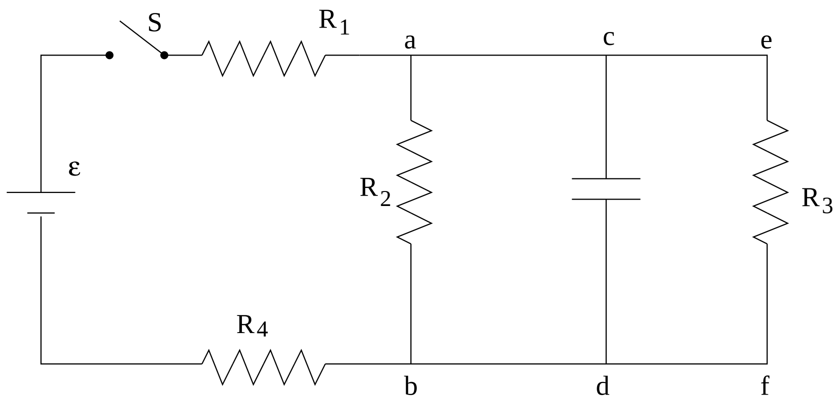
1. When switch S is closed, a current flows in the circuit which changes with time. Eventually, after a sufficiently long time, a steady state is reached in which all current is constant. Unless otherwise instructed, give all answers for this steady situation. For the circuit shown:



$\varepsilon = 10$ V
$R_1 = R_4 = 500\ \Omega$
$R_2 = 6$ kΩ
$R_3 = 1.2$ kΩ
$C = 6\ \mu$F

a) What is the battery current?

After a long time there is no longer any change in the capacitor's charge, as it has reached its final voltage. Therefore that branch of the circuit carries no current. We then have R_2 and R_3 in parallel, and this effective resistance in series with R_1 and R_4, so that

$$R_{\text{tot}} = R_1 + \frac{R_2 R_3}{R_2 + R_3} + R_4 = \left(0.5 + \frac{6 \cdot 1.2}{7.2} + 0.5\right)\ \text{k}\Omega = 2\ \text{k}\Omega,$$

so that the current is

$$I = \frac{10\ \text{V}}{2\ \text{k}\Omega} = 5\ \text{mA}.$$

b) What is the potential difference between points a and b (i.e. $V_\text{a} - V_\text{b}$)?

$$V_{ab} = I \frac{R_2 R_3}{R_2 + R_3} = 5 \cdot 1\ \text{mA} \cdot \text{k}\Omega = 5\ \text{V}.$$

c) What is the potential difference between points c and d? Between e and f?

As a, c and e are all connected by a wire, as are b, d, and f, then the potentials at a, c, and e are the same as are those at b, d, and f, so that both these potential differences are also 5 V.

d) If switch S is reopened after a long time, write the equation for the time dependence of the current in the section of the circuit containing the capacitor.

The capacitor will discharge through the two resistors R_2 and R_3, with an equivalent resistance of $R_{\rm eq} = 1$ kΩ. The equivalent circuit is a series RC circuit with an initial voltage of $V_0 = 5$ V on the capacitor. The differential equation for such a circuit is given by the capacitor relation $q = CV_C$ so that $I = dq/dt = C\,dV_C/dt$, and since $I = V_C/R_{\rm eq}$ we have

