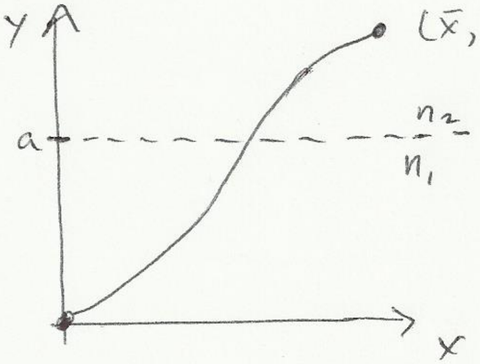
HW 2 Solutions – PHY 576 – Jeff Hyde

① a. Consider some path $x(y)$ joining the initial point $(0,0)$ and the final point $(\bar{x},\bar{y})$:



∞'ll path length is

$$ds = \sqrt{dx^2+dy^2} = dy\sqrt{1+(dx/dy)^2}$$

Use $'$ to denote d/dy: $\frac{dx}{dy} \equiv x'$.

Also, for convenience label the parts of the path in each medium: Say $x(y) = x_1(y) + x_2(y)$ with $x_1 = 0$ when $y>a$, $x_2(y)=0$ when $y<a$.

Since $v_1 = \frac{c}{n_1}$, $v_2 = \frac{c}{n_2}$, and $dt = \frac{ds}{v}$, the travel time along a given path $x(y)$ is

$$t[x(y)] = \int_{y=0}^{a} dy\, \frac{n_1}{c}\sqrt{1+x_1'(y)^2} + \int_{y=a}^{\bar{y}} dy\, \frac{n_2}{c}\sqrt{1+x_2'(y)^2}.$$

The path that extremizes the time will have

$$0 = \frac{\delta t[x(y)]}{\delta x(y)}$$

$$= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\frac{n_1}{c}\int_{y=0}^{a} dy\sqrt{1+(x_1'(y)+\epsilon\delta(a-y'))^2} - \frac{n_1}{c}\int_{y=0}^{a} dy\sqrt{1+x_1'(y)^2}\right.$$

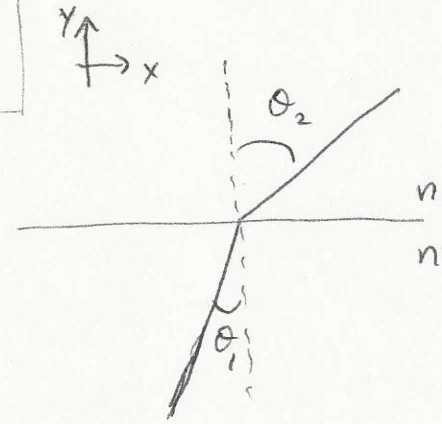
$$\left. + \frac{n_2}{c}\int_{y=a}^{\bar{y}} dy\sqrt{1+(x_2'(y)+\epsilon\delta(a-y'))^2} - \frac{n_2}{c}\int_{y=a}^{\bar{y}} dy\sqrt{1+x_2'(y)^2}\right]$$

Now expand square roots, neglecting terms $\mathcal{O}(\epsilon^2)$ which will go away in the $\epsilon\to 0$ limit.

$$0 = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\Big[\frac{n_1}{c}\int_{y=0}^{a} dy\left(1+\tfrac{1}{2}(x_1'+\epsilon\delta(a-y))^2+\cdots-1-\tfrac{1}{2}x_1'^2-\cdots\right)$$
$$+\frac{n_2}{c}\int_{y=a}^{\bar{y}} dy\left(1+\tfrac{1}{2}(x_2'+\epsilon\delta(a-y))^2+\cdots-1-\tfrac{1}{2}x_2'^2-\cdots\right)\Big]$$

$$=\frac{n_1}{2c}\int_{y=0}^{a} dy\ 2x_1'\delta(a-y') + \frac{n_2}{2c}\int_{y=a}^{\bar{y}} dy\ 2x_2'\delta(a-y)$$

$$0 = \frac{n_1}{c}x_1' + \frac{n_2}{c}x_2'.$$



Near $y=a$ define angles θ_1, θ_2 by:

