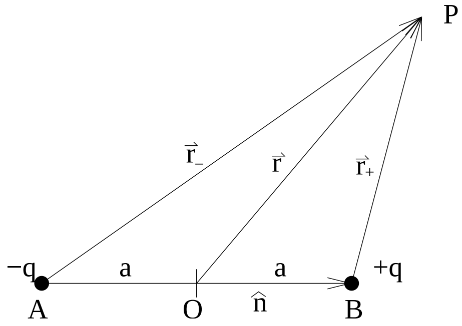
PHY6938 Proficiency Exam Spring 2003
March 28, 2003
E & M

1. **In the diagram below, an electric dipole has a charge $-q$ at point A and $+q$ at point B, a distance $2a$ apart. The vector $2a\hat{n}$ points from point A to point B.**



Assuming that $a << r$, show that the electric potential $\Phi(\vec{r})$ at point P is given by

$$\Phi(\vec{r}) = \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2},$$

where $\vec{p} = q2a\hat{n}$ is the electric dipole moment, and $\hat{r}$ is the unit vector pointing in direction of P from the origin O.

This is a standard problem whose solution can be found in many Electromagnetic text books. Start the solution by writing the potential at the point P due to charges at the points A and B. The potential of at an arbitrary point at point defined by $\vec{r}$, due to a point charge q in vacuum,is given by

$$\Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0 r}. \tag{1}$$

The total potential at a point in space due to many point charges is the sum of the potential of each point charge at that point (we assume that the point charges are fixed and do not move. Otherwise the charged particles interact with each others and the distances between them and the point whose potential we intend to calculate change).

$$\Phi(\vec{r}) = \frac{q}{4\pi\epsilon_0|\vec{r}_+|} - \frac{q}{4\pi\epsilon_0|\vec{r}_-|}, \tag{2}$$

where the first and second terms are the contributions from the positive and negative charges respectively. From the figure, it can be seen that $|\vec{r}_+|$ and $|\vec{r}_-|$ are given by

$$\vec{r}_+ = \vec{r} - a\hat{n} \tag{3}$$
$$\vec{r}_- = \vec{r} + a\hat{n}. \tag{4}$$

The magnitude of $\vec{r}_+$ or $\vec{r}_-$ can be found by taking the square root of the scalar product of Eq.3 or Eq.4 by itself. These are given by

$$|\vec{r}_+| = \sqrt{r^2 + a^2 - 2a\hat{n}\cdot\vec{r}} \tag{5}$$
$$|\vec{r}_-| = \sqrt{r^2 + a^2 + 2a\hat{n}\cdot\vec{r}} \tag{6}$$

Substitute Eq.5 and 6 in Eq.2. It yields

$$\begin{aligned}\Phi(\vec{r}) &= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{r^2+a^2-2a\hat{n}\cdot\vec{r}}}-\frac{1}{\sqrt{r^2+a^2+2a\hat{n}\cdot\vec{r}}}\right]\\ &= \frac{q}{4\pi\epsilon_0 r}\left[\frac{1}{\left(1-2\frac{a}{r}\hat{n}\cdot\hat{r}+\frac{a^2}{r^2}\right)^{\frac{1}{2}}}-\frac{1}{\left(1+2\frac{a}{r}\hat{n}\cdot\hat{r}+\frac{a^2}{r^2}\right)^{\frac{1}{2}}}\right]. \qquad (7)\end{aligned}$$

