ECE 250 Data Structures and Algorithms

Laboratory 5: Graphs

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Graphs
Outline

- In this topic, we will cover the representation of graphs on a computer
- We will examine:
  - an adjacency matrix representation
  - smaller representations and pointer arithmetic
  - sparse matrices and linked lists

Graphs
Background

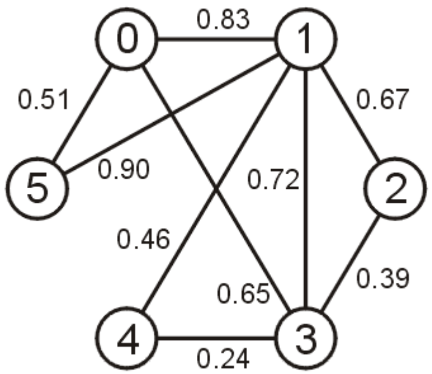
- Project 5 requires you to store a graph with a given number of vertices numbered 0 through $n - 1$
- Initially, there are no edges between these n vertices
- The **`insert`** command adds edges to the graph while the number vertices remains unchanged

Graphs
Background

- In this laboratory, we will look at techniques for storing the edges of a graph
- This laboratory will focus on weighted graphs, however, for unweighted graphs, one can easily use **`bool`** in place of **`double`**

Graphs
Background

- To demonstrate these techniques, we will look at storing the edges of the following graph:



Graphs

Adjacency Matrix

- A graph of n vertices may have up to

$$\binom{n}{2} = \frac{n(n-1)}{2} = \mathbf{O}(n^2)$$

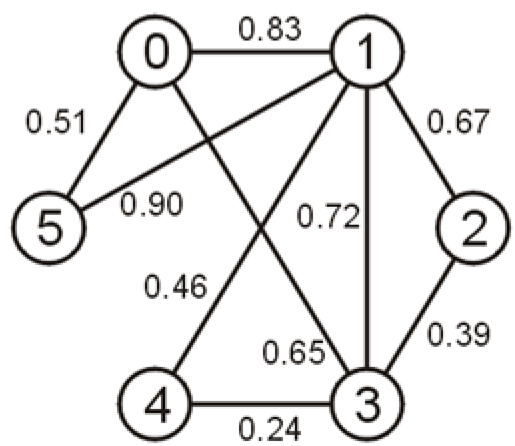
edges

- The first straight-forward implementation is an adjacency matrix

Graphs
Adjacency Matrix

- Define an $n \times n$ matrix $\mathbf{A} = (a_{ij})$ and if the vertices v_i and v_j are connected with weight w, then set $a_{ij} = w$ and $a_{ji} = w$
- That is, the matrix is symmetric, *e*.*g*.,

	0	1	2	3	4	5
0		0.83		0.65		0.51
1	0.83		0.67	0.72	0.46	0.90
2		0.67		0.39		
3	0.65	0.72	0.39		0.24	
4		0.46		0.24		
5	0.51	0.90				

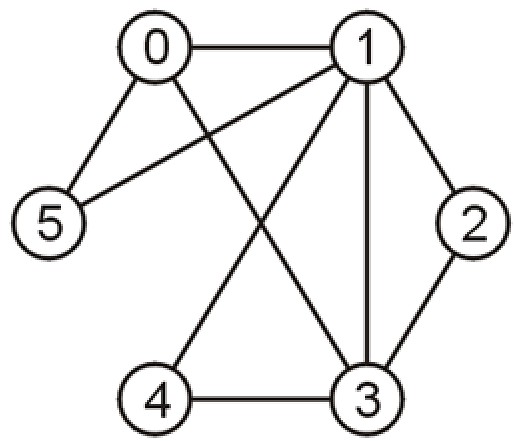


Graphs
Adjacency Matrix

- An unweighted graph may be saved as an array of Boolean values
  - vertices v_i and v_j are connected then set

$$a_{ij} = a_{ji} = true$$

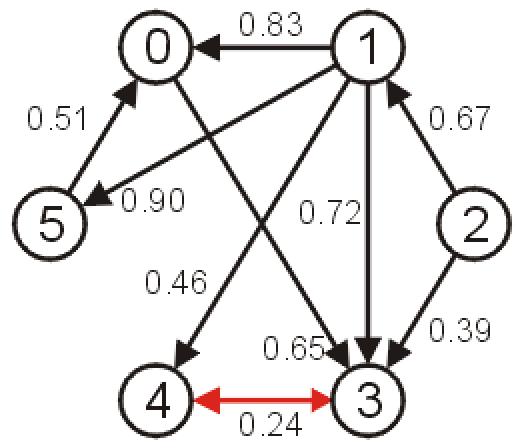
	0	1	2	3	4	5
0		T	F	T	F	T
1	T		T	T	T	T
2	F	T		T	F	F
3	T	T	T		T	F
4	F	T	F	T		F
5	T	T	F	F	F	



Graphs
Adjacency Matrix

- If the graph was directed, then the matrix would not necessarily be symmetric

	0	1	2	3	4	5
0				0.65		
1	0.83			0.72	0.46	0.90
2		0.67		0.39		
3					0.24	
4				0.24		
5	0.51					



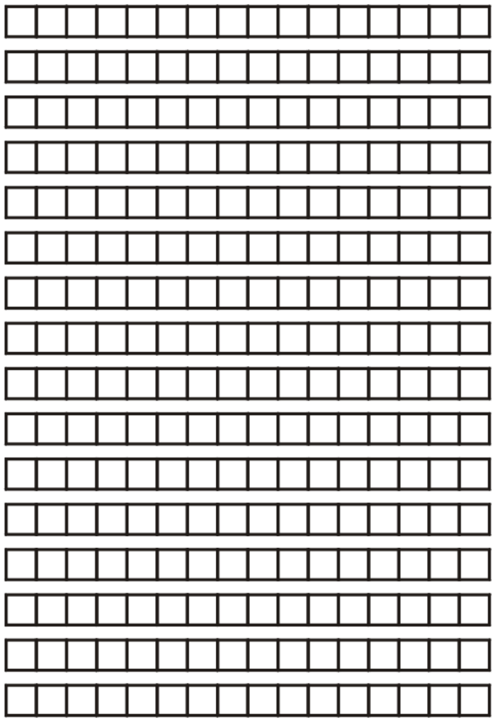
Graphs
Adjacency Matrix

- First we must allocate memory for a two-dimensional array
- C++ does not have native support for anything more than one-dimensional arrays, thus how do we store a two-dimensional array?
  - as an array of arrays

Graphs
Adjacency Matrix

- Suppose we require a 16 x 16 matrix of double-precision floating-point numbers
- Each row of the matrix can be represented by an array
- The address of the first entry must be stored in a pointer to a double:

