Proficiency Exam, Fall 2001
Friday, September 14
Part 1 – 9:00 a.m. to 1:00 p.m.

Useful constants:

- $e = 1.60 \times 10^{-19}$ C
- $hc = 1240$ eV $\cdot$ nm
- $c = 3.00 \times 10^{8}$ $\frac{\mathrm{m}}{\mathrm{s}}$
- $m_e = 0.511$ $\frac{\mathrm{MeV}}{\mathrm{c}^2}$
- $k = \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^{9}$ $\frac{\mathrm{Nm}^2}{\mathrm{C}^2}$
- $\epsilon_0 = 8.85 \times 10^{-12}$ $\frac{\mathrm{C}^2}{\mathrm{Nm}^2}$
- $\frac{\mu_0}{4\pi} = 10^{-7} \frac{\mathrm{N}}{\mathrm{A}^2}$
- $g = 9.81$ $\frac{\mathrm{m}}{\mathrm{s}^2}$

1. **Four infinitely long wires parallel to the z-axis are arranged with $(x, y) = (0, d)$, $(0, -d)$, $(d, 0)$, and $(-d, 0)$, respectively. (Here $d > 0$.) The wires at $(0, d)$ and $(0, -d)$ carry a current of magnitude i in the positive z-direction, and the wires at $(d, 0)$ and $(-d, 0)$ carry a current of magnitude i in the negative z-direction.**

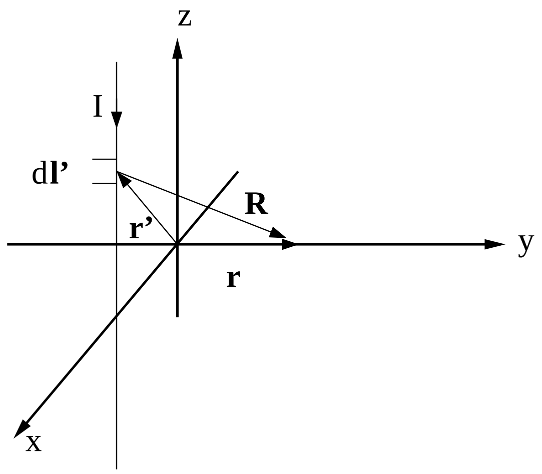
   (a) **What are the magnitude and direction of the magnetic field at any point along the x-axis?**

   For better understanding we try to explicitly calculate almost all part of the problem.

   First of all consider a point on the x-axis and a point on each one of the wires. We have to determine contribution from each wire to the total magnetic field for a point on the x-axis.

   1. First from the wire with coordinate $(0, d, z')$. Here we denote the source points with prime and the field points unprimed, see figure. The magnetic field from an element of length dl' at point $(x, 0, 0)$ is given by Biot-Savart law.

$$
\begin{aligned}
d\vec{B}_1 &= \frac{\mu_0\, I}{4\,\pi}\left(\frac{\vec{dl'} \times \vec{R}}{R^3}\right) && (1)\\
\vec{R} &= \vec{r} - \vec{r'} && (2)\\
\vec{r} &= (x, 0, 0),\ \vec{r'} = (0, d, z') && (3)\\
\Rightarrow \vec{R} &= (x, -d, z') && (4)\\
d\vec{l'} &= \hat{z}\, dl' && (5)\\
d\vec{l'} \times \vec{R} &= (\hat{x}\ d + \hat{y}\ x) && (6)
\end{aligned}
$$



