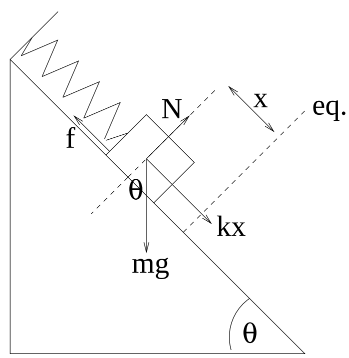
1. A block of mass m rests on an inclined plane which forms an angle of θ with the horizontal. The block is prevented from sliding down by being attached to a spring with spring constant k. The block is displaced upwards (along the plane) by a distance d, and then released at time $t = 0$.

a) Calculate the displacement of the block at time $t > 0$, ignoring friction.

Since we are not given the equilibrium length of the spring, we can safely assume it is zero. The following diagram shows the forces on the block after it is displaced by a distance d up the plane:



We can assume that the displacement of the block is from its *equilibrium* position, which means that it will oscillate around that equilibrium position due to the restoring force of the spring. Note that the component of the gravitational force along the plane serves to extend the spring to establish the equilibrium position of the block, and so if we refer all positions to this equilibrium position then it does not enter into equations of motion. This means that the block simply oscillates around the equilibrium position with a displacement d, so that

$$x(t) = d\cos(\omega\, t),$$

where $\omega = \sqrt{k/m}$.

b) How much work is required to perform the initial displacement if the coefficients of static and kinetic friction are μ_s and μ_k, respectively?

No work is done against static friction as this only applies when the block is stationary, so there can be no displacement. The work done against kinetic friction is just the magnitude $\mu_k N = \mu_k mg\cos(\theta)$ of the frictional force times the displacement d,

$$W_{\rm f} = \mu_k mgd\cos(\theta),$$

and the work done to compress the spring is $W_{\rm spr} = kd^2/2$, so that

$$W = W_{\rm f} + W_{\rm spr} = \mu_k mgd\cos(\theta) + kd^2/2.$$

c) What is the minimum angle θ, such that the block will start to move after the displacement?

Now we have to balance the frictional force, which always opposes the applied force and so points up the plane, and the force from the spring. Note that, again, the gravitational force is balanced at the equilibrium point and so does not enter into the equations. This means that

$$kd = f \leq \mu_s N = \mu_s mg\cos(\theta),$$

and the minimum angle is attained when f reaches its maximum value and

$$kd = \mu_s mg\cos(\theta),$$

so that

$$\theta = \cos^{-1}\left(\frac{kd}{\mu_s mg}\right).$$

d) What is the minimum displacement d, such that the block will start to move, independent of the angle θ?

The largest value the frictional force can have is when the plane is horizontal and $f = \mu_s N = \mu_s mg$. If d is this large or larger then the block will always move regardless of the angle θ. In this case $kd_{\text{min}} = \mu_s mg$ and

$$d_{\text{min}} = \frac{\mu_s mg}{k}.$$

2. A projectile of mass m is fired at an angle θ above the horizontal, with an initial velocity v_0. At the highest point of the trajectory, the projectile explodes into two fragments of equal mass. One of the fragments falls vertically with zero initial speed, following the explosion.

a) How far from the point of firing does the other fragment strike the level terrain?

Firstly, the vertical component of the projectile's velocity is $v_{oy} = v_0\sin(\theta)$, and so with a constant gravitational acceleration $-g$ in the y direction its vertical speed goes to zero at a time $t = v_0\sin(\theta)/g$. Since the projectile's constant horizontal velocity is $v_{0x} = v_0\cos(\theta)$, during that time it has traveled

$$x(t) = v_{0x}t = \frac{v_0^2}{g}\cos(\theta)\sin(\theta).$$

If one of the fragments falls straight down it can carry none of the projectile's initial horizontal momentum, which must be carried entirely by the second fragment, so that from conservation of momentum the horizontal component $v_{2x}(t)$ of its velocity at time t must be given by

$$\frac{m}{2}v_{2x}(t) = mv_{0x},$$

so that

$$v_{2x}(t) = 2v_{0x}.$$

Since the projectile was not moving vertically when it exploded and the first fragment is not moving vertically after the explosion, we know from conservation of momentum in the vertical direction that the second fragment cannot be moving in the vertical direction either. This means that it also simply free falls [while moving along the x direction with constant velocity $v2x(t)$], and will take the same time t to fall as the projectile took to rise. This means that it will fall a distance $v_{2x}(t) \times t$ away from the explosion, and its final distance from the point of firing is

$$\Delta x = v_{0x}t + 2v_{0x}t = 3v_{0x}t = 3\frac{v_0^2}{g}\cos(\theta)\sin(\theta).$$

b) How much energy was released during the explosion? (Assume that the energy loss in the form of heat and sound may be ignored.)

