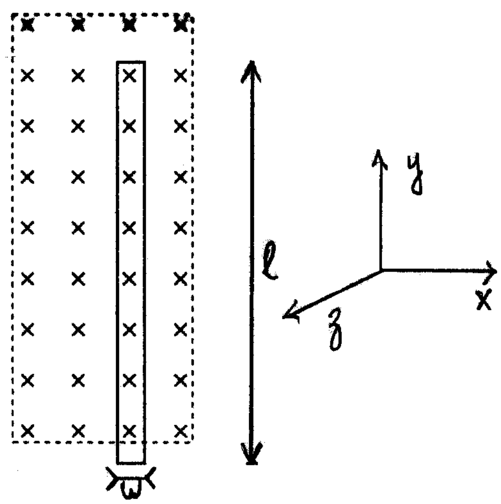
PHY6938 E & M Fall 2000

1. The accompanying figure shows a rectangular loop of wire, $w = 0.30$ m wide and $l = 1.50$ m long, in the vertical plane and perpendicular to a uniform magnetic field $B = 0.40$ T, directed into the page as shown. The magnetic field is limited to the region indicated by the dashed rectangle. The portion of the loop not in the magnetic field is 0.10 m long. The resistance of the loop is 0.20 Ω and its mass is 0.50 kg. The loop is released from rest at $t = 0$.



a) What is the magnitude and direction of the induced current when the loop has a downward velocity v?

If the loop has a downward velocity v and any part of the loop remains in the region of the magnetic field, that means that the flux cutting through the loop is decreasing. The rate of change of the area of the loop with flux through it is wv, so the rate of change of the flux is Bwv. This means that there is an induced emf of $\epsilon = Bwv$ which acts to oppose the change, so that it tries to increase the B field and so makes the current in the loop run clockwise (which makes a field into the page by the right-hand rule). Therefore, the current in the loop will have magnitude

$$I(v) = \frac{\epsilon}{R} = \frac{Bwv}{R} = \frac{0.40 \text{ T} \cdot 0.30 \text{ m} \cdot v}{0.20 \ \Omega} = 0.6 \ v \frac{\text{A}}{\text{m/s}}$$

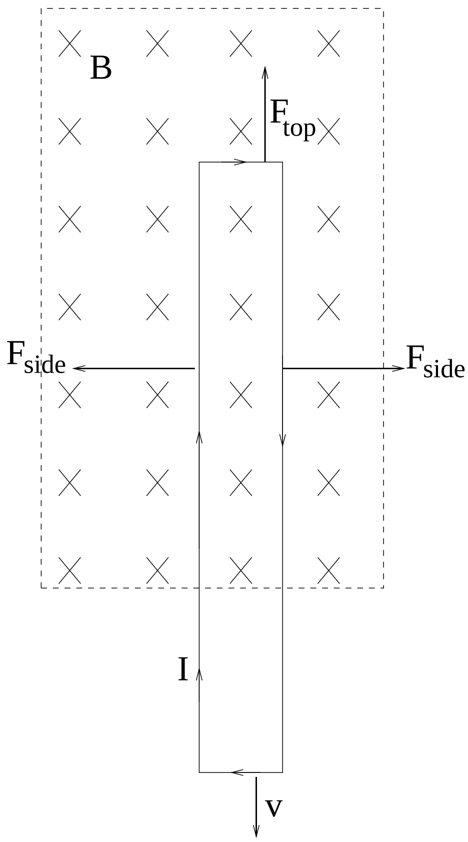
b) What is the net force acting on the loop?

There are two sources of forces on the loop: gravity, and the magnetic force on the wires due to the current in them and the external magnetic field. Part **a)** was set up to allow you to know the current in the wires. We can find the direction of the forces on the wires using the equation for a long straight wire of (directed) length $\mathbf{l}$ which is

$$\mathbf{F} = I\,\mathbf{l} \times \mathbf{B}$$

Referring to the diagram below, we see that the forces on the loop from the sides of the loop cancel, the bottom edge of the loop is outside the magnetic field and so feels no force, so that the net force on the loop is due to the force $\mathbf{F}_{\rm top}$ on the top wire of the loop, points up, and when the downward speed is v has magnitude

$$F_{\rm top} = I(v)wB = \frac{B^2w^2v}{R},$$



so that the net force in the $+y$ direction is

$$F_y = F_{\rm top} - mg = \frac{B^2w^2v}{R} - mg.$$

c) Write the equation of motion of the loop and obtain an expression for the velocity and displacement of the loop as a function of time. Your answers should include only t as an unknown. All constants should be evaluated. Consider only times short enough that part of the loop is still inside the magnetic field.

