# PHY6938 Mechanics Fall and Spring 2001
January 15, 2002

**1. A mass $m$ is hung on the end of a light unstretchable cord of length $L$, forming a simple pendulum. The mass is pulled to one side so as to make an angle $\theta_1$ with the vertical. a) Find the potential energy of the system relative to its lowest position.**

We can refer the value of the potential to any point, but a natural choice is the equilibrium point. If the pendulum is extended by an angle $\theta$, the projection of the cord onto a vertical line under the pivot point is $L\cos(\theta)$. This means that the height of the mass above the equilibrium point below the pivot point is $h = L - L\cos(\theta)$. So if the initial angle of displacement is $\theta_1$, the initial potential energy is

$$E = U_i = mgh = mgL[1 - \cos(\theta_1)]$$

Note that because the initial kinetic energy is zero, this is also the total energy of the system, and because the system is conservative, this is always the total energy.

**b) If the system is released at $\theta_1$ from rest, find the velocity of the mass when it is at its lowest position.**

If the mass is at the lowest position it is at the equilibrium position, and so the potential energy is zero. This means that the kinetic energy is simply the total energy

$$T = \frac{1}{2}mv^2 = E = mgL[1 - \cos(\theta_1)],$$

and so

$$v = \sqrt{2gL[1 - \cos(\theta_1)]}.$$

**c) Find the kinetic and potential energies at an angle $\theta_2 < \theta_1$ if the pendulum is released at $\theta_1$ from rest.**

At the angle $\theta_2$ the potential is

$$U = mgL[1 - \cos(\theta_2)]$$

and so

$$T = E - U = mgL[1 - \cos(\theta_1)] - mgL[1 - \cos(\theta_2)] = mgL[\cos(\theta_2) - \cos(\theta_1)]$$

**2. A uniform spherical ball of mass $M$ and radius $R$ is set rotating about a horizontal axis with angular speed $\omega_0$ and is placed gently on the floor. The initial center-of-mass velocity of the ball is zero. If the coefficient of sliding friction between the ball and the floor is $\mu$, find the speed of the center of mass of the ball when it begins to roll without slipping. (The moment of inertia of the ball about an axis passing through its center of mass is $I = \frac{2}{5}MR^2$.)**

Is energy conserved? The ball will continue to rotate and so slip when it contacts the floor, and the frictional force will cause it to accelerate. This means that energy is not conserved and so we will have to solve the problem using forces. First we must find the value of the frictional force. Since the normal force is $N = Mg$, we have

$$f = \mu N = \mu M g,$$

and as the ball is slipping this force points forward. This is the only horizontal force and is the net force, since the normal force balances the gravitational force (or else the ball would fall through the floor).

Newton's law then tells us that

$$M\ddot{x} = \frac{f}{M} = \mu g = \text{constant},$$

where $x$ is the center-of-mass position, and we see that as long as the ball is slipping on the floor there is a constant acceleration of the center of mass, and the speed of the center of mass is

$$v = \dot{x} = \mu g t.$$

The frictional force also causes a torque since it does not act through the center of the ball,

$$\tau = -Rf = -\mu M g R,$$

where the minus sign tells us that the torque acts to slow the rotation of the ball. The angular equivalent of Newton's law is

$$\ddot{\theta} = \dot{\omega} = \frac{\tau}{I} = \frac{\mu M g R}{2MR^2/5} = \frac{5}{2}\frac{\mu g}{R}.$$

