1 Random matrix models

Random matrix models are used in a wide variety of applications to model complex systems. In quantum chaos, the Bohigas–Giannoni–Schmit (BGS) conjecture asserts that the spectral statistics of quantum chaotic systems are described by random matrix theory. In this assignment, we will construct one of the most commonly used random matrix models and analyse its spectral statistics.

1.1 The Gaussian orthogonal ensemble

A random matrix is a matrix of size $n \times n$ where some or all of the elements are random variables. The most-commonly studied random matrix distributions are the Gaussian ensembles. Gaussian orthogonal ensemble GOE(n) is used to model chaotic Hamiltonians with time reversal symmetry. The name stems from the fact that the matrix elements are distributed according to the Gaussian (normal) distribution, and the ensemble is invariant under orthogonal transformations. Let H be a matrix from the GOE(n)., then the matrix H is a real symmetric matrix. The diagonal matrix elements H_{ii} are independent and identically distributed (i.i.d.) according to the Gaussian distribution, with zero expectation and variance equal 2. The off diagonal matrix elements H_{ij} are i.i.d, with zero expectation and variance equal 1. Your first assignment is to construct a matrix from GOE(n) and compute its eigenspectum.

Hint: Use the symmetry of the matrix. Use Julia's inbuilt matrix types to make the computation more efficient.

1.2 Eigenvalues

Next, we will analyse the spectrum of the matrix H. Let $\{\lambda_i\}$ denote the sequence of eigenvalues of the matrix H divided by $\sqrt{n}$. The distribution of eigenvalues (level density) of GOE(n) matrices follows Wigner's semicircle law

$$\rho(\lambda) = \begin{cases} \frac{1}{\pi}\sqrt{2-\lambda^2}, & \lambda \in [-\sqrt{2}, \sqrt{2}] \\ 0, & \text{otherwise} \end{cases}, \tag{1}$$

in the limit $n \to \infty$. In the finite dimensional case, the distribution of the eigenvalues will fluctuate around the theoretical prediction thet gives the local mean level

density. Compute a histogram of the eigenvalues and compare it to Wigner's semi-circle law. Aggregate the eigenvalues from many independent realizations of H and compare.

We define the integrated level density as

$$\mathcal{N}(E) = n \int_0^E \rho(\lambda) d\lambda \tag{2}$$

In the discrete case, this is known as the spectral staircase function

$$\mathcal{N}(E) = \#\{i, \quad \lambda_i < E\}, \tag{3}$$

that counts the number of levels encountered up to energy E. This is a step-like function with a step at each encountered eigenvalue. In order to be able to compare the spectra of differently sized random matrices or indeed quantum spectra of different chaotic systems, we may use the spectral staircase function to normalize the local density of states to 1. This is called the unfolding procedure. The unfolded levels are obtained by inserting the discrete eigenlevels into the formula for the integrated smooth level density. The resulting level sequence $\{e_i\}$ will on average increase by one at each step, with small fluctuations. The statistics of these fluctuations are an important property of the spectrum that is used to analyse the correlations in the spectrum.

Hint: When unfolding the spectra, the finite precision of the numerical computation may cause problems near the spectral edges. It is often better to take only the eigenvalues from the center.

1.3 Level spacing distributions

The most commonly used statistic to study the spectral fluctuations is the level spacing distribution. The level spacing is defined as the difference in energy between two consecutive levels in the unfolded spectrum $s_i = e_{i+1} - e_i$. The unfolding procedure guarantees that the mean level spacing is equal to 1. The main object of interest is the *level spacing distribution* i.e. the probability density $P(s)$ or its cumulative density $W(s) = \int_0^s P(s)\, ds$. The Wigner surmise gives a very good approximation for the level spacing distribution of the GOE matrices

$$P_W(s) = \frac{\pi s}{2} e^{-\frac{\pi s}{4} s^2}. \tag{4}$$

Compute the histogram of the level spacing distribution and compare it to the Wigner surmise. Compute also the cumulative level density and compare it with the analytical result.

1.4 Number variance

When studying medium- and long-range statistics of the spectra *number variance* (NV) Σ^2 is commonly considered. The NV is defined as

$$\Sigma^2(L, e) := \left\langle (n(L, x) - L)^2 \right\rangle_{e,w}, \; L > 0, \tag{5}$$

that is the local variance of the number $n(L, x) = N(x + L/2) - N(x - L/2)$ of unfolded energy levels in the interval $e_n \in [x - L/2, \, x + L/2]$. The brackets $\langle ... \rangle_{e,w}$ denote a local average around the central energy e and window width w, so that $x \in [e - w/2, \, e + w/2]$. In our case, the energy window is dictated by the size of the available spectrum. To compute the NV you will need to compute the local variance of the number of energies in an interval of size L and then perform a moving average of this quantity over the extent of your spectrum. The size of your energy interval L must therefore be much smaller than the biggest energy of your unfolded spectrum. A good rule of thumb is $L_{max} \approx \sqrt{w}$ You will therefore need a very large spectrum to get good results. We are interested in the dependence $\Sigma^2(L)$. You will therefore need to compute the number variance for many different L values. The RMT result for the GOE case is the following,

$$\begin{aligned}\Sigma^2_{\mathrm{GOE}}(L) = \frac{2}{\pi^2}\Bigg\{ & \ln(2\pi L) + \gamma + 1 \\ & + \frac{1}{2}\mathrm{Si}^2(\pi L) - \frac{\pi}{2}\mathrm{Si}(\pi L) - \cos(2\pi L) \\ & - \mathrm{Ci}(2\pi L) + \pi^2 L \left(1 - \frac{2}{\pi}\mathrm{Si}(2\pi L)\right) \Bigg\},\end{aligned}$$

where $\gamma = 0.5772...$ is Euler's constant and $\mathrm{Si}(x)$ and $\mathrm{Ci}(x)$ are the sine and cosine integral respectively. Compute the number variance and compare with the analytical result.

Hint: There is a very efficient algorithm to compute number variance exist that requires looping over your unfolded energy spectrum only once. Consider alternative ways to compute the local variance instead of counting the number of levels in the interval.