PHY6938 Proficiency Exam Spring 2003
March 28, 2003
Modern Physics and Quantum Mechanics

1. **Light of wavelength 300 nm strikes a metal plate, producing photoelectrons that move with speed of $0.002c$.**

   In the photoelectric effect the incoming photons remove electrons from the target. There is a minimum energy required to remove an electron from the interior of a solid to a position just outside. This minimum energy is called the work function Φ. The relation between the energy of photon E_γ, the energy of electron E_e, and the work function Φ is given by

$$E_\gamma = E_e + \Phi. \tag{1}$$

   Note that energy of electron E_e is equal to the kinetic energy of electron, since when electron is outside the solid it is not affected by any potential. In other words electron will be considered as a free particle.

   (a) **What is the work function of the metal ?**

   Use Eq.1, since the speed of the electron is much less than speed of the light we do not need to use the special relativity formula for the kinetic energy.

$$\begin{aligned}
E_e &= \frac{1}{2}\, mv^2 && (2)\\
&= \frac{1}{2}\,(511\,\frac{keV}{c^2})\,(0.002c)^2 \\
&= 1.022\, eV && (3)\\
E_\gamma &= h\nu && (4)\\
&= \frac{hc}{\lambda} \\
&= \frac{1420\, eV\; nm}{300\, nm} \\
&= 4.13\, eV && (5)\\
\Phi &= E_\gamma - E_e && (6)\\
&= 3.111\, eV. && (7)
\end{aligned}$$

   (b) **What is the critical wavelength for this metal, so that photoelectrons are produced ?**

   The critical wavelength is defined as the wavelength for photons which remove remove electrons from interior of a solid to a position outside the solid. In this case the kinetic energy of electrons outside the solid is equal zero $E_e = 0$. Using the Eq.1 and the results from part (a) we obtain

$$\begin{aligned}
E_\gamma &= \Phi && (8)\\
\frac{hc}{\lambda_{critical}} &= \Phi
\end{aligned}$$

$$\lambda_{critical} = \frac{hc}{\Phi} \tag{9}$$

$$\lambda_{critical} = \frac{1420\ eV}{3.111\ eV}$$

$$\lambda_{critical} = 398.6\ eV. \tag{10}$$

(c) What is the significance of the critical wavelength ?

Photons with a lower wavelength than the critical wavelength cannot produce photo-electrons. This led Einstein to the postulation of quantized energy for the electromagnetic fields.

2. A system consists of two distinguishable particles of mass m, bound in an infinite-strength one-dimensional square well potential of width a. They have no spin and do not interact. The one-particle states are given by

$$\Psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi}{a} x\right)$$

This problem is similar to the problem for one particle in an infinite potential well. This means that that particles can only exist inside the well in the interval $0 \leq x \leq a$. Since the particles do not interact with each other and the potential is zero inside the well, they are free particles in the potential well.

(a) Write the 2-body Hamiltonian in terms of the individual particle coordinates x_1 and x_2. Write down the wave-functions for all excited states.

As we mentioned above the particles are free inside the potential well. The Hamiltonian of the system is given by

$$H = H_1 + H_2, \tag{1}$$

where H_1 (H_2) refers to the Hamiltonian of the first (second) particle.

$$H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_1^2} - \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x_2^2}. \tag{2}$$

To find the wave function of this system we can use the method of separation of variables. This means that we assume that the total solution for the eigenvalue problem

$$\begin{aligned} H\psi(x_1, x_2) &= \lambda\psi(x_1, x_2) \\ \psi(0,0) &= 0 \\ \psi(a,a) &= 0 \end{aligned} \tag{3}$$

is given by

$$\psi(x_1, x_2) = \psi_1(x_1)\psi_2(x_2) \tag{4}$$

Substituting Eq.4 in Eq.2 and separating for each function result in

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_1^2}\psi_1(x_1) &= E_1\psi_1(x_1) & -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x_2^2}\psi_2(x_2) &= E_1\psi_2(x_2) \\
\psi_1(0) &= 0 & \psi_2(0) &= 0 \\
\psi_1(a) &= 0 \qquad (5) & \psi_2(a) &= 0 \qquad (6)
\end{aligned}
$$

Solving Eqs.5 and 6 gives the following eigenvalues and eigenvectors

$$
\begin{aligned}
\Psi_{n_1}(x_1) &= \sqrt{\frac{2}{a}}\sin\left(\frac{n_1\pi}{a}x_1\right) && (7)\\
\Psi_{n_2}(x_2) &= \sqrt{\frac{2}{a}}\sin\left(\frac{n_2\pi}{a}x_2\right) && (8)\\
E_{n,1} &= \frac{\hbar^2 n_1^2\pi^2}{2ma^2} && (9)\\
E_{n,2} &= \frac{\hbar^2 n_2^2\pi^2}{2ma^2} && (10)\\
n_1, n_2 &= 1, 2, 3, \cdots && (11)
\end{aligned}
$$

