① a. The amplitude will be

$$\left(\frac{m}{2\pi\hbar i T}\right)^2 \Big\{ \exp\left[\frac{im}{2\hbar T}\left((x_f-x_i)^2 + (\bar{y}-y)^2\right)\right]\exp\left[\frac{im}{2\hbar T}\left((x_f-x_i)^2 + (0-y-s)^2\right)\right]$$

$$+ \exp\left[\frac{im}{2\hbar T}\left((x_f-x_i)^2 + (\bar{y}-y-s)^2\right)\right]\exp\left[\frac{im}{2\hbar T}\left((x_f-x_i)^2 + (0-y)^2\right)\right]\Big\}$$

$$= \left(\frac{m}{2\pi\hbar i T}\right)^2 \exp\left(\frac{im}{\hbar T}(x_f-x_i)^2\right)$$

$$\times \Big\{ \exp\left[\frac{im}{2\hbar T}\left(2y^2 - 2(s+\bar{y})y + (\bar{y}^2+s^2)\right)\right]$$

$$\exp\left[\frac{im}{2\hbar T}\left(2y^2 - 2(s+\bar{y})y + (\bar{y}+s)^2\right)\right] \Big\}$$

$$+ \exp\left[\frac{im}{2\hbar T}\left(2y^2 - 2(s+\bar{y})y\right.\right.$$

$$\text{Prob.} = |\text{Amp.}|^2$$

$$= \left(\frac{m}{2\pi\hbar T}\right)^4 \Big\{ 2 + \exp\left[\frac{im}{2\hbar T}\left(\bar{y}^2 + s^2 - (\bar{y}+s)^2\right)\right] + \exp\left[\frac{im}{2\hbar T}\left(-\bar{y}^2 - s^2 + (\bar{y}+s)^2\right)\right]\Big\}$$

$$= \left(\frac{m}{2\pi\hbar T}\right)^4 \Big\{ 2 + \exp\left[\frac{im}{2\hbar T}\left(\bar{y}^2 + s^2 - (\bar{y}+s)^2\right)\right] + \exp\left[-\frac{im}{2\hbar T}\left(\bar{y}^2 + s^2 - (\bar{y}+s)^2\right)\right]\Big\}$$

$$= \left(\frac{m}{2\pi\hbar T}\right)^4 2\left(1 + \cos\left[\frac{m}{2\hbar T}(-2\bar{y}s)\right]\right)$$

$$\text{Prob.} = \frac{m^4}{8\pi^4\hbar^4T^4}\left(1 + \cos\left[\frac{m\bar{y}s}{\hbar T}\right]\right)$$

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b. $$P_{total} = \int_{s=0}^{D} ds \int_{y=0}^{D-s} dy \, \frac{m^4}{8\pi^4\hbar^4T^4}\left(1 + \cos\left[\frac{m\bar{y}s}{\hbar T}\right]\right)$$

$$= \frac{m^4}{8\pi^4\hbar^4T^4}\int_{s=0}^{D} ds\,(D-s)\left(1 + \cos\left[\frac{m\bar{y}s}{\hbar T}\right]\right)$$

$$= \frac{m^4}{8\pi^4\hbar^4T^4}\left[-\frac{\hbar^2T^2}{m^2\bar{y}^2}\cos\left(\frac{m\bar{y}s}{\hbar T}\right) + \frac{D\hbar T}{m\bar{y}}\sin\left(\frac{m\bar{y}s}{\hbar T}\right) - \frac{s\hbar T}{m\bar{y}}\sin\left(\frac{m\bar{y}s}{\hbar T}\right) + Ds - \frac{s^2}{2}\right]_{s=0}^{D}$$

$$= \frac{m^4}{8\pi^4 \hbar^4 T^4}\left(- \frac{\hbar^2 T^2}{m^2 \bar{y}^2} \cos\left(\frac{m\bar{y}D}{\hbar T}\right) + \frac{D^2}{2} + \frac{\hbar^2 T^2}{m^2 \bar{y}^2} \right)$$

$$= \frac{m^2}{8\pi^4 \hbar^2 T^2 \bar{y}^2}\left(1 - \cos\left(\frac{m\bar{y}D}{\hbar T}\right) + \frac{1}{2}\left(\frac{m\bar{y}D}{\hbar T}\right)^2 \right)$$

c. The functional form

Prob. of coincident detection $\sim (\ )\frac{1}{\bar{y}^2}\left(1 - \cos(k\bar{y}D) + \frac{1}{2}(k\bar{y}D)^2\right)$

parallels the HBT result for photons, which are most likely for small $\bar{y}$ and give periodic interference fringes as $\bar{y}$ is increased from zero.

