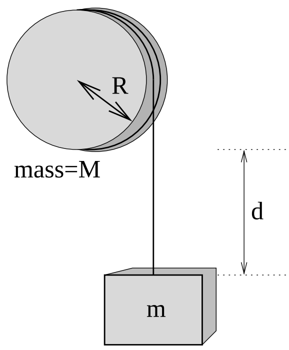
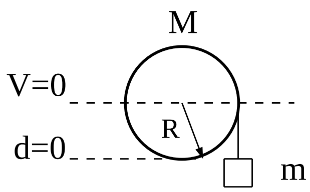
PHY6938 Proficiency Exam Spring 2003
March 28, 2003
Mechanics

1. **The drum of a winch has a mass M and a radius R. A cable wound around the drum suspends a load of mass m. The entire cable has a length L and a density (mass per unit length) λ, with a total mass $m_c = L\lambda$. The load starts from rest, at the bottom of the winch, and begins to fall toward the ground, unwinding the cable as it moves.**



(a) **Write an expression for the change in potential energy of the load and the cable that has been paid out from the drum, between the initial situation and when the load has fallen a distance d.**

To find the potential, it is useful to choose a reference level or point for which the potential is zero. This selection of the reference point is arbitrary. But it always helps to choose a point which make the solution easier. Below that point the potential energy has negative value and above that point the potential energy is positive. In this problem the center of the drum is chosen as the point with potential energy $V = 0$. With this choice the potential energy of the drum and the part of the cable wrapped around the drum will be zero. When the



load is at the highest level, its potential energy is given by

$$V_{\mathrm{l-initial}} = -mgR. \tag{1}$$

The initial potential energy of the cable is

$$V_{\mathrm{c-initial}} = 0. \tag{2}$$

When the load has fallen a distance d the final potential energy of the load is

$$V_{\text{l}-\text{final}} = -mg(R+d). \tag{3}$$

The final potential energy of the cable can be obtained as following. Since the cable is homogeneous the center of the mass of the part paid out from the drum is located at the center of this part.

$$V_{\text{c}-\text{final}} = -d\lambda\frac{d}{2} = -\frac{d^2}{2L}m_c. \tag{4}$$

The change in the potential energy for each part will be

$$\Delta V_l = V_{\text{l}-\text{final}} - V_{\text{l}-\text{initial}} = -mgd \tag{5}$$

$$\Delta V_c = V_{\text{c}-\text{final}} - V_{\text{c}-\text{initial}} = -\frac{d^2}{2L} = m_c, \tag{6}$$

and the total change in the potential is

$$\Delta V_{total} = \Delta V_l + \Delta V_c = -(m + \frac{d^2}{2L}m_c)g. \tag{7}$$

(b) Write an expression for the kinetic energy of the cable and the load. Assume that the load has speed v and there is no slipping between the cable and the drum. Since the cable is connected to the load its speed will also be v. The kinetic energy of the load and the cable are given by

$$T_l = \frac{1}{2}mv^2 \tag{8}$$

$$\begin{aligned} T_c &= \frac{1}{2}d\lambda v^2 \\ &= \frac{d}{2L}m_c v^2, \end{aligned} \tag{9}$$

and the total kinetic of the drum and the cable is

$$T_{l+c} = \frac{1}{2}(m + \frac{d}{L}m_c)v^2. \tag{10}$$

(c) Express the rotational kinetic energy of the winch in terms of I and ω and convert this expression in terms of M and v.
Since there is no slipping between the drum and the cable, the points of the drum which are in contact with the cable will rotate with the same speed v. The rotational energy of the drum is

$$T_d = \frac{1}{2}I\omega^2. \tag{11}$$

The moment of inertia I of the drum and its angular velocity are given by

$$I = \frac{1}{2}MR^2 \tag{12}$$

$$\omega = \frac{v}{R}. \tag{13}$$

Substitute Eqs.12 and 13 in Eq.11 it yields

$$T_d = \frac{1}{4}Mv^2 \tag{14}$$

(d) Using the results from above, obtain an expression for the speed of the falling load v after it has fallen the distance d, in terms of M, m, m_c, d, L and g.

