

# Study guide for qualifying exams

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## 1 Classical Mechanics Equations

### Newtonian Mechanics

#### Newton's Laws:

1. An object will maintain it's current motion unless acted upon by an external force.
2.  $\vec{F} = m\vec{a}$
3. All forces occur in equal but directionally opposite pairs.

**Second Law:**  $\vec{F} = m\vec{a} = \dot{\vec{p}}$

**Angular Position/Velocity/Acceleration:**  $\theta = s/r$ ,  $\omega = v/r$ ,  $\alpha = a/r$

**Angular Momentum:**  $\vec{L} = \vec{r} \times \vec{p}$

**Torque:**  $\vec{\tau} = \vec{r} \times \vec{F} = \dot{\vec{L}}$

**Centripital Acceleration:**  $a_c = v^2/r$

**Centrifugal/Coriolis Forces:**  $\vec{F}_{cent} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r})$ ,  $\vec{F}_{cor} = -2m\vec{\omega} \times \dot{\vec{r}}$

**Work to go from positions  $\vec{a}$  to  $\vec{b}$ :**  $W_{ab} = \int_{\vec{a}}^{\vec{b}} \vec{F} \cdot d\vec{s}$

**Conservative Force Field (2 eq):**  $W_{ab}$  is the same regardless of path so  $\oint \vec{F} \cdot d\vec{s} = 0$ , and thus we can write the force as  $\vec{F} = -\nabla V(\vec{r})$ .

### Lagrangian Formalism

**Functional Derivative:**  $\frac{\delta F}{\delta u}[x_0] = \lim_{\epsilon \rightarrow 0} \frac{F[x_0 + \epsilon u] - F[x_0]}{\epsilon} \rightarrow \frac{\delta F}{\delta x(t)}[x(t')] = \lim_{\epsilon \rightarrow 0} \frac{F[x(t') + \epsilon \delta(t' - t)] - F[x(t')]}{\epsilon}$

**Principle of Least Action:**  $\delta S = 0$ , where  $S = \int_{t_i}^{t_f} L(\vec{q}, \dot{\vec{q}}, t) dt$

**Lagranges Equation:**  $\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^A} - \frac{\partial L}{\partial x^A} = 0$

**Holonomic Constraints:**  $f_\alpha(x^A, t) = 0$ ,  $L' = L(x^A, \dot{x}^A) + \lambda_\alpha f_\alpha(x^A, t) \rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{x}^A} - \frac{\partial L}{\partial x^A} = \lambda_\alpha \frac{\partial f_\alpha}{\partial x^A}$

**Noether's Theorem:** A continuous symmetry in the Action (and thus Lagrangian) result in a conserved quantity.

**Moment of Inertia Tensor:**  $\vec{L} = \overleftrightarrow{I} \vec{\omega}$ ,  $T = \frac{1}{2} \omega_a I_{ab} \omega_b$ ,  $I_{ab} = \sum_i m_i ((\vec{r}_i \cdot \vec{r}_i) \delta_{ab} - (\vec{r}_i)_a (\vec{r}_i)_b)$

**Euler's Equations:** Only look at rotation, not translation. Conservation of Angular Momentum gives  $I_i\dot{\omega}_i + \omega_j\omega_k(I_k - I_j) = 0$ , for  $i,j,k$  being cyclic permutations of 1,2,3.

## Hamiltonian Formalism

**Generalized Momenta:**  $p_i = \frac{\partial L}{\partial \dot{q}_i}$ ,  $\dot{p}_i = -\frac{\partial L}{\partial q_i}$

**Hamiltonian:**  $H(q_i, p_i, t) = \sum_{i=1}^n p_i \dot{q}_i - L(q_i, \dot{q}_i, t)$

**Hamilton's Equations:**

1.  $\dot{p}_i = -\frac{\partial H}{\partial q_i}$
2.  $\dot{q}_i = \frac{\partial H}{\partial p_i}$
3.  $-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$

**Cyclic/Ignorable Coordinates:**  $q$  is ignorable if  $\frac{\partial L}{\partial q} = 0$ , i.e. if  $q$  does not appear in  $L$ . Thus  $p = \frac{\partial L}{\partial \dot{q}}$  is conserved.

**Liouville's Theorem:** A volume of a region of phase space remains the same, even when the region changes.  $V = dq_1 \dots dq_n dp_1 \dots dp_n$ .