so that:
$$\sin\theta_1 = -\frac{dx_1}{dy}$$
$$\sin\theta_2 = \frac{dx_2}{dy}$$

Then $\quad 0 = n_1(-\sin\theta_1) + n_2(\sin\theta_2)\quad$ or

$$\underline{n_1\sin\theta_1 = n_2\sin\theta_2.}$$

b. $$\left.\frac{\delta V[f]}{\delta u}\right|_{u(z)=z} = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left(V[f_0+\epsilon z]-V[f_0]\right)$$
$$=\lim_{\epsilon\to 0}\frac{\pi}{\epsilon}\left(\int_0^1 (f_0^2+2\epsilon f_0 z+\epsilon^2 z^2)\,dz - \int_0^1 f_0^2\,dz\right)$$
$$=\pi\int_0^1 2f_0 z\,dz$$
$$=\pi\int_0^1 2ze^z\,dz$$
$$=\underline{2\pi}$$

2.a) Free particle in $|p_i\rangle$ at $t=0$.

Prob. of position $|x_f\rangle$ at $t=t_f$?

$$\psi(x_f,t_f) = \int dx_i \; G(x_f,t_f;x_i,t_i)\,\psi(x_i,t_i)$$

$$= \int dx_i \int dp \; G_{free}(x_f,t_f;x_i,t_i)\langle x_i|p\rangle \underbrace{\langle p|\psi(t_i)\rangle}_{=\langle p|p_i\rangle}$$

$$= \int dx_i \int dp \left(\frac{m}{2\pi i T\hbar}\right)^{1/2} \exp\left[\frac{im(x_f-x_i)^2}{2\hbar(t_f-t_i)}\right] \frac{\exp[i\vec{p}\cdot\vec{x}/\hbar]}{\sqrt{2\pi\hbar}}\,\delta(p-p_i)$$

$$= \frac{1}{2\pi\hbar}\left(\frac{m}{iT}\right)^{1/2}\int dx_i \exp\left[\frac{im(x_f-x_i)^2}{2\hbar(t_f-t_i)} + \frac{i\vec{p}_i\cdot\vec{x}_i}{\hbar}\right]$$

$$= \frac{1}{2\pi\hbar}\left(\frac{m}{iT}\right)^{1/2}\int dx_i \exp\left[\frac{imx_f^2}{2\hbar T} + \frac{i}{\hbar}\left(p_i - \frac{mx_f}{T}\right)x_i + \frac{imx_i^2}{2\hbar T}\right]$$

$$= \frac{1}{2\pi\hbar}\left(\frac{m}{iT}\right)^{1/2}\exp\left[\frac{imx_f^2}{2\hbar T}\right]\int dx_i \exp\left[\frac{i}{\hbar}\left(\left(p_i-\frac{mx_f}{T}\right)x_i + \left(\frac{m}{2T}\right)x_i^2\right)\right]$$

Integral of form:

$$\int_{-\infty}^{+\infty} dx \exp\left(\tfrac{1}{2}iax^2 + ibx\right) = \left(\frac{2\pi i}{a}\right)^{1/2}\exp\left(ib^2/2a\right)$$

$$= \frac{1}{2\pi\hbar}\left(\frac{m}{iT}\right)^{1/2}\exp\left[\frac{imx_f^2}{2\hbar T}\right]\left(\frac{2\pi i\hbar T}{m}\right)^{1/2}\exp\left[\frac{i}{2\hbar^2}\left(p_i-\frac{mx_f}{T}\right)^2\frac{\hbar T}{m}\right]$$

$$= \frac{1}{2\pi\hbar}\left(2\pi\hbar\right)^{1/2}\exp\left[\frac{imx_f^2}{2\hbar T} - \frac{iT}{2m\hbar}\left(p_i-\frac{mx_f}{T}\right)^2\right]$$

$$\psi(x_f,t_f) = \frac{1}{\sqrt{2\pi\hbar}}\exp\left[\cancel{\frac{imx_f^2}{2\hbar T}} - \frac{iTp_i^2}{2m\hbar} - \cancel{\frac{iTm^2x_f^2}{2m\hbar T^2}} + \frac{iT}{m\hbar}\frac{p_i m x_f}{T}\right]$$

$$\boxed{\psi(x_f,t_f) = \frac{1}{\sqrt{2\pi\hbar}}\exp\left[\frac{i}{\hbar}\left(p_ix_f - \frac{p_i^2T}{2m}\right)\right]}$$

