HW 1 Solutions – PHY 576 – Jeff Hyde

(1) The form of the basis vectors is

$$f_1 = c_1$$
$$f_2 = c_2 x + c_3$$
$$f_3 = c_4 x^2 + c_5 x + c_6$$
$$f_4 = c_7 x^3 + c_8 x^2 + c_9 x + c_{10},$$

with c_i's constants that we will determine.

Use the inner product $\langle f|g\rangle \equiv \int_{-1}^{+1} dx\, fg$, along with requirement that the basis be orthonormal: $\langle f|g\rangle = \delta_{fg}$.

$$1 = \langle f_1|f_1\rangle = \int_{-1}^{+1} dx\ c_1^2 = 2c_1^2 \rightarrow \underline{c_1 = \frac{1}{\sqrt{2}}}$$

$$0 = \langle f_1|f_2\rangle = \int_{-1}^{+1} dx\,(c_1 c_2 x + c_1 c_3)$$
$$= \frac{1}{\sqrt{2}} c_2 (0) + \frac{1}{\sqrt{2}} c_3 (2) \rightarrow \underline{c_3 = 0}$$

$$1 = \langle f_2|f_2\rangle = \int_{-1}^{+1} dx\ c_2^2 x^2 = \frac{1}{3} c_2^2\, 2 \rightarrow \underline{c_2 = \sqrt{\frac{3}{2}}}$$

$$0 = \langle f_1|f_3\rangle = \int_{-1}^{+1} dx\left(\frac{1}{\sqrt{2}} c_4 x^2 + \frac{1}{\sqrt{2}} c_5 x + \frac{1}{\sqrt{2}} c_6\right)$$
$$= \frac{1}{3\sqrt{2}} 2c_4 + 0 + \sqrt{2}\, c_6 \quad \text{so} \quad c_6 = -\frac{c_4}{3}$$

$$0 = \langle f_2|f_3\rangle = \int_{-1}^{+1} dx\left(\sqrt{\frac{3}{2}} c_4 x^3 + \sqrt{\frac{3}{2}} c_5 x^2 - \sqrt{\frac{3}{2}}\frac{1}{3} c_4 x\right)$$
$$= \sqrt{\frac{3}{2}} c_4 \frac{1}{4}(0) + \sqrt{\frac{3}{2}} c_5 \frac{1}{3}(2) - \sqrt{\frac{3}{2}}\frac{1}{3} c_4 \frac{1}{2}(0)$$
$$\rightarrow \boxed{c_5 = 0}$$

$$1 = \langle f_3|f_3\rangle = \int_{-1}^{+1} dx\left(c_4^2 x^4 + 2c_4\left(-\frac{c_4}{3}\right)x^2 + \frac{c_4^2}{9}\right)$$
$$= c_4^2\left(\frac{1}{5}2 - \frac{2}{3}\frac{1}{3}2 + \frac{2}{9}\right)$$
$$= c_4^2\left(\frac{2}{5} - \frac{2}{9}\right) = c_4^2\,\frac{8}{45}$$
$$\rightarrow \underline{c_4 = \sqrt{\frac{45}{8}}}$$
$$\rightarrow \underline{c_6 = -\sqrt{\frac{45}{72}}}$$

$$0 = \langle f_1 | f_4 \rangle = \int_{-1}^{+1} dx \left(\frac{1}{\sqrt{2}} c_7 x^3 + \frac{1}{\sqrt{2}} c_8 x^2 + \frac{1}{\sqrt{2}} c_9 x + \frac{1}{\sqrt{2}} c_{10} \right)$$

$$= \frac{1}{\sqrt{2}} c_7 \frac{1}{4} (0) + \frac{1}{\sqrt{2}} c_8 \frac{1}{3} 2 + \frac{1}{\sqrt{2}} c_9 \frac{1}{2} (0) + \frac{1}{\sqrt{2}} c_{10} 2$$

$$0 = \frac{\sqrt{2}}{3} c_8 + \sqrt{2} c_{10} \quad \rightarrow \quad c_{10} = -\frac{1}{3} c_8$$

