计算视觉第四次作业

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本次作业

```
import torch
torch.manual_seed(0)

x = torch.randn(10,4, requires_grad=True)
W = torch.randn(4,4, requires_grad=True)
y = torch.randn(10,4, requires_grad=True)
```

目标函数: $f = ||\max(XW, 0) - Y||_F^2$

手动写出以下表达式，并用PyTorch进行验证:

$\frac{\partial f}{\partial W}$ $\frac{\partial f}{\partial X}$ $\frac{\partial f}{\partial Y}$

推导表达式

1. $\frac{\partial f}{\partial W}$

已知 F-范数的性质:

$$\|A\|_F = \sqrt{tr(A^T A)}$$

则有

$$\begin{aligned} f &= \|\max(XW, 0) - Y\|_F^2 \\ &= tr((\max(XW, 0) - Y)^T(\max(XW, 0) - Y)) \end{aligned}$$

设 $Z = \max(XW, 0)$:

$$\begin{aligned} df &= d\left[tr\left((Z - Y)^T(Z - Y)\right)\right] \\ &= tr(d[(Z - Y)^T(Z - Y)]) \\ &= tr(dZ^T(Z - Y) + (Z - Y)^T dZ) \\ &= tr(2 * (Z - Y)^T dZ) \\ &= tr(2 * (Z - Y)^T(\frac{dZ}{dXW} \odot XdW) \end{aligned}$$

$$= \mathrm{tr}\left(\left[2 * \left[(\mathrm{Z}-\mathrm{Y}) \odot \left(\frac{dZ}{dXW}\right)\right]^T X\right] dW\right)$$

$$= \mathrm{tr}(\left[2 * \mathrm{X}^T \left[(\mathrm{Z}-\mathrm{Y}) \odot \left(\frac{\mathrm{dZ}}{dXW}\right)\right]\right]^T \mathrm{dW})$$

由:

$$df = tr[\frac{\partial f}{\partial W}^T dW]$$

可得:

$$\frac{\partial f}{\partial W} = 2 * \mathrm{X}^T \left[(\mathrm{Z}-\mathrm{Y}) \odot \left(\frac{\mathrm{dZ}}{dXW}\right)\right]$$

$$= 2 * X^T \left[(\max(XW, 0) - Y) \odot \left(\frac{d(\max(XW, 0))}{dXW}\right)\right]$$

2. $\frac{\partial f}{\partial Y}$

$$df = d\left[tr\left((Z-Y)^T(Z-Y)\right)\right]$$

$$= tr(d[(Z-Y)^T(Z-Y)])$$

$$= \mathrm{tr}\left((-\mathrm{dY})^T(\mathrm{Z}-\mathrm{Y}) + (\mathrm{Z}-\mathrm{Y})^T(-\mathrm{dY})\right)$$

$$= \mathrm{tr}([-2 * (\mathrm{Z}-\mathrm{Y})^T]\mathrm{dY})$$

$$= tr([-2 * (Z-Y)]^T \mathrm{dY})$$

由于:

$$df = tr[\frac{\partial f}{\partial Y}^T dY]$$

可得:

$$\frac{\partial f}{\partial Y} = -2 * (Z-Y)$$

$$= -2 * (\max(XW, 0) - Y)$$

3. $\frac{\partial f}{\partial X}$

$$df = d\left[tr\left((Z-Y)^T(Z-Y)\right)\right]$$

$$= tr(d[(Z-Y)^T(Z-Y)])$$

$$= \mathrm{tr}\left(\left(\frac{\mathrm{dZ}}{dXW}dXW\right)^T(\mathrm{Z}-\mathrm{Y}) + (\mathrm{Z}-\mathrm{Y})^T\left(\frac{\mathrm{dZ}}{dXW}dXW\right)\right)$$

$$= tr\left(2*(Z-Y)^T\left(\frac{dZ}{dXW}\odot dX\right)W\right)$$

$$= tr\left(2*W(Z-Y)^T\left(\frac{dZ}{dXW}\odot dX\right)\right)$$

$$= tr(\left[2*W\left[(Z-Y)\odot\frac{dZ}{dXW}\right]^T\right]dX)$$

$$= tr\left(\left[2*\left[(Z-Y)\odot\frac{dZ}{dXW}\right]*W^T\right]^T dX\right)$$