Firstly, let's abandon the inconvenient choice of coordinate system in the figure and choose both v and y to be positive for motion in the downward direction. We can write Newton's second law as a differential equation in the velocity,

$$m\frac{dv}{dt} = mg - \frac{B^2w^2v}{R}.$$

The solution of this equation would be a simple increasing exponential without the constant term on the right. This term can be taken care of by adding a particular solution, which we can see is a constant (so that $dv/dt = 0$) which is fixed so that the right hand side of the equation is zero,

$$v_p = \frac{mgR}{B^2w^2},$$

so that

$$v(t) = A\exp(-B^2w^2t/mR) + \frac{mgR}{B^2w^2},$$

where A is a constant of integration set by the initial conditions. Since $v(0) = 0$ we have

$$A = \frac{mgR}{B^2w^2},$$

so that

$$v(t) = \frac{mgR}{B^2w^2}\left\{1 - \exp(-B^2w^2t/mR)\right\}.$$

Note that $v > 0$ for all positive times t, which means that the loop is moving downward.

We can find the displacement by integrating one more time,

$$y(t) = \frac{mgR}{B^2w^2}\left\{t + \frac{mR\exp(-B^2w^2t/mR)}{B^2w^2}\right\} + C$$

where C is a constant of integration fixed by the initial condition which we can choose as $y(0) = 0$, so that

$$y(0) = 0 = \frac{mgR}{B^2w^2}\left\{0 + \frac{mR}{B^2w^2}\right\} + C$$

so that

$$C = \frac{m^2gR^2}{B^4w^4}$$

and

$$y(t) = \frac{mgR}{B^2w^2}\left\{t + \frac{mR}{B^2w^2}\left[\exp(-B^2w^2t/mR) - 1\right]\right\}.$$

The quantity mR/B^2w^2 is a time, call it τ, and

$$\tau = \frac{0.50\ \text{kg} \cdot 0.20\ \Omega}{(0.40\ \text{T} \cdot 0.30\ \text{m})^2} = 6.94\ \text{s},$$

so that

$$\begin{aligned} v(t) &= 68.0\ \text{m/s}\left\{1 - \exp(-t/6.94\ \text{s})\right\} \\ y(t) &= 68.0\ \text{m/s}\left\{t + 6.94\ \text{s}\left[\exp(-t/6.94\ \text{s}) - 1\right]\right\}. \end{aligned}$$

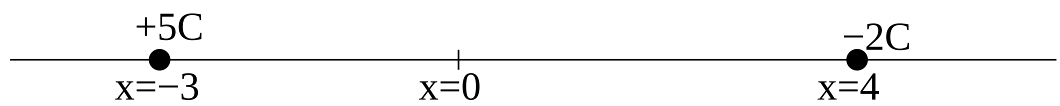
2. A point charge of $+5$ C is on the x-axis at $x = -3$ m and a second point charge of -2 C is on the x-axis at $x = +4$ m.

a) Find all the points on the x-axis where the electric potential is zero.

Pick an arbitrary point x on the x-axis. We can find the electric potential at this point, which is a scalar, by adding up the electric potential due to the charge at $x = -3$ m and that due to the charge at $x = +4$ m, i.e.

$$V = k\left(\frac{+5}{|x+3|} + \frac{-2}{|x-4|}\right),$$

where $k = 1/4\pi\epsilon_0$ and x is in meters. Note that we have used the absolute values of the distances because we need the distance to the charge to *not* change sign if x moves past the position of the charge.



Referring to the figure it is obvious that there are no positions to the left of the +5 C charge where the potential is zero, since it has a larger charge than that of the negative charge, and we will always be closer to it than the -2 C charge. There should be one point between the two charges where the electric potential is zero, which is where

$$\frac{+5}{x+3} + \frac{-2}{4-x} = 0,$$

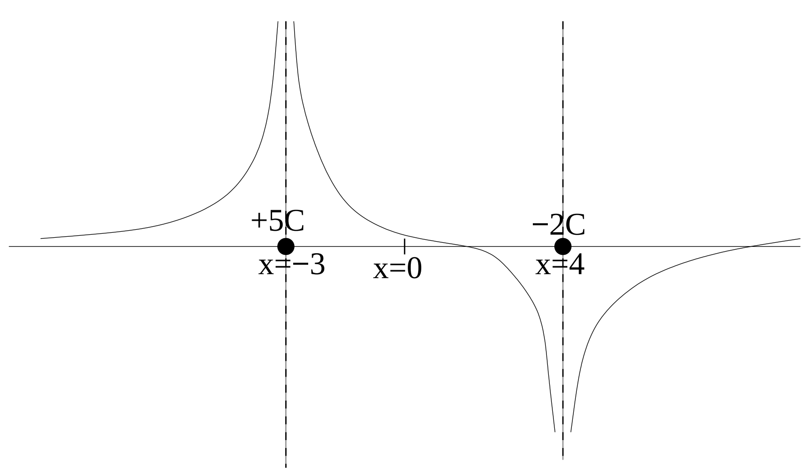
since everywhere in this region $4 - x > 0$ and $x + 3 > 0$. Solving, we find $x = 2$.

To the right of the -2 C charge we have that $x - 4 > 0$ (and $x + 3 > 0$) so that we need to solve

$$\frac{+5}{x+3} + \frac{-2}{x-4} = 0,$$

which gives $x = 26/3$.

b) Draw a graph of the electric potential $V(x)$ vs x for -10 m $< x < +10$ m.