$$0 = \langle f_2 | f_4 \rangle = \int_{-1}^{+1} dx \left(\sqrt{\tfrac{3}{2}} c_7 x^4 + \sqrt{\tfrac{3}{2}} c_8 x^3 + \sqrt{\tfrac{3}{2}} c_9 x^2 + \sqrt{\tfrac{3}{2}} c_{10} x \right)$$

$$= \sqrt{\tfrac{3}{2}} c_7 \frac{1}{5} 2 + \sqrt{\tfrac{3}{2}} c_9 \frac{1}{3} 2 \quad \rightarrow \quad c_9 = -\frac{3}{5} c_7$$

$$0 = \langle f_3 | f_4 \rangle = \int_{-1}^{+1} dx \left(c_4 x^2 - \frac{c_4}{3} \right)\left(c_7 x^3 + c_8 x^2 - \frac{3}{5} c_7 x - \frac{1}{3} c_8 \right)$$

$$= c_4 \int_{-1}^{+1} dx \left(c_7 x^5 + c_8 x^4 - \frac{3}{5} c_7 x^3 - \frac{1}{3} c_8 x^2 - \frac{1}{3} c_7 x^3 - \frac{1}{3} c_8 x^2 - \frac{1}{5} c_7 x + \frac{1}{9} c_8 \right)$$

$$= c_4 \int_{-1}^{+1} dx \left(c_8 x^4 - \frac{2}{3} c_8 x^2 + \frac{1}{9} c_8 \right) \quad \Rightarrow \quad \underline{c_8 = 0}$$

$$= c_4 c_8 [\cdots] \quad \Rightarrow \quad \underline{c_{10} = 0}$$

$$1 = \langle f_4 | f_4 \rangle = \int_{-1}^{+1} dx \left(c_7 x^3 - \frac{3}{5} c_7 x \right)^2$$

$$= c_7^2 \int_{-1}^{+1} \left(x^6 - \frac{6}{5} x^4 + \frac{9}{25} x^2 \right)$$

$$\cancel{= c_7^2 \left(\frac{1}{7} 2 - \frac{6}{5} \frac{1}{5} (2) + \frac{9}{25} \frac{1}{3} (2) \right) = c_7^2 \left(\frac{2}{7} - \frac{12}{25} \right)}$$

$$\cancel{= c_7^2 \left(\frac{2}{7} - \frac{18}{25} + \frac{6}{25} \right)}$$

$$= c_7^2 \, 2 \, \frac{1}{175} \underbrace{(25 - 42 + 21)}_{=4} = c_7^2 \frac{8}{175}$$

$$\rightarrow c_7 = \sqrt{\frac{175}{8}}$$

$$c_9 = -\frac{3}{5} \sqrt{\frac{175}{8}}$$

$$f_1 = \frac{1}{\sqrt{2}}$$
$$f_2 = \sqrt{\frac{3}{2}}\, x$$
$$f_3 = \sqrt{\frac{45}{8}}\left(x^2 - \frac{1}{3}\right)$$
$$f_4 = \sqrt{\frac{175}{8}}\left(x^3 - \frac{3}{5}x\right)$$

Legendre polynomials normalized on $[-1,1]$

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(2) a. $\hat{O}$ is Hermitian $\rightarrow$ could correspond to a physical observable.

Eigenvalues: $0 = \det(\hat{O} - \lambda \hat{I})$

$$= -\lambda(\lambda^2-1) - (1)(-1) + 0$$
$$\Rightarrow \lambda = 0, \pm\sqrt{2}$$

(Normalized) Eigenvectors:
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \lambda v_3 \end{pmatrix}$$

$\lambda = 0$: $\quad v_2 = 0, \quad v_1 + v_3 = 0 \rightarrow |0\rangle = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$

$\lambda = +\sqrt{2}$: $\quad v_2 = \sqrt{2}\, v_1, \ v_1 + v_3 = \sqrt{2}\, v_2 \rightarrow |+\rangle = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \\ 1/2 \end{pmatrix}$

