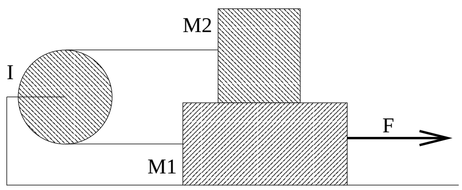
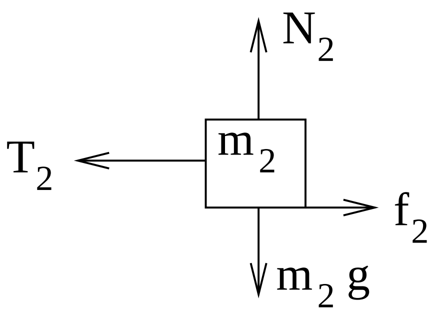
PHY6938 Proficiency Exam Spring 2002
April 5 , 2002
Mechanics

1. **As shown in the diagram, two blocks with masses M_1 and M_2 are attached by an unstretchable rope around a frictionless pulley with radius r and moment of inertia I. There is no slipping between the rope and the pulley. The coefficient of kinetic friction between the blocks is the same as between block 1 and the surface, μ. A horizontal force F is applied to M_1. Find the acceleration of M_1.**

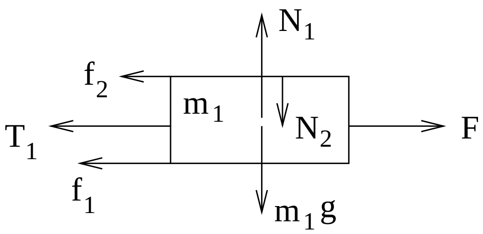


First draw a free-body diagram for M_1, M_2, and the pulley, being careful to label all the forces and find the line along which they act. Firstly let's examine M_2. It has a tension due to the string, gravity, a frictional force due its motion relative to M_1, which we can assume is to the left, and the normal force which M_1 exerts on it to stop it from moving vertically.

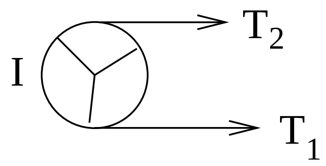


Note that the tension T_2 in the top half of the string does not have to equal that in the bottom half if we cannot ignore the pulley's moment of inertia and it is accelerating.

Now do the same for M_2. Note that the two forces on M_1 due to M_2 are prescribed by



Newton's third law to be equal and opposite to those on M_2 due to M_1. Finally, we have the free-body diagram for the pulley We can now write down the equations of



motion for these three objects. In the past we have written independent coordinates and written down constraint equations which we have formally added to our system of equations to solve for the acceleration. Here for simplicity just note that if M_1 moves to the right a distance x, which we will take as our coordinate, then M_2 moves to the left by x and the pulley rotates counterclockwise through an angle x/r. Then the equations of motion for M_1, M_2 and the pulley are

$$\begin{aligned} M_1\ddot{x} &= F - f_1 - f_2 - T_1 && (1)\\ M_2\ddot{x} &= T_2 - f_2 && (2)\\ I\frac{\ddot{x}}{r} &= rT_1 - rT_2, && (3) \end{aligned}$$

where the last involves the torques due to the two tensions. We also need to find the frictional forces by balancing all the vertical forces on M_1 and M_2

$$\begin{aligned} N_2 &= M_2 g\\ N_1 &= N_2 + M_1 g = (M_1 + M_2)g, \end{aligned}$$

so that $f_2 = \mu N_2 = \mu M_2 g$ and $f_1 = \mu N_1 = \mu(M_1 + M_2)g$. If we consider f_1 and f_2 as known quantities, we now have three equations in the three unknowns T_1, T_2 and $\ddot{x}$ which we can solve as follows:

$$\begin{aligned} (1)+(2):\ (M_1+M_2)\ddot{x} &= F - f_1 - 2f_2 - (T_1 - T_2) && (4)\\ (4)+(3)/r:\ \left(M_1 + M_2 + \frac{I}{r^2}\right)\ddot{x} &= F - f_1 - 2f_2, && (5) \end{aligned}$$

so that

$$\ddot{x} = \frac{F - f_1 - 2f_2}{M_1 + M_2 + I/r^2} = \frac{F - \mu(M_1 + M_2 + 2M_2)g}{M_1 + M_2 + I/r^2} = \frac{F - \mu(M_1 + 3M_2)g}{M_1 + M_2 + I/r^2} \qquad (6)$$

2. **The asteroid Toro was discovered in 1964. Its radius is about 5 km. Acceleration due to gravity on earth is** $\mathrm{g} = 9.81 m/s^2$**, the earth's radius is 6378 km, and the gravitational constant** $G = 6.67 \times 10^{-11} \frac{m^3}{kg\, s^2}$**.**

   (a) **Assuming the density of Toro is the same as that of Earth, find its total mass and the acceleration due to gravity at its surface.**

$$M = \left(\frac{R_{\mathrm{T}}}{R_{\mathrm{E}}}\right)^3 M_{\mathrm{E}} = \left(\frac{5.0}{6.38 \cdot 10^3}\right)^3 5.98 \cdot 10^{24}\ \mathrm{kg} = 2.88 \cdot 10^{15}\ \mathrm{kg}. \qquad (7)$$

Then the acceleration due to gravity at its surface is, according to Newton, the same as if we placed all of its mass at its center which is a distance R_{T} away,

$$g(R_{\mathrm{T}}) = \frac{GM}{R_{\mathrm{T}}^2} = \frac{6.67 \cdot 10^{-11} \cdot 2.88 \cdot 10^{15}}{(5.0 \cdot 10^3)^2} \frac{\mathrm{Nm}^2}{kg^2} \mathrm{kg}m^2 = 7.68 \cdot 10^{-3} \frac{\mathrm{m}}{s^2}. \quad (8)$$

(b) Suppose a body is placed in a circular orbit around Toro, with radius just slightly larger than the asteroid's radius. What is the speed of the body ?

