Notes on Radiation

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Radiation from arbitrary source

Let's start with the retarted sources which give us the scalar and vector potentials

$$\mathbf{\Phi}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{|\mathbf{r}-\mathbf{r}'|} \tag{1}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{|\mathbf{r}-\mathbf{r}'|}. \tag{2}$$

The first approximation for radiation is that the source is localized (also meaning that the radiation zone is far from the source $r \gg r'$). Under this assumption we get,

$$|\mathbf{r}-\mathbf{r}'| \approx r - \frac{\mathbf{r}\cdot\mathbf{r}'}{r} \tag{3}$$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3}, \tag{4}$$

using the Fourier Transform

$$f(\mathbf{r}+\mathbf{a}) \approx f(\mathbf{r}) + \mathbf{a}\cdot\nabla f(\mathbf{r}) = f(\mathbf{r}) + a_i\partial_i f(\mathbf{r}). \tag{5}$$

Now lets use the fact that the sources have harmonic time dependence.

$$\mathbf{J}(\mathbf{r}',t) = \mathbf{J}(\mathbf{r}')e^{-i\omega t} \tag{6}$$

With this we can now approximate the vector potential by

$$\mathbf{J}(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}) \approx \mathbf{J}(\mathbf{r}')e^{-i\omega t + i\omega r/c - i\omega\frac{\hat{r}\cdot\mathbf{r}'}{c}}. \tag{7}$$

Now we can plug this into equation 2, and dropping terms with higher order terms that $1/r$ to get,

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0 e^{-i\omega(t-r/c)}}{4\pi r}\int d^3r' \mathbf{J}(\mathbf{r}')e^{-i\omega\frac{\hat{r}\cdot\mathbf{r}}{c}}. \tag{8}$$

Now is we assume that the region of sources is small compared to the wavelength, $c/\omega \gg r'$, then we can make the **dipole approximation**, $e^{-i\omega\hat{r}\cdot\mathbf{r}'/c} \approx 1$, which when pluggen into equation 8 gives

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 e^{-i\omega(t-r/c)}}{4\pi r} \int d^3r' \mathbf{J}(\mathbf{r}'). \tag{9}$$

Now let's go on an aside to explore the $\mathbf{J}(\mathbf{r}')$.
Claim: $\int d^3r' \mathbf{J}(\mathbf{r}') = \frac{d\mathbf{p}}{dt}$ where

$$\mathbf{p}(\mathbf{r}', t) = \int d^3r' \mathbf{r}' \rho(\mathbf{r}', t) \tag{10}$$

is the electric dipole moment. Now let's find the integral of $\mathbf{J}(\mathbf{r}')$ in terms of this dipole moment. Let's start with the time derivative of the dipole.

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt} \int d^3r' \mathbf{r}' \rho(\mathbf{r}', t) \tag{11}$$

$$= \int d^3r' \mathbf{r}' \frac{\partial}{\partial t} \rho(\mathbf{r}', t) \tag{12}$$

$$= -\int d^3r' \mathbf{r}' \nabla \cdot \mathbf{J}(\mathbf{r}', t), \tag{13}$$

where we have used the continuity equation

$$\nabla \cdot \mathbf{J}(\mathbf{r}', \mathbf{t}) = -\frac{\partial}{\partial t} \rho(\mathbf{r}', t). \tag{14}$$

Now expand the $\mathbf{r}' \nabla \cdot \mathbf{J}(\mathbf{r}', t)$ term.

$$\nabla \cdot r_i \mathbf{J} = (\nabla r_i) \cdot \mathbf{J} + r_i \nabla \cdot \mathbf{J} \tag{15}$$

$$= \hat{r_i} \cdot \mathbf{J} + r_i \nabla \cdot \mathbf{J} \tag{16}$$

If you sum over all possible r_i's you get

$$\mathbf{r}' \nabla \cdot \mathbf{J} = \nabla \cdot (r_x \mathbf{J} + r_y \mathbf{J} + r_z \mathbf{J}) - \mathbf{J} \tag{17}$$

However when the second term is integrated over the volume it can be turned into a surface integral with the divergence theorem, but $\mathbf{J}$ is zero on the surface to we get that

$$\frac{d\mathbf{p}}{dt} = \int d^3r' \mathbf{J}. \tag{18}$$

Now applying this to equation 9 we get

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r} \frac{d}{dt} \mathbf{p} e^{-i\omega(t-r/c)} \tag{19}$$

$$= \frac{\mu_0}{4\pi r} \dot{\mathbf{p}}(t - r/c), \tag{20}$$

which is the standard result. I'm not going to worry about finding $\Phi(\mathbf{r}, t)$ because if

$$\mathbf{E} = \mathbf{E}_0 e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{21}$$
$$\mathbf{B} = \mathbf{B}_0 e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{22}$$

it can be shown with Maxwell's equations that $\mathbf{E}_0 = -c\hat{k} \times \mathbf{B}_0$, so we only need to solve for $\mathbf{B}$ using $\mathbf{B} = \nabla \times \mathbf{A}$.

$$\begin{align}
\mathbf{B} &= \nabla \times \mathbf{A} \tag{23}\\
&= \frac{\mu_0}{4\pi}\nabla \times \frac{\dot{\mathbf{p}}(t-r/c)}{r} \tag{24}\\
&= \frac{\mu_0}{4\pi r}\nabla \times \dot{\mathbf{p}}(t-r/c) \tag{25}
\end{align}$$