**Poisson Bracket:**  $\{f, g\} = \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i}$ .

**Constant of Motion from Poisson Bracket:**  $\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$ . If  $I, H = 0$ , then  $I$  is a constant of motion.

**Canonical Transformation:** Transformation  $(q_i \rightarrow Q_i(q, p), p_i \rightarrow P_i(q, p))$  that leaves Hamilton's equations invariant.

## 2 Statistical Mechanics Equations

### 2.1 Thermodynamics

**Laws of Thermodynamics:**

1. Energy conservation.  $dE = dQ - pdV$ .  $dQ$  just means that the heat is an inexact differential and the integral depends on the path.
2.  $\Delta S \geq \int \frac{dQ}{T}$ , where equality is for a process that is reversible (never leaves equilibrium).
3. Entropy at zero temperature is zero. In stat mech this means that the ground state is nondegenerate and  $S \propto \ln(W)$ , where  $W$  is the number of available states.

**Intensive vs Extensive Variables:** Intensive variables do NOT scale with system size ( $T, p, \mu$ ), while extensive do scale ( $E, S, V, N$ ).

**Thermodynamic Potentials:**

- Internal Energy:  $U(S, V, N)$

- Helmholtz Free Energy:  $F(T, V, N) = U - TS$
- Enthalpy:  $H(S, p, N) = U + pV$
- Gibbs Free Energy:  $G(T, p, N) = U - TS + pV$
- Landau(Grand) Potential:  $\Omega(T, V, \mu) = U - TS - \mu_i N_i$

### Thermodynamic Ensembles:

1. Microcanonical: Does not exchange energy or particles with environment. Fixed  $E, N$
2. Canonical: Does not exchange particles, but can exchange energy (heat bath). Fixed  $N, T$
3. Grand canonical: Can exchange energy and particles with environment. Fixed  $T, \mu$ .

### Maxwell's Relations (4 main):

- $\frac{\partial^2 U}{\partial S \partial V} = - \left( \frac{\partial p}{\partial S} \right)_V = \left( \frac{\partial T}{\partial V} \right)_S$
- $\frac{\partial^2 F}{\partial T \partial V} = \left( \frac{\partial p}{\partial T} \right)_V = \left( \frac{\partial S}{\partial V} \right)_T$
- $\frac{\partial^2 H}{\partial S \partial p} = \left( \frac{\partial V}{\partial S} \right)_p = \left( \frac{\partial T}{\partial p} \right)_S$
- $\frac{\partial^2 G}{\partial T \partial p} = \left( \frac{\partial V}{\partial T} \right)_p = - \left( \frac{\partial S}{\partial p} \right)_T$

**Engine Efficiency:**  $\eta = \frac{Q_{in} - Q_{out}}{Q_{in}} = 1 - \frac{T_{out}}{T_{in}}$

**Isobaric Thermal Expansion Coefficient:**  $\alpha = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_P$ , How much the volume changes with a change in temperature.

**Isothermal Compressibility:**  $\kappa_T = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_T$ , How much the volume changes when the pressure changes.

**Isentropic(Adiabatic) Compressibility:**  $\kappa_S = -\frac{1}{V} \left( \frac{\partial V}{\partial P} \right)_S$ , Same as above.

**Specific Heat at Constant V:**  $C_V = \left( \frac{\partial Q}{\partial T} \right)_V = \left( \frac{\partial U}{\partial T} \right)_V$ , Amount of heat per unit mass to raise the temp by 1 degree.

**Specific Heat at Constant p:**  $C_p = \left( \frac{\partial Q}{\partial T} \right)_p = \left( \frac{\partial H}{\partial T} \right)_p$ , Same as above.

