. Hence

$$\text{tr}(M) = (-\frac{1}{2}\Delta) + (+\frac{1}{2}\Delta) = 0$$

A 2×2 matrix with zero trace has eigenvalues $\pm R$, where:

$$R^2 = \left(\frac{1}{2}\Delta\right)^2 + (\lambda\sqrt{n+1})^2 = \frac{\Delta^2}{4} + \lambda^2(n+1)$$

$$\Rightarrow R = \sqrt{\frac{\Delta^2}{4} + \lambda^2(n+1)} = \frac{1}{2}\sqrt{\Delta^2 + 4\lambda^2(n+1)}$$

- It can be custom to write $\Omega_n = R = \frac{1}{2}\sqrt{\Delta^2 + 4\lambda^2(n+1)}$; Ω_n is the generalised Rabi frequency in this subspace.

- A well-known result for any 2×2 matrix M with zero trace and eigenvalues $\pm R$ is:

$$e^{-iMt} = \cos(Rt) \mathbb{I} - i\sin(Rt) \frac{M}{R}$$

Applying this identity:

$$U_n(t) = \cos(Rt) \mathbb{I} - i\sin(Rt) \frac{M}{R}$$

where \mathbb{I} is the 2×2 identity matrix.

$$\text{Let } a = \frac{1}{2}\Delta, b = \lambda\sqrt{n+1}, R = \sqrt{a^2 + b^2} = \frac{1}{2}\Omega_n$$

Then:

$$M = \begin{pmatrix} -a & b \\ b & a \end{pmatrix}, \frac{M}{R} = \begin{pmatrix} -\frac{a}{R} & \frac{b}{R} \\ \frac{b}{R} & \frac{a}{R} \end{pmatrix}$$

Hence,

$$U_n(t) = \begin{pmatrix} \cos(Rt) & 0 \\ 0 & \cos(Rt) \end{pmatrix} - i\sin(Rt) \begin{pmatrix} -\frac{a}{R} & \frac{b}{R} \\ \frac{b}{R} & \frac{a}{R} \end{pmatrix}$$

$$\Rightarrow U_n(t) = \begin{pmatrix} \cos(Rt) + i\sin(Rt) \frac{a}{R} & -i\sin(Rt) \frac{b}{R} \\ -i\sin(Rt) \frac{b}{R} & \cos(Rt) - i\sin(Rt) \frac{a}{R} \end{pmatrix}$$

Remember that $a = \frac{1}{2}\Delta$ and $b = \lambda\sqrt{n+1}$

$$\text{Recall } \Omega_n = \frac{1}{2}\sqrt{\Delta^2 + 4\lambda^2(n+1)} \Rightarrow R = \Omega_n$$

$$\text{Thus, } Rt = \Omega_n t, \frac{a}{R} = \frac{\frac{1}{2}\Delta}{\Omega_n} = \frac{\Delta}{2\Omega_n}, \frac{b}{R} = \frac{\lambda\sqrt{n+1}}{\Omega_n}$$

So the final matrix can also be written:

$$U_n(t) = \begin{pmatrix} \cos(-\Omega_n t) + i \frac{\Delta}{2\Omega_n} \sin(-\Omega_n t) & -i \frac{\lambda\sqrt{n+1}}{\Omega_n} \sin(-\Omega_n t) \\ -i \frac{\lambda\sqrt{n+1}}{\Omega_n} \sin(-\Omega_n t) & \cos(-\Omega_n t) - i \frac{\Delta}{2\Omega_n} \sin(-\Omega_n t) \end{pmatrix}$$

- Off-resonant Rabi flopping in the JC subspace. ($\Omega_n = \frac{1}{2}\sqrt{\Delta^2 + 4\lambda^2(n+1)}$)

Consequences:

In the resonant case ($\Delta=0$), we have purely sinusoidal "Rabi oscillations" with frequency $\Omega_n = \lambda\sqrt{n+1}$.

Off resonance, the atom and the field do not exchange excitation so simply. The time evolution in each 2-D subspace is still exactly solvable, but now the Rabi frequency is:

$$\Omega_n = \sqrt{\Delta^2 + 4\lambda^2(n+1)}$$

and the amplitudes have different forms. You still get oscillatory behaviour, but:

- > The atom will never be fully excited (or fully de-excited) if $\Delta \neq 0$; there is typically some population 'left behind' in each level.
- > The frequency of oscillations is higher than in the resonant case (because Δ contributes to Ω_n)
- > The phase of oscillations is also changed.

