

Study guide for qualifying exams

Cody L. Petrie

July 1, 2015

1 Classical Mechanics

1. Newtonian Mechanics

- (a) Newton's Laws/Kinematics
- (b) Energy
- (c) Momentum/Angular Momentum

2. Lagrangian Mechanics

- (a) Calculous of Variations
- (b) Principle of Least Action/Lagranges Equation
- (c) Generalized Coordinates
- (d) Holonomic/Non-Holonomic Constraints
- (e) Noether's Theorem
- (f) Rigid Body Motion
 - i. Inertia Tensor
 - ii. Euler's Equations

3. Hamiltonian Formalism

- (a) Legendre Transformation/Hamilton's Equations
- (b) Generalized Momenta/Cyclic Coordinates/Conserved Quantities
- (c) Liouville's Theorem
- (d) Poisson Brackets
- (e) Canonical Transformations

2 Statistical Mechanics

1. Thermodynamics Review
 - (a) Laws of Thermodynamics
 - (b) Intensive vs Extensive Variables
 - (c) Thermodynamic Potentials and Ensembles
 - (d) Maxwell's Relations
 - (e) Various Definitions
 - i. Compressibility
 - ii. Heat Capacity etc.
2. Statistical Mechanics
 - (a) Statistical Review
 - (b) Partition Function/Trace
 - (c) Thermodynamic Limit
 - (d) Density Matrix
 - (e) Ideal Gas
 - (f) Ideal Bose Gas
 - (g) Ideal Fermi Gas
 - (h) Photons (BB)
 - (i) Phonons
 - (j) Bose-Einstein Condensates
 - (k) Cluster Expansion

3 Quantum Mechanics

1. Shankar Math Review
2. Postulates
3. Free Particle
4. Particle in a Box
5. Harmonic Oscillator
6. Angular Momentum
7. Hydrogen Atom
8. Spin

- 9. Angular Momentum Addition
- 10. Time-Independent Perturbation Theory
- 11. Time-Dependent Perturbation Theory
 - (a) Einstein A and B Coefficients
- 12. Scattering
- 13. WKB Formula
- 14. Dirac Equation

4 Electricity and Magnetism

- 1. Electrostatics
 - (a) Coulomb's Law
 - (b) Electrostatic Potentials
 - i. Poisson/Laplace's Equations
 - (c) Boundary Conditions
 - (d) Method of Images
 - (e) Multipole Expansion
 - (f) Work and Energy
 - (g) Electric Fields in Matter
- 2. Magnetostatics
 - (a) Lorentz Force Law
 - (b) Biot-Savart Law
 - (c) Vector Potential
 - (d) Magnetic Fields in Matter
- 3. Electrodynamics
 - (a) Ohm's Law
 - (b) Maxwell's Equations
 - (c) Boundary Conditions to Maxwell's Equations
 - (d) Continuity Equation
 - (e) Poynting's Theorem
 - (f) Maxwell Stress Tensor

- (g) Electromagnetic Waves
 - i. The Wave Equation from Maxwell's Eq.
 - ii. EM Waves in Matter
 - iii. Wave Guides
- 4. Scalar and Vector Potentials
- 5. Coulomb and Lorentz Gauge
- 6. Retarded Potentials
- 7. Lienard-Wiechert Potentials
- 8. Radiation
 - (a) Electric/Magnetic Dipole Radiation
- 9. Helmholtz Theorem
- 10. Special Relativity
 - (a) Einstein's Postulates
 - (b) Lorentz Transformation
 - (c) 4-Vectors
 - (d) Field Tensor and Transformation
 - (e) Relativistic Potentials

5 Classical Mechanics Equations

Newtonian Mechanics

Newton's Laws:

1. An object will maintain it's current motion unless acted upon by an external force.
2. $\vec{F} = m\vec{a}$
3. All forces occur in equal but directionally opposite pairs.

Second Law: $\vec{F} = m\vec{a} = \dot{\vec{p}}$

Angular Position/Velocity/Acceleration: $\theta = s/r$, $\omega = v/r$, $\alpha = a/r$

Angular Momentum: $\vec{L} = \vec{r} \times \vec{p}$

Torque: $\vec{\tau} = \vec{r} \times \vec{F} = \dot{\vec{L}}$

Centripetal Acceleration: $a_c = v^2/r$

Centrifugal/Coriolis Forces: $\vec{F}_{cent} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})$, $\vec{F}_{cor} = -2m\vec{\omega} \times \dot{\vec{r}}$

Work to go from positions \vec{a} to \vec{b} : $W_{ab} = \int_{\vec{a}}^{\vec{b}} \vec{F} \cdot d\vec{s}$

Conservative Force Field (2 eq): W_{ab} is the same regardless of path so $\oint \vec{F} \cdot d\vec{s} = 0$, and thus we can write the force as $\vec{F} = -\nabla V(\vec{r})$.

