# Modern Physics and Quantum Mechanics
# Fall 2001 Proficiency and Diagnostic

**Useful constants:**

- $e = 1.60 \times 10^{-19}$ C
- $hc = 1240$ eV $\cdot$ nm
- $c = 3.00 \times 10^{8}\ \frac{\text{m}}{\text{s}}$
- $m_e = 0.511\ \frac{\text{MeV}}{\text{c}^2}$

1. Cosmic ray photons from space are bombarding your laboratory and smashing massive objects to pieces. Your detectors indicate that two fragments, each of mass $m_0$, depart such a collision moving at a speed of $0.6c$ at angles of $60°$ relative to the photon's original direction of motion.

   a. In terms of $m_0$ and $c$, what is the energy of the cosmic ray photon?

   b. In terms of $m_0$, what is the mass $M$ of the particle being struck (assumed to be initially stationary)?

2. The Pauli spin operators for a particle of spin 1/2 are given by the matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

   a. Write the normalized eigenvectors of $\sigma_z$, $|+\rangle$ and $|-\rangle$ which are defined such that $\sigma_z|+\rangle = |+\rangle$ and $\sigma_z|-\rangle = -|-\rangle$, as column vectors in the same basis as the Pauli matrices given above. (You can assume, without loss of generality, that these vectors are real.)

   b. Consider a wave vector $|\psi\rangle = a|+\rangle + b|-\rangle$. Assuming that $a$ is a real number, $0 \le a \le 1$, show that $b = e^{i\phi}\sqrt{1-a^2}$, where $\phi \in [0, 2\pi)$ is an arbitrary angle.

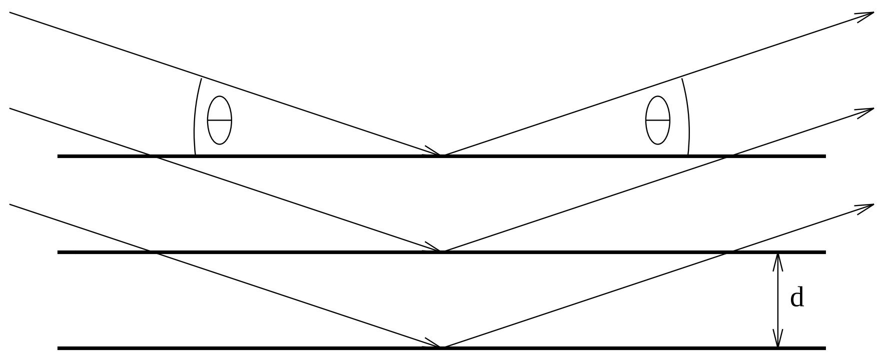
   c. Find the expectation values of $\sigma_x$, $\sigma_y$, and $\sigma_z$ in the state $|\psi\rangle$ in terms of $a$ and $\phi$.

3. Neutron scattering is often done by cooling fast neutrons, which are among the fission products from a nuclear reactor, by thermalizing them in a moderator such as solid deuterium oxide ($D_2O$ ice) held at some constant cryogenic temperature $T$. The average kinetic energy of neutrons from such a source is,

$$E = \frac{1}{2}m\langle v^2\rangle = \frac{3}{2}kT \ ,$$

   where $m$ is the neutron mass, $\langle v^2\rangle$ is the mean-square velocity, and $k$ is Boltzmann's constant.

   (a) What is the de Broglie wave length $\lambda_n$, for neutrons, in terms of their energy, mass, and Planck's constant?




(b) What is the de Broglie wave length for neutrons with the average kinetic energy in terms of the temperature of the source?

(c) Hydrogen freezes at 20.25 K and can be used to keep the $D_2O$ moderator at a fixed temperature. What is the de Broglie wave length $\lambda_n$ (in Å) for neutrons with the average energy corresponding to that temperature? What is the average neutron energy $E$ (in eV)?

(d) The neutrons are monochromatized at the 20.25 K thermal intensity maximum $\lambda_n$ and diffracted off a crystalline target with lattice spacing $d = 4.21$ Å. What is the scattering angle $\theta$ that would give Bragg diffraction of the neutrons from this crystal? Use the convention for diffraction shown in the figure.

The neutron mass is $m_{\rm n} = 1.68 \times 10^{-27}$ kg.

4. The time-independent wave function for a particle of mass $m$ which moves in a one-dimensional potential $V(x)$ has the form $\psi(x) = A\exp\left[-a^2x^2\right]$, where $a = \sqrt{m\omega/2\hbar}$ and $A$ is a normalization constant.

   a. Using the time-independent Schrödinger equation, find $V(x)$ and the energy eigenvalue for $\psi(x)$.

   b. Identify the system. Which one of its quantum states is described by $\psi(x)$?

5. The Hamiltonian for the hydrogen atom is given by

$$H = -\frac{\hbar^2}{2m}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\right] - \frac{e^2}{r}\,,$$

   where $r$, $\theta$ and $\phi$ are spherical coordinates of the position of the electron relative to the nucleus, $m$ is the reduced mass, and $e$ is the charge of the proton.

   (a) What are the quantum numbers characterizing the wave functions of the hydrogen atom?

   (b) Consider an $s$-wave state. Obtain the value of $\alpha$ for which the wave function

$$\psi(r) = C\exp(-\alpha r)$$

   is a solution of Schrödinger's equation. Find the corresponding energy eigenvalue. Calculate the normalization constant $C$.

(c) Obtain the expectation values of $\vec{p}^2$ and $1/r$ for the wave function in (b). Hint: the Hamiltonian above contains the expression for $\vec{p}^2$ in spherical coordinates.

(d) Verify that the expectation value of the kinetic energy is $-1/2$ times that of the potential energy (virial theorem).

6. Consider a quantum mechanical system with two states, $|\alpha\rangle$ and $|\beta\rangle$. In this orthonormal basis of states the Hamiltonian is given by the matrix

$$H = \begin{pmatrix} W & V \\ V & -W \end{pmatrix}.$$

(a) Obtain the exact energy eigenvalues.

(b) Consider the Hamiltonian as $H = H_W + H_V$,

$$H_W = \begin{pmatrix} W & 0 \\ 0 & -W \end{pmatrix}, \qquad H_V = \begin{pmatrix} 0 & V \\ V & 0 \end{pmatrix},$$

and assuming that $|V| \ll |W|$ obtain the energy eigenvalues to second order perturbation theory in $V$.

(c) Compare your results in (a) and (b) and verify that they agree to second order in $V$.