* Dressed Energies:

- Resonant case ($\Delta=0$):

when $\Delta=0$, the Hamiltonian is

$$\begin{pmatrix} 0 & \hbar\lambda\sqrt{n+1} \\ \hbar\lambda\sqrt{n+1} & 0 \end{pmatrix}$$

which has eigenvalues: $E_{\pm} = \pm \hbar\lambda\sqrt{n+1} = \pm \hbar\Omega_n$

Hence, the energy splitting between the 2 dressed states ($|n,+\rangle, |n,-\rangle$) is:

$$2\hbar\lambda\sqrt{n+1} = 2\hbar\Omega_n$$

- Off-resonant case ($\Delta \neq 0$)

when $\Delta \neq 0$, the 2x2 matrix $\begin{pmatrix} -\frac{1}{2}\hbar\Delta & \hbar\lambda\sqrt{n+1} \\ \hbar\lambda\sqrt{n+1} & +\frac{1}{2}\hbar\Delta \end{pmatrix}$

has eigenvalues:

$$E_{\pm} = \pm \frac{1}{2}\hbar\sqrt{\Delta^2 + 4\lambda^2(n+1)} = \pm \hbar\Omega_n$$

So, the energy splitting between these 2 dressed states is

$$\Delta E = [E_+ - E_-] = \hbar \sqrt{\Delta^2 + 4\lambda^2(n+1)} = 2\hbar\omega_n$$

> If $\Delta=0$, then $\Delta E = 2\hbar\lambda\sqrt{n+1}$, matching the resonant result.

> If $\Delta \neq 0$, you see an additional Δ^2 term under the square root, so the splitting becomes larger than $2\hbar\lambda\sqrt{n+1}$

Physical Picture:

1. Pairs for Each n : Because each 2D subspace is spanned by $\{|e,n\rangle, |g,n+1\rangle\}$, for each photon number n you get a pair of dressed states $|l\pm, n\rangle$
2. Entangled Atom-Field states: These dressed states are typically superpositions (e.g. $\alpha|e,n\rangle + \beta|g,n+1\rangle$). That means the atom and field are no longer in a simple product state - they are entangled.
3. Splitting: Off resonance, the energy separation is $\sqrt{\Delta^2 + 4\lambda^2(n+1)}$. On resonance ($\Delta=0$), it simplifies to $2\hbar\lambda\sqrt{n+1}$.
4. Rabi Oscillations: If you start in a bare state, the system will oscillate between $|e,n\rangle$ and $|g,n+1\rangle$ with the "Rabi frequency" $\omega_n = \sqrt{\Delta^2 + 4\lambda^2(n+1)}$. This is intimately related to the fact that the dressed states are the true eigenstates.

Why the name "Dressed"?

- Historically, the word "dressed" highlights that the atom (or field) is "dressed" by the presence of the other system. Instead of an atom alone and a field alone, the combined system's eigenstates are mixtures. In Quantum optics:
 - A "bare atom" has eigenstates $\{|g\rangle, |e\rangle\}$
 - A "bare field" has eigenstates $\{|n\rangle\}$.
 - When you put them in a strong enough coupling scenario, you do not see those bare states. Instead, you see the new eigenstates (superpositions) called "atom-photon" dressed states.
- The time evolution no longer corresponds to simple, complete population exchange between the states $|e,n\rangle$ and $|g,n+1\rangle$. Instead, you get partial or less efficient Rabi oscillations with a detuning-dependent frequency and phase.

- Being off-resonance ($\Delta \neq 0$) the oscillation ("Rabi") frequency becomes

$$\omega_{\text{off-res}} = \sqrt{\Delta^2 + \omega_{\text{res}}^2}, \text{ where } \omega_{\text{res}} = 2\lambda\sqrt{n+1}$$

Notice $|\omega_{\text{off-res}}| \geq |\omega_{\text{res}}|$

However, the amplitude of the population oscillations change. Off resonance, the atom does not get fully transferred from ground to excited (or vice versa); the system no longer undergoes complete population inversion. This reduced "amplitude" can be described by saying the transition is "less likely" or "less efficient".

(The period of the basic oscillation (in the 2-level subspace) is actually shorter when Δ is large. So in that sense, the system's state vector rotates faster in Hilbert space.)

- If you start the atom in the excited state it will never go all the way down to the ground state during an off-resonant cycle.
- Mathematically (for a pure 2-level system in a classical drive), the maximum excited-state population one can reach is bounded by a factor:

$$\frac{\omega_{\text{res}}^2}{\Delta^2 + \omega_{\text{res}}^2} < 1$$

when $\Delta \neq 0$. In the JC context (quantized field), you find a similar effect in each n-photon subspace: the off-resonant oscillations do not achieve full inversion between $|e, n\rangle$ and $|g, n+1\rangle$.

Hence, even though the oscillations happen at a frequency $\sqrt{\Delta^2 + (2\lambda\sqrt{n+1})^2}$, the population that actually gets transferred back and forth is less than 100%. On resonance by contrast, the system can go fully from $|e, n\rangle$ to $|g, n+1\rangle$ (complete inversion).

- In practical terms, for large detuning $|\Delta| \gg \omega_{\text{res}}$, the system stays mostly in its initial state (ground state), with only small partial oscillations of population.