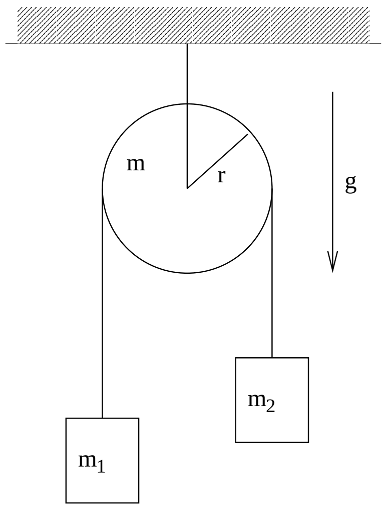
PHY6938 Proficiency Exam Fall 2002
September 13, 2002
Mechanics

1. The system (Atwood's machine) shown in the figure consists of two masses, m_1 and m_2, attached to the ends of a string of length l which hangs over a pulley. The pulley is a uniform disk of radius r and mass m. Assume the string is massless and does not slip on the pulley.



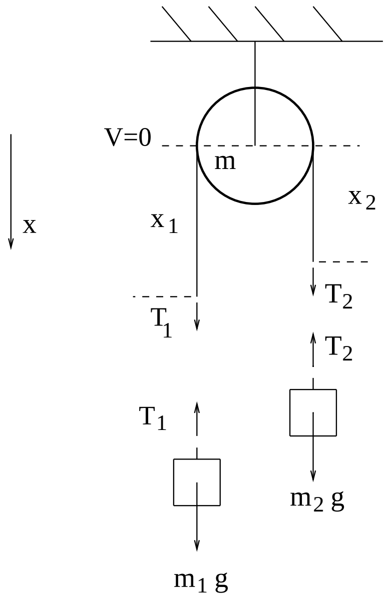
(a) What is the moment of inertia of the pulley about its axis ? (Perform a calculation to determine I; do not just write down a remembered answer.)

The pulley is homogeneous with the density ρ per unit area. The moment of inertia is give by

$$
\begin{aligned}
I &= \int dm\, r^2 && (1)\\
\text{where } dm &= \rho\, r\, dr d\phi && (2)\\
I &= \rho \int_0^{2\pi} d\phi \int_0^r dr\, r^3 && (3)\\
&= \rho\, 2\pi\, \frac{1}{4} r^4 \\
&= \frac{1}{2} \overbrace{(\rho \pi r^2)}^{m}\, r^2 \\
\text{this means } I &= \frac{1}{2}\, m r^2 && (4)\\
& && (5)
\end{aligned}
$$

(b) Obtain the acceleration of the masses.

First of all we draw free body diagram for all the parts in the system.



The equations of motion for each part will be given by

$$\begin{aligned} m_1\ddot{x}_1 &= m_1 g - T_1 && (6) \\ m_1\ddot{x}_2 &= m_2 g - T_2 && (7) \\ I\dot{\omega} &= r(T_2 - T_1) && (8) \end{aligned}$$

There are two constraints in the system. The first is that the length of the string is constant. This constraint can be written as

$$x_1 + x_2 + \pi r = l = \text{ constant,} \tag{9}$$

where l is the length of the string. Derivating Eq.9 twice with respect to time t, it yields

$$\ddot{x}_2 = -\ddot{x}_1. \tag{10}$$

The second constraint is that the string does not slip. This means that if the pulley rotates an angle α clockwise, the relation between x_2 and α is given by

$$x_2 = r\alpha \tag{11}$$

Derivating Eq.11 twice with respect to time t, we obtain

$$\begin{aligned} \ddot{x}_2 &= r\ddot{\alpha} && (12) \\ \text{let } \omega &= \dot{\alpha} && (13) \\ \ddot{x}_2 &= r\dot{\omega} && (14) \end{aligned}$$

