PHY6938 Optics and Thermodyamics Spring 2000

1. An ideal classical gas made of N point-like atoms with mass m is placed in a confining potential: $V(x,y,z) = \frac{1}{2}(k_x x^2 + k_y y^2 + k_z z^2)$. The temperature of the gas is T and there is no external container.

a) Calculate the average kinetic energy of each gas atom.

The equipartition theorem states that for every independent term quadratic in the dynamical variables (which are momenta and positions) in the Hamiltonian (energy) of the system we have an average energy of $kT/2$. The kinetic energy of each gas atom has three such terms, which are $p_x^2/2m$, $p_y^2/2m$, and $p_z^2/2m$. Each term has associated with it an average energy of $kT/2$, so the average kinetic energy of each gas atom is

$$\langle \frac{\mathbf{p}_i^2}{2m} \rangle = \frac{3}{2}kT.$$

b) Calculate the average potential energy of each gas atom.

Exactly the same argument holds for the potential energy of each particle, which contains terms quadratic in the independent position variables x, y, and z. so that for any i

$$\langle V(\mathbf{r}_i) \rangle = \frac{3}{2}kT.$$

c) Calculate the heat capacity of the gas.

For an ideal gas heated at constant volume, which can do no work, the first law of thermodynamics tells us that the heat capacity C_V is derivative of the internal energy U of the gas with temperature T. The internal energy is the sum of the average kinetic and potential energies of each gas particle multiplied by the number of such particles,

$$U = N\left(\frac{3}{2}kT + \frac{3}{2}kT\right) = 3NkT,$$

and so

$$C_V = \frac{dU}{dT} = 3Nk.$$

d) At low temperatures, the gas has to be considered as composed of quantum mechanical particles, and the fact that their energy levels are discrete is important. Briefly discuss what happens to the specific heat at low temperatures (no calculation is required).

The third law of thermodynamics states that the entropy S goes to a constant S_0 as the temperature goes to zero,

$$\lim_{T\to 0} S = S_0.$$

We can understand this microscopically by the number of available microstates of the system tending to a constant at absolute zero, and the entropy is related to the natural log of the number of available microstates. Now since the heat capacity is defined to be

$$C_V = \left(\frac{dQ}{dT}\right)_V$$

and since $dQ = T\,dS$, we have

$$C_V = T\left(\frac{dS}{dT}\right)_V.$$

But S approaches a constant as $T \to 0$, so we have

$$\lim_{T\to 0} C_V = 0.$$

Although somewhat beyond the scope of this question, we can establish *how* the specific heat tends to zero. Examine the energy of a single particle in a one-dimensional harmonic potential. The partition function is

$$Z = \sum_\alpha e^{-\beta E_\alpha},$$

where $\beta = 1/kT$ and the energy eigenvalues E_α are simply $(n + 1/2)\hbar\omega$, where ω is the harmonic oscillator frequency. For this case the partition function is just a geometric series with the sum

$$Z = \sum_{n=0}^{\infty} e^{-(n+1/2)\beta\hbar\omega} = e^{-\beta\hbar\omega/2}\sum_{n=0}^{\infty} e^{-n\beta\hbar\omega} = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega}}.$$

Now the average energy of the system is

$$\langle E\rangle = \frac{1}{Z}\sum_\alpha E_\alpha e^{-\beta E_\alpha} = \frac{1}{Z}\left(-\frac{\partial Z}{\partial\beta}\right) = -\frac{\partial \ln Z}{\partial\beta}.$$

Now we have that

$$\ln(Z) = -\frac{1}{2}\beta\hbar\omega - \ln(1 - e^{-\beta\hbar\omega}),$$

so that

$$\langle E\rangle = -\frac{\partial \ln Z}{\partial\beta} = \frac{1}{2}\hbar\omega + \frac{\hbar\omega e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} = \hbar\omega\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1}\right).$$

The first temperature-independent term is the zero-point energy of the oscillator. The generalization to N particles in three dimensions is simple–each dimension in which each particle moves behaves like an independent one-dimensional oscillator, so that the energy is just multiplied by $3N$,

$$\langle E\rangle_{3\mathrm{D}} = 3N\hbar\omega\left(\frac{1}{2} + \frac{1}{e^{\beta\hbar\omega} - 1}\right),$$

and now by a straight-forward differentiation we have that

$$C_V = 3N\frac{\hbar\omega}{kT^2}\left[\frac{e^{\beta\hbar\omega}\hbar\omega}{(e^{\beta\hbar\omega} - 1)^2}\right],$$

and as the temperature goes to zero we see that

$$\lim_{T\to 0} C_V = 3Nk\left(\frac{\hbar\omega}{kT}\right)^2 e^{-\frac{\hbar\omega}{kT}}.$$

We see that, since the exponential goes to zero faster than any power of T, the specific heat goes to zero exponentially. This is a characteristic of any system with an energy gap such as this one (which has an energy gap of $\hbar\omega$ between the ground state and first excited state).

