E & M Proficiency Exam, Spring 2002

Useful constants:

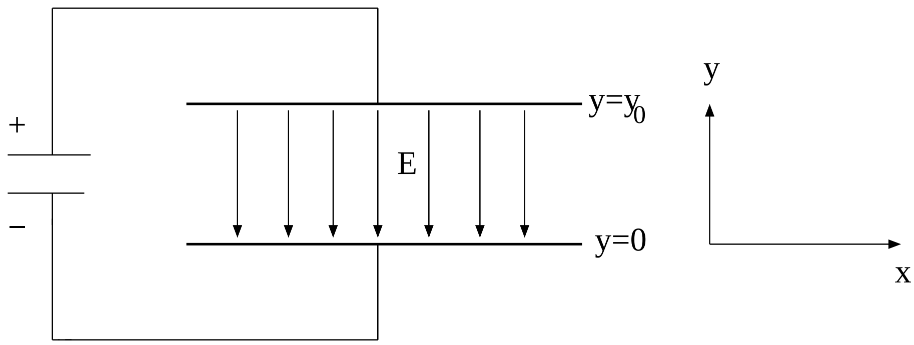
- $e = 1.60 \times 10^{-19}$ C
- $m_e = 0.511 \ \frac{\mathrm{MeV}}{\mathrm{c}^2}$
- $c = 3.00 \times 10^8 \ \frac{\mathrm{m}}{\mathrm{s}}$
- $k = \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9 \ \frac{\mathrm{Nm}^2}{\mathrm{C}^2}$
- $\epsilon_0 = 8.85 \times 10^{-12} \ \frac{\mathrm{C}^2}{\mathrm{Nm}^2}$

1. **A parallel-plate capacitor is constructed using a dielectric whose constant varies with position. The plates have area A. The bottom plate is at $y = 0$ and the top plate is at $y = y_0$. The dielectric constant is given as a function of y according to $\kappa = 1 + \frac{3}{y_0}y$.**

   (a) **What is the capacitance?** Capacitance is defined by

$$C = \frac{Q}{V}, \tag{1}$$

where Q and V are free charge and potential of the capacitor respectively. The electric field intensity points from the plate with the positive charge to the plate with the negative charge.



The potential is given by

$$\begin{aligned} V &= -\int_{y=0}^{y=y_0} d\vec{x} \cdot \vec{E} \\ &= -\int_{y=0}^{y=y_0} dy\, E. \end{aligned} \tag{2}$$

Now, we determine the electric field intensity $\vec{E}$. When an dielectric is placed in an external electric field, the external electric field polarizes the dielectric. Therefore there will be bound charges on the surface of the dielectric. In this kind of problems it is sometimes easier to work with the electric displacement vector $\vec{D}$ instead of $\vec{E}$. The reason is that the divergence of $\vec{D}$ is given by

$$\nabla \cdot \vec{D} = \sigma_f, \tag{3}$$

where σ_f is the free charge density. Because free charges only exist on the surface of the plates. The electric field will be obtained by

$$\begin{aligned}\vec{E} &= \frac{\vec{D}}{\epsilon} \\ &= \frac{\vec{D}}{\epsilon_0 \kappa},\end{aligned} \tag{4}$$

where κ is the relative permitivity of the dielectric material. Let the total surface charge density on the upper plates is σ_f then by applying Gauss's theorem we get on a small box

$$\vec{D} = -\hat{y}\,\sigma_f. \tag{5}$$

And the electric field will be

$$\vec{E} = -\hat{y}\,\frac{\sigma_f}{\epsilon_0}\,\frac{1}{\kappa(y)} \tag{6}$$

Substitute Eq.6 in Eq.2 and perform the integration. We get

$$\begin{aligned}V &= \frac{\sigma_f}{\epsilon_0}\int_{y=0}^{y=y_0} dy\,\frac{1}{\kappa(y)} \\ &= \frac{\sigma_f}{\epsilon_0}\int_{y=0}^{y=y_0} dy\,\frac{1}{(1+\frac{3}{y_0}\,y)}. \\ &= \frac{\sigma_f}{\epsilon_0}\,\frac{y_0}{3}\,\ln\left(1+\frac{3}{y_0}\,y\right) \\ &= \frac{\sigma_f}{\epsilon_0}\,\frac{y_0}{3}\ln 4\end{aligned} \tag{7}$$

The total charge is given by

$$Q = \sigma_f\,A, \quad A \text{ is the area of the plate} \tag{8}$$

Substitute Eqs.7 and 8 in Eq.1. We obtain

$$C = \frac{3A\epsilon_0}{y_0 \ln 4} \tag{9}$$

(b) Find the ratio σ_b/σ_f between the bound and free area charge densities on the the surfaces of the dielectric.