2.b

$$\Psi(x_f, t_f) = \langle x_f | e^{-iHT/\hbar} | p_i \rangle$$

$$= \langle x_f | e^{-i\hat{p}^2 T/2m\hbar} | p_i \rangle$$

$$= \langle x_f | e^{-ip_i^2 T/2m\hbar} | p_i \rangle$$

$$= \langle x_f | p_i \rangle e^{-ip_i^2 T/2m\hbar}$$

$$= \frac{1}{\sqrt{2\pi\hbar}} e^{ip_i x_f/\hbar} e^{-ip_i^2 T/2m\hbar}$$

$$\Psi(x_f, t_f) = \frac{1}{\sqrt{2\pi\hbar}} \exp\left[\frac{i}{\hbar}\left(p_i x_f - p_i^2 T/2m\right)\right]$$

$\boxed{2.c}$ w.f. at $t=0$: $\psi(x,0) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp(-x_i^2/2\sigma^2)$

w.f. at $t>0$ from free-particle propagator?

$$\psi(x_f,t) = \int dx_i \; G_{free}(x,t;x_i,0)\,\psi(x_i,0)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}}\left(\frac{m}{2\pi i T\hbar}\right)^{1/2} \int dx_i \exp\left[\frac{mi}{2\hbar}\frac{(x_f-x_i)^2}{T}\right]\exp\left[-\frac{x_i^2}{2\sigma^2}\right]$$

$$= \frac{1}{2\pi\sigma}\left(\frac{m}{iT\hbar}\right)^{1/2}\int dx_i \exp\left[\frac{mi}{2\hbar T}x_f^2 - \frac{mi}{\hbar T}x_f x_i + \frac{mi}{2\hbar T}x_i^2 - \frac{1}{2\sigma^2}x_i^2\right]$$

$$= \frac{1}{2\pi\sigma}\left(\frac{m}{iT\hbar}\right)^{1/2}\int dx_i \exp\left[\frac{mi}{2\hbar T}x_f^2 - \frac{mix_f}{\hbar T}x_i + \underbrace{\left(\frac{mi}{2\hbar T}-\frac{1}{2\sigma^2}\right)}_{=\frac{-mi\sigma^2+\hbar T}{2\sigma^2\hbar T}}x_i^2\right]$$

$$= \frac{1}{2\pi\sigma}\left(\frac{m}{iT\hbar}\right)^{1/2}\exp\left(\frac{mix_f^2}{2\hbar T}\right)\int dx_i \exp\left[-\left(\frac{\hbar T - mi\sigma^2}{2\sigma^2\hbar T}\right)x_i^2 - \left(\frac{imx_f}{\hbar T}\right)x_i\right]$$

$$= \frac{1}{2\pi\sigma}\left(\frac{m}{iT\hbar}\right)^{1/2}\exp\left[\frac{imx_f^2}{2\hbar T}\right]\left(\frac{2\pi\sigma^2\hbar T}{\hbar T - mi\sigma^2}\right)^{1/2}$$
$$\times \exp\left[\left(-\frac{imx_f}{\hbar T}\right)^2 \frac{1}{2}\frac{\sigma^2\hbar T}{\hbar T - mi\sigma^2}\right]$$

$$= \frac{1}{2\pi\sigma}\left(\frac{m}{iT\hbar}\right)^{1/2}\left(\frac{2\pi\sigma^2\hbar T}{\hbar T - mi\sigma^2}\right)^{1/2}\exp\left[\frac{imx_f^2}{2\hbar T} - \frac{m^2x_f^2}{2\hbar T}\frac{\sigma^2}{\hbar T - mi\sigma^2}\right]$$

