1. a.

$$G_{SHO}(x_f, t_f; x_i, t_i) = \left(\frac{m\omega f}{2\hbar\pi i f \sin(\omega T)}\right)^{1/2} \exp\left[\frac{i}{\hbar}\frac{m\omega}{2\sin(\omega T)}\left((x_f^2 + x_i^2)\cos(\omega T) - 2x_i x_f\right)\right]$$

Integrate prop. over closed paths w/ imag. time period $T = -i\hbar\beta$:

gaussian int.

$$Z(\beta) = \int_{-\infty}^{+\infty} G_{SHO}(x, T; x, 0)$$

$$= \left(\frac{m\omega}{2\hbar i\pi \sin(\omega T)}\right)^{1/2} \int_{-\infty}^{+\infty} \exp\left[\frac{i}{\hbar}\frac{m\omega}{2\sin(\omega T)} 2x^2(\cos(\omega T) - 1)\right] dx$$

$$= \left(\frac{m\omega}{2\hbar i\pi \sin(\omega T)}\right)^{1/2} \left(\frac{-\pi\hbar\sin(\omega T)}{im\omega(\cos(\omega T) - 1)}\right)^{1/2} = \frac{1}{\sqrt{2(\cos(\omega T) - 1)}}$$

$$= \left(\frac{1}{2}\frac{1}{\frac{e^{\omega\hbar\beta} + e^{-\omega\hbar\beta}}{2} - 1}\right)^{1/2}$$

$$= \dots = \boxed{\frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega/2}}} = \frac{1}{2\sinh(\beta\omega\hbar/2)}$$

b. For SHO, $E_n = (n + 1/2)\hbar\omega$:

$$Z(\beta) = \sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} e^{-\beta\hbar\omega/2}\left(e^{-\beta\hbar\omega}\right)^n$$

$$\sum_{r=0}^{\infty} ar^n = \frac{a}{1-r} \Rightarrow \boxed{Z(\beta) = \frac{e^{-\beta\hbar\omega/2}}{1 - e^{-\beta\hbar\omega/x}}}$$

② a. $V(x) = V_0\left(1 - \frac{|x|}{a}\right)$

Classical turning pt: $E = V$:

$$\frac{E}{V_0} = 1 - \frac{|x|}{a} \rightarrow |x| = a\left(1 - \frac{E}{V_0}\right)$$

$$\boxed{\begin{aligned} x_1 &= -a\left(1-\frac{E}{V_0}\right) \\ x_2 &= +a\left(1-\frac{E}{V_0}\right)\end{aligned}}$$

b. In imag. time, $\underset{t \equiv i\tau}{\downarrow}$ $m\frac{d^2x}{d\tau^2} = \frac{dV}{dx}$.

$$m\frac{dx}{d\tau}\frac{d^2x}{d\tau^2} = \frac{dV}{dx}\frac{dx}{d\tau}$$

$$\frac{d}{d\tau}\left(\frac{1}{2}m\left(\frac{dx}{d\tau}\right)^2\right) = \frac{d}{d\tau}V(x(\tau))$$

so $\frac{1}{2}m\left(\frac{dx}{d\tau}\right)^2 = V(x) - \underset{\text{int. const.}}{\underset{\uparrow}{E}}$

or $E = -\frac{1}{2}m\left(\frac{dx}{d\tau}\right)^2 + V(x)$.

$$\frac{dx}{d\tau} = \sqrt{\frac{2}{m}(V-E)}$$

$$\int \frac{dx}{\sqrt{\frac{2}{m}(V(x)-E)}} = \int d\tau$$

$$\int \frac{dx}{\sqrt{\frac{2V_0}{m}\left(1-\frac{|x|}{a}-E\right)}} = \int d\tau$$

$$= \int_{x=-a}^{0} \frac{dx}{\sqrt{\frac{2V_0}{m}\left(1+\frac{x}{a}-E\right)}} + \int_{x=0}^{a} \frac{dx}{\sqrt{\frac{2V_0}{m}\left(1-\frac{x}{a}-E\right)}} = \int d\tau$$

