Modern Physics and Quantum Mechanics
Spring 2001 Proficiency

Useful constants:

- $e = 1.60 \times 10^{-19}$ C
- $hc = 1240$ eV $\cdot$ nm
- $c = 3.00 \times 10^8 \ \frac{\text{m}}{\text{s}}$
- $m_e = 0.511 \ \frac{\text{MeV}}{\text{c}^2}$
- $k = \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9 \ \frac{\text{Nm}^2}{\text{C}^2}$
- $\epsilon_0 = 8.85 \times 10^{-12} \ \frac{\text{C}^2}{\text{Nm}^2}$

1. **Consider a one-dimensional step potential of the form**

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x \geq 0 \end{cases},$$

where $V_0 > 0$. A particle with total energy $E > V_0$ and mass m is incident on the step potential "from the left" (in other words: the particle starts at negative values of x and travels toward positive values of x).

(a) **Use the time-independent Schrödinger equation to determine the form of the particle's wave function in the two regions $x < 0$ and $x \geq 0$.**

In the $x < 0$ region (call this region 1) the potential is zero so the Schrödinger equation has the form

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi_1}{dx^2} = E\Psi_1, \tag{1}$$

and Ψ_1 has the form of a plane wave moving to the right (the incident wave) and another moving to the left (the reflected wave), so that

$$\Psi_1 = Ae^{ik_1x} + Be^{-ik_1x}. \tag{2}$$

Substituting this into the Schrödinger equation we see that it is a solution if

$$\begin{aligned} -\frac{\hbar^2}{2m}(-k_1^2) &= E \\ k_1 &= \sqrt{\frac{2mE}{\hbar^2}} = \frac{\sqrt{2mE}}{\hbar}. \end{aligned}$$

In the $x > 0$ region (call this region 2) we have

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\frac{d^2\Psi_2}{dx^2} + V_0\Psi_2 &= E\Psi_2. \\
-\frac{\hbar^2}{2m}\frac{d^2\Psi_2}{dx^2} &= (E - V_0)\Psi_2.
\end{aligned}
$$

and Ψ_2 has the form of a plane wave moving to the right (the transmitted wave), so that

$$\Psi_2 = Ce^{ik_2x}, \tag{3}$$

with

$$k_2 = \frac{\sqrt{2m(E - V_0)}}{\hbar}. \tag{4}$$

(b) Derive expressions for the probabilities that the particle is
(i) reflected (R), and
(ii) transmitted (T).
Hint: Recall that the probability density current is given by

$$j(x) = \mathrm{Re}\left(\Psi^* \frac{\hbar}{im}\frac{\partial\Psi}{\partial x}\right) ,$$

and that R and T are ratios of probability density currents. We have to derive the values of A, B, and C using the boundary conditions, which are that the wavefunction and its derivative must be continuous at the boundary ($x = 0$). Continuity of the wavefunction gives that

$$A + B = C, \tag{5}$$

while that of the derivative at $x = 0$ gives that

$$k_1A - k_1B = k_2C, \tag{6}$$

since taking the derivative of the plane wave just brings down a factor of $-ik$, and we have divided by $-i$. For that reason the incident probability current is simply

$$j_{\mathrm{in}} = \mathrm{Re}\left[Ae^{-k_1x}\frac{\hbar}{im}ik_1Ae^{ik_1x}\right] = \frac{\hbar}{m}k_1A^2, \tag{7}$$

and similarly

$$
\begin{aligned}
j_{\mathrm{refl}} &= \frac{\hbar}{m}k_1B^2 \\
j_{\mathrm{trans}} &= \frac{\hbar}{m}k_2C^2.
\end{aligned}
$$

Putting these together we get that the reflection probability is

$$R = \frac{j_{\mathrm{refl}}}{j_{\mathrm{in}}} = \frac{B^2}{A^2}, \tag{8}$$

and we can find B/A by combining (1) and (2) above,

$$k_1(A - B) = k_2(A + B), \tag{9}$$

so that

$$\frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2}, \tag{10}$$

and

$$R = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2. \tag{11}$$

By conservation of probability we have that

$$T = 1 - R = \frac{(k_1 + k_2)^2 - (k_1 - k_2)}{2}(k_1 + k_2)^2 = \frac{4k_1k_2}{(k_1 + k_2)}^2 \tag{12}$$

2. A monochromatic particle beam consists of particles whose total energy is 100 times their rest mass. The rest lifetime of the particles is 0.10 ns. In the laboratory, the distance between the point where the particles are generated and the detector is 6.0 m. What fraction of the generated particles reach the detector?

