Notes on Siemens Ch. 3

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The Black Sphere

- Model neutron scattering from nuclei as a particle being absorbed by spherical object.
- Start by expanding an incident plane wave in terms of spherical harmonics.

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_l C_l Y_l^0(\theta) \tag{1}$$

$$e^{ikz} \approx \sum_l \frac{\sqrt{\pi}}{kr}\sqrt{2l+1}\, i^{l+1}\left(e^{-i(kr-\frac{l\pi}{2})} - e^{i(kr-\frac{l\pi}{2})}\right) Y_l^0(\theta) \tag{2}$$

- Here we have used the fact that $\mathbf{k}\cdot\mathbf{r}$ only depends on θ, and not on ϕ, thus $m = 0$. Also, we have used various identities and the orthonormality of spherical harmonics.

The Black Sphere

- Scattering only happens for short time

$$\phi(r \to \infty) = \sum_l \frac{\sqrt{\pi}}{kr} \sqrt{2l+1}\, i^{l+1} \left(e^{-i(kr - \frac{l\pi}{2})} - \eta_l e^{i(kr - \frac{l\pi}{2})} \right) Y_l^0(\theta) \tag{3}$$

- Scattered wave is just the total wave function minus the incident wave function, $\phi_{sct} = \phi(r \to \infty) - e^{ikz}$.

$$\phi(r \to \infty) = e^{ikz} + f(\theta) \frac{e^{ikr}}{r} \tag{4}$$

$$f(\theta) = \sum_l i \frac{\sqrt{\pi}}{k} \sqrt{2l+1} Y_l^0(\theta)(1 - \eta_l) \tag{5}$$

- This looks like a scattering amplitude, $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$.

- **Approximations:** Classical turning point is where $k^2 = l(l+1)/R^2 \approx (l+\frac{1}{2})^2/R^2$. If particle passes inside the range of force (R) you get absorption ($\eta_l = 0$), but if not you get none ($\eta_l = 1$).

$$\frac{d\sigma}{d\Omega} = \frac{\pi}{k^2}\left|\sum_{l=0}^{kr-1/2}\sqrt{2l+1}Y_l^0(\theta)\right|^2 \tag{6}$$

- **More Approximations:** Here we approximate this for large and small angle scattering. I was not able to figure out the integrals so I'll just quote their answer here.

$$\frac{d\sigma}{d\Omega} \approx \begin{cases} \frac{2R}{\pi}k\theta^2\sin\theta\cos^2\left(kR\theta+\frac{\pi}{4}\right), & \text{for } kR\theta \gg 1 \\ \frac{k^2R^4}{4}(1-(kR\theta/2)^2)^2, & \text{for } kR\theta \ll 1 \end{cases} \tag{7}$$

The Black Sphere

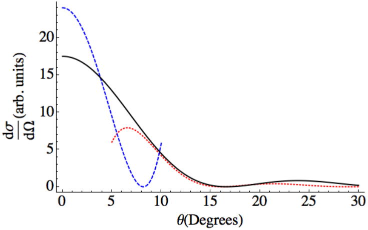


Figure : Rough reproduction of figure 3.2 in the book.

- To find the angle of minumum scattering I have taken the derivative of the high angle scattering and set it equal to zero to get.

$$\theta_{min} = \frac{\pi}{4kR}(2n - 1) \tag{8}$$

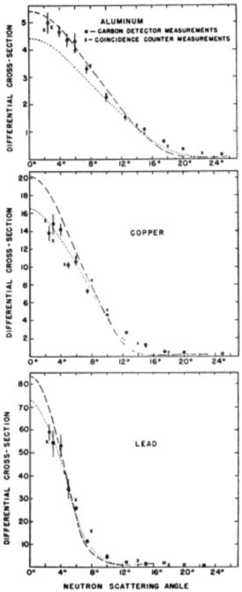
- Experiment must show that it's actually

$$\theta_{min} = \frac{5\pi}{4kR} \tag{9}$$

$$\theta_{min} = \frac{5\pi}{4kR} \tag{10}$$

- Now we can use this diffraction pattern to estimate the radius of nuclei. For Pb with $\epsilon = 84$ MeV we get $k = \sqrt{2m_N\epsilon/\hbar} \approx 2.0$ fm^{-1}. Now the graph above shows that $\theta_{min} \approx 15^\circ$. This gives us a radius of 7.5 fm.
- A quick google search gives Pb a radius of 7 fm.

Nuclear Sizes and Saturation



- You can see from this that the volume $\Omega_r \propto \theta_{min}^{-3}$, so the smaller the nuclei the bigger the scattering angles.
- Coupled with figure 3.1 in the book which shows that lighter nuclei have larger scattering angles this shows that lighter nuclei are smaller than heavier nuclei.
- In fact it turns out that

$$\Omega_r = \Omega_0 A \tag{11}$$

$$R = r_0 a^{1/3} \tag{12}$$

with $r_0 \approx 1.3$ fm and $\Omega_0 = \frac{4}{3}\pi r_0^3 \approx 9$ fm^3.