Substitute Eqs.2-5 in Eq.1 and integrate

$$\vec{B}_1 = \frac{\mu_0\, I}{4\,\pi} \int_{-\infty}^{\infty} dz' \, \frac{\hat{x}\ d + \hat{y}\ x}{\left(x^2 + d^2 + z'^2\right)^{\frac{3}{2}}} \tag{7}$$

$$\text{let } x^2 + d^2 = a^2 \quad \text{and} \quad \tan(\alpha) = \frac{z'}{a}$$

$$\Rightarrow \frac{dz'}{a} = (1 + \tan^2(\alpha))\, d\alpha$$

$$d\alpha = \frac{dz'}{a\,(1 + \tan^2(\alpha))} = \frac{dz'}{a\,(1 + (\frac{z'}{a})^2)}$$

$$\text{note also } \frac{1}{1 + (\frac{z'}{a})^2} = \frac{1}{1 + \tan^2(\alpha)} = \cos^2(\alpha)$$

$$\text{and, } z' \in]-\infty\ \infty[\quad \Rightarrow \quad \alpha \in]-\frac{\pi}{2}\ \frac{\pi}{2}[\tag{8}$$

Rewrite Eq.7 we have

$$\vec{B}_1 = \frac{\mu_0\, I}{4\,\pi\, a^2}\,(\hat{x}\ d + \hat{y}\ x) \int_{-\infty}^{\infty} \frac{dz'}{a\ (1 + (\frac{z'}{a})^2)\ \sqrt{(1 + \frac{z'}{a})^2}}$$

integrand is even function of z' $f(z) = f(-z)$

$$\Rightarrow \vec{B}_1 = \frac{\mu_0\, I}{4\,\pi}\ \frac{\hat{x}\ d + \hat{y}\ x}{x^2 + d^2}\ 2 \int_0^{\frac{\pi}{2}} d\alpha\ \cos(\alpha)$$

$$\text{note that} \quad \cos(\alpha) \geq 0 \text{ for } \alpha \in]-\frac{\pi}{2}\ \frac{\pi}{2}[\tag{9}$$

$$\Rightarrow \vec{B}_1 = \frac{\mu_0\, I}{2\,\pi}\ \frac{\hat{x}\ d + \hat{y}\ x}{x^2 + d^2} \tag{10}$$

2. For the wire at $(0, -d, z')$ the calculation will be the same except that we

have to change $d \to -d$.

$$\vec{B_2} = \frac{\mu_0\, I}{2\,\pi}\;\frac{-\hat{x}\;d + \hat{y}\;x}{x^2 + d^2} \tag{11}$$

3. For the wire at $(d, 0, z')$ we can repeat the calculation. Note that the current is in the $-\hat{z}$ direction.

$$\begin{aligned}
\vec{r} &= (x, 0, 0),\;\; \vec{r'} = (d, 0, z') && (12)\\
\Rightarrow\; \vec{R} &= (x - d, 0, z') && (13)\\
d\vec{l'} &= -\hat{z}\, dl' && (14)\\
d\vec{l'} \times \vec{R} &= -\hat{y}\;(x - d) && (15)\\
\vec{B_3} &= \frac{\mu_0\, I}{4\,\pi\, a^2}\hat{y}\;(-x + d)\int_0^{\frac{\pi}{2}} d\alpha\;\cos(\alpha) && (16)\\
\text{here}, a &= x - d\\
\vec{B_3} &= -\,\hat{y}\;\frac{\mu_0\, I}{2\,\pi\,(x - d)} && (17)
\end{aligned}$$

4. For the wire at $(-d, 0, z')$ change $d \to -d$

$$\vec{B_4} = -\,\hat{y}\;\frac{\mu_0\, I}{2\,\pi\,(x + d)}. \tag{18}$$

The total magnetic field for point on x-axis is

$$\begin{aligned}
\vec{B} &= \vec{B_1} + \vec{B_2} + \vec{B_3} + \vec{B_4}\\
&= \frac{\mu_0\, I}{2\,\pi}\;\frac{\hat{x}\;d + \hat{y}\;x}{x^2 + d^2} + \frac{\mu_0\, I}{2\,\pi}\;\frac{-\hat{x}\;d + \hat{y}\;x}{x^2 + d^2}\\
&- \;\hat{y}\;\frac{\mu_0\, I}{2\,\pi\,(x - d)} - \hat{y}\;\frac{\mu_0\, I}{2\,\pi\,(x + d)}\\
&= \;\hat{y}\;\frac{\mu_0\, I}{\pi}\left[\frac{x}{x^2 + d^2} - \frac{x}{x^2 - d^2}\right]\\
&= -\,\hat{y}\;\frac{2\,\mu_0\, I}{\pi}\;\frac{x\;d^2}{x^4 - d^4} && (19)
\end{aligned}$$

(b) What are the magnitude and direction of the magnetic field at any point along the y-axis?