$$= \left(\frac{m}{2\pi i(\hbar T - mi\sigma^2)}\right)^{1/2}\exp\left[\frac{x_f^2}{2\hbar T}\left(im - \frac{m^2\sigma^2}{\hbar T - mi\sigma^2}\right)\right]$$

$$= \left(\frac{m}{2\pi i(\hbar T - mi\sigma^2)}\right)^{1/2}\exp\left[\frac{x_f^2}{2\hbar T}\left(\frac{im\hbar T + \cancel{m^2\sigma^2} - \cancel{m^2\sigma^2}}{\hbar T - mi\sigma^2}\right)\right]$$

$$\boxed{\psi(x_f,t) = \left(\frac{m}{2\pi i(\hbar T - mi\sigma^2)}\right)^{1/2}\exp\left[\frac{imx_f^2}{2(\hbar T - mi\sigma^2)}\right]}$$

(3)

$$G_{SHO} = \left(\frac{m\omega}{2\pi\hbar i\sin\omega T}\right)^{1/2} \exp\left(\frac{im\omega}{2\hbar\sin\omega T}\left((x_f^2+x_i^2)\cos\omega T - 2x_i x_f\right)\right)$$

$$e^{i\omega T} + e^{-i\omega T} = 2\cos\omega T$$

$$e^{i\omega T} - e^{-i\omega T} = 2i\sin\omega T$$

$$G_{SHO} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\left(e^{i\omega T}-e^{-i\omega T}\right)^{-1/2}$$

$$\cdot \exp\left\{-\frac{m\omega}{\hbar}\left(e^{i\omega T}-e^{-i\omega T}\right)^{-1}\left((x_f^2+x_i^2)\tfrac{1}{2}\left(e^{i\omega T}+e^{-i\omega T}\right)-2x_ix_f\right)\right\}$$

$$= \overset{\equiv A}{\boxed{\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}}}\left(e^{i\omega T}-e^{-i\omega T}\right)^{-1/2}$$

$$\cdot \exp\left\{-\overset{\equiv B}{\boxed{\frac{m\omega}{2\hbar}(x_f^2+x_i^2)}}\,\frac{e^{i\omega T}+e^{-i\omega T}}{e^{i\omega T}-e^{-i\omega T}} + \overset{\equiv C}{\boxed{\frac{2m\omega}{\hbar}\,x_ix_f}}\,\frac{1}{e^{i\omega T}-e^{-i\omega T}}\right\}$$

$$= A\left(e^{i\omega T}-e^{-i\omega T}\right)^{-1/2}\exp\left\{-B\,\frac{e^{i\omega T}+e^{-i\omega T}}{e^{i\omega T}-e^{-i\omega T}} + \frac{C}{e^{i\omega T}-e^{-i\omega T}}\right\}$$

$$= A\,e^{-i\omega T/2}\left(1-e^{-2i\omega T}\right)^{-1/2}\exp\left\{-B\,\frac{1+e^{-2i\omega T}}{1-e^{-2i\omega T}} + Ce^{-i\omega T}\left(1-e^{-2i\omega T}\right)^{-1}\right\}$$

$$\boxed{J \equiv e^{-i\omega T}}$$

$$G_{SHO} = A\,J^{1/2}\left(1-J^2\right)^{-1/2}\exp\left\{-B\,\frac{1+J^2}{1-J^2} + CJ\left(1-J^2\right)^{-1}\right\}$$

$$(1-J^2)^{-1} = 1 + J^2 + J^4 + \cdots$$

$$(1-J^2)^{-1/2} = 1 + \tfrac{1}{2}J^2 + \cdots$$

$$(1-J^2)^{-1/2} = 1 + \cancel{J}\left(-\tfrac{1}{2}\right)(1-J^2)^{-2}\,2(0) + \underbrace{\frac{J^2}{2}\left[\frac{+1}{(1-J^2)^2} + \frac{4J\;2J}{(1-J^2)^3}\right]_{J=0}}_{=\frac{J^2}{2}(+1)}$$

