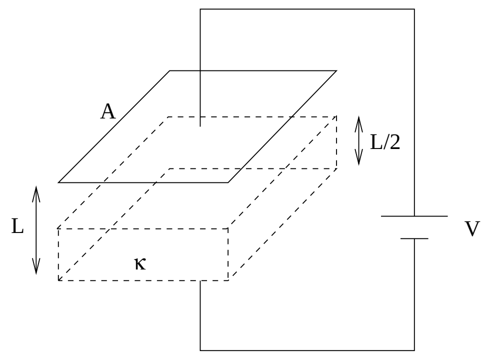
PHY6938 E & M Spring 98

1. Consider a capacitor connected to a battery of voltage V. Let the capacitor have an area A, and a distance L between the plates. Assume that the capacitor has a layer of dielectric (of dielectric constant κ, so that $\epsilon = \kappa\epsilon_0$) of thickness $L/2$ on the lower plate, as shown in the figure.



a) Calculate the capacitance of the capacitor.

To find the capacitance we would like to find the voltage between the plates for a given charge $+Q$ and $-Q$ on the two plates. We can use Gauss's law (both with and without a dielectric) to find the electric field due to the charge on the plates, using the approximation that there are no end effects. Gauss's law reads, in general

$$\oint \mathbf{E} \cdot d\mathbf{S} = \frac{Q_{\text{encl}}}{\epsilon},$$

where $\epsilon = \kappa\epsilon_0$ is the general form of the electric permitivity in the presence of a dielectric (note that in free space $\kappa = 1$).

If we use a pill-box shaped Gaussian surface with end area A' and with one end below the bottom plate, which is charged to $-Q$, and the other end inside the dielectric, then Gauss's law gives us

$$\oint \mathbf{E}_{\text{d}} \cdot d\mathbf{S} = E_{\text{d}}A' = \frac{Q_{\text{encl}}}{\epsilon} = \frac{\sigma A'}{\epsilon} = \frac{Q}{\epsilon A}A',$$

where E_{d} is the field inside the dielectric, since the field is zero outside the electric on the bottom end of the pill-box and always perpendicular to the sides of the pill-box. This gives

$$E_{\text{d}} = \frac{Q}{\epsilon A} = \frac{Q}{\kappa\epsilon_0 A}.$$

Exactly the same procedure, but with the two ends of the pillbox above and below the top plate in empty space, results in the field between the dielectric and the top plate of

$$E = \frac{Q}{\epsilon_0 A} = \frac{Q}{\epsilon_0 A}.$$

We can now find the relation between the voltage difference V between the bottom and top plates and Q by integrating the constant electric field (which points straight up) along a vertical line from the bottom plate to the top plate

$$V = \int_0^{L/2} \mathbf{E}_\mathrm{d} \cdot d\mathbf{l} + \int_{L/2}^{L} \mathbf{E} \cdot d\mathbf{l} = \frac{Q}{\kappa\epsilon_0 A}\frac{L}{2} + \frac{Q}{\epsilon_0 A}\frac{L}{2} =$$

so that we have

$$C = \frac{Q}{V} = \frac{2\epsilon_0 A}{L}\left(\frac{1}{\kappa}+1\right)^{-1} = \frac{2\epsilon_0 A}{L}\left(\frac{\kappa}{1+\kappa}\right).$$

Note that this is just the capacitance $2\epsilon_0 A/L$ of a free-space parallel-plate capacitor with area A and plate gap $L/2$ in series with that of the same capacitor with dielectric inserted, which is larger, $2\kappa\epsilon_0 AL$. The total capacitance of two capacitors in series is like that of resistors in parallel, i.e. it is given by

$$C = \left(\frac{1}{C_1}+\frac{1}{C_2}\right)^{-1}.$$

b) Calculate the charge on the capacitor.

For any capacitor $Q = CV$, so

$$Q = \frac{2\epsilon_0 AV}{L}\left(\frac{\kappa}{1+\kappa}\right).$$

c) Calculate the value of the electric displacement D in the capacitor.

The source of the electric displacement D is free charges, so we are to ignore any surface charges on the dielectric. This means that the electric displacement everywhere in the capacitor is just the electric field in the free space region,

$$D = E = \frac{Q}{\epsilon_0 A} = \frac{2V}{L}\left(\frac{\kappa}{1+\kappa}\right),$$

and it points from the lower plate to the upper plate.

d) Calculate the value of the electric field inside the dielectric layer, and in the air above it.