We can repeat this for y axis. It is possible to make proper changes in the calculation for part (a) to get the result for part (b) without calculating. But we repeat the calculation this part.

1. For the wire at $(d, 0, z')$

$$\begin{aligned}
\vec{R} &= \vec{r} - \vec{r'} && (20)\\
\vec{r} &= (y, 0, 0),\;\; \vec{r'} = (d, 0, z') && (21)\\
\Rightarrow\; \vec{R} &= (-d, y, -z') && (22)\\
d\vec{l'} &= -\hat{z}\, dl' && (23)\\
d\vec{l'} \times \vec{R} &= (\hat{x}\;y + \hat{y}\;d) && (24)
\end{aligned}$$

and we get

$$\begin{aligned}
\vec{B}_1 &= \frac{\mu_0 I}{4\pi a^2} (\hat{x}\ y + \hat{y}\ d) \int_0^{\frac{\pi}{2}} 2\ d\alpha\ \cos(\alpha) \\
& \quad \text{with } a^2 = y^2 + d^2 \\
\Rightarrow \vec{B}_1 &= \frac{\mu_0 I}{2\pi} \frac{\hat{x}\ y + \hat{y}\ d}{y^2 + d^2} \qquad (25)
\end{aligned}$$

2. For the wire at $(-d, 0, z')$ change $d \to -d$

$$\vec{B}_2 = \frac{\mu_0 I}{2\pi} \frac{\hat{x}\ y - \hat{y}\ d}{y^2 + d^2} \qquad (26)$$

3. For the wire at $(0, d, z')$ we have

$$\begin{aligned}
\vec{r} &= (0, y, 0),\ \vec{r'} = (0, d, z') && (27) \\
\Rightarrow \vec{R} &= (0, y - d, z') && (28) \\
d\vec{l'} &= \hat{z}\, dl' && (29) \\
d\vec{l'} \times \vec{R} &= -\hat{x}\ (y - d) && (30) \\
\vec{B}_3 &= -\frac{\mu_0 I}{4\pi a^2} \hat{x}\ (y - d) \int_0^{\frac{\pi}{2}} d\alpha\ \cos(\alpha) && (31) \\
\text{here, } a &= y - d \\
\vec{B}_3 &= -\hat{x}\ \frac{\mu_0 I}{2\pi (y - d)} && (32)
\end{aligned}$$