$$\left[\frac{2ma}{2V_0}\sqrt{\frac{2V_0}{m}\left(1+\frac{x}{a}-E\right)}\right]_{x=-a}^{0} + \left[-\frac{2ma}{2V_0}\sqrt{\frac{2V_0}{m}\left(1-\frac{x}{a}-E\right)}\right]_{x=0}^{a}$$

$$\int d\tau = \frac{ma}{V_0}\left(\sqrt{\frac{2V_0}{m}} - 2\sqrt{\frac{V_0}{m}} - 0 + \sqrt{\frac{2V_0}{m}}\right)$$

$$\tau = \frac{ma}{V_0}\sqrt{\frac{V_0}{m}}\left(2\sqrt{2} - 2\right) \longrightarrow \underline{\tau = 2a\sqrt{\frac{m}{V_0}}\left(\sqrt{2}-1\right)}$$

For $x < 0$:

$$\tau = \frac{\cancel{2}ma}{\cancel{2}V_0}\left(\sqrt{\frac{2V_0}{m}\left(1+\frac{x}{a}\right)} - 0\right) = a\sqrt{\frac{2m}{V_0}\left(1+\frac{x}{a}\right)}$$

For $x > 0$:

$$\tau = a\sqrt{\frac{2m}{V_0}} + \frac{ma}{V_0}\sqrt{\frac{2V_0}{m}}\left(-\sqrt{1-\frac{x}{a}} + 1\right)$$

$$\tau = a\sqrt{\frac{2m}{V_0}}\left(2 - \sqrt{1-\frac{x}{a}}\right)$$

For $x < 0$, invert to get path:

$$\frac{\tau^2}{a^2} = \frac{2m}{V_0}\left(1+\frac{x}{a}\right) \longrightarrow \frac{V_0}{2m}\frac{\tau^2}{a^2} = 1 + \frac{x}{a}$$

$$\frac{x}{a} = 1 - \frac{V_0}{2m}\frac{\tau^2}{a^2}$$

$$x(\tau) = a\left(1 - \frac{V_0}{2m}\frac{\tau^2}{a^2}\right)$$

For $x > 0$,

$$\cancel{\frac{\tau^2}{a^2}\frac{V_0}{2m} = 2 - \left(1 - \frac{x}{a} = 4 - 2\tau\right.}$$

$$\sqrt{1-\frac{x}{a}}\sqrt{\frac{2ma^2}{V_0}} = 2a\sqrt{\frac{2m}{V_0}} - \tau$$

$$\sqrt{1-\frac{x}{a}} = 2 - \sqrt{V_0/2ma^2}\,\tau$$

$$1 - \frac{x}{a} = 4 - \frac{4}{a}\sqrt{\frac{V_0}{2m}}\,\tau + \frac{V_0}{2ma^2}\tau^2$$

x

c. $S_E = \int d\tau \left(\frac{1}{2} m \left(\frac{dx}{d\tau}\right)^2 + V(x) \right)$

$$= \int_{x_1}^{x_2} dx \sqrt{2m(V(x)-E)} + i\int dt\, E$$

$$= \int_{x_1}^{x_2} dx \sqrt{2m\left(V_0\left(1-\frac{|x|}{a}\right)-E\right)} + i\int dt\, E$$

$$= \int_{-a(1-E/V_0)}^{0} dx \sqrt{2m\left(V_0\left(1+\frac{x}{a}\right)-E\right)} + \int_0^{a(1-E/V_0)} dx \sqrt{2m\left(V_0\left(1-\frac{x}{a}\right)-E\right)} + i\int dt\, E$$

$$= \frac{4}{3} a(1-E/V_0)\sqrt{2m(V_0-E)} + i\int dt\, E$$

d. Prob. amp. $\approx \int dT\, e^{-iET/\hbar}\, e^{-S_E/\hbar}$

$$= e^{-\frac{1}{\hbar}\int dx \sqrt{2m(V(x)-E)}}$$

Prob. $= |\text{amplitude}|^2$

$$= \exp\left(-\frac{8a(1-E/V_0)\sqrt{2m(V_0-E)}}{3V_0\hbar}\right)$$

③

a. $V(r) = \begin{cases} 0, & r < a \\ \frac{k}{r}, & r \geq a \end{cases}$ and $E < \frac{k}{a}$.

