Study guide for qualifying exams

Cody L. Petrie

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1 Classical Mechanics

1. Newtonian Mechanics
   (a) Newton's Laws/Kinematics
   (b) Energy
   (c) Momentum/Angular Momentum
2. Lagrangian Mechanics
   (a) Calculous of Variations
   (b) Principle of Least Action/Lagranges Equation
   (c) Generalized Coordinates
   (d) Holonomic/Non-Holonomic Constraints
   (e) Noether's Theorem
   (f) Rigid Body Motion
      i. Inertia Tensor
      ii. Euler's Equations
3. Hamiltonian Formalism
   (a) Legendre Transformation/Hamilton's Equations
   (b) Generalized Momenta/Cyclic Coordinates/Conserved Quantities
   (c) Liouville's Theorem
   (d) Poisson Brackets
   (e) Canonical Transformations

2 Statistical Mechanics

1. Thermodynamics Review
   (a) Laws of Thermodynamics
   (b) Intensive vs Extensive Variables
   (c) Thermodynamic Potentials and Ensembles
   (d) Maxwell's Relations
   (e) Various Definitions
      i. Compressibility
      ii. Heat Capacity etc.
2. Statistical Mechanics
   (a) Statistical Review
   (b) Partition Function/Trace
   (c) Thermodynamic Limit
   (d) Density Matrix
   (e) Ideal Gas
   (f) Ideal Bose Gas
   (g) Ideal Fermi Gas
   (h) Photons (BB)
   (i) Phonons
   (j) Bose-Einstein Condensates
   (k) Cluster Expansion

3 Quantum Mechanics

1. Shankar Math Review
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3. Free Particle
4. Particle in a Box
5. Harmonic Oscillator
6. Angular Momentum
7. Hydrogen Atom
8. Spin

9. Angular Momentum Addition
10. Time-Independent Perturbation Theory
11. Time-Dependent Perturbation Theory
    (a) Einstein A and B Coefficients
12. Scattering
13. WKB Formula
14. Dirac Equation

4 Electricity and Magnetism

1. Electrostatics
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    (b) Electrostatic Potentials
        i. Poisson/Laplace's Equations
    (c) Boundary Conditions
    (d) Method of Images
    (e) Multipole Expansion
    (f) Work and Energy
    (g) Electric Fields in Matter
2. Magnetostatics
    (a) Lorentz Force Law
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    (c) Vector Potential
    (d) Magnetic Fields in Matter
3. Electrodynamics
    (a) Ohm's Law
    (b) Maxwell's Equations
    (c) Boundary Conditions to Maxwell's Equations
    (d) Continuity Equation
    (e) Poynting's Theorem
    (f) Maxwell Stress Tensor

(g) Electromagnetic Waves
    i. The Wave Equation from Maxwell's Eq.
    ii. EM Waves in Matter
    iii. Wave Guides

4. Scalar and Vector Potentials
5. Coulomb and Lorentz Gauge
6. Retarted Potentials
7. Lienard-Wiechert Potentials
8. Radiation
    (a) Electric/Magnetic Dipole Radiation
9. Helmholtz Theorem
10. Special Relativity
    (a) Einstein's Postulates
    (b) Lorentz Transformation
    (c) 4-Vectors
    (d) Field Tensor and Transformation
    (e) Relativistic Potentials

5 Classical Mechanics Equations

Newtonian Mechanics

Newton's Laws:

1. An object will maintain it's current motion unless acted upon by an external force.
2. $\vec{F} = m\vec{a}$
3. All forces occur in equal but directionally opposite pairs.

Second Law: $\vec{F} = m\vec{a} = \dot{\vec{p}}$
Angular Position/Velocity/Acceleration: $\theta = s/r$, $\omega = v/r$, $\alpha = a/r$
Angular Momentum: $\vec{L} = \vec{r} \times \vec{p}$
Torque: $\vec{\tau} = \vec{r} \times \vec{F} = \dot{\vec{L}}$
Centripital Acceleration: $a_c = v^2/r$
Centrifugal/Coriolis Forces: $\vec{F}_{cent} = -m\vec{\omega} \times (\vec{\omega} \times \vec{r'})$, $\vec{F}_{cor} = -2m\vec{\omega} \times \dot{\vec{r'}}$
Work to go from positions $\vec{a}$ to $\vec{b}$: $W_{ab} = \int\limits_{\vec{a}}^{\vec{b}} \vec{F} \cdot d\vec{s}$
Conservative Force Field (2 eq): W_{ab} is the same regardless of path so $\oint \vec{F} \cdot d\vec{s} = 0$, and thus we can write the force as $\vec{F} = -\nabla V(\vec{r})$.

