

# Notes on Radiation

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## Radiation from arbitrary source

Let's start with the retarded sources which give us the scalar and vector potentials

$$\Phi(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{|\mathbf{r} - \mathbf{r}'|} \quad (1)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{c})}{|\mathbf{r} - \mathbf{r}'|}. \quad (2)$$

The first approximation for radiation is that the source is localized (also meaning that the radiation zone is far from the source  $r \gg r'$ ). Under this assumption we get,

$$|\mathbf{r} - \mathbf{r}'| \approx r - \frac{\mathbf{r} \cdot \mathbf{r}'}{r} \quad (3)$$

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3}, \quad (4)$$

using the Fourier Transform

$$f(\mathbf{r} + \mathbf{a}) \approx f(\mathbf{r}) + \mathbf{a} \cdot \nabla f(\mathbf{r}) = f(\mathbf{r}) + a_i \partial_i f(\mathbf{r}). \quad (5)$$

Now let's use the fact that the sources have harmonic time dependence.

$$\mathbf{J}(\mathbf{r}', t) = \mathbf{J}(\mathbf{r}') e^{-i\omega t} \quad (6)$$

With this we can now approximate the vector potential by

$$\mathbf{J}(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}) \approx \mathbf{J}(\mathbf{r}') e^{-i\omega t + i\omega r/c - i\omega \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}}. \quad (7)$$

Now we can plug this into equation 2, and dropping terms with higher order terms than  $1/r$  to get,

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 e^{-i\omega(t-r/c)}}{4\pi r} \int d^3r' \mathbf{J}(\mathbf{r}') e^{-i\omega \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{c}}. \quad (8)$$

Now is we assume that the region of sources is small compared to the wavelength,  $c/\omega \gg r'$ , then we can make the **dipole approximation**,  $e^{-i\omega\hat{r}\cdot\mathbf{r}'/c} \approx 1$ , which when pluggen into equation 8 gives

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 e^{-i\omega(t-r/c)}}{4\pi r} \int d^3r' \mathbf{J}(\mathbf{r}'). \quad (9)$$

Now let's go on an aside to explore the  $\mathbf{J}(\mathbf{r}')$ .

**Claim:**  $\int d^3r' \mathbf{J}(\mathbf{r}') = \frac{d\mathbf{p}}{dt}$  where

$$\mathbf{p}(\mathbf{r}', t) = \int d^3r' \mathbf{r}' \rho(\mathbf{r}', t) \quad (10)$$

is the electric dipole moment. Now let's find the integral of  $\mathbf{J}(\mathbf{r}')$  in terms of this dipole moment. Let's start with the time derivative of the dipole.

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt} \int d^3r' \mathbf{r}' \rho(\mathbf{r}', t) \quad (11)$$

$$= \int d^3r' \mathbf{r}' \frac{\partial}{\partial t} \rho(\mathbf{r}', t) \quad (12)$$

$$= - \int d^3r' \mathbf{r}' \nabla \cdot \mathbf{J}(\mathbf{r}', t), \quad (13)$$

where we have used the continuity equation

$$\nabla \cdot \mathbf{J}(\mathbf{r}', t) = - \frac{\partial}{\partial t} \rho(\mathbf{r}', t). \quad (14)$$

Now expand the  $\mathbf{r}' \nabla \cdot \mathbf{J}(\mathbf{r}', t)$  term.

$$\nabla \cdot r_i \mathbf{J} = (\nabla r_i) \cdot \mathbf{J} + r_i \nabla \cdot \mathbf{J} \quad (15)$$

$$= \hat{r}_i \cdot \mathbf{J} + r_i \nabla \cdot \mathbf{J} \quad (16)$$

If you sum over all possible  $r_i$ 's you get

$$\mathbf{r}' \nabla \cdot \mathbf{J} = \nabla \cdot (r_x \mathbf{J} + r_y \mathbf{J} + r_z \mathbf{J}) - \mathbf{J} \quad (17)$$

However when the second term is integrated over the volume it can be turned into a surface integral with the divergence theorem, but  $\mathbf{J}$  is zero on the surface so we get that

$$\frac{d\mathbf{p}}{dt} = \int d^3r' \mathbf{J}. \quad (18)$$

Now applying this to equation 9 we get

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r} \frac{d}{dt} \mathbf{p} e^{-i\omega(t-r/c)}. \quad (19)$$

Is this right?