PHY6938 Modern Physics and Quantum Mechanics: Fall 99 to Spring 97

1. Consider an electron moving in one dimension in the presence of an attractive potential of the form:

$$V(x) = \begin{cases} -V_0, & |x| \le a, \\ 0, & |x| > a. \end{cases}$$

a) Find the equation that determines the bound state energy eigenvalues. Namely, give an algebraic equation that, when solved, provides the energy as a function of V_0 and a.

Schrödinger's equation for a particle of mass m moving in one dimension looks like

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + V(x)\Psi = E\Psi,$$

where for a bound electron we must have that the energy is less than zero (the value of the potential outside the potential well). Inside the well the symmetry forces the wave function to be even under $x \to -x$, so that it looks like a cosine

$$\Psi_{\text{in}} = N\cos(k_{\text{in}}x),$$

where N_{in} is a normalization constant and k_{in} is the wavenumber which we will determine by substituting Ψ_{in} and $V(x) = -V_0$ into the Schrödinger equation, and we find

$$\frac{\hbar^2 k_{\text{in}}^2}{2m}\Psi - V_0\Psi = E\Psi,$$

so that

$$k_{\text{in}}^2 = \frac{2m(V_0 - |E|)}{\hbar^2},$$

where we have used $E = -|E|$ for a bound state.

Outside of the potential well the potential is zero so that

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi_{\text{out}}}{dx^2} = E\Psi_{\text{out}},$$

so that (recall that $E < 0$ for a bound state)

$$\frac{d^2\Psi_{\text{out}}}{dx^2} = \frac{2m|E|}{\hbar^2}\Psi_{\text{out}},$$

and so we have the general solution

$$\Psi_{\text{out}}(x) = Ae^{-k_{\text{out}}x} + Be^{+k_{\text{out}}x},$$

where $k_{\text{out}} = \sqrt{2m|E|/\hbar^2}$ Since this is a bound state we know we need the exponentially decreasing solution for both $x > a$ and $x < -a$, so that we have

$$\Psi_{\text{out}} = N_{\text{out}}e^{-k_{\text{out}}|x|}.$$

Now the normalization constants N_{in} and N_{out} can be found by matching the two wave-functions and their derivatives at one of the two boundaries between the inside and outside regions (by symmetry these must both give the same information). At $x = a$ we have

$$\begin{aligned} N_{\text{in}} \cos(k_{\text{in}} a) &= N_{\text{out}} e^{-k_{\text{out}} a} \\ -k_{\text{in}} N_{\text{in}} \sin(k_{\text{in}} a) &= -k_{\text{out}} N_{\text{out}} e^{-k_{\text{out}} a}. \end{aligned}$$

An easy way to remove the dependence on the normalization constants is to divide these two equations, which yields

$$k_{\text{in}} \tan(k_{\text{in}} a) = k_{\text{out}},$$

so that

$$\tan(k_{\text{in}} a) = \frac{k_{\text{out}}}{k_{\text{in}}} = \sqrt{\frac{|E|}{V_0 - |E|}},$$

so that, finally, we have

$$\tan\left(\sqrt{\frac{2m(V_0 - |E|)}{\hbar^2}}\, a\right) = \sqrt{\frac{|E|}{V_0 - |E|}}.$$

b) Consider the limit of a very weak bound state, i.e. $|E|/V_0 \ll 1$. Find the ground state energy and wave function (apart from an overall multiplicative constant) in this limit. How can one determine this multiplicative constant?

In the above equation we can now make the approximation $V_0 - |E| \simeq V_0$. The right hand side of the above equation is small, so that the argument of the tangent is small, and since $\tan(x) \simeq x$ when x is small we have

$$\sqrt{\frac{2mV_0}{\hbar^2}}\, a = \sqrt{\frac{|E|}{V_0}},$$

so that

$$|E| = \frac{2mV_0^2 a^2}{\hbar^2}.$$

To find the normalization constants N_{in} and N_{out} in the wave function we can use the first of the above boundary conditions to eliminate N_{out} in the wave-function,

$$\Psi(x) = \begin{cases} N_{\text{in}} \cos(k_{\text{in}} x), & |x| \le a, \\ N_{\text{in}} \cos(k_{\text{in}} a) e^{-k_{\text{out}}(|x|-a)}, & |x| > a, \end{cases}$$

and then the constant N_{in} is determined from the condition

$$\int_0^\infty |\Psi|^2 = 1.$$

2. Provide a brief qualitative description for each item listed below.

a) Heisenberg uncertainty principle

The Heisenberg uncertainty principle states that it is impossible to know simultaneously the precise position and momentum, or time coordinate and energy, of a particle. The product of the uncertainties of the two measurements must be at least $\hbar/2$. An example of the energy-time relation is the product of the lifetime of an unstable state and the uncertainty in the energy of the photon it emits (the decay line-width).

