PHY 576: Quantum Theory

Problem Set 4

Due by 1 pm on Monday, May 4 in grader's mailbox. Note the unusual deadline. No extensions! Solutions will hopefully be posted by Monday evening.

1. We heard in Frank Wilczek's lecture that the Hanbury Brown and Twiss interference effect can be used to determine the size of stars. In fact, this is nicely described by free particle propagators. Consider a two-dimensional simplification in which there is a line segment source (the star) at x_i with extent D in the y-direction. Suppose also there are two detectors both at x_f separated by $\bar{y}$ in the y-direction. The free particle propagator in two dimensions is

$$G_{\text{free}}(x_f, y_f, T; x_i, y_i) = \left(\frac{m}{2\pi\hbar i T}\right) e^{\frac{im}{2\hbar T}\left((x_f - x_i)^2 + (y_f - y_i)^2\right)}$$

which is easily derived by performing the additional path integral over $\mathcal{D}y(t)$, or just multiplying two one-dimensional propagators. Suppose the detectors can detect these particles of mass m (usually the detectors would detect light from the star, but for that we would need the propagator for a massless particle).

   (a) Consider a pair of points on the star separated by s in the y-direction from which two *identical* particles are emitted and detected at the two detectors. You can assume that the travel time for the particles to reach either of the detectors from either of the two sources points is the same, say T. Find the probability, as a function of $\bar{y}$ and s that both detectors will detect a particle.

   (b) Integrate over all pairs of points on the star to find the total probability that both detectors will detect a particle. You can again assume for simplicity that the travel time from all these points to either detector is the same T.

   (c) Given the form of the answer, how would you determine the width D of the star? You may assume that you can vary the detector separation.

2. Say whether or not each of the following is a group. If so, identify the identity element. If not, state which group properties (there could be more than one) are not obeyed.

   (a) The set of $m \times n$ matrices under addition.

   (b) The set of rational numbers under multiplication.

   (c) The set of natural numbers under exponentiation.

   (d) The set of students in our class under the binary operator *, where student1 * student2 gives the student who is not shorter in height. Assume no two students have exactly the same height.

   (e) The set of all $n \times n$ unitary matrices of determinant one under multiplication.

3. (a) Calculate ΔJ_x, ΔJ_y, and ΔJ_z in the state $|lm\rangle$ using operator methods (i.e. without doing any integrals).

   (b) Use your result above to confirm that the generalized uncertainty principle

$$\Delta A \Delta B \geq \left| \frac{1}{2i} \langle [A, B] \rangle \right|$$

   is satisfied in the $|lm\rangle$ state when the operators A and B are J_x and J_y respectively.

   (c) Using the spherical harmonics calculate $\langle J_x \rangle$, $\langle J_y \rangle$, $\langle J_z \rangle$ in the state $|lm\rangle$. Use the derivative forms of the J_i in spherical coordinates.

4. Suppose a particle in a spherically symmetric potential has the wavefunction

$$\psi(x, y, z) = N(xy + xz + yz)e^{-\alpha r^2} \ ,$$

   where N and α are constants.

   (a) What is the probability that a measurement of the square of the angular momentum gives zero?

   (b) What is the probability that it yields $6\hbar^2$?

(c) If the value of the quantum number l is found to be 2, what are the relative probabilites for $m = 2, 1, 0, -1, -2$?

5. Consider a world with four dimensions of space, x_1, x_2, x_3, x_4. The rotation group is now $SO(4)$ instead of $SO(3)$.

   (a) Write down the 4×4 matrix that rotates a vector by an angle η in the $x_3 - x_4$ plane, and find its infinitesimal generator.

   (b) Writing the generators in differential representation as

   $$J_{ij} = -i \left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i} \right) ,$$

   find the commutators $[J_{ij}, J_{kl}]$ i.e. the Lie algebra of $SO(4)$. How many mutually commuting generators can be chosen?

   (c) In four dimensions, it turns out that the eigenvalue of L^2 is $\hbar^2 l(l+2)$, rather than $\hbar^2 l(l+1)$. Consistent with Gauss' law in four dimensions, suppose there was a $1/r^2$ Coulomb-like potential between the proton and the electron. Write down the radial part of the Schrödinger equation for four-dimensional hydrogen. Using Mathematica (if necessary) argue that it has no normalizable solutions. Atoms can only exist in three dimensions.

6. Repeat Dirac's derivation, including all steps, to obtain the necessary matrices for a *massless* relativistic particle. How many components does ψ now have?

7. Find the Dirac spinor for a particle with spin-up and linear momentum p in the $+\hat{z}$ direction in two ways:

   (a) Solve the Dirac equation.

   (b) Go to the rest frame of the particle, for which we know the spin-up solution, and transform to the lab frame. Recall that infinitesimal Lorentz transformations act on the Dirac spinor by the generator $\frac{i}{4}[\gamma^\mu, \gamma^\nu]$. Thus a finite boost of η (where $\cosh \eta = \gamma$, the Lorentz factor) in the $-\hat{z}$ direction is given by $\exp\left(-\frac{1}{4}[\gamma^0, \gamma^3]\eta\right)$.