Lagrangian Formalism

Functional Derivative: $\frac{\delta F}{\delta u}[x_0] = \lim_{\epsilon \rightarrow 0} \frac{F[x_0 + \epsilon u] - F[x_0]}{\epsilon} \rightarrow \frac{\delta F}{\delta x(t)}[x(t')] = \lim_{\epsilon \rightarrow 0} \frac{F[x(t') + \epsilon \delta(t' - t)] - F[x(t')]}{\epsilon}$

Principle of Least Action: $\delta S = 0$, where $S = \int_{t_i}^{t_f} L(\vec{q}, \dot{\vec{q}}, t) dt$

Lagrange's Equation: $\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^A} - \frac{\partial L}{\partial x^A} = 0$

Holonomic Constraints: $f_\alpha(x^A, t) = 0$, $L' = L(x^A, \dot{x}^A) + \lambda_\alpha f_\alpha(x^A, t) \rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^A} - \frac{\partial L}{\partial x^A} = \lambda_\alpha \frac{\partial f_\alpha}{\partial x^A}$

Noether's Theorem: A continuous symmetry in the Action (and thus Lagrangian) result in a conserved quantity.

Moment of Inertia Tensor: $\vec{L} = \overleftrightarrow{I} \vec{\omega}$, $T = \frac{1}{2} \omega_a I_{ab} \omega_b$, $I_{ab} = \sum_i m_i ((\vec{r}_i \cdot \vec{r}_i) \delta_{ab} - (\vec{r}_i)_a (\vec{r}_i)_b)$

Euler's Equations: Only look at rotation, not translation. Conservation of Angular Momentum gives $I_i \dot{\omega}_i + \omega_j \omega_k (I_k - I_j) = 0$, for i, j, k being cyclic permutations of 1, 2, 3.

Hamiltonian Formalism

Generalized Momenta: $p_i = \frac{\partial L}{\partial \dot{q}_i}$, $\dot{p}_i = \frac{\partial L}{\partial q_i}$

Hamiltonian: $H(q_i, p_i, t) = \sum_{i=1}^n p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$

Hamilton's Equations:

1. $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

2. $\dot{q}_i = \frac{\partial H}{\partial p_i}$
3. $-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$

Cyclic/Ignorable Coordinates: q is ignorable if $\frac{\partial L}{\partial q} = 0$, i.e. if q does not appear in L .

Thus $p = \frac{\partial L}{\partial \dot{q}}$ is conserved.

Liouville's Theorem: A volume of a region of phase space remains the same, even when the region changes. $V = dq_1 \dots dq_n dp_1 \dots dp_n$.

Poisson Bracket: $\{f, g\} = \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}$.

Constant of Motion from Poisson Bracket: $\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$. If $I, H = 0$, then I is a constant of motion.

Canonical Transformation: Transformation $(q_i \rightarrow Q_i(q, p), p_i \rightarrow P_i(q, p))$ that leaves Hamilton's equations invariant.

6 Statistical Mechanics Equations

6.1 Thermodynamics

Laws of Thermodynamics:

1. Energy conservation. $dE = dQ - pdV$. dQ just means that the heat is an inexact differential and the integral depends on the path.
2. $\Delta S \geq \int \frac{dQ}{T}$, where equality is for a process that is reversible (never leaves equilibrium).
3. Entropy at zero temperature is zero. In stat mech this means that the ground state is nondegenerate and $S \propto \ln(W)$, where W is the number of available states.

Intensive vs Extensive Variables: Intensive variables do NOT scale with system size (T, p, μ) , while extensive do scale (E, S, V, N) .

