Modern Physics and Quantum Mechanics
Proficiency Exam, Spring 2002

1. Consider the Schrödinger equation for the linear harmonic oscillator,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi = E\psi\,,$$

where m is the mass of the particle and ω is the angular frequency. The wave function and energy of the ground and first excited state are given by

$$\psi_{GS}(x) = \sqrt{\frac{\alpha}{\sqrt{\pi}}}\,\exp(-\alpha^2 x^2/2)\,,\ E_{GS} = \frac{1}{2}\hbar\omega$$

$$\psi_{E1}(x) = \sqrt{\frac{\alpha}{2\sqrt{\pi}}}\,2\alpha x\,\exp(-\alpha^2 x^2/2)\,,\ E_{E1} = \frac{3}{2}\hbar\omega\,,$$

respectively, where $\alpha = \sqrt{m\omega/\hbar}$. A perturbation term $H' = -m\omega^2 x x_0$ is added to the Hamiltonian of the harmonic oscillator.

(a) Calculate the transition matrix element $\langle\psi_{GS}|H'|\psi_{E1}\rangle$.

(b) Obtain the ground state energy of the perturbed system to second order in H'.

(c) The perturbation is actually a displacement of the origin of the oscillator by x_0. Obtain the exact value of the ground-state energy by completing the square in the potential energy term of $H + H'$ and compare your result to (b).

The following integral may be useful:

$$\int_{-\infty}^{\infty} dx\, x^2 \exp(-x^2) = \sqrt{\pi}/2$$