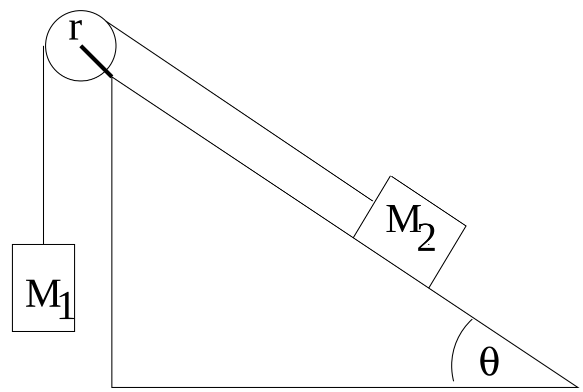
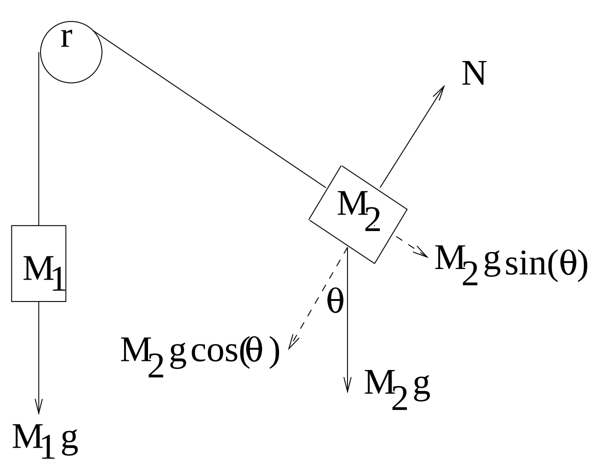
PHY6938 Mechanics Fall 2000
January 19, 2001

**1. In the figure, the fixed incline is frictionless and the massless string (which cannot be stretched) passes through the center of mass of each block. The pulley has moment of inertia $I$ and radius $r$, the two blocks have mass $m_1$ and $m_2$, respectively, and the angle of the incline is $\theta$.**

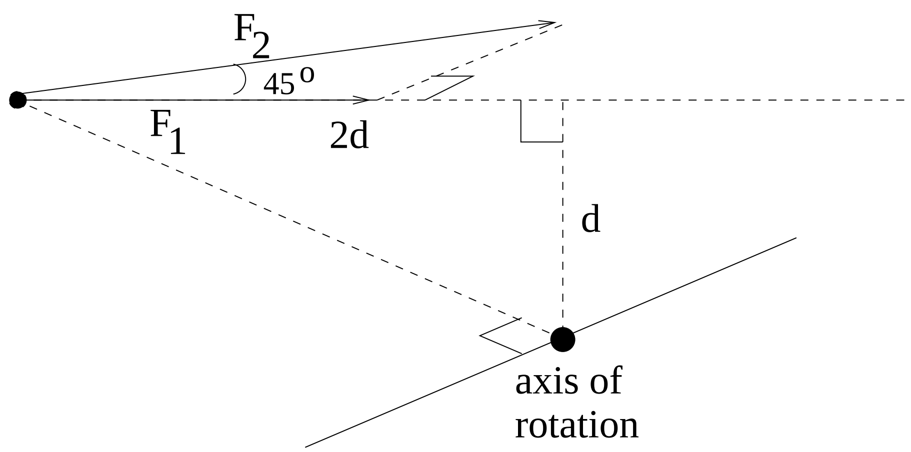



**a) Find the net torque acting on the system (including the two masses, string, and pulley) about the center of the pulley.**

It is somewhat tricky to know what is meant by this part of the question. This means find the torque on the entire system of masses, string and pulley due to the external forces on it. Because of Newton's third law, there can be no net torque on a system due to internal forces such as the forces on $M_1$, or $M_2$, or on the pulley, due to the tension in the string. So any net torques on the system have to come from the external forces, which in this case are simply the gravitational forces on the masses $M_1$ and $M_2$. There is also a normal force on the second block which cancels the component of the gravitational force on it into the plane, so we can consider the *net* external force on $M_2$ to be the component $M_2 g \sin(\theta)$ of the gravitational force on it along the string, as in the free-body diagram below.



These torques can be calculated using the usual $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{f}$ rule, or we can use a trick which engineers are very familiar with, which is that the torque due to a force applied along a line (which cannot be through the rotation axis if there is to be a torque) is equal to the component of that force in a plane perpendicular to the rotation axis, multiplied by the shortest distance of the line along which the force acts from the rotation axis. If you are not convinced that this is true, find the torque due to the forces in the following diagram by using the cross product and then by using this technique, and you should get the same answer.