b) The fundamental forces

In order of increasing strength these are: the gravitational force between any two massive objects, which falls off like $1/r^2$ and so has infinite range; the electromagnetic force between any two charged particles, which has the same force law and infinite range; the weak force responsible for, for example, the beta decay of a neutron into a proton, and electron and an anti-electron-neutrino; and the strong force which is responsible for binding quarks together into protons and neutrons and also for the nuclear force between nucleons. The weak force has extremely short range because of the large mass of the force carriers (the W and Z bosons) and the strong force has effectively a short range because confinement of colored objects requires any final-state particles in an interaction to be color neutral.

c) Michelson-Morley experiment

This experiment was set up to determine how quickly the Earth moved through the aether which was thought to be the medium in which light propagated. If light moved with a fixed speed through the aether, the apparent speed of a beam of light through an apparatus fixed to the moving Earth would depend on whether it moved perpendicular or parallel to the direction of motion of the Earth. A beam of light was split by a half-silvered mirror and sent along two paths at right angles to each other, reflected back by mirrors and then recombined. By counting the number of interference fringes moving by an observer as the apparatus was rotated by 90°, Michelson and Morley planned to measure the difference in speed of the two beams. They saw no such motion of the fringes, which invalidated the idea of the aether.

3. When light of wavelength 520 nm is incident on the surface of a metal, electrons are ejected with a maximum speed of 1.78×10^5 m/s. What wavelength is needed to give a maximum speed of 4.81×10^5 m/s?

The electrons are bound to the metal with a work function ϕ, which is the *minimum* energy required to remove an electron from the metal. Assuming an electron which requires only this minimum energy, the energy $h\nu$ of the photon goes into releasing the electron from the surface and any extra energy is kinetic energy of the electron, so that (note the electron velocity is non relativistic)

$$h\nu = \frac{hc}{\lambda} = \phi + \frac{1}{2}m_e v_{\rm max}^2.$$

For two different wavelengths we have

$$\begin{aligned} \frac{hc}{\lambda_1} &= \phi + \frac{1}{2}m_e v_1^2 \\ \frac{hc}{\lambda_2} &= \phi + \frac{1}{2}m_e v_2^2, \end{aligned}$$

where v_1 and v_2 are understood to be maximum velocities, and so we can eliminate ϕ to obtain

$$\begin{aligned}\frac{hc}{\lambda_1} - \frac{1}{2}m_e v_1^2 &= \frac{hc}{\lambda_2} - \frac{1}{2}m_e v_2^2 \\ \frac{hc}{\lambda_2} &= \frac{hc}{\lambda_1} + \frac{1}{2}m_e(v_2^2 - v_1^2)\end{aligned}$$

or

$$\begin{aligned}\lambda_2 &= \frac{hc}{hc/\lambda_1 + \frac{1}{2}m_e(v_2^2 - v_1^2)} \\ &= 1240 \text{ eV} \cdot \text{nm}/\left\{\frac{1240 \text{ eV} \cdot \text{nm}}{520 \text{ nm}} + \frac{1}{2}0.511 \times 10^6 \text{ eV}\frac{([4.81 \times 10^5]^2 - [1.78 \times 10^5]^2)}{(3 \times 10^8)^2}\right\} \\ &= 420 \text{ nm}.\end{aligned}$$

4. A particle of mass m is confined in a one-dimensional infinite square well of width a.

a) Find the energy and the wave function of the n-th level.

Inside the well the particle satisfies the free-particle Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} = E\Psi,$$

and so its wave function looks like a wave subject to the boundary conditions required by the infinite potential at $x = 0$ and $x = a$, which are that the wave function must vanish there. This means that the wave function will look like

$$\Psi_n(x) = N\sin(k_n x),$$

with $k_n a = n\pi$, $n = 1, 2, ...$ and N a normalization coefficient. Using the Schrödinger equation we find that

$$\begin{aligned}E_n\Psi_n &= -\frac{\hbar^2}{2m}(-k_n^2)\Psi_n \\ E_n &= \frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2 = \frac{n^2\pi^2\hbar^2}{2ma^2}.\end{aligned}$$

The normalization coefficient is found by simply noting that the average value of $\sin^2(kx)$ over an integral number of half wavelengths is always $1/2$, so that

$$1 = \int_0^a dx|\Psi(x)|^2 = N^2 a/2,$$

so that

$$\Psi_n(x) = N\sin(k_n x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi}{a}x\right).$$

b) A one-dimensional electron is confined between two impenetrable walls a distance 1 Å apart. What is the energy of the electron in the first excited state in electron volts? Note $m_e = 0.511$ MeV/c^2, and $\hbar c = 197.3$ MeV·fm.

