PHY531 Problem Set 2. Due February 5, 2015

1. You likely learned about general orthogonal coordinates in your undergraduate mathematical methods course. Let's apply the method to spheroidal coordinates. We use a metric tensor, but don't be afraid. The metric tensor here is your friend and you just have to plug into the standard formula, Eq 3, below. The metric tensor simplifies the ugly algebra you would otherwise get if you just ground through the transformation of derivatives using the chain rule.

    a. Calculate the metric tensor for the transformation to oblate spheroidal coordinates (ξ, γ, ϕ) defined by

    $$\begin{aligned} x &= c\sqrt{1+\xi^2}\sin\gamma\cos\phi \\ y &= c\sqrt{1+\xi^2}\sin\gamma\sin\phi \\ z &= c\,\xi\;\cos\gamma \end{aligned} \tag{1}$$

    and for prolate spheroidal coordinates defined by

    $$\begin{aligned} x &= c\xi\sin\gamma\cos\phi \\ y &= c\xi\sin\gamma\sin\phi \\ z &= c\sqrt{1+\xi^2}\cos\gamma \end{aligned} \tag{2}$$

    The metric tensor g_{ij} is defined to be

    $$g_{ij} = \sum_{k=1}^{3} \frac{\partial x_k}{\partial u_i}\frac{\partial x_k}{\partial u_j} \tag{3}$$

    where x_1, x_2, x_3 are x, y, and z, and u_1, u_2, and u_3 are ξ, η, and ϕ. Explain why a diagonal metric tensor indicates an orthogonal coordinate system.

    b. The Laplacian can be derived by using the chain rule. For an orthogonal coordinate system, the Laplacian simplifies to

    $$\nabla^2 = \frac{1}{h_1 h_2 h_3}\sum_{\text{cyclic}} \frac{\partial}{\partial u_i}\left(\frac{h_j h_k}{h_i}\right)\frac{\partial}{\partial u_i} \tag{4}$$

    where the cyclic sum means that i, j, k take on the values $(1,2,3)$, $(2,3,1)$, and $(3,1,2)$, and $h_i \equiv \sqrt{g_{ii}}$. Similarly the volume element for volume integrals are $d^3r = h_1 h_2 h_3 du_1 du_2 du_3$ and the surface elements on surfaces of constant u_i are $h_j h_k du_j du_k$. The various vector derivative operations are

    $$\begin{aligned} \boldsymbol{\nabla} F &= \sum_i \frac{1}{h_i}\frac{\partial}{\partial u_i} F\hat{\boldsymbol{u}}_i\,. \\ \boldsymbol{\nabla}\cdot\boldsymbol{A} &= \frac{1}{h_1 h_2 h_3}\sum_{\text{cyclic}}\frac{\partial}{\partial u_i}(h_j h_k A_i) \\ \boldsymbol{\nabla}\times\boldsymbol{A} &= \sum_{\text{cyclic}}\frac{1}{h_j h_k}\left[\frac{\partial}{\partial u_j}(h_k A_k) - \frac{\partial}{\partial u_k}(h_j A_j)\right]\hat{\boldsymbol{u}}_i \end{aligned} \tag{5}$$

Show for oblate spheroidal coordinates we get

$$\nabla^2 = \frac{1}{c^2(\xi^2+\cos^2\gamma)}\left\{\frac{\partial}{\partial\xi}\left[(\xi^2+1)\frac{\partial}{\partial\xi}\right]+\frac{1}{\sin\gamma}\frac{\partial}{\partial\gamma}\left[\sin\gamma\frac{\partial}{\partial\gamma}\right]\right\}+\frac{1}{c^2(\xi^2+1)\sin^2\gamma}\frac{\partial^2}{\partial\phi^2} \tag{6}$$

and for prolate spheroidal coordinates

$$\nabla^2 = \frac{\sqrt{1+\xi^2}}{c^2(\xi^2+\sin^2\gamma)\xi}\frac{\partial}{\partial\xi}\left[\xi\sqrt{1+\xi^2}\frac{\partial}{\partial\xi}\right]+\frac{1}{c^2(\xi^2+\sin^2\gamma)\sin\gamma}\frac{\partial}{\partial\gamma}\left[\sin\gamma\frac{\partial}{\partial\gamma}\right]+\frac{1}{c^2\xi^2\sin^2\gamma}\frac{\partial^2}{\partial\phi^2} \tag{7}$$

