PHY6938, Modern Physics and Quantum Mechanics, Spring 2000

1. A boy standing on a ladder drops marbles of mass M from a height H. He tries to hit a point on the ground. Show that even if he is very careful, the marbles are going to miss the point by an average distance Δx which is proportional to

$$\left(\frac{\hbar}{M}\right)^{1/2}\left(\frac{H}{g}\right)^{1/4},$$

where g is the gravitational acceleration. How large is the average distance for $M = 1$ g and $H = 2$ m? Compare this distance to the radius of an atom and the radius of a nucleus. ***Hint:*** **Use Heisenberg's uncertainty principle.**

The boy's dilemma is as follows: if he constrains the marbles to have a very small initial Δx_0, then they will have a substantial spread in their horizontal momenta

$$\Delta p_x \geq \frac{\hbar}{2\Delta x_0}$$

by the uncertainty principle. This means that as they fall the initial uncertainty in their position is going to grow linearly with time (since they start, on average, with some initial velocity in the x direction) by

$$\Delta x \geq \Delta x_0 + \frac{\Delta p_x}{m} t = \Delta x_0 + \frac{\hbar}{2M\Delta x_0} t$$

and may become large. If, on the other hand, he relaxes the restriction on the initial position of the marbles, then the initial Δx_0 will be larger and the uncertainty in the position will grow more slowly. This means that there is some ideal Δx_0 where the final Δx is minimized, which is his intent.

The next step is to minimize the value of Δx after the time $t = \sqrt{2H/g}$ which the marbles take to fall the height H, by setting the derivative of Δx with respect to Δx_0 to be zero,

$$\frac{d(\Delta x)}{d(\Delta x_0)} = 1 - \frac{\hbar}{2M(\Delta x_0)^2}\sqrt{\frac{2H}{g}} = 0,$$

so that

$$\Delta x_0 = \sqrt{\frac{\hbar}{2M}\sqrt{\frac{2H}{g}}},$$

gives the minimum final spread Δx. Substituting this back into the formula for Δx we find

$$\Delta x \geq \Delta x_0 + \frac{(\Delta x_0)^2}{\Delta x_0} = 2\Delta x_0 = 2\sqrt{\frac{\hbar}{2M}\sqrt{\frac{2H}{g}}} = 2^{3/4}\left(\frac{\hbar}{M}\right)^{1/2}\left(\frac{H}{g}\right)^{1/4},$$

as advertised.

For the mass and height given we have

$$\Delta x \geq 2^{3/4}\left(\frac{1.05\times 10^{-34}\ \mathrm{J\cdot s}}{0.001\ \mathrm{kg}}\right)^{1/2}\left(\frac{2\ \mathrm{m}}{9.8\ \mathrm{m/s^2}}\right)^{1/4} = 3.66\times 10^{-16}\ \mathrm{m},$$

which is very small compared to a typical atomic radius (of the order of 10^{-10} m) and is roughly an order of magnitude smaller than the radius of a nucleus (typically several fm=10^{-15} m).

2. A vessel holds 2 μg of tritium.

a) What is the initial decay rate of the tritium?

The number N of tritium nuclei as a function of time is

$$N = N_0 e^{-\ln(2)\, t/t_{1/2}},$$

so that the rate of decay is

$$\frac{dN}{dt} = N_0 \left(-\frac{\ln(2)}{t_{1/2}}\right) e^{-\ln(2)\, t/t_{1/2}},$$

which at time $t = 0$ is

$$\frac{dN}{dt}(0) = -N_0 \left(\frac{\ln(2)}{t_{1/2}}\right).$$

The total number of nuclei is the mass divided by the mass per nucleus, and the half life is 12.3 y, so that

$$\frac{dN}{dt}(0) = -\frac{2 \times 10^{-9}\ \text{kg}}{5.01 \times 10^{-27}\ \text{kg}} \left(\frac{\ln(2)}{12.3 \cdot 365 \cdot 24 \cdot 3600\ \text{s}}\right) = -7.13 \times 10^{8}\ \text{s}^{-1}.$$

b) How much time will elapse before the amount of tritium falls to 1% of its initial value?

If $N/N_0 = 1/100$, then we have that

$$e^{-\ln(2)\, t/t_{1/2}} = \frac{1}{100},$$

or by taking the natural log of both sides

$$\ln(2)\frac{t}{t_{1/2}} = \ln(100),$$

and so

$$t = \frac{\ln(100)}{\ln(2)} t_{1/2} = 81.7\ \text{y}.$$

3. Cosmic ray photons from space are bombarding your laboratory and smashing massive objects to pieces! Your detectors indicate that two fragments, each of mass m_0, depart such a collision (between a photon and a particle of mass M) moving at speed 0.6 c at 60° to the photon's original direction of motion.

a) In terms of m_0 and c, what is the energy of the cosmic ray photon?

We will solve this part of the problem using conservation of momentum to find the photon momentum (and so energy). The momentum of each fragment is

$$p = m_0\gamma(v)\, v$$

where

$$\gamma(v) = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{0.64}} = \frac{5}{4},$$

so that

$$p = \frac{5}{4} m_0 (0.6\ c) = \frac{3}{4} m_0 c.$$

The components perpendicular to the photon's direction of motion cancel, and those parallel to it add to

$$p_\gamma = 2p\cos(60^\circ) = p = \frac{3}{4} m_0 c,$$

where we have used conservation of momentum, so the photon energy is

$$E_\gamma = p_\gamma c = \frac{3}{4} m_0 c^2.$$

b) In terms of m_0, what is the mass M of the particle being struck (assumed originally stationary)?