We simply apply the above formula for E_2 (the energy of the first excited state above) to find (note 1Å$= 10^5$ fm)

$$E_2 = \frac{4\pi^2 \cdot (197.3 \text{ MeV} \cdot \text{fm})^2}{2(0.511 \text{ MeV})(10^5 \text{ Å})^2} = 150 \text{ eV}.$$

c) What is the average force the electron applies on each wall in this state?

If we imagine moving the walls from a to $a - da$ apart, the energy of the particle will go up by dE. By analogy with potential energy, we can find the force exerted by the walls on the particle by the rate of change of this energy with the separation of the walls (like $F = -dU/dx$)

$$F = -\frac{dE}{da} = \frac{n^2\pi^2\hbar^2}{ma^3} = \frac{2E_n}{a},$$

and if each of the walls exert this force on the particle, the particle exerts this force on each of the walls. For $n = 2$ this gives

$$F_2 = \frac{2 \cdot 150 \cdot 1.602 \times 10^{-19} \text{ J}}{10^{-10} \text{ m}} = 4.8 \times 10^{-7} \text{ N}.$$

5. Provide a brief qualitative description for each item listed below.

a) Čerenkov radiation

Čerenkov radiation occurs when a charged particle travels through a material faster than the speed of light c/n in that material. The radiation is emitted in a forward opening cone whose opening angle depends on the particle's speed. It is like the shock wave which occurs when a body moves faster than the speed of sound in air.

b) Planck's constant

Planck's constant h relates the energy of a photon to the frequency of the electromagnetic radiation, $E = h\nu$. As such, it provides the link between the wave and particle nature of the photon (and other particles). Planck introduced h to explain the failure of the Rayleigh-Jeans formula for black-body radiation, which led to an infinity (the ultraviolet catastrophe) in the amount of energy emitted by a black-body when high energy photons were considered. His suggestion was that atoms emit energy only in discrete bundles whose energy was proportional to the photon frequency.

c) Rutherford scattering

Rutherford scattered alpha particles (^{4}He nuclei) off a gold foil. It was expected that there would be few backward scattering events because it was thought that the material in atoms was roughly evenly spread out. In fact the distribution of the scattering angle showed the existence of small massive centers of positive charge in the gold foil. This led to the Rutherford scattering formula (for point like charged particles) and to the Bohr model of the atom.

6. Let $\mathbf{s}_1$ and $\mathbf{s}_2$ be the spin operators of two spin-1/2 particles.

a) Find the simultaneous eigenfunctions of the operators $\mathbf{s}^2$ and $\mathbf{s}_z$, where $\mathbf{s} = \mathbf{s}_1 + \mathbf{s}_2$.

Here the authors of the question really meant for you to *write down* the simultaneous eigenfunctions, as finding them from first principles is a laborious process when you likely already know the answer. The product space contains the four sub-states

$$|\uparrow\uparrow\rangle, \quad |\uparrow\downarrow\rangle, \quad |\downarrow\uparrow\rangle, \quad |\downarrow\downarrow\rangle,$$

with total spin projections $S_z = s_{1z} + s_{2z}$ of 1, 0, 0, and -1 respectively. We can be sure that the first and last states have total spin $S = 1$ since the only two possibilities from adding spin 1/2 to spin 1/2 are 0 and 1, and only $S = 1$ is allowed to have $M = -1, 1$. This means that

$$\begin{aligned} |SM\rangle = |11\rangle &= |\uparrow\uparrow\rangle \\ |1-1\rangle &= |\downarrow\downarrow\rangle. \end{aligned}$$

Applying a lowering operator $S_- = s_{1-} + s_{2-}$ to the top state $|11\rangle$ gives a symmetric linear combination

$$S_-|\uparrow\uparrow\rangle = a\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right),$$

with a a constant which we can determine from normalization considerations to be $a = 1/\sqrt{2}$, so that

$$|10\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right),$$

and then we know that (up to a phase which we can choose) from orthogonality of $|10\rangle$ and $|00\rangle$ we must have

$$|o0\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right).$$

b) Assume the spin of these particles are coupled by an exchange interaction of the form $H = -J\mathbf{s}_1 \cdot \mathbf{s}_2$. What are the allowed energy states of the system?

