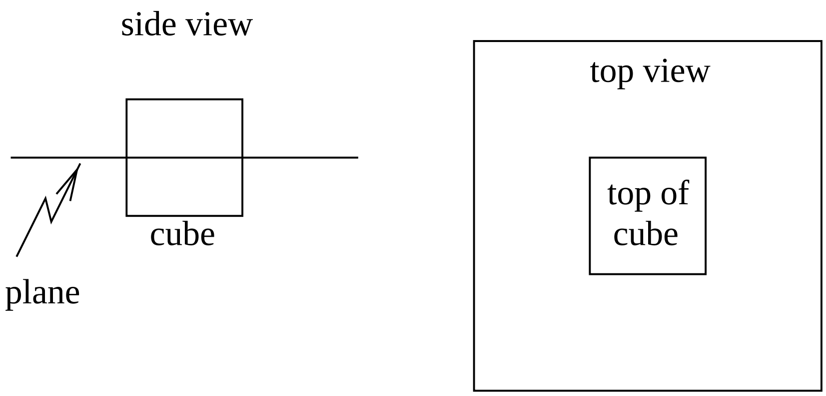
PHY6938 E & M Fall 99 February 11, 2000

1. A very large, non-conducting plane has a surface charge density of $\sigma = 2.5\ \mu\text{C}/\text{m}^2$.



a) Find the total charge enclosed within a cubic box of side $L = 0.2$ m which intersects the plane as shown.

The total charge enclosed is just the area of the intersection of the box with the plane multiplied by the surface charge density of the plane

$$Q = \sigma L^2 = 2.5\ \mu\text{C}/\text{m}^2 \cdot (0.2\ \text{m})^2 = 0.1\ \text{C}.$$

b) Using Gauss's law, find an expression for the electric flux through the cube in terms of the magnitude of the electric field E and the length of the side of the box.

Contrary to the way the question is worded we *do not* need Gauss's law to find the flux of the electric field through the surface; we simply need to exploit the symmetry. The electric field at the top of the cube is constant by symmetry, points away from the top, and is perpendicular to the surface, so that the flux through the top of the box is simply $E \cdot L^2$. The same is true of the bottom surface. The electric field is parallel to the surface on the sides so that the flux through the sides is zero, so that the total flux is

$$\phi = \oint_S \mathbf{E} \cdot \hat{\mathbf{n}} = 2EL^2.$$

c) Find the electric field (magnitude and direction, up or down) both above and below the plane.

Now we can use Gauss's law, which states that the electric flux through a closed surface is

$$\phi = \frac{1}{\epsilon_0} Q_{\text{encl}},$$

where Q_{encl} is the charge enclosed by the surface, so that

$$2EL^2 = \frac{1}{\epsilon_0} Q_{\text{encl}} = \frac{1}{\epsilon_0} \sigma L^2$$

and so

$$E = \frac{1}{2\epsilon_0}\sigma = \frac{2.5 \times 10^{-6}\ \mathrm{C/m^2}}{2 \cdot 8.85 \times 10^{-12}\,\mathrm{C^2/(N \cdot m^2)}} = 1.41 \times 10^5\ \frac{\mathrm{N}}{\mathrm{C}}.$$

d) A second non-conducting plane of charge density $-2.5\ \mu\mathrm{C/m^2}$ is placed 10 cm above and parallel to the first plane. What is the electric field at the midpoint between the two planes?

The electric field found in part c) did not depend on the distance from the plane; it is constant and points away from the plane. The electric field due to the second plane is also constant, with the same magnitude, but points towards the second plane since the charge is negative. The electric field from both planes between them therefore points in the same direction, and so is given by the principle of superposition to be

