PHY541 Problem Set 3. Due October 2, 2014.

1. The goal of this problem and the next is to calculate the plot shown in Pathria figure 8.2. To be specific, we are looking at N spin one half, unpolarized (i.e. equal number of up and down) noninteracting nonrelativistic fermions, of mass m. Parts a and b apply to any single component thermodynamic system, c, d and e are specific to the nonrelativistic ideal Fermi gas.

   a. Explain why $n = \frac{N}{V}$ and $\epsilon = \frac{E}{V}$ can be written as functions of T and μ for any thermodynamic system.

   b. Starting with
   $$\frac{C_V}{N} = \frac{T}{N}\frac{\partial S}{\partial T}\bigg|_{V,N} = \frac{1}{N}\frac{\partial E}{\partial T}\bigg|_{V,N}, \tag{1}$$
   show that for any thermodynamic system
   $$\frac{C_V}{N} = \frac{1}{n}\left[\frac{\partial \epsilon}{\partial T}\bigg|_{\mu} - \frac{\frac{\partial \epsilon}{\partial \mu}\Big|_T \frac{\partial n}{\partial T}\Big|_\mu}{\frac{\partial n}{\partial \mu}\Big|_T}\right] \tag{2}$$

   c. Without looking at any book or notes, calculate the thermodynamic potential $\Omega(T, V, \mu)$ for the ideal Fermi gas above. Take the appropriate derivatives etc. to get expressions for n and ϵ. Then look at books or your notes to make sure your results are correct. If not, practice until you can perform this calculation starting from $\mathrm{tr} e^{-\beta(H-\mu N)}$

      The point of this question is to practice answering questions about the ideal fermi gas starting from the basics of statistical mechanics. Success will also mean that you can answer these types of questions on an oral (or written) exam or when you give a research seminar.

   d. Show by either dimensional analysis and scaling with system size (or explicitly if you are unable to use dimensional analysis) that the specific heat, C_V/N, can be written as a function of the single variable
   $$\frac{T}{T_F} = \frac{2mT}{\hbar^2(3\pi^2 n)^{2/3}} = \frac{1}{(3\pi^2)^{2/3}}\frac{2mT}{\hbar^2 n^{2/3}} \tag{3}$$

   e. Show that
   $$\frac{d}{d\alpha} I_j(\alpha) = \frac{j-1}{2} I_{j-2}(\alpha) \tag{4}$$
   where
   $$I_j(\alpha) = \int_0^\infty dx \frac{x^j}{1+e^{x^2-\alpha}}. \tag{5}$$

f. For the ideal Fermi gas calculate the terms in the specific heat above and write the result in terms of the $I_j(\alpha)$ integrals. Your result should be

$$\frac{C_V}{N} = \frac{5I_4(\alpha)}{2I_2(\alpha)} - \frac{9I_2(\alpha)}{2I_0(\alpha)} \,. \tag{6}$$

Identify α in terms of the parameters of the problem. Show that $\frac{T}{T_F}$ is also a function of the same α, and give the function in terms of the $I_j(\alpha)$.

2. Use whatever numerical tools you are comfortable with e.g. Python, Maple, Matlab, Mathematica, or any programming language and/or numerical libraries, to perform the necessary integrals from problem 1, and make a plot like Pathria figure 8.2 of C_V/N as a function of $\frac{T}{T_F}$. Make a second plot of $\frac{\mu}{T_F}$ as a function of $\frac{T}{T_F}$.

   Hint. It is not necessary to calculate the specific heat as a function of $\frac{T}{T_F}$. It is easier to calculate parametrically. That is calculate both $\frac{T}{T_F}$ and $\frac{C_V}{N}$ as a function of the variable α in problem 1.

3. a. Write the 6N dimensional integrals that you would need to perform to calculate the Helmholtz Free energy of the following systems

   i. N identical nonrelativistic classical particles of mass m interacting via a pairwise central potential $\sum_{i<j} v(|\boldsymbol{r}_i - \boldsymbol{r}_j|)$.

   ii. $N/2$ identical nonrelativistic classical particles of mass m_1, and $N/2$ identical particles of mass m_2. The particles all interact via the same pairwise central potential $\sum_{i<j} v(|\boldsymbol{r}_i - \boldsymbol{r}_j|)$. An example is a mixture of ^{3}He and ^{4}He. The nuclear mass difference gives fractional corrections of order the mass ratio, $\frac{m_{e^-}}{3m_{^4\text{He}}} \sim 5 \times 10^{-5}$, on the electronic structure so the atom-atom interaction is essentially identical for any pair.

   iii. N identical noninteracting relativistic classical particles of mass m with kinetic energy $\sqrt{p^2c^2 + m^2c^4}$.

   b. Calculate the momentum integrals above for i and ii of part a, write the Helmholtz free energy as a sum of the ideal gas Helmholtz free energy F_0 for the system and a term depending on integrals over the potential energy only.

   c. For iii of part a, perform all the integrals and calculate the Helmholtz Free energy. in terms of modified Bessel functions. I found the following two relations useful

$$\begin{aligned} x^{-1}K_1(x) &= \int_1^\infty dt e^{-xt}\sqrt{t^2-1} \\ K_n'(x) &= K_{n-1}(x) - \frac{n}{x}K_n(x) \,. \end{aligned} \tag{7}$$

   Show that when $\frac{mc^2}{T} \ll 1$ that $P = \frac{1}{3}\frac{E}{V}$.
   Show that when $\frac{mc^2}{T} \gg 1$ that $P = \frac{2}{3}\frac{E-Nmc^2}{V}$.