PHY541 Problem Set 2. Due September 16, 2014.

For the following problems a volume V is divided into $2L$ regions so that $V = 2Lv$. Half the regions are labeled type 0, and the other half are type ϵ. When a particle is in a type 0 region it contributes zero to the total energy. When a particle is in a type ϵ region it contributes ϵ to the total energy. A complete set of eigenstates of the Hamiltonian and particle number are given by specifying the occupation number of each region. Such a quantum state can be represented as $|n_1^{(0)}, n_2^{(0)}, ..., n_L^{(0)}; n_1^{(\epsilon)}, n_2^{(\epsilon)}, ..., n_L^{(\epsilon)}\rangle$, where $n_j^{(0)}$ and $n_j^{(\epsilon)}$ are the occupancy numbers of the jth type 0 region and jth type ϵ region respectively. The eigenstate, eigenvalue equations are

$$H_{\rm op}|n_1^{(0)}, n_2^{(0)}, ..., n_L^{(0)}; n_1^{(\epsilon)}, n_2^{(\epsilon)}, ..., n_L^{(\epsilon)}\rangle = \epsilon \sum_{j=1}^{L} n_j^{(\epsilon)} |n_1^{(0)}, n_2^{(0)}, ..., n_L^{(0)}; n_1^{(\epsilon)}, n_2^{(\epsilon)}, ..., n_L^{(\epsilon)}\rangle$$

$$N_{\rm op}|n_1^{(0)}, n_2^{(0)}, ..., n_L^{(0)}; n_1^{(\epsilon)}, n_2^{(\epsilon)}, ..., n_L^{(\epsilon)}\rangle = \sum_{j=1}^{L} [n_j^{(\epsilon)} + n_j^{(0)}] |n_1^{(0)}, n_2^{(0)}, ..., n_L^{(0)}; n_1^{(\epsilon)}, n_2^{(\epsilon)}, ..., n_L^{(\epsilon)}\rangle \quad (1)$$

1. For the case where the occupation number in each region can be only 0 or 1.
   a. Calculate the total number of states with energy E, where E is an integer multiple of ϵ. You can assume that $L\epsilon \geq \max(E, N\epsilon - E)$ and $N\epsilon \geq E$.
   b. Using the microcanonical ensemble calculate the entropy, $S(E, V, N)$ in the thermodynamic limit.
   c. Evaluate $T(E, V, N)$, and $\mu(E, V, N)$.
   d. Solve for E and N as functions of μ and T and V.
   e. Calculate the thermodynamic potential $\Omega(T, V, \mu) = E - TS - \mu N$ from your microcanonical results above.

2. Repeat problem 1 for the case where the occupation numbers can take on any nonnegative integer value.

3. a. Calculate $\Omega(T, V, \mu)$ using the grand canonical ensemble for both cases above. Verify it agrees with your results from the microcanonical ensemble.
   b. Show that for any thermodynamic system,

$$\frac{\partial}{\partial \beta} \beta \Omega(T, V, \mu) = E - \mu N \quad (2)$$

   where $\beta = \frac{1}{T}$.
   c. Calculate E and N as functions of T, V, and μ from your Ω derived from the grand canonical ensemble and verify they all agree with the results calculated from the entropy of the microcanonical ensemble for both cases.