Substitute Eq.10 in Eq.7, and subtract Eq.7 from Eq.6. The result is

$$(m_1 + m_2)\ddot{x}_1 = (m_1 - m_2)g + (T_2 - T_1). \tag{15}$$

Use Eqs.5, 8, 10, and 14, and solve for $(T_2 - T_1)$. We get

$$(T_2 - T_1) = -I\frac{\ddot{x}_1}{r^2} = -\frac{1}{2}M\ddot{x}_1. \tag{16}$$

Substitute Eq.16 in Eq.15 we obtain

$$\ddot{x}_1 = \frac{(m_1 - m_2)}{(m_1 + m_2 + \frac{1}{2}m)} g \quad (17)$$

$$\ddot{x}_2 = \frac{(m_2 - m_1)}{(m_1 + m_2 + \frac{1}{2}m)} g \quad (18)$$

(c) Find the tension of the string on both sides of the pulley.

To find the tension in the string we use Eqs.6, 7, 17, and 18. The results will be

$$T_1 = \Big[\frac{2m_2 + \frac{1}{2}M}{(m_1 + m_2 + \frac{1}{2}m)}\Big] m_1 g \quad (19)$$

$$T_2 = \Big[\frac{2m_1 + \frac{1}{2}M}{(m_1 + m_2 + \frac{1}{2}m)}\Big] m_2 g. \quad (20)$$

(d) Write down an expression for the total energy of the system.

The total energy of the system is given by

$$E = E_{kin,1} + V_1 + E_{kin,2} + V_2 + E_{kin,m} + V_m. \quad (21)$$

By choosing the center of the pulley as the reference level for $V = 0$, each term in Eq.21 can be written as following

$$E_{kin,1} = \frac{1}{2} m_1 \dot{x}_1^2 \quad (22)$$

$$E_{kin,2} = \frac{1}{2} m_2 \dot{x}_2^2 \quad (23)$$

$$E_m = \frac{1}{2} I \omega^2 \quad (24)$$

$$V_1 = -m_1 g x_1 \quad (25)$$

$$V_2 = -m_2 g x_2 \quad (26)$$

$$V_m = 0. \quad (27)$$

We have to find the velocity $\dot{x}_1$ ($\dot{x}_2$). This can be done by using integrating Eq.17

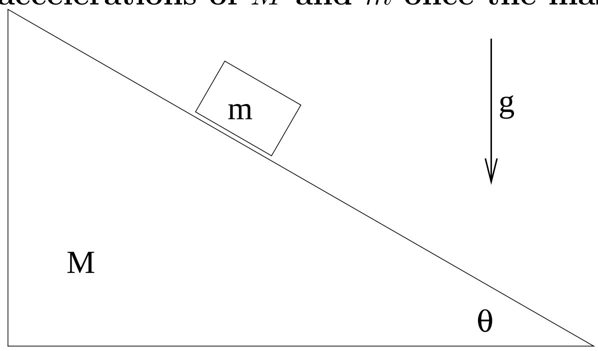
$$\int_0 \dot{x}_1 \, dt \, \ddot{x}_1 = \int_0^t dt \, \frac{(m_1 - m_2)}{(m_1 + m_2 + \frac{1}{2}m)} g \quad (28)$$

$$\dot{x}_1 = \frac{(m_1 - m_2)}{(m_1 + m_2 + \frac{1}{2}m)} g t \quad (29)$$

$$\dot{x}_2 = \dot{x}_1. \quad (30)$$

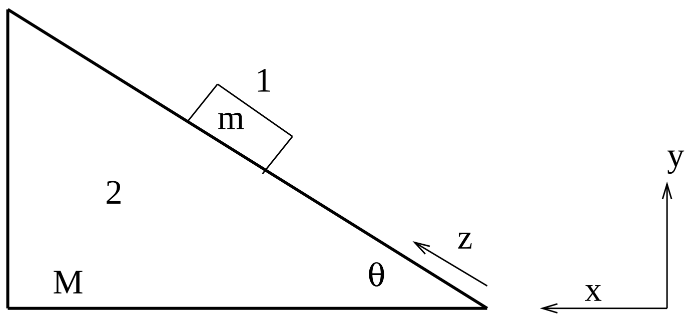
Substitute the parameters in Eq.21 we get the total energy.