$$C\,\frac{\mathrm{dV_C}}{\mathrm{dt}} = \frac{V_C}{R_{\rm eq}},$$

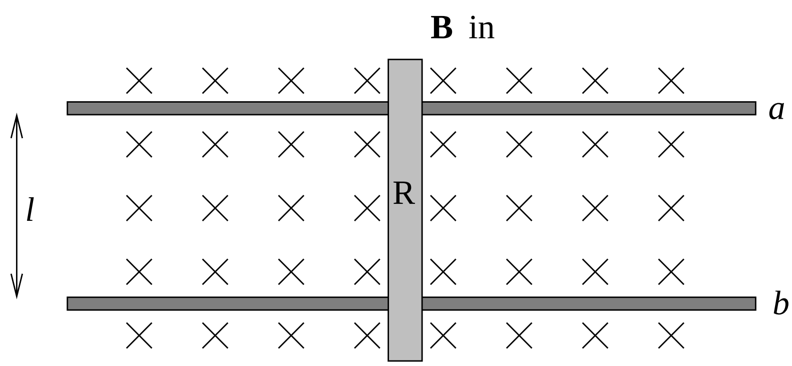
which has the solution

$$V_C = V_{C0}e^{-t/(R_{\rm eq}C)},$$

where V_{C0} is the initial voltage on the capacitor. The current is therefore

$$I = C\,\frac{dV_C}{dt} = \frac{V_{C0}}{R_{\rm eq}}e^{-t/(R_{\rm eq}C)} = \frac{5\ \mathrm{V}}{1\ \mathrm{k\Omega}}e^{-t/(1\ \mathrm{k\Omega}\cdot 6\ \mu\mathrm{F})} = 5\,e^{-t/(6\ \mathrm{ms})}\ \mathrm{mA}.$$

2. In the figure below, the rod has a resistance R and mass M and the rails have negligible resistance. A capacitor with charge Q_0 and capacitance C is connected between points a and b such that the current in the rod is downward. The rod is at rest at time $t = 0$.



a) Write the equation of motion for the rod on the rails.

Since the current in the rod is downward, by the right-hand rule the magnetic force on the rod is to the right, so use the coordinate x to give the position of the rod on the rails, where x increases going to the right. Then the force on the rod is $F = IlB$ and acts to the right, where $I = -dQ/dt$ is the rate of change of the charge on the capacitor. This means that

$$M\ddot{x} = M\frac{dv}{dt} = IlB. \tag{1}$$

Note that we can't integrate this yet as we don't know I as a function of time. Since we do not need to find the position of the rod on the rails but only its velocity in the next part, we can view this as an equation for the velocity v.

There is an induced emf in the circuit made by connecting the capacitance to the rails, because the flux cutting through the circuit is $\phi = Blx$ and so the induced emf is $\epsilon =$

$-d\phi/dt = -Blv$, where the minus sign reminds us that the emf acts to oppose the increase in the flux. This means that the emf acts to make a current counterclockwise to produce a **B** field out of the page. The equation for the circuit is

$$V_C - Blv - IR = 0,$$

where V_C is the voltage across the capacitor. Using $V_C = Q/C$ and substituting for I from (1), we have

$$\frac{Q}{C} - Blv - \frac{MR}{lB}\frac{dv}{dt} = 0. \tag{2}$$

This is not yet a diffential equation for v as it involves Q, an unknown function of time. However, we can relate $Q(t)$ to the velocity of the rod by integrating (1), which reads

$$I = -\frac{dQ}{dt} = \frac{M}{lB}\frac{dv}{dt},$$

so that by integrating we have

$$Q = Q_0 - \frac{M}{lB}v,$$

where we have used that at $t = 0$ we have $Q(0) = Q_0$ and $v(0) = 0$.

Substituting this into (2) we get an equation we can now solve for v,

$$\frac{1}{C}\left(Q_0 - \frac{M}{lB}v\right) - Blv - \frac{MR}{lB}\frac{dv}{dt} = 0,$$

so that

$$\frac{dv}{dt} = -\left(\frac{1}{RC} + \frac{B^2l^2}{MR}\right)v + \frac{Q_0lB}{MRC}. \tag{3}$$

b) Show that the terminal speed of the rod on the rails is related to the final charge on the capacitor.

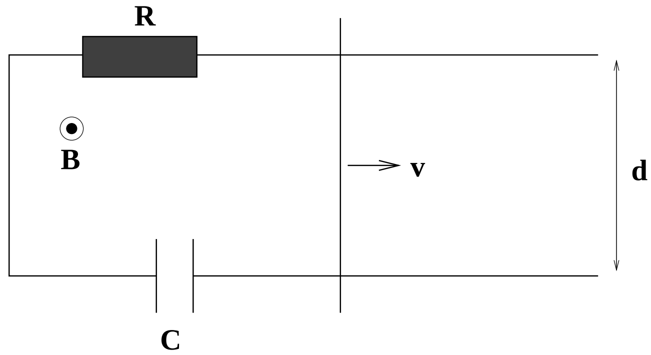
From Eq. (2) we have that when $dv/dt = 0$, i.e. when we have reached the terminal velocity, we have $Q_f/C - BLv_f = 0$, so that

$$v_f = \frac{Q_f}{BlC}.$$

We can find the terminal velocity v_f by setting $dv/dt = 0$ in Eq. (3) and solving for v, with the result

$$v_f = \frac{Q_0lB}{M + B^2l^2C}.$$

3) A closed circuit loop consists of a resistor R, a capacitor C, two parallel conducting tracks a distance d apart, and a rigid wire that can slide along the tracks. A uniform magnetic field B is applied perpendicular to the loop. Initially there is no current in the loop and the wire is at rest. At $t = 0$ it starts to move to the right with constant speed v, and it comes to a sudden stop at time $t = t_0$. Determine the charge on the capacitor for $t > 0$.



