A study of asymmetric and symmetric simple exclusion processes in the presence of stochastic resetting

Report to be submitted in partial fulfillment of the requirements for the degree

of

Integrated MSc in Sciences (Physics)

by

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CERTIFICATE

This is to certify that the project report entitled **A study of asymmetric and symmetric simple exclusion processes in the presence of stochastic resetting**, is a bona-fide work done by **C L Sriram** (Roll Number: 35218025) at the **Department of Theoretical Physics, Tata Institute of Fundamental Research, Mumbai**, under the guidance of **Prof. Shamik Gupta**, in partial fulfilment of the requirements for the award of the degree of Integrated MSc in Sciences (Physics) from Cochin University of Science and Technology.

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DECLARATION

I hereby declare that the work presented in this thesis titled "**A study of asymmetric and symmetric simple exclusion processes in the presence of stochastic resetting**" is based on the original work done by me under the guidance of **Prof. Shamik Gupta**, at the **Department of Theoretical Physics, Tata Institute of Fundamental Research, Mumbai.**

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ACKNOWLEDGEMENTS

First and foremost, I would like to express my sincere gratitude to the **Department of Theoretical Physics, Tata Institute of Fundamental Research** for extending to me the facilities and resources which helped in the completion of this work. I am deeply indebted to my project supervisor, **Prof. Shamik Gupta**, for helping me at every step, and for the numerous encouraging, inspiring and engaging conversations. I am also thankful to the whole team for the valuable suggestions and group discussions.

I am also grateful to **Prof. Titus K Mathew**, **Head, Department of Physics, Cochin University of Science and Technology**, **Dr. Sasidevan**, course coordinator and **Dr. Prasad V V**,my internal project supervisor, for their support during this endeavor.

No acknowledgement, however embellished, can convey my sense of gratitude, for the support and encouragement I received from my family and friends, **Rohitashwa**, **Saswata**, **Devika** and **Parvathy**.

I owe a lot to these people and numerous others who have helped, guided, motivated and inspired me, and I would like to take this opportunity to acknowledge all their support.

«And if we understand things too quickly,
perhaps we shan't understand them thoroughly.»

Fyodor Dostoevsky

Abstract

We study here, the symmetric and asymmetric simple exclusion processes (ASEP/SSEP) on a one dimensional periodic lattice in the presence and absence of stochastic resetting. This work considers three cases, namely, ASEP/SSEP in the absence of resetting, SSEP in the presence of resetting and ASEP in the presence of resetting. For all three cases, we attempt to obtain an expression for the average occupancy by using a mean field approximation, in the stationary state.

We show by using the mean field approximation that, the Master equation, in the stationary state, has an exact solution in the case of ASEP/SSEP in the absence of resetting. This gives the already well known result, that the average occupancy is a constant.

In the case of SSEP with stochastic resetting, we show that symmetry arguments help in simplifying the Master equation in the stationary state, leading to an exact solution. The Monte-Carlo simulations are also found to agree quite well with the theoretical result.

We found that, even using the mean-field approximation, one cannot exactly determine the average occupancy in the stationary state, in the case of ASEP with resetting. But, we show here a fixed point approach to numerically calculate the same up to very high accuracy. The numerical results are also found to be in good agreement with Monte-Carlo simulations.

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Introduction

Simple exclusion processes have been a very important paradigm in non equilibrium statistical mechanics. ASEP/TASEPs are widely regarded as the "Ising Models" of non-equilibrium statistical mechanics. ASEPs and TASEPs have been extensively studied as purely mathematical models [1][2], which have lead to the development of very interesting mathematics. This mathematical simplicity combined with the fact that these systems are integrable, make them an overwhelmingly popular choice for the study of non-equilibrium phenomenon. The model is simple enough for analytical study yet complex enough to demonstrate interesting physical phenomenon such as phase transitions.

One finds utility in the study of these systems because of their wide range of applications in areas ranging from the study of lattice gases [3], transport phenomenon [4] and traffic jams [5] to the study of protein synthesis in RNA [6]. The theoretical study of the ASEP are mainly using two approaches: a matrix ansatz by B. Derrida *et al* [7], and the Bethe ansatz by D. Dhar [8].

We in this work wish to consider the ASEP and SSEP (symmetric simple exclusion process) with periodic boundaries under the presence and absence of stochastic resetting. The aim here, is to obtain the stationary state properties of the system by employing a mean field approximation. The hope with resetting, is to induce in the system, certain properties and behaviors, which are not seen otherwise [9] (For a review of resetting, see Evans, Majumdar and Schehr [10], S. Gupta and A. Jayannavar [11]).

We in this work, were able to derive exact analytical results for the case of SSEP and numerical results for the case of ASEP in the presence of resetting. The results in both cases, when compared with Monte-Carlo simulations, are found to be in good agreement.

1. Brief overview of stochastic processes

1.1 Deterministic and stochastic processes

One deals with deterministic processes a lot in physics, a very simple example being Newton's equations of motion. The reason for calling these equations or "processes" deterministic, is that the evolution of the processes in time will always be exactly the same no matter how many times one repeats it, given that the initial conditions are not changed.

The same cannot be said for stochastic processes. They will evolve in a different trajectory even if one starts from the same initial conditions.

