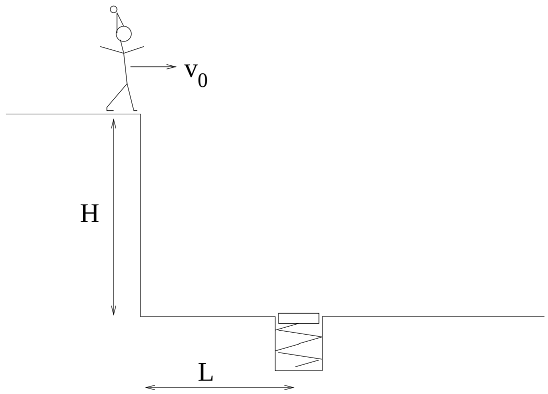
PHY6938 Mechanics Spring 98

1. A circus performer of mass M jumps onto a trampoline at a horizontal distance L from her initial position, and at a vertical distance H below it, as shown in the figure.



a) What should the initial velocity v_0 of the performer be (assume that it is in the horizontal direction) so that she lands on the trampoline?

The time taken to fall a distance H under constant vertical acceleration of g and with no initial vertical velocity is given by $H = gt^2/2$ so that $t = \sqrt{2H/g}$. Since this acceleration does not affect the horizontal motion, the horizontal distance traveled is $x = v_0 t$, so if the performer wishes to land on the trampoline at $x = L$ she must have an initial horizontal speed of

$$v_0 = \frac{L}{t} = L\sqrt{\frac{g}{2H}}.$$

b) The trampoline is supported by a huge spring of spring constant k. What is the maximum displacement h of the trampoline when compressed?

The easiest way to find this is to find the performer's vertical speed, convert this to a kinetic energy, and then use that all of this kinetic energy must be converted to potential energy in the spring at its maximum displacement. Note that the performer will also still have a horizontal speed of v_0; in the absence of other information we must assume that, as in other kinds of elastic collisions, the component of the velocity along the plane of contact, horizontal in this case, is conserved.

The vertical speed after falling a height H is $v_y = \sqrt{2gH}$, and so the kinetic energy which is converted to potential energy is $T = mv_y^2/2 = mgH$. Using that a potential energy of $kx^2/2$ is stored in a spring compressed by x, we have that at the point of maximum compression, when the performer is no longer moving vertically, $kx^2/2 = mgH$, and so

$$x = \sqrt{\frac{2mgH}{k}}.$$

c) How far away will the performer land (from the trampoline) when she hits the ground (ignore the height of the trampoline)?

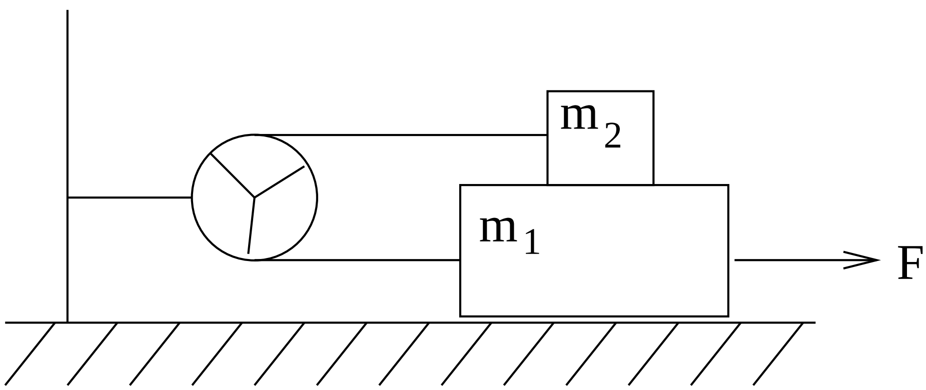
With no loss of energy during the collision she must be in flight for twice as long as she took to fall from H, so that she lands a distance $2L$ from the trampoline.

d) How long will the performer be in flight after she leaves the trampoline?