In $r \gtrsim a$,

$$E = \frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + V \rightarrow \left(\frac{dr}{dt}\right)^2 = \frac{2}{m}(E - V) < 0.$$

$\Rightarrow \frac{dr}{dt}$ imag., i.e. no real path in this region.

Use imag. time $\tau \equiv it$:

$$E = -\frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 + V \rightarrow \left(\frac{dr}{d\tau}\right)^2 = \frac{2}{m}(V - E) > 0$$

$\Rightarrow \frac{dr}{d\tau}$ real, $\exists$ a path.

---

b. classical turning pts: $r_1 = a$, $r_2 = \frac{k}{E}$.

$$\int \frac{dr}{\sqrt{\frac{2}{m}(V-E)}} = \int d\tau$$

$$\tau = \int_{r'=a}^{r} \frac{dr'}{\sqrt{\frac{2}{m}\left(\frac{k}{r'} - E\right)}} = \int_{r'=a}^{r} \frac{dr'}{\sqrt{\frac{2E}{m}\left(\frac{k}{Er'} - 1\right)}}$$

$$= \sqrt{\frac{m}{2E}} \int_{r'=a}^{r} \frac{dr'}{\sqrt{\frac{k}{Er'} - 1}}$$

Take $r' \equiv \frac{k}{E} \sin^2 u$ (valid up to $r = \frac{k}{E}$ turning pt.)

$\Rightarrow dr' = 2\frac{k}{E} \sin u \cos u \, du$. $r' = a = \frac{k}{E}\sin^2 u$

$\Rightarrow u = \sin^{-1}\left(\sqrt{\frac{aE}{k}}\right)$

$r' = \frac{k}{E} \Rightarrow u = \pi/2$

$$\tau = \frac{2k}{E}\sqrt{\frac{m}{2E}} \int_{\sin^{-1}(\sqrt{aE/k})}^{\pi/2} \left(\frac{1}{\sin^2 u} - 1\right)^{-1/2} \sin u \cos u \, du$$

$$= \sqrt{\frac{2m}{E}} \frac{k}{E} \left[\frac{(u - \sin u \cos u)}{2} \right]_{u = \sin^{-1}(\sqrt{aE/k})}^{\pi/2}$$

$$\tau = \sqrt{\frac{2m}{E}} \frac{k}{E} \left(\frac{\pi}{4} - \frac{1}{2}\sin^{-1}\left(\sqrt{aE/k}\right) + \frac{1}{2}\sqrt{\frac{aE}{k}} \cos\left(\sin^{-1}\left(\sqrt{aE/k}\right)\right)\right)$$

c. $S = \int dt \left(\frac{1}{2} m \left(\frac{dr}{dt}\right)^2 + \frac{k}{r} \right)$ (const. θ, φ)

$\tau \equiv it$

$S_E = i S[\tau] = \int d\tau \left(-\frac{1}{2} m \left(\frac{dr}{d\tau}\right)^2 + \frac{k}{r} \right)$

$S_E = \int_{r_1}^{r_2} dr \sqrt{2m\left(\frac{k}{r} - E\right)}$ w/ $r_1 = a$, $r_2 = k/E$

↓ Mathematica...

