PHY531 Problem Set 4. Due March 5, 2015

The purpose of these problems is to work through some solutions for wave equations in one space dimension to better understand the mathematics in three space dimensions.

1. An ideal coaxial cable operates in the TEM mode. The voltage between the conductors of the cable and the current on the conductors are functions of position and time. From Maxwell's equations and the waveguide methods described in Jackson chapter 8 (see e.g. page 358 and problem 8.2 for a discussion), these obey the following differential equations,

$$\begin{aligned} \frac{\partial V(x,t)}{\partial x} &= -\frac{Z_0}{c}\frac{\partial I(x,t)}{\partial t} \\ \frac{\partial I(x,t)}{\partial x} &= -\frac{1}{Z_0 c}\frac{\partial V(x,t)}{\partial t} \end{aligned} \tag{1}$$

where $c$ is the velocity of light in the cable and $Z_0$ is a real number, the characteristic impedance (its units are second/centimeter like all impedances in Heaviside-Lorentz or Gaussian units). Positive $I$ is defined to be current in the positive $x$ direction. The coaxial cable that runs to the back of your television is typically RG-6U, which nominally has a 75 Ohm (1 ohm $\simeq \frac{1}{36\pi} \times 10^{-11}$ seconds/cm in Heaviside-Lorentz units) characteristic impedance and velocity factor of 0.82 which means that electromagnetic waves travel at 0.82 of the speed of light in vacuum. This reduction is simply because the inner and outer conductors are insulated by a dielectric with an effective dielectric constant $\epsilon_{eff} = 0.82^{-2}$. Generally the insulator is polyethylene, $\epsilon = 2.2$, with air bubbles (foamed polyethylene).

   a. (5 points) Show that $V(x,t)$ and $I(x,t)$ each satisfy the one-dimensional wave equation.

   b. (20 points) In class we calculated the $\boldsymbol{E}(\boldsymbol{r},t)$ and $\boldsymbol{B}(\boldsymbol{r},t)$ fields that satisfy Maxwell's equations given the initial localized fields $\boldsymbol{E}(\boldsymbol{r},0)$ and $\boldsymbol{B}(\boldsymbol{r},0)$. Repeat that calculation for the interior of an infinitely long coaxial cable. Your result should be

$$\begin{aligned} V(x,t) &= \mathrm{Re} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\left[\tilde{V}(k,0) + \frac{k}{|k|}Z_0\tilde{I}(k,0)\right] e^{ikx - i|k|ct} \\ I(x,t) &= \mathrm{Re} \int_{-\infty}^{\infty} \frac{dk}{2\pi}\left[\tilde{I}(k,0) + \frac{k}{|k|Z_0}\tilde{V}(k,0)\right] e^{ikx - i|k|ct} \end{aligned} \tag{2}$$

   where

$$\begin{aligned} \tilde{V}(k,0) &= \int_{-\infty}^{\infty} dx V(x,0) e^{-ikx} \\ \tilde{I}(k,0) &= \int_{-\infty}^{\infty} dx I(x,0) e^{-ikx}\,. \end{aligned} \tag{3}$$

2. A cable like that of problem 1 of length $L$, $0 \leq x \leq L$, is shorted at the $x = L$ end and the $x = 0$ end is driven by a voltage source in series with a resistor of resistance $R$. The voltage source is periodic with period $T$ and in the interval $0 < t < T$ satisfies

$$V_s(t) = \begin{cases} V_i & 0 < t < \frac{T}{2} \\ 0 & \frac{T}{2} < t < T \end{cases} . \tag{4}$$

a. (20 points) Separate variables to get a Sturm-Liouville problem in one of the variables. Which variable $x$ or $t$ should you choose to satisfy the Sturm-Liouville problem for this problem and why? Calculate a series solution for $V(0,t)$ and $I(0,t)$.

b. (10 points) Repeat the calculation where the cable is open at the $x = L$ end.

c. (20 points) For the cases where $R = Z_0$, $R = 2Z_0$ and $R = \frac{Z_0}{2}$, and $cT = 20L$, plot $V(0,t)/V_i$ as a function of $\frac{t}{T}$ for the cases of part a and part b. Use sufficient terms in your series (and any necessary convergence factor to ameliorate any Gibbs phenomenon) to give an accurate plot.

   Do not turn in this part. Before plotting, sketch your prediction for how the plot should look using your understanding of wave phenomenon. Does your plot look like your prediction? If not, after you make your plot, try to understand why the plots look like they do from physical arguments about waves.

A diagram of the connections is shown in fig. 1. The end not connected to the resistor is either shorted together or left open.

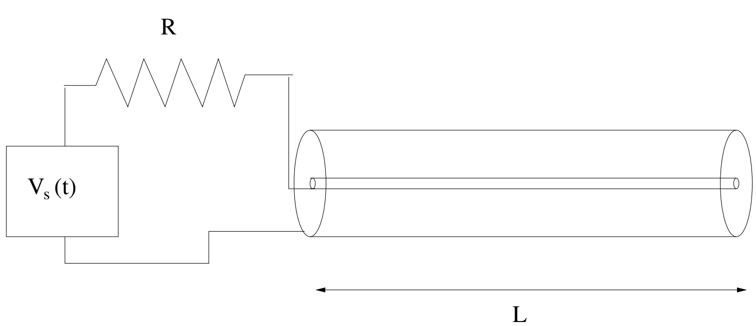


Figure 1: A diagram of the connections of the problem.