With our approach we have already done this and we have

$$E_\mathrm{d} = \frac{Q}{\kappa\epsilon_0 A} = \frac{2V}{L(1+\kappa)}, \quad E = \frac{Q}{\epsilon_0 A} = D.$$

e) Calculate the electrostatic energy stored in the system. How would it change if the dielectric is removed?

The electrostatic energy stored in a capacitor is just

$$U = \frac{Q^2}{2C} = \frac{1}{2}CV^2$$

and so we see that

$$U = \frac{1}{2}\frac{2\epsilon_0 A}{L}\left(\frac{\kappa}{1+\kappa}\right)V^2 = \frac{\epsilon_0 A V^2}{L}\left(\frac{\kappa}{1+\kappa}\right).$$

Obviously if κ is reduced to unity by removing the dielectric but the applied voltage remains the same, the capacitance is reduced and so the amount of stored energy is reduced.

2. A plasma generated inside a long hollow cylinder of radius R has the following charge distribution:

$$\rho(r) = \frac{\rho_0}{[1+(r/a)^2]^2}$$

where r is the distance to the center and ρ_0 and a are constants. Determine the electric field everywhere.

This is an exercise in the application of Gauss's law. To find the electric field for $r < R$, construct a Gaussian surface with the same symmetry as the cylinder, a cylinder of radius r and length l. Then since the electric field has to point radially outwards and have the same magnitude everywhere on the curved surface by symmetry, it also has no flux through the ends of the cylinder. Gauss's law then gives that

$$\oint \mathbf{E}\cdot d\mathbf{A} = 2\pi r l E = \frac{Q_{\text{encl}}}{\epsilon_0} = \frac{l}{\epsilon_0}\cdot\int_0^r dV'\rho(r') = \frac{\rho_0 l}{\epsilon_0}\cdot\int_0^r 2\pi r' dr' \frac{1}{[1+(r'/a)^2]^2},$$

so that

$$E = \frac{\rho_0}{2\epsilon_0 r}\cdot\int_0^r 2r'dr'\frac{1}{[1+(r'/a)^2]^2},$$

Define a new variable $x = r'/a$, then $dr' = a\,dx$, and

$$E = \frac{\rho_0 a^2}{2\epsilon_0 r}\cdot\int_0^{r/a} 2x dx\frac{1}{[1+x^2]^2},$$

which integrates to

$$E = \frac{\rho_0 a^2}{2\epsilon_0 r}\cdot\left[-\frac{1}{1+x^2}\right]_0^{r/a} = \frac{\rho_0 a^2}{2\epsilon_0 r}\cdot\left[-\frac{1}{1+r^2/a^2}+1\right] = \frac{\rho_0 a^2}{2\epsilon_0 r}\cdot\frac{r^2}{a^2+r^2} = \frac{\rho_0 a^2}{2\epsilon_0}\cdot\frac{r}{a^2+r^2},$$

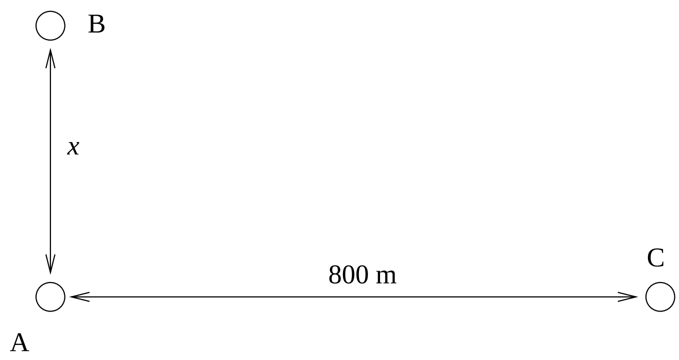
and $\mathbf{E}$ points radially outward from the center of the cylinder.

For $r > R$ we just need to find the total charge from the above integral and then Gauss's law gives us

$$E(r) = \frac{\rho_0}{2\epsilon_0 r}\cdot\int_0^R 2r'dr'\frac{1}{[1+(r'/a)^2]^2} = \frac{\rho_0 a^2}{2\epsilon_0 r}\cdot\frac{R^2}{a^2+R^2}, \quad r > R,$$

and again $\mathbf{E}$ points radially outward.