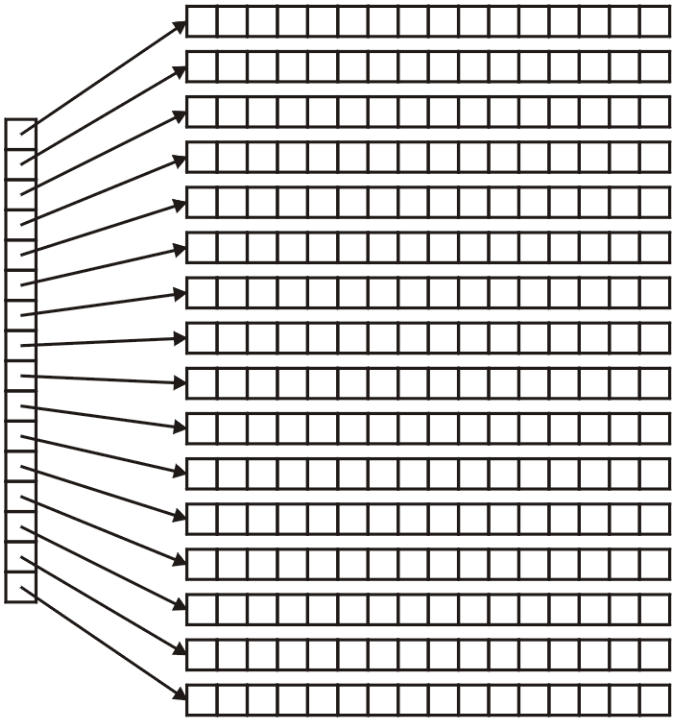
```
double *
```



Graphs
Adjacency Matrix

- However, because we must store 16 of these pointers-to-doubles, it makes sense that we store these in an array
- What is the declaration of this array?
- Well, we must store a

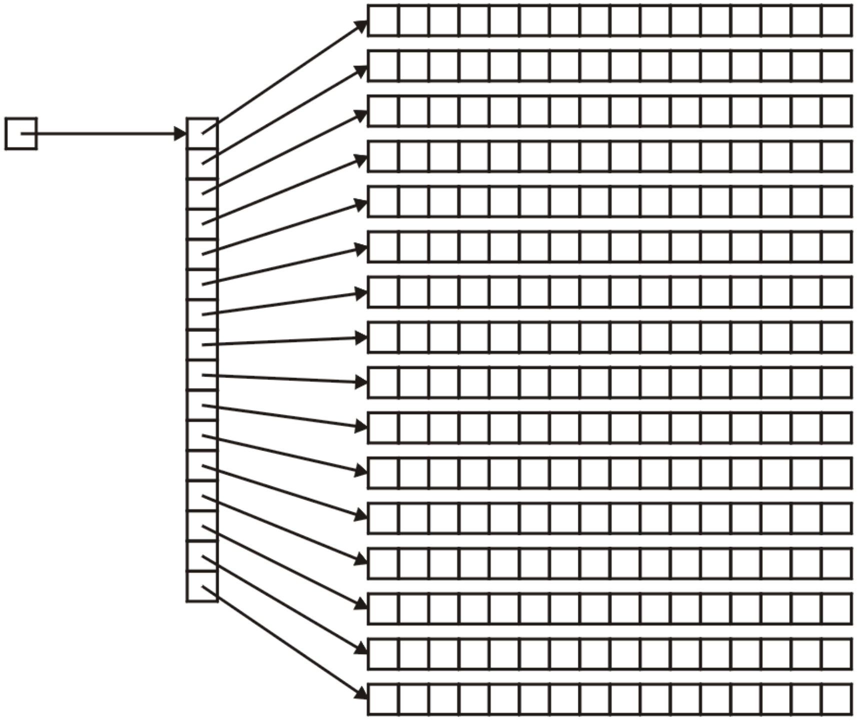
  pointer to a pointer to a double
- That is: **`double **`**



Graphs
Adjacency Matrix

- Thus, the address of the first array must be declared to be:

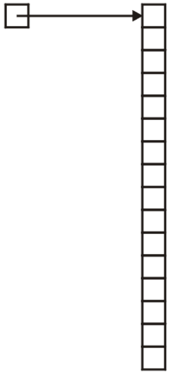
```
double **matrix;
```



Graphs
Adjacency Matrix

- The next question is memory allocation
- First, we must allocate the memory for the array of pointers to doubles:

```
matrix = new double * [16];
```

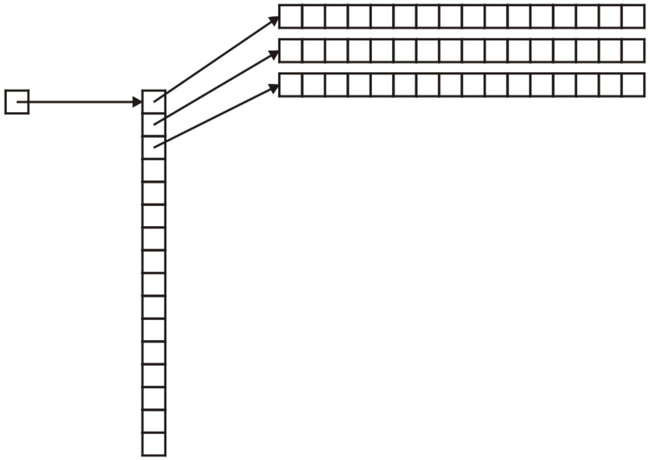


Graphs

Adjacency Matrix

- Next, to each entry of this matrix, we must assign the memory allocated for an array of doubles

```
for ( int i = 0; i < 16; ++i ) {
    matrix[i] = new double[16];
}
```

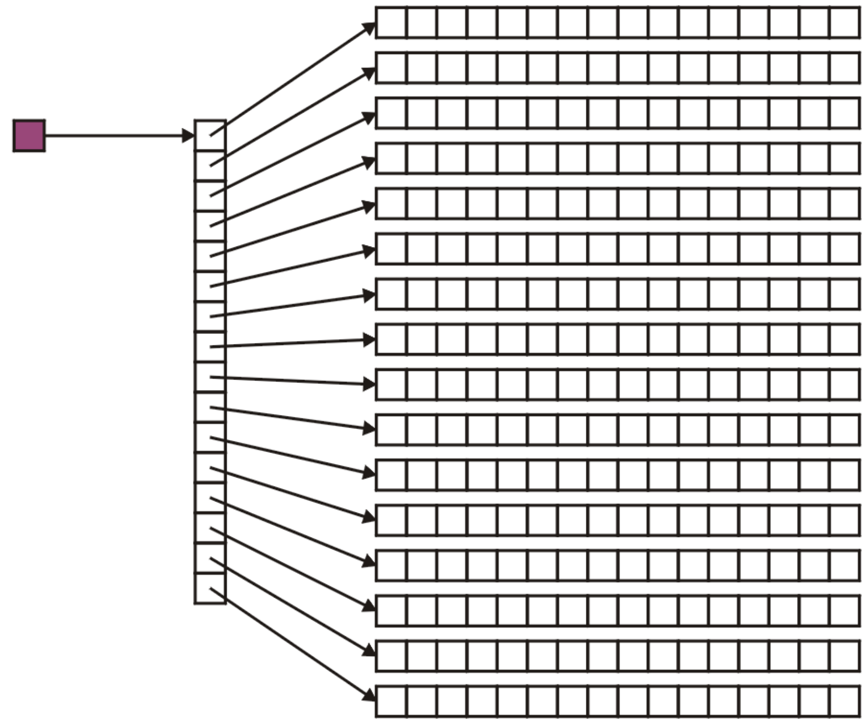


Adjacency Matrix

- Accessing a matrix is done through a double index, *e.g.*, **`matrix[3][4]`**
- You can interpret this as **`(matrix[3])[4]`**:

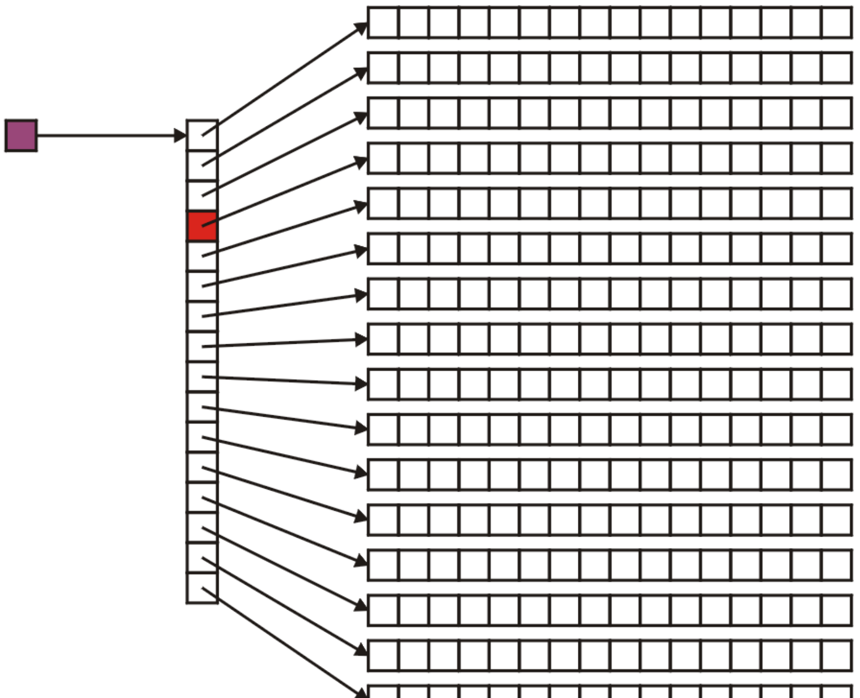
Graphs
Adjacency Matrix

- Recall that in **`matrix[3][4]`**, the variable **`matrix`** is a pointer-to-a-pointer-to-a-double:



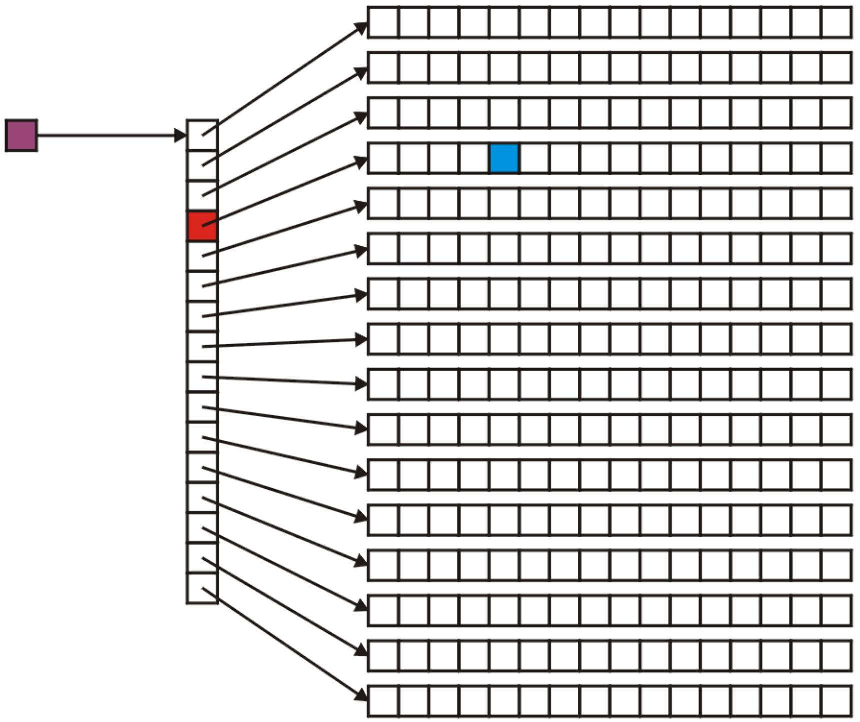
Graphs
Adjacency Matrix

- Therefore, `matrix[3]` is a pointer-to-a-double:



Graphs
Adjacency Matrix

- And consequently, `matrix[3][4]` is a double:



Graphs
C++ Notation Warning

- Do not use **`matrix[3, 4]`** because:
  - in C++, the comma operator evaluates the operands in order from left-to-right
  - the *value* is the last one
- Therefore, **`matrix[3, 4]`** is equivalent to calling **`matrix[4]`**
- Try it:

```
int i = (3, 4);
cout << i << endl;
```

Graphs
C++ Notation Warning

- Many things will compile if you try to use this notation:

```
matrix = new double[N, N];
```

  will allocate an array of N doubles, just like:

```
matrix = new double[N];
```

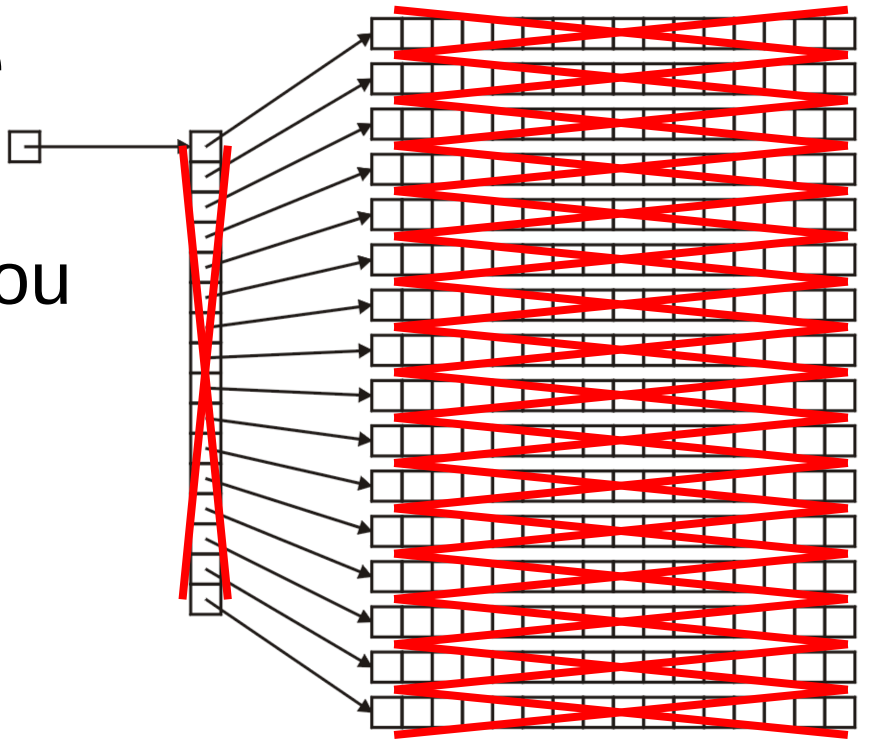
- However, this is likely not to do what you really expect...