In a circular orbit the acceleration of the body is just v^2/R_{T}, where v is the speed of the body. However this acceleration must be provided by the force of gravity, so that $mg(R_{\mathrm{T}}) = v^2/R_{\mathrm{T}}$ and so we have that

$$\begin{aligned} v^2 &= g(R_{\mathrm{T}})R_{\mathrm{T}} \\ v &= \sqrt{g(R_{\mathrm{T}})R_{\mathrm{T}}} = \sqrt{\frac{GM}{R_{\mathrm{T}}}} = \sqrt{\frac{6.67 \cdot 10^{-11} \cdot 2.88 \cdot 10^{15}}{5.0 \cdot 10^3}} \frac{\mathrm{m}}{s} = 6.20 \frac{\mathrm{m}}{s}. \end{aligned}$$

(c) If a rock is thrown into a circular orbit around Toro at a height of 200 m above the surface, what is its period of revolution ?

Here we have to be careful to take care of the reduction of g with height above the surface, as 200 m is a significant fraction of 5 km. We can again equate the acceleration die to gravity and that required to go in a circle, although using the acceleration $\omega^2 r$, where $r = R_{\mathrm{T}} + 200$ m $= 5200$ m, will get us to the period faster

$$\begin{aligned} \omega^2 r &= g(r) \\ \omega^2 &= \frac{g(r)}{r} \\ T &= \frac{2\pi}{\omega} \\ &= 2\pi\sqrt{\frac{r}{g(r)}} = 2\pi\sqrt{\frac{r^3}{GM}} = 2\pi\sqrt{\frac{(5.2 \cdot 10^3)^3}{6.67 \cdot 10^{-11} \cdot 2.88 \cdot 10^{15}}} \mathrm{~s} \\ &= 5.38 \cdot 10^3 \mathrm{~s} = 1.49 \mathrm{~hr}. \end{aligned}$$

3. Consider the off-center elastic collision of two objects of equal mass when one is initially at rest.

(a) Show that the final velocity vectors of the two objects are perpendicular to each other.

Let's call the angle between the projectile's initial and final directions the lab scattering angle ψ, and the angle between the target's final direction and the projectile's initial direction the recoil angle ζ. Our job is to show that $\psi + \zeta$

is 90°. Let the projectile's initial velocity be v, its final velocity v_1, and the target's final velocity v_2. The statement of conservation of momentum is

$$\begin{aligned} mv &= mv_1\cos(\psi) + mv_2\cos(\zeta) \\ 0 &= mv_1\sin(\psi) - mv_2\sin(\zeta), \end{aligned}$$

which simplifies to

$$v = v_1\cos(\psi) + v_2\cos(\zeta) \tag{9}$$
$$0 = v_1\sin(\psi) - v_2\sin(\zeta) \tag{10}$$

once the mass is divided out.

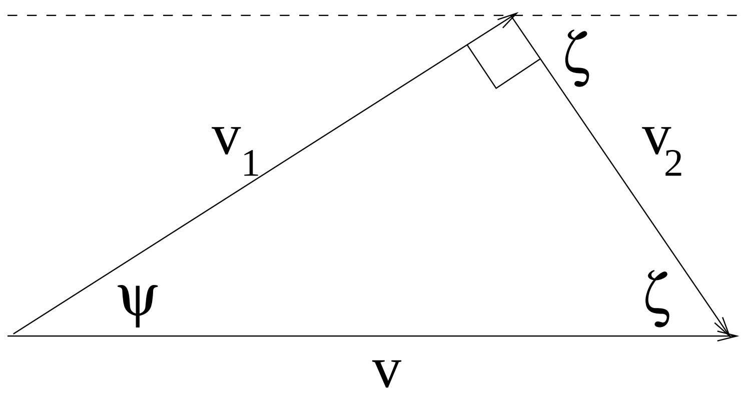
Energy conservation gives that (here dividing by $m/2$ straight away)

$$v^2 = v_1^2 + v_2^2. \tag{11}$$

Applying the Pythagorean theorem to (3) we know that a triangle with a hypotenuse of length v has sides of length v_1 and v_2. If we draw the vectors representing the velocities in order to represent the *vector* momentum conservation equation

$$\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2, \tag{12}$$

as in the following figure,



we see that it must be the case that the sum of ψ and ζ is 90°.

(b) Show that the incoming object cannot have a backward scattered component.

Neither ψ nor ζ can be negative (i.e. $\mathbf{v}_1$ and $\mathbf{v}_2$ cannot be on the same side of the projectile's initial momentum) or we would not be able to conserve momentum in the direction perpendicular the projectile's initial velocity. Since their sum is 90° this means that they both satisfy

$$0 \leq \psi, \zeta \leq 90°, \tag{13}$$

which means that the projectile cannot scatter at an angle ψ larger than 90°.