This means that the torque due to the external gravitational force on mass $M_1$ is just $M_1gr$, and the torque due to the external gravitational force on mass $M_2$ is just $M_2g\sin(\theta)$, the component of $M_2g$ along the string, times $r$, i.e. $M_2g\sin(\theta)gr$. Note that the component $M_2g\cos(\theta)$ acting into the plane is cancelled by the normal force out of the plane and so cannot provide a torque on the system. Note also that these two torques act in opposite directions, and if we take counterclockwise rotation of the pulley as being in the positive direction, the net torque is

$$\tau = M_1gr - M_2g\sin(\theta)r, \tag{1}$$

and is directed out of the page.

**b) Write an expression for the total angular momentum of the system about the center of the pulley when the blocks are moving with speed $v$.**

We can use exactly the same technique to find the angular momentum of the components of the system around the rotation axis. If the string is moving with speed $v$ the pulley is rotating with angular speed $\omega = v/r$ and so its angular momentum is

$$L_{\text{pulley}} = I\left(\frac{v}{r}\right), \tag{2}$$

and that of the two blocks (all of these angular momenta point in the same direction, positive if we assume $M_1$ falls)

$$L_{M_1} = rM_1v, \quad L_{M_2} = rM_2v. \tag{3}$$

So the total angular momentum is

$$L_{\text{tot}} = v\left[(M_1 + M_2)r + \frac{I}{r}\right]. \tag{4}$$

**c) Find the acceleration of the blocks, using your results from parts a and b.**

Using $\tau = dL/dt$ and our results from **a)** and **b)** we have

$$\tau = \frac{dL}{dt} = \dot{v}\left[(M_1 + M_2)r + I/r\right] = \left[M_1 - M_2\sin(\theta)\right]gr, \tag{5}$$

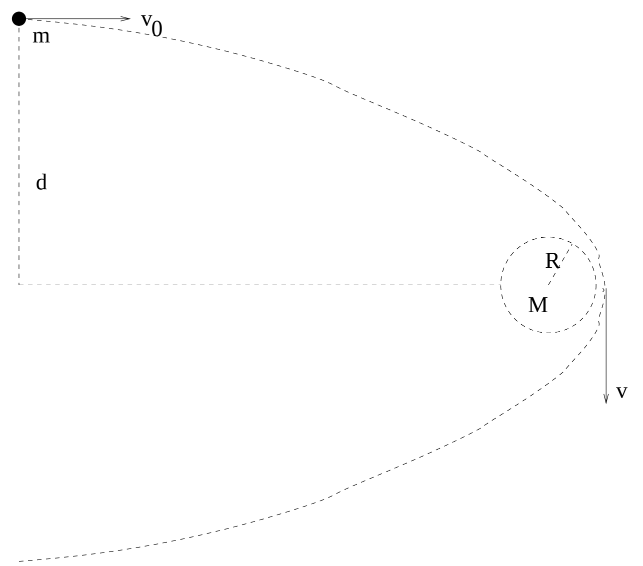
so that

$$\dot{v} = \frac{\left[M_1 - M_2\sin(\theta)\right]gr}{(M_1 + M_2)r + I/r} = \frac{\left[M_1 - M_2\sin(\theta)\right]g}{(M_1 + M_2) + I/r^2}. \tag{6}$$

**2. An asteroid is moving toward a planet from infinity with initial speed $v_0$. It its trajectory were not deflected by the planet's gravitational field, it would pass by the planet with a minimum distance $d$ from the center of the planet. The planet has a radius $R$ ($R < d$) and mass $M$. The gravitational constant is $G$. Calculate the minimum value of $v_0$ in order for the asteroid to not hit the planet's surface.**

It might be possible to solve this problem using Kepler's laws or by directly solving the equations of motion, but this is a daunting task. Almost always problems of this kind (this is almost universally true in Mechanics) are best solved by using conservation laws. So we have to recall the conserved quantities for this kind of 'planetary' motion. These are the total energy (kinetic plus gravitational potential), and the angular momentum. Using conservation of both of these quantities, and thinking hard about how the asteroid moves relative to the planet, we can easily solve this problem.

