PHY541 Problem Set 4. Due October 16, 2014.

1. Starting with $e^{-\beta\Omega} = \mathrm{tr} e^{-\beta(H_{\mathrm{op}}-\mu N_{\mathrm{op}})}$, calculate $\Omega(T,V,\mu)$ for a nonrelativistic non-interacting Fermi gas using the low temperature Sommerfeld expansion. Calculate N from your Ω, and solve for the Helmholtz free energy, $F = \Omega + \mu N$, as a function of T, V and N.

   Your result should be

$$F(T,V,N) = \frac{3}{5} N \epsilon_F \left[1 - \frac{5\pi^2 T^2}{12\epsilon_F^2} + \frac{\pi^4 T^4}{48\epsilon_F^4} + O\left(\frac{T^6}{\epsilon_F^6}\right)\right] \tag{1}$$

   where $\epsilon_F = \frac{\hbar^2 k_F^2}{2m}$, and $k_F^3 = \frac{6\pi^2 N}{s_d V}$.

2. A very simple (and oversimplified) model of a metal is as follows. Electrons are constrained to be inside a uniformly positively charged sphere of volume V. V goes to infinity in the thermodynamic limit. The positive charge density is en_b where e is the magnitude of the electronic charge.

   a. Calculate the energy of the electrons at $T = 0$ as a function of electronic density n and the volume V assuming i) the electronic density is always constant inside the sphere for large V; ii) the energy can be approximated by the sum of the ideal fermi gas energy and the electrostatic energy of the uniform electron and background densities. Note n_b is fixed, n can vary.

   b. Two different metals each initially neutral with different background densities are connected by a wire so that electrons can flow from one to the other. Label the systems by a superscript (1) or (2), so that the variables are $n^{(1)}$, $n_b^{(1)}$, $V^{(1)}$ and $n^{(2)}$, $n_b^{(2)}$, $V^{(2)}$. Show that the total free energy at $T = 0$ is minimized when the two metals' chemical potentials are equal. Evaluate the equilibrium electron densities $n^{(1)}$, $n^{(2)}$ in the thermodynamic limit.

   c. Repeat part b, but this time assume that the systems are uncharged (i.e. that $e = 0$.

   d. Do not turn in. Think about the results of parts c. and d. and how they relate to energy band bending at interfaces between solids.

3. A classical monatomic ideal gas of particles of mass m at a uniform temperature T is in a uniform gravitational field of acceleration t. You can think of it as being in a container of area A and height that goes to infinity. The density at a height $z = 0$ is n_0. Pollen grains, modeled as hard spheres of mass M and radius a, small compared to density variations of the gas, are introduced into the system. Assume the density of pollen grains is low so that not only can their interactions with each other can be neglected, but any effect a pollen grain has on the gas particles has negligible effect on the gas near other particles. This low density approximation is mathematically equivalent to looking at the statistical behavior of a single pollen grain and $N \to \infty$ gas atoms.

Calculate using the canonical ensemble for the N gas plus N_P pollen particles, $n_p(z)/n_p(0)$, where $n_p(z)$ is the pollen grain density.

Show that your result is consistent with the expected distribution of pollen if instead of the ideal gas we included a bouyancy force from Archimedes principle with the gas as the bouyant fluid. You can view your result as microscopic demonstration of Archimedes principle.

4. Give numerical estimates of the Fermi energy or temperature for the following systems

   a. The conduction electrons of sodium metal. The mass density of sodium is is $0.95\,\mathrm{g/cm}^3$. Assume there is 1 conduction electron per sodium atom. Calculate the Fermi energy in eV and in Kelvin.

   b. Nucleons (neutrons and protons) in a heavy nucleus. The charge radius shows that the number density is about $0.16\ \mathrm{fm}^{-3}$. Calculate the Fermi energy in MeV. Compare the fermi temperature to the temperature in the interior of the sun.

   c. ^{3}He atoms in the liquid phase, where the volume per atom is $50\,A^3$. At 1 atmosphere ^{3}He liquefies at 3.2 K.