PHY6938 Proficiency Exam Fall 2002
September 13, 2002
E & M

1. **A positive charge is uniformly distributed throughout a very long cylindrical volume of radius R. The charge per unit volume is ρ.**

   (a) **Find the electric field $\vec{E}$ everywhere as a function of the distance r from the axis of the cylinder.**

   Let's start by finding the electric field outside of the cylinder. Build a Gaussian surface out of a cylinder of length L and radius $r > R$. Then by symmetry the field must point radially outward on the curved part of the Gaussian surface and must be parallel to the surface on the end caps. This means that the flux of the electric field through this surface is

$$\oint \mathbf{E} \cdot d\mathbf{A} = E(2\pi r L). \tag{1}$$

   Now Gauss's law tells us that this is also $Q_{\text{encl}}/\epsilon_0$, where Q_{encl} is the enclosed charge. The volume of the charged cylinder cut out by our surface is $V = \pi R^2 L$, so the enclosed charge is

$$Q_{\text{encl}} = \pi R^2 L \rho \tag{2}$$

   and Gauss's law gives

$$E(2\pi r L) = \frac{1}{\epsilon_0}\pi R^2 L\rho, \quad E = \frac{\rho R^2}{2\epsilon_0 r}. \tag{3}$$

   Since the electric field points radially outward, we have

$$\mathbf{E} = \frac{\rho R^2}{2\epsilon_0 r}\hat{\mathbf{r}}. \tag{4}$$

   Inside the cylinder we use exactly the same technique, and everything goes as before except now the charge enclosed by the Gaussian surface is reduced to

$$Q_{\text{encl}} = (\pi r^2 L)\rho \tag{5}$$

   which means that the electric field is (take $R \to r$ in the above)

$$\mathbf{E} = \frac{\rho r}{2\epsilon_0}\hat{\mathbf{r}}. \tag{6}$$

   (b) **Find the electric potential V everywhere as a function of r. Define $V = 0$ at the surface of the cylinder.**

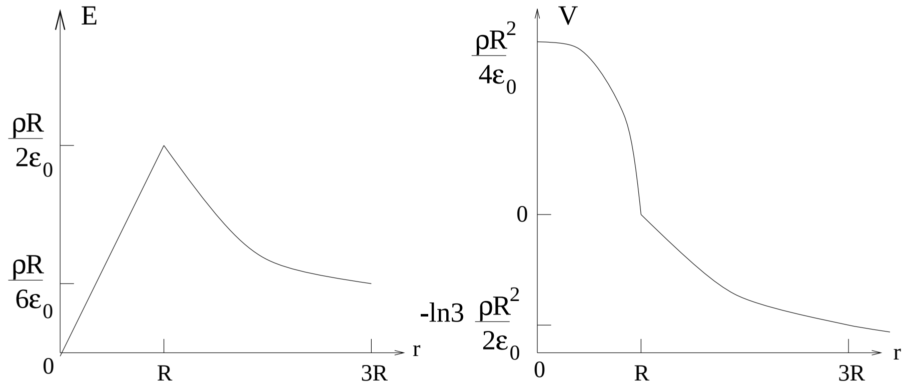
   Starting again with the potential outside the cylinder, we know that the potential will be the highest at the surface of the cylinder as the electric field always point outward (and so a charged particle will lose energy as it moves away from the cylinder). This means that

$$V(r') = -\int_R^{r'} \mathbf{E}(r) \cdot d\mathbf{r} = -\int_R^{r'} \frac{\rho R^2}{2\epsilon_0 r} dr = -\frac{\rho R^2}{2\epsilon_0} \ln\left(\frac{r'}{R}\right). \tag{7}$$

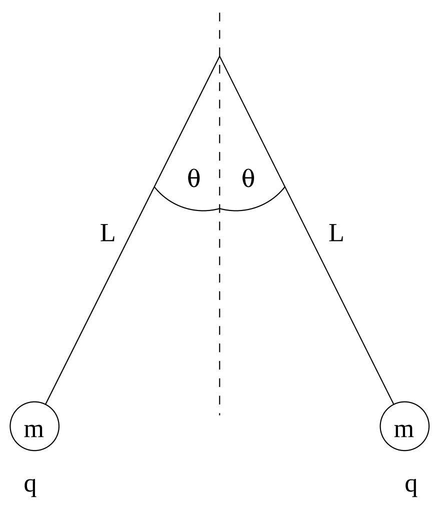
Inside the cylinder the potential increases as we move away from $r = R$, so we have

$$V(r') = \int_{r'}^{R} \mathbf{E}(r) \cdot d\mathbf{r} = \int_{r'}^{R} \frac{\rho r}{2\epsilon_0}\, dr = \frac{\rho}{4\epsilon_0}\left(R^2 - r'^2\right). \tag{8}$$

