PHY531 Problem Set 3. Due February 19, 2015

1. a. Using the method of images, calculate the solution for the Green's function in spherical coordinates for the interior of a sphere of radius a satisfying:

$$\begin{aligned} \nabla^2 G(\boldsymbol{r}, \boldsymbol{r}') &= -\delta^3(\boldsymbol{r} - \boldsymbol{r}') \\ G(a, \theta, \phi, \boldsymbol{r}') &= 0\,. \end{aligned} \tag{1}$$

b. Use the Green's function of part a to calculate the potential inside a sphere of radius a if the potential on the sphere is given as $V(a, \theta, \phi) = f(\theta, \phi)$. Show your result can be manipulated into the form

$$\Phi(r, \theta, \phi) = r^{1/2} \frac{\partial}{\partial r} r^{1/2} \int_{-1}^{1} d\cos\theta' \int_0^{2\pi} \frac{d\phi'}{2\pi} \frac{a f(\theta', \phi')}{\sqrt{r^2 + a^2 - 2ra[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')]}} \tag{2}$$

c. Calculate the potential along the z axis inside the sphere for the case where the upper hemisphere is at potential V_0 and the lower at $-V_0$ by integrating Eq. 2.

Hint: Check your result for part b for a constant f where you should know the answer, and check that your result for part c gives the correct values at $z = 0$ and $z = \pm a$.

2. Use the stress tensor to calculate the magnitude and direction of the forces below:

a. The force on the upper hemisphere of an overall neutral conducting sphere at the origin, with a charge q, a distance d along the $\hat{\boldsymbol{z}}$ axis.

b. The force per area on the current of a long solenoid with n turns per unit length and current I in the wire. You can ignore the contribution to the field from the pitch of the winding that you calculated in Jackson problem 5.2.

3. Most undergraduate electromagnetism texts include a demonstration, using the Biot-Savart law and symmetry, that the static magnetic field of a uniformly wound toroid is circumferential. Here let's show this using the vector potential.

a. Show that if the current density in cylindrical coordinates has the form

$$\boldsymbol{J}(r, \phi, z) = J_r(r, z)\hat{\boldsymbol{r}} + J_z(r, z)\hat{\boldsymbol{z}} \tag{3}$$

the vector potential has components

$$\begin{aligned} A_r(r, \phi, z) &= \frac{1}{4\pi c} \int_0^\infty r' dr' \int_0^{2\pi} d\phi' \int_{-\infty}^{\infty} dz' \frac{J_r(r', z')\cos(\phi')}{\sqrt{r^2 + r'^2 - 2rr'\cos(\phi') + (z - z')^2}} \\ A_\phi(r, \phi, z) &= \frac{1}{4\pi c} \int_0^\infty r' dr' \int_0^{2\pi} d\phi' \int_{-\infty}^{\infty} dz' \frac{J_r(r', z')\sin(\phi')}{\sqrt{r^2 + r'^2 - 2rr'\cos(\phi') + (z - z')^2}} = 0 \\ A_z(r, \phi, z) &= \frac{1}{4\pi c} \int_0^\infty r' dr' \int_0^{2\pi} d\phi' \int_{-\infty}^{\infty} dz' \frac{J_z(r', z')}{\sqrt{r^2 + r'^2 - 2rr'\cos(\phi') + (z - z')^2}} \end{aligned} \tag{4}$$

so that the form of $\boldsymbol{A}$ is

$$\boldsymbol{A} = A_r(r, z)\hat{\boldsymbol{r}} + A_z(r, z)\hat{\boldsymbol{z}}\,. \tag{5}$$

b. Explain why the uniformly wound toroid (many equally spaced, close wound single turns with equal current) has a current density of the form given in part a.

   Show that the curl of $\boldsymbol{A}$ of part a has only a ϕ component which is independent of ϕ, i.e. it is circumferential.

4. A toroidal core with unit relative permeability and permittivity (i.e. like vacuum) has an inner radius a, an outer radius b, and a height h. In cylindrical coordinates, the toroidal core is described by $0 < z < h$, $a < r < b$, and ϕ takes all values.

   Each half is wound with a large number N of equally spaced turns of wire. Each turn is separately powered to give a a static current $\frac{I}{N}$, all in the same sense. Since N is large, you can apply the result of problem 3.

   a. What is the easiest way to calculate the $\boldsymbol{B}$ field? Calculate $\boldsymbol{B}$.

   b. The toroid is cut along the plane $\phi = 0$ $\phi = \pi$. Use the stress tensor to calculate the force holding the two halves together.

   c. Verify that the effective current density for this toroid is

$$\begin{aligned} \boldsymbol{J}(r,\phi,z) &= \frac{I}{\pi}\Big\{\hat{\boldsymbol{z}}\left[a^{-1}\delta(r-a) - b^{-1}\delta(r-b)\right]\left[\Theta(z)-\Theta(z-h)\right] \\ &\quad +\hat{\boldsymbol{r}}r^{-1}\left[\delta(z-h)-\delta(z)\right]\left[\Theta(r-a)-\Theta(r-b)\right]\Big\} \end{aligned} \tag{6}$$

   and show that the force calculated from the Lorentz force law agrees with the result of part b.

