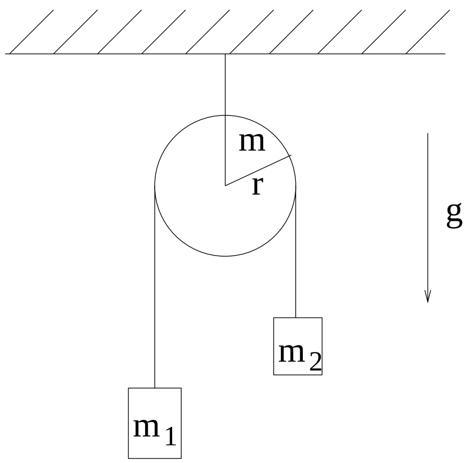
PHY6938 Mechanics Fall 99

1. The system (Atwood's machine) shown in the figure consists of two masses, M_1 and M_2, attached to the ends of a string of length l which hangs over a pulley. The pulley is a uniform disk of radius r and mass m. Assume the string is massless and does not slip on the pulley.



a) What is the moment of inertia of the pulley about its axis? (Perform a calculation to determine I; don't just write down a remembered answer).

The moment of inertia of an extended body is $I = \int dmr'^2$, where r' is the perpendicular (or closest possible) distance of an element of mass dm of the body from the rotation axis. Since the axis of the pulley goes through the center we can use plane polar coordinates to perform the integration over the area of the pulley, which has uniform density and thickness and so constant mass per unit area ρ_A. The expression for dm in plane polar coordinates is therefore

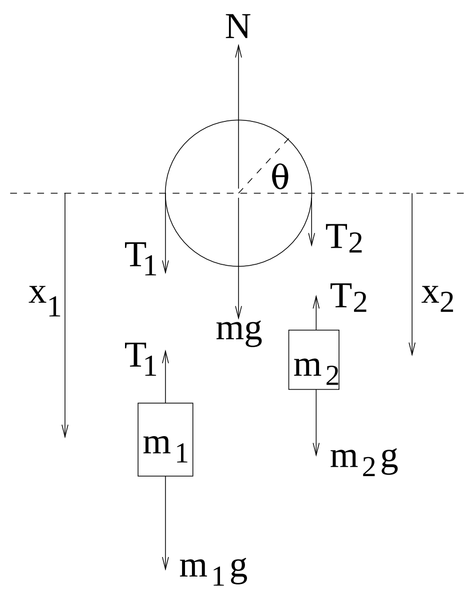
$$dm = \rho_A dA = \frac{m}{\pi r^2} \cdot r' d\theta dr',$$

so that

$$\begin{aligned} I &= \frac{m}{\pi r^2} \int_0^{2\pi} d\theta \int_0^r r'^3 dr' \\ &= \frac{2m}{r^2} \left[\frac{r'^4}{4} \right]_0^r \\ &= \frac{1}{2} m r^2. \end{aligned}$$

b) Obtain the acceleration of the masses.

Let's draw a separate free-body diagram for every moving object in the system, and define coordinates x_1, x_2 and θ to give the position of every moving object in the system:



Then the equations of motion (Newton's law) for m_1 amd m_2 are simply

$$\begin{aligned} m_1 g - T_1 &= m_1 \ddot{x}_1 && (1)\\ m_2 g - T_2 &= m_2 \ddot{x}_2. && (2) \end{aligned}$$

The equation of motion for the pulley is the angular equivalent of $\sum_i \mathbf{F}_i = m\mathbf{a}$ which is $\sum_i \boldsymbol{\tau}_i = I\boldsymbol{\alpha}$, where $\boldsymbol{\tau}$ is the torque vector and $\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}$ is the angular acceleration vector. Since in this case all torques and angular accelerations, etc., are along a single axis we can reduce this to a scalar equation if we are careful to take care of the directions of vectors with signs. The equation of motion for the pulley is, therefore,

$$rT_1 - rT_2 = I\alpha = I\ddot{\theta} = \frac{1}{2}mr^2\ddot{\theta}, \tag{3}$$

where positive rotations are treated as counterclockwise, as defined by θ in the figure.