Graphs
Adjacency Matrix

- Now, once you've used the matrix, you must also delete it...

Graphs
Adjacency Matrix

- Recall that for each call to **`new[]`**,you must have a corresponding call to **`delete[]`**
- Therefore, we must use a for-loop to delete the arrays
  - implementation up to you



Graphs

Default Values

- Question: what do we do about vertices which are not connected?
  - the value 0
  - a negative number, *e.g.*, -1
  - positive infinity: ∞
- The last is the most logical, in that it makes sense that two vertices which are not connected have an infinite distance between them

Graphs
Default Values

- To use infinity, you may declare a constant static member variable **`INF`**:

```
#include <limits>

class WeightedGraph {
    private:
        static const double INF;
        // ...
    // ...
};

const double WeightedGraph::INF =
    numeric_limits<double>::infinity();
```

Default Values

- As defined in the IEEE 754 standard, the representation of the double-precision floating-point infinity is the special double (8 bytes):

  `0x 7F F0 00 00 00 00 00 00`

- Incidentally, negative infinity is stored as:

  `0x FF F0 00 00 00 00 00 00`

Graphs
Default Values

- In this case, you can initialize your array as follows:

```
for ( int i = 0; i < N; ++i ) {
    for ( int j = 0; j < N; ++j ) {
        matrix[i][j] = INF;
    }

    matrix[i][i] = 0;
}
```

- It makes intuitive sense that the distance from a node to itself is 0

Graphs
Default Values

- If we are representing an unweighted graph, then we have Boolean values:

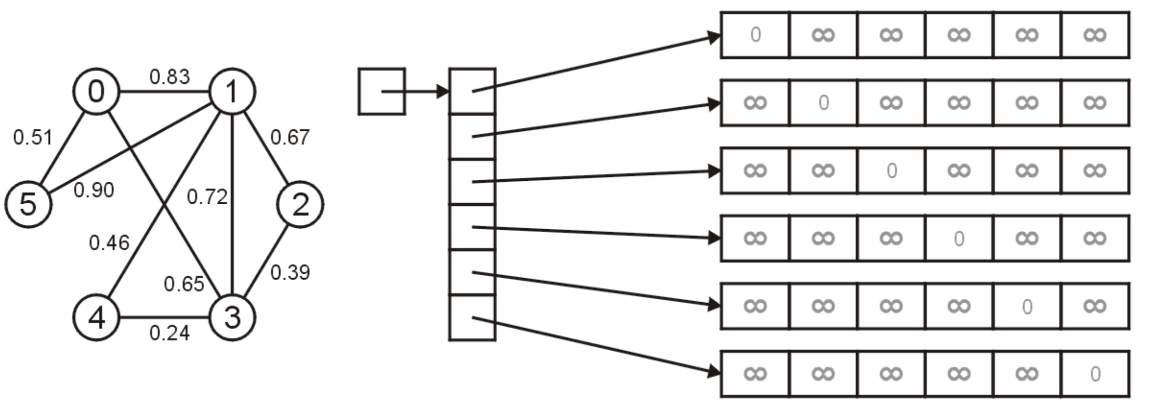
```
for ( int i = 0; i < N; ++i ) {
    for ( int j = 0; j < N; ++j ) {
        matrix[i][j] = false;
    }

    matrix[i][i] = true;
}
```

- It makes intuitive sense that a vertex is connected to itself

Graphs
Adjacency Matrix

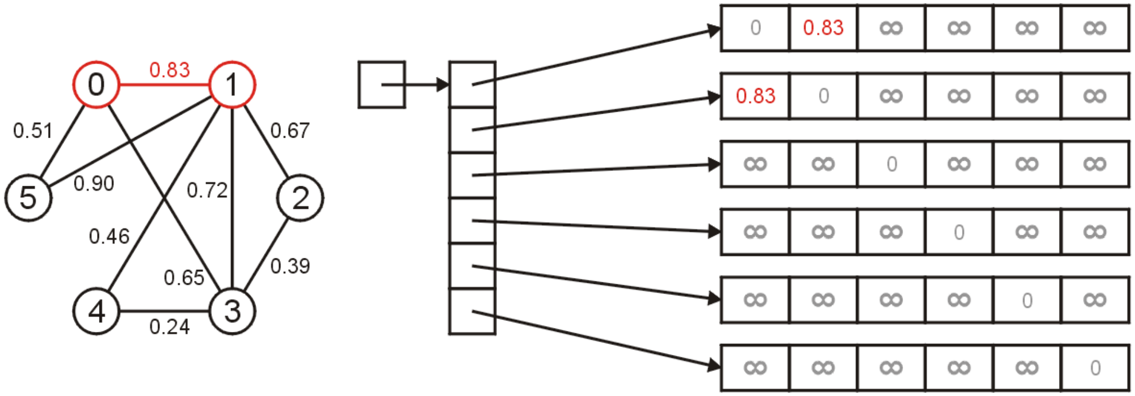
- Let us look at the representation of our example graph
- Initially none of the edges are recorded:



Graphs
Adjacency Matrix

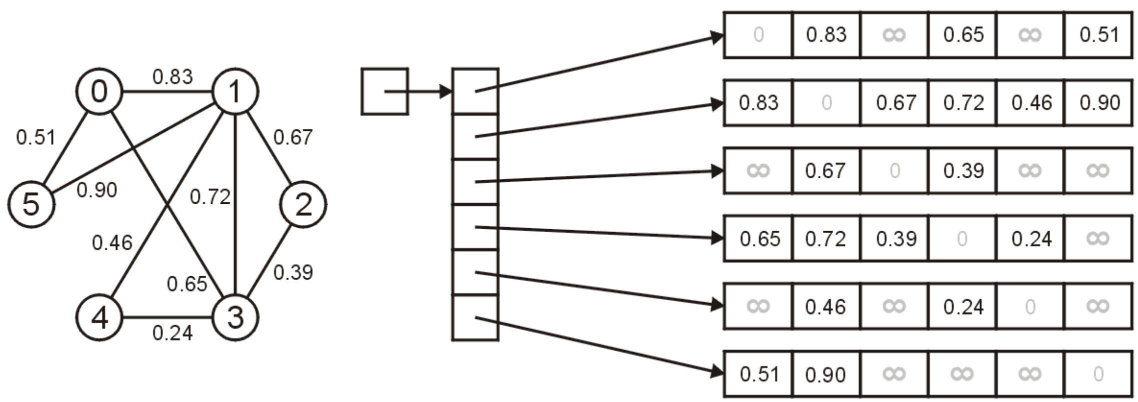
- To insert the edge between 0 and 1 with weight 0.83, we set

```
matrix[0][1] = matrix[1][0] = 0.83;
```



Graphs
Adjacency Matrix

- The final result is shown as follows
- Note, however, that these six arrays could be anywhere in memory...