Even though this looks like a very similar problem to the previous one (they were on consecutive exams, not the same one!), it is actually quite different. The solution to the equation of motion in the previous problem gives an exponentially increading v which reaches a terminal velocity. What is intended here is that *an external influence* is making the rod slide on the rails with a constant force, and then stop. Our job is therefore much simpler than before–we simply have to integrate the (clockwise) induced current in the loop to find the charge placed on the capacitor by the induced emf. Since the area of the loop and so the flux is increasing linearly with time,

$$\phi = \phi(0) + Bd(vt),$$

we have that the induced emf is a constant

$$\epsilon = -\frac{d\phi}{dt} = -Bdv.$$

The current in the loop is *not* constant since the capacitor builds up a charge and so there is a potential drop across the capacitor. The equation of the circuit is

$$|\epsilon| = Bdv = IR + \frac{q}{C},$$

so that

$$I = \frac{dq}{dt} = \frac{Bdv}{R} - \frac{q}{RC}.$$

The solution to this equation is an exponentially rising charge q. We can find the charge after a long time by setting $dq/dt = 0$, so that $q(\infty) = BCdv$. Since the time constant is RC we know that the solution is

$$q(t) = BCdv\left(1 - e^{-t/RC}\right), \quad 0 < t < t_0.$$

This is the solution only up to $t = t_0$; after that time the wire stops so that there is no source of emf and the circuit is just a charged capacitor discharging across a resistor, where the charge decreases exponentially with the *time after* t_0, with the same time constant RC. The initial charge is

$$q(t_0) = BCdv\left(1 - e^{-t_0/RC}\right),$$

so that our solution looks like

$$\begin{aligned} q(t) &= q(t_0)e^{-(t-t_0)/RC} = BCdv\left(1 - e^{-t_0/RC}\right)e^{-(t-t_0)/RC} \\ &= BCdv\left(e^{t_0/RC} - 1\right)e^{-t/RC}, \quad t_0 < t. \end{aligned}$$

4) A series AC circuit is operated at half its resonance frequency. The coil resistance is 40 Ω, its inductance is 80 mH, and the capacitance is 0.4 μF.

a) What is the resonance frequency of the circuit?

This is a series circuit so the current $I(t)$ in *every* part of the circuit is the same function of time. To analyse this circuit we have to add the voltages across the resistor, capacitor, and inductor, and the sum of these is the applied emf $\epsilon(t)$. Solve this by assuming a current $I(t) = I_0 \cos(\omega t)$ flowing into the capacitor. Then the voltage drop across the resistor is

$$V_R = IR.$$

To find the voltage drop across the capacitor we use $q = CV_C$ where $dq/dt = I$, so that V_C is given by

$$V_C = \frac{1}{C}\int dt I(t) = \frac{I_0}{\omega C}\sin(\omega t).$$

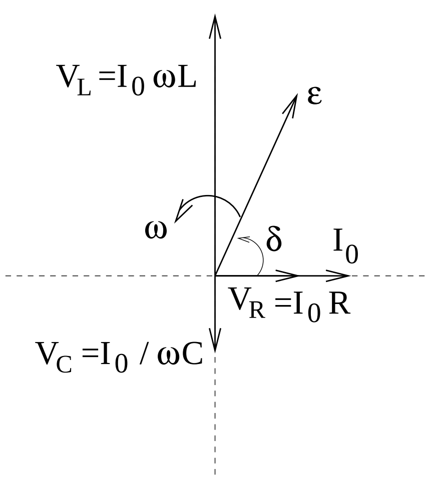
Similarly, to find the voltage drop across the the inductor we use

$$V_L = L\frac{dI}{dt} = -\omega L I_0 \sin(\omega t).$$

Then the equation for our circuit is that the applied emf $\epsilon(t)$ is related to the current $I(t) = I_0 \cos(\omega t)$ by

$$\epsilon(t) = I_0 R\cos(\omega t) + \frac{I_0}{\omega C}\sin(\omega t) - I_0 \omega L \sin((\omega t).$$