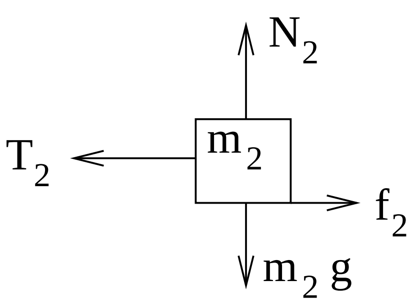
She will be in flight for

$$2t = 2\sqrt{\frac{2H}{g}}.$$

2. As shown in the diagram, two blocks with masses m_1 and m_2 are attached by an unstretchable rope around a frictionless pulley with radius r and moment of inertia I. There is no slipping between the rope and the pulley. The coefficient of kinetic friction between the blocks and between the blocks and the level surface is μ. A horizontal force F is applied to m_1. Find the acceleration.

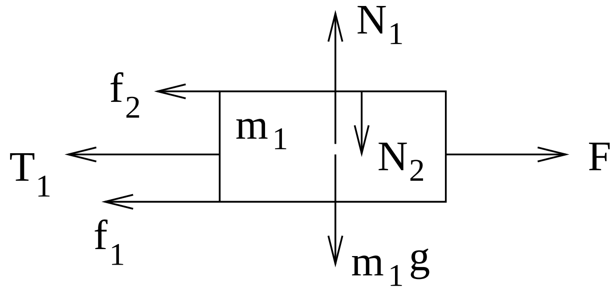


First draw a free-body diagram for m_1, m_2, and the pulley, being careful to label all the forces and find the line along which they act. Firstly let's examine m_2. It has a tension due to the string, gravity, a frictional force due its motion relative to m_1, which we can assume is to the left, and the normal force which m_1 exerts on it to stop it from moving vertically.

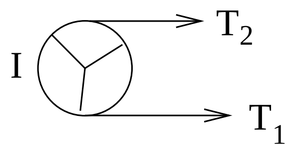


Note that the tension T_2 in the top half of the string does not have to equal that in the bottom half if we cannot ignore the pulley's moment of inertia and it is accelerating.

Now do the same for m_2.



Note that the two forces on m_1 due to m_2 are prescribed by Newton's third law to be equal and opposite to those on m_2 due to m_1. Finally, we have the free-body diagram for the pulley



We can now write down the equations of motion for these three objects. In the past we have written independent coordinates and written down constraint equations which we have formally added to our system of equations to solve for the acceleration. Here for simplicity just note that if m_1 moves to the right a distance x, which we will take as our coordinate, then m_2 moves to the left by x and the pulley rotates counterclockwise through an angle x/r. Then the equations of motion for m_1, m_2 and the pulley are

$$
\begin{aligned}
m_1\ddot{x} &= F - f_1 - f_2 - T_1 && (1)\\
m_2\ddot{x} &= T_2 - f_2 && (2)\\
I\frac{\ddot{x}}{r} &= rT_1 - rT_2, && (3)
\end{aligned}
$$

where the last involves the torques due to the two tensions. We also need to find the frictional forces by balancing all the vertical forces on m_1 and m_2

$$
\begin{aligned}
N_2 &= m_2 g\\
N_1 &= N_2 + m_1 g = (m_1 + m_2)g,
\end{aligned}
$$

so that $f_2 = \mu N_2 = \mu m_2 g$ and $f_1 = \mu N_1 = \mu(m_1 + m_2)g$. If we consider f_1 anf f_2 as known quantities, we now have three equations in the three unknowns T_1, T_2 and $\ddot{x}$ which we can solve as follows:

$$
\begin{aligned}
(1) + (2): \ (m_1 + m_2)\ddot{x} &= F - f_1 - 2f_2 - (T_1 - T_2) && (4)\\
(4) + (3)/r: \ \left(m_1 + m_2 + \frac{I}{r^2}\right)\ddot{x} &= F - f_1 - 2f_2, && (5)
\end{aligned}
$$

so that

$$
\ddot{x} = \frac{F - f_1 - 2f_2}{m_1 + m_2 + I/r^2} = \frac{F - \mu(m_1 + m_2 + 2m_2)g}{m_1 + m_2 + I/r^2} = \frac{F - \mu(m_1 + 3m_2)g}{m_1 + m_2 + I/r^2}
$$

3. A mass m at the end of a string swings in a vertical circle of radius R. The velocity v_{top} of the mass at the top of the circle is the minimum value for vertical circular motion.

a) Find $v_{\rm top}$.

