PHY6938 Proficiency Exam Fall 2002
September 13, 2002
Modern Physics and Quantum Mechanics

1. Consider a one-dimensional step potential of the form

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x \geq 0 \end{cases}$$

where $V_0 > 0$. A particle with total energy $E > V_0$ and mass m is incident on the step potential "from the left" (in other words: the particle starts at negative values of x and travels toward positive values of x).

(a) Use the time-independent Schrödinger equation to determine the form of the particle's wave function in the two regions $x < 0$ and $x \geq 0$.

In the $x < 0$ region (call this region 1) the potential is zero so the Schrödinger equation has the form

$$-\frac{\hbar^2}{2m}\frac{d^2\Psi_1}{dx^2} = E\Psi_1, \tag{1}$$

and Ψ_1 has the form of a plane wave moving to the right (the incident wave) and another moving to the left (the reflected wave), so that

$$\Psi_1 = Ae^{ik_1x} + Be^{-ik_1x}. \tag{2}$$

Substituting this into the Schrödinger equation we see that it is a solution if

$$\begin{aligned} -\frac{\hbar^2}{2m}(-k_1^2) &= E \\ k_1 &= \sqrt{\frac{2mE}{\hbar^2}} = \frac{\sqrt{2mE}}{\hbar}. \end{aligned}$$

In the $x > 0$ region (call this region 2) we have

$$\begin{aligned} -\frac{\hbar^2}{2m}\frac{d^2\Psi_2}{dx^2} + V_0\Psi_2 &= E\Psi_2. \\ -\frac{\hbar^2}{2m}\frac{d^2\Psi_2}{dx^2} &= (E - V_0)\Psi_2. \end{aligned}$$

and Ψ_2 has the form of a plane wave moving to the right (the transmitted wave), so that

$$\Psi_2 = Ce^{ik_2x}, \tag{3}$$

with

$$k_2 = \frac{\sqrt{2m(E - V_0)}}{\hbar}. \tag{4}$$

(b) Derive expressions for the probabilities that the particle is reflected (R) and transmitted (T).
Hint: Recall that the probability density current is given by

$$j(x) = \mathrm{Re}\left(\Psi^* \frac{\hbar}{im}\frac{\partial\Psi}{\partial x}\right),$$

and that R and T are ratios of probability density currents.
We have to derive the values of A, B, and C using the boundary conditions, which are that the wavefunction and its derivative must be continuous at the boundary ($x = 0$). Continuity of the wavefunction gives that

$$A + B = C, \tag{5}$$

while that of the derivative at $x = 0$ gives that

$$k_1 A - k_1 B = k_2 C, \tag{6}$$

since taking the derivative of the plane wave just brings down a factor of $-ik$, and we have divided by $-i$. For that reason the incident probability current is simply

$$j_{\mathrm{in}} = \mathrm{Re}\left[Ae^{-k_1 x}\frac{\hbar}{Am}ik_1 Ae^{ik_1 x}\right] = \frac{\hbar}{m}k_1 A^2, \tag{7}$$

and similarly

$$\begin{aligned} j_{\mathrm{refl}} &= \frac{\hbar}{m}k_1 B^2 \\ j_{\mathrm{trans}} &= \frac{\hbar}{m}k_2 C^2. \end{aligned}$$

Putting these together we get that the reflection probability is

$$R = \frac{j_{\mathrm{refl}}}{j_{\mathrm{in}}} = \frac{B^2}{A^2}, \tag{8}$$

and we can find B/A by combining (1) and (2) above,

$$k_1(A - B) = k_2(A + B), \tag{9}$$

so that

$$\frac{B}{A} = \frac{k_1 - k_2}{k_1 + k_2}, \tag{10}$$

and

$$R = \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2. \tag{11}$$

By conservation of probability we have that

$$T = 1 - R = \frac{(k_1 + k_2)^2 - (k_1 - k_2)}{2}(k_1 + k_2)^2 = \frac{4k_1 k_2}{(k_1 + k_2)}^2 \tag{12}$$

2. The nucleus ^{113}Cd captures a thermal neutron having negligible kinetic energy, producing ^{114}Cd in an excited state. The excited state of ^{114}Cd decays to the ground state by emitting a photon. Find the energy of the photon.