$$G_{SHO} = AJ^{1/2}\left(1+\tfrac{1}{2}J^2+\dots\right)\exp\Big(-B(1+J^2)(1+J^2+\dots) + CJ(1+J^2+\dots)\Big)$$

$$= AJ^{1/2}\left(1+\tfrac{1}{2}J^2+\dots\right)\exp\Big(-B\left(1+2J^2+\dots\right) + C\left(J+J^3+\dots\right)\Big)$$

$$= Ae^{-B}J^{1/2}\left(1+\tfrac{1}{2}J^2+\dots\right)\exp\left(-2J^2+\dots\right)\exp\left(CJ+CJ^3+\dots\right)$$

$$= Ae^{-B}J^{1/2}\left(1+\tfrac{1}{2}J^2+\dots\right)\left(1-2J^2+\dots\right)\left(1+CJ+CJ^3+\dots\right)$$

$$= Ae^{-B}J^{1/2}\left(1+CJ-2J^2+\tfrac{1}{2}J^2+\dots\right)$$

$$= Ae^{-B}\left(J^{1/2}+CJ^{3/2}-\tfrac{3}{2}J^{5/2}\right) + \text{(higher order terms)}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} e^{-\frac{m\omega}{2\hbar}(x_f^2+x_i^2)} e^{-i\omega T/2}$$

$$+ \left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\frac{2m\omega}{\hbar}x_i x_f\, e^{-\frac{m\omega}{2\hbar}(x_f^2+x_i^2)} e^{-i\omega T(3/2)} + \dots$$

$$\boxed{\begin{array}{l}\text{Spectral rep: } G = \sum_n \psi_n(x_f)\psi_n^*(x_i)\, e^{-iE_nT/\hbar} \\ \to \text{Identify terms of powers of exponential w/ terms in this expansion}\end{array}}$$

$$= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x_f^2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x_i^2}\, e^{-i\left(\frac{\hbar\omega}{2}\right)T/\hbar}$$

$$+ \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}\right)^{1/2} x_f\, e^{-\frac{m\omega}{2\hbar}x_f^2}\left(\frac{2m\omega}{\hbar}\right)^{1/2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} x_i \cdot e^{-\frac{m\omega}{2\hbar}x_i^2}\cdot e^{-i\frac{3\hbar\omega}{2}T/\hbar}$$

$$\Rightarrow \quad \psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} e^{-\frac{m\omega}{2\hbar}x^2}, \quad E_0 = \frac{\hbar\omega}{2}$$

$$\psi_1(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\left(\frac{2m\omega}{\hbar}\right)^{1/2} x\, e^{-\frac{m\omega}{2\hbar}x_f^2}, \quad E_1 = \frac{3}{2}\hbar\omega$$

4. Goodman, American Journal of Physics 49, 843 (1981) ...
Better solution than I would write...

5. Use spectral repn to calculate the energy spectrum & normalized w.f.s for a particle in box of length L, from the propagator

$$G_{box}(x_f, T; x_i, 0) = \sum_{n=-\infty}^{+\infty} \Big(G_{free}(2nL + x_f, T; x_i, 0) - G_{free}(2nL - x_f, T; x_i, 0) \Big)$$

$$G_{box}(x_f, T; x_i, 0) = \sum_{n=-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar} \left(\exp\left[\frac{ip}{\hbar}(2nL + x_f - x_i) - \frac{ip^2}{2m\hbar} T \right] - \exp\left[\frac{ip}{\hbar}(2nL - x_f - x_i) - \frac{ip^2}{2m\hbar} T \right] \right) \right]$$

$$= \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar} \exp\left[\frac{-ip^2}{2m\hbar} T \right] \exp\left[\frac{ip}{\hbar} 2nL \right] \times \left(\exp\left[\frac{ip}{\hbar}(x_f - x_i) \right] - \exp\left[\frac{ip}{\hbar}(-x_f - x_i) \right] \right)$$

