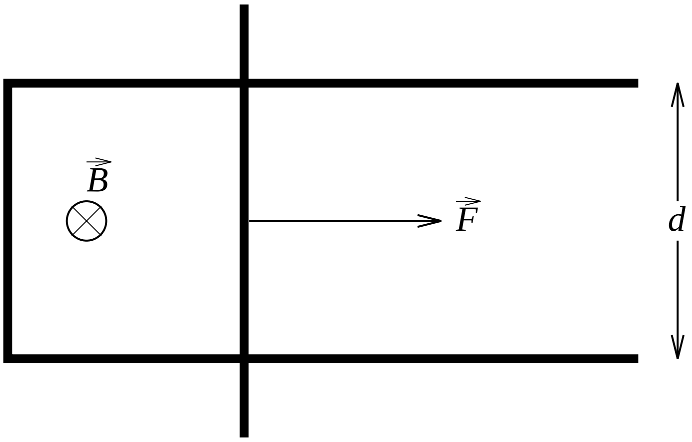


E & M Proficiency Exam, Spring 2001

Useful constants:

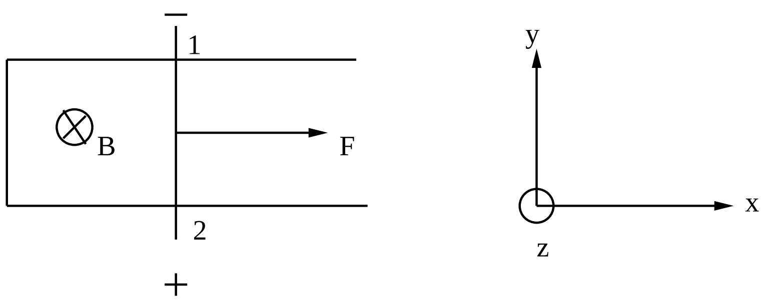
$$
\begin{aligned}
e &= 1.60 \times 10^{-19}\ \mathrm{C} \\
hc &= 1240\ \mathrm{eV} \cdot \mathrm{nm} \\
c &= 3.00 \times 10^{8}\ \frac{\mathrm{m}}{\mathrm{s}} \\
m_e &= 0.511\ \frac{\mathrm{MeV}}{\mathrm{c}^2} \\
k &= \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^{9}\ \frac{\mathrm{Nm}^2}{\mathrm{C}^2} \\
\epsilon_0 &= 8.85 \times 10^{-12}\ \frac{\mathrm{C}^2}{\mathrm{Nm}^2}
\end{aligned}
$$

1. **The two rails of a superconducting track are separated by a distance d. A conductor can slide along the track. The conductor, initially at rest, is pulled to the right by a constant force F. The friction force between the conductor and the track is directly proportional to its velocity with a proportionality constant α. The portion of the conductor between the rails has a resistance of R. The entire setup is in a uniform magnetic field $\vec{B}$ as shown in the figure. The field $\vec{B}$ points *into* the page.**

   (a) **What is the direction of the induced current in the conductor?**

   We can answer to this questions as follows. When the conductor moves to right the magnetic flux through the area enclosed by the conductor and the superconducting track increases. According to the Lenz's law a current will be induced in the closed circuit to oppose the increase of the magnetic flux. Therefore the induced magnetic field points out of paper in the closed loop. The current that produces this magnetic flux must go counterclockwise. The other way to reach this conclusion is to use the Lorentz's law. According to this law when a charged particle moves in a magnetic field a force will act on it which is given by

$$\vec{F_L} = q\,\vec{v} \times \vec{B}. \tag{1}$$



With the directions as in the figure above we get

$$\begin{aligned} \vec{F_L} &= q\,v\,B\;\hat{x} \times (-\hat{z}) \\ &= \hat{y}\,q\,v\,B. \end{aligned} \tag{2}$$

This means that free electrons in the conductor will move in the $(-\hat{y})$ direction. Therefore the induced current will be counterclockwise.

(b) Determine the magnitude of the velocity of the conductor as a function of time.