2. A wedge of mass $M = 4.5$ kg sits on a horizontal surface. Another mass $m = 2.3$ kg sits on the sloping side of the wedge. The incline is at an angle of $31.7°$ with respect to the horizontal. All surfaces are frictionless. The mass m is released from rest on mass M, which is also initially at rest. What are the accelerations of M and m once the mass is released ?



This problem can be solved in different ways. One of them is drawing free body diagram for each body, introducing a coordinate system, writing the Newton equations plus the constraints. But this can be complicated. Instead, we use Lagrange formalism to solve this problem.

Introduce the generalized coordinates as in the figure below.



The Lagrangian for this system will be given by

$$L = T_{wedge} + T_m - V_{wedge} - V_m. \tag{1}$$

and

$$T_{wedge} = \frac{1}{2}M\dot{x}_2^2 \tag{2}$$

$$V_{wedge} = \frac{1}{2}Mgy_2 = \text{ constant, since the wedge does not move in the } y \text{ direction.} \tag{3}$$

$$T_m = \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) \tag{4}$$

$$V_m = mgy_1. \tag{5}$$

But x_1 and y_1 can be written as following

$$x_1 = x_2 + z_1 cos(\theta) \tag{6}$$

$$y_1 = z_1 sin(\theta) \tag{7}$$

Substitute Eqs.6 and 7 in Eqs.4 and 5, then replace T's and V's in Eq.1., and rename $x_2 = x$, $z_1 = z$ for simplicity. The result is given by

$$\begin{aligned} L &= \frac{1}{2}m[\dot{x}^2 + \dot{z}^2 + 2\dot{x}\dot{z}cos(\theta)] \\ &+ \frac{1}{2}M\dot{x}^2 - mgzsin(\theta) \end{aligned} \tag{8}$$

The equations of motions are obtained by

$$\frac{\mathrm{d}}{\mathrm{d}t}(\frac{\partial L}{\partial \dot{q}}) - \frac{\partial L}{\partial q} = 0, \text{ where } q = x, z. \tag{9}$$

We get

$$m\ddot{x} + m\ddot{z}cos(\theta) + M\ddot{x} = 0 \tag{10}$$

$$m\ddot{z} + m\ddot{x}cos(\theta) + mgsin(\theta) = 0. \tag{11}$$

From Eq.11 solve for $\ddot{z}$, then substitute in Eq.10

$$\ddot{x} = \frac{mgsin(\theta)cos(\theta)}{m + M - mcos^2(\theta)}. \tag{12}$$

Substituting the numerical values for different parameters, we get

$$\ddot{x} = 1.9624 \ \frac{\mathrm{m}}{\mathrm{s}^2} \tag{13}$$

$$\ddot{z} = -6.82 \ \frac{\mathrm{m}}{\mathrm{s}^2} \tag{14}$$

To find the acceleration of the mass m, derivate Eqs.6 and 7 twice and use the numerical values from Eq.13 and 14

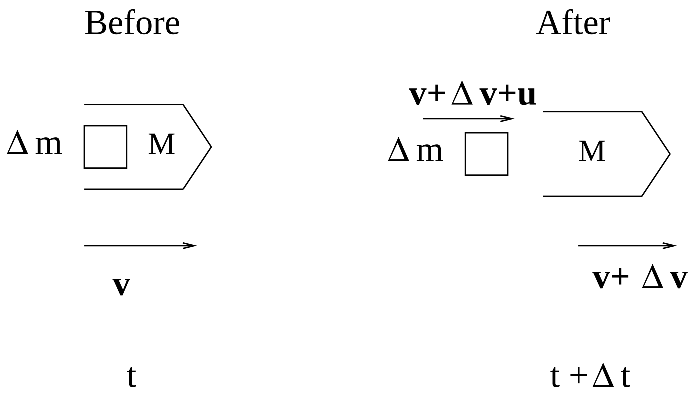
$$\ddot{x}_1 = -3.85 \ \frac{\mathrm{m}}{\mathrm{s}^2} \tag{15}$$

$$\ddot{y}_1 = -3.58 \ \frac{\mathrm{m}}{\mathrm{s}^2} \tag{16}$$

3. **A single-stage rocket of initial mass m_0 is launched vertically at constant burn rate of the fuel α. The payload mass of the rocket is m_F. The gas is exhausted at a constant speed u relative to the rocket. Neglect air resistance and assume that the acceleration of gravity is constant with height.**

   (a) **Derive an expression for the velocity as a function of mass of the rocket.**

   Consider the following figure, which shows the rocket before and after the gas is exhausted.



