PHY6938 Mechanics Spring 99

1. A 1.2 kg block moving in the $+z$ direction at 0.8c makes a head-on collision with a 1.6 kg block moving in the $-z$ direction at 0.6c. They stick together. (*Note*: this is unrealistic; it is far more likely that the blocks would both disintegrate. But carry on anyway!)

a) What is the total momentum of the system (give your answer in units of kg· c)?

The statement of this problem is entirely unphysical, as the note above suggests, and so it is somewhat difficult to know exactly was assumptions to make about the collision since we cannot use our *physical* intuition. In a real collision of this kind only momentum would be conserved and lots of mechanical energy would be lost to heat, sound, structural changes in the blocks, etc. However, the question in part c) below tends to suggest that the question wants us to assume that the collision forms a new object which has a rest mass which isn't the sum of the two initial rest masses since kinetic energy is converted to ress mass in the collision. This is exactly what happens, for example, if two particles or nuclei collide and stick together and no particles or photons are emitted after the collision.

To find the momentum we have to be aware that with speeds not small compared to the speed of light the formula for the momentum of an object moving at speed v is altered from the nonrelativistic formula $p = mv$ to $p = m\gamma(v)\,v$, where $\gamma(v) = 1/\sqrt{1 + v^2/c^2} \geq 1$. This means that the momenta of the blocks are (we treat the momenta as scalars with a sign since the motion in a head-on collision is along a single direction)

$$\begin{aligned}
p_1 &= m_1\gamma(v_1)v_1 = 1.2\frac{1}{\sqrt{1-0.8^2}} \cdot 0.8 \text{ kg} \cdot c = 1.6 \text{ kg} \cdot c \\
p_2 &= m_2\gamma(v_2)v_2 = 1.6\frac{1}{\sqrt{1-0.6^2}} \cdot (-0.6) \text{ kg} \cdot c = -1.2 \text{ kg} \cdot c,
\end{aligned}$$

so that the total momentum is $p = p_1 + p_2 = 0.4$ kg $\cdot\, c$.

b) What is the total energy of the system (give your answer in units of kg· c^2)?

There are two ways to find the total *relativistic* energy (it is implied but not stated that this is what is required here), using $E = m\gamma(v)c^2$ or $E = \sqrt{p^2c^2 + m^2c^4}$, where p is the magnitude of the three-momentum of the particle. Which you should use depends on the application. Since we already know $\gamma(v_1) = 5/3$ and $\gamma(v_2) = 5/4$, we will use

$$\begin{aligned}
E_1 &= m_1\gamma(v_1)c^2 = 1.2 \cdot \frac{5}{3} \text{ kg} \cdot \text{c}^2 = 2.0 \text{ kg} \cdot c^2 \\
E_1 &= m_2\gamma(v_2)c^2 = 1.6 \cdot \frac{5}{4} \text{ kg} \cdot c^2 = 2.0 \text{ kg} \cdot c^2,
\end{aligned}$$

so that $E = E_1 + E_2 = 4.0$ kg $\cdot\, c^2$.

c) What are the mass and speed of the combined object after the collision?

Now we know the total momentum and the total energy of the combined system, we can find its mass using the $E = \sqrt{p^2c^2 + m^2c^4}$ relation,

$$\begin{aligned} m^2c^4 &= E^2 - p^2c^2 = (16.0 - 0.16)\ \mathrm{kg}^2 \cdot c^4 \\ m &= \sqrt{16.0 - 0.16}\ \mathrm{kg} = 3.98\ \mathrm{kg}, \end{aligned}$$