We now have three equations and five unknowns T_1, T_2, $\ddot{x}_1$, $\ddot{x}_2$, and $\ddot{\theta}$, so we require two additional equations to proceed. The are the *constraints* imposed by the additional physics which we have not so far included; the fact that the string does not stretch, which gives that $x_1 + x_2 =$ constant, and that it doesn't slip, which gives that $x_1 = r\theta +$ constant. These translate to the constraints

$$\begin{aligned} \ddot{x}_2 &= -\ddot{x}_1 && (4)\\ \ddot{\theta} &= \ddot{x}_1/r. && (5) \end{aligned}$$

We now have five equations in five unknowns which we can solve as follows:

$$\begin{aligned} (1)-(2) &: \quad (m_1 - m_2)g - (T_1 - T_2) = m_1\ddot{x}_1 - m_2\ddot{x}_2 && (6)\\ (3)/r &: \quad T_1 - T_2 = \frac{1}{2}mr\ddot{\theta} && (7)\\ (6)+(7) &: \quad (m_1 - m_2)g = m_1\ddot{x}_1 - m_2\ddot{x}_2 + \frac{1}{2}mr\ddot{\theta} && (8)\\ \text{sub. (4) and (5)} &: \quad (m_1 - m_2)g = \left(m_1 + m_2 + \frac{1}{2}m\right)\ddot{x}_1 && (9) \end{aligned}$$

$$\ddot{x}_1 = \frac{(m_1 - m_2)g}{m_1 + m_2 + \frac{1}{2}m}, \tag{10}$$

From which $\ddot{x}_2$ and $\ddot{\theta}$ follow trivially from (4) and (5).

c) Find the tension in the string on both sides of the pulley.

From (1), (2), and (4) we have that

$$\begin{aligned} T_1 &= m_1 g - m_1 \ddot{x}_1 = \frac{(m_1^2 + m_1 m_2 + \frac{1}{2} m_1 m - m_1^2 + m_1 m_2) g}{m_1 + m_2 + \frac{1}{2} m} = m_1 g \frac{(2m_2 + \frac{1}{2} m)}{m_1 + m_2 + \frac{1}{2} m} \\ T_2 &= m_2 g + m_2 \ddot{x}_1 = \frac{(m_1 m_2 + m_2^2 + \frac{1}{2} m_2 m + m_1 m_2 - m_2^2) g}{m_1 + m_2 + \frac{1}{2} m} = m_2 g \frac{(2m_1 + \frac{1}{2} m)}{m_1 + m_2 + \frac{1}{2} m}. \end{aligned}$$

d) Write down an expression for the total energy of the system.

The system is conservative, so the total energy of the system is the same at all times, and is equal to the total potential energy at $t = 0$ since then nothing is moving. Taking the zero of potential to be at $x_1 = x_2 = 0$, we have

$$E = U_1(0) + U_2(0) = -m_1 g x_1(0) - m_2 g x_2(0),$$

where note the signs ensure that as the x_i increase the potential energy *decreases.*

2. Use the following data to estimate the ratio of the average density of the Earth and the Sun:

$\theta =$ the angular diameter of the Sun viewed from the Earth $\simeq 0.5^{\circ}$
$l =$ the length of each degree of latitude on the surface of the Earth $\simeq 110$ km
$T = 1$ year $\simeq 3 \times 10^7$ s
$g = 10$ m/s^2.

Here the only way to know how to start is to stare at the input data and use this to figure out the approach as we go along. The ultimate goal is to find the ratio of the average densities of the Earth and Sun, which means $M_{\mathrm{E}} R_{\mathrm{S}}^3 / (M_{\mathrm{S}} R_{\mathrm{E}}^3)$.

Firstly, the angular diameter θ of the Sun as viewed from the Earth gives us the ratio between the radius R_S of the Sun (needed to find its average density) and the distance r_{ES} between the Earth and the Sun,

$$\theta \text{ (deg)} = \frac{360}{2\pi} \frac{2R_{\mathrm{S}}}{r_{\mathrm{ES}}},$$

where we have used the small angle formula to find the angular width in radians and converted to degrees with the first factor, and so

$$\frac{R_S}{r_{\mathrm{ES}}} = \frac{\pi \theta}{360}. \tag{11}$$

The length l of a degree of latitude on the surface of the Earth gives us the radius R_{E} of the Earth,