The reaction is

$$^{113}\mathrm{Cd} + \mathrm{n} \rightarrow ^{114}\mathrm{Cd} + \gamma \tag{1}$$

The four momentum P^{μ} must be conserved. This means that

$$\begin{aligned} P^{mu}_{before} &= P^{\mu}_{after} && (2)\\ (\frac{E_{before}}{c}, \mathbf{p_{before}}) &= (\frac{E_{after}}{c}, \mathbf{p_{after}}). && (3) \end{aligned}$$

Before the reaction both $^{113}\mathrm{C}d$ and n are at rest. But after the reaction $^{114}\mathrm{C}d$ and γ are not at rest. This means

$$\begin{aligned} E_{before} &= \left[m(^{113}\mathrm{C}d) + m(\mathrm{n})\right] c^2 && (4)\\ E_{after} &= m(^{114}\mathrm{C}d)c^2 + E_{kinetic}(^{114}\mathrm{C}d) + E_{\gamma} && (5)\\ E_{kinetic}(^{114}\mathrm{C}d) + E_{\gamma} &= \left[m(^{113}\mathrm{C}d) + m(\mathrm{n}) - m(^{114}\mathrm{C}d)\right] c^2 && (6)\\ &= 9.042 \ \mathrm{M}eV. && (7) \end{aligned}$$

Conservation of momentum gives

$$\begin{aligned} \mathbf{p}_{before} &= 0 && (8)\\ \mathbf{p}_{after} &= \mathbf{p}(^{114}\mathrm{C}d) + \mathbf{p}_{\gamma} && (9)\\ \mathbf{p}(^{114}\mathrm{C}d) &= -\mathbf{p}_{\gamma} && (10)\\ \left|\mathbf{p}(^{114}\mathrm{C}d)\right| &= |\mathbf{p}_{\gamma}| && (11)\\ &\text{but} \quad |\mathbf{p}_{\gamma}| = \frac{E_{\gamma}}{c} && (12) \end{aligned}$$

The kinetic energy of $^{114}\mathrm{C}d$ is given by

$$E_{kinetic}(^{114}\mathrm{C}d) = \frac{p^2(^{114}\mathrm{C}d)}{2m(^{114}\mathrm{C}d)} \tag{13}$$

From Eqs.7, 12, and 13 we obtain

$$\frac{E_{\gamma}^2}{2m_{\mathrm{Cd}}c^2} + E_{\gamma} = 9.042 \tag{14}$$

One way to determine the energy of the photon is to calculate the roots of the Eq.14. But we can also obtain it as follows. In Eq.14 all the terms are positive. This means that $E_{\gamma} \leq 9.042$Mev. The maximum value that the first term in Eq.14 can have is

$$\begin{aligned} E_{gamma} &= 9.042\mathrm{M}eV\\ \frac{E_{\gamma}^2}{2m_{\mathrm{Cd}}c^2} &= \frac{81}{2 \times 113.9 \times 931.5}\\ &< 3.8 \times 10^{-4}\mathrm{M}eV. && (15) \end{aligned}$$

Eq.15 means that the first term is very smaller than second term. Therefore we can neglect this term. Consequently $E_{gamma} \approx 9.042$MeV.

3. A current of quantum-mechanical particles of mass m can be written as

$$\vec{J} = -\frac{i\hbar}{2m}\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right] ,$$

where ψ is the wave function. Assume that the particles move through a region of space where the potential is complex $V = V_r - iV_i$. Show that particles are being annihilated at a rate

$$R = \frac{2}{\hbar}V_i\,\psi^*\psi$$

per unit volume.
Hint: Use the time-dependent Schrödinger equation to obtain "the material derivative"

$$\frac{\partial}{\partial t}(\psi^*\psi) + \nabla\cdot\mathbf{J}\,.$$

This problem can be difficult if we do not really understand the hint. Looking at the last equation in the problem we can see that this equation is the continuity equation. The continuity equation exists in many different areas of physics. In general this equation states that

Increase (decrease) in of density at a point of space = the flow of material inside (outside) a surface surrounding that point + the amount of material which is created (annihilated) at that point.

In Electromagnetism the continuity equation is given by

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0. \tag{1}$$

This means that the increase in the charge density at any point in the space is equal to flow of charge inside a surface surrounding that point. As we can see there are no source which creates or annihilate charges. In this case the continuity equation results in the conservation of charge at any point in space. Many processes are therefore forbidden because of this equation. For example if there is no current density then a charge cannot be annihilated at a point of space and be created at another point. Although the total charge is conserved but this process is forbidden.