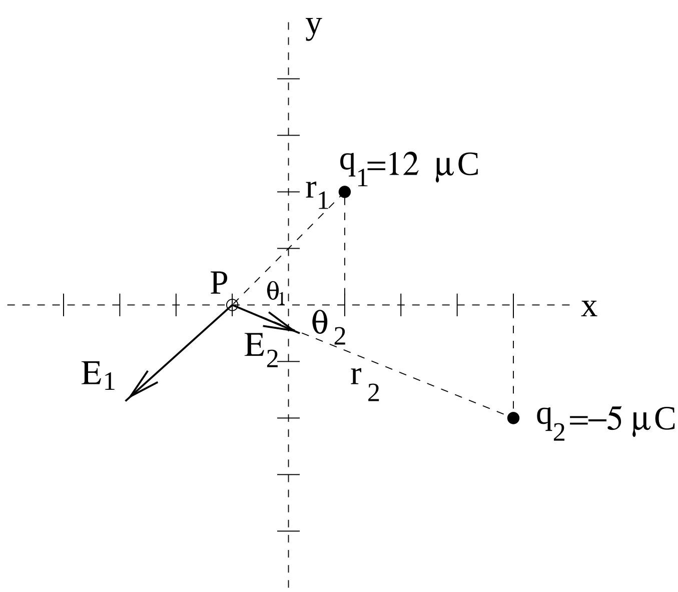
$$E_{\mathrm{tot}} = \frac{1}{2\epsilon_0}\sigma + \frac{1}{2\epsilon_0}\sigma = \frac{1}{\epsilon_0}\sigma = 2.82 \times 10^5\ \frac{\mathrm{N}}{\mathrm{C}},$$

and points from the positively charged plane towards the negatively charged plane.

2. A point charge of -5 μC is located at $x = 4$ m, $y = -2$ m. A second point charge of 12 μC is located at $x = 1$ m, $y = 2$ m.

a) Find the potential at $x = -1$ m, $y = 0$.

We will draw a diagram showing the configuration.



The potential at P$= (-1, 0)$ is given by

$$V(P) = \frac{kq_1}{r_1} + \frac{kq_2}{r_2},$$

where from the diagram we can see that

$$r_1 = \sqrt{2^2 + 2^2}\ \mathrm{m} = 2\sqrt{2}\ \mathrm{m}, \quad r_2 = \sqrt{5^2 + 2^2}\ \mathrm{m} = \sqrt{29}\ \mathrm{m},$$

so that

$$V(P) = 9 \times 10^9 \ \frac{\mathrm{N \cdot m^2}}{\mathrm{C^2}} \left(\frac{12 \times 10^{-6}\ \mathrm{C}}{2\sqrt{2}\ \mathrm{m}} - \frac{5 \times 10^{-6}\ \mathrm{C}}{\sqrt{29}\ \mathrm{m}} \right) = 2.98 \times 10^4\ \mathrm{V}.$$

b) Calculate the work required to bring an electron to $x = -1$ m, $y = 0$.

The question does not state from where the electron is brought; in the absence of further information we will assume that it is brought from infinity. Then the work done is

$$W = V(P) - V(\infty) = qV(P) - 0 = qV(P),$$

and since $q = -e$, we have

$$W = -eV(P) = -2.98 \times 10^{-4}\ \mathrm{eV}.$$

The negative sign indicates that the force is attractive and that the electron does work on us; we could convert to Joules, but why bother when we have this much nicer unit at our fingertips?

c) Find the magnitude and direction of the electric field at $x = -1$ m, $y = 0$.

For the electric field we have to find the magnitude and direction of the resultant vector of the two electric fields shown in the diagram. First lets find the (positive) magnitude of the electric field vectors

$$E_1 = \frac{k|q_1|}{r_1^2}, \quad E_2 = \frac{k|q_2|}{r_2^2}$$

and then we need to know the cosines and sines of the angles θ_1 between $\mathbf{E}_1$ and the positive x-axis, and θ_2 between $\mathbf{E}_2$ and the negative x-axis. From the diagram we see that

$$\begin{aligned}
E_{1x} &= E_1 \cos(\theta_1) = -E_1 \cdot \frac{2}{r_1} \\
E_{1y} &= -E_1 \sin(\theta_1) = -E_1 \cdot \frac{2}{r_1} \\
E_{2x} &= -E_2 \cos(\theta_2) = E_2 \cdot \frac{5}{r_2} \\
E_{2y} &= E_2 \sin(\theta_2) = -E_2 \cdot \frac{2}{r_2},
\end{aligned}$$

