PHY531 Problem Set 6. Due April 16, 2015

You may find useful the small argument expansion of the Bessel functions, Jackson Eq. 3.89. The integral representation of the Bessel function given in Jackson problem 3.16, can be used to derive

$$\int_0^{2\pi} d\phi \sin\phi e^{-ix\cos\phi+in\phi} = -\frac{2\pi n}{i^n x} J_n(x)\,, \qquad \int_0^{2\pi} d\phi \cos\phi e^{-ix\cos\phi+in\phi} = i\frac{2\pi}{i^n} J_n'(x)\,. \qquad (1)$$

1. A particle of charge q oscillates harmonically in the z direction with amplitude a and angular frequency ω_0, i.e. $\boldsymbol{r}_p(t) = \hat{\boldsymbol{z}} a\cos(\omega_0 t)$.

    a. Calculate the electric dipole moment, and give the power radiated per solid angle in the electric dipole approximation. What frequency is the radiation.

    b. Calculate the exact result for the power radiated per solid angle for each frequency. Your result for the power radiated per unit solid angle at frequency $n\omega_0$ should be

    $$\frac{dP_n}{d\Omega} = \frac{q^2 n^2 \omega_0^2}{8\pi^2 c} \tan^2\theta J_n^2\left(\frac{n\omega_0 a}{c}\cos\theta\right) \qquad (2)$$

    where $J_n(x)$ in this expression is a Bessel function.

    c. Show that in the nonrelativistic limit the result of part b goes to your result of part a.

2. A classical particle of charge q and mass m is bound to the origin by a damped harmonic force

    $$\boldsymbol{F}(t) = -m\omega_0^2 \boldsymbol{r}(t) - m\gamma \boldsymbol{v}(t) \qquad (3)$$

    where ω_0 and γ are real positive parameters. A plane electromagnetic wave of angular frequency ω and polarization along $\hat{\boldsymbol{x}}$ is incident in the $\hat{\boldsymbol{z}}$ direction. Calculate the differential scattering cross section. Assume the amplitude of the plane wave is small enough that $|\boldsymbol{v}(t)| \ll c$, and any radiation reaction can be ignored.

    Hint: First calculate the motion of the particle when acted upon by the plane wave using the nonrelativistic approximation. Then calculate the radiation from this charge/current using the same approximation. A cross section is always the ratio of something per unit time into the detector divided by something per unit area per unit time incident. In this case, the something is electromagnetic energy, so you need to calculate the electromagnetic energy per unit time per solid angle divided by the incident electromagnetic energy per unit area per unit time.

    As a check your result should go to the appropriate Rayleigh cross section for small ω and the Thomson cross section for large ω.

3. Jackson problem 14.21.

4. Revisit the harmonic oscillator particle of problem 1.

a. Calculate the instantaneous power radiated by the particle per solid angle.

As a check, your result should be

$$\frac{dP(t)}{d\Omega} = \frac{q^2 \omega_0^4 a^2}{16\pi^2 c^3} \frac{\sin^2\theta \cos^2(\omega_0 t)}{(1 + \frac{\omega_0 a}{c} \cos\theta \sin(\omega_0 t))^5} \, . \tag{4}$$

b. Time average the result of part a to find the time average power radiated per solid angle.

As a check, your result should be

$$\frac{dP}{d\Omega} = \frac{q^2 \omega_0^4 a^2}{128\pi^2 c^3} \frac{4 + \frac{\omega_0^2 a^2}{c^2} \cos^2\theta}{(1 - \frac{\omega^2 a^2}{c^2} \cos^2\theta)^{7/2}} \sin^2\theta \, . \tag{5}$$

c. Show that summing the power radiated at all frequencies in problem 1 and equating that to your result in part b, gives the identity

$$\sum_{n=1}^{\infty} n^2 J_n^2(nx) = \frac{x^2}{16} \frac{4 + x^2}{(1 - x^2)^{7/2}} \, , \tag{6}$$

for $0 < x < 1$.

Check that this identity is correct by using any numerical tool you wish to evaluate the left and write sides for x equal to 0.1, 0.5, and 0.9.

d. Plot the polar radiation pattern (i.e. like those shown in Jackson Figure 9.5) for the nonrelativistic case where $\frac{\omega_0 a}{c}$ equals 0.1 and the relativistic case where it equals 0.9.