$\lambda = -\sqrt{2}$: $\quad \ldots \quad |-\rangle = \begin{pmatrix} 1/2 \\ -1/\sqrt{2} \\ 1/2 \end{pmatrix}$

For $|1\rangle \equiv \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $|2\rangle \equiv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $|3\rangle \equiv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle) = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix},$$

$$\langle + | \psi\rangle = (1/2, 1/\sqrt{2}, 1/2)\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix} = \frac{1}{2\sqrt{2}} + \frac{1}{2}$$

$$\langle - | \psi\rangle = \frac{1}{2\sqrt{2}} - \frac{1}{2}$$

$$\langle 0 | \psi\rangle = -\frac{1}{2}$$

So the probability of measuring each eigenvalue is

$$P(+\sqrt{2}) = |\langle + | \psi\rangle|^2 \approx 0.729$$
$$P(-\sqrt{2}) = |\langle - | \psi\rangle|^2 \approx 0.021$$
$$P(0) = |\langle 0 | \psi\rangle|^2 = 0.25.$$

b. As we know, $A_1 H - H A_1 = 0$ means we can find states that are simultaneous eigenstates of A_1 and H: $|E,1\rangle$ s.t.

$$H|E,1\rangle = E|E,1\rangle, \quad A_1|E,1\rangle = a_1|E,1\rangle.$$

Similarly, $\exists$ $|E,2\rangle$ with $H|E,2\rangle = E|E,2\rangle$, $A_2|E,2\rangle = a_2|E,2\rangle$.

Are $|E,1\rangle$ and $|E,2\rangle$ the same state?

$A_1 A_2 - A_2 A_1 \neq 0 \rightarrow$ apply to e.g. $|E,2\rangle$:

$$A_1 A_2 |E,2\rangle - A_2 A_1 |E,2\rangle$$
$$= a_2 (A_1|E,2\rangle) - A_2 (A_1|E,2\rangle) \neq 0$$

i.e. the state $A_1|E,2\rangle$ is not an eigenstate of A_2, as it would be if $|E,1\rangle$ and $|E,2\rangle$ were the same state.

$\therefore$ The energy eigenstates are degenerate.

Above, we implicitly assumed that the relation $A_1 A_2 - A_2 A_1 \neq 0$ holds when applied to any states. An exception is when A_1, A_2 have eigenvalue 0.

(3) a. $\langle x\rangle = \int_{-\infty}^{+\infty} dx\ \psi^*(x)\ \hat{x}\ \psi(x)$

$= \int_{-\infty}^{+\infty} dx\ \frac{1}{d\sqrt{\pi}}\ x\ \exp[-x^2/d^2] = 0$ (odd integrand)

$\langle p\rangle = \int_{-\infty}^{+\infty} dx\ \psi^*(x)(-i\hbar\frac{\partial}{\partial x})\ \psi(x)$

$= -i\hbar\frac{1}{d\sqrt{\pi}}\int_{-\infty}^{+\infty} dx\left(\frac{ip_0}{\hbar} - \frac{x}{d^2}\right)\exp[-x^2/d^2]$

$= \frac{p_0}{d\sqrt{\pi}}\int_{-\infty}^{+\infty} dx\ \exp(-x^2/d^2)$ (again ditching the odd integrand)

$= \frac{p_0}{d\sqrt{\pi}}\sqrt{\pi}\,d = p_0$

$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$, $\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$

$\rightarrow$ Evaluate $\langle x^2\rangle$ and $\langle p^2\rangle$:

$\langle x^2\rangle = \int_{-\infty}^{+\infty} dx\ \frac{1}{d\sqrt{\pi}}\ x^2\exp(-x^2/d^2) = \frac{d^2}{2}$

$\langle p^2\rangle = \int_{-\infty}^{+\infty} dx\ \frac{1}{d\sqrt{\pi}}\left(\frac{\hbar^2}{d^2} + p_0^2 + \cancel{\frac{2i\hbar p_0 x}{\hbar d^2}} - \frac{\hbar^2 x^2}{d^4}\right)\exp[-x^2/d^2]$