$+ i \int dt E$

$S_E = \boxed{\sqrt{2mE} \left(\frac{k}{E} \cos^{-1}\left(\sqrt{\frac{Ea}{k}}\right) + \sqrt{\frac{ka}{E} - a^2} \right)}$

Tunnel Prob $= \exp(-2 S_E/\hbar)$

d. $\frac{\text{tunnel attempt}}{\text{time}} = \frac{\text{speed}}{\text{diameter}} = \frac{\sqrt{2E/m}}{2a}$

So for an ensemble of these atoms,

decay rate $= \sqrt{\frac{E}{2ma}} \exp\left(-\frac{2}{\hbar} \sqrt{2mE} \left(\frac{k}{E} \cos^{-1}\left(\sqrt{\frac{Ea}{k}}\right) + \sqrt{\frac{ka}{E} - a^2} \right) \right)$

↓

Γ

$\tau \sim \Gamma^{-1} = \sqrt{\frac{2ma^2}{E}} \exp\left(\frac{2}{\hbar} \sqrt{2mE} \left(\frac{k}{E} \cos^{-1}\left(\sqrt{\frac{Ea}{k}}\right) + \sqrt{\frac{ka}{E} - a^2} \right) \right)$

(4) $(q \equiv 2k\sin\theta/2)$

a. $$f(\theta) = -\frac{2m}{q\hbar^2}\int_0^\infty r V(r)\sin(qr)\,dr$$

$$= -\frac{2m}{q\hbar^2}\int_0^{a} r V_0 \sin(qr)\,dr$$

$$= -\frac{2mV_0}{q\hbar^2}\left[\frac{1}{q^2}\left(\sin(qr) - qr\cos(qr)\right)\right]_{r=0}^{a}$$

$$= -\frac{2mV_0}{q^3\hbar^2}\left(\sin(qa) - qa\cos(qa)\right)$$

so d.c.s. is

$$\frac{d\sigma}{d\Omega} = |f|^2 = \frac{4m^2V_0^2}{q^6\hbar^4}\left(\sin(qa) - qa\cos(qa)\right)^2$$

b. $$f(\theta) = -\frac{2m}{q\hbar^2}\int_0^\infty r V(r)\sin(qr)\,dr$$

$$= -\frac{2m}{q\hbar^2}\int_0^\infty r V_0 e^{-\mu r^2}\sin(qr)\,dr$$

$$= -\frac{2mV_0}{q\hbar^2}\,\frac{\pi^{1/2}q}{4\mu^{3/2}}\exp(-q^2/4\mu)$$

so $$\frac{d\sigma}{d\Omega} = |f|^2 = \frac{\pi m^2V_0^2}{4\mu^3\hbar^4}\exp(-q^2/2\mu)$$

Total cross section is

$$\sigma = \int \frac{d\sigma}{d\Omega}\,d\Omega$$

$$= \frac{\pi m^2 V_0^2}{4\mu^3\hbar^4}2\pi\int_{\theta=0}^{\pi}\exp\left(-2k^2\sin^2(\theta/2)/\mu\right)\sin\theta\,d\theta$$

$$= \frac{\pi^2 m^2 V_0^2}{2\mu^3\hbar^4}\left(\frac{-\mu}{2k^2}\right)\left[\exp\left(-\frac{2k^2}{\mu}\cos(\theta) - \frac{2k^2}{\mu}\right)\right]_{\theta=0}^{\pi}$$

$$= \frac{\pi^2 m^2 V_0^2}{2\mu^2\hbar^4k^2}\left(\underbrace{\exp\left(\frac{2k^2}{\mu} - \frac{2k^2}{\mu}\right)}_{=1} - \exp(-2k^2/\mu)\right)$$

$$\boxed{\sigma = \frac{\pi^2 m^2 V_0^2}{4\mu^2\hbar^4k^2}\left(1 - e^{-2k^2/\mu}\right)}$$

5. a.

$\cancel{\langle n|H'|m\rangle = At^2 e^{-(t/\tau)^2}}$

$$\vec{F} = q\vec{E} = qAt^2 e^{-(t/\tau)^2}\hat{z}$$

$$F_z = -\frac{\partial V}{\partial z} \Rightarrow V = -qAt^2 e^{-(t/\tau)^2} z$$

$$z = \sqrt{\frac{\hbar}{2m\omega}}(a_z + a_z^\dagger)$$

$$\langle n_z|H'|m_z\rangle = qAt^2 e^{-(t/\tau)^2}\sqrt{\frac{\hbar}{2m\omega}}\left(\langle n|a|m\rangle^{\,0} + \langle n|a^\dagger|m\rangle^{\,0}\right)$$

