Notes on Radiation

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Radiation from arbitrary source

Let's start with the retarted sources which give us the scalar and vector potentials

$$\boldsymbol{\Phi}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c})}{|\mathbf{r}-\mathbf{r}'|} \tag{1}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \frac{\mathbf{J}(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c})}{|\mathbf{r}-\mathbf{r}'|}. \tag{2}$$

The first approximation for radiation is that the source is localized (also meaning that the radiation zone is far from the source $r \gg r'$). Under this assumption we get,

$$|\mathbf{r}-\mathbf{r}'| \approx r - \frac{\mathbf{r}\cdot\mathbf{r}'}{r} \tag{3}$$

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} \approx \frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3}, \tag{4}$$

using the Fourier Transform

$$f(\mathbf{r}+\mathbf{a}) \approx f(\mathbf{r}) + \mathbf{a}\cdot\nabla f(\mathbf{r}) = f(\mathbf{r}) + a_i\partial_i f(\mathbf{r}). \tag{5}$$

Now lets use the fact that the sources have harmonic time dependence.

$$\mathbf{J}(\mathbf{r}',t) = \mathbf{J}(\mathbf{r}')e^{-i\omega t} \tag{6}$$

With this we can now approximate the vector potential by

$$\mathbf{J}(\mathbf{r}',t-\frac{|\mathbf{r}-\mathbf{r}'|}{c}) \approx \mathbf{J}(\mathbf{r}')e^{-i\omega t + i\omega r/c - i\omega\frac{\hat{r}\cdot\mathbf{r}'}{c}}. \tag{7}$$

Now we can plug this into equation 2, and dropping terms with higher order terms that $1/r$ to get,

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0 e^{-i\omega(t-r/c)}}{4\pi r}\int d^3r'\mathbf{J}(\mathbf{r}')e^{-i\omega\frac{\hat{r}\cdot\mathbf{r}}{c}}. \tag{8}$$

Now is we assume that the region of sources is small compared to the wavelength, $c/\omega \gg r'$, then we can make the **dipole approximation**, $e^{-i\omega\hat{r}\cdot\mathbf{r}'/c} \approx 1$, which when pluggen into equation 8 gives

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0 e^{-i\omega(t-r/c)}}{4\pi r}\int d^3r'\mathbf{J}(\mathbf{r}'). \tag{9}$$

Now let's go on an aside to explore the $\mathbf{J}(\mathbf{r}')$.
Claim: $\int d^3r'\mathbf{J}(\mathbf{r}') = \frac{d\mathbf{p}}{dt}$ where

$$\mathbf{p}(\mathbf{r}',t) = \int d^3r'\mathbf{r}'\rho(\mathbf{r}',t) \tag{10}$$

is the electric dipole moment. Now let's find the integral of $\mathbf{J}(\mathbf{r}')$ in terms of this dipole moment. Let's start with the time derivative of the dipole.

$$\begin{aligned}\frac{d\mathbf{p}}{dt} &= \frac{d}{dt}\int d^3r'\mathbf{r}'\rho(\mathbf{r}',t) &(11)\\ &= \int d^3r'\mathbf{r}'\frac{\partial}{\partial t}\rho(\mathbf{r}',t) &(12)\\ &= -\int d^3r'\mathbf{r}'\nabla\cdot\mathbf{J}(\mathbf{r}',t), &(13)\end{aligned}$$

where we have used the continuity equation

$$\nabla\cdot\mathbf{J}(\mathbf{r}',\mathbf{t}) = -\frac{\partial}{\partial t}\rho(\mathbf{r}',t). \tag{14}$$

Now expand the $\mathbf{r}'\nabla\cdot\mathbf{J}(\mathbf{r}',t)$ term.

$$\begin{aligned}\nabla\cdot r_i\mathbf{J} &= (\nabla r_i)\cdot\mathbf{J} + r_i\nabla\cdot\mathbf{J} &(15)\\ &= \hat{r_i}\cdot\mathbf{J} + r_i\nabla\cdot\mathbf{J} &(16)\end{aligned}$$

If you sum over all possible r_i's you get

$$\mathbf{r}'\nabla\cdot\mathbf{J} = \nabla\cdot(r_x\mathbf{J} + r_y\mathbf{J} + r_z\mathbf{J}) - \mathbf{J} \tag{17}$$

However when the second term is integrated over the volume it can be turned into a surface integral with the divergence theorem, but $\mathbf{J}$ is zero on the surface to we get that

$$\frac{d\mathbf{p}}{dt} = \int d^3r'\mathbf{J}. \tag{18}$$

Now applying this to equation 9 we get

$$\begin{aligned}\mathbf{A}(\mathbf{r},t) &= \frac{\mu_0}{4\pi r}\frac{d}{dt}\mathbf{p}e^{-i\omega(t-r/c)} &(19)\\ &= \frac{\mu_0}{4\pi r}\dot{\mathbf{p}}(t-r/c), &(20)\end{aligned}$$

which is the standard result. I'm not going to worry about finding $\Phi(\mathbf{r}, t)$ because if

$$\mathbf{E} = \mathbf{E}_0 e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)} \tag{21}$$

$$\mathbf{B} = \mathbf{B}_0 e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}, \tag{22}$$

it can be shown with Maxwell's equations that $\mathbf{E}_0 = -c\hat{k} \times \mathbf{B}_0$, so we only need to solve for $\mathbf{B}$ using $\mathbf{B} = \nabla \times \mathbf{A}$.

$$\begin{aligned}
\mathbf{B} &= \nabla \times \mathbf{A} && (23)\\
&= \frac{\mu_0}{4\pi}\nabla \times \frac{\dot{\mathbf{p}}(t - r/c)}{r} && (24)\\
&= \frac{\mu_0}{4\pi r}\nabla \times \dot{\mathbf{p}}(t - r/c) && (25)
\end{aligned}$$

Where the chain rule was applied here but the derivative of $1/r$ gave a term that went as $1/r^2$ so we ignored it. Now this cross product can be done if we remember the product rule $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)$. Thus $\mathbf{B}$ becomes

$$\begin{aligned}
\mathbf{B}(\mathbf{r}, t) &= -\frac{\mu_0}{4\pi r}\left[e^{-i\omega(t-r/c)}(\nabla \times \dot{\mathbf{p}}) - \dot{\mathbf{p}} \times (\nabla e^{-i\omega(t-r/c)})\right] && (26)\\
&= \frac{\mu_0}{4\pi r}\dot{\mathbf{p}} \times (\nabla e^{-i\omega(t-r/c)}) && (27)\\
&= \frac{\mu_0}{4\pi r}\frac{i\omega}{c}\hat{r} \times \dot{\mathbf{p}} e^{-i\omega(t-r/c)}. && (28)
\end{aligned}$$

but since we said that $\dot{\mathbf{p}}(t - r/c) = \dot{\mathbf{p}}e^{-i\omega(t-r/c)}$ we can say that

$$\frac{d}{dt_{ret}}\dot{\mathbf{p}} = \ddot{\mathbf{p}} = -i\omega\dot{\mathbf{p}}, \tag{29}$$

which we can use to give us

$$\mathbf{B}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi rc}\hat{r} \times \ddot{\mathbf{p}}(t - rc). \tag{30}$$

Now we can find $\mathbf{E}$ to be

$$\mathbf{E}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r}\hat{r} \times \hat{r} \times \ddot{\mathbf{p}}(\mathbf{r}, t - r/c), \tag{31}$$

using the fact that $\mathbf{E}_0 = -c\hat{k} \times \mathbf{B}_0$, and noting that $\hat{k} = \hat{r}$.