1 Propagation of the three-body commutator

The 2-π exchange part of the three-body potential is given by

$$V_{ijk} = A_{2\pi} \sum_{cyc} \Big(\{\hat{X}_{ij}, \hat{X}_{jk}\}\{\tau_i \cdot \tau_j, \tau_j \cdot \tau_k\} + [\hat{X}_{ij}, \hat{X}_{jk}][\tau_i \cdot \tau_j, \tau_j \cdot \tau_k] \Big) , \tag{1}$$

where

$$\hat{X}_{ij} = X_{ij}^{\alpha\beta} \sigma_i^\alpha \sigma_j^\beta . \tag{2}$$

The anticommutator term is a two-body operator in the spin-isospin, and thus can be easily included in the propagator:

$$A_{2\pi} \sum_{i<j<k} \sum_{cyc} \{X_{ij}^{\alpha\beta}\sigma_i^\alpha\sigma_j^\beta, X_{jk}^{\delta\gamma}\sigma_j^\delta\sigma_k^\gamma\}\{\tau_i \cdot \tau_j, \tau_j \cdot \tau_k\} = 4A_{2\pi} \sum_{i<j} \sigma_i^\alpha \sigma_j^\beta \tau_i \cdot \tau_j \sum_{k\neq i,j} X_{ij}^{\alpha\beta} X_{ij}^{\gamma\delta} . \tag{3}$$

The commutator term is more difficult. We first note that

$$[\tau_i^\eta \tau_j^\eta, \tau_j^\mu \tau_k^\mu] = \tau_i^\eta [\tau_j^\eta, \tau_j^\mu] \tau_k^\mu = 2i\epsilon^{\eta\mu\nu} \tau_i^\eta \tau_j^\nu \tau_k^\mu , \tag{4}$$

and similarly

$$[\sigma_i^\alpha X_{ij}^{\alpha\beta} \sigma_j^\beta, \sigma_j^\gamma X_{jk}^{\gamma\delta} \sigma_k^\delta] = X_{ij}^{\alpha\beta} X_{jk}^{\gamma\delta} [\sigma_i^\alpha \sigma_j^\beta, \sigma_j^\gamma \sigma_k^\delta] = 2i X_{ij}^{\alpha\beta} X_{jk}^{\gamma\delta} \epsilon^{\beta\gamma\kappa} \sigma_i^\alpha \sigma_j^\kappa \sigma_k^\delta . \tag{5}$$

We can write the total contribution of the commutator as

$$V_{[]} = A_{2\pi} \sum_{i<j<k} \sum_{cyc} [X_{ij}^{\alpha\beta}\sigma_i^\alpha\sigma_j^\beta, X_{jk}^{\gamma\delta}\sigma_j^\gamma\sigma_k^\delta][\tau_i^\eta\tau_j^\eta, \tau_j^\mu\tau_k^\mu] = -4A_{2\pi} \sum_{i<j<k} \sum_{cyc} \sigma_i^\alpha \sigma_j^\kappa \sigma_k^\delta \tau_i^\eta \tau_j^\nu \tau_k^\mu X_{ij}^{\alpha\beta} X_{jk}^{\gamma\delta} \epsilon^{\beta\gamma\kappa} \epsilon^{\eta\mu\nu} . \tag{6}$$

The propagator of the three-body commutator can be written using a Hubbard-Stratonovich-like transformation, with the combination of more auxiliary fields:

$$\exp\left[-V_{[]}\Delta t\right] \approx \prod_{i<j<k,cyc} \int dx\, dy\, dz\, \exp \frac{x^2+y^2+z^2}{2} \tag{7}$$

$$\times \left[1 - xy\Delta t^{1/3} X_{ij}^{\alpha\beta} \sigma_i^\alpha \tau_i^\eta\right] \left[1 - xz\Delta t^{1/3} \epsilon^{\beta\gamma\kappa}\epsilon^{\eta\mu\nu} \sigma_j^\kappa \tau_j^\nu\right] \left[1 - yz\Delta t^{1/3} X_{jk}^{\gamma\delta} \sigma_k^\delta \tau_k^\mu\right] = \tag{8}$$

$$\prod_{i<j<k,cyc} \int dx\, dy\, dz\, \exp \frac{x^2+y^2+z^2}{2} \tag{9}$$

$$\times \exp\left[-xy\Delta t^{1/3} X_{ij}^{\alpha\beta} \sigma_i^\alpha \tau_i^\eta\right] \exp\left[-xz\Delta t^{1/3} \epsilon^{\beta\gamma\kappa}\epsilon^{\eta\mu\nu} \sigma_j^\kappa \tau_j^\nu\right] \exp\left[-yz\Delta t^{1/3} X_{jk}^{\gamma\delta} \sigma_k^\delta \tau_k^\mu\right] , \tag{10}$$

where the factor $-4A_{2\pi}$ should also be included. I think that in the worse case, for each triple ijk we need to sample 9^3=729 auxiliary fields. Perhaps a factor of 9 can be dropped because of the antisymmetric tensors ϵ in the above equations.

We also need to cancel terms proportional to $\Delta t^{2/3}$. Such terms come from either propagating one triple, and from the propagation of different triples. For example, from the same triple:

$$xy^2 z\Delta t^{2/3} X_{ij}^{\alpha\beta} \sigma_i^\alpha \tau_i^\eta X_{jk}^{\gamma\delta} \sigma_k^\delta \tau_k^\mu . \tag{11}$$

These terms can be canceled using the plus-minus sampling. For each triple, we need to sample the X, Y and Z auxiliary fields (capital letters mean all the ones that we need). Then we can cancel all the terms proportional to $\Delta t^{2/3}$ by considering the 8 combinations by taking $(\pm X, \pm Y, \pm Z)$.

There are other terms coming from different triples:

$$xyzx'y'z'\Delta t^{2/3} X_{ij}^{\alpha\beta} \sigma_i^\alpha \tau_i^\eta X_{lm}^{\theta\phi} \sigma_l^\theta \tau_m^\phi \tag{12}$$

I have no idea how to cancel all those terms (it is certainly possible, but the question is how many times do we need to evaluate the wave function?). One way would be to sample independent triples at each time-step, but I need more thinking.