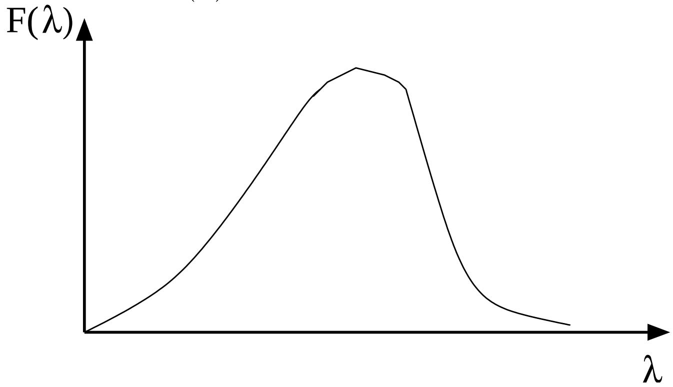
PHY6938 Proficiency Exam Fall 2002
September 13, 2002
Optics and Thermodynamics

1. **Objects at finite temperature T emit electromagnetic radiation with a continuous spectrum, called Blackbody radiation. The radiated power per unit area and unit wavelength is given by the function $F(\lambda) = \frac{2\pi hc^2}{\lambda^5}\frac{1}{\exp(hc/\lambda k_B T) - 1}$, where λ is the wavelength, $c = 3.00 \times 10^8$m/s is the speed of light, $h = 6.26 \times 10^{-34}$ Js is Planck's constant, and $k_B = 1.381 \times 10^{-23}$J/K is Boltzmann's constant.**

   (a) **Make a sketch of $F(\lambda)$ as a function of λ.**



   (b) **The power density $F(\lambda)$ has a maximum at a particular wavelength, $\lambda_{\max}$. Derive the relation**

$$T\lambda_{\max} \approx 2.90 \times 10^{-3} Km.$$

This result is known as Wien's Displacement law.
Hint: It may be useful to note that the two roots of the transcendental equation $5 - x = 5e^{-x}$ are 0 and approximately 4.965.

To find the maximum of $F(\lambda)$ we have to take the derivative of $F(\lambda)$ with respect to λ. Let $x = \frac{hc}{\lambda kT}$.

$$\begin{aligned}
0 &= \frac{dF}{d\lambda} \\
&= \frac{dF}{dx}\frac{dx}{\lambda} \\
&= -\frac{hc}{\lambda^2 kt}\frac{d}{dx}\Big[const. \times \frac{x^5}{e^x - 1}\Big] \\
&= -C'x^2\frac{d}{dx}\frac{x^5}{e^x - 1} \\
&= -C'x^2\frac{5x^4(e^x - 1) - x^5e^x}{(e^x - 1)^2}
\end{aligned}$$

$$\begin{aligned} &= \frac{-C'x^6}{(e^x-1)^2}\left(5e^x - 5 - xe^x\right) \\ &= \frac{C'x^6e^x}{(e^x-1)^2}\left(5e^{-x} + x - 5\right) \end{aligned} \tag{1}$$

We can see that the maximum is the solution of the following equation

$$e^x = 5 - x \tag{2}$$

let

$$f_1(x) = e^x \tag{3}$$

$$f_2(x) = 5 - x. \tag{4}$$

The Eq.2 can be solved by choosing different values for x and calculating both side of the equation. To minimize the work, we can see that for $x = 5$ the right hand side of Eq.2 is zero but the left hand side is non-zero. This means that the two functions $f_1(x)$ and $f_2(x)$ intersect each other for $x < 5$. Using this information we find $x_{max} \approx 4.965$ (the other root is $x = 0$).
Use the definition of x we obtain

$$\begin{aligned} \lambda_{max}T &= \frac{hc}{kx_{max}} \\ &= 2.90 \times 10^{-3}\ Km. \end{aligned} \tag{5}$$

(c) Show that the total power radiated per unit area is given by the Stefan-Boltzmann law,

$$P = \sigma T^4,$$

with the Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}\frac{\mathrm{W}}{\mathrm{m^2K^4}}$.
Hint: The following integral may be useful:

$$\int_0^{\inf} \frac{x^3dx}{e^x-1} = \frac{\pi^4}{15}$$

Power per unit area $\equiv P$ can be calculated as following

$$\begin{aligned} P &= \int_0^\infty d\lambda\ F(\lambda) \\ &= \int_0^\infty d\lambda\ \frac{2\pi hc^2}{\lambda^5}\ \frac{1}{exp\{\frac{hc}{\lambda kT}\} - 1} \\ \text{let } x &= \frac{hc}{\lambda kT} \\ \text{and } dx &= -\frac{hc}{\lambda^2 kT}\ d\lambda \\ P &= \int_0^\infty dx\ \frac{2\pi hc^2}{\lambda^5}\ \frac{\lambda^2 kT}{hc}\ \frac{1}{e^x-1} \\ &= \frac{2\pi(kT)^4c}{(hc)^3}\int_0^\infty dx\ \frac{x^3}{e^x-1} \\ &= \frac{2\pi^5k^4c}{15(hc)^3}\ T^4 \\ &= 5.67 \times 10^{-8}\ \frac{W}{m^2K^4}. \end{aligned} \tag{6}$$

2. Two lenses are separated by 35 cm. An object is placed 20 cm to the left of the first lens, which is a converging lens of focal length 10 cm. The second lens if a diverging lens of focal length −15 cm. What is the position of the final image ? Is the image real or virtual ? Erect of inverted ? What is the overall magnification of the image?

We can find everything we need from the lens formula(e)

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}, \quad m = -\frac{s'}{s}, \tag{1}$$

being careful to use a negative f for the diverging lens (since the focal point of light on the incident side is on the incident side and not the transmitted side, it is therefore negative), and use the figure to check our answers. For the first lens we have

$$\frac{1}{s_1'} = \frac{1}{f_1} - \frac{1}{s_1} = \frac{1}{10 \text{ cm}} - \frac{1}{20 \text{ cm}} = \frac{1}{20 \text{ cm}}, \tag{2}$$

so that the first real image is 20 cm from the lens and has

$$m_1 = -\frac{s_1'}{s_1} = -1, \tag{3}$$

so that the image is the same size as the object and inverted.

For the second lens we have

$$\frac{1}{s_2'} = \frac{1}{f_2} - \frac{1}{s_2} = \frac{1}{-15 \text{ cm}} - \frac{1}{15 \text{ cm}} = -\frac{2}{15 \text{ cm}}, \tag{4}$$

so that the image is virtual (it is on the incident side, since $s_2' < 0$) and 7.5 cm to the left of the second lens. The magnification of the second lens is

$$m_2 = -\frac{s_2'}{s_2} = -\frac{-7.5\text{cm}}{15 \text{ cm}} = -\frac{1}{2}, \tag{5}$$

so that the combined magnification is $m = m_1 m_2 = -0.5$ and the virtual image is inverted with an overall magnification of -0.5.