**Fermi Energy/Temperature:** Chemical potential at  $T = 0$ .  $\epsilon_F = \mu(T = 0)$

## 2.2 Statistical Mechanics

**Number of microstates in a macrostate (ways to get n heads):**  $\Omega = \frac{N!}{\prod_i n_i!}$

**Stirling's Approximation:**  $\ln n! = n \ln n - n$

**How many order important ways to order n things:**  $n!$

**How many order important ways to order n things r at a time:**  $\frac{n!}{(n-r)!}$

**How many NOT order important ways to order n things r at a time:**  $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

**Microcanonical(Classical) Partition Function:**  $Z_m = \sum_s g_s e^{-\beta E_s}$

**Canonical Partition Function:**  $Z_c = \text{tr} \left( e^{-\beta \hat{H}} \right)$

**Grand Canonical Partition Function:**  $Z_{gc} = \text{tr} \left( e^{-\beta(\hat{H} - \mu \hat{N})} \right)$

**Geometric Series:**  $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

**Classical limit of the trace of an operator:**  $\text{tr}(\mathcal{O}) = \frac{1}{N!(2\pi\hbar)^{3N}} \int d^3r_1 \dots d^3r_N \int d^3p_1 \dots d^3p_N \mathcal{O}$ ,  $N!$  is for identical particles.

**Thermodynamic Limit:**  $T \rightarrow \infty, V \rightarrow \infty, N/V = \text{const}$

**Expectation value for pure/mixed:**  $\langle \mathcal{O} \rangle_p = \langle \psi | \mathcal{O} | \psi \rangle, \langle \mathcal{O} \rangle_m = \sum_i P_i \langle \psi_i | \mathcal{O} | \psi_i \rangle$

**Density Matrix (ex. Canonical Ensemble):**  $\rho = \sum_n P_n |\psi_n\rangle \langle \psi_n|, \rho_c = \frac{e^{-\beta \hat{H}}}{\text{tr} e^{-\beta \hat{H}}}$

**Expectation value with Density Matrix:**  $\langle \mathcal{O} \rangle = \text{tr}(\mathcal{O} \rho)$

**Trace of Density matrix:**  $\text{tr}(\rho) = 1$

**Time evolution of density matrix:**  $\frac{\partial}{\partial t} \rho = \frac{1}{i\hbar} [\hat{H}, \hat{\rho}]$

$Z_{gc}$  for an ideal gas:  $Z_{gc} = \frac{V^N (2mT\pi)^{3N/2}}{N! (2\pi\hbar)^{3N}} e^{\beta\mu}$

$Z_{gc}$  for ideal fermi gas:  $Z_{gc} = \prod_k (1 + e^{-\beta(\epsilon_k - \mu)})$

$Z_{gc}$  for ideal bose gas:  $Z_{gc} = \prod_k \frac{1}{(1 - e^{-\beta(\epsilon_k - \mu)})}$

Stuff here for black-body and phonons and bose condensates.

**Explain Bose-Condensates with Bose statistics:**  $\lim_{T \rightarrow 0} n(p) = \lim_{\beta \rightarrow \infty} \frac{1}{1 - e^{\beta(\epsilon - \mu)}} \rightarrow 0$  un-

less  $\epsilon \rightarrow \mu$ , which happens at the ground state.

**What is cluster expansion used for?:** Systems of interacting particles.

### 3 Quantum Mechanics Equations

**Properties of a vector space:**

- Sum  $|V\rangle + |W\rangle$
- Scalar product with properties
  1. closure: results in another vector in the space.
  2. distributive:  $a(|V\rangle + |W\rangle) = a|V\rangle + a|W\rangle, (a+b)|V\rangle = a|V\rangle + b|V\rangle$
  3. associative:  $a(b|V\rangle) = ab|V\rangle, |V\rangle + (|W\rangle + |Z\rangle) = (|V\rangle + |W\rangle) + |Z\rangle$
  4. commutative:  $|V\rangle + |W\rangle = |W\rangle + |V\rangle$
  5. additive inverse:  $|V\rangle + |-V\rangle = |0\rangle$
  6. null vector:  $|V\rangle + |0\rangle = |V\rangle$

**Hilbert space:** Vector space with defined inner product.

**Expand in orthonormal basis:**  $|V\rangle = \sum_i v_i |i\rangle$

**Hermitian operator:**  $\mathcal{O}^\dagger = \mathcal{O}$

**Anti-Hermitian operator:**  $\mathcal{O}^\dagger = -\mathcal{O}$

**Unitary operator:**  $UU^\dagger = \mathbb{I}$

**Orthogonality:**  $\langle i | j \rangle = \delta_{ij}$

**Completeness:**  $\sum_i i = \mathbb{I}$

**Postulates of QM:**

1. The state of a physical system, at some fixed time, is given by a normalized ray in a Hilbert space over the complex numbers. (ray is vector whose norm doesn't matter)
2. The ray evolves deterministically in time according to Schrödinger's equation.
3. Observables correspond to self-adjoint (hermitian) operators.
4. If a particle is in the state  $|\psi\rangle$  then a measurement of  $\mathcal{O}$  will yield one of the eigenvalues of  $\mathcal{O}$ ,  $\omega$ . The state of the system changes to an eigenstate of  $\mathcal{O}$ ,  $|\omega\rangle$ .

