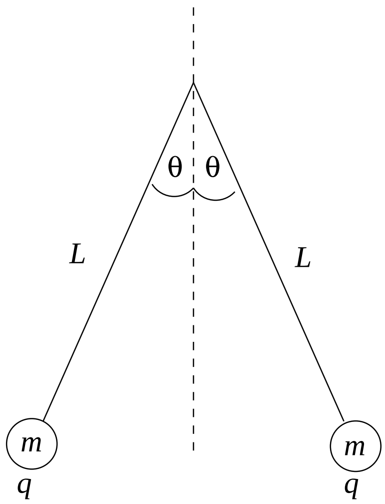
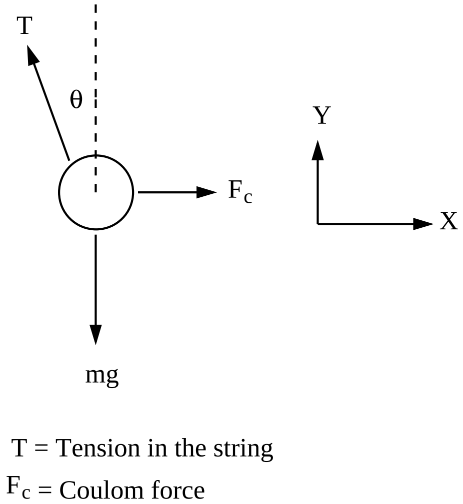
PHY6938 Proficiency Exam Spring 1997
February 25, 1997
E & M

1. **Two small spheres of mass m are suspended from a common point by threads of length L. When each sphere carries a charge q, each thread makes an angle θ with the vertical as shown in the figure. Find an expression for the charge q at equilibrium in terms of L, m, g, θ, and the Coulomb constant k.**



The spheres are in equilibrium, this means that the total force affecting these two spheres is equal zero. To solve the problem we draw free body diagram for one of the spheres.



Write equations of motion for one of the spheres. Using figure above we obtain

$$\begin{aligned} m\ddot{x} &= F_c - Tsin(\theta) \\ &= \frac{kq^2}{r^2} - Tsin(\theta) \\ &= \{r = 2Lsin(\theta)\} \end{aligned}$$

$$
\begin{aligned}
&= \frac{kq^2}{(2Lsin(\theta))^2} - Tsin(\theta) \\
&= 0
\end{aligned}
\tag{1}
$$

and

$$
\begin{aligned}
m\ddot{y} &= Tcos(\theta) - mg \\
&= 0.
\end{aligned}
\tag{2}
$$

From Eq.1 and 2 we obtain

$$
\begin{aligned}
Tsin(\theta) &= \frac{kq^2}{(2Lsin(\theta))^2} \\
T &= \frac{kq^2}{(2Lsin(\theta))^2 sin(\theta)} \qquad (3) \\
Tcos(\theta) &= mg \\
T &= \frac{mg}{cos(\theta)}. \qquad (4)
\end{aligned}
$$

From Eq.3 and 4 we get

$$
\begin{aligned}
\frac{kq^2}{(2Lsin(\theta))^2 sin(\theta)} &= \frac{mg}{cos(\theta)} \\
q^2 &= \frac{(2Lsin(\theta))^2 mgsin(\theta)}{kcos(\theta)}. \\
q &= \pm\, 2Lsin(\theta)\sqrt{\frac{mg}{kcos(\theta)}}.
\end{aligned}
\tag{5}
$$

Note, here $sin(\theta)$ and $cos(\theta)$ are always positive, since $\theta \leq 90$.

2. **After improvements in magnet technology reduce the cost, physicists propose building a circular accelerator around the Earth's circumference using bending magnets that provide a magnetic field of 2 T.**

   Remember that $c = 3 \times 10^8$ m/s, 1 eV $= 1.6 \times 10^{-19}$ J, the rest mass of the proton is 938 MeV/c^2, the radius of the Earth is 6.4×10^6 m, and the charge of a proton is 1.6×10^{-19} m.

