PHY6938 E & M Spring 2000 Proficiency Solutions

1. A non-uniform, non-conducting disk of mass M, radius R, and total charge Q has a surface charge density $\sigma = \sigma_0 r/R$, where r is the distance from the center of the disk, and a mass per unit area $\sigma_m = M\sigma/Q$. The disk rotates with angular velocity ω about its axis.

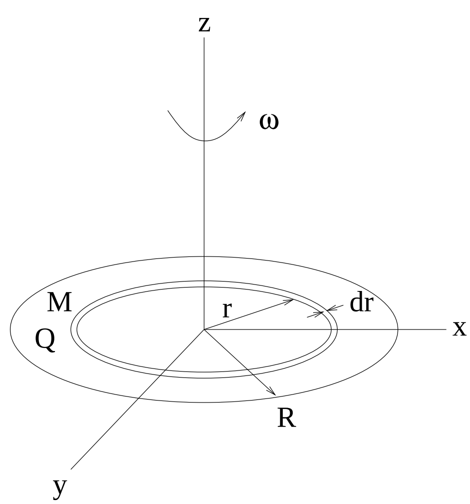
a) Calculate the magnetic moment μ of the disk.

Break the disk up into a series of concentric rings at radial distance r from the rotation axis, and with thickness dr (see figure). The magnetic moment of a current loop is defined to be the current flowing around the loop multiplied by the area of the loop. Firstly, we can find the current due to our ring of charge by noting that the charge of the ring is

$$dQ = \sigma dA = \sigma 2\pi r\, dr = \sigma_0 \frac{r}{R} 2\pi r\, dr.$$

Now all of this charge passes a fixed point in the plane of the disk and at radius r (say on the x axis) in a time $T = 2\pi/\omega$, so we have that the current from this loop is

$$dI = \frac{dQ}{T} = \sigma_0 \frac{r}{R} 2\pi r\, dr \cdot \frac{\omega}{2\pi} = \sigma_0 \omega \frac{r^2}{R} dr.$$



This current loop has an area πr^2 so we have an element of the magnetic moment of

$$d\mu = dI \cdot \pi r^2,$$

and so the magnetic moment is

$$\mu = \frac{\sigma_0 \pi \omega}{R} \int_0^R r^4 dr = \frac{\sigma_0 \pi \omega}{R} \frac{R^5}{5} = \frac{\pi}{5} \sigma_0 \omega R^4.$$

Although it is not necessary we can also find the total charge and so find the magnetic moment in terms of the total charge,

$$Q = \int_0^R \frac{\sigma_0 r}{R} 2\pi r\, dr = \frac{2\pi}{3}\sigma_0 R^2,$$

so that $\sigma_0 = 3Q/2\pi R^2$ and

$$\boldsymbol{\mu} = \frac{3}{10} Q R^2 \omega\ \hat{\mathbf{z}}.$$

b) Show that the magnetic moment $\boldsymbol{\mu}$ and the angular momentum L are related by $\boldsymbol{\mu} = Q\mathbf{L}/2M$, where $\mathbf{L} = I\boldsymbol{\omega}$ and I is the moment of inertia.

To find the angular momentum we need the moment of inertia, which we find using the mass density and by doing the integral in the same way as before. Note that since the mass and charge densities are proportional, we have the mass density in terms of the total mass

$$\sigma_m = \frac{3M}{2\pi R^2} \frac{r}{R}$$

so that

$$I = \int dM\, r^2 = \sigma_m \int dA\, r^2 = \frac{3M}{2\pi R^2} \int_0^R 2\pi r\, dr \cdot \frac{r}{R} r^2 = \frac{3M}{R^3} \int_0^R r^4\, dr = \frac{3}{5} M R^2,$$

and we have that

$$\mathbf{L} = I\boldsymbol{\omega} = I\omega\ \hat{\mathbf{z}} = \frac{3}{5} M R^2 \omega\ \hat{\mathbf{z}},$$

and we see that $\boldsymbol{\mu} = Q\mathbf{L}/2M$.

2. A series LCR circuit is driven by an emf with angular frequency ω, so that the voltage across the resistor is

$$V_R(t) = V_0 \sin(\omega t)$$

In answering the following questions, express everything in terms of the given quantities: L, C, R, ω, V_0, and ε_0.