The magnitude of the electric field intensity, given by Eq.6, is given by

$$E = \frac{\sigma_f}{\epsilon_0}\,\frac{1}{\kappa(y)} \tag{10}$$

Application of the Gauss's law, very close to the surface of the dielectric on a small box, and using $\vec{E}$ instead of $\vec{D}$ results in

$$\begin{aligned}\nabla\cdot\vec{E} &= \frac{\sigma}{\epsilon_0} \\ &= \frac{\sigma_f+\sigma_b}{\epsilon_0},\end{aligned} \tag{11}$$

where σ_b is the bound charge density. Integration gives

$$E = \frac{\sigma_f + \sigma_b}{\epsilon_0} \tag{12}$$

From Eqs.10 and 12 we obtain

$$\frac{\sigma_b}{\sigma_f} = \frac{1}{\kappa(y)} - 1. \tag{13}$$

For two surfaces we get

$$\begin{aligned} y = 0 \quad &\Rightarrow \quad \frac{\sigma_b}{\sigma_f} = 0 \\ y = y_0 \quad &\Rightarrow \quad \frac{\sigma_b}{\sigma_f} = -0.75 \end{aligned} \tag{14}$$

(c) Use Gauss's law to find the induced volume charge density $\rho(y)$ within this dielectric.

To find the volume charge density inside the dielectric material we apply Gauss's law to $\vec{E}$ inside the dielectric.

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{\rho}{\epsilon_0} \\ \text{use Eq.6} \\ \frac{dE}{dy} &= -\frac{\sigma_f}{\epsilon_0} \left(\frac{3}{y_0} \frac{1}{(1+3\frac{y}{y_0})^2} \right) \\ \Rightarrow \quad \rho(y) &= \frac{3\,\sigma_f}{y_0\,(1+3\frac{y}{y_0})^2} \end{aligned} \tag{15}$$

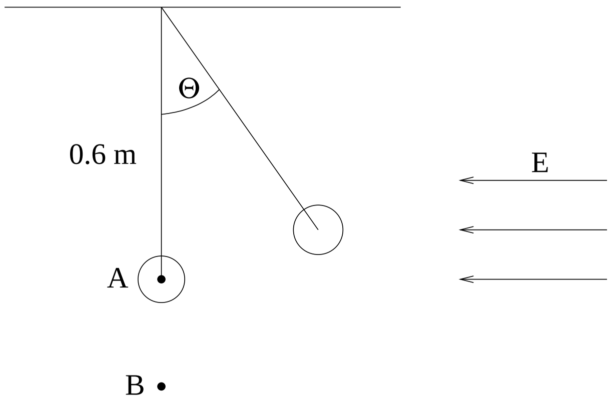
(d) Integrate the expression for the volume charge density found in (c) over the dielectric and show that the total induced bound charge, including that on the surfaces, is zero.

Integrate Eq.15 gives

$$\begin{aligned} Q_b &= \int_0^{y_0} dy\ \rho(y) \\ &= \frac{3\,\sigma_f}{y_0} \int_0^{y_0} dy\, \frac{1}{(1+3\frac{y}{y_0})^2} \\ &= -\,\sigma_f \left[\frac{1}{(1+3\frac{y}{y_0})} \right]_0^{y_0} \\ &= 0.75\,\sigma_f \end{aligned} \tag{16}$$