Thermodynamic Potentials:

- Internal Energy: $U(S, V, N)$
- Helmholtz Free Energy: $F(T, V, N) = U - TS$
- Enthalpy: $H(S, p, N) = U + pV$
- Gibbs Free Energy: $G(T, p, N) = U - TS + pV$
- Landau(Grand) Potential: $\Omega(T, V, \mu) = U - TS - \mu_i N_i$

Thermodynamic Ensembles:

1. Microcanonical: Does not exchange energy or particles with environment. Fixed E, N
2. Canonical: Does not exchange particles, but can exchange energy (heat bath). Fixed N, T

3. Grand canonical: Can exchange energy and particles with environment. Fixed T, μ .

Maxwell's Relations (4 main):

- $\frac{\partial^2 U}{\partial S \partial V} = - \left(\frac{\partial p}{\partial S} \right)_V = \left(\frac{\partial T}{\partial V} \right)_S$
- $\frac{\partial^2 F}{\partial T \partial V} = \left(\frac{\partial p}{\partial T} \right)_V = \left(\frac{\partial S}{\partial V} \right)_T$
- $\frac{\partial^2 H}{\partial S \partial p} = \left(\frac{\partial V}{\partial S} \right)_p = \left(\frac{\partial T}{\partial p} \right)_S$
- $\frac{\partial^2 G}{\partial T \partial p} = \left(\frac{\partial V}{\partial T} \right)_p = - \left(\frac{\partial S}{\partial p} \right)_T$

Engine Efficiency: $\eta = \frac{Q_{in} - Q_{out}}{Q_{in}} = 1 - \frac{T_{out}}{T_{in}}$

Isobaric Thermal Expansion Coefficient: $\alpha = \frac{1}{V} \left(\frac{\partial V}{\partial T} \right)_P$, How much the volume changes with a change in temperature.

Isothermal Compressibility: $\kappa_T = -\frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_T$, How much the volume changes when the pressure changes.

Isentropic(Adiabatic) Compressibility: $\kappa_S = -\frac{1}{V} \left(\frac{\partial V}{\partial P} \right)_S$, Same as above.

Specific Heat at Constant V: $C_V = \left(\frac{\partial Q}{\partial T} \right)_V = \left(\frac{\partial U}{\partial T} \right)_V$, Amount of heat per unit mass to raise the temp by 1 degree.

Specific Heat at Constant p: $C_p = \left(\frac{\partial Q}{\partial T} \right)_p = \left(\frac{\partial H}{\partial T} \right)_p$, Same as above.

Fermi Energy/Temperature: Chemical potential at $T = 0$. $\epsilon_F = \mu(T = 0)$

6.2 Statistical Mechanics

Number of microstates in a macrostate (ways to get n heads): $\Omega = \frac{N!}{\prod_i n_i!}$

Stirling's Approximation: $\ln n! = n \ln n - n$

How many order important ways to order n things: $n!$

How many order important ways to order n things r at a time: $\frac{n!}{(n-r)!}$

How many NOT order important ways to order n things r at a time: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

Microcanonical(Classical) Partition Function: $Z_m = \sum_s g_s e^{-\beta E_s}$

Canonical Partition Function: $Z_c = \text{tr} \left(e^{-\beta \hat{H}} \right)$

Grand Canonical Partition Function: $Z_{gc} = \text{tr} \left(e^{-\beta(\hat{H} - \mu \hat{N})} \right)$

Geometric Series: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

Classical limit of the trace of an operator: $\text{tr}(\mathcal{O}) = \frac{1}{N!(2\pi\hbar)^{3N}} \int d^3r_1 \dots d^3r_N \int d^3p_1 \dots d^3p_N \mathcal{O}$, $N!$ is for identical particles.

Thermodynamic Limit: $T \rightarrow \infty, V \rightarrow \infty, N/V = \text{const}$

Expectation value for pure/mixed: $\langle \mathcal{O} \rangle_p = \langle \psi | \mathcal{O} | \psi \rangle, \langle \mathcal{O} \rangle_m = \sum_i P_i \langle \psi_i | \mathcal{O} | \psi_i \rangle$

Density Matrix (ex. Canonical Ensemble): $\rho = \sum_n P_n |\psi_n\rangle \langle \psi_n|, \rho_c = \frac{e^{-\beta \hat{H}}}{\text{tr} e^{-\beta \hat{H}}}$

Expectation value with Density Matrix: $\langle \mathcal{O} \rangle = \text{tr}(\mathcal{O} \rho)$

Trace of Density matrix: $\text{tr}(\rho) = 1$

Time evolution of density matrix: $\frac{\partial}{\partial t} \rho = \frac{1}{i\hbar} [\hat{H}, \hat{\rho}]$

Z_{gc} for an ideal gas: $Z_{gc} = \frac{V^N (2mT\pi)^{3N/2}}{N! (2\pi\hbar)^{3N}} e^{\beta\mu}$

Z_{gc} for ideal fermi gas: $Z_{gc} = \prod_k (1 + e^{-\beta(\epsilon_k - \mu)})$

Z_{gc} for ideal bose gas: $Z_{gc} = \prod_k \frac{1}{(1 - e^{-\beta(\epsilon_k - \mu)})}$

Stuff here for black-body and phonons and bose condensates.