Here we use a famous and very useful trick. We need to find the expectation value of $\mathbf{s}_1 \cdot \mathbf{s}_2$, which is a scalar operator in the total spin space (it is the dot product of two vectors) and so links only states with the same total S and M (it is diagonal in the total spin space). If we square the total spin operator we find

$$\mathbf{S}^2 = (\mathbf{s}_1 + \mathbf{s}_2)^2 = \mathbf{s}_1^2 + 2\mathbf{s}_1 \cdot \mathbf{s}_2 + \mathbf{s}_2^2,$$

so that

$$\mathbf{s}_1 \cdot \mathbf{s}_2 = \mathbf{S}^2 - \mathbf{s}_1^2 - \mathbf{s}_2^2.$$

This means that

$$\langle SM|\mathbf{s}_1\cdot\mathbf{s}_2|SM\rangle = \frac{1}{2}\left[S(S+1)-\frac{1}{2}\left(\frac{1}{2}+1\right)-\frac{1}{2}\left(\frac{1}{2}+1\right)\right] = \begin{cases} 1/4, & S=1,\\ -3/4, & S=0,\end{cases}$$

since the eigenvalue of the operator $\mathbf{s}^2$ is $s(s+1)$ for any (spin) angular momentum $\mathbf{s}$. This means that

$$\langle SM| - J\mathbf{s}_1\cdot\mathbf{s}_2|SM\rangle = \begin{cases} -J/4, & S=1,\\ +3J/4, & S=0.\end{cases}$$

c) Assuming $J>0$, what is the degeneracy of the ground and the first excited state?

We have just derived that the ground state is a triplet ($M=-1,0,1$) and the first excited state a singlet ($M=0$).

d) If an external magnetic field H is turned on, how are the allowed energy states modified? (*Hint:* the coupling of the field to the spins is of the Zeeman type, $H_Z = -g\mu_B\mathbf{S}\cdot\mathbf{H}$).

Due to the interaction there is a preferred direction along which the spins will be quantized (call this now the z-axis), so that

$$\langle SM|\mathbf{S}\cdot\mathbf{H}|SM\rangle = H\langle SM|S_z|SM\rangle = MH.$$

Note that the interaction, which is proportional to S_z, commutes with the total spin operator $\mathbf{S}^2$ and so cannot link the $|00\rangle$ state to the $|10\rangle$ state. This means that the only effect is to symmetrically split the ground state triplet by

$$\langle SM| - g\mu_B\mathbf{S}\cdot\mathbf{H}|SM\rangle = -g\mu_B MH,$$

with the lowest energy state now having $M=+1$.

7 a). How fast does a muon have to travel to have the same energy as a charged pion at rest?

The mass of the muon is $m_\mu = 106$ MeV/c^2, and that of the pion is $m_\pi = 140$ MeV/c^2. The (total relativistic) energy of the moving muon is therefore

$$m_\mu\gamma c^2 = m_\pi c^2,$$

so that

$$\gamma = \frac{m_\pi}{m_\mu},$$

and so

$$\begin{aligned} 1-\frac{v^2}{c^2} &= \frac{1}{\gamma^2} = \left(\frac{m_\mu}{m_\pi}\right)^2 \\ \frac{v}{c} &= \sqrt{1-\left(\frac{m_\mu}{m_\pi}\right)^2} = 0.65. \end{aligned}$$

b) What is the de Broglie wavelength of the muon?

The de Broglie wavelength is given by

$$\lambda = \frac{h}{p} = \frac{hc}{pc}.$$

Now the momentum of a relativistic particle is $p = m\gamma v$, so we have

$$\lambda = \frac{hc}{pc} = \frac{hc}{m\gamma vc} = \frac{hc\sqrt{1-(v/c)^2}}{mc^2(v/c)} = \frac{1240 \text{ MeV} \cdot \text{fm}\sqrt{1-(0.65)^2}}{106 \text{ MeV} \cdot 0.65} = 13.7 \text{ fm}.$$

c) If you could measure the momentum of the muon with 10% accuracy, how well could you measure its position along its direction of motion?

