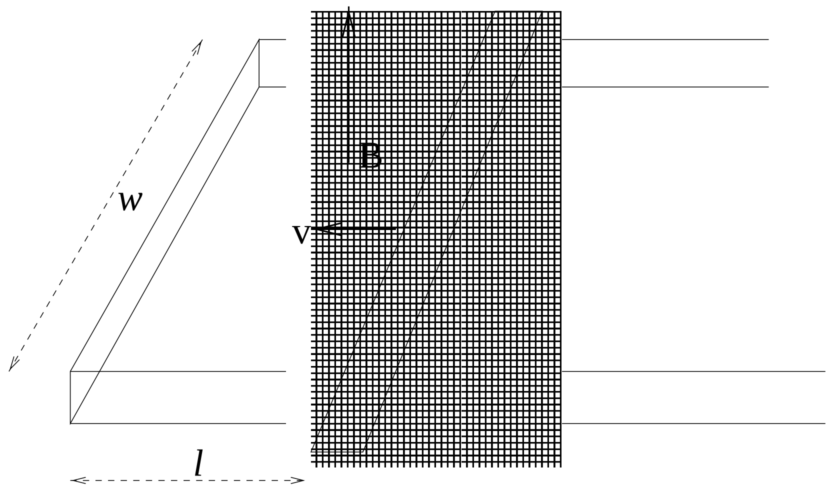
1. A bar slides along two parallel rails separated by a distance w, with no friction and with velocity v. The parallel bars, connector, and sliding bar are all made of a conductor with resistance r per unit length. The bars are in a uniform magnetic field B which is perpendicular to the plane of the bars and directed upward. Assume the bar starts at a distance l from the end.



a) Show the direction of the current and the force on the bar due to the field.

By Lenz's law, the current acts to oppose the change in the flux due to the motion of the bar. Since the flux is decreasing, the induced current must be in the direction which *adds* to the magnetic field B, so that by the right-hand-rule we must have a counter-clockwise current. Such a current produces, by using the Lorentz force law

$$d\mathbf{F} = I d\mathbf{l} \times \mathbf{B},$$

a force which opposes the motion of the bar (so it points to the right in the figure). Note we could have also found the *force* by Lenz's law, i.e. it also acts to oppose the change caused by the motion of the rod.

b) Find the current in the bar as a function of time. Since the bf B field is constant we know that the magnetic flux ϕ is proportional to the rate at which the area is decreasing,

$$\frac{d\phi}{dt} = \frac{d}{dt} BA = B\frac{dA}{dt} = Bw\frac{dl}{dt} = -Bwv,$$

where the sign tells us that l is decreasing. Then the induced emf in the loop is

$$\varepsilon = -\frac{d\phi}{dt} = Bwv,$$

and since the resistance of the loop is

$$R(t) = [2w + 2l(t)]r = [2w + 2(l - vt)]r$$

then we have that the current is

$$I(t)=\frac{\varepsilon}{R(t)}=\frac{Bwv}{r[2w+2(l-vt)]}.$$

c) Find the electrical power dissipated in the loop as a function of time.

This is simply $P(t)=\varepsilon I(t)$ which is

$$P(t)=\frac{B^2w^2v^2}{r[2w+2(l-vt)]}.$$

d) Find the mechanical power which must be provided to move the bar with velocity v.

The work dW required to move the bar a distance dx is given by $dW=F\,dx$ since here the force required and the displacement are in the same direction. From the Lorentz force law we have that the force required is

$$F(t)=I(t)wB$$

so that

$$dW=F\,dx=I(t)wB\,dx,$$

and the power required is

$$P=\frac{dW}{dt}=I(t)wB\frac{dx}{dt}=I(t)wBv=\frac{B^2w^2v^2}{r[2w+2(l-vt)]},$$

the same as the electrical power dissipated.

2. A superconducting (SC) solenoid is to be wound from superconducting wire having a critical current density $J_c=10^5$ A/cm^2 at 15 T (J_c is the current above which the resistance of the wire suddenly becomes very large compared to the resistance of the copper) and a diameter of 2 mm. The wire is clad with a copper sheath of thickness 0.5 mm. The magnet has a field of 15 T, a bore (inner diameter) of 1 m, and is 2 m long. Remember that the resistance of a superconductor in the superconducting state is exactly zero.

Use the long solenoid approximation to determine the requested characteristics including the approximation that the flux is confined to the innermost winding. You may also assume the layer thickness is very small compared with the inner diameter of the solenoid, obviously a good approximation. (Note this magnet is approximately the size of the SC outsert of the new hybrid under construction at the NHMFL. These are to be only "back of the envelope" calculations to give some idea of the parameters.)

$m_0=4\pi\times10^{-7}$ T·m/A; the resistivity of Cu is 1.7×10^{-8} Ω·m.

a) What is the maximum current the SC wire can carry and remain superconducting at 15 T?

This is just the critical current density times the cross sectional area of the superconducting part of the wire, which is

$$I_c = J_c A = J_c \pi r^2 = 10^5\ \mathrm{A/cm^2} \cdot \pi (0.1\ \mathrm{cm})^2 = 3.14 \times 10^3\ \mathrm{A}.$$

b) How many layers of wire are needed to produce the 15 T field, and how many total turns?

