PHY6938 Optics and Thermodynamics Fall 2000

1. A solid contains N magnetic atoms having spin $\frac{1}{2}$. At sufficiently high temperature, each spin is completely randomly oriented, i.e. equally likely to be in either of its two possible states. But at sufficiently low temperature, the interaction between the magnetic atoms causes them to exhibit ferromagnetism, with the result that all their spins become oriented along the same direction as $T \to 0$. A very crude approximation suggests that the spin-dependent contribution $C(T)$ to the heat capacity of this solid has an approximate temperature dependence given by

$$C(T) = \begin{cases} C_1 \left(\frac{2T}{T_1} - 1\right) & \textbf{if } T_1/2 < T < T_1 \\ 0 & \textbf{otherwise} \end{cases}$$

Use entropy considerations to find the maximum possible value of C_1.

If we know the specific heat we can find the entropy by using the specific heat to find the heat added, and then using that the entropy is related to the heat added by $dQ = TdS$. For a system of spins we can also find the entropy using the number of microstates available to the system. If we relate the entropies found in these two ways we can find a relation for C_1.

If $dQ = TdS$ and $dQ = C(T)dT$, then we have that the relationship between entropy and specific heat is that

$$dS = \frac{C(T)}{T} dT,$$

so that we can integrate to find the entropy S

$$S(T') = \int_0^{T'} \frac{C(T)}{T} dT.$$

If we choose a final temperature T' large compared to T_1, then we have

$$\begin{aligned} S(T') &= \int_0^{T'} \frac{C(T)}{T} dT \\ &= \int_{T_1/2}^{T_1} C_1 \left(\frac{2}{T_1} - \frac{1}{T}\right) dT \\ &= C_1 \left[\frac{2T}{T_1} - \ln(T)\right]_{T_1/2}^{T_1} \\ &= C_1 \left[1 - \ln(2)\right]. \end{aligned}$$

Now when the system is at very low temperature the spins are all aligned and the number of microstates available to the system is very small–just 2! This means the entropy is simply

$$S = k_{\mathrm{B}} \ln(\Omega) = k_{\mathrm{B}} \ln(2)$$

at very low temperature. As we raise the temperature a few of the spins will be able to be misaligned, until at very high temperature all spin configurations (including half of the spins aligned one way and half the other way, for example, and the states where all the spins are aligned) are possible. If the spins are aranged in a regular lattice and can be labeled, then the largest possible number of available microstates of the system is given by two choices for spin 1, times two for spin 2, etc, for a total of $\Omega = 2^N$, so that the maximum value of the entropy is

$$S = k_B \ln(\Omega) \leq k_B \ln(2^N) = N k_B \ln(2).$$

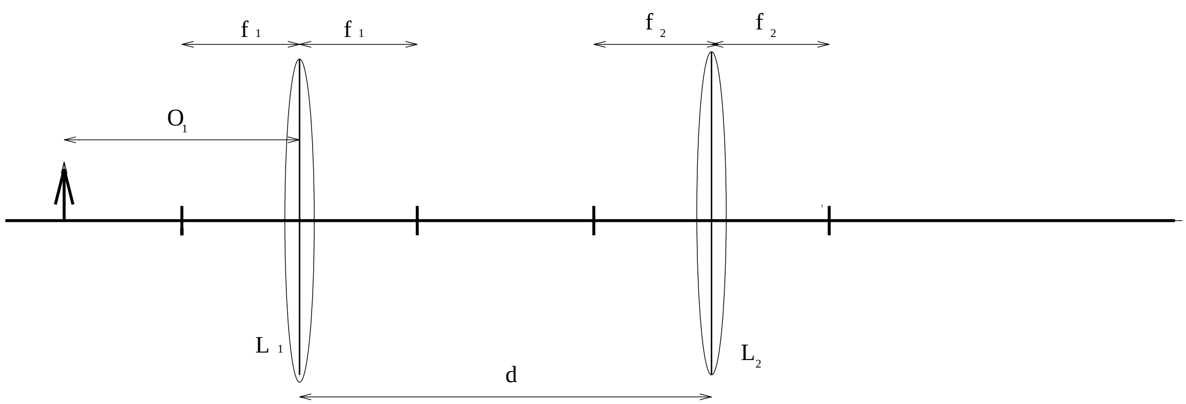
Comparing this to our result above, which is also for high temperature, we have