**Schrödinger equation:**  $i\hbar \frac{\partial}{\partial t} \Psi = \hat{H} \Psi$

**Free particle  $\psi_p$  and  $E_p$ :**  $\psi_p = Ae^{ikx} + Be^{-ikx}$ ,  $k^2 = \frac{2mE_p}{\hbar^2}$ ,  $E_p = \frac{p^2}{2m}$

**Particle in a box  $\psi_n$  and  $E_n$ :**  $\psi_n = \sqrt{\frac{2}{L}} \sin(k_n x)$ ,  $k_n = \frac{n\pi}{L}$ ,  $E_n = \frac{\hbar^2 \pi^2 n^2}{2mL^2}$

**Harmonic Oscillator  $\hat{H}$ ,  $\psi_n$  and  $E_n$ :**  $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 \hat{x}^2$ ,  $\psi_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{2^n n!} H_n(x) e^{-x^2/2}$ ,  $E_n = (n + \frac{1}{2})\hbar\omega$

**Raising and lowering operators and how to affect  $|n\rangle$  (3-2):**

- $a = \left(\frac{m\omega}{2\hbar}\right)^{1/2} \left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$ ,  $a|n\rangle = \sqrt{n}|n-1\rangle$ ,  $a|0\rangle = 0$
- $a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2} \left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$ ,  $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

**$\hat{H}$  in terms of  $a$  and  $a^\dagger$ :**  $\hat{H} = \hbar\omega(a^\dagger a + 1/2)$

**Commutation relations for  $\hat{H}$ ,  $a$ ,  $a^\dagger$ :**

- $[\hat{H}, a] = -\hbar\omega a$
- $[\hat{H}, a^\dagger] = \hbar\omega a^\dagger$
- $[a, a^\dagger] = 1$

**$J^2$  and  $J_z$  on the angular momentum state  $|jm_j\rangle$ :**

- $J^2|jm_j\rangle = j(j+1)\hbar^2|jm_j\rangle$
- $J_z|jm_j\rangle = m_j\hbar|jm_j\rangle$

**Commutation relations for  $J_i$  and  $J_j$  and for  $J^2$  and  $J_i$ :**

- $[J_i, J_j] = i\hbar J_k$
- $[J^2, J_i] = 0$

**$J_z$  and  $J^2$  in position basis:**

- $J_z = -i\hbar \frac{\partial}{\partial t}$
- $\mathbf{J}^2 = -\hbar^2 \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right]$

**Raising and Lowering Angular Momentum Operators on  $|j, m\rangle$ :**

$$J_{\pm} |j, m\rangle = \hbar[j(j+1) - m(m \pm 1)]^{1/2} |j, m \pm 1\rangle$$

$$J_x \text{ and } J_y \text{ in terms of } J_+ \text{ and } J_-: J_x = \frac{1}{2}(J_+ + J_-), J_y = \frac{1}{2i}(J_+ - J_-)$$

$$\text{Momentum eigenstate, } \langle x|p\rangle: \langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

$$\text{Hydrogen Atom } V(r), \psi_n, E_n(\mathbf{x4}): V(r) = -\frac{Ze^2}{4\pi\epsilon_0 r}, \psi_n = \text{stuff} * L_{n-l-1}^{2l+1}(\rho) Y_l^m(\theta, \phi) (\text{Laguerre})$$

$$E_n = -\frac{1}{2n^2} \left( \frac{Ze^2}{4\pi\epsilon_0 \hbar} \right)^2 m_e = -\frac{1}{2n^2} \alpha^2 m_e c^2 = -\frac{1}{n^2} 13.6 \text{ eV} = -\frac{1}{2n^2} \left( \frac{1}{4\pi\epsilon_0} \frac{e^2}{a_0} \right),$$

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c}, a_0 = \frac{4\pi\epsilon_0 \hbar^2}{m_e e^2}$$