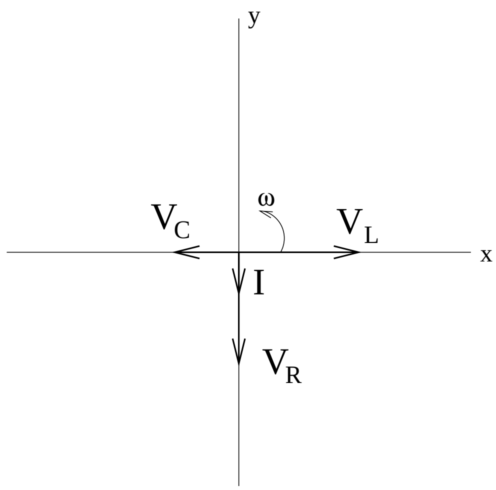
a) Draw a phasor diagram for V_R, V_L, and V_C, and specify the angles between the phasors.

This is a series circuit, so the current in every element of the circuit is the same common function of time $I_R(t) = I_L(t) = I_C(t) = I(t)$. The solution of the problem lies in finding the relationships between the voltage across each of the circuit elements and this current.

Using Warren's mnemonic 'ELI the ICE man' we have that E leads I in the inductor L, and I leads E in the capacitor C. Now we are given that the voltage across the resistor, which is in phase with and proportional to the current through it, is $V_0 \sin(\omega t)$, which means that the total current in the circuit is

$$I(t) = \frac{V_R(t)}{R} = \frac{V_0}{R} \sin(\omega t).$$

So we can represent the current in the circuit and the voltage $V_R(t)$ by phasors lying along the negative y axis and rotating counterclockwise, so that they start with x-projection zero and increase like $\sin(\omega t)$. This means that the inductor voltage $V_L(t)$, which leads $I(t)$ by $90°$, lies along the positive x-axis. The capacitor voltage $V_C(t)$ is led by the current by $90°$, so it must lie along the negative y-axis, as shown in the figure.



b) What is the current $I_L(t)$ through the inductance?

The current through the inductance is just

$$I_L(t) = I(t) = \frac{V_0}{R}\sin(\omega t).$$

c) Find the voltage $V_L(t)$ across the inductance.

The magnitude of the voltage across the inductor is

$$V_L = X_L I = \omega L \frac{V_0}{R},$$

and since it leads the current it is

$$V_L(t) = \omega L \frac{V_0}{R}\cos(\omega t).$$

d) Find the voltage $V_C(t)$ across the capacitance.

The magnitude of the voltage across the capcacitor is

$$V_C = X_C I = \frac{1}{\omega C}\frac{V_0}{R},$$

and since it lags the current it is

$$V_C(t) = -\frac{1}{\omega C}\frac{V_0}{R}\cos(\omega t).$$

e) Determine the angular frequency, ω_0, for which the voltage $V_R(t)$ across the resistance is the same as that across the emf source, i.e. $V_R(t) = \varepsilon(t)$.

From the above we see that the voltages across the inductor and capacitor will cancel when

$$\omega_0 L = \frac{1}{\omega_0 C}, \quad \omega_0 = \frac{1}{\sqrt{LC}},$$

and when this is true the voltage across the resistor will simply be the applied voltage $\varepsilon(t)$.

f) If the emf source is suddenly short-circuited, what will be the frequency of the current $I_L(t)$? What is $I_L(t)$ long after the emf is short-circuited?

If the applied voltage is removed (but the series circuit is still intact) the system is a damped harmonic oscillator without a driving emf. If we assume that it is an underdamped oscillator, it will oscillate at a frequency which is reduced from ω_0 by the damping term. We can find this reduced frequency ω_1 by making an analogy between the electrical differential equation

$$L\frac{dI}{dt} + IR + \frac{Q}{C} = 0$$

and the mechanical one

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = 0,$$

which has the underdamped frequency

$$\omega_1 = \sqrt{\omega_0^2 - \beta^2},$$

to write that for the electrical circuit

$$\omega_1 = \sqrt{\omega_0^2 - \left(\frac{R}{2L}\right)^2}.$$

Since there is no source of energy for the circuit and there is dissipation of energy in the resistor, eventually the current will go to zero.

3. A positive charge is uniformly distributed throughout a very long cylindrical volume of radius R. The charge per unit volume is ρ.

a) Find the electric field $\vec{E}$ everywhere as a function of the distance r from the axis of the cylinder.