If the configurations attainable by a system are labelled by $C_1, C_2, \ldots, C_n$ and we represent time instances by $t_1, t_2, \ldots$, then, the n-point probability for the occurrence of C_1 at time t_1, C_2 at time t_2, ... etc are represented by the probability $P_n(C_n, t_n; C_{n-1}, t_{n-1}; \ldots; C_1, t_1)$. One may express this as:

$$P_n(C_n, t_n; C_{n-1}, t_{n-1}; \ldots; C_1, t_1) = \mathcal{P}(C_n, t_n | C_{n-1}, t_{n-1}; \ldots; C_1, t_1)\ P_{n-1}(C_{n-1}, t_{n-1}; \ldots; C_1, t_1). \tag{1.1}$$

where $\mathcal{P}(C_n, t_n | C_{n-1}, t_{n-1}; \ldots; C_1, t_1)$ is the conditional probability that the system is in a configuration C_n at time t_n given that the system was in configuration C_{n-1} at time t_{n-1}, in configuration C_{n-2} at time t_{n-2} ..., etc.

1.2 Stationary, Homogeneous and Markov processes

A stochastic process is said to be **stationary** when the whole process is time translation invariant, i.e, the origin of time doesn't matter. Written mathematically, it would read:

$$P_n(C_n, t_n; C_{n-1}, t_{n-1}; \ldots; C_1, t_1) = P_n(C_n, t_n + \tau; C_{n-1}, t_{n-1} + \tau; \ldots; C_1, t_1 + \tau). \tag{1.2}$$

For $n = 1$ and 2, we will have:

$$P_1(C_1, t_1) = P_1(C_1), \tag{1.3}$$

$$P_2(C_2, t_2; C_1, t_1) = f_{t_2 - t_1}(C_2|C_1). \tag{1.4}$$

For a **homogeneous** stochastic process, the conditional probabilities obey the following property:

$$\mathcal{P}(C_2, t_2|C_1, t_1) = f_{t_2 - t_1}(C_2|C_1). \tag{1.5}$$

It is useful to note at this point that a stationary process is homogeneous but not vice-versa. A stochastic process is said to be **Markovian** when it has a lack of memory. In this case the conditional probability may be written as :

$$\mathcal{P}(C_n, t_n|C_{n-1}, t_{n-1}; \ldots; C_1, t_1) = \mathcal{P}(C_n, t_n|C_{n-1}, t_{n-1}). \tag{1.6}$$

This will help reduce the n-point probability to the form:

$$P_n(C_n, t_n; C_{n-1}, t_{n-1}; \ldots; C_1, t_1) = \prod_{i=2}^{n} \Big[\mathcal{P}(C_i, t_i|C_{i-1}, t_{i-1})\Big] P_1(C_1, t_1). \tag{1.7}$$

1.3 The Chapman-Kolmogorov and Master equations

Considering a stationary Markovian stochastic process, one may write the following equation for the conditional probability $\mathcal{P}(C_n, t_n|C_m, t_m)$:

$$\mathcal{P}(C_n, t_n|C_m, t_m) = \sum_{C_l} \mathcal{P}(C_n, t_n|C_l, t_l)\ \mathcal{P}(C_l, t_l|C_m, t_m). \tag{1.8}$$

where $t_m < t_l < t_n$. This is the **Chapman-Kolmogorov** equation.
In the limit of continuous time, the Chapman-Kolmogorov equation gives the **Master equation**:

$$\frac{\mathrm{d}\mathcal{P}(C,t)}{\mathrm{d}t} = \sum_{C' \neq C} \Big[W(C|C')\mathcal{P}(C',t) - W(C'|C)\mathcal{P}(C,t) \Big]. \tag{1.9}$$

where $\mathcal{P}(C,t) = \mathcal{P}(C,t|C_0,t_0)$, and $t > t_0$.

1.4 Stationary states and the condition of detailed balance

A stationary state of a stochastic process may be described as a state where the probability of the system attaining a particular configuration becomes time independent. Written mathematically, $\mathcal{P}(C,t) \to \mathcal{P}^{\mathrm{St}}(C)$ as $t \to \infty$. Thus, looking at (1.9), we see that in a stationary state, we will have:

$$\sum_{C' \neq C} W(C|C')\mathcal{P}^{\mathrm{St}}(C') = \sum_{C' \neq C} W(C'|C)\mathcal{P}^{\mathrm{St}}(C) \quad \forall\ C. \tag{1.10}$$

The stationary states so obtained are called **non-equilibrium stationary states (NESS)**. A very special kind of stationary state is obtained when the sufficient but not necessary condition:

$$W(C|C')\mathcal{P}^{\mathrm{St}}(C') = W(C'|C)\mathcal{P}^{\mathrm{St}}(C) \quad \forall\ C, C', \tag{1.11}$$

is satisfied. This condition is called the **condition of detailed balance** which gives rise to a very special kind of stationary state called, an **equilibrium stationary state**. Equilibrium stationary states will have time reversal invariance, i.e, the probability of occurrence of events $C_1, C_2, \ldots, C_k$ and the probability of occurrence of events in reverse, $C_k, C_{k-1}, \ldots, C_1$ will be

the same.

$$P_{\text{forward}} = W(C_k|C_{k-1})W(C_{k-1}|C_{k-2})\dots W(C_2|C_1)\ P^{\text{St}}(C_1), \tag{1.12}$$

$$P_{\text{reverse}} = W(C_1|C_2)W(C_2|C_3)\dots W(C_{k-1}|C_k)\ P^{\text{St}}(C_k). \tag{1.13}$$

Usually it is very difficult to check the condition of detailed balance. One may instead use the Kolmogorov condition to check if detailed balance is satisfied:

$$\begin{aligned} W(C_2|C_1)W(C_3|C_2)\dots W(C_k|C_{k-1})W(C_1|C_2) = \\ W(C_k|C_1)W(C_{k-1}|C_k)\dots W(C_1|C_2). \end{aligned} \tag{1.14}$$

2. Simple exclusion processes

2.1 The system

The system under consideration is a one dimensional lattice containing L sites. N sites are occupied by interacting particles wherein the interaction is hard-core exclusion, i.e, a particle may occupy a site if and only if the site is empty. The general idea is to define the dynamics for the time evolution of the system (done in the following section), and then look at the system's stationary state.