The forces on the mass when it is at the top of the circle are the tension T in the string, which points downward, and the gravitational force mg acting on the mass. These are the only forces on the mass, and are not balanced–they are causing an acceleration of the mass which is moving in a circle. This acceleration is $v_{\rm top}^2/R$, so that

$$m\frac{v_{\rm top}^2}{R} = T + mg.$$

Obviously the minimum $v_{\rm top}$ attains when the T is a minimum, i.e. $T = 0$, so that the minimum speed is

$$v_{\rm top} = \sqrt{gR}.$$

b) Find the velocity of the mass when the string makes an angle of 120° with a vertical line through the center of the circle.

Assume the mass is rotating counterclockwise, then the angular position of the mass is at 60° before the 6 o'clock or vertically downward position. We are not going to be able to find the speed of the mass in the same way as above, because we do not have a way to specify the tension in the string. This is a conservative system, so we can find the energy of the mass using its kinetic energy at the top of its path and the potential energy it has lost in falling to this position,

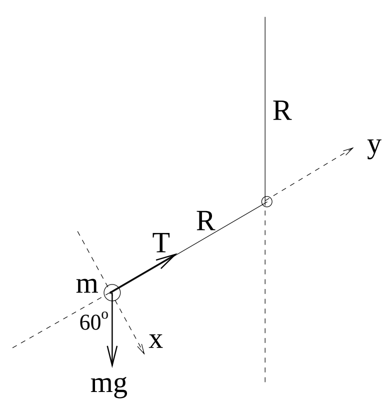
$$T_{\rm f} = \frac{1}{2}mv^2 = T_{\rm i} + U_{\rm i} - U_{\rm f} = \frac{1}{2}mv_{\rm top}^2 + 0 - \left\{-mgR[1+\cos(60°)]\right\},$$

where we have defined the potential to be zero at the top of the path and where $R[1-\cos(60°)]$ is the height through which the mass has fallen. Solving for v we have

$$\begin{aligned} v^2 &= v_{\rm top}^2 + 2gR\,[1+\cos(60°)] = gR + 2\frac{3}{2}gR = 4gR \\ v &= 2\sqrt{gR}. \end{aligned}$$

c) Find the tension in the string for the position of part b).

Let's draw a free-body diagram for the mass at this position.



Using a coordinate system as shown in the diagram, with y along the string and x along the direction of travel of the mass, we have

$$\begin{aligned} y:\ m\ddot{y} &= T - mg\cos(60°) \\ x:\ m\ddot{x} &= mg\sin(60°). \end{aligned}$$

Now since the mass is moving in a circle we know that $\ddot{y} = v^2/R$, where V is the velocity from part c). Solving for T, we have

$$T = m\frac{v^2}{R} + mg\cos(60°) = mg\left(4 + \frac{1}{2}\right) = \frac{9}{2}mg.$$

d) Find the acceleration of the mass at the position of part b).

We already know the y component of the acceleration, which is $\ddot{y} = v^2/R = 4mg$; the x component is given by the x equation above, $\ddot{x} = g\sin(60°) = g\sqrt{3}/2$. The magnitude of the acceleration is

$$a = \sqrt{16 + 3/4}\,g = \sqrt{67}\,g/2 = 4.09\,g,$$

and the angle it makes with the x axis is $\tan^{-1}(a_y/a_x) = \tan^{-1}(8/\sqrt{3}) = 70.5°$.

e) What is the angular momentum of the mass about the center of the circle when it is in the position of part b)?

The angular momentum is just

$$\mathbf{L} = \mathbf{R} \times \mathbf{p}$$

where $\mathbf{p} = m\mathbf{v}$ is tangential to the path and $\mathbf{R}$ points out along the string. Then the magnitude of $\mathbf{L}$ is

$$L = mvR = 2m\sqrt{gR^3}$$

and the direction is (assuming the mass is moving clockwise) out of the page.

f) What is torque about an axis through the center of the circle when the mass is at the position of part b)?