$$= qAt^2 e^{-(t/\tau)^2}\sqrt{\frac{\hbar}{2m\omega}}\,\delta_{n,1}$$

$\Rightarrow$ For first excited state, $\hbar\omega(1+\frac{1}{2}-\frac{1}{2})$

$$d_f(t) = -\frac{i}{\hbar}\int_{-\infty}^{+\infty}\langle 1|H'|0\rangle e^{i(E_1-E_0)t/\hbar}\,dt$$

$$= -\frac{i}{\hbar}\sqrt{\frac{\hbar}{2m\omega}}\,qA\int_{-\infty}^{+\infty} t^2 e^{-(t/\tau)^2} e^{i\omega t}\,dt$$

$$\cancel{= \frac{-iqA}{\sqrt{2\hbar m\omega}}\int_{-\infty}^{+\infty} t^2 e^{-(t/\tau)^2}\cos(\omega t)\,dt + \frac{qA}{\sqrt{2\hbar m\omega}}\int_{-\infty}^{+\infty} t^2 e^{-(t/\tau)^2}\sin(\omega t)\,dt}$$

$$= \frac{-i}{\sqrt{2\hbar m\omega}}qA\int_{-\infty}^{+\infty} t^2 e^{-\alpha t^2 + \beta t} \quad \text{w/}\quad \alpha = \frac{1}{\tau^2},\ \beta = i\omega$$

$$\int_{-\infty}^{+\infty} dx\, x^2 e^{-ax^2+bx} = 2\left(\frac{1}{4a}\sqrt{\frac{\pi}{a}}\, e^{-\frac{b^2}{4a}+c}\right)$$

$$= \frac{-iqA}{\sqrt{2\hbar m\omega}}\,\frac{1}{2}\tau^3\sqrt{\pi}\,\exp(\omega^2\tau^2/4)$$

$$\Rightarrow \boxed{P_{10} = \frac{q^2A^2\tau^6\pi}{8\hbar m\omega}\exp(\omega^2\tau^2/2)}$$

since $\to 0$ δ_{n1}

$\boxed{P_{20} = 0}$

b. 1-d box, length L. $n=1$ init.

$$V_{pert} = \frac{1}{2} k x^2 e^{-\lambda t} \quad \text{for } t \geq 0.$$

$P_n(t=+\infty)$.

$$d_n = -\frac{i}{\hbar} \int_0^\infty \langle n | \frac{1}{2} k x^2 e^{-\lambda t} | 1 \rangle e^{i(E_n - E_1)t/\hbar} \, dt$$

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2mL^2}$$

$$d_n = -\frac{i}{\hbar} \frac{k \cancel{2}}{\cancel{2} L} \int_0^\infty dt \, e^{-\lambda t} e^{i\hbar\pi^2(n^2-1)t/2mL^2} \times \boxed{\int_0^L \sin(n\pi x/L) \sin(\pi x/L) x^2 \, dx}$$

$y \equiv \pi x / L \rightarrow dy = \frac{\pi}{L} dx$

$x = \frac{L}{2} \Leftrightarrow y = \pi/2$

$$= \left(\frac{L}{\pi}\right)^3 \int_0^{+\pi} \sin(ny) \sin(y) \, y^2 \, dy$$

$$= \frac{L^3}{\pi^3} \left[\frac{((n-1)^2 x^2 - 2) \sin((n-1)x) + 2(n-1)x\cos((n-1)x)}{2(n-1)^3} - \frac{((n+1)^2 x^2 - 2) \sin((n+1)x) + 2(n+1)x\cos((n+1)x)}{2(n+1)^3} \right]_{x=0}^{\pi}$$

$$= \frac{L^3}{\pi^3} \left(\frac{2(n-1)\pi(-1)^{n-1}}{2(n-1)^3} - \frac{2(n+1)\pi(-1)^{n+1}}{2(n+1)^3} \right)$$

$$= \frac{L^3}{\pi^2} (-1)^{n+1} \left(\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} \right)$$

