PHY6938 Mechanics Fall 98

1. The asteroid Toro was discovered in 1964. Its radius is about 5 km.

a) Assuming the density of Toro is the same as that of Earth, find its total mass and the acceleration due to gravity at its surface.

Note that although it is not stated, we must assume that Toro is spherically symmetric to proceed; note part b) mentions Toro's 'radius'. From the 'Useful constants' given with the test we can calculate the mass of Toro from that of the Earth assuming the same densities, simply by scaling by the cube of the radius

$$M = \left(\frac{R_{\rm T}}{R_{\rm E}}\right)^3 M_{\rm E} = \left(\frac{5.0}{6.38 \cdot 10^3}\right)^3 5.98 \cdot 10^{24}\ \text{kg} = 2.88 \cdot 10^{15}\ \text{kg}.$$

Then the acceleration due to gravity at its surface is, according to Newton, the same as if we placed all of its mass at its center which is a distance $R_{\rm T}$ away,

$$g(R_{\rm T}) = \frac{GM}{R_{\rm T}^2} = \frac{6.67 \cdot 10^{-11} \cdot 2.88 \cdot 10^{15}}{(5.0 \cdot 10^3)^2}\ \frac{\text{Nm}^2}{\text{kg}^2}\ \frac{\text{kg}}{\text{m}^2} = 7.68 \cdot 10^{-3}\ \frac{\text{m}}{\text{s}^2}.$$

b) Suppose a body is to be placed in a circular orbit around Toro, with radius just slightly larger than the asteroid's radius. What is the speed of the body?

In a circular orbit the acceleration of the body is just $v^2/R_{\rm T}$, where v is the speed of the body. However this acceleration must be provided by the force of gravity, so that $mg(R_{\rm T}) = v^2/R_{\rm T}$ and so we have that

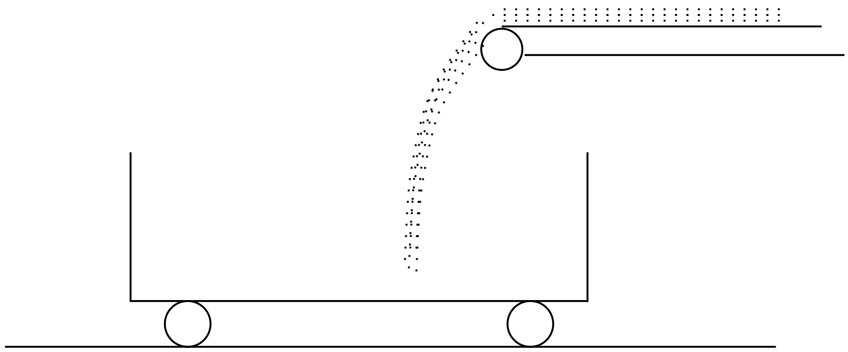
$$\begin{aligned} v^2 &= g(R_{\rm T})R_{\rm T} \\ v &= \sqrt{g(R_{\rm T})R_{\rm T}} = \sqrt{\frac{GM}{R_{\rm T}}} = \sqrt{\frac{6.67 \cdot 10^{-11} \cdot 2.88 \cdot 10^{15}}{5.0 \cdot 10^3}}\ \frac{\text{m}}{\text{s}} = 6.20\ \frac{\text{m}}{\text{s}}. \end{aligned}$$

c) If a rock is thrown into a circular orbit around Toro at a height 200 m above the surface, what is its period of revolution?

Here we have to be careful to take care of the reduction of g with height above the surface, as 200 m is a significant fraction of 5 km. We can again equate the acceleration die to gravity and that required to go in a circle, although using the acceleration $\omega^2 r$, where $r = R_{\rm T} + 200\ \text{m} = 5200\ \text{m}$, will get us to the period faster

$$\begin{aligned} \omega^2 r &= g(r) \\ \omega^2 &= \frac{g(r)}{r} \\ T = \frac{2\pi}{\omega} &= 2\pi\sqrt{\frac{r}{g(r)}} = 2\pi\sqrt{\frac{r^3}{GM}} = 2\pi\sqrt{\frac{(5.2 \cdot 10^3)^3}{6.67 \cdot 10^{-11} \cdot 2.88 \cdot 10^{15}}}\ \text{s} = 5.38 \cdot 10^3\ \text{s} = 1.49\ \text{hr}. \end{aligned}$$

2. A conveyor unloads sand with a horizontal velocity of v_0 into a cart at rest on a level track with a mass rate of m_0 per unit time. The top of the conveyor is a height h above the floor of the cart. Determine the force that the sand applies to the cart at time $t > 0$. Ignore the height of the sand pile.