$= \frac{1}{d\sqrt{\pi}}\left(\frac{\hbar^2}{d^2} + p_0^2\right)\int_{-\infty}^{+\infty} dx\ \exp(-x^2/d^2) + \frac{-\hbar^2}{d^5\sqrt{\pi}}\int_{-\infty}^{+\infty} dx\, x^2\exp\left(\frac{-x^2}{d^2}\right)$

$= \frac{1}{\cancel{d\sqrt{\pi}}}\left(\frac{\hbar^2}{d^2} + p_0^2\right)\cancel{\sqrt{\pi}d} - \frac{\hbar^2}{d^5\sqrt{\pi}}\frac{d^3\sqrt{\pi}}{2}$

$= \frac{\hbar^2}{2d^2} + p_0^2$

$\Rightarrow \Delta x = \frac{d}{\sqrt{2}}$, $\Delta p = \frac{\hbar}{\sqrt{2}d}$

So $\Delta x\,\Delta p = \frac{\hbar}{2}$, i.e. equals the uncertainty bound,

"minimum uncertainty wave packet"

b. Take $\Delta x \Delta p = \frac{\hbar}{2}$, $p = mv$ so $\Delta x \Delta v = \hbar/2m$

or $v_{max} = \Delta v = \frac{\hbar}{2m\Delta x}$.

($v_{max} = \Delta v$ since you want to argue that $v=0$ was plausible so $v_{max} - 0$ falls w/in one stdev Δv of measurement unc.)

(i) $m = 1000$ kg :

$$v_{max} = \frac{10^{-34}\ \cancel{kg}\, m^{\cancel{2}} s^{-1}}{2\times 10^3\ \cancel{kg} \times 1\ \cancel{m}} = 5\times 10^{-38}\ m/s$$

$$\approx 5 \times 10^{-38} \times \frac{3600}{1000}\ \frac{km}{hr}$$

$$\underline{\sim 2\times 10^{-37}\ km/hr}$$

(ii) $m = 1$ eV :

$$v_{max} = \frac{\hbar}{2\,eV/c^2\, 1m} = \frac{c}{2\pi}\,\frac{hc}{2eVm} = \frac{c}{2\pi}\,\frac{1240\ eV\,nm}{2\,eV\,m}$$

$$= \frac{3\times 124}{4\pi}\ m/s$$

$$\sim 30\ m/s \sim \underline{100\ km/hr}$$

(4) a. $\frac{d}{dt}\int dx\, \psi^* \psi$

$= \int dx \left(\frac{\partial \psi^*}{\partial t}\psi + \psi^* \frac{\partial \psi}{\partial t} \right)$

Use Schrodinger eqn. $i\hbar \frac{\partial \psi}{\partial t} = H\psi \rightarrow \frac{\partial \psi}{\partial t} = -\frac{i\hbar}{2m}\frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar}V\psi$

$\frac{\partial \psi^*}{\partial t} = \frac{i\hbar}{2m}\frac{\partial^2 \psi^*}{\partial x^2} + \frac{i}{\hbar}V\psi^*$

Then $\frac{d}{dt}\int dx\, \psi^*\psi = \int_{-\infty}^{+\infty} dx \left(\frac{i\hbar}{2m}\frac{\partial^2\psi^*}{\partial x^2}\psi + \cancel{\frac{i}{\hbar}V\psi^*\psi} - \frac{i\hbar}{2m}\psi^*\frac{\partial^2\psi}{\partial x^2} - \cancel{\frac{i}{\hbar}V\psi^*\psi} \right)$

$= \frac{i\hbar}{2m}\int_{-\infty}^{+\infty} dx \frac{\partial}{\partial x}\left(\frac{\partial \psi^*}{\partial x}\psi - \psi^*\frac{\partial \psi}{\partial x} \right)$

$= \frac{i\hbar}{2m}\left[\frac{\partial \psi^*}{\partial x}\psi - \psi^*\frac{\partial \psi}{\partial x} \right]_{x=-\infty}^{+\infty} = 0$

b. From the integrand above, we had

$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}(\psi^*\psi) = \frac{\partial}{\partial x}\left[\frac{i\hbar}{2m}\left(\frac{\partial \psi^*}{\partial x}\psi - \psi^*\frac{\partial \psi}{\partial x} \right)\right]$