According to the uncertainty principle we have that

$$\Delta p \Delta x \geq \frac{\hbar}{2} = \frac{h}{4\pi}.$$

This means that

$$\Delta x \geq \frac{h}{4\pi(0.1p)} = \frac{h}{4\pi(0.1h/\lambda)} = \frac{\lambda}{0.4\pi} = 10.9 \text{ fm}.$$

8. Provide a brief qualitative description for each item listed below.

a) Compton effect

Compton scattering is the process of the scattering of a photon by an electron (although it can also happen for other charged particles). The collision conserves energy and momentum, so the electron recoils and the photon is emitted, in general, at an angle to its original direction and with a reduced energy. The net effect is that the photon appears to scatter from the electron with a reduced final energy and so longer wavelength. The change in the wavelength depends only on the scattering angle and not on the original wavelength.

b) the difference between bosons and fermions

Bosons are particles like the photon which have integral spin (or intrinsic angular momentum), while fermions are particles like the electron that have half-integral spin. Fermions satisfy Fermi-Dirac statistics, because they obey the Pauli exclusion principle, which states that no two indistinguishable fermions can occupy the same quantum mechanical state. Bosons satisfy Bose-Einstein statistics, which means that they do not obey the Pauli exclusion principle.

c) Rutherford scattering

See **5c**).

9. Provide a brief qualitative description for each item listed below.

a) Hyperfine structure in a hydrogen atom

The proton that makes up the hydrogen nucleus has a spin, and since it is a charged particle it has a magnetic moment. The same is true of the electron. These two magnetic moments interact with each other (one moves in the magnetic field generated by the other) and the resulting interaction Hamiltonian cause a splitting of the ground state (and other energy levels) which depends on the projection of the spin of the electron. The energy difference is very small, of the order of 10^{-7} eV, and wavelength of the light from this transition is 21 cm, in the radio part of the spectrum, which is important for radio astronomy.

b) Rayleigh scattering

Rayleigh scattering is a single-step process where a photon is absorbed by an atom and promotes an electron into an excited state, which then decays back to the ground state. The incident and scattered photons are correlated, and have the same energy.

c) The four basic forces (describe each force)

See **2b**).

10. Monochromatic blue light with a wavelength of 434.2 nm is incident on a sample of cesium. Electrons emitted from the cesium surface are observed to have velocities ranging up to 5.491×10^5 m/s. Note $m_e = 0.511$ MeV/c^2, $q_e = 1.602 \times 10^{-19}$ C, $hc = 1240$ eV·nm.

a) What is the work function for this sample of cesium?

The maximum kinetic energy of the emitted electron plus the work function of the cesium is equal to the energy of the photon. This means that we can find the work function by

$$\begin{aligned}
\phi &= E_\gamma - T\mathrm{max} = h\nu - T_{\max} = \frac{hc}{\lambda} - \frac{1}{2} m_e v_{\max}^2 \\
&= \frac{1240 \text{ eV} \cdot \text{nm}}{434.2 \text{ nm}} - \frac{1}{2} \cdot 0.511 \times 10^6 \text{ eV} \cdot (\frac{5.491 \times 10^5}{3.0 \times 10^8})^2 \\
&= 2.0 \text{ eV}.
\end{aligned}$$

b) Explain why there is a *range* of emitted electron velocities.

Electrons near the surface absorb all of the energy of the incident photon and then leave the surface with the maximum energy available, which is the photon energy minus the work function. Electrons deeper inside the metal are more likely to interact with the electrons around other atoms on their way out of the sample before leaving the surface, and these interactions will reduce their kinetic energy.

c) What is the wavelength of the fastest emitted electrons?

The de Broglie wavelength is

$$\lambda = \frac{h}{p} = \frac{hc}{mvc} = \frac{1240 \text{ eV} \cdot \text{nm}}{0.511 \times 10^6 \text{ eV}(5.491 \times 10^5 / 3.0 \times 10^8)} = 1.33 \text{ nm}.$$

d) Now assume that the hydrogen discharge lamp produces 2.0 μW of power radiated in this particular blue Balmer series spectral line. If the lamp can be considered to be a point source and emits the light isotropically, estimate how many of these blue photons per second strike a circular cesium sample 7.5 cm in diameter and placed 10 cm from the lamp.

It is actually a difficult geometrical problem to find the area cut out of a sphere by a circular disk which has its circumference touching the inside of the disk. We need this to find the solid angle subtended by the cesium sample and so the fraction of the emitted energy which is absorbed by the sample. If the radius of the disk is small compared to that of the sphere we can approximate the solid angle by the area of the disk divided by the surface area of the sphere which goes through the center of the disk, but that is not the case here. This problem was not intended to be that hard, so we will use this approximation even though it is not justified. The solid angle is, with this approximation,

$$\Omega = \frac{\pi r^2}{4\pi R^2} = \frac{1}{4}\left(\frac{7.5/2}{10.0}\right)^2 = 0.0351$$

and so the energy E per second which goes into this solid angle is this fraction of the total emitted energy per second,

$$E = 0.0351 \cdot P_{\text{tot}} \cdot 1 \text{ s} = 0.0351 \cdot 2 \times 10^{-6} \text{ J} = \frac{0.0702 \times 10^{-6}}{1.602 \times 10^{-19} \text{ J/eV}} = 4.38 \times 10^{11} \text{ eV},$$

and the number of photons per second is this energy divided by the energy $h\nu = hc/\lambda$ per photon,

$$N = \frac{2.19 \times 10^11 \text{ eV}}{1240 \text{ eV} \cdot \text{nm}/434.2 \text{ nm}} = 1.53 \times 10^{11} \text{ photons}.$$

