PHY541 Problem Set 7. Due December 4, 2014.

1. Pathria and Beale problem 12.6.

   I solved this by writing $H(L)$, calculating the number of states $\mathcal{N}(L)$ for a particular L, and writing $e^{-\beta F} = \sum_L \mathcal{N}(L) e^{-\beta H(L)}$. Using the thermodynamic limit to convert this to an integral and integrating using the same steepest descent method we used to derive Stirling's approximation gave me the same equation as mean field theory.

2. The Hamiltonian of a system can be written as

$$H = -J \sum_{\langle ij \rangle} \hat{\boldsymbol{\epsilon}}_i \cdot \hat{\boldsymbol{\epsilon}}_j - \sum_i \boldsymbol{h} \cdot \hat{\boldsymbol{\epsilon}}_i \tag{1}$$

   where $\boldsymbol{h}$ is the scaled magnetic field, $J > 0$, and the sum is over the nearest neighbor spins. A spin has $2d$ nearest neighbors. The classical "spins" $\hat{\boldsymbol{\epsilon}}_i$ are unit vectors given in terms of the usual spherical angles θ_i and ϕ_i

$$\hat{\boldsymbol{\epsilon}}_i = \cos\theta_i \hat{\boldsymbol{z}} + \sin\theta_i \cos\phi_i \hat{\boldsymbol{x}} + \sin\theta_i \sin\phi_i \hat{\boldsymbol{y}} \,. \tag{2}$$

   The exact Helmholtz free energy would be given by

$$e^{-\beta F} = \int_{-1}^{1} d\cos\theta_1 \int_0^{2\pi} d\phi_1 \int_{-1}^{1} d\cos\theta_2 \int_0^{2\pi} d\phi_2 ... \int_{-1}^{1} d\cos\theta_N \int_0^{2\pi} d\phi_N e^{-\beta H} \,. \tag{3}$$

   a. Approximate the density matrix using the mean field approximation

$$e^{-\beta(H-F)} \simeq \prod_{i=1}^{N} \rho^{(1)}(\theta_i, \phi_i) \tag{4}$$

   Extremize the variational expression for the trial Helmholtz free energy using the methods of the calculus of variations you learned in classical mechanics. Write the form of $\rho^{(1)}(\theta, \phi)$, that your solution gives, along with an algebraic equation (or equations) that must be solved for any unknown parameters in your solution.

   b. For the case where $\boldsymbol{h} = 0$, find the transition temperature(s) if any. Verify that, for T near the transition temperature that if there is more than one solution of your equations of part a, that you have chosen the free energy global minimum (or minima).