2.2 Dynamics in the absence of resetting

The dynamics for the evolution of a particular configuration, in continuous time, is as follows: A particle moves between times t and $t + \mathrm{d}t$ to the nearest-neighbour site to the right (respectively, to the left) with rate p (respectively, with rate q).
We define for a site i, the indicator variable n_i as :

$$n_i = \begin{cases} 1 & \text{if site } i \text{ is occupied,} \\ 0 & \text{if site } i \text{ is unoccupied.} \end{cases} \tag{2.1}$$

This allows us to specify a configuration of the system as $\{n_0, n_1, \ldots, n_{L-1}\}$. In discrete time, we will have the following:

$$\begin{aligned} P(\{n_i\}, t + \Delta t) = P(\{n_i\}, t)\Bigg[1 - \sum_{j=0}^{L-1}\Big(p\Delta t\ \delta_{n_j,1}(1 - \delta_{n_{j+1},1}) + q\Delta t\ \delta_{n_j,1}(1 - \delta_{n_{j-1},1})\Big)\Bigg] \\ + \sum_{j=0}^{L-1}\Big(q\Delta t\ \delta_{n_j,1}(1 - \delta_{n_{j+1},1})P(\{n_0, \ldots, n_j - 1, n_{j+1} + 1, \ldots\}, t) \\ + p\Delta t\ \delta_{n_j,1}(1 - \delta_{n_{j-1},1})P(\{n_0, \ldots, n_{j-1} + 1, n_j - 1, \ldots\}, t)\Big). \end{aligned} \tag{2.2}$$

We now convert this to a differential equation by dividing throughout by Δt an taking the limit $\Delta t \to 0$.

$$\frac{\mathrm{d}P(\{n_i\},t)}{\mathrm{d}t} = \sum_{j=0}^{L-1}\Bigg[\Big(qP(\{n_0,\ldots,n_j=0,n_{j+1}=1,\ldots\},t) + pP(\{n_0,\ldots,n_{j-1}=1,n_j=0,\ldots\},t)\Big) - P(\{n_i\},t)\Big(p\ \delta_{n_j,1}(1-\delta_{n_{j+1},1}) + q\ \delta_{n_j,1}(1-\delta_{n_{j-1},1})\Big)\Bigg]. \tag{2.3}$$

Simplifying this, we get:

$$\frac{\mathrm{d}P(\{n_i\},t)}{\mathrm{d}t} = \sum_{j=0}^{L-1}\Big[qP^j_{j+1}(t) + pP^j_{j-1}(t) - pP^{j+1}_j(t) - qP^{j-1}_j(t)\Big], \tag{2.4}$$

where $P_i(t)$ is the probability that the system is in a configuration in which the i^{th} site is occupied, and $P^j_i(t)$ is the joint probability that the system is in a configuration in which the i^{th} site is occupied and the j^{th} site is unoccupied. We observe at this point that:

$$\langle n_i\rangle = \sum_{n_i=0,1} n_i P(\{n_j\},t) = P(\{n_0,\ldots,n_i=1\ \ldots\},t) \equiv P_i(t), \tag{2.5}$$

$$\langle(1-n_{i-1})n_i\rangle = \sum(1-n_{i-1})n_iP(\{n_j\},t) = P(n_0,\ldots,n_{i-1}=0,n_i=1,\ldots,t) \equiv P^{i-1}_i(t). \tag{2.6}$$

Multiplying both sides of (2.4) by n_k and summing:

$$\frac{\mathrm{d}\langle n_k\rangle}{\mathrm{d}t} = q\langle n_{k+1}(1-n_k)\rangle + p\langle n_{k-1}(1-n_k)\rangle - p\langle(1-n_{k+1})n_k\rangle - q\langle(1-n_{k-1})n_k\rangle. \tag{2.7}$$

We change the variable from k to i and look at the form of $\langle n_i \rangle$ in the stationary state, i.e, when $\frac{\mathrm{d}\langle n_i \rangle}{\mathrm{d}t} = 0$. Setting the LHS as 0 in (2.7) we get:

$$0 = q\Big(\langle n_{i+1}\rangle - \langle n_i n_{i+1}\rangle - \langle n_i\rangle + \langle n_{i-1}n_i\rangle\Big) - p\Big(\langle n_i\rangle - \langle n_{i+1}n_i\rangle - \langle n_{i-1}\rangle + \langle n_i n_{i-1}\rangle\Big). \quad (2.8)$$

To further simplify the above, we use a mean field approximation $\langle n_i n_j \rangle = \langle n_i \rangle \langle n_j \rangle$ for $i \neq j$. Using this in (2.8) we obtain:

$$\langle n_i \rangle = \frac{q\langle n_{i+1}\rangle + p\langle n_{i-1}\rangle}{\Big\{q + p + (q-p)\Big[\langle n_{i+1}\rangle - \langle n_{i-1}\rangle\Big]\Big\}}. \quad (2.9)$$

Looking at the system, we recognize that no lattice site is special. This would mean that the solution to (2.9) would be of the form $\langle n_i \rangle =$ a constant C. The constant C can be found using the normalization condition:

$$\sum_{i=0}^{L-1} \langle n_i \rangle = N. \quad (2.10)$$

which will give us $\langle n_i \rangle = \frac{N}{L}$.