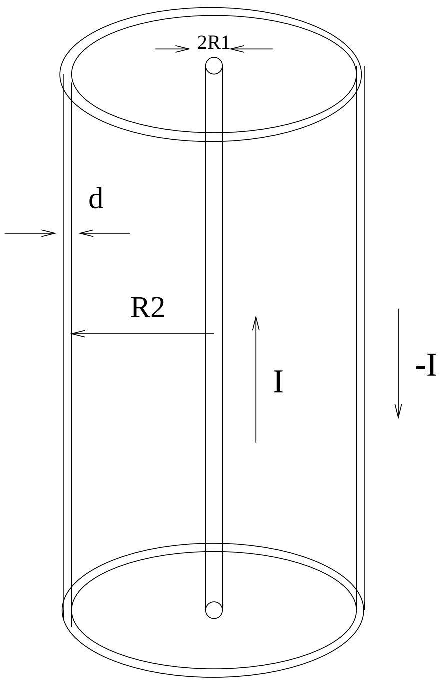
c) How much work is needed to bring a third charge of $+3$ C from infinity to the point $x = +1$ m on the x-axis?

This is simple: at infinity the charge has zero potential energy, while at $x = +1$ m on the

x-axis it has potential energy

$$\begin{aligned} U = qV &= +3 \cdot 9 \times 10^9 \left(\frac{+5}{1+3} + \frac{-2}{4-1} \right) \text{J} \\ &= 1.575 \times 10^{10} \text{ J.} \end{aligned}$$

3. Consider a coaxial cable made of a hollow cylindrical wire of inner radius R_2 and thickness d and a wire in the center of radius R_1 as shown in figure. The wire axis is labeled the z-axis. A total current I flows in the inner wire along the positive z direction and a total current I flows in the outer hollow cylindrical wire in the negative z direction. The current densities in both conductors are constant throughout their volumes.



a) Find the magnetic field inside the inner wire, i.e. for $r < R_1$.

This is a straight-forward application of Ampère's law, which states that for a closed path within a magnetic field $\mathbf{B}$, we have

$$\oint \mathbf{B} \cdot d\boldsymbol{l} = \mu_0 I_{\text{encl}},$$

where I_{encl} is the current enclosed by the wire and $d\boldsymbol{l}$ is a vector pointing along the line element dl. The direction of the current and the direction of the integration around the closed path are given by the right-hand rule (thumb along I, fingers show which way to integrate around the closed path).

Since we have cylindrical symmetry we can embed a circular closed path inside the inner wire at a distance $r < R_1$ away from its center, and in a plane perpendicular to the wire, and everywhere along this path the field $\mathbf{B}$ will have the same magnitude and will (again by the right hand rule) lie tangent to the path. Then

$$\oint \mathbf{B} \cdot d\boldsymbol{l} = B(2\pi r) = \mu_0 I_{\text{encl}}.$$

To find I_{encl} we use the fact that the current density is uniform, which means that the current enclosed by our loop is the total current scaled by the enclosed area,

$$I_{\text{encl}} = I\left(\frac{r^2}{R_1^2}\right),$$

so that

$$\begin{aligned} B(2\pi r) &= \mu_0 I\left(\frac{r^2}{R_1^2}\right) \\ B(r < R_1) &= \frac{\mu_0 I r}{2\pi R_1^2}, \end{aligned}$$

and $\mathbf{B}$ points tangential to the circular path in the counterclockwise direction.

b) Find the magnetic field in the space between the wires, i.e. for $R_1 < r < R_2$.

This works exactly as before except now the enclosed current is all of I, so that

$$\begin{aligned} B(2\pi r) &= \mu_0 I \\ B(R_1 < r < R_2) &= \frac{\mu_0 I}{2\pi r}, \end{aligned}$$

with the same direction as above.

c) Find the magnetic field inside the outer wire, i.e. for $R_2 < r < R_2 + d$.

Again this works the same way, except now we have to include current moving in the opposite direction in the outer cylindrical shell. First let's find the current enclosed by a loop of radius r, with $R_2 < r < R_2 + d$: the cross-sectional area of the cylindrical shell is

$$A = \pi\left[(R_2 + d)^2 - R_2^2\right] = \pi d(2R_2 + d),$$

while the area inside a circle of radius r is

$$A(r < R_2 + d) = \pi(r^2 - R_2^2),$$

so that

$$I_{\text{encl}} = I - I\left[\frac{r^2 - R_2^2}{d(2R_2 + d)}\right],$$

and so

$$\begin{aligned} B(2\pi r) &= \mu_0 I\left\{1 - \left[\frac{r^2 - R_2^2}{d(2R_2 + d)}\right]\right\} \\ B(R_2 < r < R_2 + d) &= \frac{\mu_0 I}{2\pi r} - \frac{\mu_0 I(r - R_2^2/r)}{2\pi d(2R_2 + d)}, \end{aligned}$$

with the same direction as above.

d) Find the magnetic field in the space for $r > R_2 + d$.

Since for a loop of radius $r > R_2 + d$ we have $I_{\text{encl}} = I - I = 0$, then

$$B(r > R_2 + d) = 0.$$