This means that the angular speed of the ball decreases linearly from its initial value of $\omega_0$ according to

$$\omega = \omega_0 - \frac{5}{2}\frac{\mu g t}{R}.$$

How do we know when the ball starts to roll without slipping? The condition that it is not slipping is that the rate of linear motion of the center of mass is equal to $\omega R$, the speed at which the edge of the ball moves because of its rotation, so that

$$\begin{aligned}
\omega R &= \omega_0 R - \frac{5}{2}\mu g t = v = \mu g t \\
\omega_0 R &= \frac{7}{2}\mu g t \\
t &= \frac{2}{7}\frac{\omega_0 R}{\mu g}.
\end{aligned}$$

and at this time

$$v = \mu g t = \frac{2}{7}\omega_0 R$$

**3. Consider the Earth as a frame rotating about its axis with angular velocity $\vec{\omega}$. Particles moving relative to the Earth's surface are then subject to the Coriolis**

**force given by $\vec{F} = -2m(\vec{\omega} \times \vec{v}_r)$, where $\vec{v}_r$ is the velocity of the particle relative to the rotating frame, and $m$ is the mass of the particle. Choose the $z$–axis along the upwards vertical direction, the $y$–axis pointing North, and the $x$–axis to the East. Assume you are in the Northern hemisphere at a latitude such that the angle of the $z$–axis with $\vec{\omega}$ is $\theta$.**

**a) A particle moves under the influence of gravitation and the Coriolis force. Write the equations of motion in the $x$, $y$, and $z$ directions. Use the approximation that the component of the velocity in the $z$ direction is much larger than the components in the $x$ and $y$ directions, i.e. $|v_z| \gg |v_x|,\ |v_y|$.**

Since $\boldsymbol{\omega}$ points at an angle of $\theta$ to the $z$-axis (or "up"), and since the other component of $\omega$ is must be North, or along the $y$ axis, we must have that

$$\boldsymbol{\omega} = \omega \sin(\theta)\hat{\boldsymbol{y}} + \omega \cos(\theta)\hat{\boldsymbol{z}}.$$

Using the expression for the Coriolis force given above, we must have that

$$\mathbf{F}_c = -2m[\omega \sin(\theta)\hat{\boldsymbol{y}} + \omega \cos(\theta)\hat{\boldsymbol{z}}] \times [\dot{x}\hat{\boldsymbol{x}} + \dot{y}\hat{\boldsymbol{y}} + \dot{z}\hat{\boldsymbol{z}}].$$

We can evaluate this force using the rules $\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}} = \hat{\boldsymbol{z}}$ (and cyclic permutations, non-cyclic permutations have a minus sign), which gives

$$\begin{aligned}\mathbf{F}_c &= -2m\omega[-\sin(\theta)\dot{x}\hat{\boldsymbol{z}} + \sin(\theta)\dot{z}\hat{\boldsymbol{x}} + \cos(\theta)\dot{x}\hat{\boldsymbol{y}} - \cos(\theta)\dot{y}\hat{\boldsymbol{x}}] \\ &= -2m\omega\left\{[\sin(\theta)\dot{z} - \cos(\theta)\dot{y}]\hat{\boldsymbol{x}} + \cos(\theta)\dot{x}\hat{\boldsymbol{y}} - \sin(\theta)\dot{x}\hat{\boldsymbol{z}}\right\}.\end{aligned}$$

The only other force on the particle is the gravitational force $\mathbf{F}_g = -mg\hat{\boldsymbol{z}}$.

Newton's law for the particle's motion, in three dimensions is, therefore,

$$\begin{aligned}m\ddot{x} &= -2m\omega[\sin(\theta)\dot{z} - \cos(\theta)\dot{y}] \\ m\ddot{y} &= -2m\omega\cos(\theta)\dot{x} \\ m\ddot{z} &= -mg - 2m\omega[-\sin(\theta)\dot{x}].\end{aligned}$$

Now the only one of the components of the velocity of the particle which becomes appreciable is the $z$ component, because of the large gravitational force. Since $\omega$ is only one revolution per day, the Coriolis force is small and the other components of the velocity stay small. If we neglect all but $\dot{z}$ in the above expression we have the result

$$\begin{aligned}m\ddot{x} &= -2m\omega\sin(\theta)\dot{z} \\ m\ddot{y} &= 0 \\ m\ddot{z} &= -mg,\end{aligned}$$

and we see that the particle moves only vertically and in the $x$ direction in this approximation.

