PHY 576: Quantum Theory

Problem Set 2

Due in TA's mailbox by 4 pm on Friday, February 27.

1. (a) Fermat's principle of least time: Consider a light source at the origin of the $x - y$ plane. Suppose that for $y < a$ the medium has a refractive index n_1, while for $y > a$ the refractive index is n_2. For a given path in space (not spacetime) between the origin and some point (x, y) for $y > a$, define a functional for the time taken by light to traverse that path. Functionally differentiate (using the physicists' functional derivative) your time-functional to find the path that light actually takes and thereby derive Snell's law.

   (b) Using the definition of the Gateaux derivative (see notes), calculate the functional derivative of the volume functional

   $$V[f] = \pi \int_0^1 f(z)^2 dz$$

   in the "direction" of $u(z) = z$ at the "point" (in function space) $f_0(z) = e^z$, i.e. evaluate $\frac{\delta V[f_0]}{\delta u}$.

2. (a) Suppose a free particle is in a momentum eigenstate $|p_i\rangle$ at time $t = 0$. Using the free-particle propagator, calculate the probability amplitude for the particle to be in the position eigenstate $|x_f\rangle$ at time $t = t_f$?

   (b) Calculate the same amplitude directly in the canonical formalism.

   (c) A free particle is prepared with a Gaussian wave function centered around the origin with standard deviation σ at time $t = 0$. Using the free-particle propagator, calculate the wave function at time $t > 0$.

3. Find the first-excited-state energy and wavefunction from the propagator of the simple harmonic oscillator, $G_{\text{SHO}}(x_f, T; x_i, 0)$. (This is messy. Reduce clutter by defining $y = e^{-i\omega T}$ and expanding the propagator as a power series in y.)

4. Consider the free-particle propagator, $G_{\text{free}}(x_f, T; x_i, 0)$. This is expressed as a sum over all paths, with no restrictions on the paths.

   (a) Find the propagator, $G(x_f, T; x_i, 0)$, for a particle confined to the half-line $x > 0$. Not only do we have $x_f, x_i > 0$ but only those paths that stay in the region $x > 0$ are to be considered. (Hint: Consider an image point in the region $x < 0$. Show that for every path from x_i to x_f that crosses the line $x = 0$, there exists a path *with the same action* from the image point to x_f. Where is the image point? Find the propagator $G(x_f, T; x_i, 0)$ by subtracting another propagator from G_{free}. No integrals are necessary here!)

   (b) (Hard) Consider a particle in a box of length L i.e. $V(x) = 0$ for $0 < x < L$ and $V(x) = \infty$ elsewhere. This is like the previous problem, but with two boundaries. Show that the propagator for a particle in a box can be written as

$$G_{\text{box}}(x_f, T; x_i, 0) = \sum_{n=-\infty}^{+\infty} \left[G_{\text{free}}(2nL + x_f, T; x_i, 0) - G_{\text{free}}(2nL - x_f, T; x_i, 0)\right]$$

5. Use the spectral representation to calculate the energy spectrum and normalized wavefunctions for a particle in a box of length L from the propagator, given by the infinite sum above.

6. The action for a free, massive, spinless relativistic particle is $S = -mc\tau$ where τ is the proper time. This can be written $S = -mc \int \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{dt} \frac{dx^\nu}{dt}}\, dt$, where the 4×4 matrix $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ is the Minkowski metric, and $x^0 = ct, x^1 = x, x^2 = y, x^3 = z$ with the repeated indices μ, ν summed from 0 to 3.

   (a) Verify that S reduces to the non-relativistic action for $v/c \ll 1$.

   (b) Write down the classical equation of motion.

   (c) From S, find the amplitude for the particle to go from $\vec{x}_i = t_i = 0$ to $\vec{x}_f = \vec{r}$, $t_f = t$ when $|\vec{r}| > ct$. Assume that only the action for the straight-line path contributes to G; also, ignore the pre-factor in G. Quantum mechanics allows for faster-than-light travel!