so that the total field $\mathbf{E}$ has components

$$\begin{aligned}
E_x &= E_{1x} + E_{2x} = -\frac{2k|q_1|}{r_1^3} + \frac{5k|q_2|}{r_2^3} \\
&= 9 \times 10^9 \ \frac{\mathrm{Nm^2}}{\mathrm{C^2}} \left(-2 \cdot \frac{12 \times 10^{-6}\ \mathrm{C}}{16\sqrt{2}} + 5 \cdot \frac{5 \times 10^{-6}\ \mathrm{C}}{29\sqrt{29}} \right) \\
&= -8.10 \times 10^3\ \frac{\mathrm{N}}{\mathrm{C}} \\
E_y &= E_{1y} + E_{2y} = -\frac{2k|q_1|}{r_1^3} - \frac{2k|q_2|}{r_2^3}
\end{aligned}$$

$$
\begin{aligned}
&= 9 \times 10^9 \, \frac{\mathrm{Nm}^2}{\mathrm{C}^2} \left(-2 \cdot \frac{12 \times 10^{-6} \ \mathrm{C}}{16\sqrt{2}} - 2 \cdot \frac{5 \times 10^{-6} \ \mathrm{C}}{29\sqrt{29}} \right) \\
&= -1.012 \times 10^4 \ \frac{\mathrm{N}}{\mathrm{C}},
\end{aligned}
$$

so that the magnitude of $\mathbf{E}$ is $E = \sqrt{E_x^2 + E_y^2} = 1.30 \times 10^4$ N/C, and it points at an angle of

$$\theta = \tan^{-1}\left(\frac{E_y}{E_x}\right) = 51.3^\circ$$

to the negative x axis.

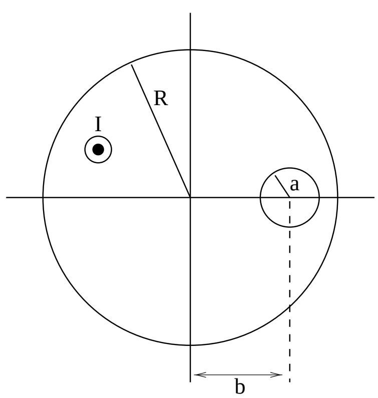
d) Calculate the magnitude and direction of the force on an electron at $x = -1$ m, $y = 0$.

This is just $\mathbf{F} = -e\,\mathbf{E}$, and so the force has magnitude

$$F = 1.6 \times 10^{-19} \ \mathrm{C} \cdot 1.30 \times 10^4 \ \frac{\mathrm{N}}{\mathrm{C}} = 2.08 \times 10^{-15} \ \mathrm{N},$$

and points at 51.3° to the positive x axis.

3. A very long, straight conductor with a circular cross section of radius R carries a current I. Inside the conductor is a cylindrical hole of radius a whose axis is parallel to the axis of the conductor but offset a distance b from the axis of the conductor. The current I is uniformly distributed across the cross section of the conductor and is directed out of the page. Find the magnetic field everywhere outside the conductor.



We solve this problem by the principle of superposition. The magnetic field is identical to that of a solid conductor of radius R with the same current density as the cylindrical conductor with the hole in it, superimposed on another solid conductor with radius a and the same magnitude of the current density but with the current in the opposite direction at the position of the hole. First let's find the current in each. The current density is $J = I/[\pi(R^2 - a^2)]$, so that the magnitude of the current in the first solid conductor should be