3. A radio transmitter at position A operates at a wavelength of 20 m.



a) Assuming the antenna is arranged so an approximately plane wave is emitted and the maximum electric field at the transmitter is 600 V/m, write the equation for the electric field for all time and space.

This is just a plane wave

$$\mathbf{E} = \mathbf{E}_0\, e^{i[kx-\omega t]},$$

where the real part is understood, $\mathbf{E}_0$ is the maximum electric field which is transverse to the direction of propagation of the wave, $k = 2\pi/\lambda$ and $\omega = 2\pi\nu = 2\pi c/\lambda$, so that

$$\mathbf{E} = \mathbf{E}_0\, e^{i2\pi[x-ct]/\lambda} = \mathbf{E}_0 \cos\left(2\pi[x-ct]/\lambda\right).$$

b) A second, identical transmitter is located at a distance x from the first transmitter, at position B. The transmitters are phase locked together such that the second transmitter is lagging $\pi/2$ out of phase with the first. Find the minimum distance between them, x, such that the electric field will be a maximum at position C, 800 m from the transmitter at position A.

The phase of the second plane wave is the quantity $2\pi(x-ct)/\lambda$, and differs from that of the first by $\pi/2$, so that the electric field from it looks like

$$\mathbf{E}' = \mathbf{E}_0'\, e^{i(2\pi[x-ct]/\lambda-\pi/2)}.$$

Note that the direction $\mathbf{E}_0'$ of the electric field will be different from that of the first transmitter, in general.

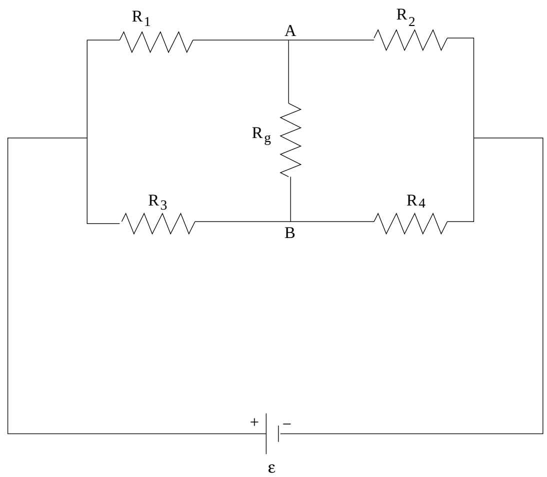
The two waves will interfere constructively when the phase lag due to the extra distance the second wave has to go is an integral number times 2π. The minimum such lag is 2π, so we need an *additional* phase lag of $3\pi/2$. This will be brought about by having an extra distance given by

$$\delta d = \frac{3\pi/2}{2\pi}\lambda = \frac{3}{4}\lambda = 15 \text{ m}.$$

This extra distance is along the hypotenuse of the right triangle BAC, which must therefore be 815 m, so that the distance x is given by

$$x = \sqrt{815^{\,2} - 800^{\,2}} \text{ m} = 156 \text{ m}.$$

4. Consider the Wheatstone bridge resistor circuit shown in the diagram below. The values of the resistors are $R_1 = 100\ \Omega$, $R_2 = 200\ \Omega$, $R_3 = 100\ \Omega$, and $R_4 = 400\ \Omega$. The galvanometer has a resistance of $R_g = 10\ \Omega$, and the battery supplies a voltage of $\varepsilon = 12$ V.



a) What is the current through the galvanometer?

We need to apply Kirchoff's laws to the circuit, which are that the current going into a branch in the circuit is the sum of the currents in the two branches (conservation of charge), and that the voltage drops around any loop in the circuit add to zero. Label by I_i the current in resistor i, assuming each goes to the right except the galvanometer current I_g, which we can assume goes down. Then we have five unknown currents. There are two branch points at A and B, and the two conservation of charge equations are

$$I_1 = I_2 + I_g \tag{1}$$
$$I_3 + I_g = I_4. \tag{2}$$

This leaves us with three equations necessary to solve for the five currents.