11. A photon with wavelength 24.8 fm strikes a proton at rest. The photon undergoes Compton scattering, and the scattered photon is seen by an observer in the lab to be emitted at 180° with respect to the direction of the incident photon. The mass of the proton is $M_p = 938$ MeV/c^2, and $hc = 1240$ MeV·fm.

a) What is the energy of the incident photon? What name would typically be given to classify this "type" of photon? Give a very brief explanation for your choice.

The energy of the photon is

$$E = h\nu = \frac{hc}{\lambda} = \frac{1240 \text{ eV} \cdot \text{nm}}{24.8 \times 10^-6 \text{ nm}} = 50 \text{ MeV},$$

which is a gamma ray. Gamma rays are typically emitted during transitions between excited states and ground states of nuclei, and typical nuclear excitation energies are in the MeV range.

b) Using relativistic kinematics, find (i) the wavelength of the scattered photon and (ii) the de Broglie wavelength of the recoiling proton.

We have to conserve relativistic energy and momentum in the collison. The initial energy and momentum of the photon are related by $E_\gamma = pc$, or we can write that the photon four-momentum is (assume all motion is along the z-axis)

$$p_\gamma = (E_\gamma, \mathbf{p}_\gamma) = (pc, 0, 0, p),$$

while the initial proton four momentum is

$$P = (m_p c^2, 0, 0, 0).$$

The final photon four momentum is

$$p'_\gamma = (p'c, 0, 0, -p'),$$

since the photon back scatters, and by conservation of four-momentum (i.e. energy *and* momentum) we know the final four momentum of the proton looks like

$$P' = (m_p c^2 + pc - p'c, 0, 0, p + p').$$

This takes care of conservation of energy and momentum, and we can now use this to find p' in terms of p if we use that the square of the proton four-momentum is

$$P'^2 = m_p^2 c^2$$

which is equivalent to the relation

$$E'^2 = \mathbf{P}'^2 c^2 + m_p^2 c^4.$$

This gives the relation

$$\begin{aligned}
m_p^2 c^4 &= (m_p c^2 + pc - p'c)^2 - (p + p')^2 c^2 \\
&= m_p^2 c^4 + 2 m_p c^3 (p - p') + (p - p')^2 c^2 - (p + p')^2 c^2 \\
4pp'c^2 &= 2 m_p c^3 (p - p') \\
2pp' &= m_p c (p - p') \\
p' &= \frac{m_p c p}{m_p c + 2p} \\
p'c &= \frac{m_p c^2 pc}{m_p c^2 + 2pc} = 50\ \text{MeV} \cdot \frac{938}{938 + 100} = 45.2\ \text{MeV}
\end{aligned}$$

so that the photon wavelength is

$$\lambda' = \frac{h}{p'} = \frac{hc}{p'c} = \frac{hc}{pc}\left(\frac{m_p c^2 + 2pc}{m_p c^2}\right) = \lambda \cdot \frac{938 + 100}{938} = 27.4\ \text{fm}.$$

Now we know that the momentum of the proton is $p + p'$, so that the de Broglie wavelength of the proton is

$$\lambda_p = \frac{hc}{pc + p'c} = \frac{1240\ \text{eV} \cdot \text{nm}}{(50.0 + 45.2)\ \text{MeV}} = 13.0\ \text{fm}.$$

c) If we could observe this reaction occuring in the center-of-mass frame instead of the lab frame, what would we then see as the difference between the wavelengths of the incoming and scattered photons? Explain your answer.

In the center of mass frame the initial momentum of the photon and proton must be equal and opposite, and the same is true of the final momenta. Then the conservation of energy condition looks like

$$pc + \sqrt{m_p^2c^4 + p^2c^2} = p'c + \sqrt{m_p^2c^4 + p'^2c^2}.$$

Since the left and right sides of this equation are the same function of p (or p') then this can only be true if $p = p'$, so the incident and scattered photons must have the same momentum.

12. Consider a one dimensional step potential of the form:

$$V(x) = \begin{cases} 0, & x < 0, \\ +V_0, & x \geq 0. \end{cases}$$

A particle with total energy E and mass m is incident on the step potential "from the left" (in other words, the particle starts at negative values of x and travels towards positive values of x). The particle's energy E is greater than V_0.

a) Use the time-independent Schrödinger equation to determine the form of the particle's wave function in the two regions $x < 0$ and $x \geq 0$.