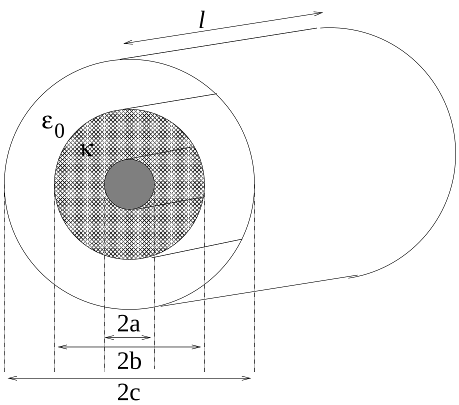
Since $a \ll r$ the $\frac{a^2}{r^2}$ can be neglected from the denominators of Eq.7. Then Taylor expand these terms. We obtain

$$\begin{aligned}\Phi(\vec{r}) &= \frac{q}{4\pi\epsilon_0 r}\left[\frac{1}{1-2\frac{1}{2}\frac{a}{r}\hat{n}\cdot\hat{r}}-\frac{1}{1+2\frac{1}{2}\frac{a}{r}\hat{n}\cdot\hat{r}}\right]\\ &= \frac{q}{4\pi\epsilon_0 r}\left[\frac{1}{1-\frac{a}{r}\hat{n}\cdot\hat{r}}-\frac{1}{1+\frac{a}{r}\hat{n}\cdot\hat{r}}\right]. \qquad (8)\end{aligned}$$

Since $a \ll r$ the denominators in Eq.8, they can be expanded using Taylor expansion. It yields

$$\begin{aligned}\Phi(\vec{r}) &= \frac{q}{4\pi\epsilon_0 r}\left[(1+\frac{a}{r}\hat{n}\cdot\hat{r})-(1-\frac{a}{r}\hat{n}\cdot\hat{r})\right]\\ &= \frac{q}{4\pi\epsilon_0 r}\left[1+\frac{a}{r}\hat{n}\cdot\hat{r}-1+\frac{a}{r}\hat{n}\cdot\hat{r}\right]\\ &= \frac{2aq\hat{n}\cdot\hat{r}}{4\pi\epsilon_0 r^2}\\ &= \frac{\hat{P}\cdot\hat{r}}{2\pi\epsilon_0 r^2} \qquad (9)\end{aligned}$$

2. A coaxial capacitor of length l consists of an inner conductor of radius a, a cylindrical dielectric with dielectric constant κ of inner radius a and outer radius b, and a shell of free space of inner radius b and outer radius c. There is a conducting surface holding total charge q at the radius c. The inner conductor carries charge $-q$.



(a) Find the radial component of the electric field E in all regions of space.

We will use Gauss's law, suitably modified in the presence of a dielectric. This is

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{encl}}}{\epsilon} = \frac{Q_{\text{encl}}}{\kappa\epsilon_0}, \tag{1}$$

where κ is the dielectric constant. Note that since $\kappa > 1$ the presence of the dielectric has the effect of reducing the effective charge to Q_{encl}/κ in Gauss's law.

For $r < a$ we are inside the central conductor and there can be no electric field. For the same reason all of the charge of the central conductor has to lie on its surface.

For $a < r < b$ we are inside the dielectric, and if we make a cylindrical Gaussian surface S with a radius r and length l around the central conductor, we have that the enclosed charge is the total charge on the central conductor and

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = 2\pi r l E(r) = \frac{Q_{\text{encl}}}{\kappa\epsilon_0} = -\frac{qX}{\kappa\epsilon_0}, \tag{2}$$

where we used that the field is purely radial, i.e. we have ignored edge effects. This gives

$$E(r) = -\frac{q}{2\pi l \kappa \epsilon_0 r}, \quad a < r < b. \tag{3}$$

For $b < r < c$ we are outside the dielectric and so we can use the usual form of Gauss's law with the same shape of Gaussian surface, so that by analogy

$$E(r) = -\frac{q}{2\pi l \epsilon_0 r}, \quad b < r < c. \tag{4}$$

(b) Find the potential V in each region and the total potential across the capacitor (ignore all edge effects). Now that we know the field is radial and its form as a function of the radius in the two regions, we can find the potential difference between the inner and outer conductors by simply integrating the field with respect to r,

$$V_{\text{ca}} := V_{\text{c}} - V_{\text{a}} = V_{\text{cb}} + V_{\text{ba}} = -\int_{\text{b}}^{\text{c}} E(r)dr - \int_{\text{a}}^{\text{b}} E(r)dr \tag{5}$$

$$= \frac{q}{2\pi l\epsilon_0}\left(\int_{\text{b}}^{\text{c}} \frac{1}{r}dr + \frac{1}{\kappa}\int_{\text{a}}^{\text{b}} \frac{1}{r}dr\right) = \frac{q}{2\pi l\epsilon_0}\left[\ln\left(\frac{c}{b}\right) + \frac{1}{\kappa}\ln\left(\frac{b}{a}\right)\right]. \tag{6}$$

c) Find the capacitance.

