PHY 576: Quantum Theory

Problem Set 1

Due in TA's mailbox by 4 pm on Friday, February 6.

1. Consider P_3, the vector space of all polynomials (with real coefficients) of degree at most three (i.e. up to and including cubics). We can turn P_3 into a (real) Hilbert space by giving it an inner product. Choose the inner product to be

$$\langle f|g\rangle \equiv \int_{-1}^{+1} dx f g \ ,$$

where f and g are polynomials in x. With respect to this inner product, find an orthonormal basis for which the first basis vector is a constant, the second basis vector is a linear polynomial, the third basis vector is a quadratic polynomial, and the fourth basis vector is a cubic polynomial. What are these polynomials called?

2. (a) Consider a 3-dimensional quantum-mechanical Hilbert space and the operator O represented in some basis by the matrix:

$$O = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Could O correspond to a physical observable? Why or why not? Find the eigenvalues and eigenvectors of O. What is the probability for measuring each of these eigenvalues if the state-vector for the system is $|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$?

(b) Two observables A_1 and A_2, which do not involve time explicitly, are known not to commute:

$$[A_1, A_2] \neq 0 \ .$$

Suppose we also know that they both commute with the Hamiltonian, H: $[A_i, H] = 0$; $i = 1, 2$. Prove that the energy eigenstates are in general degenerate. Are there any exceptions?

3. (a) Consider a particle with Gaussian wavefunction

$$\psi(x) = \frac{1}{\sqrt{d\sqrt{\pi}}} e^{-\frac{x^2}{2d^2}} e^{ip_0 x/\hbar}$$

where p_0 is its momentum and d is a parameter with dimensions of length. What are the expectation values of x and p for the particle? Calculate $\Delta x \Delta p$. With respect to the uncertainty principle, what is special about a Gaussian?

(b) You run a STOP sign and get pulled over by a police officer. Suppose the law requires that your vehicle stop within one meter of the STOP sign. What is the maximum speed you can go (in km per hour) for you to still be able to argue that, by the uncertainty principle, you were actually at rest at the STOP sign? Take the mass of the vehicle to be (i) 1000 kg, (ii) the possible mass of a neutrino, 1 eV/c^2.

4. To have a probabilistic interpretation of the wave function $\psi(x,t)$ it is normalized: $\int dx \psi^*(x,t)\psi(x,t) = 1$. The RHS is a constant ($= 1$) and does not change with time. However, it is not obvious that the LHS is constant in time because the wave function depends on time.

(a) Prove (using the Schrödinger equation) that the normalization of the wave function is indeed independent of time.

(b) We have established that a wave function, once normalized, will remain normalized. However, conservation of total probability could in principle be achieved by having probability jump between very distant points. This is unphysical: we would like probability to be conserved locally. That is, we want the probability to increase at say point x only if the probability simultaneously decreases at some adjacent point $x \pm \epsilon$. A local conservation law is expressed in physics as a continuity equation. Show that in quantum mechanics, the probability density, $\rho(x,t) = \psi^*(x,t)\psi(x,t)$ also satisfies a continuity equation: $\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial x}$. Find $J(x,t)$.

5. Consider a particle of mass m confined to a three-dimensional cube of side a. Suppose that there is secretly an extra (fourth) spatial dimension. The extra dimension is curled-up into a circle of circumference $2\pi R$ i.e. functions on this space are periodic with period $2\pi R$.

   (a) Solve the stationary state Schrödinger equation for the wave function, $\psi(\vec{x}, \xi)$, where ξ ($\xi \sim \xi + 2\pi R$) is the coordinate over the extra dimension. Besides the usual spectrum of a particle in a box, there are now additional energy eigenvalues corresponding to excitations in the hidden extra dimension.

   (b) Determine the lowest new energy level that indicates the presence of the extra dimension. Compare this energy level with the energy levels of a particle in a box of size a without the extra dimension. If $R \ll a$, argue that one can observe the effects of this extra curled-up dimension if the probing energy is sufficiently high.

6. Consider a harmonic oscillator in two dimensions with

$$\hat{H} = \frac{\hat{p}_x^2 + \hat{p}_y^2}{2m} + \frac{1}{2}k\hat{x}^2 + \frac{1}{2}k\hat{y}^2$$

   Define

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + i\frac{\hat{p}_x}{m\omega}\right) \qquad \hat{b} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{y} + i\frac{\hat{p}_y}{m\omega}\right)$$

   (a) Evaluate all six commutators between $\hat{a}$, $\hat{a}^\dagger$, $\hat{b}$, and $\hat{b}^\dagger$. Express $\hat{H}$ in terms of $\hat{a}$, $\hat{a}^\dagger$, $\hat{b}$, and $\hat{b}^\dagger$.