The periodicity coming from the cosine will recur every $\Delta\bar{y} \sim \frac{2\pi\hbar T}{mD}$, allowing D to be determined.

(2) a. Yes, with $m \times n$ matrix full of zero entries as the identity.

b. No, zero has no inverse.

c. No, associativity fails.

d. No, there is no inverse element.

e. Yes, with $\mathrm{diag}(\underbrace{1, \dots, 1}_{n \text{ times}})$ as the identity elt.

(3) a. $\langle J_x \rangle = \langle lm | \frac{1}{2}(J_+ + J_-) | lm \rangle = 0$. Also $\langle J_y \rangle = 0$.

$$\langle J_x^2 \rangle = \frac{1}{4} \langle lm | (J_+ + J_-)^2 | lm \rangle$$

$$= \frac{1}{4} \langle lm | (J_+ J_- + J_- J_+) | lm \rangle$$

$$= -\frac{1}{4} \langle lm | (-J_+ J_- - J_- J_+) | lm \rangle$$

$$= \langle lm | (\frac{1}{2i}(J_+ - J_-))^2 | lm \rangle$$

$$= \langle J_y^2 \rangle$$

$\langle J_z \rangle = m\hbar$, $\langle J_z^2 \rangle = m^2 \hbar^2$

$\langle J^2 \rangle = \hbar^2 l(l+1)$

Then since $J^2 = J_x^2 + J_y^2 + J_z^2$,

$$\hbar^2 l(l+1) = 2\langle J_x^2 \rangle + m^2 \hbar^2$$

$$\text{or } \langle J_x^2 \rangle = \langle J_y^2 \rangle = \frac{\hbar^2}{2}(l(l+1) - m^2)$$

So finally,

$$\Delta J_x = \sqrt{\langle J_x^2 \rangle - \langle J_x \rangle^2} = \sqrt{\frac{\hbar^2}{2}(l(l+1) - m^2)} = \Delta J_y$$

$$\Delta J_z = \sqrt{m^2 \hbar^2 - m^2 \hbar^2} = 0.$$

b. LHS: $\Delta J_x \Delta J_y = \frac{\hbar^2}{2}(l(l+1) - m^2)$

RHS: $\frac{1}{2i}\langle [J_x, J_y] \rangle = \frac{1}{2i}\langle lm | i\hbar J_z | lm \rangle$

$= \frac{\hbar^2}{2} m$

Since $l \geq |m|$, it follows that

$l(l+1) \geq m^2 + |m|$

or $\frac{\hbar^2}{2}(l(l+1) - m^2) \geq \left| \frac{\hbar^2}{2} m \right|$

which means that $\Delta J_x \Delta J_y \geq \left| \frac{1}{2i}\langle [J_x, J_y] \rangle \right|$

is satisfied here.

c.

$$\langle lm | J_x | lm \rangle = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, \langle lm | \theta\, \phi \rangle \, J_x \, \langle \theta\, \phi | lm \rangle$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, Y^*_{lm}(\theta, \phi) \underbrace{\left[i\hbar \left(\sin\phi \frac{\partial}{\partial\theta} + \cos\phi \cot\theta \frac{\partial}{\partial\phi} \right) \right]}_{=\hat{J}_x} \times Y_{lm}(\theta, \phi)$$

Use $Y_{lm}(\theta, \phi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} \, P_l^m(\cos\theta) \exp(im\phi)$

and $\frac{d}{d\theta} P_l^m(\cos\theta) = \frac{l \cos\theta \, P_l^m(\cos\theta) - (l+m) P_{l-1}^m(\cos\theta)}{\sin\theta}$

$\Longrightarrow \langle lm | J_x | lm \rangle = 0$.

Similarly, plug in $J_y = -i\hbar \left(\cos\phi \frac{\partial}{\partial\theta} - \sin\phi \cot\theta \frac{\partial}{\partial\phi} \right)$

to show $\langle J_y \rangle = 0$ and $J_z = -i\hbar \, \partial/\partial\phi$ to

show $\langle J_z \rangle = m\hbar$.