Note that the cart *remains* at rest and we are to find the force on it due to the sand at any time $t > 0$ after the first sand starts to fall. First we must find the velocity of the sand when it hits the cart floor. Define the x axis pointing horizontally away from the conveyor and the y axis pointing down. Then since gravity points directly along the y axis the x motion is unaffected during the falling of the sand, so that $v_x(t) = v_0$. The velocity in the y direction is just given by the constant acceleration formula $v_y = gt = \sqrt{2gh}$, since the time taken to fall the height h is $\sqrt{2h/g}$.

To find the force exerted by the sand on the cart we first need to find the force that the cart must exert on the sand in order to brake its fall. In a time dt an amount $dm = m_0 dt$ falls and comes to rest in the cart. The change in momemntum of the piece dm of the sand is $d\mathbf{p} = dm(-v_x, -v_y) = -dm(v_0, \sqrt{2gh})$. This change in momentum is caused by a force from the cart of

$$\mathbf{F}_{\text{cart}} = \frac{d\mathbf{p}}{dt} = -\frac{dm}{dt}\left(v_0, \sqrt{2gh}\right) = -m_0\left(v_0, \sqrt{2gh}\right),$$

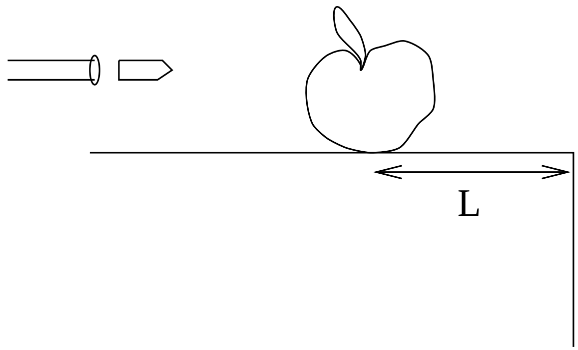
and by Newton's third law the sand must exert an equal and opposite force on the cart

$$\mathbf{F}_{\text{sand}} = m_0\left(v_0, \sqrt{2gh}\right).$$

We have to be careful to add the weight of the sand already in the cart to this reaction force. so that at time $t > 0$ we have a force on the cart due to the sand of

$$\mathbf{F}_{\text{sand}} = m_0\left(v_0, \sqrt{2gh}\right) + m_0 t g(0, 1) = m_0\left(v_0, gt + \sqrt{2gh}\right).$$

3. An apple of mass M is placed on a table. Let the coefficient of kinetic friction between the apple and the table be μ_k. A gun is fired at the apple, and a bullet of mass m and initial velocity v_0 hits the apple, and eventually ends up stuck in it.



a) What is the initial velocity of the apple?

The collision between the bullet and apple is inelastic, since we know that at the least structural damage is done to the apple, which will dissipate energy. We will *assume* that momentum is conserved even though there is an external force on the apple/bullet system during the collision, which is that of friction. This is a good approximation since the collision takes place over a very short time dt and so the change $d\mathbf{p} = \mathbf{F}_\mathrm{f}dt$ in the momentum of the apple/bullet system due to the frictional force $\mathbf{F}_\mathrm{f}$ will necessarily be small.

The initial momentum is mv_0 and so the velocity v of the apple/bullet system is given by

$$\begin{aligned}(m+M)v &= mv_0 \\ v &= \frac{mv_0}{m+M}.\end{aligned}$$

b) After what time will the apple stop (if it placed far from the edge of the table)?

The acceleration due to the frictional force is given by $F_\mathrm{f}/(m+M)$, and since the frictional force is just $F_\mathrm{f} = \mu_\mathrm{k}N = \mu_\mathrm{k}(m+M)g$ we have that the acceleration is just $a = \mu_\mathrm{k}g$. Then the time taken to stop is simply

$$t = v/a = \frac{mv_0}{(m+M)\mu_\mathrm{k}g}.$$

c) If the apple is initially placed at a distance L from the edge of the table, what is the minimum velocity v_min of the bullet, such that the apple falls from the table?