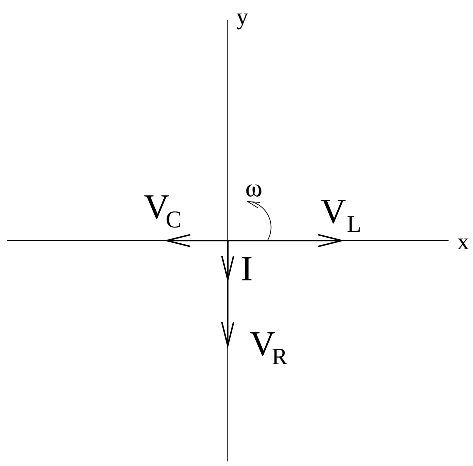
4. For the wire $(0, -d, z')$ we get

$$\vec{B}_4 = -\hat{x}\ \frac{\mu_0 I}{2\pi (y + d)} \qquad (33)$$

The total magnetic field at a point on y-axis is given by

$$\begin{aligned}
\vec{B} &= \vec{B}_1 + \vec{B}_2 + \vec{B}_3 + \vec{B}_4 \\
&= \frac{\mu_0 I}{2\pi} \frac{\hat{x}\ y + \hat{y}\ d}{y^2 + d^2} + \frac{\mu_0 I}{2\pi} \frac{\hat{x}\ y - \hat{y}\ d}{y^2 + d^2} \\
&+ \hat{x}\ \frac{\mu_0 I}{2\pi (x - d)} + \hat{x}\ \frac{\mu_0 I}{2\pi (x + d)} \\
&= \hat{x}\ \frac{\mu_0 I}{\pi} \left[\frac{y}{y^2 + d^2} - \frac{y}{y^2 - d^2} \right] \\
&\Rightarrow \quad \vec{B} = -\hat{x}\ \frac{2\mu_0 I}{\pi} \frac{y\, d^2}{y^4 - d^4} \qquad (34)
\end{aligned}$$

2. A series LCR circuit is driven by an emf with angular frequency ω, so that the voltage across the resistor is $V_R(t) = V_0 \sin(\omega t)$. In answering the following questions, express everything in terms of the given quantities: L, C, R, ω, V_0, and ε_0.



(a) Draw a phaser diagram for V_R, V_L, and V_C, and specify the angles between the phasers.

This is a series circuit, so the current in every element of the circuit is the same common function of time $I_R(t) = I_L(t) = I_C(t) = I(t)$. The solution of the problem lies in finding the relationships between the voltage across each of the circuit elements and this current.

Using Warren's mnemonic 'ELI the ICE man' we have that E leads I in the inductor L, and I leads E in the capacitor C. Now we are given that the voltage across the resistor, which is in phase with and proportional to the current through it, is $V_0 \sin(\omega t)$, which means that the total current in the circuit is

$$I(t) = \frac{V_R(t)}{R} = \frac{V_0}{R}\sin(\omega t). \tag{1}$$

So we can represent the current in the circuit and the voltage $V_R(t)$ by phasers lying along the negative y axis and rotating counterclockwise, so that they start with x-projection zero and increase like $\sin(\omega t)$. This means that the inductor voltage $V_L(t)$, which leads $I(t)$ by 90°, lies along the positive x-axis. The capacitor voltage $V_C(t)$ is led by the current by 90°, so it must lie along the negative y-axis, as shown in the figure.

(b) What is the current $I_L(t)$ through the inductance?
The current through the inductance is just

$$I_L(t) = I(t) = \frac{V_0}{R}\sin(\omega t). \tag{2}$$

Find the voltage $V_L(t)$ across the inductance.

The magnitude of the voltage across the inductor is

$$V_L = X_L I = \omega L \frac{V_0}{R}, \tag{3}$$

and since it leads the current it is

$$V_L(t) = \omega L \frac{V_0}{R}\cos(\omega t). \tag{4}$$

(d) **Find the voltage $V_C(t)$ across the capacitance.** The magnitude of the voltage across the capacitor is

$$V_C = X_C I = \frac{1}{\omega C}\frac{V_0}{R}, \tag{5}$$

and since it lags the current it is

$$V_C(t) = -\frac{1}{\omega C}\frac{V_0}{R}\cos(\omega t). \tag{6}$$

(e) **Determine the angular frequency, ω_0, for which the voltage $V_R(t)$ across the resistance is the same as that across the emf source, i.e. $V_R(t) = \varepsilon(t)$.**

From the above we see that the voltages across the inductor and capacitor will cancel when

$$\omega_0 L = \frac{1}{\omega_0 C}, \quad \omega_0 = \frac{1}{\sqrt{LC}}, \tag{7}$$

and when this is true the voltage across the resistor will simply be the applied voltage $\varepsilon(t)$.