$= -\frac{\partial}{\partial x}\left[\frac{\hbar}{2mi}\left(\frac{\partial \psi^*}{\partial x}\psi - \psi^*\frac{\partial \psi}{\partial x} \right)\right]$

or $\frac{\partial \rho}{\partial t} = -\frac{\partial J}{\partial x}$ with

$J = \frac{\hbar}{2mi}\left(\frac{\partial \psi^*}{\partial x}\psi - \psi^*\frac{\partial \psi}{\partial x} \right)$.

(5) a. $-\frac{\hbar^2}{2m}\vec{\nabla}^2\psi(\vec{x},\xi) + V(\vec{x},\xi)\psi(\vec{x},\xi) = E\psi(\vec{x},\xi)$

with $V(\vec{x},\xi) = \begin{cases} 0 & \text{for } 0 \le x \le a,\ 0 \le y \le a,\ 0 \le z \le a \quad \forall \xi \\ \infty & \text{else} \end{cases}$,

i.e. $V = 0$ inside box, so

$\vec{\nabla}^2\psi(\vec{x},\xi) = -\frac{2mE}{\hbar^2}\psi(\vec{x},\xi)$. Start as usual

with separation of variables:

$\psi(\vec{x},\xi) = X(x) + Y(y) + Z(z) + \Xi(\xi)$

$\Rightarrow \underbrace{\frac{d^2X}{dx^2} = -\frac{2mE}{\hbar^2}X, \ \ldots,}_{} \ \underbrace{\frac{d^2\Xi}{d\xi^2} = -\frac{2mE}{\hbar^2}\Xi,}_{(*)}$

These have b.c.s

$X(0) = 0 = X(a)$,

$\rightarrow$ usual w.f.s & energy

$X(x) = \sqrt{\frac{2}{a}}\sin(nkx)$, $E_n = \frac{n^2\pi^2\hbar^2}{2ma^2}$, etc

However the topology of the ξ-coordinate means that Ξ has periodic b.c.s:

$\Xi(0) = \Xi(2\pi R)$, $\frac{d\Xi}{d\xi}(0) = \frac{d\Xi}{d\xi}(2\pi R)$.

Enforcing these on $\Xi(\xi) = A\sin(k\xi) + B\cos(k\xi)$.

(general soln. of $(*)$) gives

$B = A\sin(k2\pi R) + B\cos(k2\pi R)$

$kA = kA\cos(k2\pi R) - kB\sin(k2\pi R)$

Can't uniquely determine A, B but can eliminate to get

$-1 + 2\cos(k2\pi R) - \cos^2(k2\pi R) = \sin^2(k2\pi R)$

or $\cos(k2\pi R) = 1 \Rightarrow k2\pi R = 2\pi n$ for $n = 0, 1, 2, \ldots$

or $k = \frac{n}{R}$

~~⇒ The w.f.s are~~

So $\sqrt{2mE} = \frac{n\hbar}{R}$ or $E_n = \frac{n^2\hbar^2}{2mR^2}$

$\therefore$ The w.f.s are

$$\psi = \left(\frac{2}{a}\right)^{3/2} \sin\left(\frac{n_x \pi x}{a}\right) \sin\left(\frac{n_y \pi y}{a}\right) \sin\left(\frac{n_z \pi z}{a}\right)\left[A\cos(n\xi/R) + B\sin(n\xi/R)\right]$$

and the energy is

$$E = \frac{n_x^2 \pi^2 \hbar^2}{2ma^2} + \frac{n_y^2 \pi^2 \hbar^2}{2ma^2} + \frac{n_z^2 \pi^2 \hbar^2}{2ma^2} + \frac{n_\xi^2 \hbar^2}{2mR^2}$$

$$= \frac{\pi^2 \hbar^2}{2ma^2}\left(n_x^2 + n_y^2 + n_z^2 + \left(\frac{a}{\pi R}\right)^2 n_\xi^2\right)$$

b. Would we see excitations in the ξ-direction?
Look at ratio of one quantum of energy in the extra dimension to one excitation in "usual" 3D space:

$$\frac{E_{extra}}{E_{normal}} = \frac{\left(\frac{a}{\pi R}\right)^2 1^2}{1^2} \sim \left(\frac{a}{R}\right)^2$$

So even if $R \ll a$, an excitation of order $\frac{a^2}{R^2} \times$ (typical energies) could excite this mode.