Using $C = q/V$ we have

$$C = 2\pi l\epsilon_0\left[\ln\left(\frac{c}{b}\right) + \frac{1}{\kappa}\ln\left(\frac{b}{a}\right)\right]^{-1}. \tag{7}$$

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(d) Find the stored energy in the capacitor. Is the energy greater than or less than the energy stored in a similar capacitor with no dielectric?

The energy stored in the capacitor is the work done to build up the charge against the potential difference caused by the charge already there,

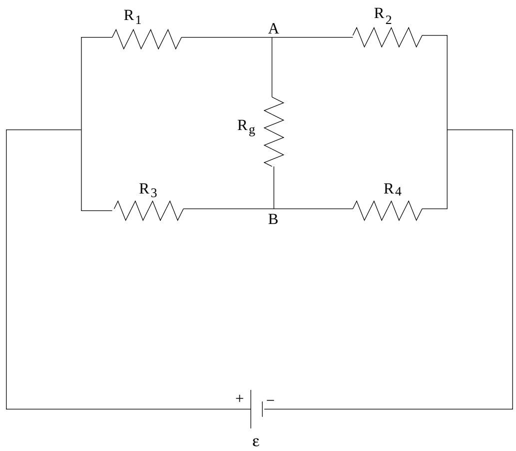
$$U = W = \int dW = \int dq V_C = \int dq \frac{q}{C} = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2} V_C q, \tag{9}$$

so that

$$U = \frac{1}{2} V_{\text{ca}} q = \frac{q^2}{4\pi l \epsilon_0}\left[\ln\left(\frac{c}{b}\right) + \frac{1}{\kappa}\ln\left(\frac{b}{a}\right)\right]. \tag{10}$$

Since $\kappa > 1$, we see that U is smaller than it would be if $\kappa = 1$, i.e. if there were no dielectric present.

3. Consider the Wheatstone bridge resistor circuit shown in the diagram below.



(a) Use Kirchhoff's laws to derive a condition for the values of R_1, R_2, R_3 and R_4, under which the galvanometer does not measure a current and the Wheatstone bridge is "balanced".

We need to apply Kirchoff's laws to the circuit, which are that the current going into a branch in the circuit is the sum of the currents in the two branches (conservation of charge), and that the voltage drops around any loop in the circuit add to zero. Label by I_i the current in resistor i, assuming each goes to the right except the galvanometer current I_{g}, which we can assume goes down. Then we have five unknown currents. There are two branch points at A and B, and the two conservation of charge equations are

$$\begin{aligned} I_1 &= I_2 + I_{\text{g}} & (1)\\ I_3 + I_{\text{g}} &= I_4. & (2) \end{aligned}$$

This leaves us with three equations necessary to solve for the five currents.

There are three loops in the circuit; the first is the loop made up by the three resistors on the left, which has the energy conservation equation (start at A and add voltage *rises* clockwise)

$$-I_{\text{g}} R_{\text{g}} + I_3 R_3 - I_1 R_1 = 0. \tag{3}$$

Similarly for the loop of resistors on the right (start at A and add voltage rises clockwise)

$$-I_2R_2 + I_4R_4 + I_gR_g = 0. \tag{4}$$

We have one more equation which is for a loop containing ε; note we can choose any of a number of loops and because we have already written down the equations for the interior loops they will all amount to the same thing, so choose the top branch of the circuit and start at the $-$ side of the emf,

$$\varepsilon - I_1R_1 - I_2R_2 = 0. \tag{5}$$

We now have five equations in five unknowns which we can solve, however tedious that process is, for the five currents. It is better to do this without substituting for the resistors so that we have the general formula for the galvanometer current.