5. The interior of a perfectly conducting circularly cylindrical cavity of radius R and height L has fields described by the Coulomb gauge potentials, $\boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$,

$$\begin{aligned} \Phi(\boldsymbol{r},t) &= 0 \\ \boldsymbol{A}(\boldsymbol{r},t) &= \mathrm{Re}\left[\boldsymbol{A}_c(\boldsymbol{r})e^{-i\omega t}\right] \\ \boldsymbol{A}_c(r,\phi,z) &= A_0 J_0\left(\frac{\gamma r}{R}\right)\hat{\boldsymbol{z}} \end{aligned} \tag{7}$$

   where I use *cylindrical coordinates* $x = r\cos\phi$, $y = r\sin\phi$, and γ is the first zero of the Bessel function $J_0(\gamma) = 0$, i.e. $\gamma \sim 2.404825558$. *Note* A_0 *is complex.*

   a. Explain why the $\boldsymbol{E}$, $\boldsymbol{B}$ fields as well as the surface charge density and the surface current density can all be written as the real part of space dependent complex fields multiplied by $e^{-i\omega t}$, i.e. like $\boldsymbol{A}$ in terms of $\boldsymbol{A}_c$ above.

   Calculate the complex fields $\boldsymbol{E}_c(\boldsymbol{r})$ and $\boldsymbol{B}_c(\boldsymbol{r})$ and solve for the value of ω needed to satisfy Maxwell's equations.

   b. Verify that all the boundary conditions for a perfect conductor are satisfied.

   c. Explain why all quantities quadratic in the fields can be written as a part independent of time, and a part that oscillates at a an angular frequency 2ω.

   Calculate the total electromagnetic energy inside the cavity.

d. Use the stress tensor to calculate the time averaged outward force per unit area on the constant r wall, and then on the constant z walls.

e. Calculate the complex σ_c needed to give the surface charge density on the cavity surface as a function of position and time. Calculate the complex $\boldsymbol{K}_c$ needed to give the surface charge current density on the cavity surface as a function of position and time.

Hint: *Inside a perfect conductor, ohms law $\boldsymbol{J} = \sigma \boldsymbol{E}$ with $\sigma \to \infty$ tells you that $\boldsymbol{E} = 0$. Since $\boldsymbol{E}$ is zero, $\boldsymbol{B}$ must have no time derivative, so any nonzero $\boldsymbol{B}$ must be static. The Bessel function recursion relations given in Jackson, and the Lommel integral notes on the class blackboard site may be helpful.*

6. You have studied black-body radiation in statistical mechanics and quantum mechanics. Obviously, since Planck invented quantum mechanics to describe black-body radiation, classical theory fails. However, it does give the correct relationship between the pressure, volume and total energy. If we look at radiation in a cavity surrounded by a perfect conductor (as in the previous problem) in thermal equilibrium and write the momentum conservation equation as in Jackson Eq. 6.121, we find the differential form

$$\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{E}(\boldsymbol{r},t) \times \boldsymbol{B}(\boldsymbol{r},t) + \boldsymbol{f}_{\text{mech}}(\boldsymbol{r},t) = \boldsymbol{\nabla} \cdot \overleftrightarrow{T}(\boldsymbol{r},t) \tag{8}$$

where $\boldsymbol{f}_{\text{mech}}(\boldsymbol{r},t)$ is the mechanical force per unit volume at $\boldsymbol{r}$ and time t.

a. Show that the time average of the time derivative of a periodic function is identically zero when averaged over a period. Show that the time average of the time derivative of a bounded function goes to zero when averaged over long times.

b. In thermal equilibrium the time-averaged mechanical force on the walls from the electromagnetic field balances the pressure P. Therefore, the mechanical force is an outward force normal to the surface of the cavity with magnitude $P\hat{\boldsymbol{n}}dS$ over an infinitesimal surface. Convince yourself that

$$\int_V d^3r\ \boldsymbol{r} \cdot \boldsymbol{f}_{\text{mech}}(\boldsymbol{r},t) = P \int_S dS\ \boldsymbol{r} \cdot \hat{\boldsymbol{n}} \tag{9}$$

where V is the volume of the cavity extending infinitesimally into the metal walls so that f_{mech} acts. Show using the divergence theorem that

$$\begin{aligned} P \int_S dS\ \boldsymbol{r} \cdot \hat{\boldsymbol{n}} &= 3PV \\ \int_V d^3r\ \boldsymbol{r} \cdot \left[\boldsymbol{\nabla} \cdot \overleftrightarrow{T}(\boldsymbol{r},t)\right] &= -\text{tr} \int_V d^3r \overleftrightarrow{T}(\boldsymbol{r},t) \end{aligned} \tag{10}$$

where tr is the trace, $\text{tr}\overleftrightarrow{T} = \sum_\alpha T_{\alpha\alpha}$.

c. Put these pieces together to show that the equation of state is

$$PV = \frac{1}{3}W_{\text{em}}\,, \tag{11}$$

where W_{em} is the time averaged electromagnetic energy in the cavity.