What will the trajectory of the asteroid be if it has much more energy than the minimum required to miss the planet? It has positive energy since at infinity, where its potential energy $U = 0$, it has a non-zero kinetic energy $T = mv_0^2/2$. We know that the path of a particle with positive energy in a gravitational field is a hyperbola. This hyperbola will be close to a straight line in the limit of a lot of kinetic energy, and as the kinetic energy is decreased it will start to wrap a little around the planet, with the distance of closest approach decreasing from the straight-line limit of $d$. When we reach the smallest kinetic energy which will prevent a collision with the planet, by symmetry the path of the asteroid will have to be the hyperbola shown in the figure.




When the asteroid is a long way from the planet its angular momentum, found by the method introduced in the previous question, is

$$l_0 = mv_0 d, \tag{7}$$

and its energy is just $E_0 = mv_0^2/2$. When it is at its distance of closest approach (the other point on the path where it is simple to calculate the angular momentum and energy!) it has

$$l = mvR, \quad E = \frac{1}{2}mv^2 - \frac{GmM}{R}. \tag{8}$$

We now have two equations given by our two conservation principles, which we can use to eliminate $v$ and find $v_0$ in terms of the other given quantities. The angular momentum condition tells us that

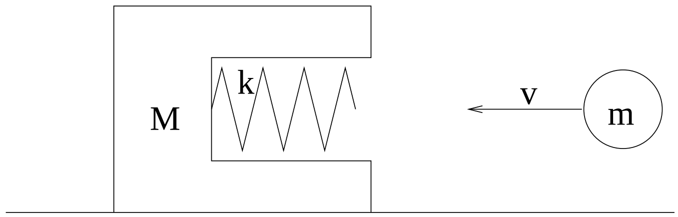
$$l_0 = mv_0 d = l = mvR, \quad v = v_0\frac{d}{R}. \tag{9}$$

Substitute the linear relationship (the angular momentum) into the quadratic one (and **not** the other way around!) to find

$$\begin{aligned} \frac{1}{2}mv_0^2 &= \frac{1}{2}m\left(v_0^2\frac{d^2}{R^2}\right) - \frac{GmM}{R} \\ v_0^2\left(\frac{d^2}{R^2} - 1\right) &= \frac{2MG}{R} \end{aligned}$$

$$v_0^2 = \frac{2MGR}{d^2 - R^2}$$

**3. A small ball of mass $m$ is fired into an opening in a block of mass $M$ with a horizontal velocity of $v$. Inside the opening is a spring with spring constant $k$ attached to the block, as shown in the diagram. The block is initially at rest and it can slide along the floor without friction. Determine the maximum compression of the spring, $\Delta x$.**




The way to solve problems involving collisions of this sort is to exploit the conservation of momentum, and (sometimes also) energy. We are instructed in this case to assume there is no friction between the block and the floor, and it is safe to assume also that there are no energy losses when the ball compresses the spring, since we are not instructed otherwise. As a result we can assume that energy is conserved in the collision.

The key to solving this problem is to ask ourselves what the state of the system is when the spring has reached its point of maximum compression. At this point both the ball and the block are moving, since the compressed spring has had time to act with a forward pointing force (to the left in the diagram) on the block. However, when the spring is at its point of maximum compression it is not moving **relative to the block**. This means that we can calculate the momentum and kinetic energy of the block and ball using just one velocity, call it $v_{\mathrm{f}}$.

We can find this velocity using conservation of momentum. Note that since some of the kinetic energy is converted to energy stored in the spring, we will not use conservation of energy to find this velocity, but we will use it to find the amount by which the spring is compressed.

From conservation of momentum we have

$$p_{\mathrm{f}} = (M+m)v_{\mathrm{f}} = mv \tag{10}$$

so that

$$v_{\mathrm{f}} = \frac{mv}{M+m}. \tag{11}$$

We can now substitute this into our condition for the conservation of energy, which is that the initial kinetic energy is equal to the final plus the energy stored in the spring,

$$\begin{aligned}
\frac{1}{2}mv^2 &= \frac{1}{2}(M+m)v_{\mathrm{f}}^2 + \frac{1}{2}k(\Delta x)^2 \\
(\Delta x)^2 &= \frac{mv^2 - (M+m)v_{\mathrm{f}}^2}{k} \\
&= \frac{v^2}{k}\left(m - \frac{m^2}{M+m}\right) \\
&= \frac{v^2}{k(M+m)} \cdot Mm \\
&= \frac{Mmv^2}{k(M+m)}
\end{aligned}$$