To find the velocity as a function of time we have set up the equations of motion for the conductor. There are three forces which affect the on the conductor. The first one is $\vec{F}$, since the conductor carries current in a magnetic field a force will act on it, and the third is the friction force between the track and the conductor.

$$m\frac{d\vec{v}}{dt} = \vec{F} + \vec{F_B} + \vec{F_f} \tag{3}$$

The conductor moves only in x direction. The force $\vec{F} = F\,\vec{x}$. The second force is given by

$$\begin{aligned} \frac{\vec{F_B}}{dl} &= \vec{I} \times \vec{B} \\ &= I\,B\;\hat{y} \times (-\hat{z}) \\ &= -\hat{x}\,B\,I \\ \Rightarrow \vec{F_B} &= -\hat{x}\int_0^d dl\,I\,B \\ \vec{F_B} &= -\hat{x}\,I\,B\,d. \end{aligned} \tag{4}$$

The induced current is given by

$$I = \frac{V_{ind}}{R}, \tag{5}$$

where V_{ind} is the induced electric potential. The electric potential is given by

$$\begin{aligned} V_{ind} &= \oint_C (\vec{v} \times \vec{B}) \cdot d\vec{l} \\ &= \int_1^2 dl\,v\,B\,(\hat{x} \times \hat{-z}).(-\hat{y}) \\ &= d\,v\,B. \end{aligned} \tag{6}$$

From Eqs. 4, 5 and 6 we obtain

$$\vec{F_B} = -\hat{x}\,\frac{v\,B^2\,d^2}{R}. \tag{7}$$

Using this equation the equation of the motion for the conductor is given by

$$\begin{aligned}
m\frac{dv}{dt} &= F - \frac{B^2\,d^2}{R}\,v - \alpha\,v \\
&= F - (\alpha + \frac{B^2\,d^2}{R})v \\
\Rightarrow m\,\frac{dv}{F - (\alpha + \frac{B^2\,d^2}{R})v} &= dt \\
m\int_0^v \frac{dv}{F - (\alpha + \frac{B^2\,d^2}{R})v} &= \int_0^t dt \\
\text{let}\quad C &= (\alpha + \frac{B^2\,d^2}{R})
\end{aligned}$$

$$\Rightarrow -\frac{m}{C}\,[\ln(F - C\,v)]_0^v = t - \frac{m}{C}\,\ln\left(1 - \frac{C}{F}\,v\right) = t$$

$$\Rightarrow v(t) = \frac{F}{C}\left(1 - \exp[-\frac{C}{m}\,t]\right) \tag{8}$$

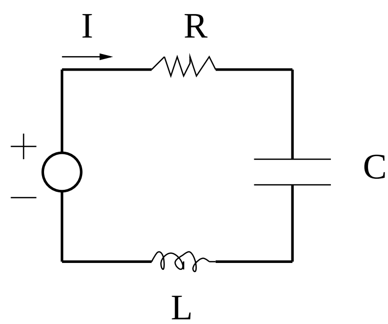
(c) Determine the magnitude of the induced current as a function of time. From Eqs. 5, 6 and 8 we get

$$I(t) = \frac{B\,d}{R}\,\frac{F}{C}\,\left(1 - \exp[-\frac{C}{m}\,t]\right). \tag{9}$$

(d) Determine the terminal velocity of the conductor.
when $t \to \infty$, from Eq.8 we get

$$v = \frac{F}{C}. \tag{10}$$

2. Consider a circuit consisting of a resistance $R = 6\ \Omega$, an inductance $L = 0.5$ H and a capacitance $C = 0.02$ F, all connected in series. A sinusoidal voltage $V(t) = V_0 \sin(\omega t)$ is applied to this LCR circuit.



(a) **What is the impedance of the circuit as a function of ω?**

To find the impedance we use phaser notation. In other words

$$\begin{aligned} I &= I_0 \exp[i\omega t + \phi] \\ V &= V_0 \exp[i\omega t + \theta] \end{aligned} \tag{1}$$

Apply KVL we obtain

$$\begin{aligned} -V + V_R + V_C + V_L &= 0 \\ V_R = R\,I &= R\,I_0 \exp[i\omega t + \phi] \\ I_C = C\,\frac{dV_C}{dt} &= i\,\omega\,C V_C \\ V_L = L\,\frac{dI_L}{dt} &= i\,\omega\,L I_L \\ \Rightarrow V &= \left[R + i\left(\omega L - \frac{1}{\omega C}\right)\right] I. \end{aligned} \tag{2}$$

From Eq.2 we get

$$\begin{aligned} Z &= R + i\left(\omega L - \frac{1}{\omega C}\right) \\ &= 6 + i\left(0.5\omega - \frac{50}{\omega}\right) \end{aligned} \tag{3}$$

(b) **Find the resonance frequency.**

Resonance frequency is defined as the frequency when impedance is real. In other words the second term in Eq.3 is equal zero.

$$\omega = \frac{1}{\sqrt{L\,C}} = 100\ (\frac{rad}{s})^2. \tag{4}$$

(c) **Assume now that $V_0 = 240$ V and $\omega = 10$ rad/s. Assume that both the current and the charge vanish at $t = 0$. Obtain the charge on the capacitor, $Q(t)$, for these initial conditions.**