   (a) **Find the momentum of a proton going around this accelerator.**
   The relativistic generalization of Newton's second law is

$$
\vec{F} = \frac{d\vec{p}}{dt}. \tag{6}
$$

   So for our accelerating proton we have

$$
\frac{d}{dt}(m\,\gamma(v)\,\vec{v}) = q\,\vec{v} \times \vec{F}. \tag{7}
$$

If we had used the non-relativistic formula we would find that

$$
\begin{aligned}
\frac{m\,v^2}{r} &= q\,v\,B \\
v &= \frac{e \times 2 \times 6.4 \times 10^6}{\dfrac{938 \times 10^6 \times e\,V}{(3 \times 10^8)^2}} \\
&= 1.29 \times 10^{15} \text{ m/s}, \qquad (8)
\end{aligned}
$$

which is not correct. Since v we have

$$
\begin{aligned}
\frac{d\vec{v}}{dt} &= -\frac{v^2}{r}\,\hat{r} \\
m\,\gamma(v)\,\frac{v^2}{r} &= q\,v\,B \\
m\,\gamma(v)\,v &= q\,B\,R_e \\
p &= q\,B\,R_e \\
&= 2.05 \times 10^{-12} \text{ kg m/s}. \qquad (9)
\end{aligned}
$$

(b) What is the kinetic energy of the protons orbiting in this accelerator? State any assumptions that you make.

To find kinetic energy T we need the total relativistic energy

$$
\begin{aligned}
E &= \left(p^2\,c^2 + m^2\,c^4\right)^{\frac{1}{2}} \\
T &= E - m\,c^2 \\
\text{but } p\,c &= 6.14 \times 10^{-4} \text{ J} = 3.84 \times 10^9 \text{ MeV} \\
\text{and } m\,c^2 &= 938 \times \text{MeV} \\
\text{this means that } p\,c &\gg m\,c^2 \\
\text{therefore } T \simeq E \simeq p\,c &= 3.84 \times 10^9 \text{ MeV} \qquad (10)
\end{aligned}
$$

(c) Find the period of rotation of the protons.

If we calculate the velocity we find that the velocity of the proton is very close the velocity of light c. Therefore

$$
T = \frac{2\,\pi\,R_e}{c} = 0.13 \text{ s}. \qquad (11)
$$

3. A sphere of radius a carries a charge density proportional to the distance from the center of the sphere, $\rho(r) = \kappa r$.

(a) Derive expressions for the electric field, both inside and outside of the sphere.

Outside the sphere we can use Newton's result for the field outside a spherically symmetric distribution of charge (he dealt with gravity, a different $\frac{1}{r^2}$ law force), which is that the field is the same as if all of the charge were at the center of the sphere

$$\vec{E} = \frac{k\,Q}{r^2}\,\hat{r}. \tag{1}$$

The total charge Q is given by

$$\begin{aligned} Q &= \int_0^a dV\,\rho(r) \\ &= 4\,\pi \int_0^a dr\,r^2\,\rho(r) \\ &= 4\,\pi\,\kappa \int_0^a dr\,r^3 \\ &= \pi\,\kappa\,a^4. \end{aligned} \tag{2}$$

Then

$$\vec{E} = k\,\frac{\pi\,\kappa\,a^4}{r^2}\,\hat{r} = \frac{\kappa\,a^4}{4\,\epsilon_0 r^2}\,\hat{r},\ \ r > a. \tag{3}$$

For $r < a$ this is modified to

$$\vec{E} = k\frac{Q_{end}}{r^2}\,\hat{r}, \tag{4}$$

where Q_{end} is the charge inside the radius r, as the field inside a shell of inner radius r and outer radius a is zero.

$$\begin{aligned} Q_{end} &= \int_0^r dV\,\rho(r') \\ &= 4\,\pi \int_0^r dr'\,r'^2\,\rho(r') \\ &= 4\,\pi\,\kappa \int_0^r dr'\,r'^3 \\ &= \pi\,\kappa\,r^4 \end{aligned} \tag{5}$$

so that

$$\vec{E} = k\,\frac{\pi\,\kappa\,r^4}{r^2}\,\hat{r} = \frac{\kappa\,r^2}{4\,\epsilon_0}\,\hat{r},\ \ r \le a. \tag{6}$$

(b) Derive expressions for the electric potential, again both inside and outside of the sphere. Use infinity as your reference point (i.e. the electric potential at $r = \infty$ is zero).