$$I_R = J \cdot \pi R^2 = I \frac{R^2}{R^2 - a^2},$$

and that in the second conductor should be

$$I_a = J \cdot \pi a^2 = I \frac{a^2}{R^2 - a^2}.$$

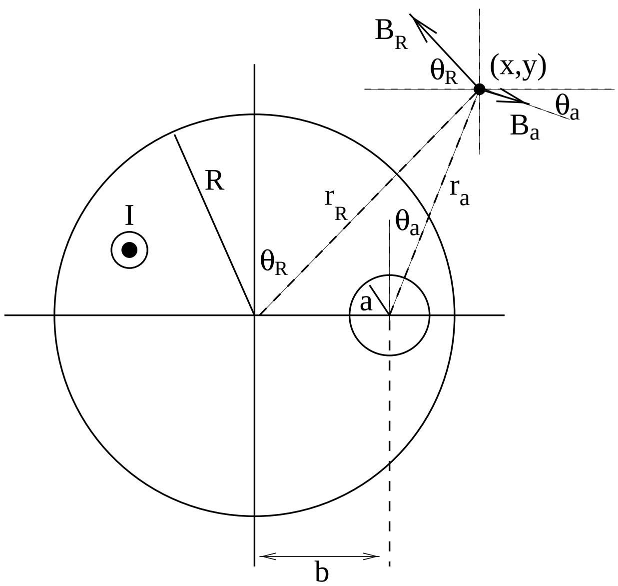
Now we can use Ampère's law to find the magnetic field at a general point $P = (x, y)$ outside the two conductors, because each is now a symmetric conductor and we can do the integral of the magnetic field around a circle centered at the center of each wire. Formally, we have

$$\begin{aligned}
\oint \mathbf{B}_R \cdot \mathbf{dl} &= \mu_0 I_R \\
\oint \mathbf{B}_a \cdot \mathbf{dl} &= \mu_0 I_a
\end{aligned}$$

where the integrals are over circles centered at the center of the solid conductor of radius R and that of radius a, respectively. This means that

$$\begin{aligned}
B_R &= \frac{\mu_0}{2\pi} \frac{I_R}{r_R} \\
B_a &= \frac{\mu_0}{2\pi} \frac{I_a}{r_a},
\end{aligned}$$

where r_R and r_a are the distances to the centers of the two conductors from the field point $P = (x, y)$. The direction of each $\mathbf{B}$ is different; the large conductor has a current out of the page so its field points along $\hat{\boldsymbol{\theta}}_R$, and that from the small conductor points along $-\hat{\boldsymbol{\theta}}_a$, as shown below.



From the diagram we can see that

$$\begin{aligned}
B_{Rx} &= -B_R \cos(\theta_R) = -\frac{\mu_0}{2\pi} \frac{I_R}{\sqrt{x^2 + y^2}} \frac{y}{\sqrt{x^2 + y^2}} = -\frac{\mu_0}{2\pi} \frac{I_R\, y}{x^2 + y^2} \\
B_{Ry} &= B_R \sin(\theta_R) = \frac{\mu_0}{2\pi} \frac{I_R}{\sqrt{x^2 + y^2}} \frac{x}{\sqrt{x^2 + y^2}} = \frac{\mu_0}{2\pi} \frac{I_R\, x}{x^2 + y^2} \\
B_{ax} &= B_a \cos(\theta_a) = \frac{\mu_0}{2\pi} \frac{I_a}{\sqrt{(x-b)^2 + y^2}} \frac{y}{\sqrt{(x-b)^2 + y^2}} = \frac{\mu_0}{2\pi} \frac{I_a\, y}{(x-b)^2 + y^2} \\
B_{ay} &= -B_a \sin(\theta_a) = -\frac{\mu_0}{2\pi} \frac{I_a}{\sqrt{(x-b)^2 + y^2}} \frac{(x-b)}{\sqrt{(x-b)^2 + y^2}} = -\frac{\mu_0}{2\pi} \frac{I_R\, (x-b)}{(x-b)^2 + y^2},
\end{aligned}$$

so that

$$\begin{aligned}
B_x &= B_{Rx} + B_{ax} = -\frac{\mu_0}{2\pi} I y \left(\frac{R^2}{R^2 - a^2} \frac{1}{x^2 + y^2} - \frac{a^2}{R^2 - a^2} \frac{1}{(x-b)^2 + y^2} \right) \\
B_y &= B_{Ry} + B_{ay} = -\frac{\mu_0}{2\pi} I \left(\frac{R^2}{R^2 - a^2} \frac{x}{x^2 + y^2} - \frac{a^2}{R^2 - a^2} \frac{x-b}{(x-b)^2 + y^2} \right).
\end{aligned}$$