Area of a Circle

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1 Introduction

During the time of Ancient Greece, calculus did not exist, and many modern concepts were thought of in terms of geometry. A main example is the area of a circle: the mathematicians approximated the area of a circle by looking at a polygon that is inscribed in a circle, that is, the vertices of the polygon rest on the edge of the circle. It can be noticed that as the amount of sides n of the polygon increases, the relative error between the area of the polygon and the area of the circle goes down. It follows that as the amount of sides n limits to infinity, the area of the polygon is therefore equal to that of the circle

2 The Area of a Regular n-gon Inscribed in a Circle with Radius r.

A regular n-gon is defined to be an n sided polygon, with all angles equal to one another. Being inscribed in a circle shows that all the vertices of the n-gon are touching the circle. If we take n line segments originating in the center that connect to the n vertices of the n-gon, we will split this n-gon into n isosceles triangles, with the two legs of each triangle equal to the radius of the circle r, and the respective edge of the polygon acting as the base. We can further simplify the arithmetic of the area of each isosceles triangle by splitting each isosceles triangle into two equal right triangles, by connecting the center of the circle to the midsection of the respective edge of the polygon, creating a right angle. There are now $2n$ right triangles, all of which have equal area and dimensions. Now, we must look at the angle in each triangle whose vertex is connected to the center point of the triangle. For simplicity, let's call this angle θ. Since we know all θ of these $2n$ triangles must add to equal 360, it follows that each θ can be represented by $\theta = \frac{180}{n}$. The base of one of the right triangles will be represented by b while the height will be represented by h. By using θ, and trigonometry, the equation for the base is derived in equations 1-2 below, and the equation for the height is represented in equations 3-4.

$$\frac{b}{r} = \sin\theta \tag{1}$$

$$b = r * \sin\theta \tag{2}$$

$$\frac{h}{r} = \cos\theta \tag{3}$$

$$h = r * \cos\theta \tag{4}$$

The general equation for a triangle is represented in equation 5, and then plugging in to equation 5 we can obtain the equation for each of the right triangles, shown in equation 6, which simplifies to become equation 7.

$$A = \frac{1}{2}bh \tag{5}$$

$$A = \frac{1}{2} * r\sin\theta * r\cos\theta \tag{6}$$

$$A = \frac{r^2}{2}\sin\theta\cos\theta \tag{7}$$

Equation 7 is the equation for one of the right triangles; however, there are $2n$ of these right triangles. The 2's in the numerator and the denominator will cancel, therefore, the area of the entire n-gon is represented by:

$$A_{n-gon} = nr^2 * \sin\theta\cos\theta \tag{8}$$

3 Area of Circle with radius r

Now that we know the area of the inscribed n-gon, it follows that the way to determine the area of the circle would be to have the amount of sides, n, limit to infinity, represented in equation 9.

$$A_{Circle} = \lim_{n\to\infty} nr^2\sin\theta\cos\theta \tag{9}$$

The r term can be pulled out, and we can also simplify through the multiplication law of limits:

$$A_{Circle} = r^2 \lim_{n\to\infty} (n \sin\theta) \lim_{n\to\infty} (\cos\theta) \tag{10}$$

Since from before, $\theta = 180/n$, it follows that $\lim_{n\to\infty} \theta = 0$, so $\lim_{n\to\infty}(\cos\theta) = 1$. However, when we do the same to $\lim_{n\to\infty}(n\sin\theta)$, we get $\infty * 0$ which is undefined, so we must do L'Hopitals rule, as shown below with the subsequent simplifications by taking the derivative of the numerator and denominator, and taking the limit of this.

$$\lim_{n\to\infty} (n \sin\theta) = \lim_{n\to\infty} \frac{\sin\theta}{\frac{1}{n}} \tag{11}$$

$$\lim_{n\to\infty} (n \sin\theta) = \lim_{n\to\infty} \frac{\frac{\mathrm{d}\sin\theta}{\mathrm{d}n}}{\frac{\mathrm{d}\frac{1}{n}}{\mathrm{d}n}} \tag{12}$$

$$\frac{\mathrm{d}\sin\theta}{\mathrm{d}n} = \frac{\mathrm{d}\sin\frac{180}{n}}{\mathrm{d}n} = -\frac{180}{n^2} * \cos\frac{180}{n} \tag{13}$$

$$\frac{\mathrm{d}\frac{1}{n}}{\mathrm{d}n} = -\frac{1}{n^2} \tag{14}$$

$$\lim_{n\to\infty} (n \sin\theta) = \lim_{n\to\infty} \frac{-\frac{180}{n^2} * \cos\frac{180}{n}}{-\frac{1}{n^2}} \tag{15}$$

$$\lim_{n\to\infty} (n \sin\theta) = \lim_{n\to\infty} 180\cos\frac{180}{n} = 180\cos 0 = 180 \tag{16}$$

We know that 180 degrees is the same as π radians, so plugging these values into the area of a circle equation we get the final area equation:

$$A_{Circle} = r^2 * 1 * 180 \tag{17}$$

$$A_{Circle} = \pi r^2 \tag{18}$$