Here are some general notes on physics

Cody L. Petrie

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1 Expand the plane wave in terms of spherical harmonics

Here I'm going to show how to expand a plane wave in terms on spherical harmonic s (or legendre polynomials) and then show what this goes to as $r \to \infty$. Expanding a plane wave in the spherical harmonic basis gives

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_l C_l Y_l^0(\theta). \tag{1}$$

There is no ϕ dependance here because the plane wave only depend on θ the angle between $\mathbf{k}$ and $\mathbf{r}$, since the plane wave is propagating in the z direction. Now solving for the expansion coefficients C_l using the orthogonality relationship

$$\int_0^\pi \int_0^{2\pi} Y_l^m Y_{l'}^{m'*} d\Omega = \delta_{ll'}\delta_{mm'}, \tag{2}$$

gives us

$$C_l = 2\pi \int_0^\pi e^{ikr\cos\theta} Y_l^0(\theta) \sin\theta d\theta. \tag{3}$$

Now writting the spherical harmonics in terms of legendre polynomials

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1(l-m)!}{4\pi(l+m!)}} P_l^m(\cos\theta) e^{im\phi}, \tag{4}$$

or for $m = 0$,

$$Y_l^0(\theta) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta) \tag{5}$$

we get,

$$C_l = \pi \int_0^\pi e^{ikr\cos\theta} \sqrt{\frac{2l+1}{4\pi}} P_l(\cos\theta) \sin\theta d\theta. \tag{6}$$

Now if you use an identity relating the spherical bessel functions of the first kind to the legendre polynomials (an identity which I found online and proved with Mathematica)

$$j_l(kr) = \frac{1}{2i^l} \int_0^\pi e^{ikr\cos\theta} P_l \cos\theta, \tag{7}$$

we can get an expansion of the plane wave in terms of spherical bessel functions and legendre polynomials.

$$\boxed{e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_l 2i^l \sqrt{\pi}\sqrt{2l+1} j_l(kr) Y_l^0(\theta)} \tag{8}$$

Often we want to look at these things in the scattering or radiation limit where r is large. We can use the expansion of the spherical bessel function as given by Jackson eq. 9.89 to be

$$\lim_{r->\infty} j_l(kr) = \frac{1}{kr} \sin\left(kr - \frac{l\pi}{2}\right) = \frac{i}{2kr}(e^{-i(kr-\frac{l\pi}{2})-e^{i(kr-\frac{l\pi}{2})}}). \tag{9}$$

We can thus write the expansion in the asymptotic limit as

$$\boxed{e^{i\mathbf{k}\cdot\mathbf{r}} \approx \sum_l \frac{\sqrt{\pi}}{kr}\sqrt{2l+1}\, i^{l+1} \left(e^{-i(kr-\frac{l\pi}{2})} - e^{i(kr-\frac{l\pi}{2})}\right) Y_l^0(\theta)}. \tag{10}$$

This is equation 3.1.1 in Siemens.