# PHY6938 Spring 2000 Proficiency Solutions

**1. A uniform bowling ball with mass $M$ and radius $R$ is hit by a stick and acquires an initial speed $v_0$ but no angular velocity ($\omega_0 = 0$) at $t = 0$. The coefficient of the kinetic friction between the ball and the horizontal surface on which it is moving is $\mu$, and the acceleration due to gravity is $g$.** ***Hint:*** **$\int \sin^3\theta d\theta = -\frac{1}{3}\cos\theta(\sin^2\theta + 2)$.**

**a) Calculate the moment of inertia of the ball about an axis that passes through the center of the ball.**

Assume the ball rotates about the $z$ axis (the result is independent of the axis about which it rotates, by symmetry); then the moment of inertia is the sum of $dM \cdot (x^2 + y^2)$ (the perpendicular distance to the $z$-axis squared times the mass element) over the mass distribution

$$\begin{aligned} I &= \int dM(x^2+y^2) = \int \rho d^3\mathbf{r}\,(x^2+y^2) = \rho \int d^3\mathbf{r}\,(r^2 - z^2) \\ &= \rho \int d^3\mathbf{r}\,[r^2 - r^2\cos^2(\theta)] = \rho \int d^3\mathbf{r}\, r^2 \sin^2(\theta), \end{aligned}$$

which we can do in spherical polar coordinates

$$I = 2\pi\rho \int_0^R r^2 dr \int_0^\pi sin(\theta) d\theta\, r^2 \sin^2(\theta) = 2\pi\rho \frac{R^5}{5}\left[-\frac{1}{3}\cos\theta(\sin^2\theta + 2)\right]_0^\pi = 2\pi\rho\frac{R^5}{5}\left[\frac{4}{3}\right],$$

and substituting $\rho = M/[4\pi R^3/3]$ we find

$$I = \frac{2}{5}MR^2.$$

**b) Calculate the time $t_1$ after which the ball rolls without slipping on the surface.**

The only forces on the ball after it starts to translate are the frictional force $f$ pointing backward along the surface, and the gravitational force and the normal force. The latter two balance so that

$$N = Mg, \;\; f = \mu N = \mu M g.$$

The only unbalanced force on the ball is, therefore, the friction, and it acts to both slow down the ball and to cause it to rotate. The other two forces could have a vector sum of zero and still produce a torque, but as they both act along a line through the center of mass they produce no torque around the rotation axis which passes through the CM. Note that the friction is a constant force which means that the acceleration of the ball is constant and equal to $-\mu g$ and that torque on the ball is constant and equal to $\tau = Rf = \mu mgR$. This means that the CM velocity will drop linearly, and the angular velocity will increase linearly, so that at some point the condition

$$v(t_1) = R\omega(t_1)$$

will be met, and the ball will then start to roll without slipping *if the static friction is larger than the kinetic friction.* To find the angular acceleration $\alpha$ we need to divide the torque by the moment of inertia found in a), so that

$$\alpha = \frac{\tau}{I} = \frac{R\mu M g}{2MR^2/5} = \frac{5}{2}\frac{\mu g}{R},$$

and so the above condition is

$$v(t_1) = v_0 - \frac{f}{M}t_1 = v_0 - \mu g t_1 = R\omega(t_1) = R\frac{5}{2}\frac{\mu g}{R}t_1 = \frac{5}{2}\mu g t_1$$

and so we have

$$t_1 = \frac{v_0}{\mu g(1+5/2)} = \frac{2}{7}\left(\frac{v_0}{\mu g}\right),$$

and note

$$v(t_1) = \frac{5}{2}\mu g \cdot \frac{2}{7}\left(\frac{v_0}{\mu g}\right) = \frac{5}{7}v_0.$$