What is cluster expansion used for?: Systems of interacting particles.

7 Quantum Mechanics Equations

Properties of a vector space:

- Sum $|V\rangle + |W\rangle$
- Scalar product with properties
 1. closure: results in another vector in the space.
 2. distributive: $a(|V\rangle + |W\rangle) = a|V\rangle + a|W\rangle$, $(a + b)|V\rangle = a|V\rangle + b|V\rangle$
 3. associative: $a(b|V\rangle) = ab|V\rangle$, $|V\rangle + (|W\rangle + |Z\rangle) = (|V\rangle + |W\rangle) + |Z\rangle$
 4. commutative: $|V\rangle + |W\rangle = |W\rangle + |V\rangle$
 5. additive inverse: $|V\rangle + |-V\rangle = |0\rangle$
 6. null vector: $|V\rangle + |0\rangle = |V\rangle$

Hilbert space: Vector space with defined inner product.

Expand in orthonormal basis: $|V\rangle = \sum_i v_i |i\rangle$

Hermitian operator: $\mathcal{O}^\dagger = \mathcal{O}$

Anti-Hermitian operator: $\mathcal{O}^\dagger = -\mathcal{O}$

Unitary operator: $UU^\dagger = \mathbb{1}$

Orthogonality: $\langle i | j \rangle = \delta_{ij}$

Completeness: $\sum_i |i\rangle\langle i| = \mathbb{1}$

Postulates of QM:

1. The state of a physical system, at some fixed time, is given by a normalized ray in a Hilbert space over the complex numbers. (ray is vector whose norm doesn't matter)
2. The ray evolves deterministically in time according to Schrödinger's equation.
3. Observables correspond to self-adjoint (hermitian) operators.
4. If a particle is in the state $|\psi\rangle$ then a measurement of \mathcal{O} will yield one of the eigenvalues of \mathcal{O} , ω . The state of the system changes to an eigenstate of \mathcal{O} , $|\omega\rangle$.

Schrödinger equation: $i\hbar \frac{\partial}{\partial t} \Psi = \hat{H} \Psi$

Free particle ψ_p and E_p : $\psi_p = Ae^{ikx} + Be^{-ikx}$, $k^2 = \frac{2mE_p}{\hbar^2}$, $E_p = \frac{p^2}{2m}$

Particle in a box ψ_n and E_n : $\psi_n = \sqrt{\frac{2}{L}} \sin(k_n x)$, $k_n = \frac{n\pi}{L}$, $E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$

Harmonic Oscillator \hat{H} , ψ_n and E_n : $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 \hat{x}^2$, $\psi_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{2^n n!} H_n(x) e^{-x^2/2}$, $E_n = (n + \frac{1}{2})\hbar\omega$

Raising and lowering operators and how to affect $|n\rangle$ (3-2):

- $a = \left(\frac{m\omega}{2\hbar}\right)^{1/2} \left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$, $a|n\rangle = \sqrt{n}|n-1\rangle$, $a|0\rangle = 0$
- $a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2} \left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

\hat{H} in terms of a and a^\dagger : $\hat{H} = \hbar\omega(a^\dagger a + 1/2)$

Commutation relations for \hat{H} , a , a^\dagger :

- $[\hat{H}, a] = -\hbar\omega a$
- $[\hat{H}, a^\dagger] = \hbar\omega a^\dagger$
- $[a, a^\dagger] = 1$

J^2 and J_z on the angular momentum state $|jm_j\rangle$:

- $J^2|jm_j\rangle = j(j+1)\hbar^2|jm_j\rangle$
- $J_z|jm_j\rangle = m_j\hbar|jm_j\rangle$

Commutation relations for J_i and J_j and for J^2 and J_i :

- $[J_i, J_j] = i\hbar J_k$
- $[J^2, J_i] = 0$

J_z and J^2 in position basis:

- $J_z = -i\hbar \frac{\partial}{\partial \phi}$
- $J^2 = -\hbar^2 \left[\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2}{\partial \phi^2} \right]$

Raising and Lowering Angular Momentum Operators on $|j, m\rangle$:

$J_\pm|j, m\rangle = \hbar[j(j+1) - m(m \pm 1)]^{1/2}|j, m \pm 1\rangle$

J_x and J_y in terms of J_+ and J_- : $J_x = \frac{1}{2}(J_+ + J_-)$, $J_y = \frac{1}{2i}(J_+ - J_-)$

Momentum eigenstate, $\langle x|p\rangle$: $\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$

Hydrogen Atom $V(r)$, ψ_n , $E_n(\mathbf{x4})$: $V(r) = -\frac{Ze^2}{4\pi\epsilon_0 r}$, $\psi_n = \text{stuff} * L_{n-l-1}^{2l+1}(\rho) Y_l^m(\theta, \phi)$ (Laguerre)

$E_n = -\frac{1}{2n^2} \left(\frac{Ze^2}{4\pi\epsilon_0 \hbar} \right)^2 m_e = -\frac{1}{2n^2} \alpha^2 m_e c^2 = -\frac{1}{n^2} 13.6 \text{ eV} = -\frac{1}{2n^2} \left(\frac{1}{4\pi\epsilon_0} \frac{e^2}{a_0} \right)$,

$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c}$, $a_0 = \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2}$

Pauli matrices and commutation relations:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_a \sigma_b] = 2i\epsilon_{abc}\sigma_c$$

Non-Deg Time-Ind Perturbation, $E_n^{(1)}$, $|n^{(1)}\rangle$, $E_n^{(2)}$:

$$E_n^{(1)} = H'_{nn} = \langle n^{(0)} | H' | n^{(0)} \rangle$$

$$|n^{(1)}\rangle = \sum_{m \neq n} \frac{\langle n^{(0)} | H' | n^{(0)} \rangle}{(E_n^{(0)} - E_m^{(0)})} |m^{(0)}\rangle$$

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle n^{(0)} | H' | n^{(0)} \rangle|^2}{(E_n^{(0)} - E_m^{(0)})^2}$$

Deg Time-Ind Perturbation, $E_n^{(1)}$: Diagonalize the perturbation hamiltonian in the degenerate subspace.

Time-Dep Perturbation, $P_{i \rightarrow f}(t)$: $P_{i \rightarrow f}(t) = \frac{1}{\hbar^2} \left| \int_0^t dt' \langle f | H'(t') | i \rangle e^{i(E_f - E_i)t'/\hbar} \right|^2$

Fermi's golden rule, and $g(E_f)as\delta$: $R_{i \rightarrow f} = \frac{2\pi}{\hbar} |\langle f | H' | i \rangle|^2 g(E_f)$, $g(E_f) \approx \delta(E_f^{(0)} - E_f^{(0)} - \hbar\omega)$

Einstein's Stimulated/Spontaneous emission coefficients:

$$\text{Stimulated: } B_{if} = \frac{\pi e^2}{3\epsilon_0 \hbar^2} |\langle f | \mathbf{r} | i \rangle|^2$$

$$\text{Spontaneous: } A_{if} = \frac{e^2 \omega_{21}^3}{3\pi \epsilon_0 \hbar c^3} |\langle f | \mathbf{r} | i \rangle|^2$$

Total $\psi(\mathbf{r})$ in scattering problem:

$$\psi(\mathbf{r}) = \psi_{inc}(\mathbf{r}) + \psi_s(\mathbf{r}) = \psi_{inc}(\mathbf{r}) + f(\theta, \phi) \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{r}$$

$$f(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \int d^3r' e^{-i\mathbf{k}' \cdot \mathbf{r}'} V(\mathbf{r}') \psi(\mathbf{r}')$$

Differential Cross Section: $\frac{d\sigma}{d\Omega} = |f(\theta, \phi)|^2$

Born Approximation: In the above integral for $f(\theta, \phi)$ let $\psi \rightarrow \psi_{inc}$.

Dirac Equation:

$$(i\hbar\gamma^\mu \partial_\mu - mc)\psi = 0$$

$$\gamma^0 = \beta, \gamma^i = \beta\alpha_i$$

$$\beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

8 Electricity and Magnetism Equations

9 Miscellaneous Physics

Taylor Expansion: $f(\vec{x} + \vec{a}) = f(\vec{x}) + a_i \partial_i f(\vec{x}) + \mathcal{O}(\vec{a}^2)$

Gaussian Integral: $\int_{-\infty}^{\infty} dx e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)+c}$

Value of fine structure constant: $\alpha \approx \frac{1}{137}$

Mass of electron in eV: $m_e c^2 = 0.511 \text{ eV}$

Value of the Bohr radius: $a_0 = 0.529\text{\AA}$