Graphs
Adjacency Matrix

- We have now looked at how we can store an adjacency graph in C++
- Next, we will look at:
  - two improvements for the array-of-arrays implementations, including:
    - allocating the memory for the matrix in a single contiguous block of code, and
    - a lower-triangular representation; and
  - a sparse linked-list implementation

Graphs
Adjacency Matrix Improvement

- To begin, we will look at the first improvement:
  - allocating all of the memory of the arrays in a single array with n^2 entries

Graphs
Adjacency Matrix Improvement

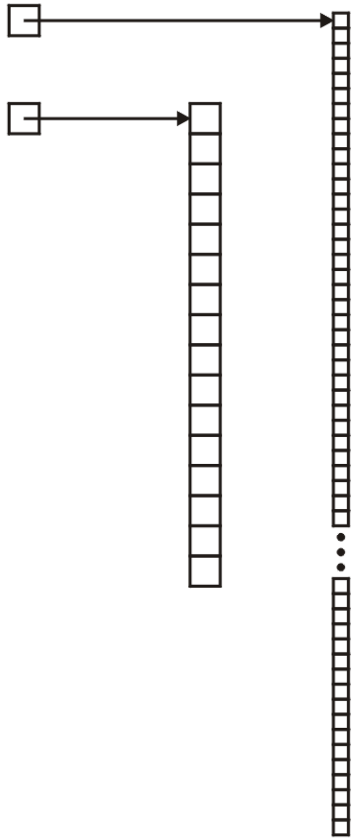
- For those of you who would like to reduce the number of calls to **`new`**, consider the following idea:
  - allocate an array of 16 pointers to doubles
  - allocate an array of $16^2 = 256$ doubles
- Then, assign to the 16 pointers in the first array the addresses of entries

0, 16, 32, 48, 64, ..., 240

Graphs
Adjacency Matrix Improvement

- First, we allocate memory:

```
matrix = new double * [16];
double * tmp = new double[256];
```



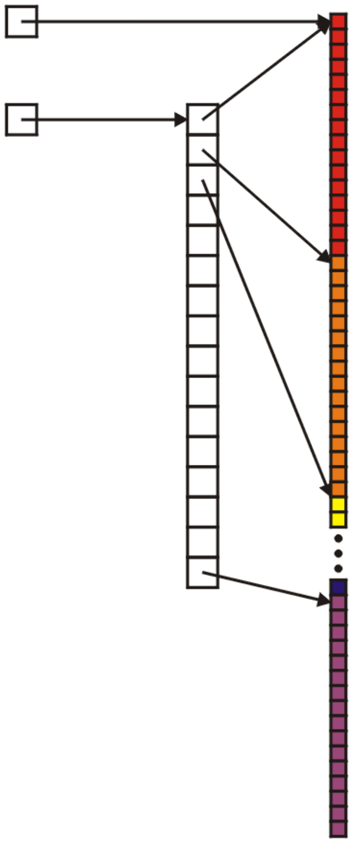
Graphs

Adjacency Matrix Improvement

- Next, we allocate the addresses:

```
matrix = new double * [16];
double * tmp = new double[256];

for ( int i = 0; i < 16; ++i ) {
   matrix[i] = &( tmp[16*i] );
}
```



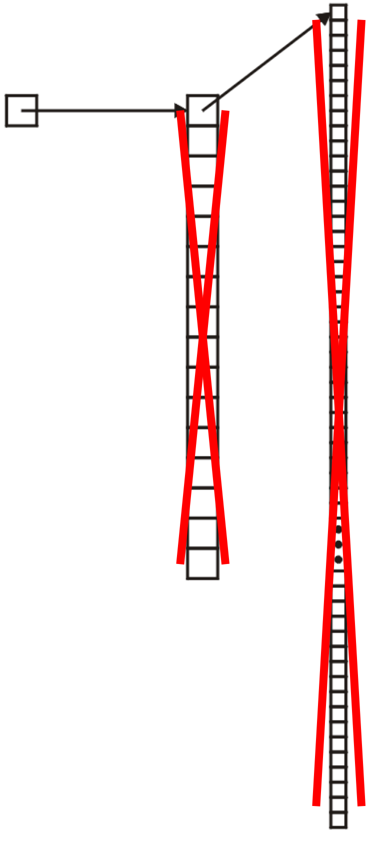
- This assigns:

```
 matrix[0] = &( tmp[  0] );
 matrix[1] = &( tmp[ 16] );
 matrix[2] = &( tmp[ 32] );
                .
                .
                .
matrix[15] = &( tmp[240] );
```

Graphs
Adjacency Matrix Improvement

- Deleting this array is easier:

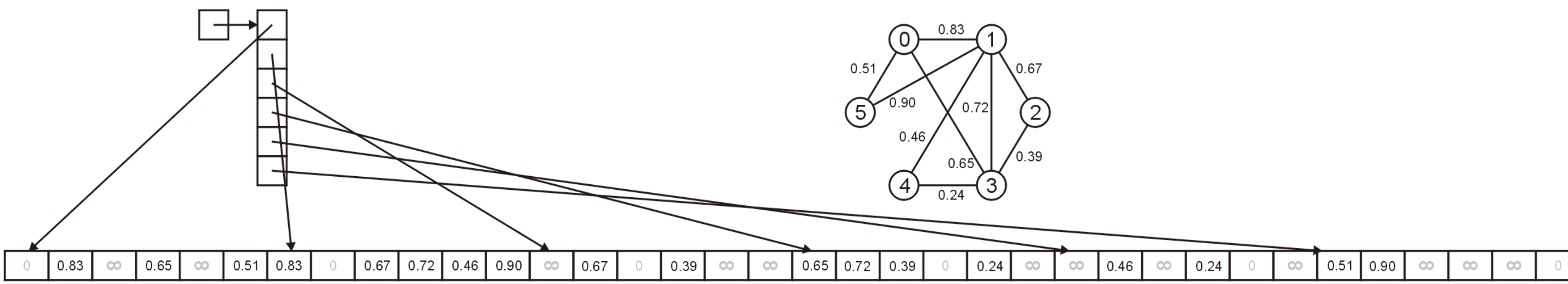
```
delete [] matrix[0];
delete [] matrix;
```



Graphs

Adjacency Matrix Improvement

- Our sample graph would be represented as follows:

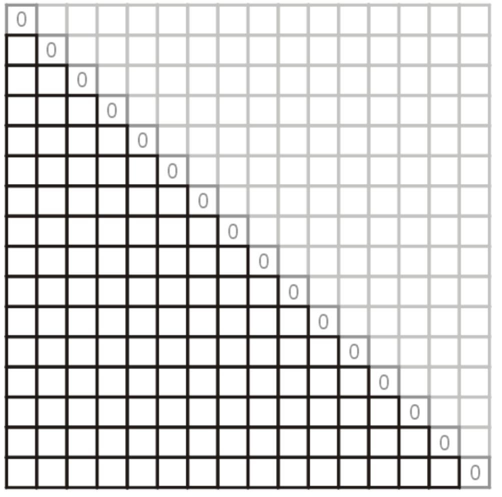


Graphs
Lower-triangular Adjacency Matrix

- Next we will look at another improvement which can be used for undirected graphs
- We will store only half of the entries
  - to do this, we must also learn about pointer arithmetic

Graphs
Lower-triangular Adjacency Matrix

- Note also that we are not storing a directed graph: therefore, we really need only store half of the matrix
- Thus, instead of 256 entries, we really only require 120 entries



Graphs

Lower-triangular Adjacency Matrix

- The memory allocation for this would be straight-forward, too:

```
matrix = new double * [16];
matrix[0] = 0;
matrix[1] = new double[120];

for( int i = 2; i < 16; ++i ) {
    matrix[i] = matrix[i – 1] + i - 1;
}
```

Graphs
Lower-triangular Adjacency Matrix

- What we are using here is pointer arithmetic:
  - in C/C++, you can add values to a pointer
  - the question is, what does it mean to set:

```
ptr = ptr + 1;
```

or

```
ptr = ptr + 2;
```

Graphs
Lower-triangular Adjacency Matrix

- Suppose we have a pointer-to-a-double:

```
double * ptr = new double( 3.14 );
```

where:

- the pointer has a value of **`0x53A1D780`**, and
- the representation of 3.14 is **`0x40091Eb851EB851F`**

53A1D780	53A1D781	53A1D782	53A1D783	53A1D784	53A1D785	53A1D786	53A1D787	53A1D788
40	09	1E	B8	51	EB	85	1F	??

Graphs
Lower-triangular Adjacency Matrix

- If we just added one to the address, then this would give us the value **`0x53A1D781`**, but this contains no useful information...

53A1D780	53A1D781	53A1D782	53A1D783	53A1D784	53A1D785	53A1D786	53A1D787	53A1D788
40	09	1E	B8	51	EB	85	1F	??

Graphs

Lower-triangular Adjacency Matrix

- The only logical interpretation of **`ptr + 1`** is to go to the *next* location a different double could exist, *i.e.*, **`0x53A1D788`**

53A1D780	53A1D781	53A1D782	53A1D783	53A1D784	53A1D785	53A1D786	53A1D787	53A1D788
40	09	1E	B8	51	EB	85	1F	??

Lower-triangular Adjacency Matrix

- Therefore, if we define:

  double * array = new double[4];

  then the following are all equivalent:

```
array[0]      *array
array[1]      *(array + 1)
array[2]      *(array + 2)
array[3]      *(array + 3)
```

Graphs
Lower-triangular Adjacency Matrix

- Thus, the following code simply adds appropriate amounts to the pointer:

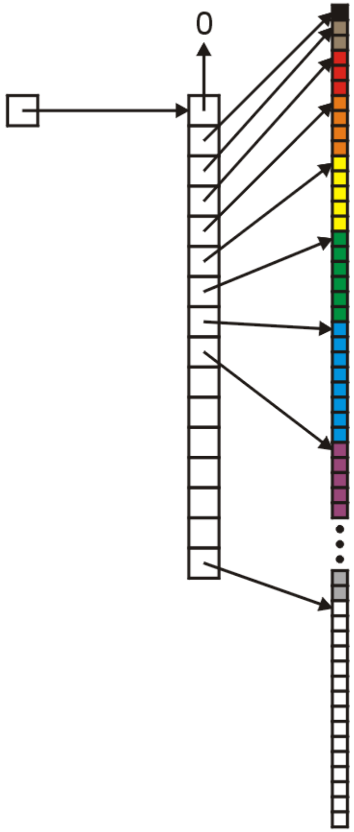
```
matrix = new double * [N];
matrix[0] = 0;
matrix[1] = new double[N*(N – 1)];

for( int i = 2; i < N; ++i ) {
    matrix[i] = matrix[i – 1] + i - 1;
}
```

Graphs
Lower-triangular Adjacency Matrix

- Visually, we have, for $N = 16$, the following:

```
matrix[0] = 0;
matrix[1] = &( tmp[0] );
matrix[2] = &( tmp[1] );
matrix[3] = &( tmp[3] );
matrix[4] = &( tmp[6] );
matrix[5] = &( tmp[10] );
matrix[6] = &( tmp[15] );
matrix[7] = &( tmp[21] );
matrix[7] = &( tmp[28] );
            .
            .
            .
matrix[15] = &( tmp[105] );
```



Graphs
Lower-triangular Adjacency Matrix

- The only thing that we would have to do is ensure that we always put the larger number first:

```
void insert( int i, int j, double w ) {
    if ( j < i ) {
        matrix[i][j] = w;
    } else {
        matrix[j][i] = w;
    }
}
```

Graphs
Lower-triangular Adjacency Matrix

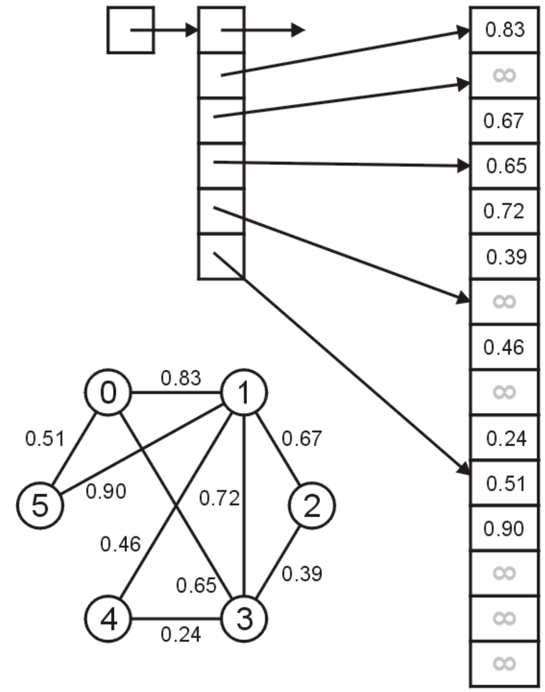
- A slightly less efficient way of writing this would be:

```
void insert( int i, int j, double w ) {
    matrix[max(i,j)][min(i,j)] = w;
}
```

- The benefits (from the point-of-view of clarity) are much more significant...