We rewrite KVL in terms of Q. We obtain

$$\begin{aligned} V_R + V_C + V_L &= V \\ V_R = R\,I &= R\,\frac{dQ}{dt} \\ V_C &= \frac{Q}{R} \\ V_L = R\,\frac{dI}{dt} &= L\,\frac{d^2Q}{dt^2} \\ \Rightarrow L\,\frac{d^2Q}{dt^2} + R\,\frac{dQ}{dt} + \frac{Q}{R} &= V. \end{aligned} \tag{5}$$

Eq.5 has two solutions. One is the particular solution Q_p and the other one is the homogen solution, i.e. without any source $V = 0$, Q_h.

$$0.5\ddot{\phi}_p + 6\dot{\phi}_p + 50\phi_p = 240\ \sin(\omega t) = 240\ \text{Im}\ \exp[i\,10t]$$

$$\begin{aligned}
\text{let } \phi_p(t) &= A\ \exp[i\,10t] \\
(-50+i60+50)\phi_p(t) &= \mathrm{Im}240\ \ \exp[i\,10t] \\
\Rightarrow\ \phi_p(t) &= \mathrm{Im}\frac{240\ \ \exp[i10t]}{i60} \\
\phi_p(t) &= -\,4\ \cos(10t) \qquad (6)
\end{aligned}$$

The homogen solution is given by

$$\begin{aligned}
0.5\ddot{\phi}_h + 6\dot{\phi}_h + 50\phi_h &= 0 \\
\text{let } \phi_h(t) &\sim \exp[i\,\omega t] \\
-0.5\,\omega^2 + i\,6\,\omega + 50 &= 0 \\
\Rightarrow\ \omega_\pm &= \pm 8 + 6i \\
\Rightarrow\ \phi_h(t) &= Re[A'\,e^{i8t} + B'\,e^{-i8t}]\,e^{-6t} \\
\Rightarrow\ \phi_h(t) &= e^{-6t}\ [A\ \cos(8t) + B\ \sin(8t)] \\
\text{initial condition} & \quad Q(0)=0,\ \ \dot{Q}(0)=0 \\
Q(t) &= -\,4\ \cos(10t) + e^{-6t}\ [4\ \cos(8t) + 3\ \sin(8t)]\,. \qquad (7)
\end{aligned}$$

3. A parallel plate capacitor with capacitance C is charged to a potential difference V and is then disconnected from the charging source. The capacitor has an area A and a plate separation z.

(a) What is the force acting on the upper capacitor plate as a function of z?

The force is given by

$$F = -\frac{dW}{dz}\Big|Q = const. \qquad (1)$$

where W is the stored energy in the capacitor. W is given

$$\begin{aligned}
C &= \epsilon_0\,\frac{A}{z} \\
Q &= C\,V \\
W &= \frac{1}{2}\,\frac{Q^2}{C} \\
&= \frac{1}{2}\,\frac{Qz}{\epsilon_0\,A}. \qquad (2)
\end{aligned}$$

Substitute Eq.2 in Eq.1 we obtain

$$\begin{aligned}
F &= -\,\frac{Q^2}{2\epsilon_0\,A} \\
&= -\,\frac{\epsilon_0\,A\,V^2}{2z^2} \\
&= -\,\frac{C^2\,V^2}{2\epsilon_0\,A} \qquad (3)
\end{aligned}$$

(b) Assume now that a glass plate of the same area A completely fills the space between the plates. The glass has a dielectric constant κ. How much work is required to pull the glass plate out of the capacitor?

When the glass plate is inserted between the two plates the capacitance and the potential between the plates will be changed but the total charge is constant.

$$\begin{aligned} C' &= \kappa\,\epsilon_0\,\frac{A}{z} \\ Q &= C'\,V' \end{aligned} \tag{4}$$

When the glass plate is removed the capacitance is given

$$\begin{aligned} C &= \epsilon_0\,\frac{A}{z} \\ Q &= C\,V \end{aligned} \tag{5}$$

This means that

$$\frac{V}{V'} = \kappa. \tag{6}$$

The energy stored in the capacitance when the glass is between the plates is given by

$$\begin{aligned} W_i' &= \frac{1}{2}\,\kappa\,\epsilon\,\frac{A}{z}\,V'^2 \\ &= \frac{1}{2}\,\frac{1}{\kappa}\,\epsilon\,\frac{A}{z}\,V^2 \end{aligned} \tag{7}$$

When the glass is removed the stored energy is given by

$$W_f = \frac{1}{2}\,\epsilon\,\frac{A}{z}\,V^2. \tag{8}$$

The required energy for removing the glass plate is

$$\begin{aligned} W_{req.} &= W_f - W_i \\ &= \frac{Q^2}{2C}\left(1 - \frac{1}{\kappa}\right). \end{aligned} \tag{9}$$