Lagrangian Formalism

Functional Derivative: $\frac{\delta F}{\delta u}[x_0] = \lim\limits_{\epsilon \to 0} \frac{F[x_0 + \epsilon u] - F[x_0]}{\epsilon} \to \frac{\delta F}{\delta x(t)}[x(t')] = \lim\limits_{\epsilon \to 0} \frac{F[x(t') + \epsilon\delta(t'-t)] - F[x(t')]}{\epsilon}$
Principle of Least Action: $\delta S = 0$, where $S = \int_{t_i}^{t_f} L(\vec{q}, \dot{\vec{q}}, t)dt$
Lagranges Equation: $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^A} - \frac{\partial L}{\partial x^A} = 0$
Holonomic Constraints: $f_\alpha(x^A, t) = 0$, $L' = L(x^A, \dot{x}^A) + \lambda_\alpha f_\alpha(x^A, t) \to \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^A} - \frac{\partial L}{\partial x^A} = \lambda_\alpha \frac{\partial f_\alpha}{\partial x^A}$
Noether's Theorem: A continuous symmetry in the Action (and thus Lagrangian) result in a conserved quantity.
Moment of Inertia Tensor: $\vec{L} = \overleftrightarrow{I}\vec{\omega}$, $T = \frac{1}{2}\omega_a I_{ab}\omega_b$, $I_{ab} = \sum_i m_i((\vec{r}_i \cdot \vec{r}_i)\delta_{ab} - (\vec{r}_i)_a(\vec{r}_i)_b$
Euler's Equations: Only look at rotation, not translation. Conservation of Angular Momentum gives $I_i\dot{\omega}_i + \omega_j\omega_k(I_k - I_j) = 0$, for i,j,k being cyclic permutations of 1,2,3.

Hamiltonian Formalism

Generalized Momenta: $p_i = \frac{\partial L}{\partial \dot{q}_i}$, $\dot{p}_i = \frac{\partial L}{\partial q_i}$

Hamiltonian: $H(q_i, p_i, t) = \sum\limits_{i=1}^{n} p_i\dot{q}_i - L(q_i, \dot{q}_i, t)$

Hamilton's Equations:

1. $\dot{p}_i = -\frac{\partial H}{\partial q_i}$

2. $\dot{q}_i = \frac{\partial H}{\partial p_i}$

3. $-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$

Cyclic/Ignorable Coordinates: q is ignorable if $\frac{\partial L}{\partial q} = 0$, i.e. if q does not appear in L. Thus $p = \frac{\partial L}{\partial \dot{q}}$ is conserved.
Liousille's Theorem: A volume of a region of phase space remains the same, even when the refion changes. $V = dq_1 \dots dq_n dp_1 \dots dp_n$.
Poisson Bracket: $\{f, g\} = \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}$.
Constant of Motion from Poisson Bracket: $\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}$. If $I, H = 0$, then I is a constant of motion.
Canonical Transformation: Transformation $(q_i \to Q_i(q,p), p_i \to P_i(q,p))$ that leaves Hamilton's equations invariant.

6 Statistical Mechanics Equations

6.1 Thermodynamics

Laws of Thermodynamics:

1. Energy conservation. $dE = đQ - pdV$. $đQ$ just means that the heat is an inexact differential and the integral depends on the path.

2. $\Delta S \geq \int \frac{đQ}{T}$, where equality is for a process that is reversible (never leaves equilibrium).

3. Entropy at zero temperature is zero. In stat mech this means that the ground state is nondegenerate and $S \propto \ln(\mathrm{W})$, where W is the number of available states.

Intensive vs Extensive Variables: Intensive variables do NOT scale with system size (T, p, μ), while extensive do scale (E, S, V, N).
Thermodynamic Potentials:

- Internal Energy: $U(S, V, N)$
- Helmholtz Free Energy: $F(T, V, N) = U - TS$
- Enthalpy: $H(S, p, N) = U + pV$
- Gibbs Free Energy: $G(T, p, N) = U - TS + pV$
- Landau(Grand) Potential: $\Omega(T, V, \mu) = U - TS - \mu_i N_i$

Thermodynamic Ensembles:

1. Microcanonical: Does not exchange energy or particles with environment. Fixed E, N

2. Canonical: Does not exchange particles, but can exchange energy (heat bath). Fixed N, T

3. Grand canonical: Can exchange energy and particles with environment. Fixed T, μ.

Maxwell's Relations (4 main):

- $\frac{\partial^2 U}{\partial S \partial V} = -\left(\frac{\partial p}{\partial S}\right)_V = \left(\frac{\partial T}{\partial V}\right)_S$
- $\frac{\partial^2 F}{\partial T \partial V} = \left(\frac{\partial p}{\partial T}\right)_V = \left(\frac{\partial S}{\partial V}\right)_T$
- $\frac{\partial^2 H}{\partial S \partial p} = \left(\frac{\partial V}{\partial S}\right)_p = \left(\frac{\partial T}{\partial p}\right)_S$
- $\frac{\partial^2 G}{\partial T \partial p} = \left(\frac{\partial V}{\partial T}\right)_p = -\left(\frac{\partial S}{\partial p}\right)_T$

Engine Efficience: $\eta = \frac{Q_{in} - Q_{out}}{Q_{in}} = 1 - \frac{T_{out}}{T_{in}}$
Isobaric Thermal Expansion Coeffieicnt: $\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P$, How much the volume changes with a change in termperature.
Isothermal Compressibility: $\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_T$, How much the volume changes when the pressure changes.
Isentropic(Adiabatic) Compressibility: $\kappa_S = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_S$, Same as above.
Specific Heat at Constant V: $C_V = \left(\frac{\partial Q}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V$, Amount of heat per unit mass to raise the temp by 1 degree.
Specific Heat at Constant p: $C_p = \left(\frac{\partial Q}{\partial T}\right)_p = \left(\frac{\partial H}{\partial T}\right)_p$, Same as above.
Fermi Energy/Temperature: Chemical potential at $T = 0$. $\epsilon_F = \mu(T = 0)$

6.2 Statistical Mechanics

Number of microstates in a mactostate (ways to get n heads): $\Omega = \frac{N!}{\prod_i n_i!}$
Stirling's Approximation: $\ln n! = n \ln n - n$
How many order important ways to order n things: $n!$
How many order important waus to order n things r at a time: $\frac{n!}{(n-r)!}$
How many NOT order important ways to order n things r at a time: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$
Microcanonical(Classical) Partition Function: $Z_m = \sum_s g_s e^{-\beta E_s}$
Canonical Partition Function: $Z_c = \text{tr}\left(e^{-\beta \hat{H}}\right)$
Grand Canonical Partition Function: $Z_{gc} = \text{tr}\left(e^{-\beta(\hat{H} - \mu \hat{N})}\right)$
Geometric Series: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$
Classical limit of the trace of an operator: $\text{tr}(\mathcal{O}) = \frac{1}{N!(2\pi\hbar)^{3N}} \int d^3r_1 \ldots d^3r_N \int d^3p_1 \ldots d^3p_N \mathcal{O}$, $N!$ is for identical particles.
Thermodynamic Limit: $T \to \infty, V \to \infty, N/V = const$
Expectation value for pure/mixed: $\langle \mathcal{O} \rangle_p = \langle \psi | \mathcal{O} | \psi \rangle, \langle \mathcal{O} \rangle_m = \sum_i P_i \langle \psi_i | \mathcal{O} | \psi_i \rangle$
Density Matrix (ex. Canonical Ensemble): $\rho = \sum_n P_n |\psi_n\rangle \langle \psi_n|, \rho_c = \frac{e^{-\beta \hat{H}}}{\text{tr} e^{-\beta \hat{H}}}$
Expectation value with Density Matrix: $\langle \mathcal{O} \rangle = \text{tr}(\mathcal{O}\rho)$
Trace of Density matrix: $\text{tr}(\rho) = 1$
Time evolution of density matrix: $\frac{\partial}{\partial t}\rho = \frac{1}{i\hbar}\left[\hat{H}, \hat{\rho}\right]$

Z_{gc} for an ideal gas: $Z_{gc} = \frac{V^N (2mT\pi)^{3N/2}}{N!(2\pi\hbar)^{3N}} e^{\beta\mu}$
Z_{gc} for ideal fermi gas: $Z_{gc} = \prod_k \left(1 + e^{-\beta(\epsilon_k - \mu)}\right)$
Z_{gc} for ideal bose gas: $Z_{gc} = \prod_k \frac{1}{\left(1-e^{-\beta(\epsilon_k - \mu)}\right)}$
Stuff here for black-body and phonons and bose condensates.
What is cluster expansion used for?: Systems of interacting particles.