Poisson resummation formula: $\sum_{n=-\infty}^{+\infty} \exp(-2\pi i n x) = \sum_{m=-\infty}^{+\infty} \delta(x - m)$

$$\rightarrow \sum_{n=-\infty}^{+\infty} \exp\left[\frac{ip}{\hbar} 2nL \right]$$

$$= \sum_{n=-\infty}^{+\infty} \exp\left[2\pi i n \frac{Lp}{\pi\hbar} \right]$$

$$= \sum_{m=-\infty}^{+\infty} \delta\left(\frac{Lp}{\pi\hbar} - m \right)$$

So

$\rightarrow p = \frac{\pi\hbar k}{L}$

$$G_{box}(x_f, T; x_i, 0) = \int_{-\infty}^{+\infty} \frac{dp}{2\pi\hbar} \exp\left[\frac{-ip^2}{2m\hbar}T\right] \sum_{k=-\infty}^{+\infty} \delta\left(\frac{Lp}{\pi\hbar} - k\right)$$
$$\times \left(\exp\left[\frac{ip}{\hbar}(x_f - x_i)\right] - \exp\left[\frac{ip}{\hbar}(-x_f - x_i)\right]\right)$$

$$= \frac{\pi\hbar}{L} \sum_{k=-\infty}^{+\infty} \frac{1}{2\pi\hbar} \exp\left[-\frac{iT}{2m\hbar}\frac{\pi^2\hbar^2k^2}{L^2}\right]$$
$$\times \left(\exp\left[\frac{i\pi k}{L}(x_f - x_i)\right] - \exp\left[\frac{i\pi k}{L}(-x_f - x_i)\right]\right)$$

Express exponentials in terms of sines/cosines:

$$(\ldots\ldots) = \cos\left[\frac{\pi k}{L}(x_f - x_i)\right] + i\sin\left[\frac{\pi k}{L}(x_f - x_i)\right]$$
$$- \cos\left[\frac{\pi k}{L}(-x_f - x_i)\right] - i\sin\left[\frac{\pi k}{L}(-x_f - x_i)\right]$$

For $k = 0$: $(\ldots) = 1 - 1 = 0$

$$G_{box} = \frac{\pi\hbar}{L} \sum_{k=1}^{\infty} \frac{1}{2\pi\hbar} \exp\left[\frac{-iT\pi^2\hbar k^2}{2mL^2}\right] \left(2\cos\left[\frac{\pi k}{L}(x_f - x_i)\right] + \cancel{i\sin\left[\frac{\pi k}{L}(x_f - x_i)\right]}\right.$$
$$\left. - 2\cos\left[\frac{\pi k}{L}(-x_f - x_i)\right] - \cancel{i\sin\left[\frac{\pi k}{L}(-x_f - x_i)\right]}\right)$$

$2\cos(\ldots)$ since $\cos(\ldots k \ldots) = \cos(\ldots(-k)\ldots)$

$\sin \to 0$ since $\sin(\ldots k \ldots) + \sin(\ldots(-k)\ldots) = \sin(\ldots k) - \sin(\ldots k \ldots) = 0$.

$$G_{box} = \frac{1}{\pi\hbar} \sum_{k=1}^{\infty} \exp\left[\frac{-iT\pi^2\hbar k^2}{2mL^2}\right] \left(\cos\left[\frac{\pi k}{L}(x_f - x_i)\right] - \cos\left[\frac{\pi k}{L}(-x_f - x_i)\right]\right)$$

Use $\cos u - \cos v = -2\sin\left(\frac{u+v}{2}\right)\sin\left(\frac{u-v}{2}\right)$:

$$\cos\left[\frac{\pi k}{L}(x_f - x_i)\right] - \cos\left[\frac{\pi k}{L}(-x_f - x_i)\right]$$

$$= -2\sin\left[\frac{1}{2}\frac{\pi k}{L}(x_f - x_i - x_f - x_i)\right]\sin\left[\frac{1}{2}\frac{\pi k}{L}(x_f - x_i + x_f + x_i)\right]$$

$$= +2\sin\left[\frac{\pi k x_i}{L}\right]\sin\left[\frac{\pi k x_f}{L}\right]$$

So

$$G_{box}(x_f, T; x_i, 0) = \frac{2}{L}\sum_{k=1}^{\infty}\sin\left[\frac{\pi k x_i}{L}\right]\sin\left[\frac{\pi k x_f}{L}\right]\exp\left[\frac{-iT\hbar\pi^2 k^2}{2mL^2}\right]$$