This equation can be represented by a diagram with vectors rotating counterclockwise at frequency ω whose x components are the physical quantities in the circuit, as follows:



We can now add the voltages like vectors. Note that the **voltage in the inductor leads the current (ELI)** and that the **voltage in the capacitor lags the current (ICE).** The 'effective' resistance of the inductor is $X_L = \omega L$ and that of the capacitor is $X_C = 1/\omega C$. The voltage in the circuit is, therefore,

$$\epsilon(t) = I_0\left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2}\cos(\omega t + \delta) =: I_0 Z\cos(\omega t + \delta),$$

where the impedance Z and the phase angle δ are

$$\begin{aligned} Z &= \left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2} \\ \delta &= \tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right). \end{aligned}$$

The resonant frequency of the circuit is the frequency ω_0 at which the impedance is a minimum, i.e. when

$$\omega_0 L = \frac{1}{\omega_0 C},$$

so that

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{32\times 10^{-9}s}} = 890 \text{ Hz}.$$

b) What is the impedance of the circuit at half the resonance frequency?

At $\omega = \omega_0/2$ the net reactance is

$$X = \frac{\omega_0 L}{2} - \frac{2}{\omega_0 C} = \omega_0\left(\frac{L}{2} - \frac{2}{\omega_0^2 C}\right) = \frac{1}{\sqrt{LC}}\left(\frac{L}{2} - 2L\right) = -\frac{3}{2}\sqrt{\frac{L}{C}} = -1.5\sqrt{2\times 10^5}\,\Omega = -671\,\Omega,$$

and so the impedance is

$$Z = \sqrt{40^2 + 671^2}\,\Omega = 672\,\Omega.$$

c) What is the phase angle between the voltage and the current at half the resonance frequency?

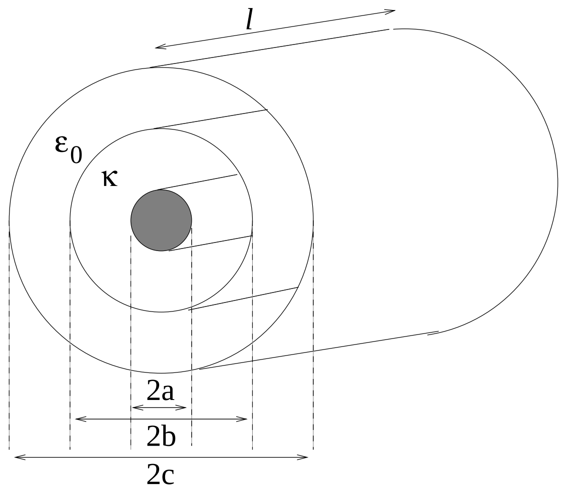
Since the net reactance is negative, referring to our diagram we see that the voltage across the capacitor *exceeds* that across the inductor, not as we have assumed in the diagram. That means that the voltage *lags* the current by the phase angle

$$\delta = \tan^{-1}\left(\frac{X}{R}\right) = \tan^{-1}\left(\frac{671}{40}\right) = 86.6^{\circ}.$$

d) What should the impedance of the source feeding this cicruit be for maximum power transfer?

The maximum power is transferred to the circuit if the impedance of the source matches that of the circuit, i.e. it also has the magnitude 672 Ω, with the same X and R components.

5) A coaxial capacitor of length l consists of an inner conductor of radius a, a cylindrical dielectric with dielectric constant k of inner radius a and outer radius b, and a shell of free space of inner radius b and outer radius c. Beyond c is a conducting surface holding total charge q. The inner conductor carries charge $-q$.



a) Find the radial component of the electric field E in all regions of space.

We will use Gauss's law, suitably modified in the presence of a dielectric. This is

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{encl}}}{\epsilon} = \frac{Q_{\text{encl}}}{\kappa \epsilon_0},$$

where κ is the dielectric constant. Note that since $\kappa > 1$ the presence of the dielectric has the effect of reducing the effective charge to Q_{encl}/κ in Gauss's law.

For $r < a$ we are inside the central conductor and there can be no electric field. For the same reason all of the charge of the central conductor has to lie on its surface.