   (b) Label the states by eigenvalues of $\hat{a}^\dagger\hat{a}$ and $\hat{b}^\dagger\hat{b}$. i.e. $\hat{a}^\dagger\hat{a}|m,n\rangle = m|m,n\rangle$ and $\hat{b}^\dagger\hat{b}|m,n\rangle = n|m,n\rangle$. What is the energy of the state $|m,n\rangle$? What is its degeneracy?

   (c) Degenerate energies indicate the presence of a symmetry. That is, there exists an operator that commutes with $\hat{H}$ but not with $\hat{a}^\dagger\hat{a}$ and $\hat{b}^\dagger\hat{b}$. Consider the operator

$$i\hbar\left(\hat{b}^\dagger\hat{a} - \hat{a}^\dagger\hat{b}\right)$$

Show that this operator commutes with the Hamiltonian. How does it transform the state $|mn\rangle$? Express the operator in terms of $\hat{x}$, $\hat{p}_x$, $\hat{y}$, and $\hat{p}_y$ and thereby explain what operator it is and the physical reason for why it commutes with the Hamiltonian.

7. Suppose a particle can be in only two possible positions, given by the position eigenstates $|1\rangle$ and $|2\rangle$. Suppose the Hamiltonian is

$$\hat{H} = E(|1\rangle\langle 2| + |2\rangle\langle 1|)$$

where E is a real constant with dimensions of energy.

(a) Express the operator $\hat{H}$ as a matrix in the $\{|1\rangle, |2\rangle\}$ basis.

(b) Find the energy eigenvalues and the corresponding normalized energy eigenstates.

(c) Suppose that, at time $t = 0$, the particle is known to be in position 2. What is the probability of finding the particle in position 1 as a function of time?

(d) For a general state $|\Psi\rangle$, how would you define the position-space wave function for this discrete system?

8. Consider the Mathematica output below. As used below, the command DSolve solves the differential equation in its first argument for $psi(z)$ (e is the energy, not Euler's number), while the command Series gives a series expansion of its first argument about either $z = 0$ or $z = \infty$, as indicated by its second argument. $C[1]$ and $C[2]$ are arbitrary constants that one always gets in solving linear differential equations, while AiryAi and AiryBi are special functions whose properties you do not need to know.

DSolve$[-psi''[z] + 2z\, psi[z] == 2\, e\, psi[z], psi[z], z]$
Output: $\left\{psi[z] \to \text{AiryAi}\left[\frac{-2e+2z}{2^{2/3}}\right] C[1] + \text{AiryBi}\left[\frac{-2e+2z}{2^{2/3}}\right] C[2]\right\}$
Series $\left[\text{AiryAi}\left[\frac{-2e+2z}{2^{2/3}}\right], \{z, 0, 1\}\right]$
Output: AiryAi $\left[-2^{1/3}e\right] + 2^{1/3}$AiryAiPrime $\left[-2^{1/3}e\right] z + O[z]^2$
Series $\left[\text{AiryBi}\left[\frac{-2e+2z}{2^{2/3}}\right], \{z, 0, 1\}\right]$

Output: $\text{AiryBi}\left[-2^{1/3}e\right] + 2^{1/3}\text{AiryBiPrime}\left[-2^{1/3}e\right] z + O[z]^2$

Series $\left[\text{AiryAi}\left[\frac{-2e+2z}{2^{2/3}}\right], \{z, \text{Infinity}, 1\}\right]$

Output: $e^{-\frac{2}{3}\sqrt{2}(-e+z)^{3/2}} \left(\frac{\left(\frac{1}{z}\right)^{1/4}}{2 2^{1/12}\sqrt{\pi}} + O\left[\frac{1}{z}\right]^{5/4} \right)$

Series $\left[\text{AiryBi}\left[\frac{-2e+2z}{2^{2/3}}\right], \{z, \text{Infinity}, 1\}\right]$

Output: $e^{\frac{2}{3}\sqrt{2}(-e+z)^{3/2}} \left(\frac{\left(\frac{1}{z}\right)^{1/4}}{2^{1/12}\sqrt{\pi}} + O\left[\frac{1}{z}\right]^{5/4} \right)$

Let the roots (zeroes) of AiryAi$[z]$ be α_n for $n = 0, 1, 2, 3...$ and of AiryBi$[z]$ be β_r for $r = 0, 1, 2, 3....$ (All values of α_n and β_r are negative.) Suppose the particle is restricted to the region $z \geq 0$.

(a) What familiar physical situation does this Schrödinger equation describe?

(b) What are the boundary conditions on the function $psi[z]$?

(c) What are the energy eigenfunctions?

(d) Find the energy eigenvalues in terms of α_n and/or β_r. Be sure to include the correct factors of all dimensionful constants that have been set to 1 in the Mathematica output.