This means that the eigenfunctions of the system are given by

$$
\Psi_{n_1,n_2}(x_1,x_2) = \frac{2}{a}\sin\left(\frac{n_1\pi}{a}x_1\right)\sin\left(\frac{n_2\pi}{a}x_2\right) \qquad (12)
$$

(b) Verify that they are eigenfunctions of the Hamiltonian and find their energy.

This part can be done very easily. Substitute Eq.12 in Eq.3 the result is

$$
\begin{aligned}
H\psi(x_1,x_2) &= \frac{\hbar^2\pi^2}{2ma^2}(n_1^2+n_2^2)\psi(x_1,x_2) && (13)\\
E &= \frac{\hbar^2\pi^2}{2ma^2}(n_1^2+n_2^2) && (14)
\end{aligned}
$$

(c) Consider now that both particles are identical bosons. Write down the excited state wave-functions. What changes with respect to part (b) ? You may ignore the normalization of the wave function.

If the particles are bosons the wave function should be symmetric with respect to x_1 and x_2 this means that

$$
\begin{aligned}
\Psi_{n_1,n_2}(x_1,x_2) &= \frac{2}{\sqrt{2}a}\left[\sin\left(\frac{n_1\pi}{a}x_1\right)\sin\left(\frac{n_2\pi}{a}x_2\right)\right.\\
&+ \left.\sin\left(\frac{n_1\pi}{a}x_2\right)\sin\left(\frac{n_2\pi}{a}x_1\right)\right] && (15)
\end{aligned}
$$

(d) What would change if the particles were identical spin 1/2 fermions ? Write down the ground state and first excited state wave-functions, in terms of eigenfunctions of the individual spins.

For fermions the wave function should be antisymmetric. The wave function for two fermions with spin-$\frac{1}{2}$ can be separated in the space part times the spin part. This means that either the space part or the spin part should be antisymmetric. The wave function can be written as

$$\begin{aligned}\Psi_{n_1,n_2}(x_1,x_2) &= \frac{2}{\sqrt{2}a}\left[\sin\left(\frac{n_1\pi}{a}x_1\right)\sin\left(\frac{n_2\pi}{a}x_2\right)\right.\\ &\pm \left.\sin\left(\frac{n_1\pi}{a}x_2\right)\sin\left(\frac{n_2\pi}{a}x_1\right)\right]\\ & \frac{1}{2}\left(\chi_1(1)\chi_2(2)\mp\chi_1(2)\chi_2(1)\right) \qquad (16)\end{aligned}$$

The ground state has the lowest energy. From Eq.14 we can see that the lowest energy is for $n_1 = n_2 = 1$. Since the space part for the ground state is symmetric the spin part must be antisymmetric. This means that

$$\psi_g = \sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{\pi}{a}x_2\right)\ |\uparrow\downarrow\rangle \qquad (17)$$

The first exited state is $n_1 = 1, n_2 = 2$ or $n_1 = 2, n_2 = 1$. This means that the space part is antisymmetric therefore the spin part must be symmetric.The wave function will be given by

$$\begin{aligned}\psi &= \frac{2}{\sqrt{2}a}\left[\sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{2\pi}{a}x_2\right)\right.\\ &- \left.\sin\left(\frac{\pi}{a}x_2\right)\sin\left(\frac{2\pi}{a}x_1\right)\right]|\uparrow\uparrow\rangle \qquad (18)\\ \psi &= \frac{2}{\sqrt{2}a}\left[\sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{2\pi}{a}x_2\right)\right.\\ &- \left.\sin\left(\frac{\pi}{a}x_2\right)\sin\left(\frac{2\pi}{a}x_1\right)\right]|\downarrow\downarrow\rangle. \qquad (19)\\ \psi &= \frac{2}{\sqrt{2}a}\left[\sin\left(\frac{\pi}{a}x_1\right)\sin\left(\frac{2\pi}{a}x_2\right)\right.\\ &- \left.\sin\left(\frac{\pi}{a}x_2\right)\sin\left(\frac{2\pi}{a}x_1\right)\right]\frac{1}{2}\left(|\uparrow\downarrow\rangle+|\downarrow\uparrow\rangle\right) \qquad (20)\end{aligned}$$

3. **Consider an electron in a hydrogen atom that has the following wave function at a particular time, $t = 0$:**

$$|\psi(0)\rangle = A\left(|100\rangle + 2i|210\rangle + 2|322\rangle\right).$$

Here, each of the individual eigenvector terms are denoted by their quantum numbers N (principal), L (angular momentum), and M (angular momentum projection) in the following manner: $|NLM\rangle$.