2.3 Dynamics in the presence of resetting

The position of the walker is reset in time with a constant rate r to the origin $i = 0$. Thus, in an infinitesimal time interval between $[t, t + \mathrm{d}t]$, the walker may perform either (a) biased hop with probability $1 - r\mathrm{d}t$, or, (b) reset with probability $r\mathrm{d}t$. Similar to hopping, resetting is permitted only if the target site (here, the origin) is vacant.

With the new dynamics in mind, we write the modified Master equation in terms of the joint

probabilities defined in (2.5) and (2.6):

$$\frac{\mathrm{d}P_i(t)}{\mathrm{d}t} = \left[q\Big(P^i_{i+1}(t) - P^{i-1}_i(t)\Big) + p\Big(P^i_{i-1}(t) - P^{i+1}_i(t)\Big)\right] - rP^0_i(t) + r\delta_{i,0}\sum_{j=1}^{L-1} P^0_j(t). \quad (2.11)$$

We again would like to represent this in terms of the indicator variable n_i:

$$\frac{\mathrm{d}\langle n_i\rangle}{\mathrm{d}t} = \left[q\Big(\langle(1-n_i)n_{i+1}\rangle - \langle(1-n_{i-1})n_1\rangle\Big) + p\Big(\langle n_{i-1}(1-n_i)\rangle - \langle n_i(1-n_{i+1})\rangle\Big)\right] - r\langle(1-n_0)n_i\rangle + r\delta_{i,0}\sum_{j=1}^{L-1}\langle(1-n_0)n_j\rangle. \quad (2.12)$$

Again invoking the mean field approximation to find the average occupancy in the stationary state, we have:

$$\langle n_i\rangle = \frac{q\langle n_{i+1}\rangle + p\langle n_{i-1}\rangle + r\delta_{i,0}\Big[(1-\langle n_0\rangle)(N-\langle n_0\rangle)\Big]}{q+p+(q-p)\Big(\langle n_{i+1}\rangle - \langle n_{i-1}\rangle\Big) + r(1-\langle n_0\rangle)}, \quad (2.13)$$

where we have made us of the normalization condition (2.10) to evaluate the last term in (2.12).

2.3.1 Symmetric hopping

We first consider the special case in which the hopping rates q and p are the same. Thus we set $q = p = \lambda$. Defining the quantity $\kappa = \lambda/r$, we can rewrite (2.13) as:

$$\langle n_i\rangle = \frac{\kappa\langle n_{i+1}\rangle + \kappa\langle n_{i-1}\rangle + \delta_{i,0}\Big[(1-\langle n_0\rangle)(N-\langle n_0\rangle)\Big]}{2\kappa + (1-\langle n_0\rangle)}. \quad (2.14)$$

Considering the case when $i = 0$, we will have:

$$\langle n_0 \rangle = \frac{\kappa\langle n_1 \rangle + \kappa\langle n_{L-1} \rangle + \Big[(1 - \langle n_0 \rangle)(N - \langle n_0 \rangle)\Big]}{2\kappa + (1 - \langle n_0 \rangle)}. \tag{2.15}$$

Since, the hopping is symmetric, the average occupancy of both the neighbors of $i = 0$ will be the same because a particle is equally likely to hop to the left as well as to the right from $i = 0$. Thus, setting $\langle n_1 \rangle = \langle n_{L-1} \rangle = \rho$ in (2.15), we can get and equation for ρ in terms of $\langle n_0 \rangle$ as:

$$\rho = \langle n_0 \rangle \Big[1 + \frac{1}{2\kappa}(1 - \langle n_0 \rangle)\Big] - \frac{1}{2\kappa}\Big[(N - \langle n_0 \rangle)(1 - \langle n_0 \rangle)\Big]. \tag{2.16}$$

Using $i \neq 0$ in (2.14):

$$\langle n_{i+1} \rangle = \omega\langle n_i \rangle - \langle n_{i-1} \rangle, \tag{2.17}$$

where $\omega = 2 + \frac{1}{\kappa}(1 - \langle n_0 \rangle)$. The above equations enable us to determine $\langle n_i \rangle$s in terms of $\langle n_0 \rangle$. This along with (2.10) gives us exact values for the $\langle n_i \rangle$s.

2.3.2 Asymmetric hopping

In the case where $p \neq q$, defining $\kappa_1 = q/r$ and $\kappa_2 = p/r$, (2.13) will read:

$$\langle n_i \rangle = \frac{\kappa_1\langle n_{i+1} \rangle + \kappa_2\langle n_{i-1} \rangle + \delta_{i,0}\Big[(1 - \langle n_0 \rangle)(N - \langle n_0 \rangle)\Big]}{\kappa_1 + \kappa_2 + (\kappa_1 - \kappa_2)\Big(\langle n_{i+1} \rangle - \langle n_{i-1} \rangle\Big) + (1 - \langle n_0 \rangle)}. \tag{2.18}$$

Using the above, we can write the recursion relation:

$$\langle n_{i+1}\rangle = \frac{\kappa_2\langle n_{i-1}\rangle - \langle n_i\rangle\Big[\kappa_1+\kappa_2-(\kappa_1-\kappa_2)\langle n_{i-1}\rangle + (1-\langle n_0\rangle)\Big]}{\Big[\langle n_i\rangle(\kappa_1-\kappa_2)-\kappa_1\Big]}. \tag{2.19}$$

Unfortunately in this case, the equations can only be solved numerically. The fixed point approach described below, can be used to calculate the $\langle n_i\rangle$s up to great numerical precision (Refer A) .