In the $x < 0$ region (call this region 1) the potential is zero so the Schrödinger equation has the form

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi_1}{dx^2} = E\Psi_1,$$

and Ψ_1 has the form of a plane wave moving to the right (the incident wave) and another moving to the left (the reflected wave), so that

$$\Psi_1 = Ae^{ik_1x} + Be^{-ik_1x}.$$

Substituting this into the Schrödinger equaton we see that it is a solution if

$$\begin{aligned} -\frac{\hbar^2}{2m}(-k_1^2) &= E \\ k_1 &= \sqrt{\frac{2mE}{\hbar^2}} = \frac{\sqrt{2mE}}{\hbar}. \end{aligned}$$

In the $x > 0$ region (call this region 2) we have

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2\Psi_2}{dx^2} + V_0\Psi_2 &= E\Psi_2 \\ -\frac{\hbar^2}{2m}\frac{d^2\Psi_2}{dx^2} &= (E - V_0)\Psi_2 \end{aligned}$$

and Ψ_2 has the form of a plane wave moving to the right (the transmitted wave), so that

$$\Psi_2 = Ce^{ik_2x},$$

with

$$k_2 = \frac{\sqrt{2m(E - V_0)}}{\hbar}.$$

b) Derive expressions for the probabilities that the particle is (i) reflected (R), and (ii) transmitted (T). (*Hint*: recall that the probability density current is given by

$$j(x) = \mathrm{Re}\left(\Psi^* \frac{\hbar}{im}\frac{\partial \Psi}{\partial x}\right),$$

and that R and T are ratios of probability density currents.)

We have to derive the values of A, B, and C using the boundary conditions, which are that the wavefunction and its derivative must be continuous at the boundary ($x = 0$). Continuity of the wavefunction gives that

$$A + B = C, \tag{1}$$

while that of the derivative at $x = 0$ gives that

$$k_1 A - k_1 B = k_2 C, \tag{2}$$

since taking the derivative of the plane wave just brings down a factor of $-ik$, and we have divided by $-i$. For that reason the incident probability current is simply

$$j_{\mathrm{in}} = \mathrm{Re}\left[Ae^{-k_1 x}\frac{\hbar}{im} ik_1 A e^{ik_1 x}\right] = \frac{\hbar}{m} k_1 A^2,$$

and similarly

$$\begin{aligned} j_{\mathrm{refl}} &= \frac{\hbar}{m} k_1 B^2 \\ j_{\mathrm{trans}} &= \frac{\hbar}{m} k_2 C^2. \end{aligned}$$

Putting these together we get that the reflection probability is

$$R = \frac{j_{\mathrm{refl}}}{j_{\mathrm{in}}} = \frac{B^2}{A^2},$$

and we can find B/A by combining (1) and (2) above,

$$k_1(A - B) = k_2(A + B),$$

so that

$$\frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2},$$

and

$$R = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2.$$

By conservation of probability we have that

$$T = 1 - R = \frac{(k_1 + k_2)^2 - (k_1 - k_2)^2}{(k_1 + k_2)^2} = \frac{4k_1 k_2}{(k_1 + k_2)^2}.$$

13. A particle of mass m is confined to a one-dimensional, square potential well with infinitely high potential walls at $x = 0$ and L.

a) Find the ground state energy and wave function of the problem.

We simply have to fit a wave function $\Psi(x)$ into the well which satisfies the boundary conditions $\Psi(0) = \Psi(L) = 0$ proscribed by the infinite potential walls. The longest wavelength and so shortest energy is where

$$\Psi_0(x) = N \sin\left(\frac{\pi x}{L}\right).$$

Since the integral of $\sin^2(x)$ over an integral number of half wavelengths is $1/2$, we know that

$$\int_0^L \Psi_0^2 = N\frac{L}{2} = 1,$$

so that $N = \sqrt{2/L}$ and

$$\Psi_0(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right).$$

The energy is found using Schrödinger's equation to be (note that the wavelength of this wave is $2L$)

$$E_0 = \frac{\hbar^2}{2m}k_0^2 = \frac{\hbar^2}{2m}\left(\frac{2\pi}{2L}\right)^2 = \frac{\hbar^2\pi^2}{2mL^2}.$$

b) From arguments based on the uncertainty principle, estimate a lower bound for the smallest energy that a particle in this potential can have, and compare it with the result of part a).