2. Find values corresponding to constant ξ, γ, or ϕ along with the value of c and either prolate or oblate spheroidal coordinates that correspond to

   a. A prolate spheroid, defined by the equation

   $$\frac{x^2+y^2}{b^2}+\frac{z^2}{a^2}=1 \tag{8}$$

   where $a > b$.

   b. A plane with a circular hole of radius a.

   c. A oblate spheroid, defined by the equation

   $$\frac{x^2+y^2}{a^2}+\frac{z^2}{b^2}=1 \tag{9}$$

   where $a > b$.

   d. A disk of radius a.

3. Calculate the self capacitance (i.e. the charge on the conductor such that the potential on the conductor is one and the potential at an infinite distance is zero.) of the oblate spheroid and the prolate spheroid of problem 2. Plot the capacitance divided by $4\pi a$ as a function of the eccentricity $e = \sqrt{1-b^2/a^2}$. Take the appropriate limits to calculate the self capacitance of an infinitely thin disk of radius a and the sphere of radius a.

4. If you know one solution to a second order linear ordinary differential equation, you can find the general solution by integration. To show this, assume that $y_1(x)$ is a nontrivial solution to

   $$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)y=0 \tag{10}$$

   where $p(x)$ and $q(x)$ are arbitrary functions of x. Write the general solution as $y(x) = f(x)y_1(x)$, and by substituting into the the differential equation, show that it can be integrated to give the general form for $f(x)$ which gives the general solution. (Note, anyone who mentions or uses the Wronskian gets zero credit.)

   As a check, (i.e. substituting this into the differential equation and showing it is the answer gets you zero credit), your result should be

   $$y(x) = Ay_1(x)+By_1(x)\int dx\frac{e^{-\int dx p(x)}}{y_1^2(x)} \tag{11}$$

where A and B are arbitrary constants. Notice $q(x)$ does not appear.

Verify that the method works by applying it to the ordinary Legendre equation

$$\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \ell(\ell+1)P = 0 \tag{12}$$

for the case where $\ell = 1$, and the Legendre polynomial solution is $P_1(x) = x$, and compare your general solution to

$$AP_1(x) + BQ_1(x) \tag{13}$$

where $Q_1(x)$ is the Legendre function of the second kind (defined, for example, in Abramowtiz and Stegun 8.4.4).

5. The polarizability tensor is defined as the induced electric dipole moment when an object is introduced into a previously constant electric field,

$$p_i = \sum_{j=1}^{3} \alpha_{ij} E_j \tag{14}$$

where the subscripts are the cartesian components, and the the dipole moment can be idenitified by expanding the potential at large distance,

$$\lim_{r\to\infty} \Phi(\boldsymbol{r}) = -\boldsymbol{E}_0 \cdot \boldsymbol{r} + \frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{4\pi r^2}\,. \tag{15}$$

The spheroids of problem 2 are aligned with their axes along the cartesian directions. In this coordinate system, α_{ij} is diagonal, and from symmetry $\alpha_{xx} = \alpha_{yy}$. Calculate the polarizabilities α_{xx} and α_{zz} for the oblate and prolate spheroids. You can assume, correctly, that the angular, γ, ϕ, part of the solution is the same as that of the uniform electric field. Using the result of problem 4 and the known solution for a constant electric field, find the general solution with this angular part and use the boundary conditions to calculate the polarizabilities.

6. Far from a plane conducting surface at $z = 0$ with a circular hole of radius a centered at the origin, the electric field is $E_0\hat{\boldsymbol{z}}$ for large positive z and zero for large negative z. Use oblate spheroidal coordinates and assuming, correctly, that the potential has the same angular, γ, ϕ, dependence as a constant field in the z direction, find the effective dipole moments by looking at the potential at large positive and large negative z. You can check your result by comparing with Jackson's Eq. 3.183 after converting it to Heaviside-Lorentz units.

   Note: Make sure your potential is continuous with continuous derivatives as you cross through the hole from positive to negative z.