1. A meteoroid of mass M happens to be falling directly towards the Earth, and eventually hits Lake Ella in Tallahassee, Florida. Assume that the initial velocity of the meteoroid is negligible (when it is very far from the Earth) and ignore the air friction.

a) What is the kinetic energy of the meteoroid upon impact? (The radius of the Earth is $R_{\rm Earth} \simeq 6.4 \times 10^3$ km.)

The gravitational potential energy of the meteorite is

$$U(R) = -\frac{GMM_{\rm E}}{R},$$

Where R is the distance from the **center** of the Earth. This follows because Newton showed that if we are outside a spherically symmetric mass distribution we can find the gravitational potential by placing all of the mass at the center of the sphere.

If the meteorite starts at a long (infinite) distance from the Earth, then the change in potential energy is

$$U(R_{\rm E}) - U(\infty) = -\frac{GMM_{\rm E}}{R_{\rm E}} + 0 == -\frac{GMM_{\rm E}}{R_{\rm E}}.$$

We do not need to know G or the mass of the Earth $M_{\rm E}$ to find this, since we know the value $g(R_{\rm E})$ of the gravitational acceleration at the surface of the Earth, which is

$$g(R_{\rm E}) = \frac{GM_{\rm E}}{R_{\rm E}^2},$$

so that

$$-\frac{GMM_{\rm E}}{R_{\rm E}} = -Mg(R_{\rm E})R_{\rm E}.$$

Since all of the potential energy which is lost is converted to kinetic energy, the kinetic energy is

$$T = Mg(R_{\rm E})R_{\rm E}.$$

b) What is the minimum mass of the meteoroid necessary to completely vaporize the water from Lake Ella? Assume that the total volume of water in Lake Ella is $V = 10^5$ m^3, and that the ambient temperature is $T_{\rm air} = 20°$ C. (The specific heat of water is $C = 1$ cal/g·K; 1 cal = 4.186 Joules; the heat of vaporization is $Q_{\rm vapor} = 539$ cal/g.)

We will assume that all of the kinetic energy is converted to heat after the collision. We need to find the energy required to vaporize the lake, which is

$$\begin{aligned} E &= \rho V\left(C[T_{\rm boil} - T] + Q_{\rm vapor}\right) \\ &= 10^3\,\frac{\rm kg}{\rm m^3}\cdot 10^5\ {\rm m}^3\left(1\times 10^3\,\frac{\rm cal}{\rm kg\,K}\cdot[80\ {\rm K}] + 539\times 10^3\frac{\rm cal}{\rm kg}\right)\cdot 4.186\frac{\rm J}{\rm cal} \\ &= 2.59\times 10^14\ {\rm J}. \end{aligned}$$

Comparing this with our kinetic energy we see that we need the meteoroid to have mass

$$M = \frac{2.59 \times 10^14 \text{ J}}{9.8 \frac{\text{m}}{\text{s}^2} \cdot 6.4 \times 10^6 m} = 4.13 \times 10^6 \text{ kg}.$$

c) If the typical density of meteoritic material is $\rho \simeq 5 \times 10^3$ kg/m^3, estimate the diameter of the meteoroid.

The volume is $V = \pi d^3/6$ so that $d^3 = 6V/\pi = 6M/(\pi\rho)$ and so we have

$$d = \left(\frac{6 \cdot 4.13 \times 10^6 \text{ kg}}{\pi \cdot 5 \times 10^3}\right)^{\frac{1}{3}} = 19 \text{ m}.$$

2. Water enters a house through a pipe with an inside diameter of 2.0 cm, at an absolute pressure of 4×10^5 Pa (about 4 atm). The pipe leading to a second floor bathroom faucet 5 m above has an inside diameter of 1 cm. When the flow velocity at the inlet pipe is 4 m/sec find:

a) The flow velocity at the open bathroom's faucet.

For this part we need the equation of continuity for this incompressible fluid, which simply states that the product of the cross sectional area of the pipe and the velocity of the fluid is a constant. Since the cross sectional area of the pipe goes down a factor of four, the velocity must go up a factor of four, so the velocity is

$$v_2 = \frac{A_1}{A_2} v_1 = 4v_1 = 16 \ \frac{\text{m}}{\text{s}}.$$

b) The pressure in the bathroom pipe.