Let's start by finding the electric field outside of the cylinder. Build a Gaussian surface out of a cylinder of length L and radius $r > R$. Then by symmetry the field must point radially outward on the curved part of the Gaussian surface and must be parallel to the surface on the end caps. This means that the flux of the electric field through this surface is

$$\oint \mathbf{E} \cdot d\mathbf{A} = E(2\pi r L).$$

Now Gauss's law tells us that this is also $Q_{\text{encl}}/\epsilon_0$, where Q_{encl} is the enclosed charge. The volume of the charged cylinder cut out by our surface is $V = \pi R^2 L$, so the enclosed charge is

$$Q_{\text{encl}} = \pi R^2 L \rho$$

and Gauss's law gives

$$E(2\pi r L) = \frac{1}{\epsilon_0}\pi R^2 L\rho, \quad E = \frac{\rho R^2}{2\epsilon_0 r}.$$

Since the electric field points radially outward, we have

$$\mathbf{E} = \frac{\rho R^2}{2\epsilon_0 r}\hat{\mathbf{r}}.$$

Inside the cylinder we use exactly the same technique, and everything goes as before except now the charge enclosed by the Gaussian surface is reduced to

$$Q_{\text{encl}} = (\pi r^2 L)\rho$$

which means that the electric field is (take $R \to r$ in the above)

$$\mathbf{E} = \frac{\rho r}{2\epsilon_0}\hat{\mathbf{r}}.$$

b) Find the electric potential V everywhere as a function of r. Define $V = 0$ at the surface of the cylinder.

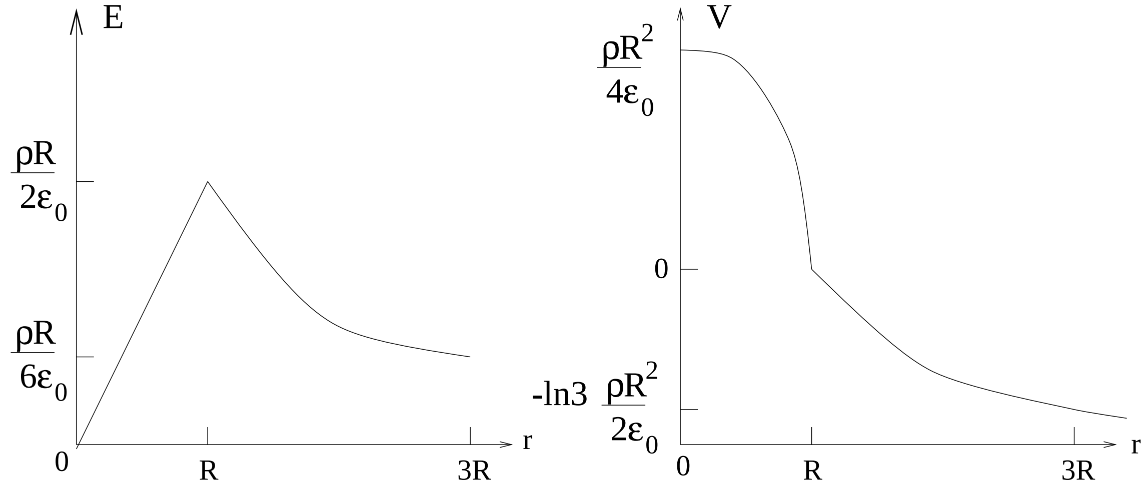
Starting again with the potential outside the cylinder, we know that the potential will be the highest at the surface of the cylinder as the electric field always point outward (and so a charged particle will lose energy as it moves away from the cylinder). This means that

$$V(r') = -\int_R^{r'} \mathbf{E}(r)\cdot d\mathbf{r} = -\int_R^{r'} \frac{\rho R^2}{2\epsilon_0 r}dr = -\frac{\rho R^2}{2\epsilon_0}\ln\left(\frac{r'}{R}\right).$$

Inside the cylinder the potential increases as we move away from $r = R$, so we have

$$V(r') = \int_{r'}^{R} \mathbf{E}(r)\cdot d\mathbf{r} = \int_{r'}^{R} \frac{\rho r}{2\epsilon_0}dr = \frac{\rho}{4\epsilon_0}\left(R^2 - r'^2\right).$$

c) Sketch E and V as functions of r, from $r = 0$ to $r = 3R$, showing the values of each at $r = 0$, R, and $3R$.



Note that the potential falls without bound as we move away from the wire, essentially because the wire has an infinite charge.