2.3.2.1 Fixed point approach for calculating $\langle n_i\rangle$s

1. First, begin with a set of $\langle n_i\rangle$s, $i = 0, 1, \ldots, L-1$, which obey the conditions $0 \leq \langle n_i\rangle \leq 1 \quad \forall\ i$ and (2.10).
2. Compute new $\langle n_i\rangle$s according to (2.19) using the given $\langle n_i\rangle$s.
3. Compare with the old $\langle n_i\rangle$ values to see if they have settled into the stationary state values, i.e, have reached a fixed point (stable).
4. If not, repeat the above steps until the profile reaches a fixed point.

2.4 Monte-Carlo simulations

The general algorithm for Monte-Carlo simulation of the dynamics of the system is given here. C++ code implementing the same can be found in (B).

1. Choose parameters L, N, r, q, p and a time step $\mathrm{d}t$. The time step should be such that $(p+q)\mathrm{d}t = 1$ and $r\mathrm{d}t \leq 1$.
2. Consider a lattice of any length L and occupy it randomly with N particles.
3. For the dynamics, choose a random number R_1 between 0 and 1. If $R_1 < r\mathrm{d}t$, pick a site at random and attempt to reset provided the following conditions hold.
   (a) The site chosen is occupied.
   (b) The site $i = 0$ is unoccupied.
4. If $R_1 > r\mathrm{d}t$, then choose another random number R_2 between 0 and 1. If $R_2 < q\mathrm{d}t$,

choose a site at random and attempt to make a hop to the left subject to the following conditions:

(a) The site chosen is occupied.

(b) If the site chosen is the site $i = 0$, then site $L - 1$ is unoccupied. If the chosen site is not $i = 0$, then site $i - 1$ is unoccupied.

5. If $R_2 > q\mathrm{d}t$, choose a site at random and attempt to make a hop to the right subject to the following conditions:

   (a) The chosen site is occupied.

   (b) If the site chosen is $i = L - 1$, then the site $i = 0$ is unoccupied. If the chosen site is not $i = L - 1$, then site $i + 1$ is unoccupied.

6. In each time step, L such attempts need to be made. This is because, according to the dynamics defined for this system, the probability for a hop to the left (say) is:

$$P = P_i^{i-1}(t)q\mathrm{d}t \tag{2.20}$$

   While simulating the dynamics, we go through the following process:

   (a) Choosing a site at random with probability $1/L$.

   (b) Hopping according to probability $q\mathrm{d}t$ provided that the system allows it.

   Thus, the probability for a left hop will be $\frac{1}{L}P_i^{i-1}(t)q\mathrm{d}t$. Thus we need to make L such attempts in one time step $\mathrm{d}t$ so that we neutralize the $1/L$ factor.

7. The system needs to evolve for a series of time steps, t_{stat}, until it reaches the stationary state.

8. For performing another realization, instead of beginning again from a random configuration and letting the system evolve for a time t_{sat}, we could start with the configuration at time t_{stat} and let the system further evolve for a decorrelation time t_{decorr} where $t_{decorr} < t_{stat}$. This saves computation time.

9. We average over a suitable number of realizations to find the average occupancy.

3. Results

3.1 Dynamics in the absence of resetting

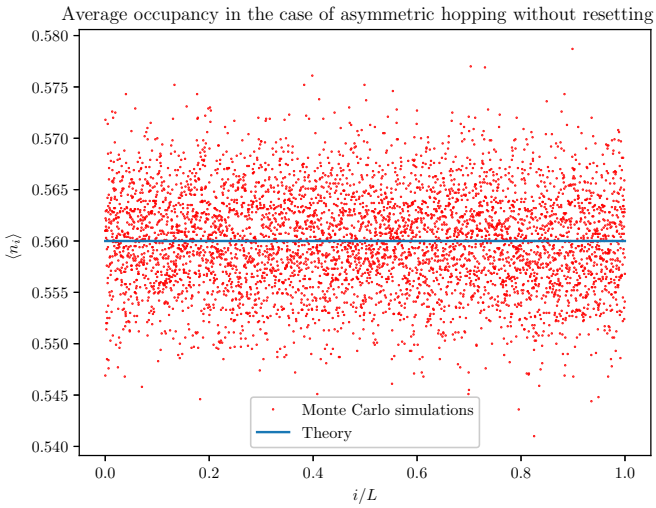


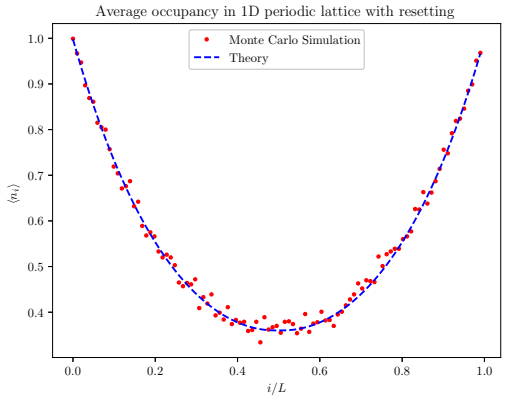
Figure 1: The average occupancy $\langle n_i \rangle$ as a function of i/L for a system with parameters $L = 5000$, $N = 2800$ and $q = 0.3$.