$$l = R_{\mathrm{E}} \cdot 1 \cdot \frac{2\pi}{360},$$

where the last factor converts from degrees to radians, so

$$R_E = \frac{360}{2\pi}\, l \tag{12}$$

The orbital period is independent of the Earth's mass to a very good approximation and can tell us about the Sun's mass. Assume a circular orbit (another good approximation) and use that the centripetal acceleration $\omega^2 r_{\mathrm{ES}}$, where $\omega = 2\pi/T$ is the orbital angular frequency of the Earth, is provided by the gravitational field of the sun

$$g_{\mathrm{S}} = \frac{GM_{\mathrm{S}}}{r_{\mathrm{ES}}^2},$$

so that

$$\frac{GM_{\mathrm{S}}}{r_{\mathrm{ES}}^2} = \omega^2 r_{\mathrm{ES}} = \frac{(2\pi)^2 r_{\mathrm{ES}}}{T^2}$$

which can be solved for M_{S}

$$M_{\mathrm{S}} = \frac{4\pi^2 r_{\mathrm{ES}}^3}{GT^2}. \tag{13}$$

Exactly the same approach can be used to relate the gravitational acceleration at the surface of the earth, g, to its mass and radius

$$g = \frac{GM_{\mathrm{E}}}{R_{\mathrm{E}}^2}$$

so that

$$M_{\mathrm{E}} = \frac{gR_{\mathrm{E}}^2}{G}. \tag{14}$$

Putting (2) and (4) and together we have

$$\frac{M_{\mathrm{E}}}{R_{\mathrm{E}}^3} = \frac{g}{GR_{\mathrm{E}}} = \frac{2\pi g}{360\, G\, l},$$

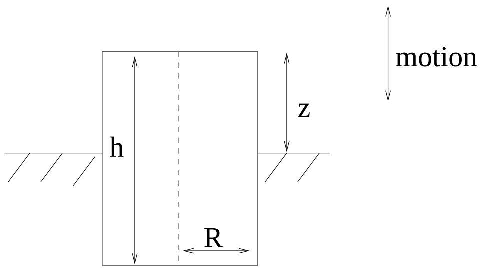
and putting (1) and (3) together we have

$$\frac{M_{\mathrm{S}}}{R_{\mathrm{S}}^3} = \frac{4\pi^2}{GT^2}\left(\frac{360}{\pi\theta}\right)^3$$

so, *finally*, we have

$$\begin{aligned}
\frac{M_{\mathrm{E}}/R_{\mathrm{E}}^3}{M_{\mathrm{S}}/R_{\mathrm{S}}^3} &= \frac{2\pi g}{360\, G\, l}\frac{GT^2}{4\pi^2}\left(\frac{\pi\theta}{360}\right)^3 \\
&= \frac{\pi^2 g T^2 \theta^3}{2(360)^4\, l} \\
&= \frac{\pi^2 \cdot 10 \cdot (3\cdot 10^7)^2 \cdot (0.5)^3\ (\mathrm{m/s})^2 \cdot \mathrm{s}^2}{2\cdot(360)^4 \cdot 110 \cdot 10^3\ \mathrm{m}} \\
&= 3.00
\end{aligned}$$

3. A cylindrical block of wood of density ρ_w, radius R, and height h is partially immersed in a liquid of density ρ_l and then released, as shown in the figure.

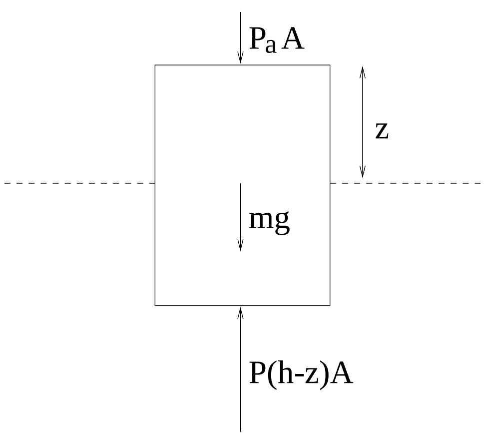


a) What is the condition for the block to keep afloat?

The block will float as long as its density does not exceed that of the liquid, $\rho_{\mathrm{w}} \leq \rho_l$.

b) What is the equilibrium height of the block above the water level z_{eq}?