**Pauli matrices and commutation relations:**

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[\sigma_a \sigma_b] = 2i\epsilon_{abc} \sigma_c$$

**Non-Deg Time-Ind Perturbation,  $E_n^{(1)}, |n^{(1)}\rangle, E_n^{(2)}$ :**

$$E_n^{(1)} = H'_{nn} = \langle n^{(0)} | H' | n^{(0)} \rangle$$

$$|n^{(1)}\rangle = \sum_{m \neq n} \frac{\langle n^{(0)} | H' | n^{(0)} \rangle}{(E_n^{(0)} - E_m^{(0)})} |m^{(0)}\rangle$$

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle n^{(0)} | H' | n^{(0)} \rangle|^2}{(E_n^{(0)} - E_m^{(0)})}$$

**Deg Time-Ind Perturbation,  $E_n^{(1)}$ :** Diagonalize the perturbation hamiltonian in the degenerate subspace.

$$\text{Time-Dep Perturbation, } P_{i \rightarrow f}(t): P_{i \rightarrow f}(t) = \frac{1}{\hbar^2} \left| \int_0^t dt' \langle f | H'(t') | i \rangle e^{i(E_f - E_i)t'/\hbar} \right|^2$$

$$\text{Fermi's golden rule, and } g(E_f) \text{ as } \delta: R_{i \rightarrow f} = \frac{2\pi}{\hbar} |\langle f | H' | i \rangle|^2 g(E_f), g(E_f) \approx \delta(E_f^{(0)} - E_f^{(0)} - \hbar\omega)$$

**Einstein's Stimulated/Spontaneous emission coefficients:**

$$\text{Stimulated: } B_{if} = \frac{\pi e^2}{3\epsilon_0 \hbar^2} |\langle f | \mathbf{r} | i \rangle|^2$$

$$\text{Spontaneous: } A_{if} = \frac{e^2 \omega_{if}^3}{3\pi \epsilon_0 \hbar c^3} |\langle f | \mathbf{r} | i \rangle|^2$$

**Total  $\psi(\mathbf{r})$  in scattering problem:**

$$\psi(\mathbf{r}) = \psi_{inc}(\mathbf{r}) + \psi_s(\mathbf{r}) = \psi_{inc}(\mathbf{r}) + f(\theta, \phi) \frac{e^{i\mathbf{k} \cdot \mathbf{r}}}{r}$$

$$f(\theta, \phi) = -\frac{m}{2\pi\hbar^2} \int d^3r' e^{-i\mathbf{k}' \cdot \mathbf{r}'} V(\mathbf{r}') \psi(\mathbf{r}')$$

$$\text{Differential Cross Section: } \frac{d\sigma}{d\Omega} = |f(\theta, \phi)|^2$$

**Born Approximation:** In the above integral for  $f(\theta, \phi)$  let  $\psi \rightarrow \psi_{inc}$ .

**Dirac Equation:**

$$(i\hbar\gamma^\mu \partial_\mu - mc)\psi = 0$$

$$\gamma^0 = \beta, \gamma^i = \beta\alpha_i$$

$$\beta = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}, \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

## 4 Electricity and Magnetism Equations

**Maxwell's Equations in Vacuum (SI):**

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0, \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

**Maxwell's Equations in Matter (SI), and  $\mathbf{D}$  and  $\mathbf{H}$ :**

$$\nabla \cdot \mathbf{D} = \rho, \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathbf{D} = \epsilon_0 \mathbf{E}, \mathbf{B} = \mu_0 \mathbf{H}$$

$$\text{Continuity Equation: } \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

$$\text{Lorentz Force: } \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$$\text{Coulomb's Law (x2): } \mathbf{F} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2}, \mathbf{F} = Q\mathbf{E}$$

$$\text{Gauss' Law: } \oint \mathbf{E} \cdot d\mathbf{A} = q/\epsilon_0$$

$$\text{Electrostatic Potential (x2): } \mathbf{E} = -\nabla\Phi, \Phi = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

$$\text{Laplace's Equation \& General Solution(Spherical Coordinates, no } \phi): \nabla^2\Phi = 0$$

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)$$

$$\text{Poisson's Equation: } \nabla^2\Phi = -\rho/\epsilon_0$$

**Explain the Method of Images:** Because of the uniqueness theorem you can add charges OUTSIDE of the computational area to meet the same boundary conditions. A solution to this new configuration is also a solution to the initial configuration.