The calculation of the electric potential goes exactly similarly: using the Newton result outside the charge distribution we obtain

$$V = \frac{k\,Q}{r} = k\,\frac{\pi\,\kappa\,a^4}{r} = \frac{\kappa\,a^4}{4\,\epsilon_0\,r},\ \ r > a. \tag{7}$$

To find the potential inside we need to integrate from radius r to radius a,

$$V = V(r = a) + \int_r^a d\vec{r}\cdot\vec{E}, \tag{8}$$

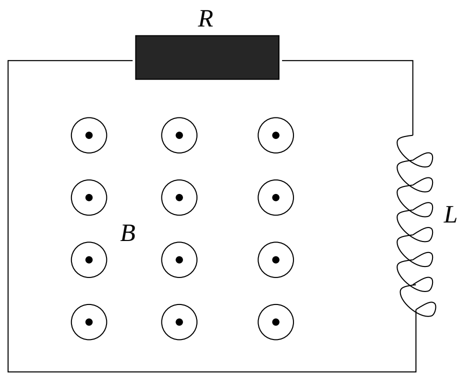
where $d\vec{r}\cdot\vec{E} > 0$ as we are imaging moving a test charge with $\vec{E}$, and so $V > V(r=a)$, as it should be.

$$\begin{aligned} V &= V(r=a) + \int_r^a dr' \, \frac{\kappa\, r'^2}{4\,\epsilon_0} \\ &= V(r=a) + \frac{\kappa}{4\,\epsilon_0}\,(\frac{a^3}{3} - \frac{r^3}{3}) \\ &= \frac{\kappa}{4\,\epsilon_0}\,(a^3 + \frac{a^3}{3} - \frac{r^3}{3}) \\ &= \frac{\kappa}{12\,\epsilon_0}\,(4\,a^3 - r^3). \end{aligned} \tag{9}$$

(c) If the sphere's radius is $a = 2.0$ cm, and the total charge carried by the sphere is 50 μC, find the magnitude of the electric potential at the surface of the sphere [note: $\epsilon_0 = 8.85\times 10^{-12}$ C^2/(Nm^2)].

$$\begin{aligned} V(a) &= \frac{\kappa\, a^3}{4\,\epsilon_0} \\ &= -\frac{k\,Q}{r} \\ &= 2.25\times 10^7 \text{ V}. \end{aligned} \tag{10}$$

4. A loop of wire of resistance R, containing a coil of self-inductance L encloses an area A. A spatially uniform magnetic field is applied perpendicular to the plane of the loop with the following time dependence: for $t<0$ the field is zero; for $0<t<t_0$, $B(t) = kt$, while for $t>t_0$ the field now remains constant at $B_0 = kt_0$. Calculate the current I in the loop for all time $t>0$, given that $I=0$ for $t=0$.



The flux Φ is given by

$$\Phi = BA = \begin{cases} k\,t\,A & t<t_0 \\ k\,t_0\,A & t>t_0 \end{cases}$$

Therefore we have

$$\varepsilon = -\frac{d\Phi}{dt} = \begin{cases} k\,A & t<t_0 \\ 0- & t>t_0 \end{cases}$$

where the minus sign reminds us that the current flows to oppose the increasing B, or in the clockwise direction. The total voltage Kirchoff equation is

$$
\begin{aligned}
V_R + V_L &= |\varepsilon| \\
I\,R + L\,\frac{dI}{dt} &= |\varepsilon| \qquad (11)
\end{aligned}
$$

We can solve this by adding a particular solution $I_p = \dfrac{|\varepsilon|}{R}$ to the solution $I_h = k\,\exp[\dfrac{-R\,t}{L}]$ of the homogeneous equation $L\,\dfrac{dI}{dt} = -\,R\,I$, and then matching to initial conditions which are $I(t=0) = 0$, we have

$$
\begin{aligned}
I &= \frac{|\varepsilon|}{R} + k\,\exp[\frac{-R\,t}{L}] \\
I(0) &= \frac{|\varepsilon|}{R} + k \Rightarrow\ k = -\,\frac{|\varepsilon|}{R} \\
I(t) &= \frac{k\,A}{R}\left(1 - e^{-\dfrac{R\,t}{L}}\right),\ 0 < t < t_0 \qquad (12)
\end{aligned}
$$

Once there is no longer an emf the current simply decays exponentially from its value at $t = t_0$, $I(t_0) = \frac{k\,A}{R}\,(1 - \exp[-\dfrac{R\,t}{L}])$, so

$$
\begin{aligned}
I(t) &= \frac{k\,A}{R}\left(1 - \exp[-\frac{R\,t_0}{L}]\right)\,\exp[-\frac{R\,(t-t_0)}{L}] \\
I(t) &= \frac{k\,A}{R}\left(\exp[\frac{R\,t_0}{L}] - 1\right)\,\exp[-\frac{R\,t}{L}] \qquad (13)
\end{aligned}
$$