For $a < r < b$ we are inside the dielectric, and if we make a cylindrical Gaussian surface S with a radius r and length l around the central conductor, we have that the enclosed charge is the total charge on the central conductor and

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = 2\pi r l E(r) = \frac{Q_{\text{encl}}}{\kappa \epsilon_0} = -\frac{q}{\kappa \epsilon_0},$$

where we used that the field is purely radial, i.e. we have ignored edge effects. This gives

$$E(r) = -\frac{q}{2\pi l \kappa \epsilon_0 r}, \quad a < r < b.$$

For $b < r < c$ we are outside the dielectric and so we can use the usual form of Gauss's law with the same shape of Gaussian surface, so that by analogy

$$E(r) = -\frac{q}{2\pi l \epsilon_0 r}, \quad b < r < c.$$

b) Find the potential V in each region and the total potential across the capacitor (ignore all edge effects).

Now that we know the field is radial and its form as a function of the radius in the two regions, we can find the potential difference between the inner and outer conductors by simply integrating the field with respect to r,

$$\begin{aligned} V_{\text{ca}} &:= V_{\text{c}} - V_{\text{a}} = V_{\text{cb}} + V_{\text{ba}} = -\int_{\text{b}}^{\text{c}} E(r)dr - \int_{\text{a}}^{\text{b}} E(r)dr \\ &= \frac{q}{2\pi l \epsilon_0}\left(\int_{\text{b}}^{\text{c}} \frac{1}{r}dr + \frac{1}{\kappa}\int_{\text{a}}^{\text{b}} \frac{1}{r}dr\right) = \frac{q}{2\pi l \epsilon_0}\left[\ln\left(\frac{c}{b}\right) + \frac{1}{\kappa}\ln\left(\frac{b}{a}\right)\right]. \end{aligned}$$

c) Find the capacitance.

Using $C = q/V$ we have

$$C = 2\pi l \epsilon_0 \left[\ln\left(\frac{c}{b}\right) + \frac{1}{\kappa} \ln\left(\frac{b}{a}\right) \right]^{-1}.$$

d) Find the stored energy in the capacitor. Is the energy greater than or less than the energy stored in a similar capacitor with no dielectric?

The energy stored in the capacitor is the work done to build up the charge against the potential difference caused by the charge already there,

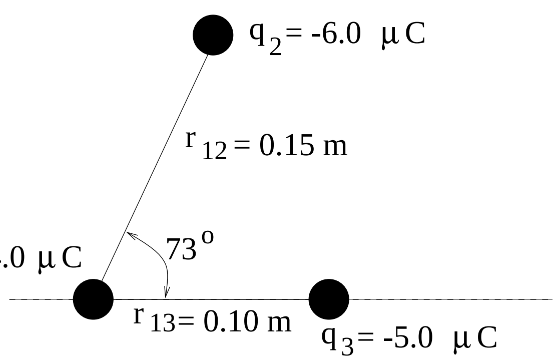
$$U = W = \int dW = \int dq V_C = \int dq \frac{q}{C} = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2} V_C q,$$

so that

$$U = \frac{1}{2} V_{\text{ca}} q = \frac{q^2}{4\pi l \epsilon_0} \left[\ln\left(\frac{c}{b}\right) + \frac{1}{\kappa} \ln\left(\frac{b}{a}\right) \right].$$

Since $\kappa > 1$, we see that U is smaller than it would be if $\kappa = 1$, i.e. if there were no dielectric present.

6) What is the net electric force on charge q_1?



The force due to q_2 has magnitude

$$F_2 = \frac{k q_1 |q_2|}{r_{12}^2} = \frac{9 \times 10^{-9} \cdot 24 \times 10^{-12}}{0.15^2} N = 9.6\ N$$

and lies at 73° to that due to q_3, which is horizontal and has magnitude

$$F_2 = \frac{k q_1 |q_3|}{r_{13}^2} = \frac{9 \times 10^{-9} \cdot 20 \times 10^{-12}}{0.10^2} N = 18.0\ N.$$

The net force in the x direction is

$$F_x = F_2 \cos(73^o) + F_3 = [9.6 \cos(73^o) + 18.0] = 20.8\ N,$$

and that in the y direction is

$$F_y = F_2 \sin(73^o) = 9.6 \sin(73^o) = 9.2\ N.$$

The net force has magnitude

$$F = \sqrt{F_x^2 + F_y^2} = 22.7\ N,$$

and lies in the first quadrant at an angle of

$$\theta = \tan^{-1}(F_y/F_x) = 23.9^{\circ}$$

to the positive x axis.