$$C_1 \left[1 - \ln(2)\right] \leq N k_B \ln(2),$$

so that

$$C_1 \leq N k_B \frac{\ln(2)}{1 - \ln(2)}.$$

2. Two converging thin lenses, L_1 and L_2, each of focal length $f_1 = f_2 = 10$ cm, are separated by the distance $d = 35$ cm. An object (the upright arrow) is placed a distance $O_1 = 20$ cm to the left of the left-hand lens (L_1). See the figure. The check marks on the horizontal axis represent the focal points of the two lenses.



a) Draw a ray diagram, and also find the position of the final image using the thin-lens equation.

Since the two focal lengths are both the same, call them both f. Then we can use the thin lens equation $1/s + 1/s' = 1/f$ to find the exact positions of the two images, which are shown in the following ray diagram. Note we have used the two rays from the tip of the object through the focal point (which comes out parallel) and through the center (which is undeflected) of the lens to locate the tip of the image, and the base of the image is located by a ray along the axis of the two lenses.

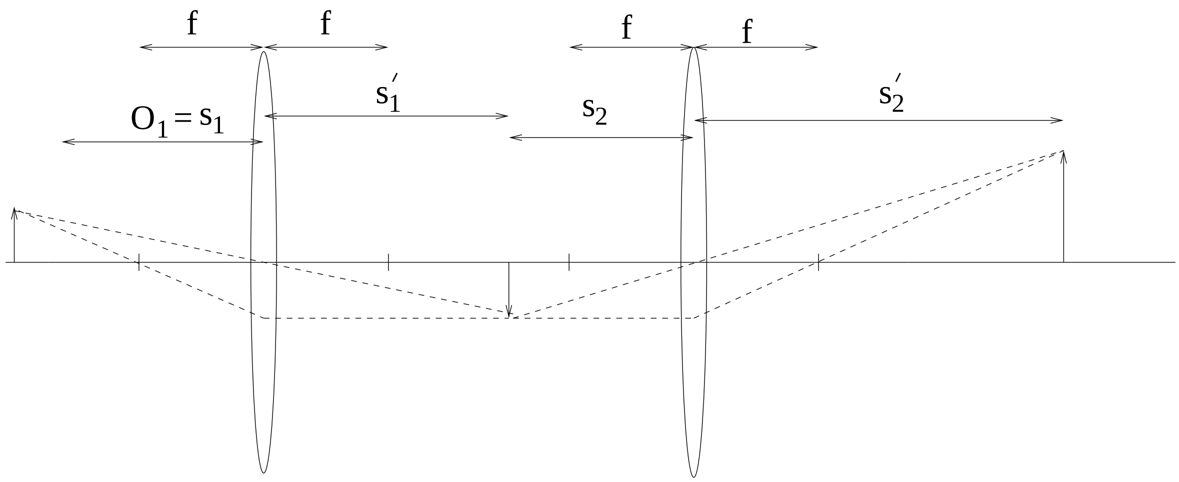
From the lens equation, using the distances labeled on the figure, we have

$$1/s_1' = 1/f - 1/s_1 = (1/10 - 1/20)\ \mathrm{cm}^{-1},$$

so that $s_1' = 20$ cm. This means that $s_2 = 35 - 20 = 15$ cm, and so

$$1/s_2' = (1/10 - 1/15)\ \mathrm{cm}^{-1} = \left(\frac{3-2}{30}\right)\ \mathrm{cm}^{-1} = 1/30\ \mathrm{cm}^{-1},$$

so that $s_2' = 30$ cm.



b) Is the final image real or virtual?

Obviously the final image is real, as rays of light converge to a point at the image where we can hold a screen.

c) Is the final image upright or inverted?.

The final image is obviously upright.

d) What is the magnification of the final image?

For this we need to calculate the size of the intermediate image from the magnification $m_1 = -s_1'/s_1 = -20/20 = -1$, so we see this image is of the same size as the object but inverted. The second image has magnification $m_2 = -s_2'/s_2 = -30/15 = -2$, so the combined magnification is $m = m_1 \cdot m_2 = 2$.