and then its velocity using $E = m\gamma(v)c^2$ so that

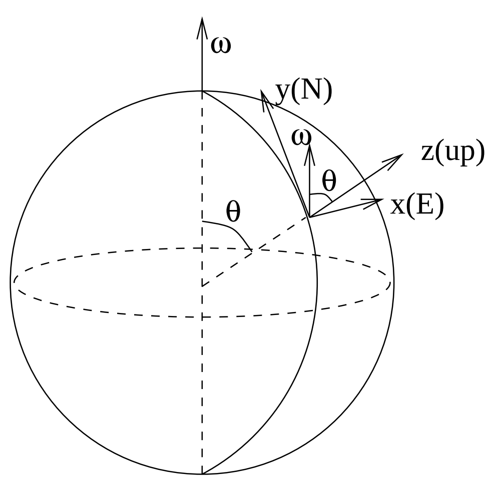
$$\gamma(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{4}{3.98} = 1.005,$$

so that $v^2/c^2 = 1 - 1/1.005^2 = 0.01$ and $v/c = 0.1$.

2. Consider the Earth as a frame rotating about its axis with frequency ω. Particles are then subject to the Coriolis force given by $\mathbf{F} = -2m(\boldsymbol{\omega} \times \mathbf{v}_r)$, where $\mathbf{v}_r$ is the velocity of the particle relative to the frame and m is the mass of the particle. Choose the z–axis along the upwards vertical direction, the y–axis pointing North, and the x–axis to the East. Assume you are in the Northern hemisphere at a colatitude θ (angle of the z–axis with $\boldsymbol{\omega}$).

a) A particle moves under the influence of gravitation and the Coriolis force. Write the equations of motion in the x, y, and z directions. Use the approximation that the component of velocity in the z direction is much larger than the components in the x and y directions, i.e. $|v_z| \gg |v_x|,\ |v_y|$.

Referring to the diagram below, we see that in this coordinate frame the vector $\boldsymbol{\omega}$ has a component $\omega\cos(\theta)$ in the up direction and a component $\omega\cos(\theta)$ in the North direction. Suppose that the velocity vector relative to the moving coordinate frame is $\mathbf{v}_r = (0, 0, \dot{z})$. Then assuming $\dot{z} > 0$ the vector $\boldsymbol{\omega} \times \mathbf{v}_r$ points East (along $\hat{\mathbf{x}}$) and has magnitude $\omega\dot{z}\sin(\theta)$, so the Coriolis force $-2m\boldsymbol{\omega} \times \mathbf{v}_r$ points West with magnitude $2m\omega\dot{z}\sin(\theta)$.



If we combine this with a gravitational force pointing down (along $-\hat{\mathbf{z}}$), the equations of motion become

$$m\ddot{x} = -2m\omega\dot{z}\sin(\theta)$$

$$\begin{aligned} m\ddot{y} &= 0 \\ m\ddot{z} &= -mg. \end{aligned}$$

b) The particle is dropped at rest from a height h above the ground; it arrives at the ground with velocity $v_0 = \sqrt{2gh}$. Find the magnitude and direction of the Coriolis deflection.

Because the Coriolis force is so weak compared to the gravitational force [ω is $2\pi/(24\cdot3600) = 7.27\cdot10^{-5}$], we can safely assume that during the entire flight we can continue to neglect the Coriolis forces in the $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ directions. First solve the z equation of motion to find

$$\dot{z}(t) = -gt$$

and then substitute $\dot{z}$ into the x equation to find

$$\ddot{x} = -2\omega\sin(\theta)(-gt) = 2\omega\sin(\theta)gt.$$

Integrating this over time twice and using the initial conditions $\dot{x}(0) = x(0) = 0$, we find

$$\begin{aligned} \dot{x} &= \dot{x}(0) + 2\omega\sin(\theta)g\frac{t^2}{2} \\ x(t) &= x(0) + \omega\sin(\theta)g\frac{t^3}{3}. \end{aligned}$$

To find the Coriolis deflection we need to find the time at which the particle hits the ground, which is (to a good approximation) $t_f = \sqrt{2h/g}$, just an object falling in constant gravity. The deflection is therefore towards the East (along $\hat{\mathbf{x}}$) with magnitude

$$x(t_f) = \frac{1}{3}\omega\sin(\theta)g\left(\frac{2h}{g}\right)^{\frac{3}{2}} = \frac{1}{3}\omega\sin(\theta)\sqrt{\frac{8h^3}{g}}$$

c) The particle is now thrown vertically upward with an initial speed v_0, so that it reaches the maximum height h [the same h as in part (b)], and then it falls back to the ground. Find the magnitude and direction of the Coriolis deflection.