**Method of Images (plane, sphere, hem boss):**

plane: add one charge below plane.

sphere: 1 test charge inside sphere.

hem boss: 3 test charges.

$$\text{Multipole Expansion of } \Phi(\mathbf{r}): \Phi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \int d^3r' \frac{P_n(\cos \alpha)}{r^{n+1}} \rho(\mathbf{r}')$$

**Work and Energy in Electrostatics:** The Energy of a system is the work it requires to assemble the system.

$$\text{Atomic Polarizability } (\alpha): \mathbf{p} = \alpha \mathbf{E}$$

$$\text{Polarization: Electric dipole moment per unit volume. } \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

$$\text{Magnetization: Magnetic dipole moment per unit volume } \mathbf{H} = \frac{1}{\mu_0} \mathbf{B} - \mathbf{M}$$

$$\text{Bound Charge: } \rho_b = -\nabla \cdot \mathbf{P}$$

$$\text{Bound Current: } \mathbf{J}_b = \nabla \times \mathbf{M}$$

$$\text{Linear Media x2: } \mathbf{P} = \epsilon_0 \chi_e \mathbf{E}, \mathbf{M} = \chi_m \mathbf{H}$$

$$\text{Biot-Savart Law: } \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}') \times |\mathbf{r}-\mathbf{r}'|}{|\mathbf{r}-\mathbf{r}'|^3}$$

$$\text{Ohm's Law: } \mathbf{J} = \sigma \mathbf{E}, \text{ where } \sigma \text{ is the conductivity}$$

$$\text{Resistivity: } \rho = 1/\sigma$$

**Boundary Conditions:**

$$D_1^\perp - D_2^\perp = \sigma_f$$

$$B_1^\perp - B_2^\perp = 0$$

$$\mathbf{E}_1^\parallel - \mathbf{E}_2^\parallel = 0$$

$$\mathbf{H}_1^\parallel - \mathbf{H}_2^\parallel = \mathbf{K}_f \times \hat{\mathbf{n}}$$

**Poynting's Theorem, units of S:**  $\mathbf{S} = \frac{\text{energy}}{\text{time} \cdot \text{energy}}$

$$\frac{dW}{dt} = -\frac{d}{dt} \int d^3r \frac{1}{2} \left( \epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) - \frac{1}{\mu_0} \oint d\mathbf{a} \cdot (\mathbf{E} \times \mathbf{B}) = -\frac{d}{dt} (W_e + W_m) - \oint d\mathbf{a} \cdot \mathbf{S}$$

**Maxwell Stress Tensor and Static Force:**

$$T_{ij} = \epsilon_0 \left( E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left( B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right)$$

$$\mathbf{F} = \oint d\mathbf{a} \cdot \overleftrightarrow{\mathbf{T}}$$

$$\text{Index of Refraction: } n = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$$

**What is a Waveguide:** A waveguide is a conductor pipe such that  $\mathbf{E}^\parallel = 0$  and  $B^\perp = 0$  on the surface. Also the transverse components of the fields (x and y) can be determined from derivatives of the axial components (z).

**Transverse electric/magnetic and TEM**

$$\text{TE: } E_z = 0$$

$$\text{TM: } B_z = 0$$

$$\text{TEM: both}$$

$$\mathbf{E} \text{ and } \mathbf{B} \text{ in terms of } \mathbf{A} \text{ and } \Phi: \mathbf{E} = -\nabla\Phi - \frac{\partial\mathbf{A}}{\partial t}, \mathbf{B} = \nabla \times \mathbf{A}$$

$$\text{Coulomb/Lorentz Gauge: } \nabla \cdot \mathbf{A} = 0, \nabla \cdot \mathbf{A} = -\mu_0\epsilon_0 \frac{\partial\Phi}{\partial t}$$

$$\text{Retarded Scalar and Vector Potentials: } \Phi = \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|}, \mathbf{A} = \frac{\mu_0}{4\pi} \int d^3r' \frac{\mathbf{J}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|}$$