**c) Calculate the work done by friction between $t = 0$ and $t = t_1$.**

To find the work by the friction we could find the distance that the two surfaces slip against each other. This is not the same thing as the distance traveled by the ball; for example, if the ball is rolling without slipping the frictional force does no work but the ball still translates. Instead of doing this it is, as usual, easier to use the fact that the frictional force is dissipating the kinetic energy so that the work done *by* the frictional force is the change in the kinetic energy of the ball. The initial kinetic energy is

$$T_{\mathrm{i}} = \frac{1}{2}Mv_0^2,$$

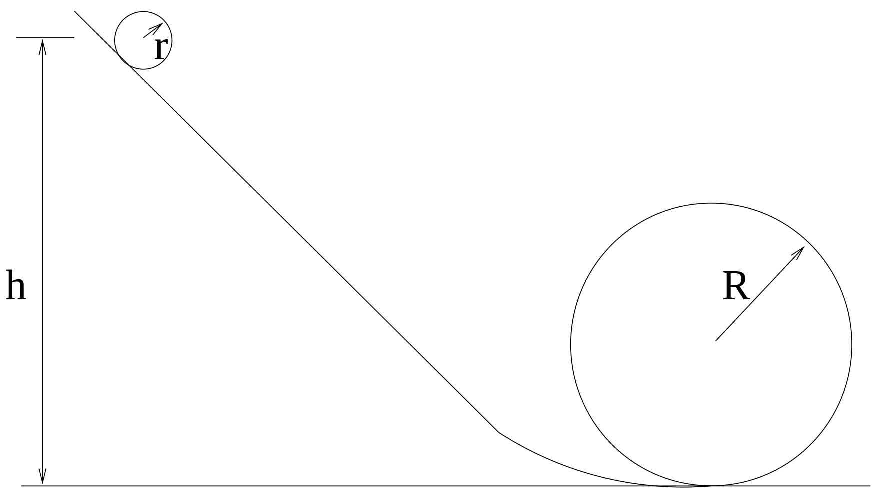
and the final kinetic energy is

$$\begin{aligned} T_{\mathrm{f}} &= \frac{1}{2}Mv(t_1)^2 + \frac{1}{2}I\omega(t_1)^2 = \frac{1}{2}Mv(t_1)^2 + \frac{1}{2}\cdot\frac{2}{5}MR^2\left(\frac{v(t_1)}{R}\right)^2 \\ &= \frac{1}{2}Mv(t_1)^2\left(1+\frac{2}{5}\right) = \frac{1}{2}\cdot\frac{7}{5}Mv(t_1)^2 = \frac{1}{2}\cdot\frac{5}{7}v_0^2, \end{aligned}$$

so the work done *by* the friction is negative and has the value

$$W = T_{\mathrm{f}} - T_{\mathrm{i}} = \frac{1}{2}Mv_0^2\left[\frac{5}{7} - 1\right] = -\frac{1}{7}Mv_0^2.$$

**2. A small uniform sphere with radius $r$ rolls without slipping along a track as shown in the diagram. The radius of the loop is $R$.**




**a) Calculate the moment of inertia of the sphere about an axis through its center.**

See 2 a); $I = 2MR^2/5$.

**b) What is the minimum height at which the ball must be released so that it will not fall off the track in the loop?**

We will use conservation of energy. At the top of the track the center of mass of the sphere is at a height $2R - r$ so that the potential energy has decreased by

$$\Delta U = mg[h - (2R - r)].$$

Call the velocity at which the center of mass is translating at this point $v$. In order to find the kinetic energy in terms of $v$ we need to find the rate $\omega$ at which the sphere is rotating in terms of $v$,

$$\omega = \frac{v}{r}.$$

Then the rotational kinetic energy is

$$\frac{1}{2}I\omega^2 = \frac{1}{2} \cdot \frac{2}{5}mr^2 \left(\frac{v}{r}\right)^2 = \frac{1}{2} \cdot \frac{2}{5}mv^2,$$

and the total kinetic energy is

$$T = \frac{1}{2}mv^2 \left(1 + \frac{2}{5}\right) = \frac{7}{10}mv^2.$$

Now using conservation of energy we can relate the speed $v$ of translation of the sphere at the top of the loop to the height $h$ by

