

Notes on Siemens Ch. 3

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The Black Sphere

- Model neutron scattering from nuclei as a particle being absorbed by spherical object.
- Start by expanding an incident plane wave in terms of spherical harmonics.

$$e^{i\mathbf{k}\cdot\mathbf{r}} = \sum_l C_l Y_l^0(\theta) \quad (1)$$

$$e^{ikz} \approx \sum_l \frac{\sqrt{\pi}}{kr} \sqrt{2l+1} i^{l+1} \left(e^{-i(kr - \frac{l\pi}{2})} - e^{i(kr - \frac{l\pi}{2})} \right) Y_l^0(\theta) \quad (2)$$

- Here we have used the fact that $\mathbf{k} \cdot \mathbf{r}$ only depends on θ , and not on ϕ , thus $m = 0$. Also, we have used various identities and the orthonormality of spherical harmonics.

The Black Sphere

- Scattering only happens for short time

$$\phi(r \rightarrow \infty) = \sum_l \frac{\sqrt{\pi}}{kr} \sqrt{2l+1} i^{l+1} \left(e^{-i(kr - \frac{l\pi}{2})} - \eta_l e^{i(kr - \frac{l\pi}{2})} \right) Y_l^0(\theta) \quad (3)$$

- Scattered wave is just the total wave function minus the incident wave function, $\phi_{sct} = \phi(r \rightarrow \infty) - e^{ikz}$.

$$\phi(r \rightarrow \infty) = e^{ikz} + f(\theta) \frac{e^{ikr}}{r} \quad (4)$$

$$f(\theta) = \sum_l i \frac{\sqrt{\pi}}{k} \sqrt{2l+1} Y_l^0(\theta) (1 - \eta_l) \quad (5)$$

- This looks like a scattering amplitude, $\frac{d\sigma}{d\Omega} = |f(\theta)|^2$.

The Black Sphere

- **Approximations:** Classical turning point is where $k^2 = l(l+1)/R^2 \approx (l + \frac{1}{2})^2/R^2$. If particle passes inside the range of force (R) you get absorption ($\eta_l = 0$), but if not you get none ($\eta_l = 1$).

$$\frac{d\sigma}{d\Omega} = \frac{\pi}{k^2} \left| \sum_{l=0}^{kr-1/2} \sqrt{2l+1} Y_l^0(\theta) \right|^2 \quad (6)$$

- **More Approximations:** Here we approximate this for large and small angle scattering. I was not able to figure out the integrals so I'll just quote their answer here.

$$\frac{d\sigma}{d\Omega} \approx \begin{cases} \frac{2R}{\pi} k\theta^2 \sin\theta \cos^2(kR\theta + \frac{\pi}{4}), & \text{for } kR\theta \gg 1 \\ \frac{k^2 R^4}{4} (1 - (kR\theta/2)^2)^2, & \text{for } kR\theta \ll 1 \end{cases} \quad (7)$$

The Black Sphere

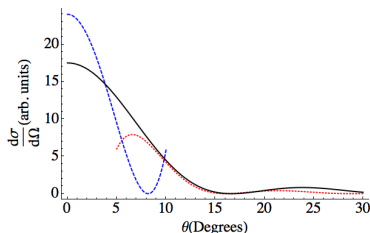


Figure: Rough reproduction of figure 3.2 in the book.

- To find the angle of minimum scattering I have taken the derivative of the high angle scattering and set it equal to zero to get.

$$\theta_{min} = \frac{\pi}{4kR}(2n - 1) \quad (8)$$

- Experiment must show that it's actually

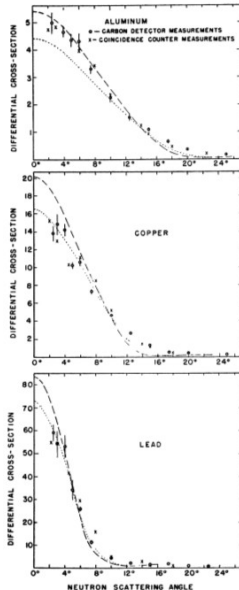
$$\theta_{min} = \frac{5\pi}{4kR} \quad (9)$$

The Black Sphere

$$\theta_{min} = \frac{5\pi}{4kR} \quad (10)$$

- Now we can use this diffraction pattern to estimate the radius of nuclei. For Pb with $\epsilon = 84$ MeV we get $k = \sqrt{2m_N\epsilon/\hbar} \approx 2.0 \text{ fm}^{-1}$. Now the graph above shows that $\theta_{min} \approx 15^\circ$. This gives us a radius of 7.5 fm.
- A quick google search gives Pb a radius of 7 fm.

Nuclear Sizes and Saturation



- You can see from this that the volume $\Omega_r \propto \theta_{min}^{-3}$, so the smaller the nuclei the bigger the scattering angles.
- Coupled with figure 3.1 in the book which shows that lighter nuclei have larger scattering angles this shows that lighter nuclei are smaller than heavier nuclei.
- In fact it turns out that

$$\Omega_r = \Omega_0 A \quad (11)$$

$$R = r_0 a^{1/3} \quad (12)$$

with $r_0 \approx 1.3$ fm and $\Omega_0 = \frac{4}{3}\pi r_0^3 \approx 9$ fm³.