It was shown in (2.2) that for the ASEP/SSEP in the absence of resetting, the density profile will be a constant given by $\langle n_i \rangle = \frac{N}{L}$. It is evident from Figure 1, that the theory and simulations agree quite well.

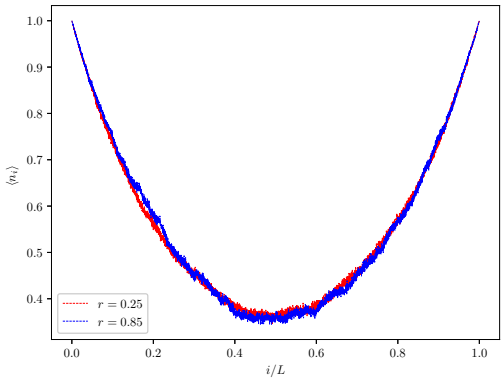
3.2 Symmetric hopping in the presence of resetting

As shown in (2.3), the recursive equations to determine the average occupancy $\langle n_i \rangle$ could be exactly solved for the SSEP, i.e, $p = q$. From the symmetry in the dynamics, one would assume that the density profile will be symmetric about the point $i = 0$, because all particles are reset to this site. We expect the average occupancy to have a maximum at $i = 0$ and then to decrease toward either side of $i = 0$, symmetrically. We see in Figure 2a, that this is the case. One interesting feature here is the dependence of the profile on the value of r as shown in Figure 2b.

Figure 2: Average occupancy $\langle n_i \rangle$ for the case of SSEP with resetting.



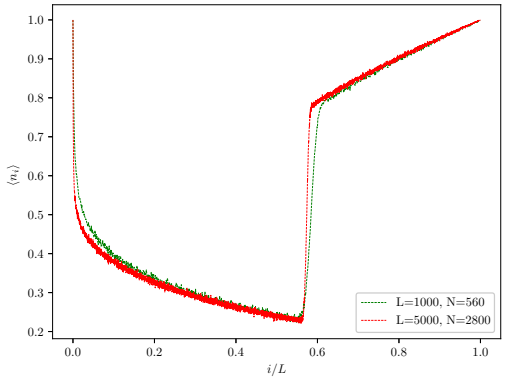
(a) The average occupancy $\langle n_i \rangle$ as a function of i/L for a system with $L = 101$, $N = 56$ and $q = 0.5$ and $r = 0.25$.



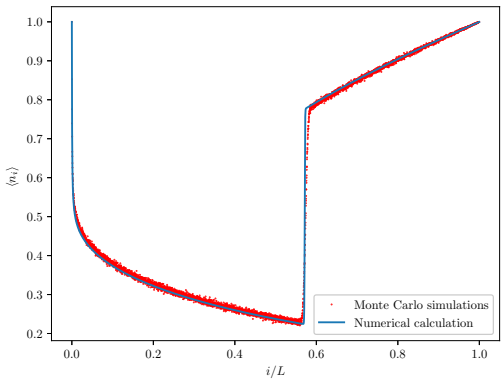
(b) The average occupancy for two different values of r, for a lattice of length $L = 5000$ and number of particles $N = 2800$.

3.3 Asymmetric hopping with resetting

In the case of asymmetric hopping in the presence of resetting, one expects the density profile to decay differently toward the left and right of the site $i = 0$. Because particles get reset to the site $i = 0$, we still expect $\langle n_0 \rangle$ to be the largest of the $\langle n_i \rangle$s. An interesting feature in this case is the shock in the density profile. We expect that the density profile decays asymmetrically to the left and right from $i = 0$, but the enforced periodic boundary condition dictates that the profile will have to match up at some point on the lattice. This causes the shock.

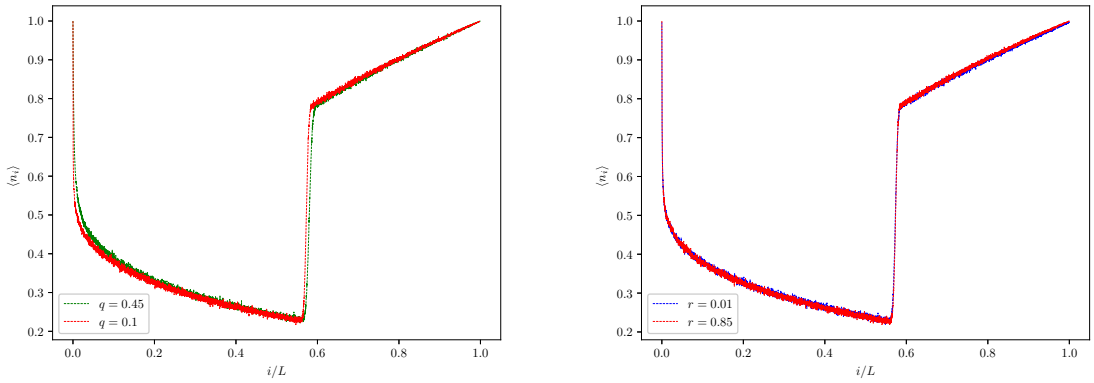


(a) A comparison of the average occupancy $\langle n_i \rangle$, as a function of i/L for two lattices of sizes $L = 1000,\ 5000$ and number of particles $N = 560,\ 2800$. The simulations were done for $q = 0.3$ and $r = 0.25$.