$$\Delta U = mg[h - (2R - r)] = \frac{7}{10}mv^2,$$

so that

$$v^2 = \frac{10}{7}g[h - (2R - r)].$$

We are now almost done, as we know the forces on the sphere at the top of the track are the frictional force which is keeping it rolling, the normal force pointing down, and the gravitational force $mg$ also pointing down. The frictional force does no work and acts horizontally so it does not affect the condition for staying on the track. The minimum velocity $v$ is attained when the sphere is just about to leave the track so that the normal force is zero. At this point the only force on the sphere is gravity, and it is this force which must be providing the centripetal acceleration $v^2/(R-r)$ required to make the sphere travel in a circle of radius $R-r$. This means that

$$mg = \frac{mv^2}{R-r} = m\frac{10}{7}g\left[\frac{h-(2R-r)}{R-r}\right]$$

Solving, we find

$$h = \frac{27}{10}R - \frac{17}{10}r.$$

**c) Should this height be increased or decreased if the solid sphere is replaced by a hollow one with the same mass and radius?**

A hollow sphere of the same mass and radius has a larger moment of inertia and so has a larger part of its total energy in the rotational kinetic energy, and so will have a lower translational speed for the same total energy. This means that we will have to *increase* the height $h$ in order for it to have the translational velocity required to keep it on the track.

**3. Consider the off-center elastic collision of two objects of equal mass when one is initially at rest.**

**a) Show that the final velocity vectors of the two objects are perpendicular to each other.**

Let's call the angle between the projectile's initial and final directions the lab scattering angle $\psi$, and the angle between the target's final direction and the projectile's initial direction the recoil angle $\zeta$. Our job is to show that $\psi+\zeta$ is 90°. Let the projectile's initial velocity be $v$, its final velocity $v_1$, and the target's final velocity $v_2$. The statement of conservation of momentum is

$$\begin{aligned} mv &= mv_1\cos(\psi) + mv_2\cos(\zeta) \\ 0 &= mv_1\sin(\psi) - mv_2\sin(\zeta), \end{aligned}$$

which simplifies to

$$\begin{aligned} v &= v_1\cos(\psi) + v_2\cos(\zeta) \qquad (1)\\ 0 &= v_1\sin(\psi) - v_2\sin(\zeta) \qquad (2) \end{aligned}$$

once the mass is divided out.

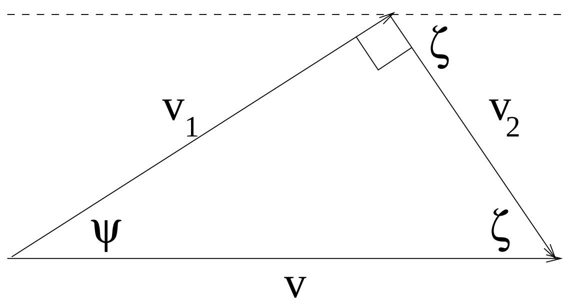
Energy conservation gives that (here dividing by $m/2$ straight away)

$$v^2 = v_1^2 + v_2^2. \qquad (3)$$

Applying the Pythagorean theorem to (3) we know that a triangle with a hypotenuse of length $v$ has sides of length $v_1$ and $v_2$. If we draw the vectors representing the velocities in order to represent the *vector* momentum conservation equation

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2,$$

as in the following figure,




we see that it must be the case that the sum of $\psi$ and $\zeta$ is $90°$.

**b) Show that the incoming object cannot have a backward scattering component.**

Neither $\psi$ nor $\zeta$ can be negative (i.e. $\mathbf{v}_1$ and $\mathbf{v}_2$ cannot be on the same side of the projectile's initial momentum) or we would not be able to conserve momentum in the direction perpendicular the projectile's initial velocity. Since their sum is $90°$ this means that they both satisfy

$$0 \leq \psi, \zeta \leq 90°,$$

which means that the projectile cannot scatter at an angle $\psi$ larger than $90°$.