⑥ a. Using $[\hat{x},\hat{p}_x]=i\hbar$, $[\hat{y},\hat{p}_y]=i\hbar$, $[\hat{p}_x,\hat{p}_y]=0$, $[\hat{x},\hat{y}]=0$

we get $[a,a^+]=1$, $[a,b]=0$, $[a,b^+]=0$, $[a^+,b]=0$,

$[a^+,b^+]=0$, $[b,b^+]=1$.

$$\hat{H}=\frac{\hat{p}_x^2+\hat{p}_y^2}{2m}+\frac{1}{2}k\hat{x}^2+\frac{1}{2}k\hat{y}^2$$

Rearrange $\hat{a}=\ldots$, $\hat{a}^+=\ldots$ to get

$$\hat{x}=\sqrt{\frac{\hbar}{2m\omega}}(a+a^+),\quad \hat{p}_x=-i\sqrt{\frac{\hbar m\omega}{2}}(a-a^+)$$

and sim. for $\hat{y},\hat{p}_y$.

Plug in to H, do some algebra $\Rightarrow$

$$\hat{H}=\hbar\omega(a^+a+b^+b+1).$$

b. $E_{m,n}=\langle m,n|H|m,n\rangle=\hbar\omega(m+n+1)$.

Degeneracy: there are $\underline{m+n+1}$ states with the same energy.

e.g. $0+1+1=1+0+1$ — 2 states

$1+1+1=2+0+1=0+2+1$ — 3 states

$\vdots$

c. For $\hat{O}\equiv i\hbar(\hat{b}^+\hat{a}-\hat{a}^+\hat{b})$,

$[\hat{H},\hat{O}]=\ldots$ algebra $=0$.

$\downarrow$

just evaluate a lot of commutators

$$\hat{O}|m,n\rangle=i\hbar b^+a|m,n\rangle-i\hbar a^+b|m,n\rangle$$

$$=i\hbar\sqrt{m}\sqrt{n+1}\,|m-1,n+1\rangle-i\hbar\sqrt{n}\sqrt{m+1}\,|m+1,n-1\rangle.$$

Plugging in expressions for a,a^+,b,b^+ in terms of x,p_x,y,p_y, we do algebra and get

$$\hat{O}=\hat{p}_x\hat{y}-\hat{p}_y\hat{x}=\hat{L}_z$$

i.e. operator for z-component of angular momentum. This is not a surprise since we have a rotational symmetry in the xy plane.

7 a. $H_{11} = \langle 1|H|1\rangle = E\langle 1|1\rangle\langle 2|1\rangle + E\langle 1|2\rangle\langle 1|1\rangle = 0$

$H_{12} = \langle 1|H|2\rangle = E$, $H_{21} = \langle 2|H|1\rangle = E$,

$H_{22} = \langle 2|H|2\rangle = 0$,

So in the basis where $|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}$, $|2\rangle = \begin{pmatrix}0\\1\end{pmatrix}$,

$$\hat{H} = \begin{pmatrix}0 & E\\ E & 0\end{pmatrix}.$$

b. $\det(\hat{H} - \lambda\hat{I}) = 0 \rightarrow \lambda = \pm E$.