$$d_n = -\frac{ikL^2\cancel{\pi}}{\hbar\cancel{k}\pi^2}(-1)^{n+1}\left(\frac{1}{(n-1)^2}-\frac{1}{(n+1)^2}\right)\frac{1}{\lambda + i\hbar\pi^2(n^2-1)/2mL^2}$$

$$= \frac{-ik}{\hbar\cancel{p}}\frac{\cancel{\pi}L^2}{\pi^2}(-1)^{n+1}\frac{(n+1)^2-(n-1)^2}{[(n+1)(n-1)]^2}\frac{\lambda - i\hbar\pi^2(n^2-1)/2mL^2}{\lambda^2 + \hbar^2\pi^4(n^2-1)^2/4m^2L^4}$$

$$P(n) = \frac{k^2\cancel{\pi^2}}{\hbar^2\cancel{p^2}}\frac{L^4}{\pi^4}\frac{16n^2}{(n^2-1)^2}\frac{\lambda^2 + \hbar^2\pi^4(n^2-1)^2/4m^2L^4}{[\lambda^2 + \hbar^2\pi^4(n^2-1)^2/4m^2L^4]^2}$$

6. a. $H' = V(\vec{r})\cos(\omega t)$

$$c(t) = -\frac{i}{\hbar}\langle 2|V(\vec{r})|1\rangle \int_0^t dt' \cos(\omega t')\, e^{i(E_2-E_1)t'/\hbar}$$

$\hookrightarrow E_2 - E_1 \equiv \hbar\omega_0$

$$= -\frac{i}{2\hbar}\langle 2|V(\vec{r})|1\rangle \int_0^t dt' \left(e^{i(\omega+\omega_0)t'} + e^{i(-\omega+\omega_0)t'}\right)$$

$$= -\frac{1}{2\hbar}\langle 2|V(\vec{r})|1\rangle \left[\frac{e^{i(\omega_0+\omega)t'}}{\omega_0+\omega} + \frac{e^{i(\omega_0-\omega)t'}}{\omega_0-\omega}\right]_{t'=0}^{t}$$

$$= -\frac{1}{2\hbar}\langle 2|V(\vec{r})|1\rangle \left[\frac{e^{i(\omega_0+\omega)t'}-1}{\omega_0+\omega} + \frac{e^{i(\omega_0-\omega)t}-1}{\omega_0-\omega}\right]$$

For $\omega \approx \omega_0$, $\omega_0+\omega \gg |\omega_0-\omega| \Rightarrow \frac{1}{\omega_0+\omega} \ll \left|\frac{1}{\omega_0-\omega}\right|$;

$$c(t) \approx \frac{-1}{2\hbar}\langle 2|V(\vec{r})|1\rangle \frac{e^{i(\omega_0-\omega)t}-1}{\omega_0-\omega}$$

$$= \frac{-1}{2\hbar}\langle 2|V|1\rangle \frac{1}{\omega_0-\omega} e^{i(\omega_0-\omega)t/2}\left(e^{i(\omega_0-\omega)t/2} - e^{-i(\omega_0-\omega)t/2}\right)$$

$$= \frac{-1}{2\hbar}\langle 2|V|1\rangle \frac{1}{\omega_0-\omega} e^{i(\omega_0-\omega)t/2}\, 2i\sin((\omega_0-\omega)t/2)$$

$$c(t) = -\frac{i}{\hbar}\langle 2|V|1\rangle \frac{1}{\omega_0-\omega} e^{i(\omega_0-\omega)t/2}\sin((\omega_0-\omega)t/2)$$

$$\Rightarrow P_{abs} = |c|^2 = \left|\frac{\langle 2|V(\vec{r})|1\rangle}{\hbar}\right|^2 \frac{\sin^2((\omega_0-\omega)t/2)}{(\omega_0-\omega)^2}$$

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b. For $2 \to 1$ probability, calculation is the same except with $\langle 2|V|1\rangle \rightarrow \langle 1|V|2\rangle$ & a $-$ in the exponential. Both of these are irrelevant since we are $|\ldots|^2$-ing the amplitude.

$\therefore$ Transition rates will also match.