(b) A comparison of the average occupancy $\langle n_i \rangle$, as a function of i/L with the numerical calculations for the case of a system with $L = 5000$, $N = 2800$, $q = 0.3$ and $r = 0.25$.

Figure 3: Average occupancy $\langle n_i \rangle$ for the case of ASEP with resetting.



(a) Simulated for $r = 0.25$.

(b) Simulated for $q = 0.3$.

Figure 4: Monte-Carlo simulation of the average occupancy for a lattice of size $L = 5000$, number of particles, $N = 2800$.

One could look at how this density profile varies with q and r (Figures 4a, 4b). It is clear from Figure 4a that there is no appreciable variation of the density profile with q. It is also apparent that the profile remains practically the same with variation in the resetting rate r, which was also seen in the case of SSEP with resetting. This could be because, looking at the dynamics, the chief purpose of r is to cause a pile-up near the site $i = 0$. The height of the shock remains the same, because, it is mostly caused by the hard-core exclusion between the particles.

One other feature of interest, which is apparent also from the way the dynamics is defined, is that, we expect the density profile for the case $q = \kappa$ and $q = 1 - \kappa$, to be mirror images of each other about $i = 0$. This was confirmed by performing simulations (Figure 5).

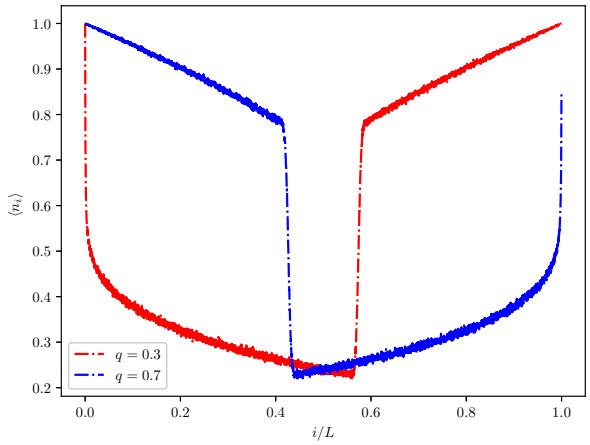


Figure 5: Average occupancy $\langle n_i \rangle$ for the case of ASEP with resetting simulated in a lattice with $L = 5000$ sites and $N = 2800$ particles for $r = 0.25$.

Conclusions

We have studied here the case of simple exclusion processes in the presence and absence of stochastic resetting.The importance of ASEPs and SSEPs as a model for the study of non-equilibrium phenomenon can hardly be overstated.

We first showed, using the mean-field approximation, that the density profile of an ASEP is a constant in the stationary state, i.e, $\langle n_i \rangle = N/L$.

In the case of SSEP with resetting, we found, using the mean-field approximation, exact values for the average occupancies in the stationary state. The incorporation of resetting in the dynamics, created a peak for the occupancy at the site $i = 0$. We also showed using simulations, the weak dependance of the profile on the value of the resetting rate r.

For the ASEP in the presence of resetting, we were able to obtain recursive equations for the average occupancy in terms of two unknowns. We have shown that numerical calculations of the same, using the already discussed fixed point method, yield results that are in excellent agreement with the simulations. We further explored some interesting features of the profile of average occupancy, namely, its dependence on the values of q and r. Unsurprisingly, the profile for the ASEP also, showed negligible dependance on the value of r. The dependence on the value of q is also minute, but more significant when compared with the dependance on r.

In the future, we would like to look at the case of the same system where the hopping is site dependant. Another interesting avenue, is to study this system and try to establish a correspondence with the ASEP with a slow site, where a similar shock in the density profile is observed [12].

Appendix

A. Python code for implementing the fixed point approach for calculating the average occupancies in the case of ASEP with resetting

```
# Program to use a fixed point approach for finding the occupation
# probabilities in the assymetric hopping case.
# @author : C L Sriram
# Created on 30/04/2023 1514 hrs


# Importing the necessary files
import numpy as np
import matplotlib.pyplot as plt
import pandas as pd

# Writing necessary functions
def comparison(n_Array_t, n_Array, Err):
    L = len(n_Array)
    Flag = 1
    for i in range(len(n_Array)):
        if(abs(n_Array[i] - n_Array_t[i]) > Err):
            Flag = 0
    return Flag

# Defining the variables.
q = 1.8
p = 2.2
Err = 1e-11
n_Array = """ Input data for SSEP with resetting for same L and N """
```

```
L = len(n_Array)
n_Array_t = np.zeros(L)
N = np.sum(n_Array)
Flag = 0
while(1):
    n0 = n_Array[0]
    n_1 = n_Array[1]
    nL_1 = n_Array[-1]
    for i in range(L):
        if(i==0):
            n_Array_t[i] = ((2 + p - n_1*p + N + p*nL_1 + q + n_1*q - nL_1*q)
                           - np.sqrt(-8*(N + n_1*q + nL_1*p) + ((-2 - p + n_1*p -
                              nL_1*p - N - q - n_1*q + nL_1*q)**2)))/4
        elif(i==(L-1)):
            n_Array_t[i] = (q*n_Array[0] + p*n_Array[i-1])/(q+p+(q-p)*(n_Array[0] -
                n_Array[i-1]) + (1-n_Array[0]))
        else:
            n_Array_t[i] = (q*n_Array[i+1] +
                p*n_Array[i-1])/(q+p+(q-p)*(n_Array[i+1] - n_Array[i-1]) +
                (1-n_Array[0]))
    if(comparison(n_Array_t, n_Array, Err)):
        break
    else:
        n_Array = n_Array_t
        n_Array_t = np.zeros(L)

# Save the data n_Array
```