This goes exactly as before, except now the vertical velocity is given by

$$\dot{z} = v_0 - gt,$$

where $v_0 = \sqrt{2gh}$ is the velocity required to get the particle to height h if it is thrown vertically upward. Substituting this into the x equation as before we get

$$\ddot{x} = -2\omega\sin(\theta)[v_0 - gt]1.$$

Integrating this over time twice and again using the initial conditions $\dot{x}(0) = x(0) = 0$, we find

$$\dot{x} = \dot{x}(0) - 2\omega\sin(\theta)\left[v_0 t - g\frac{t^2}{2}\right]$$

$$\begin{aligned}
x(t) &= x(0) - 2\omega\sin(\theta)\left[v_0\frac{t^2}{2} - g\frac{t^3}{6}\right] \\
x(t) &= -\omega\sin(\theta)\left[v_0 t^2 - g\frac{t^3}{3}\right]
\end{aligned}$$

Now we have to find the time of flight, which is simply twice the value we had before, $t_f = 2\sqrt{2h/g}$, so that the final x displacement is

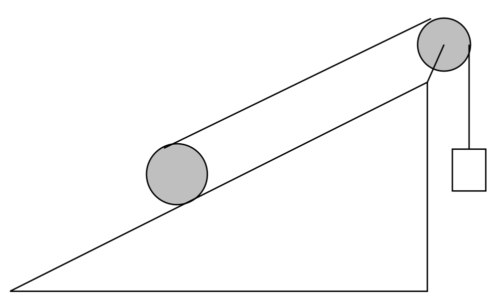
$$\begin{aligned}
x(t_f) &= -\omega\sin(\theta)\left[\sqrt{2gh}\,t_f^2 - g\frac{t_f^3}{3}\right] \\
&= -\omega\sin(\theta)\left[\sqrt{2gh}\,\frac{8h}{g} - g\frac{16h}{3g}\sqrt{\frac{2h}{g}}\right] \\
&= -\omega\sin(\theta)\left[4 - \frac{8}{3}\right]\sqrt{\frac{8h^3}{g}} \\
&= -\frac{4}{3}\omega\sin(\theta)\sqrt{\frac{8h^3}{g}},
\end{aligned}$$

where the negqtive sign means that the displacement is to the West.

d) Compare your results for parts (b) and (c).

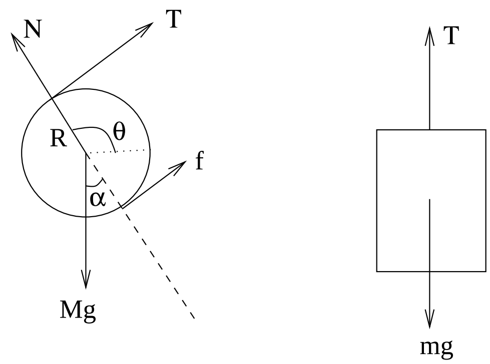
The direction of the deflection is opposite and it is four times as large.

3. A cylinder with a mass of 10 kg and a radius of 0.07 cm can roll without slipping on a 30° ramp. An unstretchable rope is wrapped around the cylinder while the other end is tied to a 2.0 kg block over a massless frictionless pulley, as shown in the diagram.



a) Draw the free-body diagrams for the cylinder and the block.

Label the mass of the cylinder M and its radius R, the angle of the ramp α, and the mass of the block m. The free-body diagram is as follows:



Deciding the direction of the frictional force **f** in this case involves some thought. It is best visualized by looking at the *static* situation and asking yourself which way **f** has to point to prevent the cylinder from rotating. Note that as long as you know the line along which the force **f** points you will get the correct answer: if you choose the wrong direction then at some point the magnitude f will turn out to be negative, which will tell you that it actually points the opposite way from the way you assumed.

b) If the block drops down by 1 m, how much does the cylinder move up vertically?

