PHY6938 Modern Physics and Quantum Mechanics, Fall 2000

1. During a photoelectric effect experiment, sodium metal is illuminated with light of wavelength 4.20×10^2 nm. The stopping potential is found to be 0.65 V. When the wavelength is changed to 3.10×10^2 nm, the stopping potential is found to be 1.69 V. Using only these data and the values of the speed of light, $c = 3.00 \times 10^8$ m/s, and the elementary charge, $e = 1.60 \times 10^{-19}$ C, find a value for Planck's constant.

Recall that in such experiments, photons are abosorbed by the conduction electrons in the surface of the metal, which then have enough kinetic energy that they are no longer bound to the metal. The minimum energy required to remove an electron from the surface (which is then moving with zero velocity and so no kinetic energy) is the work function ϕ. This is measured by detecting a current flowing out of the surface through a vacuum into an electrode, where a voltage is applied which drives the electrons back into the metal's surface. When this voltage reaches the largest kinetic energy the electrons have been given from the photons the current stops.

From this and conservation of energy we know that

$$h\nu = \phi + eV,$$

where V is the stopping potential. If we have two frequencies ν and two values of the stopping potential we can eliminate the work function by writing

$$h\nu_1 - eV_1 = h\nu_2 - eV_2,$$

so that

$$h(\nu_1 - \nu_2) = hc(1/\lambda_1 - 1/\lambda_2) = e(V_1 - V_2).$$

Solving for h we find

$$\begin{aligned} h &= \frac{e(V_1 - V_2)}{c(1/\lambda_1 - 1/\lambda_2)} \\ &= \frac{1.6 \times 10^{-19}\ \mathrm{C} \cdot (0.65 - 1.69)\ \mathrm{V}}{3.0 \times 10^8\ \mathrm{m/s} \cdot (1/4.2 \times 10^{-7} - 1/3.10 \times 10{-}7)\ \mathrm{m}^{-1}} \\ &= 6.6 \times 10^{-34}\ \mathrm{J \cdot s}. \end{aligned}$$

2. The Hamiltonian of a quantum-mechanical harmonic oscillator is given by

$$\mathcal{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + \frac{1}{2}m\omega^2 x^2 .$$

a) Obtain the value of α for which the wave function

$$\psi(x) = C \exp\left(-\alpha^2 x^2\right) ,$$

where C is the normalization constant, is a solution of Schrödinger's equation. Find the corresponding energy eigenvalue.

We need to substitute the trial wavefunction into Schrödinger's equation and find the value of α^2 for which it is a solution. Let's first take the derivatives of the wavefunction

$$\frac{d}{dx}\psi(x) = C(-2\alpha^2 x)\exp\left(-\alpha^2 x^2\right) = -2\alpha^2 x\psi(x),$$

and so

$$\frac{d^2}{dx^2}\psi(x) = -2\alpha^2\psi(x) + (-2\alpha^2 x)^2\psi(x).$$

This means that

$$\mathcal{H}\psi(x) = -\frac{\hbar^2}{2m}\left[-2\alpha^2 + (-2\alpha^2 x)^2\right]\psi(x) + \frac{1}{2}m\omega^2 x^2\psi(x).$$

The only way that this can be a constant E is if the x^2 terms cancel, which will happen if

$$\frac{\hbar^2}{2m}4\alpha^4 = \frac{1}{2}m\omega^2,$$

which gives

$$\alpha^2 = \sqrt{\frac{m^2\omega^2}{4\hbar^2}} = \frac{m\omega}{2\hbar}.$$

Under this condition we have that

$$\mathcal{H}\psi(x) = \frac{\hbar^2\alpha^2}{m}\psi(x) = \frac{\hbar\omega}{2}\psi(x); \quad E = \frac{\hbar\omega}{2}.$$

b) Obtain the expectation values of x, x^2, p, and p^2 for the wave function in (a). *Hint:* **The definition of the gamma function,**

$$\Gamma(y) = \int_0^\infty \mathrm{d}t\, t^{y-1}e^{-y}, \quad \Gamma(1+y) = y\Gamma(y)\ ,$$

could be helpful in the integration.

This question is easy to answer if you *don't* use the hint, which requires knowing how to transform Gaussian integrals into the Gamma function form given, *and* knowing the value of $\Gamma(1/2)$. Firstly we can dispose of the expectation values of x and p very simply, since we know that the potential is symmetric in x. Since the wavefunction is also symmetric in x [note $\psi(x) = \psi(-x)$] then the integral

$$\langle x\rangle = \int_{-\infty}^{+\infty} dx|\psi(x)|^2 x$$

must be zero, since it is of an odd integrand over an even region. For exactly the same reasons the average value of the momentum must also be zero (this must be true for a particle localized in a potential well like this).

We could find the expectation values of x^2 and p^2 by finding the integral

$$I(2\alpha^2) = \int_{-\infty}^{+\infty} dx\exp(-2\alpha^2 x^2),$$

which can be accomplished by noticing that I^2 is the integral

$$I^2(2\alpha^2) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \exp(-2\alpha^2[x^2+y^2]),$$

and then doing the integration in spherical polar coordinates. Once we have $I(2\alpha^2)$ we can take a derivative with respect to α^2 under the integration sign to get the integral we want (this brings down a factor of $-2x^2$).