To find the fraction of particles which reach the detector, we have to find the time that it takes for particles to reach the detector. Let N_0 be the number of the generated particles and N be the number of particles which reach the detector. The relation between them is given by

$$N = N_0 \exp\left[-\frac{t}{\tau}\right]. \tag{1}$$

Eq.1 is the formula which is used to find the number of particles after time t when their life time is τ. Therefore we have to find t and τ.

The generated particles have a life time $\tau_0 = 100$ ns, in their rest frame. This means that If we "travel" with the particle the particle will decay after 10 ns. But this is not the time which is measured in the laboratory. Since the rest frames of the particles move with respect to the lab-frame. According to the special relativity there is dilation of time between the time measured in the lab-frame and the one in the rest frame of the particle. The measured time τ is related to τ_0 by

$$\tau = \gamma\tau_0 = \frac{\tau_0}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{2}$$

where v is the velocity of the particles. Here the velocity of the particles is unknown. To find the velocity we use the information given in the problem. We know the total energy of the particle. The relation between the total energy and the velocity is given by

$$
\begin{aligned}
E^2 &= p^2c^2 + m_0^2c^4 && (3)\\
\text{but, } p &= \gamma m_0 v && (4)\\
E^2 &= \gamma^2 m_0^2 c^2 + m_0^2 c^4 \\
&= \left[\gamma^2 v^2 + c^2\right] m_0^2 c^2 \\
\text{use the definition of } \gamma &= \frac{1}{1-\dfrac{v^2}{c^2}} m_0^2 c^4 \\
&= \gamma^2 m_0^2 c^4 \\
\text{or } E &= \gamma m_0 c^2 && (5)
\end{aligned}
$$

The total energy is

$$E = 100 m_0 c^2 \;\; \Rightarrow \;\gamma = 100. \tag{6}$$

Substitute Eq.6 in Eq.2 we obtain

$$\tau = 10^{-8}\,\mathrm{s}. \tag{7}$$

From Eq.6 and the definition of γ we get

$$v \approx c. \tag{8}$$

The time for particles to reach the detector is given by

$$t = \frac{l}{v} \approx \frac{l}{c} = 2 \times 10^{-8}\,\mathrm{s}. \tag{9}$$

Substitute Eq.7 and 9 in Eq.1 it yields

$$\frac{N}{N_0} = e^{-2} \approx 13.5\%. \tag{10}$$

3. **In a hydrogen atom the electron spin $\vec{S}_e$ and the proton spin $\vec{S}_p$ interact with each other via the hyperfine interaction and with an external magnetic field through the Zeeman effect. The magnetic field is applied along the z-direction, $\vec{B} = B\hat{z}$. The Hamiltonian of the system is given by**

$$H = J\vec{S}_e \cdot \vec{S}_p + (\gamma_e \vec{S}_e + \gamma_p \vec{S}_p) \cdot \vec{B} \;,$$

where J is the hyperfine coupling, and γ_e and γ_p are the gyromagnetic ratios of the electron and proton, respectively.

(a) **State the energy eigenvalues and eigenstates for this Hamiltonian in the zero-field limit, $B = 0$.**

With out magnetic field the Hamiltonian is

$$H = J\vec{S}_e \cdot \vec{S}_p. \tag{1}$$

Note, that there are two spin-$\frac{1}{2}$ particles. This means that the wave function must be antisymmetric with respect to exchange of the particles. The eigenfunctions of this Hamiltonian has are combination of spin and space functions of the particles. For two fermions it is possible to separate the spin part and the space part of the eigenfunctions. This means if the space part of the eigenfunction is symmetric (antisymmetric) the spin part of the eigenfunction must be antisymmetric (symmetric). The Hamiltonian only has spin operators, therefore we consider the symmetric and antisymmetric spin configurations.