This part of the question asks us to establish the *constraints* on the motion of the block and the cylinder imposed by the condition that the string does not stretch or slip on the cylinder, and by the rolling without slipping of the cylinder. If you think of a rolling car wheel on a car moving at speed v, you can convince yourself that because the tire rolls without slipping, the bottom of the tire is not moving with respect to the road, the center of the wheel is moving at speed v, and the top of the tire is moving at speed $2v$ with respect to the road. This means that for every 1 m the top of the tire moves, the center of the wheel moves 1/2 m. The fact that it is rolling without slipping means that if the center of the wheel moves through x, the wheel rotates through an angle of $\theta = x/R$.

So we know the cylinder moves 1/2 m along the ramp. The change in its height is therefore $1\sin(30^o)$ m=1/4 m.

c) Determine the magnitude and direction of the acceleration of the cylinder.

Referring to our free-body diagrams, we can write equations of motion for the cylinder in the x direction (up along the plane), the y direction (out of the plane) and the rotation θ of the cylinder (take the positive sense of rotation to be clockwise, and note $I = MR^2/2$), and of course for the vertical motion z of the block. These are

$$\begin{aligned} M\ddot{x} &= T + f - Mg\sin(\alpha) \\ 0 &= N - mg\cos(\alpha) \\ \frac{1}{2}MR^2\ddot{\theta} &= RT - Rf \\ m\ddot{z} &= mg - T \end{aligned}$$

Since we are not given the coefficient of friction we cannot calculate the magnitude of the frictional force f from N, so we will ignore the second equation above and think of these as three equations in the five unknowns f, T, $\ddot{x}$, $\ddot{\theta}$, and $\ddot{z}$.

Our constraints now give us that $\ddot{z} = 2\ddot{x}$, and that $\ddot{\theta} = \ddot{x}/R$, so now we have five equations in five unknowns: writing everything in terms of $\ddot{x}$ we have

$$
\begin{aligned}
M\ddot{x} &= T + f - Mg\sin(\alpha) && (1)\\
\frac{1}{2}M\ddot{x} &= T - f && (2)\\
2m\ddot{x} &= mg - T, && (3)
\end{aligned}
$$

which can be solved as follows:

$$
\begin{aligned}
(1)+(2) &: \quad \frac{3M}{2}\ddot{x} = 2T - Mg\sin(\alpha) && (4)\\
(4)+2(3) &: \quad \left(\frac{3M}{2} + 4m\right)\ddot{x} = -Mg\sin(\alpha) + 2mg, && (5)
\end{aligned}
$$

so that

$$\ddot{x} = \frac{m - \frac{1}{2}M\sin(\alpha)}{\frac{3M}{4} + 2m}g.$$

Substituting the values given in the problem and $g = 9.8\ \mathrm{m/s^2}$ gives $\ddot{x} = -0.43\ \mathrm{m/s^2}$. Note the cylinder actually rolls *down* the ramp, not up as we originally assumed, but the solution is still correct.

d) What are the magnitude and direction of the static friction between the cylinder and the ramp?

We can eliminate T rather than f from (1) and (2) by subtracting (2) from (1), with the result

$$\frac{1}{2}M\ddot{x} = 2f - Mg\sin(\alpha),$$

so that

$$f = \frac{1}{4}M\ddot{x} + \frac{1}{2}Mg\sin(\alpha) = \frac{1}{4}M\,\frac{m - \frac{1}{2}M\sin(\alpha)}{\frac{3M}{4} + 2m}g + \frac{1}{2}Mg\sin(\alpha) = 23.4\ \mathrm{N},$$

which is positive, so the frictional force does indeed point up the incline even when the cylinder is rolling down.