Before the explosion the kinetic energy of the projectile was $T_\mathrm{i} = mv_{0x}^2/2$, while after the explosion the kinetic energy of the first fragment was zero, and that of the second fragment is

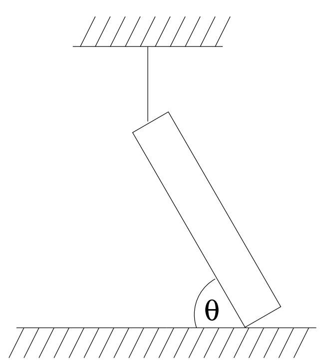
$$T_2 = \frac{1}{2}\frac{m}{2}(2v_{0x})^2 = mv_{0x}^2.$$

This means that an amount

$$Q = T_\mathrm{f} - T_\mathrm{i} = T_2 - T_\mathrm{i} = \frac{1}{2}mv_{0x}^2 = \frac{1}{2}mv_0^2\cos^2(\theta)$$

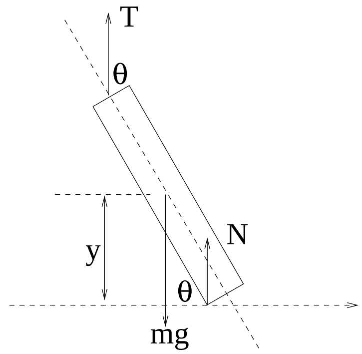
of energy is released.

3. A uniform rod of mass m and length L is suspended on one end by a string, while the other end rests on the ground, such that it forms an angle θ with the horizontal.



a) What is the tension in the string?

Let's draw a free-body diagram for the rod.



To find the tension in the string we have to balance the vertical and horizontal components of the forces on the rod, and the torques on the rod about its center of mass. The vertical equation is

$$T + N = mg \tag{1}$$

and there are no horizontal components if we ignore friction, which we must do in the absence of information about the coefficient of friction. The torque equation is (take counterclockwise as positive)

$$\frac{L}{2} N \sin(90^\circ - \theta) - \frac{L}{2} T \sin(90^\circ - \theta) = 0,$$

so that $N = T$ and from (1) we have that $N = T = mg/2$. Note we also have to assume the rod is thin compared to its length (unlike in the diagram) in the absence of information about its diameter.

b) Now imagine the string is cut, and the rod starts to fall. Calculate the vertical component of the acceleration of the center of mass of the rod, at the moment $t = 0$ immediately following the cut. Ignore friction with the ground.

Even though we are looking immediately after the string is cut, the tension force is gone so that the normal force is going to be different than it was in part a). If there is no friction there are still no horizontal components to the forces and so the center of mass of the rod simply falls straight down. The fact that it does so allows us to write *two* equations for the motion, even though it takes place in only one dimension, because the angle through which the rod rotates, which is related to the net torque, is related to the distance through which it has fallen by an equation of constraint. First deal with the vertical equation of motion,

$$m\ddot{y} = N - mg,$$

where y is the height of the center of mass. We can set up the equation of motion for the state of rotation around any point, so choose the point of contact with the ground. Then the equation of motion of the state of rotation is

$$I_{\text{end}}\ddot{\theta} = \frac{L}{2} mg \sin(90^\circ + \theta) = -\frac{L}{2} mg \cos(\theta),$$

where note the sign is negative because the torque reduces θ.

We need two things to proceed: I_{end} for the rod, and the relation between y and θ. The first is given by

$$I_{\text{end}} \;=\; \int_0^L dm\, x^2$$

$$
\begin{aligned}
&= \int_0^L \rho\, dx\, x^2 \\
&= \frac{m}{L}\int_0^L dx\, x^2 \\
&= \frac{m}{L}\left[\frac{x^3}{3}\right]_0^L = \frac{m}{L}\frac{L^3}{3} = \frac{mL^2}{3},
\end{aligned}
$$

and the second is given by the relation from the triangle made by the rod, the ground and y,

$$
y = \frac{L}{2}\sin(\theta).
$$

We need to convert this to an equation relating $\ddot{y}$ and $\ddot{\theta}$, which is found by differentiating twice

$$
\begin{aligned}
\dot{y} &= \frac{L}{2}\cos(\theta)\,\dot{\theta} \\
\ddot{y} &= \frac{L}{2}\left[\cos(\theta)\,\ddot{\theta} - \dot{\theta}\sin(\theta)\,\dot{\theta}\right] \\
\ddot{y}(0) &= \frac{L}{2}\cos(\theta)\,\ddot{\theta}
\end{aligned}
$$

where we have used that at $t = 0$ we have $\dot{\theta} = 0$.

