PHY6938 Proficiency Exam Spring 2003
March 28, 2003
Optics and Thermodynamics

1. **Light of wavelength 300 nm strikes a metal plate, producing photoelectrons that move with speed of $0.002c$.**

   In the photoelectric effect the incoming photons remove electrons from the target. There is a minimum energy required to remove an electron from the interior of a solid to a position just outside. This minimum energy is called the work function Φ. The relation between the energy of photon E_γ, the energy of electron E_e, and the work function Φ is given by

   $$E_\gamma = E_e + \Phi. \tag{1}$$

   Note that energy of electron E_e is equal to the kinetic energy of electron, since when electron is outside the solid it is not affected by any potential. In other words electron will be considered as a free particle.

   (a) **What is the work function of the metal ?**

   Use Eq.1, since the speed of the electron is much less than speed of the light we do not need to use the special relativity formula for the kinetic energy.

   $$\begin{aligned} E_e &= \frac{1}{2}\, mv^2 & (2)\\ &= \frac{1}{2}\,(511\,\frac{keV}{c^2})\,(0.002c)^2 & \\ &= 1.022\, eV & (3)\\ E_\gamma &= h\nu & (4)\\ &= \frac{hc}{\lambda} & \\ &= \frac{1420\, eV\ nm}{300\, nm} & \\ &= 4.13\, eV & (5)\\ \Phi &= E_\gamma - E_e & (6)\\ &= 3.111\, eV. & (7)\end{aligned}$$

   (b) **What is the critical wavelength for this metal, so that photoelectrons are produced ?**

   The critical wavelength is defined as the wavelength for photons which remove remove electrons from interior of a solid to a position outside the solid. In this case the kinetic energy of electrons outside the solid is equal zero $E_e = 0$. Using the Eq.1 and the results from part (a) we obtain

   $$\begin{aligned} E_\gamma &= \Phi & (8)\\ \frac{hc}{\lambda_{critical}} &= \Phi & \end{aligned}$$

$$\lambda_{critical} = \frac{hc}{\Phi} \tag{9}$$

$$\lambda_{critical} = \frac{1420\ eV}{3.111\ eV}$$

$$\lambda_{critical} = 398.6\ eV. \tag{10}$$

(c) What is the significance of the critical wavelength ?

Photons with a lower wavelength than the critical wavelength cannot produce photoelectrons. This led Einstein to the postulation of quantized energy for the electromagnetic fields.

2. **Two lenses are separated by 35 cm. An object is placed 20 cm to the left of the first lens, which is a converging lens of focal length 10 cm. The second lens is a diverging lens of focal length −15 cm. What is the position of the final image? Is the image real or virtual? Erect or inverted? What is the overall magnification of the image?**

We can find everything we need from the lens formula(e)

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}, \quad m = -\frac{s'}{s}, \tag{1}$$

being careful to use a negative f for the diverging lens (since the focal point of light on the incident side is on the incident side and not the transmitted side, it is therefore negative), and use the figure to check our answers. For the first lens we have

$$\frac{1}{s_1'} = \frac{1}{f_1} - \frac{1}{s_1} = \frac{1}{10\text{ cm}} - \frac{1}{20\text{ cm}} = \frac{1}{20\text{ cm}}, \tag{2}$$

so that the first real image is 20 cm from the lens and has

$$m_1 = -\frac{s_1'}{s_1} = -1, \tag{3}$$

so that the image is the same size as the object and inverted.

For the second lens we have

$$\frac{1}{s_2'} = \frac{1}{f_2} - \frac{1}{s_2} = \frac{1}{-15\text{ cm}} - \frac{1}{15\text{ cm}} = -\frac{2}{15\text{ cm}}, \tag{4}$$

so that the image is virtual (it is on the incident side, since $s_2' < 0$) and 7.5 cm to the left of the second lens. The magnification of the second lens is

$$m_2 = -\frac{s_2'}{s_2} = -\frac{-7.5\text{cm}}{15\text{ cm}} = -\frac{1}{2}, \tag{5}$$

so that the combined magnification is $m = m_1 m_2 = -0.5$ and the virtual image is inverted with an overall magnification of -0.5.

3. Consider a system consisting of N noninteracting spins $S = 1/2$ in a magnetic field H. The Helmholtz free energy due to the Zeeman splitting $g\mu_B H$ is given by

$$F = -NkT \ln[2\cosh(g\mu_B H/2kT)] \ ,$$

where k is the Boltzmann constant and T is the temperature.

This is a typical statistical problem. In these type of problems, first of all we have to determine the partition function Z, and from that different thermodynamical properties of the system are derived. Here the Helmholtz free energy F is already given. This means that we just have to calculate what is required in each part.

(a) Obtain the magnetization using $M = -\partial F/\partial H$.

Derivating gives

$$\begin{aligned} M &= NkT\, Tanh(\frac{g\mu_B H}{2kT})\, \frac{g\mu_B}{2kT} \\ &= \frac{Ng\mu_B}{2}\, Tanh(\frac{g\mu_B H}{2kT}). \end{aligned} \tag{1}$$

(b) What is the maximum magnetization?

The only function which varies in Eq.1 in $Tanh$. The range of it is $-1 \leq Tanh(x) \leq 1$. This means that

$$M_{max} = \frac{Ng\mu_B}{2}. \tag{2}$$

(c) For $g\mu_B H \ll kT$ obtain the susceptibility defined as $\chi = M/H$.

When $x \ll 1$ then $Tanh(x) \approx x$. This means that

$$M \approx \frac{Ng\mu_B}{2}\, \frac{g\mu_B H}{2kT} \tag{3}$$

$$\chi = N\frac{(g\mu_B)^2}{4kT} \tag{4}$$

Eq.4 is know as Curie's law.

(d) Obtain the entropy using $S = -(\partial F/\partial T)_H$.

$$S = Nk\Big\{\ln[2cosh(\frac{g\mu_B H}{2kT})] - \frac{g\mu_B H}{2kT}\, Tanh(\frac{g\mu_B H}{2kT})\Big\}. \tag{5}$$

(e) Discuss the entropy in the limits $H = 0$ and $g\mu_B H \gg kT$.

When $H \to 0$, then $Tanh(\frac{g\mu_B H}{2kT}) \to 0$ and $cosh(\frac{g\mu_B H}{2kT}) \to 1$ this means that

$$S = Nk\ln(2), \quad \text{Two degrees of freedom for each spin.} \tag{6}$$

When $g\mu_B H \gg kT$ then $Tanh(\frac{g\mu_B H}{2kT}) \rightarrow 1$. For the $cosh$ we can argue as follows

$$\begin{aligned}
cosh(x) &= \frac{e^x + e^{-x}}{2} \\
\text{when } x \rightarrow \infty \text{ , then} & \\
cosh(x) \approx \frac{e^x}{2} & \\
\text{this means that} & \\
cosh(\frac{g\mu_B H}{2kT}) &\approx \frac{1}{2}\,\mathrm{Exp}[\frac{g\mu_B H}{2kT}].
\end{aligned} \tag{7}$$

Substituting these results in Eq.5 yields

$$S = Nk\left\{\frac{g\mu_B H}{2kT} - \frac{g\mu_B H}{2kT}\right\} \rightarrow 0. \tag{8}$$

(f) Obtain the specific heat and sketch C/Nk as a function of $g\mu_B H/kT$.
The specific heat is given by

$$C = T\frac{\partial S}{\partial T} = Nk\,\frac{(\frac{g\mu_B H}{2kT})^2}{cosh^2(\frac{g\mu_B H}{2kT})} \tag{9}$$