Graphs
Lower-triangular Adjacency Matrix

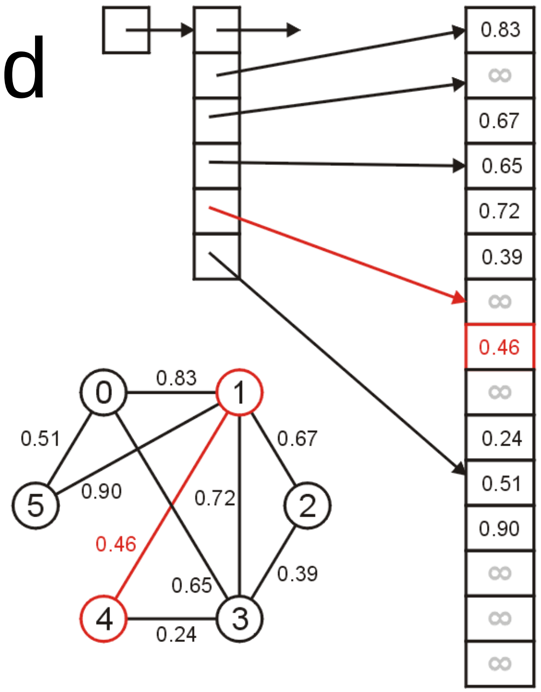
- Our example graph is stored using this representation as shown here
- Notice that we do not store any 0’s, nor do we store any duplicate entries
- The second array has only 15 entries, versus 36



Graphs
Lower-triangular Adjacency Matrix

- To determine the weight of the edge connecting vertices 1 and 4, we must look up the entry

  `matrix[4][1]`



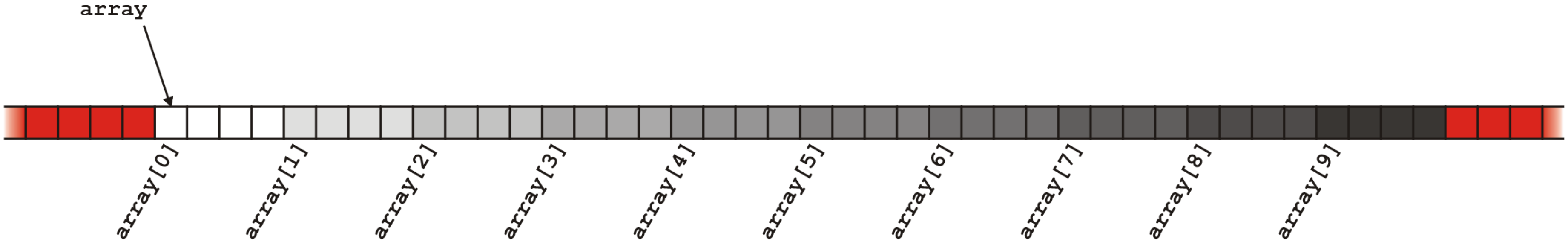
Graphs

Beyond Array Bounds

- Until now, some of you may have gone beyond array bounds accidentally
- Recall that

```
int * array = new int[10];
```

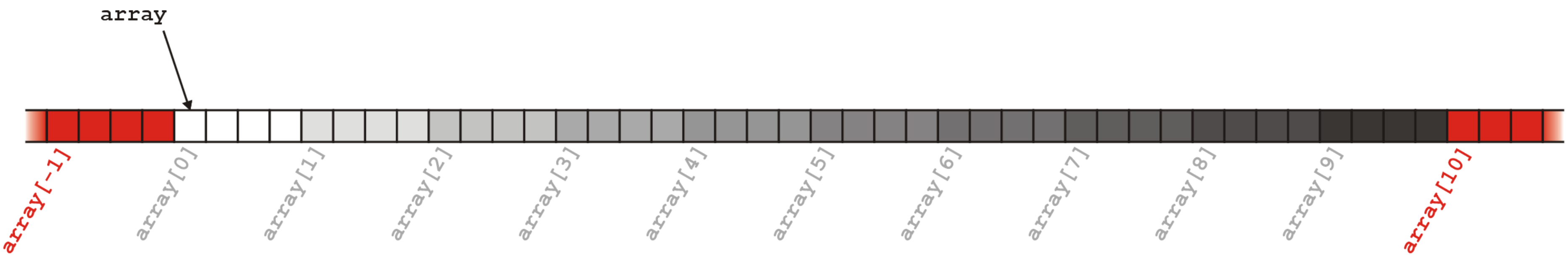
allocates 40 bytes (4 bytes/int) and the entries are accessed with `array[0]` through `array[9]`



Graphs

Beyond Array Bounds

- If you try to access either **`array[10]`** or **`array[-1]`**, you are accessing memory which has not been allocated for this array



Beyond Array Bounds

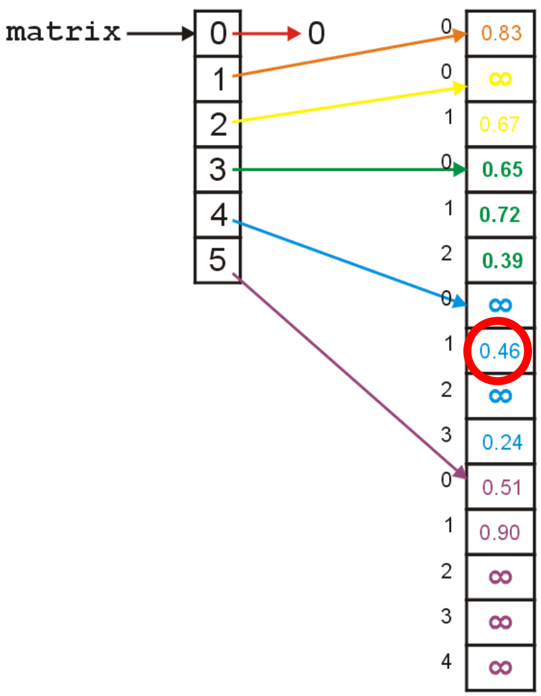
- This memory may be used:
  - for different local variables, or
  - by some other process
- In the first case, you will have a bug which is very difficult to track down
  - *e*.*g*., a variable will appear to change its value without an explicit assignment
- In the second case, the OS will terminate your process (segmentation fault)

Graphs
Beyond Array Bounds

- Now we have a very explicit example of what happens if you go outside your expected array bounds
- Notice that the value stored at **`matrix[4][1]`** is 0.46
- We can also access it using either:

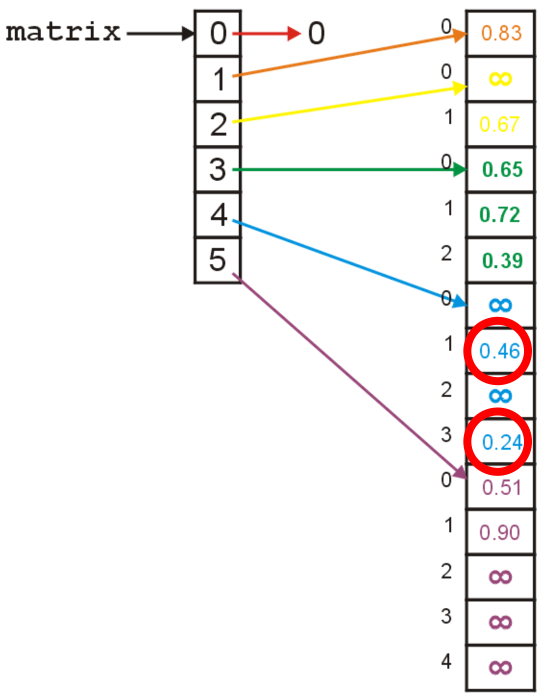
`matrix[3][4]`

`matrix[5][-3]`



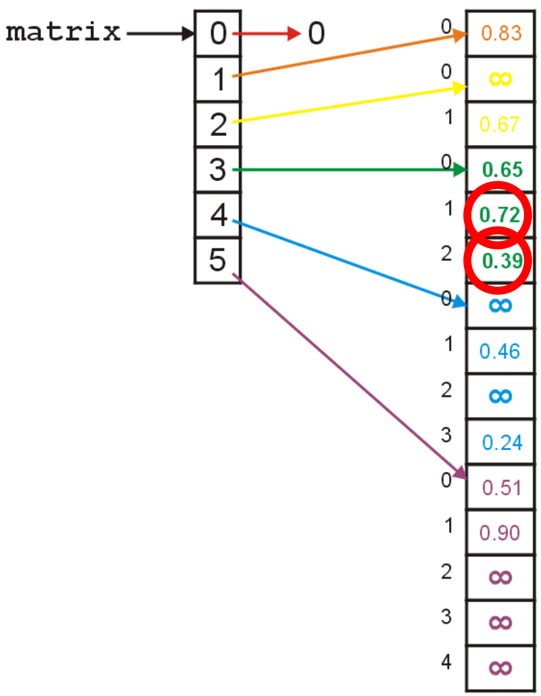
Graphs
Beyond Array Bounds

- Thus, if you wanted to find the distance between vertices 3 and 4, if you access **`matrix[4][3]`**, you get is 0.24
- If, however, you access **`matrix[3][4]`**, you get 0.46



Beyond Array Bounds

- Similarly, if you wanted to find the distance between vertices 2 and 3, if you access **`matrix[3][2]`**, you get is 0.39
- If, however, you access **`matrix[2][3]`**, you get 0.72



Graphs
Sparse Matrices

- Finally we will consider the problem with sparse matrices and we will look at one implementation using linked lists

Graphs

Sparse Matrices

- The memory required for creating an $n \times n$ matrix using an array-of-arrays is:

$$4 \text{ bytes} + 4n \text{ bytes} + 8n^2 \text{ bytes} = \Theta(n^2) \text{ bytes}$$

- This could potentially waste a significant amount of memory:
  - consider all intersections in Canada as vertices and streets as edges
  - how could we estimate the number of intersections in Canada?

Graphs
Sparse Matrices

- The population of Canada is ~33 million
- Suppose we have one intersection per 10 houses and four occupants per house
- Therefore, there are roughly

$$33 \text{ million} / 10 / 4 \approx 800\ 000$$

  intersections in Canada which would require 4.66 TiB of memory

Graphs

Sparse Matrices

- Assume that each intersection connects, on average, four other intersections
- Therefore, less than 0.0005% of the entries of the matrix are used to store connections
  - the rest are storing the value *infinity*

Graphs
Sparse Matrices

- Matrices where less than 5% of the entries are not the default value (either infinity or 0, or perhaps some other default value) are said to be *sparse*
- Matrices where most entries (25% or more) are not the default value are said to be *dense*
- Clearly, these are not hard limits

Graphs
Sparse Matrices

- We will look at a very efficient sparse-matrix implementation with the last topic
- Here, we will consider a simpler implementation:
  - use an array of linked lists to store edges
- Note, however, that each node in a linked list must store two items of information:
  - the connecting vertex and the weight

Graphs
Sparse Matrices

- One possible solution:
  – modify the **SingleNode** data structure to store both an integer and a double:

```
class SingleNode {
    private:
        int adacent_vertex;
        double edge_weight;
        SingleNode * next_node;
    public:
        SingleNode( int, double SingleNode = 0 );
        double weight() const;
        int vertex() const;
        SingleNode * next() const;
};
```

  – exceptionally stupid and inefficient

Graphs
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- A better solution is to create a new class which stores a vertex-edge pair

```
class Pair {
    private:
        double edge_weight;
        int adacent_vertex;
    public:
        Pair( int, double );
        double weight() const;
        int vertex() const;
};
```

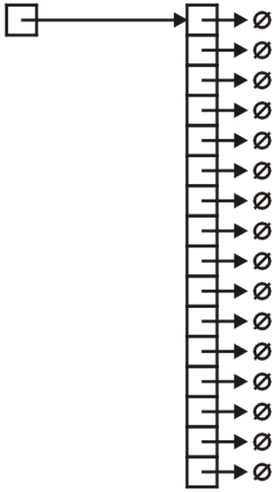
- Now create an array of linked-lists storing these pairs

Graphs
Sparse Matrices

- Thus, we define and create the array:

```
SingleList<Pair> * array;

array = new SingleList<Pair>[16];
```



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- As before, to reduce redundancy, we would only insert the entry into the entry corresponding with the larger vertex

```
void insert( int i, int j, double w ) {
    if ( i < j ) {
        array[j].push_front( Pair(i, w) );
    } else {
        array[i].push_front( Pair(j, w) );
    }
}
```

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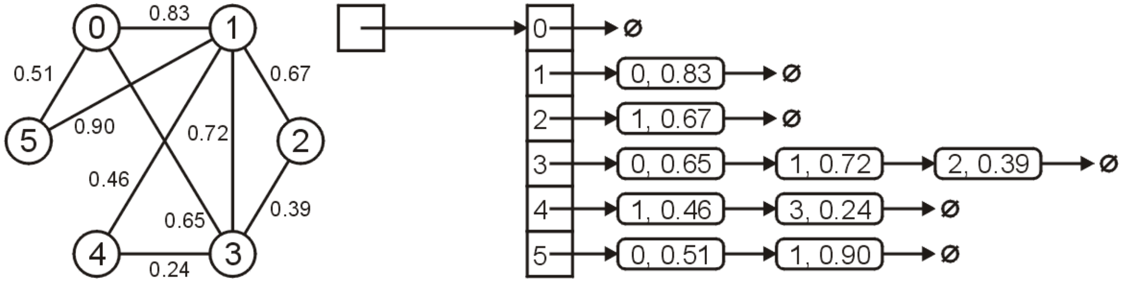
- For example, the graph shown below would be stored as



Graphs
Summary

- In this laboratory, we have looked at a number of graph representations
- C++ lacks a *matrix* data structure
  - must use array of arrays
- The possible factors affecting your choice of data structure are:
  - weighted or unweighted graphs
  - directed or undirected graphs
  - dense or sparse graphs

Usage Notes

- These slides are made publicly available on the web for anyone to use
- If you choose to use them, or a part thereof, for a course at another institution, I ask only three things:
  - that you inform me that you are using the slides,
  - that you acknowledge my work, and
  - that you alert me of any mistakes which I made or changes which you make, and allow me the option of incorporating such changes (with an acknowledgment) in my set of slides

Sincerely,

Douglas Wilhelm Harder, MMath

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