Here we must use Bernoulli's equation which expresses conservation of energy for the fluid at a particular point in the fluid. If we label the point in the 2 cm pipe just before the 1 cm pipe as position (1), and the point just after this as position (2), and the point at the end of the 1 cm pipe (before the faucet) as position (3), then we have

$$P_3 + \frac{1}{2}\rho v_3^2 + \rho g h_3 = P_2 + \frac{1}{2}\rho v_2^2 + \rho g h_2 = P_1 + \frac{1}{2}\rho v_1^2 + \rho g h_1$$

Now using $h_2 = h_1$, $v_2 = v_3 = 4v_1$ we have

$$\begin{aligned} P_3 &= P_2 + \rho g (h_2 - h_3) \\ P_2 &= P_1 + \frac{1}{2}\rho\left(v_1^2 - v_2^2\right), \end{aligned}$$

so that

$$\begin{aligned} P_3 &= P_1 + \frac{1}{2}\rho\left(v_1^2 - v_2^2\right) + \rho g (h_2 - h_3) \\ &= 4 \times 10^5 \text{ Pa} + \frac{1}{2}10^3 \ \frac{\text{kg}}{\text{m}^3}\left(4^2 - 16^2\right) \ \frac{\text{m}}{\text{s}^2} + 10^3 \ \frac{\text{kg}}{\text{m}^3} \cdot 9.8 \ \frac{\text{m}}{\text{s}^2} \cdot (-5) \text{ m} = 2.31 \times 10^5 \text{ Pa}. \end{aligned}$$

c) The pressure in the bathroom's pipe when the faucet is turned off.

Here we use exactly the same formula above but with the velocity terms set to zero, so

$$P_3 = P_1 + \rho g(h_2 - h_3) = 4 \times 10^5 \text{ Pa} + 10^3 \, \frac{\text{kg}}{\text{m}^3} \cdot 9.8 \, \frac{\text{m}}{\text{s}^2} \cdot (-5) \text{ m} = 3.51 \times 10^5 \text{ Pa}.$$

3a) Is the force $\mathbf{F} = (-6xyz,\ z(z^2 - 3x^2),\ 3y(z^2 - x^2))$ conservative?

A sufficient condition for a force field to be conservative is that it can be written as the gradient of a potential, and that potential is independent of the time; this is true if a time independent force field is *irrotational*, i.e. it has zero curl

$$\boldsymbol{\nabla} \times \mathbf{F} = \mathbf{0}.$$

Let's calculate the curl of $\mathbf{F}$:

$$\begin{aligned}
(\boldsymbol{\nabla} \times \mathbf{F})_x &= \nabla_y F_z - \nabla_z F_y = \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} = 3(z^2 - x^2) - (3z^2 - 3x^2) = 0 \\
(\boldsymbol{\nabla} \times \mathbf{F})_y &= \nabla_z F_x - \nabla_x F_z = \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} = -6xy - (-6xy) = 0 \\
(\boldsymbol{\nabla} \times \mathbf{F})_z &= \nabla_x F_y - \nabla_y F_x = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = -6xz - (-6xz) = 0,
\end{aligned}$$

so that indeed $\mathbf{F}$ is irrotational, can be written as minus the gradient of a potential U, and so is conservative.

b) What is the potential energy associated with this force?

We need to find U from the three conditions $-\boldsymbol{\nabla} U(x, y, z) = \mathbf{F}$, which are

$$\begin{aligned}
-\frac{\partial U}{\partial x} &= -6xyz \\
-\frac{\partial U}{\partial y} &= z(z^2 - 3x^2) \\
-\frac{\partial U}{\partial x} &= 3y(z^2 - x^2).
\end{aligned}$$