There are three loops in the circuit; the first is the loop made up by the three resistors on the left, which has the energy conservation equation (start at A and add voltage *rises* clockwise)

$$-I_g R_g + I_3 R_3 - I_1 R_1 = 0. \tag{3}$$

Similarly for the loop of resistors on the right (start at A and add voltage rises clockwise)

$$-I_2 R_2 + I_4 R_4 + I_g R_g = 0. \tag{4}$$

We have one more equation which is for a loop containing ε; note we can choose any of a number of loops and because we have already written down the equations for the interior loops they will all amount to the same thing, so choose the top branch of the circuit and start at the $-$ side of the emf,

$$\varepsilon - I_1 R_1 - I_2 R_2 = 0. \tag{5}$$

We now have five equations in five unknowns which we can solve, however tedious that process is, for the five currents. It is better to do this without substituting for the resistors so that we have the general formula for the galvanometer current.

Before embarking on what is formally the job of inverting a 5×5 matrix, it helps to get organized. Equation (1) can be used to eliminate I_1 from *every* equation where it appears. The same is true of Eq. (2) and I_4. Once we have used these we have only three equations left. Our new system of equations looks like

$$\begin{aligned} -I_{\rm g}R_{\rm g} + I_3R_3 - I_2R_1 - I_{\rm g}R_1 = -I_{\rm g}(R_{\rm g}+R_1) - I_2R_1 + I_3R_3 &= 0 && (6)\\ -I_2R_2 + I_3R_4 + I_{\rm g}R_4 + I_{\rm g}R_{\rm g} = \quad I_{\rm g}(R_g+R_4) - I_2R_2 + I_3R_4 &= 0 && (7)\\ \varepsilon - I_2R_1 - I_{\rm g}R_1 - I_2R_2 = \varepsilon - I_gR_1 - I_2(R_1+R_2) &= 0 && (8) \end{aligned}$$

where it obviously pays to collect terms and organize the coefficients in terms of the unknown currents.

The best way to proceed now is to combine (6) and (7) to eliminate I_3 which does not appear in (8), i.e.

$$\begin{aligned} R_4 \times (6): \quad -I_{\rm g}\left[(R_{\rm g}+R_1)R_4\right] - I_2R_1R_4 + I_3R_3R_4 &= 0 && (9)\\ R_3 \times (7): \quad I_g\left[(R_{\rm g}+R_4)R_3\right] - I_2R_2R_3 + I_3R_3R_4 &= 0 && (10)\\ (10)-(9): \quad I_{\rm g}\left[R_{\rm g}(R_3+R_4) + R_4(R_1+R_3)\right] - I_2\left[R_2R_3 - R_1R_4)\right] &= 0. && (11) \end{aligned}$$

Substituting the value of I_2 in terms of $I_{\rm g}$ from (8), we have

$$I_{\rm g}\left[R_{\rm g}(R_3+R_4) + R_4(R_1+R_3)\right] - \frac{\varepsilon - I_gR_1}{R_1+R_2}\left[R_2R_3 - R_1R_4)\right] = 0,$$

or that

$$I_{\rm g}\left\{(R_1+R_2)\left[R_{\rm g}(R_3+R_4) + R_4(R_1+R_3)\right] + R_1(R_2R_3 - R_1R_4)\right\} = \varepsilon\left[R_2R_3 - R_1R_4)\right],$$

so that, *finally*,

$$I_{\rm g} = \frac{\varepsilon\left[R_2R_3 - R_1R_4)\right]}{(R_1+R_2)\left[R_{\rm g}(R_3+R_4) + R_4(R_1+R_3)\right] + R_1(R_2R_3 - R_1R_4)} = -10.2 \text{ mA}.$$

b) What is the voltage difference between points A and B? What is the requirement on this voltage difference if the bridge is to be "balanced" (i.e. when no current flows through the galvanometer?

Obviously

$$V_{\rm A} - V_{\rm B} = V_{\rm g} = I_{\rm g}R_{\rm g} = -0.102 \text{ V},$$

and the bridge is balanced if

$$R_2R_3 = R_1R_4,$$

which can be seen easily by breaking opening the galvanometer circuit and balancing the voltages at A and B

$$\begin{aligned} V_{\rm A} &= \varepsilon\left(1 - \frac{R_1}{R_1+R_2}\right)\\ V_{\rm B} &= \varepsilon\left(1 - \frac{R_3}{R_3+R_4}\right) \end{aligned}$$

so that the voltages are equal when

$$\frac{R_1}{R_1+R_2} = \frac{R_3}{R_3+R_4} \quad \rightarrow \quad R_1R_4 = R_2R_3.$$