The minimum velocity is attained when the apple just stops when it reaches the edge of the table. This is still constant acceleration with acceleration $\mu_\mathrm{k}g$. The initial velocity is $v' = mv_\mathrm{min}/(m+M)$, so that the time required to stop is $t' = v'/(\mu_\mathrm{k}g)$, and so using the formula for the displacement under constant acceleration

$$x(t') = L = v't' - \frac{1}{2}\mu_\mathrm{k}gt'^2 = \frac{v'^2}{\mu_\mathrm{k}g} - \frac{1}{2}\mu_\mathrm{k}g\frac{v'^2}{(\mu_\mathrm{k}g)^2} = \frac{v'^2}{2\mu_\mathrm{k}g},$$

so that $v' = \sqrt{2\mu_k gL}$ and solving for v_min we have

$$v_\mathrm{min} = \frac{m+M}{m}v' = \left(1+\frac{M}{m}\right)\sqrt{2\mu_k gL}.$$

d) If a second apple is initially placed exactly half-way between the first apple and the end of the table (a distance $L/2$ from both), describe what will happen to the two apples following the bullet impact. Assume that $v_0 > v_{\rm min}$, and that both apples have the same mass M.

This is a head-on collision between two objects of the same mass. *If* the collision is elastic, then to conserve energy and momentum the first apple must stop and the second must travel on with the same speed the first had before the collision (and it will then fall off the table). However, the collision between two soft objects like apples is *very unlikely* to be elastic. An inelastic collision between two objects of the same mass is best analysed in the center of mass frame, where the relative velocity of the two masses before the collision (just half the velocity v of the first apple before the collision) is reduced by the coefficient of restitution ϵ. Converting back to the lab frame, which still moves at $v/2$ by conservation of momentum, we have that immediately after the collision the first apple moves forward with a speed $v(1-\epsilon)/2$, and the second moves forward with a speed $v(1+\epsilon)/2$. Depending on the size of v_0 and ϵ we may have neither, the second, or both apples falling off the table.

4a) A cylindrically shaped log is placed in water. The log is 15.0 cm in diameter and 1.0 m long, and is placed in the water oriented with its length along the surface. Additionally, a 1.0 m long uniform rod with a mass of 2.5 kg is placed along the top of the log. With this rod in place, the log settles at an equilibrium position floating in the water with exactly 7.5 cm (half its diameter) exposed above the surface, and 7.5 cm below the surface. What is the density of the log?

According to Archimedes' principle, the weight of the log plus rod is equal to the weight of the water they displace when floating. This can be found using the volume of the water displaced multiplied by its density and g, so that

$$\begin{aligned}(M_{\rm log} + M_{\rm rod})g &= V_{\rm displ}\rho_{\rm w} g \\ M_{\rm log} &= V_{\rm displ}\rho_{\rm w} - M_{\rm rod}.\end{aligned}$$

Dividing by the volume of the log $V_{\rm log} = \pi r^2 l$ where $r = 7.5$ cm, we find

$$\rho_{\rm log} = \frac{M_{\rm log}}{V_{\rm log}} = \frac{V_{\rm displ}}{V_{\rm log}}\rho_{\rm w} - \frac{M_{\rm rod}}{V_{\rm log}} = \frac{\rho_{\rm w}}{2} - \frac{M_{\rm rod}}{V_{\rm log}} = \left(500 - \frac{2.5}{\pi(0.075)^2\cdot 1.0}\right)\frac{\rm kg}{\rm m^3} = 359\ \frac{\rm kg}{\rm m^3}.$$

b) Consider the exact same situation as described above, except that instead of the log being placed into water, it is now placed in a mixture of alcohol and water. The mixture is 18.0 percent (by weight) alcohol and 82.0 percent water. What would the density of the log now need to be in order to make it float as described in part a)?

We simply have to adjust the density of the liquid from that of the water to a weighted average of the densities of the water and alcohol (given with the 'Useful constants'),

$$\rho_l = (0.82\cdot 1000 + 0.18\cdot 806)\ \frac{\rm kg}{\rm m^3} = 965\ \frac{\rm kg}{\rm m^3},$$

so that now

$$\rho_{\text{log}} == \frac{\rho_l}{2} - \frac{M_{\text{rod}}}{V_{\text{log}}} = \left(483 - \frac{2.5}{\pi(0.075)^2 \cdot 1.0}\right)\ \frac{\text{kg}}{\text{m}^3} = 341\ \frac{\text{kg}}{\text{m}^3}.$$