**b) The particle is dropped at rest from a height $h$ above the ground; it arrives at the ground with speed $v_0 = \sqrt{2gh}$. Find the magnitude and direction of the Coriolis deflection.**

We can easily solve for the velocity along the $z$ axis, $\dot{z} = -gt$, so that the differential equation for the $x$ motion becomes

$$\ddot{x} = -2\omega\sin(\theta)(-gt) = 2\omega\sin\theta gt,$$

and so integrating with $\dot{x}(0) = 0$ and $x(0) = 0$ yields

$$\begin{aligned} \dot{x} &= \omega \sin\theta g t^2 \\ x &= \frac{1}{3}\omega \sin\theta g t^3. \end{aligned}$$

Since the time taken to fall from a height $h$ is given by $gt^2/2 = h$, i.e. $t = \sqrt{2h/g}$, then we can evaluate $x$ at this time to give

$$x = \frac{1}{3}\omega \sin\theta g \sqrt{\frac{8h^3}{g}}.$$

Since this number is positive, the particle moves towards the East.

**c) The particle is next thrown vertically upward with an initial speed $v_0$, so that it reaches the maximum height $h$ [the same $h$ as in part (b)], and then falls back to the ground. Find the magnitude and direction of the Coriolis deflection in this case.**

The velocity required to reach a height $h$ is the same as that reached after falling from rest from a height $h$ (this is because the motion is *reversible*), then we know that during the flight of the particle

$$\dot{z} = v_0 - gt$$

and that $v_0 = g\sqrt{2h/g} = \sqrt{2gh}$. Then the above equation of motion in the $z$ direction becomes

$$\begin{aligned} \ddot{x} &= -2\omega \sin(\theta)[v_0 - gt] \\ \dot{x} &= -2\omega \sin(\theta) v_0 t + \omega \sin(\theta) g t^2 \\ x &= -\omega \sin(\theta) v_0 t^2 + \frac{1}{3}\omega \sin(\theta) g t^3. \end{aligned}$$

The time taken to come back to the ground is twice the time required to stop, or $t = 2\sqrt{2h/g}$, so

$$\begin{aligned} x &= -\omega \sin(\theta)\sqrt{2gh}\, 4\frac{2h}{g} + \frac{8}{3}\omega \sin(\theta) g (\frac{2h}{g})^{3/2} \\ &= -4\omega \sin(\theta)\sqrt{\frac{8h^3}{g}} + \frac{8}{3}\omega \sin(\theta)\sqrt{\frac{8h^3}{g}} = -\frac{4}{3}\omega \sin(\theta)\sqrt{\frac{8h^3}{g}}. \end{aligned}$$

Since this number is negative the particle was deflected to the West.

**d) Compare your results for parts (b) and (c).**

The particle went four times as far to the West as it went to the East while falling. Why wasn't it deflected the distance from (b) towards the West on its way up and the same distance to the East on its way down, resulting in no net displacement?

**4. A skydiver jumps from an airplane. We shall consider his "free" fall, before the parachute is opened. During the free fall, the magnitude of the frictional drag from the air is $F_{\rm d} = Kv^2$, where $v$ is the diver's speed.**

**a) Derive an expression for the terminal velocity of the skydiver, $v_T$. Express your answer in terms of the proportionality constant $K$, the acceleration of gravity, $g$, and the mass of the skydiver, $m$.**