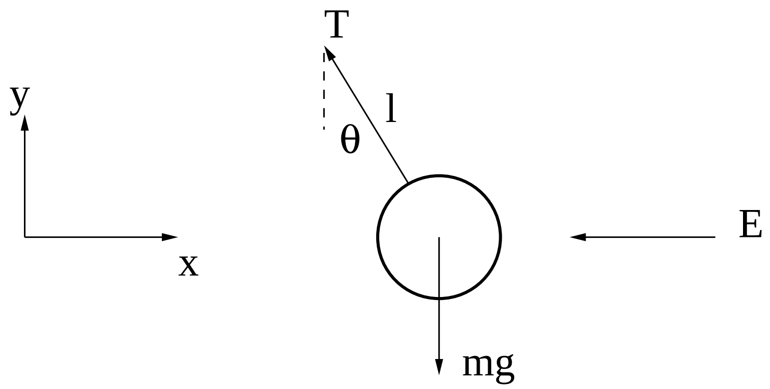
The total induced charge is given as the sum of the integral of the volume charge + the surface charges. Add Eqs.16 and 14 we get zero. Therefore the total induced charge in the dielectric material is equal zero.

2. A non-conducting sphere of mass $m = 1.3\,\mathrm{g}$ hangs vertically from a massless string of length $l = 0.6\,\mathrm{m}$. The sphere carries a charge of -3.4×10^{-4} C. A constant, uniform electric field of strength $E = 5.4$ V/m is oriented horizontally from right to left. (See the figure.)



(a) What is the equilibrium position (θ) of the sphere ?

First of all we draw free body diagram for the sphere as follows



The equations of motion for the sphere are given by

$$\begin{aligned}
\hat{x} &: \quad -T\sin(\theta) - qE = 0 \Rightarrow T\sin(\theta) = mg \\
\hat{y} &: \quad T\cos(\theta) - mg = 0 \Rightarrow T\cos(\theta) = qE \\
&\Rightarrow \tan(\theta) = -\frac{q\,E}{m\,g} \\
&\Rightarrow \tan(\theta) = 0.1441 \\
&\Rightarrow \theta = 8.2^\circ. \qquad (1)
\end{aligned}$$

(b) If the electric field is then removed, write an equation approximately describing the position of the sphere as a function of time [i.e., $\theta(t)$].

When the electric field is removed there is no force that keeps the sphere angle θ. The sphere will start a its motion whose equation is given by

$$\begin{aligned}
\hat{r} &: \quad -m\,l\,\dot{\theta}^2 = mg\cos(\theta) - T \\
\hat{\theta} &: \quad m\,l\,\ddot{\theta} = -m\,g\,\sin(theta) \qquad (2)
\end{aligned}$$

Since the sphere only moves in θ direction, Eq.3 is the equation of motion for the sphere. From Eq.2 we can calculate T.

$$\begin{aligned}
\ddot{\theta} + \frac{g}{l}\sin(\theta) &= 0 \\
\theta \text{ is small} \Rightarrow \sin(\theta) \approx \theta & \\
\ddot{\theta} + \frac{g}{l}\theta &= 0 \\
&\Rightarrow \theta(t) = A\cos(\omega t) + B\sin(\omega t),\ \omega = \sqrt{\frac{g}{l}} \\
\theta(0) = \theta_0 = 8.2^\circ,\ \dot{\theta}(0) = 0 & \\
&\Rightarrow \theta(t) = \theta_0 \cos(\omega t) \qquad (3)
\end{aligned}$$

(c) When the sphere is at point A and swinging from left to right, what is the direction of the magnetic field at point B ?

When the sphere swings from left to write it accelerates and since it has charge q, this accelerated charge produces a magnetic field. The magnetic field can be calculated using Biot-Savarts law given by

$$\vec{B} = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\vec{j(\vec{r'})} \times \vec{r}}{r^3} \qquad (4)$$

Here the sphere moves in $-\hat{\theta}$ direction. This means that current is in $-\hat{\theta}$. From this we can see that $\vec{B}$ points in $\hat{z}$ direction, out of the paper.

(d) Qualitatively describe the motion of the sphere if, instead of removing the electric field, it is quickly reduced to half its original value.