We have to assume the outer edges of the wires are just touching so that we can figure out how many turns of wire we can have per unit length; this is

$$\frac{N}{l} = \frac{1\ \mathrm{m}}{3 \times 10^{-3}\ \mathrm{m}} = 333\ \mathrm{turns/m}.$$

Then the magnitude of B inside the solenoid due to a single layer of wire is given by Ampère's law applied to a rectangular current loop cutting through the coil to be simply $\mu_0 I\, N/l$, so that from n layers (we will neglect their thickness) we have

$$B = \mu_0 I \frac{N}{l} n = 15\ \mathrm{T}.$$

This gives that

$$n = \frac{15\ \mathrm{T}}{4\pi \times 10^{-7}\ \frac{\mathrm{T \cdot m}}{\mathrm{A}} \cdot 333\ \mathrm{m}^{-1} \cdot 3.14 \times 10^3 \mathrm{A}} \simeq 11\ \mathrm{layers}.$$

c) What is the resistance of the Cu sheath?

The length of the wire is the circumference πb of each turn, where b is the inner diameter (bore) of the magnet, multiplied by the number of turns per unit length, and by the length, and finally by the number of layers,

$$L = 2\pi b \frac{N}{l}\, l\, n = \pi \cdot 333 \cdot 2 \cdot 11\ \mathrm{m} = 22{,}990\ \mathrm{m}.$$

The cross-sectional area of the Cu sheath is

$$A_{\mathrm{Cu}} = \pi[(1.5)^2 - 1^2]\ \mathrm{mm}^2 = 4 \times 10^{-6}\ m^2$$

and so the resistance is given in terms of the resistivity ρ by

$$R = \frac{\rho L}{A} = \frac{1.7 \times 10^{-8}\ \Omega \cdot \mathrm{m} \cdot 22{,}990\ \mathrm{m}}{4 \times 10^{-6}\ \mathrm{m}^2} = 96.8\ \Omega.$$

d) What is the inductance of the solenoid?

The self-inductance L of a coil is defined by $N_{\mathrm{tot}}\phi = LI$, where ϕ is the magnetic flux through the coil, I is the current, and N_{tot} is the total number of turns of wire. For our coil we have

$$L = \frac{N_{\mathrm{tot}}\phi}{I} = \frac{n \frac{N}{l}\, l\, B\, \pi b^2}{I} = \frac{11 \cdot 333 \cdot 2 \cdot 15 \cdot \pi (0.5)^2}{3.14 \times 10^3} = 27.5\ \mathrm{H}.$$

e) How much energy is stored in the solenoid when it is at full field and superconducting?

The energy strored in an inductor carrying a current I is simply $E = LI^2/2$, so we have

$$E = \frac{1}{2} \cdot 27.5 \text{ H} \cdot (3.14 \times 10^2 \text{ A})^2 = 1.36 \times 10^8 \text{ J}.$$

For the parts below assume the Cu part of the coil is shorted so the circuit is a resistance and inductance in series.

f) If the magnet quenches, that is if the resistance of the SC suddenly becomes very large, what will be the maximum voltage across the solenoid (and thus the Cu sheath)?

This is just the resistance of the Cu sheath times the critical (maximum) current allowed, which is

$$V = RI_c = 96.8 \cdot 3.14 \times 10^3 \text{ V} = 3.04 \times 10^5 \text{ V}.$$

g) Find the current in the Cu sheath as a function of time after the quench. Assume the equivalent circuit is just the Cu sheath of the solenoid shorted with a conductor of negligible resistance.

This is just an inductor-resistor circuit with an initial current of I_c. The voltage equation for this circuit is

$$L\frac{dI}{dt} + IR = 0,$$

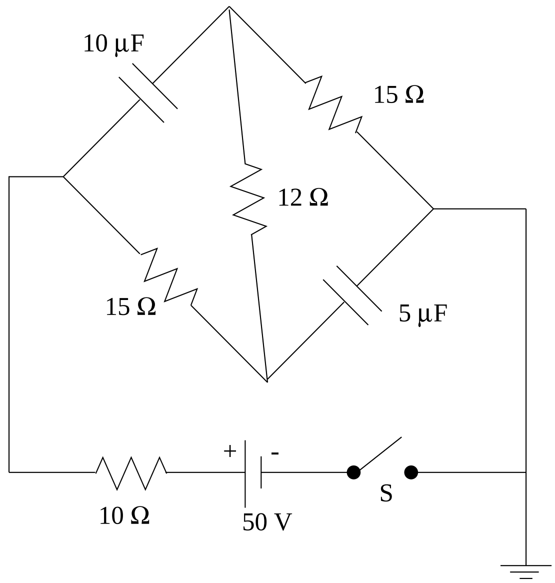
which has the solution

$$I(t) = I_0\, e^{-Rt/L},$$

so the current decays exponentially from $I_0 - I_c$ with time constant

$$\tau = \frac{L}{R} = \frac{27.5 \text{ H}}{96.8\ \Omega} = 0.284 \text{ s}.$$

3. **The capacitors in the circuit shown in the figure are initially uncharged.**



a) What is the initial value of the current drawn from the battery when the switch S is closed?