由于:

$$df = tr[\frac{\partial f}{\partial X}^T dX]$$

可得:

$$\frac{\partial f}{\partial X} = 2*(\mathrm{Z}-\mathrm{Y})\odot\left(\frac{\mathrm{dZ}}{dXW}\right)\mathrm{W}^T$$

$$= 2*\left[(\max(XW,0)-Y)\odot\left(\frac{d(\max(XW,0))}{dXW}\right)\right]*W^T$$

Pytorch 验证

```python
import torch
import torch.nn.functional as F

torch.manual_seed(0)

X = torch.randn(10, 4, requires_grad=True)
W = torch.randn(4, 4, requires_grad=True)
Y = torch.randn(10, 4, requires_grad=True)

f = (F.relu(X @ W) - Y).t() @ (F.relu(X @ W) - Y)
f = torch.trace(f)
f.backward(retain_graph=True)
print(W.grad)
print(Y.grad)
print(X.grad)

grad_W = 2 * X.t() @ ((F.relu(X @ W) - Y) * ((X @ W) > 0))
grad_Y = -2 * (F.relu(X @ W) - Y)
grad_X = 2 * ((F.relu(X @ W) - Y) * ((X @ W > 0))) @ W.t()
print(grad_W)
print(grad_Y)
print(grad_X)

print(torch.allclose(W.grad, grad_W))
print(torch.allclose(Y.grad, grad_Y))
print(torch.allclose(X.grad, grad_X))
```

图 1. Pytorch 验证代码

输出结果为:

```
tensor([[ 18.2980,   2.7573,   2.3914,  -0.1974],
        [ 11.0817,   6.6428,   2.5163, -20.3225],
        [ -8.6662,   3.4506,  -1.8979,  -3.3608],
        [-21.1681,  -6.6739,  -1.0693,  27.0278]])
```

图 2. W 的梯度

```
tensor([[  1.1002,   0.0860,   5.3377,   0.2788],
        [  0.9583,  10.4633, -13.5234, -16.3639],
        [ -0.8712,  -0.9272,  -0.7764,   2.0790],
        [ -1.4504,   5.6914,   0.7613,  -0.9693],
        [ -1.2892,  -3.4714,  -1.9788,   4.8091],
        [ -4.0523,  -4.3127,  -3.6114,   9.6703],
        [ -0.7312,  -0.7782,  -0.6516,   1.7449],
        [ -0.8191,  -0.8718,  -0.7300,   1.9547],
        [  1.0350,   2.9930,  -6.6743,  -7.5333],
        [ -2.4616,  -2.4243,  -2.1164,   5.7128]])
```

图 3. X 的梯度

```
tensor([[ 2.8885e+00,  4.1639e+00,  3.4134e+00,  3.0501e+00],
        [-1.0589e+01, -2.7045e+00, -2.1849e+00, -1.7039e-01],
        [ 6.5523e-01, -1.5214e+00, -3.1982e+00, -1.5687e+00],
        [-1.5009e+00, -3.8551e+00,  4.9843e-01,  1.2764e+00],
        [-6.6077e-03, -1.0689e+00,  1.8791e+00, -4.2604e+00],
        [ 3.8829e+00,  1.5830e+00, -4.0504e-02, -7.2968e+00],
        [-4.3767e-01, -4.8701e+00, -1.4583e-01, -1.3166e+00],
        [ 1.9250e+00,  6.9834e-01, -1.8429e+00, -1.4750e+00],
        [-5.0359e+00, -9.2744e-01,  3.8436e+00, -8.0509e-01],
        [ 2.4780e-01,  2.3296e+00, -1.7491e-01, -4.2519e+00]])
```

图 4. Y 的梯度