The continuity equation in Quantum mechanics states the continuity of the probability density $\psi^*\psi$ at any point in space. In the introductory Quantum mechanics courses we alway see that the continuity equation is given by

$$\frac{\partial}{\partial t}(\psi^*\psi) + \nabla\cdot\mathbf{J} = 0. \tag{2}$$

This is valid when there is no source. If the right hand side of Eq.2 is non-zero this means even in the absence of density current $\mathbf{J}$ particles can be created or annihilated. Therefore our task in this problem is to derive continuity equation for this system and note that the right hand side of the continuity equation is non-zero

First of all we calculate $\nabla \cdot \mathbf{J}$

$$
\begin{aligned}
\nabla \cdot \mathbf{J} &= -\frac{i\hbar}{2m}\left[\nabla \cdot (\psi^* \nabla \psi) - \nabla \cdot (\psi \nabla \psi^*)\right] \\
&= -\frac{i\hbar}{2m}\left[\nabla \psi^* \cdot \nabla \psi + \psi^* \nabla^2 \psi - \nabla \psi \cdot \nabla \psi * + \psi \nabla^2 \psi^*\right] \\
&= -\frac{i\hbar}{2m}\left[\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*\right] \qquad (3)
\end{aligned}
$$

Time- dependent Schrödinger equation gives

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\nabla^2 \psi + (V_r - iV_i) &= i\hbar \frac{\partial \psi}{\partial t} \qquad (4) \\
-\frac{\hbar^2}{2m}\nabla^2 \psi^* + (V_r + iV_i) &= -i\hbar \frac{\partial \psi^*}{\partial t} \qquad (5)
\end{aligned}
$$

Multiply Eq.4 from right with ψ^* and Eq.5 with ψ, and subtract Eq.5 from Eq.4 we get

$$
\begin{aligned}
-\frac{\hbar^2}{2m}\left[\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*\right] - 2iV_i \psi^* \psi &= i\hbar\left[\psi^* \frac{\partial \psi}{\partial t} + \psi \frac{\partial \psi^*}{\partial t}\right] \\
&= i\hbar \frac{\partial}{\partial t}(\psi^* \psi)
\end{aligned} \qquad (6)
$$

Multiply Eq.6 with $\frac{i}{\hbar}$ and use Eq.1

$$
\begin{aligned}
-\frac{i\hbar}{2m}\left[\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^*\right] + \frac{2}{\hbar} V_i \psi^* \psi &= -\frac{\partial}{\partial t}(\psi^* \psi) \\
\frac{\partial}{\partial t}(\psi^* \psi) + \nabla \cdot \mathbf{J} &= -\frac{2}{\hbar} V_i \psi^* \psi \qquad (7) \\
&= -R \qquad (8)
\end{aligned}
$$

The right hand side of Eq.8 is non-zero. This means that there is a source at any point in space. For instance assume that $\mathbf{J} = 0$, then the rate of the change of the probability density is equal to a negative quantity. In another words particles are being annihilated.

4. **Cosmic ray photons from space are bombarding your laboratory and smashing massive objects to pieces. Your detectors indicate that two fragments, each of mass m_0, depart such a collision moving at speed $0.6\,c$ at angles of 60° relative to the photon's original direction of motion.**

(a) In terms of m_0 and c, what is the energy of the cosmic ray photon ?

We will solve this part of the problem using conservation of momentum to find the photon momentum (and so energy). The momentum of each fragment is

$$p = m_0 \gamma(v), \tag{1}$$

where

$$\gamma(v) = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{0.64}} = \frac{5}{4}, \tag{2}$$

so that

$$p = \frac{5}{4} m_0 (0.6\ c) = \frac{3}{4} m_0 c. \tag{3}$$

The components perpendicular to the photon's direction of motion cancel, and those parallel to it add to

$$p_\gamma = 2p\cos(60^\circ) = p = \frac{3}{4} m_0 c, \tag{4}$$

where we have used conservation of momentum, so the photon energy is

$$E_\gamma = p_\gamma c = \frac{3}{4} m_0 c^2. \tag{5}$$

(b) In terms of m_0, what is the mass M of the particle being struck (assumed to be initially stationary) ?

The initial total relativistic energy is therefore

$$E_\mathrm{i} = Mc^2 + \frac{3}{4} m_0 c^2, \tag{6}$$

and the final total relativistic energy is

$$E_\mathrm{f} = 2E = 2\sqrt{p^2c^2 + m_0^2 c^4} = 2\sqrt{\frac{16+9}{16} m_0^2 c^4} = \frac{5}{2} m_0 c^2, \tag{7}$$

and equating these gives that

$$Mc^2 = \left(\frac{5}{2} - \frac{3}{4}\right) m_0 c^2 = \frac{7}{4} m_0 c^2, \tag{8}$$

so that

$$M = \frac{7}{4} m_0. \tag{9}$$