The antisymmetric eigenfunction is given by ($|S_{ez}, S_{pz}\rangle$)

$$|\psi_A\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle - \downarrow\uparrow\rangle\right]. \tag{2}$$

The symmetric eigenfunctions are given by

$$\begin{aligned}
|\psi_S\rangle &= |\uparrow\uparrow\rangle \\
|\psi_S\rangle &= \frac{1}{\sqrt{2}}\left[|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle\right] \\
|\psi_S\rangle &= |\downarrow\downarrow\rangle
\end{aligned} \tag{3}$$

As we can see the antisymmetric state is singlet and the symmetric state is triplet. To find the eigenvalues we have to rewrite the Hamiltonian as follows

$$\vec{S}_e \cdot \vec{S}_p = \frac{S^2 - S_e^2 - S_p^2}{2} \tag{4}$$

where S denotes the total spin. Further we have

$$\vec{S}_e \cdot \vec{S}_p = \frac{l(l+1) - l_e(l_e+1) - l_p(l_p+1)}{2}. \tag{5}$$

The energy eigenvalue for the singlet state is

$$l = 0,\, l_e = \frac{1}{2},\, l_p = \frac{1}{2} \;\Rightarrow\; E_A = -\frac{3}{4}\,J. \tag{6}$$

and for the triplet state is

$$l = 1,\, l_e = \frac{1}{2},\, l_p = \frac{1}{2} \;\Rightarrow\; E_S = \frac{1}{4}\,J. \tag{7}$$

(b) State the energy eigenvalues and eigenstates for the $J = 0$ case (no hyperfine interaction).

Since the magnetic field is in z-direction the Hamiltonian in this case is

$$H = B\left(\gamma_e S_{ez} + \gamma_p S_{pz}\right) \tag{8}$$

Note that in this case the eigenstates are not the same as part (a). Since there are only four different configurations for $|S_{ez}, S_{pz}\rangle$, it can be shown that the eigenstates and the corresponding eigenvalues are given by

$$\begin{aligned}
|\psi_{\uparrow\uparrow}\rangle &= |\uparrow\uparrow\rangle,\ E_{\uparrow\uparrow} = \frac{B}{2}\,(\gamma_e + \gamma_p) \\
|\psi_{\uparrow\downarrow}\rangle &= |\uparrow\downarrow\rangle,\ E_{\uparrow\downarrow} = \frac{B}{2}\,(\gamma_e - \gamma_p) \\
|\psi_{\downarrow\uparrow}\rangle &= |\downarrow\uparrow\rangle,\ E_{\downarrow\uparrow} = \frac{B}{2}\,(-\gamma_e + \gamma_p) \\
|\psi_{\downarrow\downarrow}\rangle &= |\downarrow\downarrow\rangle,\ E_{\downarrow\downarrow} = -\frac{B}{2}\,(\gamma_e + \gamma_p)
\end{aligned} \tag{9}$$

(c) Obtain the energy eigenvalues for the general case ($B \neq 0$ and $J \neq 0$).

This problem can be solved in " different" ways. But to make the solution easy to understand for everybody we choose a longer solution with more details.

To find the eigenvalues and eigenfunctions we have to choose a base. Then, diagonalizing the Hamiltonian the eigenvalues can be find. We choose the following base

$$|S_{ez}, S_{pz}\rangle = |\uparrow\uparrow\rangle,\ |\downarrow\downarrow\rangle,\ |\uparrow\downarrow\rangle,\ |\downarrow\uparrow\rangle. \tag{10}$$

We have to calculate the matrix elements of the Hamiltonian in this base. Note, that the spin of the electron and the proton belongs to different Hilbert space. This means that operators acting on the electron commute with operators acting on the proton.For example

$$O_e\,O_p\,|S_{ez}, S_{pz}\rangle = O_e\,|S_{ez}\rangle\ O_p|S_{pz}\rangle = |S_e', S_p'\rangle \tag{11}$$

The Hamiltonian can be written as

$$H = J\,(S_{ex}\,S_{px} + S_{ey}\,S_{py} + S_{ez}\,S_{pz} +\ B\,(\gamma_e\,S_{ez} + \gamma_p\,S_{pz}). \tag{12}$$

Rewrite S_x, and S_y and S_z in terms of the $|\uparrow\rangle_z$ and $|\downarrow\rangle_z$. We obtain

$$\begin{aligned}
S_x &= \frac{1}{2}\,[|\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow|] \\
S_y &= \frac{-i}{2}\,[|\uparrow\rangle\langle\downarrow| - |\downarrow\rangle\langle\uparrow|] \\
S_x &= \frac{1}{2}\,[|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|]
\end{aligned} \tag{13}$$