(4) a. Can write spherical harmonics in Cartesian coordinates & get:

$$yz = i r^2 \sqrt{\frac{2\pi}{15}} \left(Y_2^{-1} + Y_2^{1} \right)$$

$$xy = i r^2 \sqrt{\frac{2\pi}{15}} \left(Y_2^{-2} - Y_2^{2} \right)$$

$$xz = r^2 \sqrt{\frac{2\pi}{15}} \left(Y_2^{-1} - Y_2^{1} \right)$$

So $\psi = N r^2 \sqrt{\frac{2\pi}{15}} e^{-\alpha r^2} \left[(1+i) Y_2^{-1} + (i-1) Y_2^{1} + i Y_2^{-2} - i Y_2^{2} \right]$

$\langle 00 | \psi \rangle = (\ldots) \int_0^\infty dr \int d\Omega \; Y_0^0 \; \psi$

$\uparrow$ linear combination of Y_ℓ^m's with $\ell \neq 0$

$\Rightarrow P(\ell = 0) = |\langle 00 | \psi \rangle|^2 = \underline{\underline{0}}$.

by orth. of sph. harmonics.

b. Similarly, since ψ is a linear combination of all Y_ℓ^m's with $\ell = 2$, any measurement of J^2 will give $\hbar^2 2(2+1) = 6\hbar^2$.

$\therefore \; \underline{\underline{P(6\hbar^2) = 1}}$.

c. Look at coeffs in linear combn, (square mag.)

	Rel. Prob:	Prob:
$m = -2$:	1	1/6
-1 :	2	1/3
0 :	0	0
1 :	2	1/3
2 :	1	1/6
	6	

(5) a.

$$R_{34} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\eta & \sin\eta \\ 0 & 0 & -\sin\eta & \cos\eta \end{pmatrix}$$

This is generated by $\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \equiv \xi_{34}$

since $R_{34} = \exp(i\eta\xi_{34})$ as can be verified by series expansion.

b. $[J_{ij}, J_{kl}] = -[x_i\partial_j - x_j\partial_i,\ x_k\partial_l - x_l\partial_k]$

$$= (x_j\partial_i - x_i\partial_j)(x_k\partial_l - x_l\partial_k) - (x_k\partial_l - x_l\partial_k)(x_j\partial_i - x_i\partial_j)$$

$$= x_j\partial_i x_k\partial_l - x_j\partial_i x_l\partial_k - x_i\partial_j x_k\partial_l + x_i\partial_j x_l\partial_k$$

$$- x_k\partial_l x_j\partial_i + x_k\partial_l x_i\partial_j + x_l\partial_k x_j\partial_i - x_l\partial_k x_i\partial_j$$

$$= x_j\delta_{ik}\partial_l \cancel{+ x_j x_k\partial_i\partial_l} - x_j\delta_{il}\partial_k \cancel{- x_j x_l\partial_i\partial_k}$$

$$- x_i\delta_{jk}\partial_l \cancel{- x_i x_k\partial_j\partial_l} + x_i\delta_{jl}\partial_k \cancel{+ x_i x_l\partial_j\partial_k}$$

$$- x_k\delta_{lj}\partial_i \cancel{- x_k x_j\partial_l\partial_i} + x_k\delta_{li}\partial_j \cancel{+ x_k x_i\partial_l\partial_j}$$

$$+ x_l\delta_{kj}\partial_i \cancel{+ x_l x_j\partial_k\partial_i} - x_l\delta_{ki}\partial_j \cancel{- x_l x_i\partial_k\partial_j}$$

$$= \delta_{ik}(x_j\partial_l - x_l\partial_j) + \delta_{jk}(x_l\partial_i - x_i\partial_l)$$

$$+ \delta_{jl}(x_i\partial_k - x_k\partial_i) + \delta_{il}(x_k\partial_j - x_j\partial_k)$$

i.e. commute only when all of i, j, k, l are different $\rightarrow$ can choose at most 2 mutually commuting generators.

c. $$\left(-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)+\frac{\hbar^2 l(l+2)}{2mr^2}-\frac{Q^2}{r^2}-E\right)R(r)=0.$$

Define dimensionless vars → put into Mathematica → solutions will have divergent enough behavior to not be normalizable.

(6) Massless case → set $\beta = 0$ in original derivation.