Since there is no dissipation , i.e. slipping, in the system the total energy of the system is conserved. This means that

$$T_{total,initial} + V_{total,initial} = T_{total,final} + V_{total,final}. \tag{15}$$

But the system starts it motion from rest. Therefore $T_{total,initial} = 0$ and $\Delta V = V_{total,final} - V_{total,initial}$ is given by Eq.7, therefore

$$\begin{aligned} -(V_{total,final} V_{total,initial}) &= T_{total,final}. &(16)\\ -\Delta V &= T_{total,final} &(17)\end{aligned}$$

Substitute Eqs.7 and 10, and 14 in Eq.17 and solve v

$$v = \sqrt{\frac{2m + \frac{d^2 m_c}{L}}{m + m_c + \frac{M}{2}} g}. \tag{18}$$

2. An artificial satellite is observed to have a maximum velocity v_1 and a minimum velocity v_2 during one orbit of the earth. Find its maximum and minimum distance from the center of the earth in terms of v_1, v_2, the mass of the Earth M, and the gravitational constant G.

The satellite moves in a elliptic orbit around the earth, since its orbit is closed, and there is a minimum and maximum distance from the earth. The satellite and the earth interacts only with the gravitational force $f(r)\hat{\mathbf{r}}$. This type of motion is called central force motion, since there is only force in radial direction. In the central motion 1. the total energy E_{tot} and the angular momentum $\mathbf{L}$ are conserved.

Write conservation of the angular momentum for the minimum and the maximum distances between the satellite and the earth

$$\begin{aligned} L &= m r_{min} v_2 &(1)\\ L &= m r_{max} v_1. &(2)\end{aligned}$$

Divide Eq.1 by Eq.2, it yields

$$\frac{r_{min} v_2}{r_{max} v_1} = 1 \text{ , or, } r_{min} = \frac{v_1}{v_2} r_{max}. \tag{3}$$

The total energy for the satellite at the maximum and the minimum distances are give by

$$\begin{aligned} E_{tot} &= T_{min} + V_{min} = \frac{1}{2} m v_2^2 - \frac{G M_{earth} m}{r_{min}} &(4)\\ E_{tot} &= T_{max} + V_{max} = \frac{1}{2} m v_1^2 - \frac{G M_{earth} m}{r_{max}}. &(5)\end{aligned}$$

From Eqs.3, 4, and 5

$$\frac{1}{2}mv_2^2 - \frac{GM_{earth}m}{\frac{v_1}{v_2}r_{max}} = \frac{1}{2}mv_1^2 - \frac{GM_{earth}m}{r_{max}} \tag{6}$$

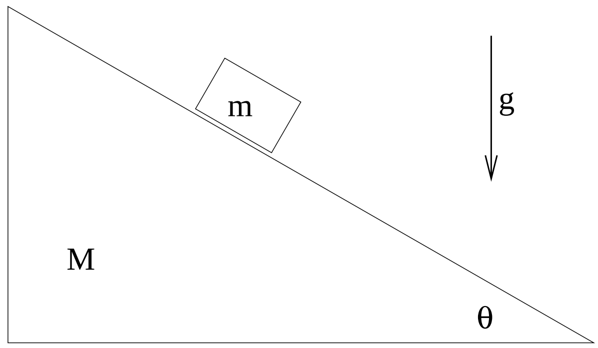
$$v_2^2 - v_1^2 = \frac{2GM_{earth}}{r_{max}}\frac{v_2 - v_1}{v_1} \tag{7}$$