When the electric field is quickly reduced to half then the equilibrium position of the sphere will change. If we go back and replace $\vec{E}$ by $\frac{\vec{E}}{2}$ we can easily see that the equilibrium angle will be $\frac{\theta}{2}$. If we want to derive the equation of motion for the sphere in this case will be given by

$$\theta(t) = \frac{\theta_0}{2}\left[1 + \cos(\omega t)\right] \qquad (5)$$

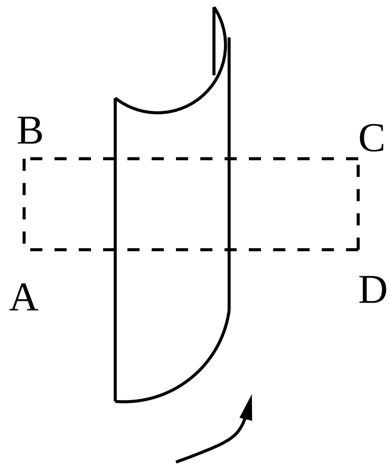
In another words the effect of the electric field is that the equilibrium position is moved.

3. **An infinitely long cylinder of radius R has a uniform charge density σ deposited on its outer surface. The charges are fixed to the surface and do not move with respect to it. However, the cylinder itself is rotating at an angular speed ω about its axis. Find the magnetic field $\vec{B}$, as a function of the distance r from the cylinder axis, both inside and outside the cylinder.**

When the cylinder rotates it produces current which move is $\hat{\theta}$ direction. This current produces magnetic field in space which we want to determine as a function of r, distance from the cylinder. To solve this types of problems when a current is given we can use the Ampere's law or the Biot-Savarts law. In this case it is easier to use the Ampere's law. This law in free space is given by

$$\oint d\vec{l} \cdot \vec{B} = \mu_0 I. \tag{1}$$

In Eq.1 we must choose a path and also calculate I before we can determine the magnetic field.



We choose the path l as in the figure above. The paths $A-B$ and $C-D$ are parallel and the paths $B-C$ and $D-A$ are perpendicular to the axis of the cylinder. The integral can be written as

$$\begin{aligned} \oint d\vec{l} \cdot \vec{B} &= \int_A^B d\vec{l} \cdot \vec{B} + \int_B^C d\vec{l} \cdot \vec{B} \\ &+ \int_C^D d\vec{l} \cdot \vec{B} + \int_C^D d\vec{l} \cdot \vec{B} \end{aligned} \tag{2}$$

The contributions from the paths $B-C$ and $D-A$ are zero. Since these paths are perpendicular to the direction of the magnetic field. The path $C-D$ can be chosen very far away from the cylinder i.e. at infinity. The field will be zero very far from the sources. The only contribution comes from the path $A-B$. The magnetic field points in the z-direction. This can be seen by using the right hand rule which says if fingers of the right hand are in the direction of the current then the magnetic field points in the direction of the thumb. Another way to see this is that the cylinder is infinite long. The current is in ϕ-direction. According to the Biot-Savarts law

$$\vec{B} = \frac{\mu_0}{4\pi} \int d^3 r' \; \frac{\vec{j(\vec{r'})} \times \vec{r}}{r^3} \tag{3}$$

where $\vec{r} = r\,\hat{r} + z\,\hat{z}$. Since the cylinder is infinite there are the same number of points with positive z as negative z coordinate. The contributions from these points cancel out each others. There are only contributions from the r coordinate do not sum to zero. This means that B field points in z-direction. This means that

$$\oint d\vec{l} \cdot \vec{B} = \int_A^B d\vec{l} \cdot \vec{B} = B\,l. \tag{4}$$

The current is given by

$$I = q\, v \tag{5}$$

The total electric charge on the cylinder is given by

$$q = 2\pi\, R\, l\, \sigma \tag{6}$$

The velocity of an infinitesimal surface area will be

$$v = R\omega. \tag{7}$$

Substitute Eqs.6 and 7 in Eq.5. It yields

$$I = 2\pi R^2 \sigma\, l. \tag{8}$$

From Eqs.4 and 8 we get

$$B = 2\pi R^2 \sigma,\ r < R. \tag{9}$$

If we choose the path completely outside the cylinder the total current passing through it will be zero. For the same reason as above the contributions from $B - C$ and $D - A$ are zero.If we again let $C - D$ be at infinity, only the contribution from $A - B$ is left which is equal zero. In other words $B = 0$. If we choose the path completely inside there will be no current. Again $B - C$ and $D - A$ give no contribution and we are left with $B_{A-B} = B_{C-D}$, which does not give any further information.