Then (*) $i\hbar \frac{\partial \psi}{\partial t} = -i\hbar \alpha_j \partial_j \psi$. Apply $i\hbar \frac{\partial}{\partial t}$:

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = \hbar^2 \alpha_j \partial_j \frac{\partial \psi}{\partial t}$$

eliminate using (*)

(implied sum over 1,2,3 for repeated indices)

$$\Rightarrow -\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = \hbar^2 \alpha_j \partial_j \left(-\alpha_k \partial_k \psi \right)$$

$$= -\hbar^2 \alpha_j \alpha_k \partial_j \partial_k \psi$$

Schrodinger eqn / relativistic relation $E^2 = p^2 c^2$

$\Rightarrow$ equate RHS with $\hat{p}^2 c^2 \psi$:

$$-\hbar^2 \alpha_j \alpha_k \partial_j \partial_k \psi = -\hbar^2 c^2 \nabla^2 \psi$$

Since ∇^2 contains the $\partial_j \partial_k$ terms with $j = k$, this requires:

$$\alpha_j^2 = c^2$$

$$\alpha_j \alpha_k = -\alpha_k \alpha_j \quad (j \neq k).$$

Obvi not satisfied by real numbers. The Pauli spin matrices work, though $\Rightarrow \psi$ has 2 components.

[One can also get this from the Dirac eqn by taking $m \to 0$ and seeing that it decouples into two eqns for 2-component spinors.]

(7) a. Solutions will have plane-wave form (solns to Dirac also solve K.G.)

$$\psi(x) = \psi(0)\, e^{-ip\cdot x}.$$

Writing $\psi(0) = N\begin{pmatrix} u \\ v \end{pmatrix}$, and in this case using

$$\not{p} = E\gamma^0 - p^1\gamma^1 - p^2\gamma^2 - p^3\gamma^3$$
$$= E\gamma^0 - p\gamma^3,$$

the Dirac eqn gives

$$\begin{pmatrix} (E-m)\mathbb{1} & -p\sigma_3 \\ p\sigma_3 & -(E+m)\mathbb{1} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0$$

or

$$(E-m)\, u - p\sigma_3 v = 0$$
$$p\sigma_3 u - (E+m) v = 0$$

$$\Rightarrow \quad v = \frac{p}{E+m}\sigma_3 u$$

For $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\quad v = \frac{p}{E+m}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$= \begin{pmatrix} p/(E+m) \\ 0 \end{pmatrix}$$

Then $\quad \psi = N\begin{pmatrix} 1 \\ 0 \\ p/(E+m) \\ 0 \end{pmatrix}$

Normalize by $\quad \psi^\dagger\psi = 2E$

$$\Rightarrow \quad 2E = N^2\left(1 + \frac{p^2}{(E+m)^2}\right)$$

$$\text{or} \quad N^2 = \frac{2E}{1 + \frac{p^2}{(E+m)^2}} = \frac{2E\,(E+m)^2}{(E+m)^2 + p^2}$$
$$= \frac{2E(E+m)^2}{2E(E+m)}$$
$$= E+m$$

So, $N = \sqrt{E+m}$

& the normalized soln. is $\psi = \begin{pmatrix} \sqrt{E+m} \\ 0 \\ p/\sqrt{E+m} \\ 0 \end{pmatrix} e^{-ip\cdot x}$

b. $\exp\left(-\frac{1}{4}[\gamma^0,\gamma^3]\eta\right)\begin{pmatrix}\sqrt{2m}\\0\\0\\0\end{pmatrix}$

$$= \exp\left(\begin{bmatrix}\frac{\eta}{2}\sigma_3 & 0\\ 0 & -\frac{\eta}{2}\sigma_3\end{bmatrix}\right)\begin{pmatrix}\sqrt{2m}\\0\\0\\0\end{pmatrix}$$

mathematica

$$= \left(\cosh(\eta/2)\begin{pmatrix}1&0\\0&1\end{pmatrix} - \sinh(\eta/2)\begin{pmatrix}0&\sigma_3\\-\sigma_3&0\end{pmatrix}\right)\begin{pmatrix}\sqrt{2m}\\0\\0\\0\end{pmatrix}$$

$$= \cosh(\eta/2)\begin{pmatrix}\sqrt{2m}\\0\\0\\0\end{pmatrix} - \sinh(\eta/2)\begin{pmatrix}0\\0\\-\sqrt{2m}\\0\end{pmatrix}$$

$$= \sqrt{\frac{E+m}{2m}}\begin{pmatrix}\sqrt{2m}\\0\\0\\0\end{pmatrix} + \sqrt{\frac{E-m}{2m}}\begin{pmatrix}0\\0\\\sqrt{2m}\\0\end{pmatrix}$$

$$= \begin{pmatrix}\sqrt{E+m}\\0\\\sqrt{E-m}\\0\end{pmatrix}$$

To compare w/ part a, note that

$$\sqrt{E-m} = \sqrt{\frac{E^2-m^2}{E+m}} = \frac{p}{\sqrt{E+m}}$$

so

$$= \begin{pmatrix}\sqrt{E+m}\\0\\p/\sqrt{E+m}\\0\end{pmatrix}$$