Consider the rocket at time t. Between t and $t + \Delta t$ a mass of fuel Δm is burned and expelled as gas with velocity $\mathbf{u}$ relative the rocket. The initial and the final momentums are given by

$$\begin{aligned} \mathbf{P}(t) &= (M + \Delta m)\mathbf{v} & (1) \\ \mathbf{P}(t + \Delta t) &= M(\mathbf{v} + \Delta \mathbf{v}) + \Delta m(\mathbf{v} + \Delta \mathbf{v} + \mathbf{u}), & (2) \end{aligned}$$

where M is the remaining mass of the rocket. The change in the momentum is

$$\Delta \mathbf{P} = M \Delta \mathbf{v} + \Delta m \mathbf{u}. \tag{3}$$

In the limit $\Delta t \to 0$

$$\frac{d\mathbf{P}}{dt} = \mathbf{F}^{external} = M \frac{d\mathbf{v}}{dt} + \mathbf{u} \frac{dm}{dt}. \tag{4}$$

From the beginning

$$\begin{aligned} M + m &= \text{ mass of the rocket + mass of the burned fuel} \\ &= \text{total initial mass of the rocket + fuel} \\ &= \text{constant} & (5) \end{aligned}$$

Derivate Eq.5 with respect to time

$$\frac{dm}{dt} = -\frac{dM}{dt}. \tag{6}$$

Substitute Eq.6 in Eq.4 we obtain

$$M \frac{d\mathbf{v}}{dt} = \mathbf{u} \frac{dM}{dt} + \mathbf{F}^{external}. \tag{7}$$

(b) Calculate the time t_b at which the fuel is burnt out.

Eq.7 is in vector form. When the rocket moves upward the external force, in this case the gravitational force, points downward and also the burned fuel moves downward. The equation of motion is

$$M \frac{dv}{dt} = -u \frac{dM}{dt} - Mg. \tag{8}$$

Multiply Eq.8 by dt and divide by M we get

$$dv = -u\frac{dM}{M} - gdt. \tag{9}$$

Integrate Eq.9 from $t = 0$ to $t = t_b$

$$\begin{aligned} \int_0^{v_b} dv &= -u\int_{M_0}^{M_F} \frac{dM}{M} - g\int_0^{t_b} dt & (10)\\ t_b &= \frac{1}{g}(u\ln\frac{M_0}{M_f} - v_b) & (11) \end{aligned}$$

(c) For the first stage of a Saturn V rocket for the Apollo moon program, the initial mass is $m_0 = 2.8 \times 10^6\text{kg}, m_F = 7 \times 10^5\text{kg}, u = 2600\text{m/s}$ and assume a mean thrust of 37×10^6 N. Obtain the final speed at burnout and the burnout time t_b.

Thrust is the force that drives the rocket forward. In Eq.8 the left hand side is equal the force that drive the rocket. This means

$$\langle M\frac{dv}{dt}\rangle = T = 37 \times 10^6\ N. \tag{12}$$

Substitute Eq.12 in Eq.8.

$$\begin{aligned} T &= -u\frac{dM}{dt} - Mg & (13)\\ dt &= -u\frac{dM}{T + Mg} & (14)\\ \int_0^{t_b} dt &= -u\int_{M_0}^{M_F} \frac{dt}{T + Mg} & (15)\\ t_b &= \frac{u}{g}\ln\Big(\frac{T + M_0 g}{T + M_F g}\Big). & (16) \end{aligned}$$

Substitute numerical values we get

$$t_b = 14.36\ S,\ \ v_b = 3463.43\ m/s. \tag{17}$$