Where the chain rule was applied here but the derivative of $1/r$ gave a term that went as $1/r^2$ so we ignored it. Now this cross product can be done if we remember the product rule $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$. Thus $\mathbf{B}$ becomes

$$\begin{align}
\mathbf{B}(\mathbf{r}, t) &= -\frac{\mu_0}{4\pi r}\left[e^{-i\omega(t-r/c)}(\nabla \times \dot{\mathbf{p}}) - \dot{\mathbf{p}} \times (\nabla e^{-i\omega(t-r/c)})\right] \tag{26}\\
&= \frac{\mu_0}{4\pi r}\dot{\mathbf{p}} \times (\nabla e^{-i\omega(t-r/c)}) \tag{27}\\
&= \frac{\mu_0}{4\pi r}\frac{i\omega}{c}\hat{r} \times \dot{\mathbf{p}}e^{-i\omega(t-r/c)}. \tag{28}
\end{align}$$

but since we said that $\dot{\mathbf{p}}(t - r/c) = \dot{\mathbf{p}}e^{-i\omega(t-r/c)}$ we can say that

$$\frac{d}{dt_{ret}}\dot{\mathbf{p}} = \ddot{\mathbf{p}} = -i\omega\dot{\mathbf{p}}, \tag{29}$$

which we can use to give us

$$\mathbf{B}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi rc}\hat{r} \times \ddot{\mathbf{p}}(t - rc). \tag{30}$$

Now we can find $\mathbf{E}$ to be

$$\mathbf{E}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r}\hat{r} \times \hat{r} \times \ddot{\mathbf{p}}(\mathbf{r}, t - r/c), \tag{31}$$

using the fact that $\mathbf{E}_0 = -c\hat{k} \times \mathbf{B}_0$, and noting that $\hat{k} = \hat{r}$.

Power Radiated

The Poynting vector gives us the the energy that is radiated per time per unit area.

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} \tag{32}$$

This points in the $\hat{k}$ of $\hat{r}$ direction but if you use the relation used above you can get the magnitude in terms of only $\mathbf{E}$ or $\mathbf{B}$.

$$\mathbf{S} = \frac{c}{\mu_0}|\mathbf{B}|^2\hat{r} \tag{33}$$

$$= \frac{\mu_0}{16\pi^2 cr^2}|\hat{r} \times \ddot{\mathbf{p}}|^2\hat{r} \tag{34}$$

We can now get the total radiated power by integrating this over an entire sphere since the Poynting vector has units of power per area. Let's let $\ddot{\mathbf{p}} = \ddot{p}\hat{z}$ so that $|\hat{r} \times \ddot{\mathbf{p}}|^2 = |\ddot{\mathbf{p}}|^2 \sin^2\theta$, so we get

$$\mathbf{S} = \frac{\mu_0}{16\pi^2 cr^2}|\ddot{\mathbf{p}}|^2 \sin^2\theta\hat{r}, \tag{35}$$

where θ is the typical spherical angle. Now let's integrate this over the full sphere to get the total power radiated.

$$P = \int r^2\mathbf{\Omega} \cdot \mathbf{S} \tag{36}$$

$$= \int r^2\Omega S\hat{r} \cdot \hat{r} \tag{37}$$

$$= \int r^2 \sin\theta d\theta d\phi \frac{\mu_0}{16\pi^2 cr^2}|\ddot{\mathbf{p}}|^2 \sin^2\theta \tag{38}$$

$$= \int_0^{2\pi}\int_0^{\pi} d\phi d\theta \frac{\mu_0}{16\pi^2 c}|\ddot{\mathbf{p}}|^2 \sin^3\theta \tag{39}$$

$$= \int_0^{2\pi} d\theta \frac{\mu_0}{8\pi c}|\ddot{\mathbf{p}}|^2 \sin^3\theta \tag{40}$$

Now we just need to do the θ integral. To do this use a u-sum with $u = \cos\theta$ so that $du = -\sin\theta d\theta$.

$$\int_0^{\pi} d\theta \sin^3\theta = \int_0^{\pi} d\theta \sin\theta(1 - \cos^2\theta) \tag{41}$$

$$= -\int_1^{-1} du(1 - u^2) \tag{42}$$

$$= 1 - \frac{1}{3}u^3\Big|_{-1}^{1} \tag{43}$$

$$= \frac{4}{3} \tag{44}$$

This then gives us for the total power radiated

$$P = \frac{\mu_0}{6\pi c}|\ddot{\mathbf{p}}|^2 \tag{45}$$

Time Averaged Power Radiated/Larmor Formula

Now to get the time averaged power let's assume the form $\mathbf{p}(t)) = p\hat{\mathbf{z}}e^{i\omega t}$ for the dipole moment so that we get $\ddot{\mathbf{p}} = -\omega^2 p\hat{\mathbf{z}}e^{i\omega t}$, where the period is $T = 2\pi/\omega$. To time agerage we integrate over a period and divide by the period.

$$\bar{P} = \frac{1}{T}\int_0^T dtP \tag{46}$$

$$= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} dt\frac{\mu_0\omega^4 p^2}{6\pi c} \tag{47}$$

$$= \frac{\omega}{2\pi}\frac{\mu_0\omega^4 p^2}{12\pi c}t\bigg|_0^{2\pi/\omega} \tag{48}$$

$$= \frac{\mu_0\omega^4 p^2}{12\pi c} \tag{49}$$

Now if we say that $p = Qd$ we can write this as

$$\bar{P} = \frac{\mu_0\omega^4 p^2}{12\pi c} = \frac{\mu_0\omega^4 Q^2 d^2}{12\pi c}. \tag{50}$$

This is the Larmor formula. Notice the strong dependence on the frequency ω.