PHY 576: Quantum Theory

Problem Set 3

Due in TA's mailbox by 4 pm on Monday, March 30.

1. Consider the propagator for the simple harmonic oscillator.

   (a) Obtain the thermal partition function, $Z(\beta)$, at inverse temperature $\beta = (k_B T)^{-1}$ by integrating the propagator over all closed paths with imaginary time period $-i\hbar\beta$.

   (b) Show that this exactly equals the partition function obtained from the statistical mechanical formula $Z(\beta) = \sum_E e^{-\beta E}$.

2. Consider the triangular potential

$$V(x) = \begin{cases} V_0(1 - |x|/a) \quad , & |x| \leq a \\ 0 \quad , & |x| \geq a \end{cases}$$

   where V_0 and a are positive (dimensionful) constants. Suppose a particle of energy $E < V_0$ is incident upon the barrier from the left.

   (a) Determine the classical turning points for a particle of energy E.

   (b) Write down the equation of motion in imaginary time. Solve it to find a path in imaginary time that takes the particle from the left turning point to the right.

   (c) Evaluate the Euclidean action.

   (d) Using the Euclidean action, find the probability for the particle to tunnel across the barrier.

3. Consider an alpha particle with energy $E > 0$ inside a nucleus of radius a. The alpha particle can be regarded as a free particle inside the nucleus. However, outside the nucleus, it encounters a potential $V(r) = \frac{k}{r}$ with $\frac{k}{a} > E$. Classically, the particle is trapped inside the nucleus by the barrier.

   (a) Show that the forbidden region becomes accessible if the particle is allowed to use Euclidean time ($\tau = it$).

(b) Solve the imaginary time equation of motion with the right boundary conditions.

(c) Compute the Euclidean action S_E and hence calculate the tunneling probability for the radioactive alpha-decay of the nucleus.

(d) To turn this into a decay rate (expressed as probability per time), imagine that the particle bounces back and forth inside the nucleus. Given that it has energy E, determine the time interval between tunneling attempts and use that to predict the half-life of the nucleus.

4. (a) A particle of mass m and momentum $\vec{p} = \hbar k \hat{z}$ is incident on a spherical step potential:

$$V(r) = \begin{cases} V_0 & r \leq a \\ 0 & r > a \end{cases}$$

Find the differential cross-section in the first Born approximation.

(b) Calculate the total cross section for scattering off the potential $V(r) = V_0 e^{-\mu r^2}$, where V_0 and μ are constants. Assume the validity of the first Born approximation.

5. (a) A particle of mass m and charge q sits in a harmonic oscillator potential $V = k(x^2 + y^2 + z^2)/2$. At time $t = -\infty$ the particle is in its ground state. It is then gradually disturbed by a spatially uniform time-dependent electric field

$$\vec{E}(t) = At^2 e^{-(t/\tau)^2} \hat{z}$$

where A and τ are constants. Calculate in first-order time-dependent perturbation theory the probability that, at $t = \infty$, the particle is in

i. the first excited state

ii. the second excited state

(b) A particle of mass m in a one-dimensional box of length L is initially in the ground state. At time $t = 0$, it is abruptly perturbed by a decaying harmonic oscillator potential:

$$V_{\text{pert}}(x,t) = \frac{1}{2}kx^2 e^{-\lambda t} \quad , \quad t \geq 0$$

where λ and k are dimensionful constants. Calculate the probability that, at $t = \infty$, the particle will be found in the nth state for $n \neq 1$ (where $n = 1$ is the ground state).

6. In a 1917 letter to his friend Michele Besso, Einstein wrote "A splendid light has dawned on me about the absorption and emission of radiation." He went on to describe his famous A and B coefficients, which underlie the workings of lasers. Consider, for simplicity, a two-state system, with stationary states $|1\rangle$ and $|2\rangle$ for which $E_2 > E_1$, perturbed by a time-varying electromagnetic field. Let B_{12} be the probability per time (the rate) that the particle will jump from $|1\rangle$ to $|2\rangle$ by absorbing a photon (stimulated absorption), let B_{21} be the rate that it will go from $|2\rangle$ to $|1\rangle$ because of the electromagnetic field (stimulated emission), and A, the rate for the transition $|2\rangle$ to $|1\rangle$, even in the absence of a background electromagnetic field (spontaneous emission). (The reason there is spontaneous emission at all for a particle in a stationary state is that, even in the absence of a classical electromagnetic field, there are vacuum fluctuations. Rate A can be calculated in quantum field theory, which takes into account transient virtual photons in the vacuum; see Weisskopf-Wigner decay if you're interested.) Einstein was able to infer equations relating A, B_{12}, and B_{21} even before the development of modern quantum mechanics. Now, however, we can determine the B coefficients directly from time-dependent perturbation theory.

(a) By linearity, the background electromagnetic field can be decomposed into modes. Consider a perturbation by the electric field $\vec{E} = E_0\hat{z}\cos(kx - \omega t)$. For wavelengths that are much longer than the size of the atom, we can ignore the x-dependence. Thus, consider the perturbation $\mathcal{V}(\vec{r})\cos\omega t$. Suppose the particle is in state

$|1\rangle$. Show that when $\omega \approx \omega_0$, where $E_2 - E_1 \equiv \hbar\omega_0$, the transition probability for stimulated absorption is roughly

$$P(t) = \left|\frac{\langle 2|\mathcal{V}(\vec{r})|1\rangle}{\hbar}\right|^2 \frac{\sin^2((\omega_0 - \omega)t/2)}{(\omega_0 - \omega)^2}$$

This can be used to calculate B_{12} by integrating over all perturbing frequencies ω.

(b) Prove that, whatever the perturbation, the rate of stimulated emission equals the rate of stimulated absorption i.e. $B_{12} = B_{21}$.