From Eq.7 solve for r_{max}, it yields

$$r_{max} = \frac{2GM}{v_1(v_2+v_1)} \tag{8}$$

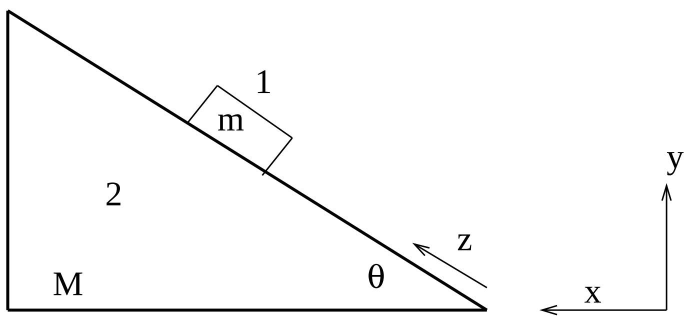
$$r_{min} = \frac{2GM}{v_2(v_2+v_1)} \tag{9}$$

3. A wedge of mass $M = 4.5$ kg sits on a horizontal surface. Another mass $m = 2.3$ kg sits on the sloping side of the wedge. The incline is at an angle of $31.7°$ with respect to the horizontal. All surfaces are frictionless. The mass m is released from rest on mass M, which is also initially at rest. What are the accelerations of M and m once the mass is released ?



This problem can be solved in different ways. One of them is drawing free body diagram for each body, introducing a coordinate system, writing the Newton equations plus the constraints. But this can be complicated. Instead, we use Lagrange formalism to solve this problem.

Introduce the generalized coordinates as in the figure below.



The Lagrangian for this system will be given by

$$L = T_{wedge} + T_m - V_{wedge} - V_m. \tag{1}$$

and

$$\begin{aligned}
T_{wedge} &= \frac{1}{2}M\dot{x}_2^2 && (2)\\
V_{wedge} &= \frac{1}{2}Mgy_2 = \text{ constant, since the wedge does not move in the } y \text{ direction.} && (3)\\
T_m &= \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) && (4)\\
V_m &= mgy_1. && (5)
\end{aligned}$$

But x_1 and y_1 can be written as following

$$\begin{aligned}
x_1 &= x_2 + z_1 cos(\theta) && (6)\\
y_1 &= z_1 sin(\theta) && (7)
\end{aligned}$$

Substitute Eqs.6 and 7 in Eqs.4 and 5, then replace T's and V's in Eq.1., and rename $x_2 = x$, $z_1 = z$ for simplicity. The result is given by

$$\begin{aligned}
L &= \frac{1}{2}m[\dot{x}^2 + \dot{z}^2 + 2\dot{x}\dot{z}cos(\theta)]\\
&+ \frac{1}{2}M\dot{x}^2 - mgzsin(\theta) && (8)
\end{aligned}$$

The equations of motions are obtained by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0, \text{ where } q = x, z. \qquad (9)$$

We get

$$\begin{aligned}
m\ddot{x} + m\ddot{z}cos(\theta) + M\ddot{x} &= 0 && (10)\\
m\ddot{z} + m\ddot{x}cos(\theta) + mgsin(\theta) &= 0. && (11)
\end{aligned}$$

From Eq.11 solve for $\ddot{z}$, then substitute in Eq.10

$$\ddot{x} = \frac{mgsin(\theta)cos(\theta)}{m + M - mcos^2(\theta)}. \qquad (12)$$

Substituting the numerical values for different parameters, we get

$$\begin{aligned}
\ddot{x} &= 1.9624 \ \frac{\mathrm{m}}{\mathrm{s}^2} && (13)\\
\ddot{z} &= -6.82 \ \frac{\mathrm{m}}{\mathrm{s}^2} && (14)
\end{aligned}$$

To find the acceleration of the mass m, derivate Eqs.6 and 7 twice and use the numerical values from Eq.13 and 14

$$\begin{aligned}
\ddot{x}_1 &= -3.85 \ \frac{\mathrm{m}}{\mathrm{s}^2} && (15)\\
\ddot{y}_1 &= -3.58 \ \frac{\mathrm{m}}{\mathrm{s}^2} && (16)
\end{aligned}$$