2. Four equally spaced coherent light sources with wavelength of 500 nm are separated by a distance of $d = 0.1$ mm. The interference pattern is viewed on a screen at a distance of 1.4 m. Finf the positions of the principal interference maxima and compare their width with that for just two sources with the same spacing..

Start by looking again at the case of two light sources. In this case when the path difference $d\sin(\theta)$, where θ is the angle from the normal to the plane of the light sources, is an integral number of wavelengths we have constructive interference and so an interference maximum. If this path difference is a half-integral number of wavelengths we have destructive interference and so an interference minimum. This means that the interference maxima are at angular positions

$$\sin(\theta) = m\frac{\lambda}{d} \quad \text{(maximum, two sources)}$$

and the minima are at

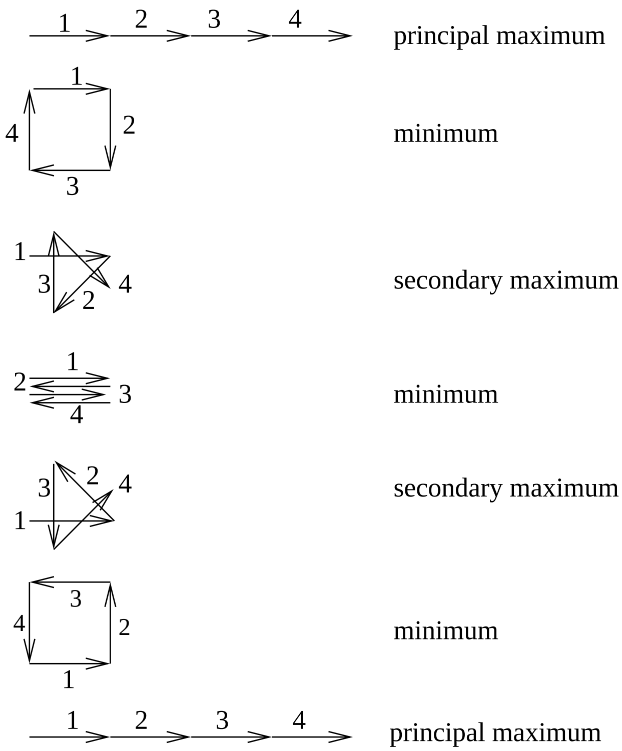
$$\sin(\theta) = \left(m+\frac{1}{2}\right)\frac{\lambda}{d} \quad \text{(minimum, two sources)},$$

so that the maxima have angular width equal to their angular spacing, λ/d.

In the case of four sources, the path difference between adjacent sources is still $d\sin(\theta)$, and the condition for a *principal* interference maximum (intensity 16 times that of a single slit) is the same, but the condition for an interference minimum changes. There is obviously an inteference maximum for $d\sin(\theta) = 0$; the first minimum will occur when the four phasors from each source form a square (see diagram), and so the phase angle between adjacent phasors is 90°. This means that we have

$$d\sin(\theta) = \frac{\lambda}{4} \quad \text{(first minimum, four sources)}.$$

By the same idea there are obviously interference minima when $d\sin(\theta) = \lambda/2$ and $3\lambda/4$. In between these angles we have secondary maxima at $d\sin(\theta) = 3\lambda/8$ and $5\lambda/8$ (where the intensity is roughly 1/16 times that of the principal maxima, see the diagram below). The next principal maximum occurs when the phase angle between each phasor is again zero, so $d\sin(\theta) = \lambda$.



This means that the principal interference maxima are at angles

$$\sin(\theta) = m\frac{\lambda}{d},\ m = 1, 2, 3...$$

and have half the width that they have in the case of two sources. Converting this to distances we can use the small angles approximation $\sin(\theta) = \theta$ to write that positions of the principal maxima

$$y_n = L\sin(\theta_n) \simeq L\theta_n = L\frac{n\lambda}{d} = (1.4\ \mathrm{m})\frac{n \cdot 5 \times 10^{-7}\ \mathrm{m}}{1 \times 10^{-4}\ \mathrm{m}} = n(7\ \mathrm{mm}),$$

and so the principal maxima are separated by 7 mm, and their width is one half of this distance, 3.5 mm. For two sources this width would be 7 mm.