The terminal velocity $v_t$ (this term has a poetic side to it, especially if the parachute doesn't open) is reached when the gravitational force (down) balances the drag force (up) so that

$$mg = Kv_t^2, \quad v_t = \sqrt{\frac{mg}{K}}.$$

**b) Derive an expression for the speed of the skydiver as a function of time, $t$.**

Newton's law can be thought of as a first order differential equation for $v$, which is the most useful form when you have a velocity-dependent force like the drag force. Written this way it reads (take $v > 0$ going downwards)

$$\begin{aligned} m\dot{v} &= mg - Kv^2 \\ \dot{v} &= g - \frac{K}{m}v^2 \\ \frac{dv}{dt} &= g - \frac{K}{m}v^2 = g\left(1 - \frac{K}{mg}v^2\right) \\ &= g\left(1 - v^2/v_t^2\right). \end{aligned}$$

This cannot be integrated in its present form since the unknown function $v(t)$ appears on the right-hand side. It is necessary to separate the dependence on $v$ and on $t$, by writing

$$\frac{dv}{1 - v^2/v_t^2} = g\, dt,$$

which can now be integrated by using tables or by noting that

$$\frac{1}{1 - v^2/v_t^2} = \frac{1}{2}\left(\frac{1}{1 - v/v_t} + \frac{1}{1 + v/v_t}\right)$$

and integrating to find (use $v(0) = 0$ at $t = 0$)

$$\int \frac{dv}{1 - v^2/v_t^2} = -v_t \ln(1 - v/v_t) + v_t \ln(1 + v/v_t) g \int dt = gt,$$

so that

$$\begin{aligned} \ln\left(\frac{1 + v/v_t}{1 - v/v_t}\right) &= 2\frac{gt}{v_t} \\ 1 + v/v_t &= e^{2gt/v_t}(1 - v/v_t) \\ v/v_t\left(1 + e^{2gt/v_t}\right) &= e^{2gt/v_t} - 1 \\ v &= v_t\left(\frac{e^{2gt/v_t} - 1}{e^{2gt/v_t} + 1}\right) \\ v &= v_t\left(\frac{1 - e^{-2gt/v_t}}{1 + e^{-2gt/v_t}}\right), \end{aligned}$$

where the last expression is the most useful as it obviously has $v(0) = 0$ and $v = v_t$ at large times.

**c) Find $v_T$ for a skydiver of mass $m = 80$ kg when $K = 0.25$ kg/m and $g = 9.8$ m/s$^2$.**

Plugging in yields 56 m/s for $v_t$ (which will be no fun if your parachute doesn't open).

**5. A curve on a highway has a radius of curvature $r = 100$ m. The road in the curve is banked at an angle of $\theta = 30°$ with the horizontal. If the coefficient of static friction is $\mu = 0.3$, what is the maximum speed with which a car can go through the curve without skidding?**

(solved in class).

**6. A satellite of mass $m$ is in a circular orbit of radius $R$ around the Earth. Assume $m$ is much smaller than the mass of the Earth.**
Useful constants:
Mass of the Earth: $M_E = 5.98 \times 10^{24}$ kg
Gravitational constant: $G = 6.67 \times 10^{-11}$ m$^3$kg$^{-1}$sec$^{-2}$

**a) What are the forces acting on the satellite? State the equilibrium condition.**

There is no equilibrium in the conventional sense of the word, as the satellite is always falling towards the center of the Earth, but there will be no change in the radius of the satellite (and so a circular orbit) if

$$\begin{aligned} \frac{GmM_E}{R^2} &= \frac{mv^2}{R} \\ v^2 &= \frac{GM_E}{R} \\ v &= \sqrt{\frac{GM_E}{R}}. \end{aligned}$$

**b) Obtain an expression for the period $\tau$ (the time it takes to make one complete orbit) as a function of the constants of the problem, e.g. the radius of the orbit.**

This is simply the distance traveled divided by the speed we just found,

$$\tau = \frac{2\pi R}{v} = \frac{2\pi R}{\sqrt{gM_E/R}} = 2\pi\sqrt{\frac{R^3}{gM_E}}.$$

**c) Briefly state and discuss the conditions for a geosynchronous orbit.**

To stay in the same place over the Earth all of the time, **two** conditions must be met; the first is that the period of the orbit must be exactly one day, and the second is that the satellite has to sit over the equator, or else it would be in the Northern hemisphere for half of the day and in the Southern hemisphere for the other half!