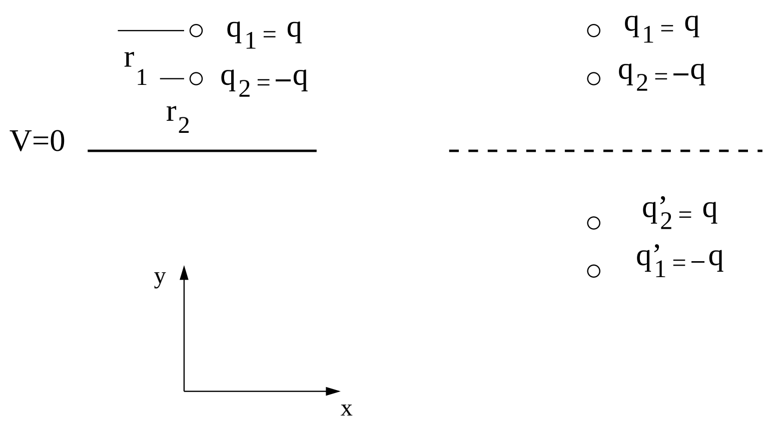
3. **A cloud is 200 m thick (200 m from the top to the bottom of the cloud). There is a positive point charge $+q$ at the top of the cloud and the same negative point charge $-q$ at the bottom. When the cloud is 300 m directly above a certain point on the ground (as measured from the bottom), the electric field at that point on the ground is measured to be 100 V/m. Assume that there are no other charges around in space, and that the ground is a perfect conductor.**

   (a) **What is the magnitude of the point charges?**

   Since the ground is considered as a perfect conductor the charges in above it induce charges in the ground. In the same way that we hold a positive charge close to a piece of metal then we can feel that there is an attracting force between the positive charge and the piece of metal. There reason is that the positive charge attracts free electrons inside the metal and therefore there will be an attractive Coulomb force between these two. The other point is that the potential on the ground is zero. Using these two facts we use method of images to solve this problem. In this method we remove the earth and replace it with the image of charges above it but with the opposite sign, in the way to make the potential on the earth zero. In this problem the images will be as following

   Using the notations in the figure, the electric field at the point right under the charges will be given by

$$\begin{aligned}
\vec{E} &= -\hat{y}\frac{k\,q_1}{r_1^2} - \hat{y}\frac{k\,q_2}{r_2^2} + \hat{y}\frac{k\,q_1'}{r_1'^2} + \hat{y}\frac{k\,q_2'}{r_2'^2} \\
&= \hat{y}\,k\,q\left[-\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_2^2} - \frac{1}{r_1^2}\right]
\end{aligned}$$



$$
\begin{aligned}
&= \hat{y}\,\frac{9\times 10^{9}\times q\times 2}{10^{4}}\left[\frac{1}{9}-\frac{1}{25}\right]\\
&= \hat{y}\,128\times 10^{3}\times q\\
\text{but } \vec{E} &= \hat{y}\,100\,\mathrm{V/m}\\
\Rightarrow q &= \frac{1}{1280}\\
&= 7.8125\times 10^{-4}\,\mathrm{C}. \qquad (8)
\end{aligned}
$$

(b) What is the electric force on the cloud, due to the Earth?

The force is given by

$$
\begin{aligned}
\vec{F} &= \hat{y}\,\frac{k\,q'_1\,q_1}{(2r_1)^2}+\hat{y}\,\frac{k\,q'_1\,q_2}{(r_1+r_2)^2}+\hat{y}\,\frac{k\,q'_2\,q_2}{(2r_2)^2}+\hat{y}\,\frac{k\,q'_2\,q_1}{(r_1+r_2)^2}\\
&= \hat{y}\;k\,q^2\left[-\frac{1}{4\,r_1^2}+\frac{1}{(r_1+r_2)^2}-\frac{1}{4\,r_2^2}+\frac{1}{(r_1+r_2)^2}\right]\\
&= \hat{y}\;\frac{9\times 10^{9}}{(1280)^2\times 10^{4}}\left[\frac{2}{64}-\frac{1}{100}-\frac{1}{36}\right]\\
&= \hat{y}\,\frac{9\times 10^{3}}{1280^2}\left[\frac{225-72-200}{7200}\right]\\
&= -\hat{y}\;\frac{9\times 47}{16384\times 72\times 10}\\
&= -\hat{y}\;3.585\times 10^{-5}\ \mathrm{N}. \qquad (9)
\end{aligned}
$$