Our three equations in the three unknowns $\ddot{y}$, $\ddot{\theta}$, and N are now

$$
\begin{aligned}
m\ddot{y} &= N - mg && (2) \\
\frac{mL^2}{3}\ddot{\theta} &= -\frac{L}{2}mg\cos(\theta) && (3) \\
\ddot{y} &= \frac{L}{2}\cos(\theta)\,\ddot{\theta}. && (4)
\end{aligned}
$$

Substituting (4) into (3) we find:

$$
\begin{aligned}
\frac{mL^2}{3}\frac{2\ddot{y}}{L\cos(\theta)} &= -\frac{L}{2}mg\cos(\theta) \\
\ddot{y} &= -\frac{3}{4}g\cos^2(\theta).
\end{aligned}
$$

c) What is the force F_N that the rod exerts on the ground at time $t = 0$, immediately after the cut? (*Hint*: It is different than before the string was cut.)

We can find N from (2) using $\ddot{y}$, so that

$$
N = mg + m\ddot{y} = mg\left[1 - \frac{3}{4}\cos^2(\theta)\right].
$$

4) A fountain designed to spray a column of water 12 m into the air has a 1 cm diameter nozzle at ground level. The water pump is 3 m below the ground. The

pipe to the nozzle has a diameter of 2 cm. What pump pressure is required? $P_{\text{atm}} = 101$ kPa. (Neglect the viscosity of the water).

Firstly, we need the state of the fluid in four distinct regions: at the water pump (1), just before the nozzle (2), just outside the nozzle (3), and at the top the column of water. To solve this we need a relation which relates the velocity of an incompressible fluid, its pressure in two places, and the height difference between those two places. This is Bernoulli's equation, which is a statement of conservation of energy in the fluid,

$$P_1 + \rho g y_1 + \frac{1}{2}\rho v_1^2 = P_2 + \rho g y_2 + \frac{1}{2}\rho v_2^2.$$

We also need the equation of continuity of the fluid, which is a statement of the lack of sources or sinks of fluid,

$$v_1 A_1 = v_2 A_2,$$

where, for example, A_1 is the cross sectional area of the pipe at position (1).

(1): At the water pump we have P_1, $y_1 = -3$ m, v_1.

(2) Just before the nozzle we have P_2, $y_2 = 0$, and $v_2 = v_1$.

(3) Just after the nozzle we have $P_3 = P_{\text{atm}}$, $y_3 = 0$, and $v_3 = 4v_2 = 4v_1$, because the cross sectional area went down by four going through the nozzle.

(4) At the top of the column of water we have $P_4 = P_{\text{atm}}$, $y_4 = 12$ m, and $v_4 = 0$.

Use Bernoulli's equation to work backwards from the top of the column of water,

$$\begin{aligned}
P_4 + \rho g y_4 + \frac{1}{2}\rho v_4^2 &= P_3 + \rho g y_3 + \frac{1}{2}\rho v_3^2 \\
\rho g y_4 &= \frac{1}{2}\rho v_3^2 \\
v_3^2 &= 2 g y_4,
\end{aligned}$$

which is just the speed required to project any object up a height y_4. Working backwards we have

$$\begin{aligned}
P_2 + \rho g y_2 + \frac{1}{2}\rho v_2^2 &= P_3 + \rho g y_3 + \frac{1}{2}\rho v_3^2 \\
P_2 + \frac{1}{2}\rho \left(\frac{v_3}{4}\right)^2 &= P_3 + \frac{1}{2}\rho v_3^2 \\
P_2 &= P_{\text{atm}} + \frac{1}{2}\rho\left(\frac{15}{16}\right) v_3^2 = P_{\text{atm}} + \left(\frac{15}{16}\right)\rho g y_4.
\end{aligned}$$

Finally, we can work backwards to P_1 by simply adding the pressure due to the 3 m column of water since $v_1 = v_2$,

$$\begin{aligned}
P_1 + \rho g y_1 + \frac{1}{2}\rho v_1^2 &= P_2 + \rho g y_2 + \frac{1}{2}\rho v_2^2 \\
P_1 + \rho g y_1 &= P_2 + \rho g y_2 \\
P_1 &= P_2 + \rho g (y_2 - y_1) = P_{\text{atm}} + \rho g\left[\left(\frac{15}{16}\right) y_4 + (y_2 - y_1)\right] \\
P_1 &= 101 \text{ kPa} + 1000\,\frac{\text{kg}}{\text{m}^3} \cdot 9.8\,\frac{m}{s^2}\left[\left(\frac{15}{16}\right)\cdot 12 + 3\right]\, m \\
P_1 &= 241 \text{ kPa}.
\end{aligned}$$