Optical Model

- Compare this black sphere model to the potential model with a nice Hermitial potential

$$H = \frac{\mathbf{p}^2}{2m_N} + U(\mathbf{r}). \quad (13)$$

- Let's compare this to the black sphere model. Experiment tells us that σ_{el} and σ_{abs} should be comparable, where.

$$\sigma_{tot} = \sigma_{el} + \sigma_{abs} \quad (14)$$

$$\sigma_{el} = \int d\Omega \left(\frac{d\sigma}{d\Omega} \right)_{el} = \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) |1 - \eta_l|^2 \quad (15)$$

$$\sigma_{abs} = \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) (1 - |\eta_l|^2) \quad (16)$$

Optical Model

$$\sigma_{el} = \int d\Omega \left(\frac{d\sigma}{d\Omega} \right)_{el} = \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) |1 - \eta_l|^2 \quad (17)$$

$$\sigma_{abs} = \frac{\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) (1 - |\eta_l|^2) \quad (18)$$

- For the black sphere model ($\eta_l = 0, 1$)

$$\sigma_{el} = \sigma_{abs} = \pi R^2 \text{ or } 0 \quad (19)$$

- For the potential model

$$\sigma_{abs} = 0 \quad (20)$$

Optical Model

- However we can alter the potential model (not Hermitian anymore, losing C.M. energy particles)

$$H = \frac{\mathbf{p}^2}{2m_N} + U(\mathbf{r}) - iW(\mathbf{r}) \quad (21)$$

- Solving the Schrödinger eq. for a stream of particles of energy ϵ moving in the x direction we get

$$\phi(\mathbf{r}, t) = \text{const} \times e^{(ik - \kappa)x} e^{-i\epsilon t/\hbar} \quad (22)$$

$$\frac{\hbar^2}{2m_N}(k^2 - \kappa^2) = \epsilon - U \quad (23)$$

$$\frac{\hbar^2}{m_N}\kappa k = W \quad (24)$$

Optical Model

- Now you can look at probability density and see how fast it attenuates.

$$|\phi(\mathbf{r})|^2 \sim e^{-2\kappa x} \quad (25)$$

- This gives us a mean free path for absorption of a nucleon of (falls of by factor $1/e$)

$$\lambda = \frac{1}{2\kappa}. \quad (26)$$

- This will be used later.

Optical Model

- Now if we add in the main spin dependant effect we get the **Phenomenological Optical Model**.

$$H^{POM} = \frac{\mathbf{p}^2}{2m_H} + U(r) + \mathbf{l} \cdot \mathbf{s} U^{ls}(r) - iW(r) \quad (27)$$

- Fitting the results to various forms for these potentials shows that the following gives good results.

$$U(r) = U_0 f((r - R(A))/a_u) + U_C(r) \quad (28)$$

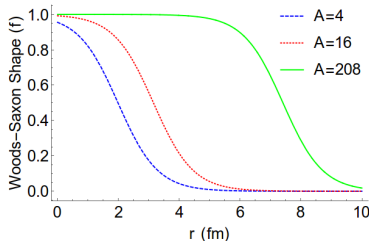
$$W(r) = \left(W_0 - 4W_1 a_W \frac{\partial}{\partial r} \right) f((r - R(A))/a_W) \quad (29)$$

$$U^{ls}(r) = U_0^{ls} \frac{1}{r} \frac{\partial}{\partial r} f((r - R(A))/a_{ls}) \quad (30)$$

Optical Model

- See the book for constants. The f is called the Woods-Saxon shape.

$$f(x) = (1 + \exp(x))^{-1}, \quad x = (r - R(A))/a_U \quad (31)$$



- The key features are that it approaches 1 inside the nucleus and falls from 0.9 to 0.1 as r varies from $R - 2.2a_U$ to $R + 2.2a_U$. Surface thickness of $4.4a_U$ or 2.9 fm, about the range of the force.

Optical Model

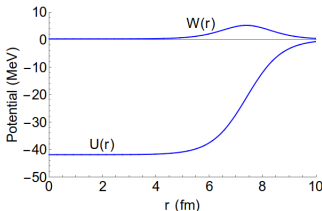
- It is interesting to note that some of these potentials depend on the energy of the incident nucleon (ϵ). The more energetic the more likely to be absorbed by exciting another nucleon.

$$W_0(\epsilon) \approx \max(0.22\epsilon - 2\text{MeV}, 0) \quad (32)$$

$$W_1(\epsilon) \approx \max \left[12\text{MeV} - 0.25\epsilon + 24\text{MeV} \cdot t_3 \frac{N - Z}{A}, 0 \right] \quad (33)$$

$$U_0(\epsilon) \approx -50\text{MeV} - 48\text{MeV} \cdot t_3 \frac{N - Z}{A} + 0.3(\epsilon - U_C(R)) \quad (34)$$

$$U_0^{ls} \approx 30\text{MeVfm}^2/\hbar^2 \quad (35)$$



Optical Model

- Riddle to be solved later. Imagine a 40 MeV neutron begin scattered. By the equations before ($\lambda = 1/2\kappa$) we get $\lambda \approx 5$ fm. However when we use the cross section to calculate it with $\sigma \approx 4\pi d\sigma/d\Omega$ we get

$$\lambda = (n\sigma)^{-1} \approx 0.4\text{fm}. \quad (36)$$