The initial total relativistic energy is therefore

$$E_{\rm i} = Mc^2 + \frac{3}{4} m_0 c^2,$$

and the final total relativistic energy is

$$E_{\rm f} = 2E = 2\sqrt{p^2c^2 + m_0^2c^4} = 2\sqrt{\frac{16+9}{16} m_0^2 c^4} = \frac{5}{2} m_0 c^2,$$

and equating these gives that

$$Mc^2 = \left(\frac{5}{2} - \frac{3}{4}\right) m_0 c^2 = \frac{7}{4} m_0 c^2,$$

so that

$$M = \frac{7}{4} m_0.$$

4. A particle of mass m moves in a one-dimensional potential

$$V(x) = \begin{cases} V_0\,\delta(x) & \textbf{for } |x| < a \\ \infty & \textbf{for } |x| \geq a \end{cases}$$

(i.e an infinite potential well with a δ-potential at the center. Find the energy levels for this particle and discuss the cases of very large and very small V_0. .

The Schrödinger equation inside the potential well is

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi}{dx^2} + V_0\,\delta(x)\Psi = E\Psi.$$

Recall that the Dirac delta function $\delta(x)$ only has a meaning as a distribution function, i.e. when it multiplies a function under the integral sign

$$\int \delta(x)\, f(x)\, dx = f(0),$$

so this suggests that we should integrate the Schrödinger equation once over x near $x = 0$ (say from $-\varepsilon$ to $+\varepsilon$) to find the effect of this term,

$$-\frac{\hbar^2}{2m}\int_{-\varepsilon}^{+\varepsilon}\frac{d^2\Psi}{dx^2}dx + V_0\int_{-\varepsilon}^{+\varepsilon}\delta(x)\Psi\, dx = E\int_{-\varepsilon}^{+\varepsilon}\Psi dx,$$

or

$$-\frac{\hbar^2}{2m}\left[\frac{d\Psi}{dx}(+\varepsilon) - \frac{d\Psi}{dx}(-\varepsilon)\right] + V_0\Psi(0) = E\left[\Psi(+\varepsilon) - \Psi(-\varepsilon)\right],$$

Since the wavefunction must be continuous the RHS term tends to zero as ε tends to zero, and so we see that the effect of the delta-function term is to create a discontinuity in the first derivative of the wavefunction,

$$\frac{d\Psi}{dx}(+\varepsilon) - \frac{d\Psi}{dx}(-\varepsilon) = \frac{2m}{\hbar^2}V_0\Psi(0).$$

Away from $x = 0$ the Schrödinger equation is that of a free particle

$$\frac{d^2\Psi}{dx^2} = -\frac{2mE}{\hbar^2}\Psi$$

with wavenumber $k = \sqrt{2mE/\hbar^2}$, and with boundary conditions

$$\Psi(-a) = \Psi(a) = 0.$$

Solutions to the square well potential are either even functions which satisfy $\Psi(x) = \Psi(-x)$ (in this case they are cosine functions) or odd functions which satisfy $\Psi(x) = -\Psi(-x)$ (in this case they are sine functions). The odd functions necessarily have $\Psi(0) = 0$, and so from the above we see that the delta-function potential causes no discontinuity in the derivative of the wavefunction at the origin and so has no effect, and so the odd solutions are the same as that of the infinite square well potential. These solutions have

$$\lambda_n = 2a, 4a, 6a, ... = 2na, \;\; n = 1, 2, 3, ...,$$

so that

$$k_n = \frac{2\pi}{\lambda_n} = \frac{\pi}{na},$$

and

$$E_n = \frac{1}{2m}\frac{\hbar^2\pi^2}{n^2a^2}, \;\; n = 1, 2, 3... \text{ (odd solutions)}.$$

The even solutions must still be sinusoidal to satisfy the wave equation away from $x = 0$, and so must be of the form

$$\Psi(x) = \begin{cases} A\cos(kx) + B\sin(kx) & 0 < x < a, \\ A\cos(kx) - B\sin(kx) & -a < x < 0, \end{cases}$$

where the sign change reflects the fact that these functions must be even. Applying the boundary conditions at $x = \pm a$ we find

$$\Psi(a) = A\cos(ka) + B\sin(ka) = \Psi(-a) = 0.$$

Applying the condition for the change in the derivative we see that

$$kB - (-kB) = \frac{2m}{\hbar^2}V_0 A, \text{ or } B = \frac{m}{\hbar^2 k}V_0 A.$$

Putting these together we find that

$$A\cos(ka) + \frac{m}{\hbar^2 k}V_0 A\sin(ka) = 0,$$

or

$$\tan(ka) = \frac{\hbar^2}{mV_0 a}(ka).$$

This is a transcendental equation for ka which must be solved numerically for the energies of the even states.

For very large $V_0 \gg \hbar^2 k/m \simeq \pi\hbar^2/(ma)$ the RHS of the energy equation is zero and ka must be an integer multiple of π, so that the energy of the even solutions is

$$E_n = \frac{1}{2m}\left(\frac{n\pi\hbar}{a}\right)^2, \quad n = 1, 2, 3....$$

which are the energy levels for an infinite square well of width a.

For very small V_0 we can invert the above equation to get

$$\frac{1}{\tan(ka)} \simeq 0,$$

which occurs whenever ka is an half-integral multiple of π (when the tangent is infinite) and so

$$ka = \left(n + \frac{1}{2}\right)\pi = (2n+1)\frac{\pi}{2},$$

and so

$$E_n = \frac{1}{2m}\left(\frac{[2n+1]\pi\hbar}{2a}\right)^2, \quad n = 1, 2, 3...,$$

which are the energy levels of the *even* solutions of the infinite square well of width $2a$ without the presence of the delta-function potential, which now has a negligible effect.