However, there is a much simpler way to do this which is less prone to error. The virial theorem tells us that a harmonic oscillator has its average potential energy equal to its average kinetic energy (and so each is half the total energy). This means that for our ground state wavefunction

$$\left\langle \frac{1}{2} m\omega^2 x^2 \right\rangle = \frac{\hbar\omega}{4},$$

so that

$$\left\langle x^2 \right\rangle = \frac{\hbar}{2m\omega},$$

and that

$$\left\langle \frac{p^2}{2m} \right\rangle = \frac{\hbar\omega}{4},$$

so that

$$\left\langle p^2 \right\rangle = \frac{\hbar m\omega}{2},$$

c) Verify that the Heisenberg uncertainty relation for p and x is satisfied.

This relation tells us that $\Delta p \Delta x \geq \hbar/2$, where the uncertainty Δp is defined by

$$\Delta p := \sqrt{\langle p^2 \rangle - \langle p \rangle^2},$$

and similarly for Δx. Since $\langle p \rangle = \langle x \rangle = 0$ then we have that

$$\Delta p \Delta x = \sqrt{\langle p^2 \rangle \langle x^2 \rangle} = \sqrt{\frac{\hbar^2}{4}} = \frac{\hbar}{2},$$

so we see that for this system in this state the equality holds.

3. In nuclear beta decay, electrons are observed to be ejected from the atomic nucleus. Assume that electrons are somehow trapped within the nucleus and that occasionally one escapes.

a) Estimate the kinetic energy that such an electron must have. Assume a nuclear diameter of 1.0×10^{-14} m.

The statement of this problem is confusing. In beta decay electrons are ejected forcibly from the nucleus, since they actually come from beta decay of a neutron into a proton and an electron (and an anti-neutrino), which releases energy. A part of this energy is in the form of rest and kinetic energy of the electron, another (smaller) part goes into the recoil energy of the proton and the neutrino energy. Since for a free neutron about 1.3 MeV is available in total for this, and we need 0.5 MeV for the electron rest energy, one might guess that in the absence of all other effects the kinetic energy of the electron would be very roughly one MeV. However, one also has to take into account the difference in the nuclear binding energy of the initial and final nuclei, and nuclear structure effects (the initial neutron may need to be in a higher energy level than the final proton because of the Pauli principle and the fact that large nuclei have more neutrons than protons). In addition, the ejected electron is attracted to the positive charge of the nucleus and so a (possibly substantial) part of its initial kinetic energy is required to overcome this Coulomb barrier if it is to become free.

What we are asked to do is to suspend all of our knowledge of the physics of this process and imagine an electron which is confined within the nucleus all of the time. Since we are not given anything except the size of the nucleus, in particular we do not know its charge, we cannot evaluate the energy required to overcome the Coulomb barrier. If all we know is the size of the (roughly spherical) nucleus which contains the electron, then all we can do is to estimate its momentum and so kinetic energy using the uncertainty principle.

If the electron is confined to a region of size 10^{-14} m, then by the uncertainty principle it must have a Δx smaller than this, but how much smaller? This is the essential problem with using the uncertaintly principle for doing physics. I got curious about this and decided to find Δx for a particle in its ground state in a one-dimensional box $x \in [-a, a]$, which has wavefunction $\psi(x) = (1/\sqrt{a}) \cos(\pi x/2a)$. The result is that

$$\left\langle x^2 \right\rangle = \frac{1}{a} \int_{-a}^{+a} x^2 \cos^2(\pi x/2a) = a^2 \frac{(\pi^2 - 6)}{3\pi^2},$$

which since $\langle x \rangle = 0$ gives

$$\Delta x = \sqrt{\langle x^2 \rangle} = \frac{a}{\pi} \sqrt{\frac{\pi^2 - 6}{3\pi^2}} = 0.361\ a.$$

So if, as is usually done, we say that the uncertainty Δx in the electron's position is the width $2a$ of the box, we make a mistake of a factor of $2a/(0.361a) = 5.5$ in Δx, and so greatly underestimate Δp. Perhaps this approach to estimating particle energies should be called the 'uncertain principle', good for an order of magnitude ar best.

If we forge ahead regardless of this problem and use the diameter d of the nucleus for Δx then we will very roughly estimate the momentum of the electron in the nucleus by using

$$\Delta p \geq \frac{\hbar/2}{\Delta x} = \frac{\hbar}{2d} = \frac{\hbar c}{2dc} = \frac{197.32\ (\mathrm{MeV/c}) \cdot \mathrm{fm}}{20\ \mathrm{fm}} = 9.85\ \frac{\mathrm{MeV}}{\mathrm{c}}.$$

where we have used the convenient unit $\hbar c = 197.32$ MeV·fm and that the nuclear diameter given is 10 fm (1 fm=10^{-15} m).

Since pc is larger (by a factor of 20) than the rest energy 0.511 MeV of the electron we know that this is a relativistic electron, and we have to use the relation (assume, again probably erroneously, that we can use Δp for the r.m.s. momentum)

$$E = (p^2c^2 + m^2c^4) = 9.86\ \mathrm{MeV} \simeq pc.$$

b) Electrons emitted from the nucleus in nuclear beta decay typically have kinetic energies of about 1 MeV. Comment on the difference between the actual kinetic energy and your result for part a.

See all the physics reasons above for why this is bound to be a poor estimate of the energy of a beta-decay electron.