Before embarking on what is formally the job of inverting a 5×5 matrix, it helps to get organized. Equation (1) can be used to eliminate I_1 from *every* equation where it appears. The same is true of Eq. (2) and I_4. Once we have used these we have only three equations left. Our new system of equations looks like

$$\begin{aligned}
-I_gR_g + I_3R_3 - I_2R_1 - I_gR_1 = -I_g(R_g + R_1) - I_2R_1 + I_3R_3 &= 0 \quad (6)\\
-I_2R_2 + I_3R_4 + I_gR_4 + I_gR_g = \quad I_g(R_g + R_4) - I_2R_2 + I_3R_4 &= 0 \quad (7)\\
\varepsilon - I_2R_1 - I_gR_1 - I_2R_2 = \varepsilon - I_gR_1 - I_2(R_1 + R_2) &= 0 \quad (8)
\end{aligned}$$

where it obviously pays to collect terms and organize the coefficients in terms of the unknown currents.

The best way to proceed now is to combine (6) and (7) to eliminate I_3 which does not appear in (8), i.e.

$$\begin{aligned}
R_4 \times (6) \;&:\; -I_g\left[(R_g + R_1)R_4\right] - I_2R_1R_4 + I_3R_3R_4 = 0 && (9)\\
R_3 \times (7) \;&:\; I_g\left[(R_g + R_4)R_3\right] - I_2R_2R_3 + I_3R_3R_4 = 0 && (10)\\
(10) - (9) \;&:\; I_g\left[R_g(R_3 + R_4) + R_4(R_1 + R_3)\right] - I_2\left[R_2R_3 - R_1R_4)\right] = 0. && (11)
\end{aligned}$$

Substituting the value of I_2 in terms of I_g from (8), we have

$$I_g\left[R_g(R_3 + R_4) + R_4(R_1 + R_3)\right] - \frac{\varepsilon - I_gR_1}{R_1 + R_2}\left[R_2R_3 - R_1R_4)\right] = 0, \tag{12}$$

or that

$$I_g\left\{(R_1 + R_2)\left[R_g(R_3 + R_4) + R_4(R_1 + R_3)\right] + R_1(R_2R_3 - R_1R_4)\right\} = \varepsilon\left[R_2R_3 - R_1R_4)\right], \tag{13}$$

so that, *finally*,

$$I_g = \frac{\varepsilon\left[R_2R_3 - R_1R_4\right]}{(R_1 + R_2)\left[R_g(R_3 + R_4) + R_4(R_1 + R_3)\right] + R_1(R_2R_3 - R_1R_4)} = -10.2\,\mathrm{mA}. \tag{14}$$

b) What is the voltage difference between points A and B? What is the requirement on this voltage difference if the bridge is to be "balanced" (i.e. when no current flows through the galvanometer?

Obviously

$$V_{\mathrm{A}} - V_{\mathrm{B}} = V_{\mathrm{g}} = I_{\mathrm{g}} R_{\mathrm{g}} = -0.102 \text{ V}, \tag{15}$$

and the bridge is balanced if

$$R_2 R_3 = R_1 R_4, \tag{16}$$

which can be seen easily by breaking opening the galvanometer circuit and balancing the voltages at A and B

$$\begin{aligned} V_{\mathrm{A}} &= \varepsilon(1 - \frac{R_1}{R_1 + R_2}) \\ V_{\mathrm{B}} &= \varepsilon(1 - \frac{R_3}{R_3 + R_4}) \end{aligned}$$

so that the voltages are equal when

$$\frac{R_1}{R_1 + R_2} = \frac{R_3}{R_3 + R_4} \quad \rightarrow \quad R_1 R_4 = R_2 R_3. \tag{17}$$