7 Quantum Mechanics Equations

Properties of a vector space:

- Sum $|V\rangle + |W\rangle$
- Scalar product with properties
  1. closure: results in another vector in the space.
  2. distributive: $a(|V\rangle + |W\rangle = a|V\rangle + a|W\rangle)$, $(a+b)|V\rangle = a|V\rangle + b|V\rangle$
  3. associative: $a(b|V\rangle) = ab|V\rangle$, $|V\rangle + (|W\rangle + |Z\rangle) = (|V\rangle + |W\rangle) + |Z\rangle$
  4. commutative: $|V\rangle + |W\rangle = |W\rangle + |V\rangle$
  5. addative inverse: $|V\rangle + |-V\rangle = |0\rangle$
  6. null vector: $|V\rangle + |)\rangle = |V\rangle$

Hilbert space: Vector space with defined inner product.
Expand in orthonormal basis: $|V\rangle = \sum_i vi\,|i\rangle$
Hermitian operator: $\mathcal{O}^\dagger = \mathcal{O}$
Anti-Hermitian operator: $\mathcal{O}^\dagger = \mathcal{O}$
Unitary operator: $UU^\dagger = \mathbb{1}$
Orthogonality: $\langle i|j\rangle = \delta_{ij}$
Completeness: $\sum_i i = \mathbb{1}$
Postulates of QM:

1. The state of a physical system, at some fixed time, is given by a normalized ray in a Hilbert space over the complex numbers. (ray is vector whose norm doesn't matter)
2. The ray evolves deterministically in time according to Schrödingers equation.
3. Observables correspond to self-adjoint (hermitian) operators.
4. If a particle is in the state $|\psi\rangle$ then a measurement of $\mathcal{O}$ will yield one of the eigenvalues of $\mathcal{O}$, ω. The state of the system changes to an eigenstate of $\mathcal{O}$, $|\omega\rangle$.

Schrödinger equation: $i\hbar\frac{\partial}{\partial t}\Psi = \hat{H}\Psi$
Free particle ψ_p and E_p: $\psi_p = Ae^{ikx} + Be^{-ikx}$, $k^2 = \frac{2mE_n}{\hbar^2}$, $E_p = \frac{p^2}{2m}$
Particle in a box ψ_n and E_n: $\psi_n = \sqrt{\frac{2}{L}}\sin(k_n x)$, $k_n = \frac{n\pi}{L}$, $E_n = \frac{\hbar^2\pi^2 n^2}{2mL^2}$
Harmonic Oscillator $\hat{H}$, ψ_n and E_n: $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$, $\psi_n = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{2^n n!}H_n(x)e^{-x^2/2}$, $E_n = (n+\frac{1}{2})\hbar\omega$
Raising and lowering operators and how to affect $|n\rangle$ (3-2):

- $a = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)$, $a|n\rangle = \sqrt{n}|n-1\rangle$, $a|0\rangle = 0$
- $a^\dagger = \left(\frac{m\omega}{2\hbar}\right)^{1/2}\left(\hat{x} - \frac{i}{m\omega}\hat{p}\right)$, $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$

$\hat{H}$ in terms of a and $a^\dagger$: $\hat{H} = \hbar\omega(a^\dagger a + 1/2)$
Commutation relations for $\hat{H}$, a, $a^\dagger$:

- $[\hat{H}, a] = -a$
- $[\hat{H}, a^\dagger] = a^\dagger$
- $[a, a^\dagger] = 1$

$\mathbf{J}^2$ and J_z on the angular momentum state $|jm_j\rangle$:

- $\mathbf{J}^2|{=}\rangle\, j(j+1)\hbar^2|jm_j\rangle$
- $J_z|jm_j\rangle = m_j\hbar|jm_j\rangle$

Commutation relations for J_i and J_j and for $\mathbf{J}^2$ and J_i:

- $[J_i, J_j] = i\hbar J_k$
- $[\mathbf{J}^2, J_i] = 0$

J_z and $\mathbf{J}^2$ in position basis:

- $J_z = -i\hbar\frac{\partial}{\partial t}$
- $\mathbf{J}^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$

8 Electricity and Magnetism Equations

9 Miscellaneous Physics

Taylor Expansion: $f(\vec{x}+\vec{a}) = f(\vec{x}) + a_i\partial_i f(\vec{x}) + \mathcal{O}(\vec{a}^2)$