At equilibrium the block is not moving so this is a statics problem, where the condition that the vector sum of the external forces on the block is zero gives us what we want. Let's draw a free-body diagram for the block:



Here we have not included any pressure forces on the sides of the block, which cancel by symmetry, and we have included the force due to the atmospheric pressure P_{a} on the top and the force due to the liquid's pressure at the bottom of the block, $P(h-z)$. Here m is the mass of the block. The pressure a distance $h-z$ below the surface of the liquid is

$$P(h-z) = P_{\mathrm{a}} + \rho_l(h-z),$$

as can be seen by examining the additional force on an object of cross-sectional area A due to the column of water of volume $A(h-z)$ above it, and then dividing by the area A. This means that the buoyant force due to the liquid, which is the vector sum of the forces due to the air and the water, is

$$F_l = [P(h-z) - P_{\mathrm{a}}]A = \rho_l(h-z)Ag.$$

Therefore $mg = \rho_l(h - z_{\mathrm{eq}})A$, which using $m = \rho_{\mathrm{w}}Ah$ gives

$$\rho_{\mathrm{w}}Ahg \quad = \quad \rho_l(h - z_{\mathrm{eq}})Ag$$

$$
\begin{aligned}
h - z_{\text{eq}} &= \frac{\rho_{\text{w}}}{\rho_l} h \\
z_{\text{eq}} = h(1 - \frac{\rho_{\text{w}}}{\rho_l}).
\end{aligned}
$$

Note as ρ_{w} approaches ρ_l, we have that z_{eq} approaches zero, as it should.

c) If the block was initially slightly raised, so that $z(t = 0) > z_{\text{eq}}$, and then released, calculate $z(t)$ assuming no friction.

Since the system will be near equilibrium and *all* mechanical systems near their equilibrium are harmonic oscillators, then we know the system is a frictionless harmonic oscillator. We know the amplitude is $z(t = 0)$, the initial velocity (zero), and the mass m, and so all we know to completely specify the motion is the oscillator constant k. Once we have k we can find the frequency ω_0 of oscillation.

To find k we need to find the restoring force $F = -k\Delta z$ on the block when it is raised a distance Δz. This force is due to the fact that the buoyant force is lowered when the block is raised, to

$$F_l = P(h - [z + \Delta z])A = \rho_l(h - [z + \Delta z])Ag,$$

so that the buoyant force is reduced by

$$\Delta F_l = \rho_l \Delta z A g$$

so that

$$k = \frac{\Delta F_l}{\Delta z} = \rho_l A g,$$

and

$$\omega_0^2 = \frac{k}{m} = \frac{\rho_l A g}{\rho_{\text{w}} A h} = \frac{\rho_l}{\rho_{\text{w}}} \frac{g}{h}.$$

Therefore the motion of the block is simple harmonic motion with this frequency

$$z(t) = z(0)\cos(\omega_0 t).$$

d) Now assume that the liquid is very viscous, and that the viscous force is proportional to the velocity, as given by $\mathbf{F}_v = -b\mathbf{v}$. How is the motion of the block modified?

Without knowing the value of b we cannot say whether the motion is underdamped, critically damped, or overdamped. The equation of motion of the oscillator now becomes

$$
\begin{aligned}
m\ddot{z} &= -b\dot{z} - m\omega_0^2 z \\
\ddot{z} + 2\left(\frac{b}{2m}\right)\dot{z} + \omega_0^2 z &= 0 \\
\ddot{z} + 2\beta\dot{z} + \omega_0^2 z &= 0,
\end{aligned}
$$

where we have used the usual definition $\beta \equiv \frac{b}{2m}$.

If $\beta < \omega_0$ the motion is underdamped and the system oscillates with frequency $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$ and with exponentially decreasing amplitude $z(t = 0)e^{-\beta t}$.

If $\beta = \omega_0$ the motion is critically damped and the system does not oscillate but has the form $(A + Bt)e^{-\beta t}$, where A and B are amplitudes fixed by the initial conditions.

If $\beta > \omega_0$, the motion is overdamped and the system does not oscillate but has the form $e^{-\beta t}[Ae^{-\omega_2 t} + Be^{\omega_2 t}]$ where $\omega_2 = \sqrt{\beta^2 - \omega_0^2}$ and again A and B are amplitudes fixed by the initial conditions.

e) What is the condition for the oscillatory motion to be over-damped?

From the above we have

$$\begin{aligned} \beta &> \omega_0 \\ \frac{b}{2m} &> \sqrt{\frac{k}{m}} \\ b &> 2\sqrt{km}. \end{aligned}$$