The uncertainty principle says

$$\Delta p \Delta x \geq \hbar,$$

and here $\Delta x = L$. Since the average value of the momentum for the particle is always zero (regardless of its momentum) we have that

$$\Delta p = \sqrt{\langle p^2 \rangle}.$$

In the box $E = \langle p^2\rangle/2m$ since the energy is all kinetic, so that $\Delta p = \sqrt{2mE}$ and in the ground state

$$L \cdot \sqrt{2mE_0} \geq \hbar,$$

and so

$$E_0 \geq \frac{\hbar^2}{2mL^2},$$

which is missing a factor of π^2 compared to **a**).

c) Write the ground state wave function if two identical particles are introduced in the well. Suppose the particles carry spin 1/2 and do not interact with each other.

The spatial wave functions of each particle is the same, but the spins must have different projections. The combined wave function must change sign under exchange of the particles, so it must have the form

$$\Psi(x_1, x_2) = \Psi_0(x_1)\Psi_0(x_2)\frac{1}{\sqrt{2}}\left[\uparrow\downarrow - \downarrow\uparrow\right].$$

d) Now consider a square-well-like potential of depth $V_0 = 1000$ eV. Suppose it is given that the first four eigenstates of the problem have an energy $E_1 = 20$ eV, $E_2 = 70$ eV, $E_3 = 200$ eV, and $E_4 = 500$ eV, measured from the bottom of the well. Introduce eight electrons in the problem (they have spin). What is the minimum energy necessary to remove an electron from the ground state of the well and move it to an infinite distance? Assume the electrons do not interact among themselves.

We simply arrange for at most two electrons to be in each level. This means that the first four levels will be occupied. The last electron to be added has an energy of 500 eV above the bottom of thr well, i.e. -500 eV relative to the exterior of the well. Then it requires 500 eV to free this last electron.

14. Consider an electron in a hydrogen atom that has the following wave function at a particular time, $t = 0$:

$$|\psi(0)\rangle = A\left(|100\rangle + 2i|210\rangle + 2|322\rangle\right).$$

Here, each of the individual eigenvector terms are denoted by their quantum numbers N (principal), L (angular momentum), and M (angular momentum projection) in the following manner: $|NLM\rangle$.

a) Calculate the value of the normalization constant A.

Since each of the eigenstates is normalized and orthogonal to every other, the modulus squared of the wavefunction is the sum of the moduli squared of each coefficient, so that we need

$$A^2(1 + 4 + 4) = 1,$$

and so $A = 1/3$.

b) Find the expectation value of the energy of this electron at $t = 0$. Express your answer in units of eV.

The energy of each substate is $-13.6/N^2$ eV, where N is the principal quantum number. Since for each state the Hamiltonian operator gives back the state times this energy, and each of the states is orthogonal, we simply have to add the energies of each state with the appropriate normalization factors (in other words the Hamiltonian is diagonal in this basis)

$$\begin{aligned}\langle\Psi(0)|H|\Psi(0)\rangle &= A^2\left[\langle 100|H|100\rangle + 4\langle 210|H|210\rangle + 4\langle 322|H|322\rangle\right] \\ &= \frac{1}{9}\left[1 + 4\cdot\frac{1}{4} + 4\cdot\frac{1}{9}\right](-13.6 \text{ eV}) = -\frac{22}{81}(13.6 \text{ eV}) = -3.69 \text{ eV}.\end{aligned}$$

c) If a measurement of the z-projection of the electron's angular momentum is made at $t = 0$, then with what probability are the results 0, $h/2\pi$, h/π, and $3h/2\pi$ obtained?

The probability that we measure $M = 0$ is simply the sum of the probabilities that we find it in a substate with $M = 0$,

$$P(M=0) = \sum_{NL} \mid \langle NL0 | \Psi(0) \rangle \mid^2 = \frac{1}{9}(1+4) = \frac{5}{9},$$

and similarly

$$P(M=1) = 0, \; P(M=2) = \frac{4}{9}, \; P(M=3) = 0.$$

d) Write the expression for the wave function $|\Psi(t)\rangle$ at any time t after $t = 0$.

The time evolution of stationary states is given by

$$\Psi(x,t) = e^{-i\hat{H}t/\hbar}\Psi(x,0),$$

and since the action of the operator $\hat{H}$ on each substate is to give back the substate times its energy, we have

$$\Psi(x,t) = \frac{1}{3}\left[e^{i(13.6 \text{ eV})t/\hbar}|100\rangle + 2ie^{i(13.6 \text{ eV})t/4\hbar}|210\rangle + 2e^{i(13.6 \text{ eV})t/9\hbar}|322\rangle\right].$$