Use Eq.13 for both proton and electron and calculate the matrix elements of the Hamiltonian

$$H \doteq \begin{bmatrix}
\frac{J}{4} + \frac{B}{2}(\gamma_e + \gamma_p) & 0 & 0 & 0 \\
0 & \frac{J}{4} - \frac{B}{2}(\gamma_e + \gamma_p) & 0 & 0 \\
0 & 0 & -\frac{J}{4} + \frac{B}{2}(\gamma_e + \gamma_p) & \frac{J}{2} \\
0 & 0 & \frac{J}{2} & -\frac{J}{4} - \frac{B}{2}(\gamma_e + \gamma_p)
\end{bmatrix}$$

As we can see the Hamiltonian is in Block-Diagonal form. The firs two eigenvalues and eigenfunctions can be decided directly from the Hamiltonian

$$\begin{aligned} E_1 &= \frac{J}{4} + \frac{B}{2}(\gamma_e - \gamma_p),\ \psi_1 = \mid\uparrow\uparrow\rangle \\ E_2 &= \frac{J}{4} - \frac{B}{2}(\gamma_e - \gamma_p),\ \psi_2 = \mid\downarrow\downarrow\rangle \end{aligned} \tag{14}$$

The other two eigenvalues and eigenfunctions are obtained by diagonalizing the following matrix

$$\begin{bmatrix} -\frac{J}{4} + \frac{B}{2}(\gamma_e - \gamma_p) & \frac{J}{2} \\ \frac{J}{2} & -\frac{J}{4} - \frac{B}{2}(\gamma_e - \gamma_p) \end{bmatrix}$$

The eigenvalues of the matrix above can be calculated easily. The final results will be given by

$$\begin{aligned} E_3 &= -\frac{J}{4} + \sqrt{\frac{J^2}{4} + (\gamma_e - \gamma_p)^2 \frac{B^2}{4}} \\ E_4 &= -\frac{J}{4} - \sqrt{\frac{J^2}{4} + (\gamma_e - \gamma_p)^2 \frac{B^2}{4}} \end{aligned} \tag{15}$$

The eigenfunctions will be linear combinations of $\mid\uparrow\downarrow\rangle$ and $\mid\downarrow\uparrow\rangle$.

4. **Light of wavelength 300 nm strikes a metal plate, producing photoelectrons that move with a speed of 0.002 c.**

In the photoelectric effect the incoming photons remove electrons from the target. There is a minimum energy required to remove an electron from the interior of a solid to a position just outside. This minimum energy is called the work function Φ. The relation between the energy of photon E_γ, the energy of electron E_e, and the work function Φ is given by

$$E_\gamma = E_e + \Phi. \tag{1}$$

Note that energy of electron E_e is equal to the kinetic energy of electron, since when electron is outside the solid it is not affected by any potential. In other words electron will be considered as a free particle.

(a) **What is the work function of the metal?**

Use Eq.1, since the speed of the electron is much less than speed of the light we do not need to use the special relativity formula for the kinetic energy.

$$\begin{aligned} E_e &= \frac{1}{2}\, mv^2 \\ &= \frac{1}{2}\,(511\,\frac{keV}{c^2})\,(0.002c)^2 \end{aligned} \tag{2}$$

$$
\begin{aligned}
&= 1.022\, eV \qquad (3)\\
E_\gamma &= h\nu \qquad (4)\\
&= \frac{hc}{\lambda}\\
&= \frac{1420\, eV\ nm}{300\, nm}\\
&= 4.13\, eV \qquad (5)\\
\Phi &= E_\gamma - E_e \qquad (6)\\
&= 3.111\, eV. \qquad (7)
\end{aligned}
$$

(b) What is the critical wavelength for this metal, so that photoelectrons are produced?

The critical wavelength is defined as the wavelength for photons which remove remove electrons from interior of a solid to a position outside the solid. In this case the kinetic energy of electrons outside the solid is equal zero $E_e = 0$. Using the Eq.1 and the results from part (a) we obtain

$$
\begin{aligned}
E_\gamma &= \Phi \qquad (8)\\
\frac{hc}{\lambda_{critical}} &= \Phi\\
\lambda_{critical} &= \frac{hc}{\Phi} \qquad (9)\\
\lambda_{critical} &= \frac{1420\, eV}{3.111\, eV}\\
\lambda_{critical} &= 398.6\, eV. \qquad (10)
\end{aligned}
$$

(c) What is the significance of the critical wavelength?

Photons with a lower wavelength than the critical wavelenght cannot produce photo-electrons. This led Einstein to the postulation of quantized energy for the electromagnetic fields.