When the switch is initially closed the capacitors act like wires, as they have no charge built up on them yet and so no voltage crop across them. The equivalent circuit is three resistors in parallel, so

$$R_{eq} = (2\frac{1}{15} + \frac{1}{12})^{-1}\ \Omega = 4.61\ \Omega,$$

so that

$$I = \frac{50\ \mathrm{V}}{10\ \Omega + 4.61\ \Omega} = 3.4\ \mathrm{A}.$$

b) What is the current drawn from the battery a long time after the switch is closed?

A long time after the switch is closed the capacitors are in their steady state and fully charged, and no current flows through them, so they act like open circuits. The effective resistance is just three resistors in series, so

$$I_{\mathrm{f}} = \frac{50\ \mathrm{V}}{10\ \Omega + 42\ \Omega} = 0.96\ \mathrm{A}.$$

c) What are the final charges on the capacitors?

The final charges are given by the voltage drops across each capacitor, which we can find by looking at the voltage drops across the resistors which span each capacitor. For the 10 μF capacitor this is

$$V_1 = I_{\mathrm{f}}(15\ \Omega + 12\ \Omega) = 26.0\ \mathrm{V}$$

so that

$$q_1 = V_1 C_1 = 25.9 \cdot 10 \times 10^{-6}\ \mathrm{C} = 260\ \mu\mathrm{C},$$

and similarly

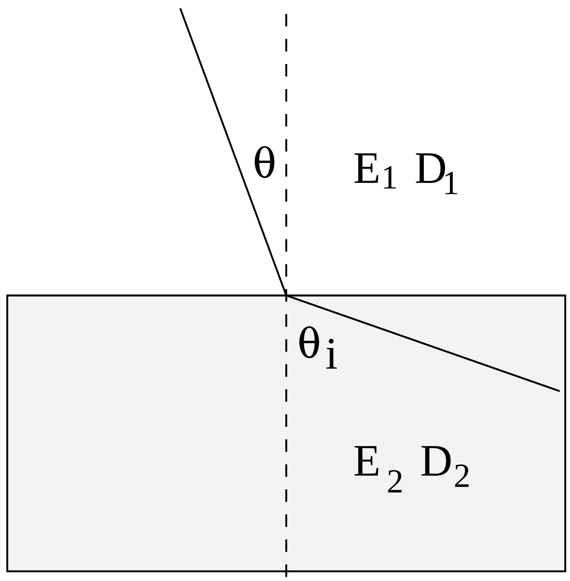
$$V_2 = I_{\mathrm{f}}(15\ \Omega + 12\ \Omega) = 26.0\ \mathrm{V}$$

so that

$$q_2 = V_2 C_2 = 25.9 \cdot 5 \times 10^{-6}\ \mathrm{C} = 130\ \mu\mathrm{C}.$$

4) A large slab of dielectric material with a plane surface is inserted into an electric field E, such that E is at an angle θ with the normal to the surface of the dielectric. The dielectric constant of the dielectric is $\epsilon = \kappa\epsilon_0$.

a) Find the magnitude of E and the magnitude of the displacement vector D inside the dielectric.



Referring to the above diagram, we have the boundary condition that the component $D_\perp$ of the electric displacement perpendicular to the surface between the air and the dielectric is the same inside and outside the boundary (this can be found using Gauss's law). We also have, using the voltage drop around a small loop through the surface, that the tangential component of the electric field is the same on both sides of the boundary. So we have that

$$D_1 \cos(\theta) = D_2 \cos(\theta_i) \qquad (1)$$
$$E_1 \sin(\theta) = E_2 \sin(\theta_i) \qquad (2)$$

and also that $D_1 = \epsilon_0 E_1$, while $D_2 = \epsilon E_2$. These give that

$$\cos(theta_i) = \cos(\theta)\frac{E_1\epsilon_0}{E_2\epsilon} \qquad (3)$$
$$\sin(theta_i) = \sin(\theta)\frac{E_1}{E_2}, \qquad (4)$$

so that by dividing these two equations we get

$$\tan\theta_i = \frac{\epsilon_0}{\epsilon}\tan(\theta) = \frac{1}{\kappa}\tan(\theta).$$

This answers part (b). Now we can find E_2 in terms of $E_1 = E$ using

$$E_2 = E_1 \frac{\sin(\theta)}{\sin(\theta_i)} = E \frac{\sin(\theta)}{\sin(\theta_i)}$$

and we can find D_2 in terms of $D_1 = \epsilon E$ by using

$$D_2 = D_1 \frac{\cos(\theta)}{\cos(\theta_i)}.$$

b) Find a relationship between the dielectric constant and the angle E and D make with the normal inside the dielectric. Sketch E and D inside the dielectric assuming $\epsilon = 3\epsilon_0$ and $\theta = 30°$.

See above.