Optical Model

- Compare this black sphere model to the potential model with a nice Hermitial potential

$$H = \frac{\mathbf{p}^2}{2m_N} + U(\mathbf{r}). \tag{13}$$

- Let's compare this to the black sphere model. Experiment tells us that σ_{el} and σ_{abs} should be comparable, where.

$$\sigma_{tot} = \sigma_{el} + \sigma_{abs} \tag{14}$$

$$\sigma_{el} = \int d\Omega \left(\frac{d\sigma}{d\Omega}\right)_{el} = \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \left|1 - \eta_l\right|^2 \tag{15}$$

$$\sigma_{abs} = \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \left(1 - |\eta_l|^2\right) \tag{16}$$

Optical Model

$$\sigma_{el} = \int d\Omega \left(\frac{d\sigma}{d\Omega} \right)_{el} = \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \left| 1 - \eta_l \right|^2 \tag{17}$$

$$\sigma_{abs} = \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \left(1 - \left| \eta_l \right|^2 \right) \tag{18}$$

- For the black sphere model $(\eta_l = 0, 1)$

$$\sigma_{el} = \sigma_{abs} = \pi R^2 \text{ or } 0 \tag{19}$$

- For the potential model

$$\sigma_{abs} = 0 \tag{20}$$

Optical Model

- However we can alter the potential model (not Hermitian anymore, losing C.M. energy particles)

$$H = \frac{\mathbf{p}^2}{2m_N} + U(\mathbf{r}) - iW(\mathbf{r}) \tag{21}$$

- Solving the Schrödinger eq. for a stream of particles of energy ϵ moving in the x direction we get

$$\phi(\mathbf{r}, t) = \text{const} \times e^{(ik-\kappa)x} e^{-i\epsilon t/\hbar} \tag{22}$$

$$\frac{\hbar^2}{2m_N}(k^2 - \kappa^2) = \epsilon - U \tag{23}$$

$$\frac{\hbar^2}{m_N}\kappa k = W \tag{24}$$

Optical Model

- Now you can look at probability density and see how fast it attenuates.
$$|\phi(\mathbf{r})|^2 \sim e^{-2\kappa x} \tag{25}$$
- This gives us a mean free path for absorption of a nucleon of (falls of by factor $1/e$)
$$\lambda = \frac{1}{2\kappa}. \tag{26}$$
- This will be used later.

Optical Model

- Now if we add in the main spin dependant effect we get the **Phenomenologibal Optical Model**.

$$H^{POM} = \frac{\mathbf{p}^2}{2m_H} + U(r) + \mathbf{l} \cdot \mathbf{s} U^{ls}(r) - iW(r) \tag{27}$$

- Fitting the results to various forms for thse potentials shows that the following gives good results.

$$U(r) = U_0 f((r - R(A))/a_u) + U_C(r) \tag{28}$$

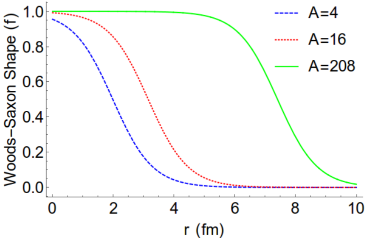
$$W(r) = \left(W_0 - 4W_1 a_W \frac{\partial}{\partial r} \right) f((r - R(A))/a_W) \tag{29}$$

$$U^{ls}(r) = U_0^{ls} \frac{1}{r} \frac{\partial}{\partial r} f((r - R(A))/a_{ls}) \tag{30}$$

Optical Model

- See the book for constants. The f is called the Woods-Saxon shape.

$$f(x) = (1 - \exp(x))^{-1}, \; x = (r - R(A))/a_U \tag{31}$$



- The key features are that it approaches 1 inside the nucleus and falls from 0.9 to 0.1 as r varies from $R - 2.2a_U$ to $R + 2.2a_U$. Surface thickness of $4.4a_U$ or 2.9 fm, about the range of the force.

Optical Model

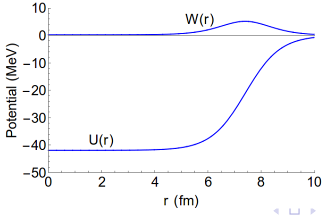
- It is interesting to note that some of these potentials depend on the energy of the incident nucleon (ϵ). The more energetic the more likely to be absorbed by exciting another nucleon.

$$W_0(\epsilon) \approx \max(0.22\epsilon - 2\text{MeV}, 0) \tag{32}$$

$$W_1(\epsilon) \approx \max\left[12\text{MeV} - 0.25\epsilon + 24\text{MeV} \cdot t_3 \frac{N-Z}{A}, 0\right] \tag{33}$$

$$U_0(\epsilon) \approx -50\text{MeV} - 48\text{MeV} \cdot t_3 \frac{N-Z}{A} + 0.3(\epsilon - U_C(R)) \tag{34}$$

$$U_0^{ls} \approx 30\text{MeVfm}^2/\hbar^2 \tag{35}$$



- Riddle to be solved later. Imagine a 40 MeV neutron begin scattered. By the equations before ($\lambda = 1/2\kappa$) we get $\lambda \approx 5$ fm. However when we use the cross section to calculate it with $\sigma \approx 4\pi d\sigma/d\Omega$ we get

$$\lambda = (n\sigma)^{-1} \approx 0.4\text{fm}. \tag{36}$$