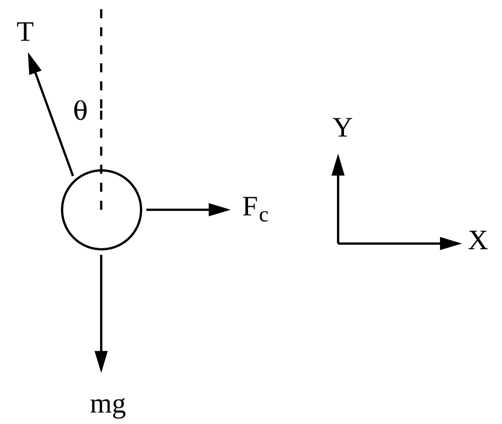
(c) Sketch E and V as function of r, from $r = 0$ to $r = 3R$, showing the values of each at $r = 0$, R, and $3R$.



2. Two small spheres of mass m are suspended from a common point by threads of length L. When each sphere carries a charge q, each thread makes an angle θ with the vertical as shown in the figure. Find an expression for the charge q at equilibrium in terms of L, m, g, θ and the Coulomb constant k.



The spheres are in equilibrium, this means that the total force affecting these two spheres is equal zero. To solve the problem we draw free body diagram for one of the spheres.



T = Tension in the string

F_c = Coulom force

Write equations of motion for one of the spheres. Using figure above we obtain

$$\begin{aligned} m\ddot{x} &= F_c - Tsin(\theta) \\ &= \frac{kq^2}{r^2} - Tsin(\theta) \\ &= \{r = 2Lsin(\theta)\} \\ &= \frac{kq^2}{(2Lsin(\theta))^2} - Tsin(\theta) \\ &= 0 \end{aligned} \tag{1}$$

and

$$\begin{aligned} m\ddot{y} &= Tcos(\theta) - mg \\ &= 0. \end{aligned} \tag{2}$$

From Eq.1 and 2 we obtain

$$\begin{aligned} Tsin(\theta) &= \frac{kq^2}{(2Lsin(\theta))^2} \\ T &= \frac{kq^2}{(2Lsin(\theta))^2 sin(\theta)} \qquad (3) \\ Tcos(\theta) &= mg \\ T &= \frac{mg}{cos(\theta)}. \qquad (4) \end{aligned}$$

From Eq.3 and 4 we get

$$\begin{aligned} \frac{kq^2}{(2Lsin(\theta))^2 sin(\theta)} &= \frac{mg}{cos(\theta)} \\ q^2 &= \frac{(2Lsin(\theta))^2 mgsin(\theta)}{kcos(\theta)}. \\ q &= \pm 2Lsin(\theta)\sqrt{\frac{mg}{kcos(\theta)}}. \end{aligned} \tag{5}$$

Note, here $sin(\theta)$ and $cos(\theta)$ are always positive, since $\theta \leq 90$.

3. A proton traveling with a velocity of $\vec{v} = 1\times10^4\text{m/s}\,\hat{\imath}+2\times10^4\text{m/s}\,\hat{\jmath}$ is located at $x = 3$m, $y = 4$m at some time t. Find the magnetic field at time t at the following positions:

Since the proton moves very fast, for an observer it seems that there is a "constant current". But for this case there is only one charged particle which moves with constant speed. The current density for this motion is given by

$$\mathbf{J} = e\mathbf{v}. \tag{1}$$

The magnetic field due to a current-carrying circuit is given by the Biot-Savarts law

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi}\left(\frac{\mathbf{d}l' \times \mathbf{R}}{R^3}\right), \tag{2}$$

where dl' is the element of the circuit, $\mathbf{R}$ is the vector which point from dl' to field point. In this problem there is only one charged particle. This means that we do not need to integrate to find $\mathbf{B}$. The current I is replaced by the current density J. The modified Biot-Savart's law will be

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left(\frac{\mathbf{J} \times \mathbf{R}}{R^3}\right). \tag{3}$$

Substitute Eq.1 in Eq.3 we obtain

$$\mathbf{B} = \frac{\mu_0 e}{4\pi}\left(\frac{\mathbf{v} \times \mathbf{R}}{R^3}\right). \tag{4}$$

For each case calculate $\mathbf{R}$ and substitute in Eq.4.

(a) $x = 2$m,$y = 2$m

$$\begin{aligned} \mathbf{R}_a &= -(\hat{\imath} + 2\hat{\jmath}) && (5)\\ \mathbf{B}_a &= 0\ T. && (6) \end{aligned}$$

(b) $x = 6$m,$y = 4$m

$$\begin{aligned} \mathbf{R}_b &= 3\hat{\imath} && (7)\\ \mathbf{B} &= -3.56\times10^{-23}\hat{k}\ T. && (8) \end{aligned}$$

(c) $x = 3$m,$y = 6$m

$$\begin{aligned} \mathbf{R}_c &= 2\hat{\jmath} && (9)\\ \mathbf{B} &= 4\times10^{-23}\hat{k}\ T. && (10) \end{aligned}$$