**What are the Liénard-Wiechert Potentials?:** Retarded potentials of a point charge with a specific trajectory.

$$\text{Radiation Estimate } |\mathbf{r} - \mathbf{r}'| \text{ and } \frac{1}{|\mathbf{r} - \mathbf{r}'|}: |\mathbf{r} - \mathbf{r}'| \approx r - \frac{\mathbf{r} \cdot \mathbf{r}'}{r}, \frac{1}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3}$$

$$\text{Radiation Dipole Approximation } e^{-i\omega\hat{\mathbf{r}} \cdot \mathbf{r}'/c} \approx 1$$

$$\text{Electric Dipole Moment: } \mathbf{p}(\mathbf{r}, t) = \int d^3r' \mathbf{r}' \rho(\mathbf{r}', t)$$

$$\text{Larmor Formula: } P = \frac{\mu_0}{6\pi c} q^2 a^2$$

**Helmholtz Theorem:** If you know the divergence ( $D$ ) and the curl ( $C$ ) of a function  $\mathbf{F}$  then  $\mathbf{F} = -\nabla U + \nabla \times \mathbf{W}$  where

$$U(\mathbf{r}) = \frac{1}{4\pi} \int d^3r' \frac{D(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

$$W(\mathbf{r}) = \frac{1}{4\pi} \int d^3r' \frac{C(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}$$

**Einstein's Postulates of Special Relativity:**

1. The laws of physics are the same in all inertial frames of reference.
2. The speed of light in free space has the same value  $c$  in all inertial frames of reference.

**Boost in the x-direction in terms of  $x_i$ ,  $\gamma$  and  $\beta$ :**

$$x'_0 = \gamma(x_0 - \beta x_1)$$

$$x'_1 = \gamma(x_1 - \beta x_0)$$

$$x'_2 = x_2$$

$$x'_3 = x_3$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}, \beta = \frac{v}{c}$$

$$\text{Boost in the x-direction in matrix form: } x'^\mu = \Lambda^\mu_\nu x^\nu, \Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

**Covariant vs. Contravariant:** covariant:  $a_\mu$ , contravariant:  $a^\mu$ ,  $(a_0, a_1, a_2, a_3) = (-a^0, a^1, a^2, a^3)$



**Minkowski metric:**  $a_\mu = g_{\mu\nu}a^\nu$ ,  $g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

**Four-( $\mathbf{v}$ ,  $\mathbf{p}$ ,  $\mathbf{J}$ ,  $\mathbf{A}$ ):**

$$v^\mu = \frac{dx^\mu}{d\tau} = (\gamma c, \gamma \mathbf{v})$$

$$p^\mu = mv^\mu = (E/c, \mathbf{p})$$

$$J^\mu = (c\rho, \mathbf{J})$$

$$A^\mu = (\Phi/c, \mathbf{A})$$

**Relativistic Energy x2:**  $E = \gamma mc^2 = \sqrt{p^2 c^2 + m^2 c^4}$

**Field Tensor and Transformation:**  $F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}$ ,  $F^{ij} = \Lambda_k^i \Lambda_l^j F^{kl} \rightarrow$

$$F' = \Lambda F \Lambda^T$$

**Maxwell's Equations with d'Alembertian:**  $\square^2 A^\mu = -\mu_0 J^\mu$ ,  $\square^2 = \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}$

## 5 Miscellaneous Physics

**Taylor Expansion:**  $f(\vec{x} + \vec{a}) = f(\vec{x}) + a_i \partial_i f(\vec{x}) + \mathcal{O}(\vec{a}^2)$

**Gaussian Integral:**  $\int_{-\infty}^{\infty} dx e^{-ax^2+bx+c} = \sqrt{\frac{\pi}{a}} e^{b^2/(4a)+c}$

**3 types of Boundary Conditions:**

Dirichlet:  $\Phi(\mathbf{a}) = \text{const}$

Neumann:  $\frac{\partial \Phi(\mathbf{a})}{\partial \mathbf{n}} = \hat{\mathbf{n}} \cdot \nabla \Phi = \text{const}$

Robin: Linear combination of the first two

**Value of fine structure constant:**  $\alpha \approx \frac{1}{137}$

**Mass of electron in eV:**  $m_e c^2 = 0.511 \text{ eV}$

**Value of the Bohr radius:**  $a_0 = 0.529 \text{ \AA}$

**Wave Equation:**  $\nabla^2 u - \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = 0$

**Diffusion Equation:**  $\nabla^2 u - \frac{1}{D} \frac{\partial u}{\partial t} = 0$