**d) What is the period of a geosynchronous orbit? At what radius $R$ from the center of the Earth can a satellite be placed in a geosynchronous orbit?**

According to our answer from (b) we have that

$$\begin{aligned}
\frac{R^3}{GM_E} &= \left(\frac{\tau}{2\pi}\right)^2 \\
R &= \left[GM_E\left(\frac{\tau}{2\pi}\right)^2\right]^{1/3} = 42,250 \text{ km} = 6.6\ R_E.
\end{aligned}$$

**7. A bucket of mass $M$ (when empty) is initially at rest and has a mass $m$ of water in it. The bucket is pulled straight upwards with a constant force $F$. The water in the bucket leaks out at a constant rate and with zero velocity relative to the bucket. At time $T$, the bucket becomes empty. Find the velocity of the bucket at time $T$.**

This is governed by the same kind of equation as problems with rockets rising under gravity, since the mass of the water+bucket changes and their is an external force (applied force plus gravity) acting on the system (bucket plus water).

The way to solve problems like these is to examine the system at some time $t$, and then again a time $dt$ later. Then if there is an external force $F_{\text{tot}}$ acting on the system during that time $dt$, the total momentum of the system changes by $F_{\text{tot}}dt$, since Newton's law can be written $dp/dt = F_{\text{tot}}$ or $dp = F_{\text{tot}}dt$.

Assume that the rate at which the water leaks out is $dm/dt = -u$, where $u$ is a positive number. At time $t$ the mass of the bucket plus water is then $M_{\text{tot}} = M + m - ut$, and assume that the velocity is $v$, so that the initial total momentum is

$$P(t) = M_{\text{tot}}v = (M + m - ut)v.$$

At time $t + dt$ the system consists of a mass $dm$ which has leaked out of the bucket and is still moving at speed $v$, and the remainder $M_{\text{tot}} - dm$ of the mass which is moving at speed $v + dv$, so

$$P(t + dt) = (M_{\text{tot}} - dm)(v + dv) + dm\, v.$$

If we neglect quantities second order in the differentials, then we find

$$P(t + dt) = M_{\text{tot}}(v + dv),$$

so that the change in the momentum is

$$dP = P(t + dt) - P(t) = M_{\text{tot}}dv.$$

Now use Newton's law as stated above to relate this change to the external force $F_{\text{tot}} = F - M_{\text{tot}}g$ acting on the system,

$$dP = F_{\text{tot}}dt = (F - M_{\text{tot}}g)dt = M_{\text{tot}}dv.$$

If we divide by $M_{\text{tot}}$ we get a left-hand side that depends only on $t$ and a right-hand side that depends only on $v$

$$\left(\frac{F}{M_{\text{tot}}} - g\right)dt = dv,$$

so that the result can be integrated to find

$$\int_0^T \left(\frac{F}{M+m-ut} - g\right) dt = \int_0^{v_f} dv,$$

where we have incorporated the initial condition $v(0) = 0$ in the lower limits and the final condition $v(T) = v_f$ in the upper limit. One other constraint has to be added to find the rate $u$ at which the mass changes: if in time $T$ all of the water of mass $m$ leaks out, then $u = m/T$. The integral is a logarithm and yields

$$\begin{aligned}
\int_0^T \left(\frac{F}{M+m-mt/T} - g\right) dt &= v_f \\
-\frac{FT}{m} \ln\left(M+m-mt/T\right)\Big|_0^T - gT &= v_f \\
-\frac{FT}{m} \ln\left(\frac{M}{M+m}\right) - gT &= v_f,
\end{aligned}$$

or

$$v_f = \frac{FT}{m} \ln\left(\frac{M+m}{m}\right) - gT.$$