B. C++ code for Monte Carlo simulation of SSEP and ASEP in the presence of resetting

```
// Program to simulate the dynamics of N interacting particles on a 1D periodic
    lattice
// @author : C L Sriram
// Created on 11 April 2023 1518 hrs

// Importing the necessary header files
#include <iostream>
#include <vector>
#include<cstring>
#include <random>
#include <fstream>
#include <chrono>
#include <ctime>
using namespace std::chrono;
using namespace std;
random_device rd;
mt19937 mt(rd());
uniform_real_distribution<double> dist(0.0, 1.0); //range is 0 to 1

// Defining useful functions
// Printing a vector
void print_vector(vector<vector<int>> &A)
{
    for (int i = 0; i <A.size(); i++)
    {
        for (int j = 0; j < A.at(i).size(); j++)
            cout << A[i][j] << " ";
        cout << endl;
```

```
    }
}

void print_vector(vector<vector<double>> &A)
{
    for (int i = 0; i <A.size(); i++)
    {
        for (int j = 0; j < A.at(i).size(); j++)
            cout << A[i][j] << " ";
        cout << endl;
    }
}

void print_vector(vector<double> &A)
{
    for(int i = 0; i< A.size(); i++)
        cout<< A[i] << " ";
    cout<< endl;
}

void print_vector(vector<int> &A)
{
    for(int i = 0; i< A.size(); i++)
        cout<< A[i] << " ";
    cout<< endl;
}

// Addtion of two vectors
vector<long double> vector_add(vector<long double> &A, vector<int> &B)
{
    vector<long double> Sum(A.size(), 0);
    for (int i =0; i< A.size(); i++)
        Sum[i] += A[i] + B[i];
    return Sum;
}
```

```
// Functions for saving data
void save(const char* Filename, vector<long double> &Data)
{
    fstream f;
    f.open(Filename, ios::out|ios::binary);
    for (auto content:Data)
        f<<content<<endl;
    f.close();
}

// Simulating the dynamics
vector<int> dynamics (vector<int> Lattice, int Iter, double q, double p, double r,
    double dt = 1.0)
{
    int Jump;
    int L = Lattice.size(), Site, NN;
    double Temp, Temp2;
    for (int i=0; i<Iter; i++)
    {
        Jump = 0;
        while(Jump <L)
        {
            Site = rand() %L;
            Temp = dist(mt);
            Jump ++;
            if(Temp < r*dt)
            {
                if(Lattice[Site] == 1)
                {
                    Jump ++;
                    if(Lattice[0] == 0)
                    {
                        Lattice[0] = 1;
                        Lattice[Site] = 0;
```

```
            }
        }
    }
    else
    {
        Temp2 = dist(mt);
        if(Temp2 < q*dt)
        {
            if(Site != 0)
            {
                NN = Site-1;

            }
            else
            {
                NN = L-1;
            }
            if((Lattice[Site] == 1) && (Lattice[NN] == 0))
            {
                Lattice[Site] = 0;
                Lattice[NN] = 1;
            }
        }
        else
        {
            if(Site != (L-1))
            {
                NN = Site+1;
            }
            else
            {
                NN = 0;
            }
            if((Lattice[Site] == 1) && (Lattice[NN] == 0))
            {
```

```
                    Lattice[Site] = 0;
                    Lattice[NN] = 1;
                }
            }
        }
    }
    return Lattice;
}

// Main function
int main()
{
    // Defining variables
    double Rho = 0.5600;  // Particle density N/L
    int L = 101;   // Number of lattice sites
    int N = Rho*L;   // Number of particles
    //double dt = 1.0/L;  // Small time step
    double q = 0.5000;  // Rate of hopping to the left
    double r = 0.2500;  // Rate of resetting
    double p = 1.0000-q;  // Rate of hopping to the right
    int Iter1 = L*L;  // Number of iterations
    int Iter2 = 1000*L; // Decorrelation time
    int Iter3 = 1000; // Number of realisations over which averaging is done
    vector<long double> Densities(L, 0);

    // Seeding the random number generator
    srand(time(NULL));
    // Creating the array
    vector<int> Lattice(L, 0);

    // Populating the lattice
    int Site, k=0;
    while (k < N)
    {
```

```
        Site = rand() % L;
        if (Lattice[Site] == 0)
        {
            Lattice[Site] = 1;
            k++;
        }

    }

    // Simulating the dynamics until the stationary state
    Lattice = dynamics(Lattice, Iter1, q, p, r);

    // Simulating the dynamics until decorrelation and then averaging the
        occupancies
    auto Start = high_resolution_clock::now();
    for (int j=0; j<Iter3; j++)
    {
        Lattice = dynamics(Lattice, Iter2, q, p, r);
        Densities = vector_add(Densities, Lattice);
    }
    auto Stop = high_resolution_clock::now();
    auto Duration = duration_cast<microseconds> (Stop-Start);
    cout<< "Total runtime = "<< Duration.count()<< " microseconds"<< endl;

    // Calculating the site occupation densities
    for (int i = 0; i < Densities.size(); i++)
        Densities[i] = Densities[i]/Iter3;
    //print_vector(Densities);
    save(/* Insert Filename*/, Densities);
    return 0;
}
```

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