Compare with spectral repn:

$$G_{box} = \sum_{n=1}^{\infty}\psi_n(x_f)\psi_n^*(x_i)\exp\left[-iE_n T/\hbar\right]$$

$$\Rightarrow \boxed{\begin{aligned}\psi_n(x) &= \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\\ E_n &= \frac{n^2\pi^2\hbar^2}{2mL^2}\end{aligned}}$$

(6) a. To take $\frac{v}{c} \ll 1$ limit, rewrite action as follows:

$$S = -mc \int \sqrt{-\eta_{\mu\nu} u^\mu u^\nu}\, dt \qquad \text{w/ } u^\mu = (c, \vec{v})$$

$$= -mc \int \sqrt{-(-c^2 + \vec{v}\cdot\vec{v})}\, dt$$

$$= -mc^2 \int \sqrt{1 - v^2/c^2}\, dt.$$

For $x \ll 1$, $\sqrt{1+x} \approx 1 + \frac{1}{2}x + O(x^2)$, so for $\frac{v}{c} \ll 1$:

$$S \approx -mc^2 \int \left(1 - \frac{v^2}{2c^2}\right) dt$$

$$= \text{const} + \int \frac{1}{2} m v^2\, dt$$

which is the usual NR action for a free particle (+ a constant, which doesn't affect the eqns of motion).

b. $$0 = \delta S = -mc \int \delta(ds).$$

$$\delta(ds^2) = -\eta_{\mu\nu} \frac{dx^\mu}{d\tau} d(\delta x^\nu) - \eta_{\mu\nu} d(\delta x^\mu) \frac{dx^\nu}{d\tau}$$

$$= -2\eta_{\mu\nu} \frac{dx^\mu}{d\tau} d(\delta x^\nu)$$

Also, $\delta(ds^2) = 2\, ds\, \delta(ds)$. So

$$0 = +mc \int \eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{d}{d\tau}(\delta x^\nu)\, d\tau$$

$$= mc \left[\cancel{\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \delta x^\nu}\right]_{\text{boundaries}}^{\to 0} - mc \int \eta_{\mu\nu} \frac{d^2 x^\mu}{d\tau^2} \delta x^\nu\, d\tau$$

$\Rightarrow$ eqn. of motion $\underline{\dfrac{d^2 x^\mu}{d\tau^2} = 0}$.

c. $G \sim \exp\left(iS[x_{cl}]/\hbar\right) = \exp\left(-\frac{i}{\hbar}mc^2\int_0^T\sqrt{1-\frac{v^2}{c^2}}\,dt\right)$

Assume a path from $x_i = 0,\ t_i = 0$ to $x_f = r,\ t_f = T$ that goes as $x(t) = rt/T$, so $\dot{x} = r/T$. Then

$$G \sim \exp\left(-\frac{i}{\hbar}mc^2\int_0^T\sqrt{1-\frac{r^2}{c^2T^2}}\,dt\right)$$

$$= \exp\left(-\frac{i}{\hbar}mc^2\sqrt{1-\frac{r^2}{c^2T^2}}\,T\right).$$

For $r > cT$, $\frac{r^2}{c^2T^2} < 1 \rightarrow 1-\frac{r^2}{c^2T^2} < 0$

so $\sqrt{1-\frac{r^2}{c^2T^2}} = \sqrt{-1}\underbrace{\sqrt{\frac{r^2}{c^2T^2}-1}}_{\text{argument a +ve real}} = \pm i\sqrt{\frac{r^2}{c^2T^2}-1}$

$$\therefore\ G \sim \exp\left(\pm\frac{mc^2}{\hbar}T\sqrt{\frac{r^2}{c^2T^2}-1}\right)$$

and we take the physically reasonable sign;

$$G \sim \exp\left(-\frac{mc^2}{\hbar}T\sqrt{\frac{r^2}{c^2T^2}-1}\right)$$

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i.e. a nonzero amplitude, but decays exponentially.