We can indefinitely integrate these equations in turn, being careful to note that the constant of integration in the x integral, for example, is in general a function $f(y, z)$ of y and z:

$$\begin{aligned}
U(x, y, z) &= \int 6xyz \, dx + f(y, z) = 3x^2yz + f(y, z) \\
U(x, y, z) &= \int z(3x^2 - z^2) \, dy + g(x, z) = zy(3x^2 - z^2) + g(x, z) = 3x^2yz - yz^3 + g(x, z) \\
U(x, y, z) &= \int 3y(x^2 - z^2) \, dz + h(x, y) = 3x^2yz - yz^3 + h(x, y).
\end{aligned}$$

These three results for $U(x, y, z)$ have to be the same function: equating the last two forms we have $g(x, z) = h(x, y)$ so that both have to independent of both y and z if this relation is to be true for all y and z. So write $g(x,z) = h(x,y) = i(x)$. Now equating the first two forms we have that

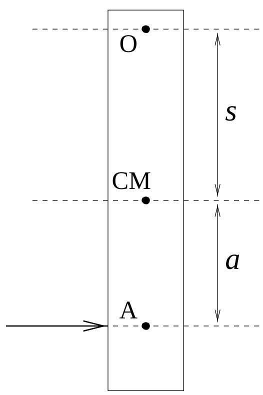
$$f(y,z) = -yz^3 + i(x),$$

which shows immediately that $i(x)$ must be a constant if this is to hold for all values of x, and so finally we have that

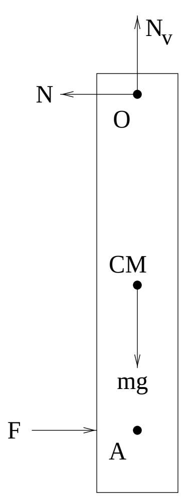
$$U(x,y,z) = 3x^2yz - yz^3 + c,$$

where c is a constant.

4 A uniform rod of mass M and length $2b$ is pivoted at a point O, a distance s above the center of mass. The rod is struck with a rapid impulsive force perpendicular to the rod at a point A, a distance a below the center of mass. The magnitude of the impulse is $P = F\Delta t$. Find the value of a such that there is no reaction at the pivot during the impact. (The moment of inertia of a uniform rod about an axis through its center perpendicular to its length is $I = ML^2/12$.



As usual, we should draw a free body diagram for the rod, which is shown below. Note that the pivot is going to impart a vertical normal force $\mathbf{N}_\mathrm{v}$ on the rod which stops it from falling, as well as a normal force $\mathbf{N}$ counter to the direction of the applied (impulsive) force $\mathbf{F}$, as shown.



Now we will write equations which for the translational acceleration of the center of mass in terms of the applied forces, and for the angular acceleration about the fixed point, which is the pivot point O. Note that in general the angular acceleration can be worked out around any point, but if there is a fixed point it is usually best to work around that point. Take the positive x direction to be as shown, and positive rotations θ around the pivot to be counterclockwise.

$$\begin{aligned} M\ddot{x} &= F - N & (1) \\ I_{\mathrm{O}}\ddot{\theta} &= (s+a)F, & (2) \end{aligned}$$

where I_{O} is the moment of inertia around the pivot point. We can find this in terms of the moment of inertia I around the center of mass using the parallel axis theorem,

$$I_{\mathrm{O}} = I + Ms^2 = M\left[\frac{(2b)^2}{12} + s^2\right] = M\left[\frac{b^2}{3} + s^2\right],$$

where the sign of the extra term is fixed by knowing that the moment of inertia around O is larger than that around the center of mass.

We have two equations in the three unknowns $\ddot{x}$, $\ddot{\theta}$, and N. To solve these for N we need to add a third equation, which is the constraint $\theta = sx$, so that

$$\ddot{\theta} = s\ddot{x}. \tag{3}$$

This now gives three equations in three unknowns that we can solve for N,

$$N = F - M\ddot{x} = F - Ms\ddot{\theta} = F - \frac{Ms(s+a)F}{M\left[\frac{b^2}{3} + s^2\right]} = F\,\frac{\frac{b^3}{3} + s^2 - s^2 - sa}{\frac{b^2}{3} + s^2} = F\,\frac{\frac{b^3}{3} - sa}{\frac{b^2}{3} + s^2},$$

so that we see that we can make the hroizontal normal force zero if we choose

$$a = \frac{b^2}{3s}.$$