This corresponds to normalized energy eigenstates $|+\rangle = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$, $|-\rangle = \begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix}$.

c. In terms of the position eigenstates $|1\rangle, |2\rangle$, these are

$$|+\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle).$$

$$\Rightarrow |2\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle).$$

So here,

$$|\psi(t)\rangle = e^{-i\hat{H}t/\hbar}|2\rangle$$

$$= \frac{1}{\sqrt{2}}\left(e^{-i\hat{H}t/\hbar}|+\rangle - e^{-i\hat{H}t/\hbar}|-\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(e^{-iEt/\hbar}|+\rangle - e^{+iEt/\hbar}|-\rangle\right)$$

So the probability of finding in position 1 as a fn of time is

$$P_1(t) = |\langle 1|\psi\rangle|^2$$

$$= \left[\frac{1}{2}(\langle +| + \langle -|)(e^{-iEt/\hbar}|+\rangle - e^{+iEt/\hbar}|-\rangle)\right]^2$$

uses $\langle +|+\rangle = 1$ $\langle +|-\rangle = 0$ etc... $\longrightarrow$

$$= \frac{1}{4}\left|e^{-iEt/\hbar} - e^{+iEt/\hbar}\right|^2$$

$$= \frac{1}{4}\left|-2i\sin(Et/\hbar)\right|^2$$

$$\boxed{P_1(t) = \sin^2(Et/\hbar)}$$

d. $\psi(\vec{x}) = \langle \vec{x} | \psi \rangle = \langle \vec{x} | 1 \rangle \langle 1 | \psi \rangle + \langle \vec{x} | 2 \rangle \langle 2 | \psi \rangle$

$= \delta(x - x_1) \langle 1 | \psi \rangle + \delta(x - x_2) \langle 2 | \psi \rangle.$

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(8) a. Particle in a linear potential - e.g. gravitational field that is uniform near surface of Earth.

b. Restricted to $z \geq 0 \Rightarrow \underline{\psi(0) = 0}$

(otherwise discontinuous fn $\rightarrow$ δ-fn derivative $\rightarrow$ ∞'te gradient energy)

Also $\underline{\psi(\infty) = 0}$ (so that it is normalizable)

c. From the general solution to the energy eigenvalue problem, $\psi(z) = c_1 \, Ai\left(\frac{-2e + 2z}{2^{2/3}}\right) + c_2 \, Bi\left(\frac{-2e+2z}{2^{2/3}}\right)$,

$\psi(\infty) = 0$ requires $c_2 = 0$, so the energy eigenfunctions are

$$\psi(z) = c_1 \, Ai\left(\frac{-2e+2z}{2^{2/3}}\right).$$

d. From the expansion near $z = 0$,

$$\psi(0) \simeq c_1 \, Ai\left(-\frac{2e + 2(0)}{2^{2/3}}\right) = c_1 \, Ai\left(\frac{-2e}{2^{2/3}}\right)$$

$\Rightarrow$ need $Ai(-2^{1/3} e) = 0.$

The zeroes of Ai are $Ai(\alpha_n) = 0$ for $n = 0, 1, 2, \ldots$

$\rightarrow$ Allowed energies given by

$$e = -\alpha_n / 2^{1/3}.$$

Now we need to restore the dimensionful constants.

(BTW you can check for yourself that if you set $\hbar^2/m = 1$ in the S.E. for a particle in a grav. field, you do not get the dimensionless eqn. we are using here.)

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \alpha x \psi(x) = E\,\psi(x).$$

$$\cancel{\frac{d^2\psi}{dx^2} + \frac{2m}{\hbar^2}(E-\alpha x)\psi = 0}$$

$$x \equiv \beta z \quad \rightarrow \quad -\frac{d^2\psi}{dz^2} + \frac{2m\beta^3\alpha}{\hbar^2} z\psi = \frac{2m\beta^2 E}{\hbar^2}\psi$$

Now define β s.t. $\frac{m\beta^3\alpha}{\hbar^2} = 1$

$$e \equiv \frac{m\beta^2 E}{\hbar^2}$$

Then $e = \frac{Em}{\hbar^2}\frac{\hbar^{4/3}}{m^{2/3}\alpha^{2/3}} = E\, m^{1/3}\alpha^{-2/3}\hbar^{-2/3}$

or $E = e\, m^{-1/3}\alpha^{2/3}\hbar^{2/3}$.

So in dimensionful variables, the energies are $E_n = -\alpha_n\left(\frac{\hbar^2\alpha^2}{2m}\right)^{1/3}$.