(a) **Calculate the value of the normalization constant A.**

To find the constant A we have to normalize the wave function. Since states with different n are perpendicular to each other this can be easily determined.

$$\begin{aligned}\langle\psi(0)|\psi(0)\rangle &= 1\\ &= A^2\ (1+4+4)\\ \text{or} \quad & A = \frac{1}{3}. \qquad (1)\end{aligned}$$

(b) Find the expectation value of the energy of this electron at $t = 0$. Express your answer in units of eV.

The expectation value for the energy is given by

$$E = \langle\psi(0)|H|\psi(0)\rangle \tag{2}$$

In the absence of any external field the eigenstates with the same n are degenerated. The energy for a state with quantum number n is given by

$$E_n = \frac{n^2}{m} \tag{3}$$

Using Eq.3 we obtain

$$E = c\left(1 + 1 + \frac{4}{9}\right) = -3.69\ eV \tag{4}$$

(c) If a measurement of the z-projection of the electron's orbital angular momentum is made at $t = 0$, then with what probability are the results 0, $\hbar$, $2\hbar$, and $3\hbar$ obtained?

The probability for measuring l_z is given by

$$P = \left|\left(\sum_{n,l} \angle n, l, l_z|\right)|\psi(0)\rangle\right|^2 \tag{5}$$

Using Eq.5 we get

$$\begin{aligned} P_{l_z=0} &= \frac{5}{9} \\ P_{l_z=\hbar} &= 0 \\ P_{l_z=2\hbar} &= \frac{4}{9} \\ P_{l_z=3\hbar} &= 0. \end{aligned} \tag{6}$$

(d) Write the expression for the wave function $|\Psi(t)\rangle$ at any time t after $t = 0$.

The wave function at time t when the wave function at time t_1 is $\psi(t_1)$ is given by

$$|\psi(T)\rangle = \exp(\frac{-iH(t - t_1)}{\hbar})\,|\psi(t_1)\rangle \tag{7}$$

Use Eq.7 we get

$$\begin{aligned} |\psi(T)\rangle &= \frac{1}{3}\left\{\exp(\frac{-iE_1t}{\hbar})\,|1,0,0\rangle + 2i\exp(\frac{-iE_2t}{\hbar})\,|2,1,0\rangle\right. \\ &+ \left. 2\exp(\frac{-iE_3t}{\hbar})\,|3,2,2\rangle\right\} \end{aligned} \tag{8}$$

4. **Cosmic ray photons from space are bombarding your laboratory and smashing massive objects to pieces! Your detectors indicate that two fragments, each of mass m_0, depart such a collision (between a photon and a particle of mass M) moving at speed 0.6 c at 60° to the photon's original direction of motion.**

(a) In terms of m_0 and c, what is the energy of the cosmic ray photon?

We will solve this part of the problem using conservation of momentum to find the photon momentum (and so energy). The momentum of each fragment is

$$p = m_0 \gamma(v), \tag{1}$$

where

$$\gamma(v) = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{0.64}} = \frac{5}{4}, \tag{2}$$

so that

$$p = \frac{5}{4} m_0 (0.6\ c) = \frac{3}{4} m_0 c. \tag{3}$$

The components perpendicular to the photon's direction of motion cancel, and those parallel to it add to

$$p_\gamma = 2p\cos(60^\circ) = p = \frac{3}{4} m_0 c, \tag{4}$$

where we have used conservation of momentum, so the photon energy is

$$E_\gamma = p_\gamma c = \frac{3}{4} m_0 c^2. \tag{5}$$

(b) In terms of m_0, what is the mass M of the particle being struck (assumed originally stationary)?

The initial total relativistic energy is therefore

$$E_{\mathrm{i}} = Mc^2 + \frac{3}{4} m_0 c^2, \tag{6}$$

and the final total relativistic energy is

$$E_{\mathrm{f}} = 2E = 2\sqrt{p^2c^2 + m_0^2 c^4} = 2\sqrt{\frac{16+9}{16} m_0^2 c^4} = \frac{5}{2} m_0 c^2, \tag{7}$$

and equating these gives that

$$Mc^2 = \left(\frac{5}{2} - \frac{3}{4}\right) m_0 c^2 = \frac{7}{4} m_0 c^2, \tag{8}$$

so that

$$M = \frac{7}{4} m_0. \tag{9}$$