The torque is due to the tangential force $F_{\mathrm{t}} = mg\sin 60°$ which is applied at R from the rotation axis, so

$$\tau = mgR\sin 60° = \sqrt{3}mgR/2,$$

and $\boldsymbol{\tau} = \mathbf{R} \times \mathbf{F}_{\mathrm{t}}$ also points out of the page, so that the angular momentum and speed are increasing.

4. A spring of constant $k = 10$ N/m is attached to a wall and to a block of mass 1 kg. The coefficients of static and kinetic friction between the block and the floor are $\mu_{\mathrm{s}} = 0.25$ and $\mu_{\mathrm{k}} = 0.2$, respectively. The spring is compressed by 0.5 m and the block is released. How far does the block travel before coming to rest for the first time?

Firstly, let's check to see that the block moves! The force due to the spring is 5 N, and that due to friction is $\mu_s mg = 0.25 \cdot 1 \cdot 9.8$ N < 5 N, so the block does move.

This is **not** a damped harmonic oscillator. The general equation for such a system is

$$m\ddot{x} = -kx - b\dot{x},$$

where $b\dot{x}$ is the resistive or frictional force, but in our case we have a constant frictional force (once the block starts to move) of $f = \mu_k mg$. Our equation of motion is, therefore,

$$m\ddot{x} = -kx - \mu_k mg.$$

There are two approaches to solving this. One is to find a particular solution $x_{\rm p}(t)$ of the equation, essentially by guessing, and then add this to the general solution of the equation $m\ddot{x} = -kx$ without the inhomogeneity so that

$$x(t) = A\cos(\omega\, t) + B\sin(\omega\, t) + x_{\rm p}(t).$$

If we guess $x_{\rm p}(t) =$ constant, the LHS of the differential equation is zero and we can arrange the constant so the RHS is also zero, i.e. $x_{\rm p}(t) = -\mu_k mg/k$.

Another approach is to factor the RHS so that it reads

$$m\ddot{x} = -k\left(x + \frac{\mu_k mg}{k}\right),$$

and then note that if we define $x' = x + \mu_k mg/k$ we have $\ddot{x}' = \ddot{x}$ and then x' is just an oscillator.

Both of these approaches give the general solution

$$x(t) + \frac{\mu_k mg}{k} = A\cos(\omega\, t) + B\sin(\omega\, t),$$

where A and B are fixed by the initial conditions and $\omega^2 = k/m$. Since the initial velocity is zero we know that $B = 0$; call the initial displacement $x(0) = -D$ (the spring is compressed, hence the choice of sign gives x increasing as the spring uncompresses), then

$$x(0) + \frac{\mu_k mg}{k} = -D + \frac{\mu_k mg}{k} = A,$$

so that $A = -D + \mu_k mg/k$ and

$$x(t) = \left(-D + \frac{\mu_k mg}{k}\right)\cos(\omega\ t) - \frac{\mu_k mg}{k}.$$

The block stops when $\omega\, t = \pi$, so that the cosine is -1, and there we have

$$x(t) = -\left(-D + \frac{\mu_k mg}{k}\right) - \frac{\mu_k mg}{k} = D - 2\frac{\mu_k mg}{k}$$

so that the block has traveled

$$2D - 2\mu_k mg/k = \left(1.0 - 2\frac{0.2 \cdot 1 \cdot 9.8}{10}\right)\ \text{m} = 0.608\ \text{m}.$$

An alternate solution uses energy. The initial potential energy is $U_{\rm i} = kD^2/2$. Call the final displacement $x_{\rm f}$, then the final potential energy is $U_{\rm f} = kx_{\rm f}^2/2$. The energy lost to friction in going from a displacement of $-D$ to one of x_f is just the magnitude of the frictional force, which is a constant, times the distance traveled, or

$$W = \mu_k mg(D + x_{\rm f}).$$

Then we have

$$\begin{aligned} U_{\rm i} &= U_{\rm f} + W \\ \frac{kD^2}{2} &= \frac{kx_{\rm f}^2}{2} + \mu_k mg(D + x_{\rm f}). \end{aligned}$$

This is a quadratic equation which can be solved for x_f.