STAT 639V - Assignment 2

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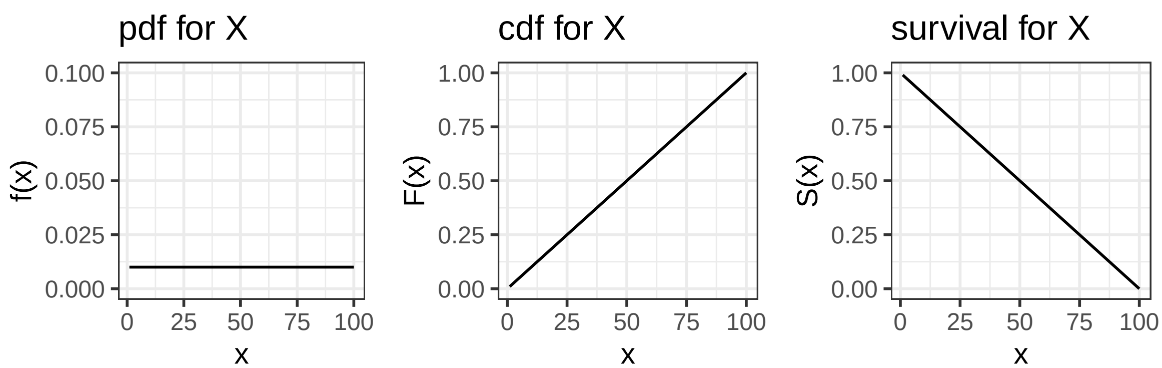
1. Let X have a uniform distribution on 0 to 100 days with probability density function $f(x) = 1/100$ for $0 < x < 100$. (10 points)



(a) Find the survival function at 25, 50 and 75 days.

$$S(x) = 1 - F(x)$$

The survival function at 25 days:

$$S(X = 25) = 1 - F(X = 25) = 1 - P(X \leq 25) = 1 - \frac{25 - 1 + 1}{100} = 1 - \frac{25}{100} = \boxed{0.75}$$

The survival function at 50 days:

$$S(X = 50) = 1 - F(X = 50) = 1 - P(X \leq 50) = 1 - \frac{50 - 1 + 1}{100} = 1 - \frac{50}{100} = \boxed{0.50}$$

The survival function at 75 days:

$$S(X = 75) = 1 - F(X = 75) = 1 - P(X \leq 75) = 1 - \frac{75 - 1 + 1}{100} = 1 - \frac{75}{100} = \boxed{0.25}$$

(b) Find the mean residual lifetime at 25, 50 and 75 days.

We know that

$$mrl(X = k) = \frac{\int_k^\infty (t - k) \times f(t)dt}{S(X = k)}$$

The mean residual lifetime at 25 days:

$$mrl(X = 25) = \frac{\int_{25}^{100} (t - 25) \times \frac{1}{100} dt}{\frac{3}{4}} = \frac{1}{75} \int_{25}^{100} (t - 25)dt = \frac{1}{75} (\frac{1}{2} t^2 \mid_{25}^{100} - 25t|_{25}^{100})$$

$$= \frac{1}{75}(\frac{1}{2} \times 100^2 - \frac{1}{2} \times 25^2 - 25 \times 100 + 25 \times 25) = 37\frac{1}{2} = \boxed{37.5} days$$

The mean residual lifetime at 50 days:

$$mrl(X = 50) = \frac{\int_{50}^{100}(t-50) \times \frac{1}{100}dt}{\frac{1}{2}} = \frac{1}{50}\int_{50}^{100}(t-50)dt = \frac{1}{50}(\frac{1}{2}t^2 \mid_{50}^{100} - 50t|_{50}^{100})$$
$$= \frac{1}{50}(\frac{1}{2} \times 100^2 - \frac{1}{2} \times 50^2 - 50 \times 100 + 50 \times 50) = \boxed{25} days$$

The mean residual lifetime at 75 days:

$$mrl(X = 75) = \frac{\int_{75}^{100}(t-75) \times \frac{1}{100}dt}{\frac{1}{4}} = \frac{1}{25}\int_{75}^{100}(t-25)dt = \frac{1}{25}(\frac{1}{2}t^2 \mid_{75}^{100} - 75t|_{75}^{100})$$
$$= \frac{1}{25}(\frac{1}{2} \times 100^2 - \frac{1}{2} \times 75^2 - 75 \times 100 + 75 \times 75) = \boxed{12.5} days$$

2. Let X have a uniform distribution on [0,θ], find the hazard function and mean residual life function of X. (10 points)

given: $h(x) = \frac{f(x)}{S(x)}$. Assuming that X is continuous, we know that $f(x) = \frac{1}{\theta - 0} = \frac{1}{\theta}$ and $S(x) = 1 - \frac{x}{\theta} = \frac{\theta - x}{\theta}$ for $x \in [0, \theta]$.

Therefore, $h(x) = \frac{\frac{1}{\theta}}{\frac{\theta - x}{\theta}} = \frac{1}{\theta} \times \frac{\theta}{\theta - x} = \boxed{\frac{1}{\theta - x}}$

To calculate the mean residual life for a given x, we have:

$$mrl(x) = \frac{\int_x^\infty (t-x)f(t)dt}{S(x)} = \frac{\int_x^\theta (t-x)\frac{1}{\theta}dt}{\frac{\theta - x}{\theta}} = \frac{\frac{1}{\theta}\int_x^\theta (t-x)dt}{\frac{\theta - x}{\theta}}$$

$$= \frac{\frac{1}{\theta}(\frac{1}{2}t^2 \mid_t^\theta - xt|_t^\theta)}{\frac{\theta - x}{\theta}} = \frac{1}{\theta - x}(\frac{1}{2}t^2 \mid_t^\theta - xt|_t^\theta)$$

$$= \frac{1}{\theta - x}(\frac{1}{2} \times \theta^2 - \frac{1}{2} \times x^2 - \theta x + x^2) = \frac{1}{\theta - x}(\frac{\theta^2}{2} + \frac{x^2}{2} - \theta x) = \frac{1}{2}\frac{1}{\theta - x}(\theta - x)^2 = \boxed{\frac{\theta - x}{2}}$$

3. The lifetime of light bulbs follows an exponential distribution with a hazard rate of 0.001 failures per hour of use ($\lambda = 0.001$). (10 points) (a) Find the mean lifetime of a randomly selected light bulb.

The mean lifetime of a randomly selected lightbulb whose lifetime follows an exponential distribution with $h(x) = \lambda = 0.001$ is the expectation of the r.v. X, the lifetime of lightbulbs.

Because $\lambda = \frac{1}{1000}$, the $E(X) = \frac{1}{\lambda} = \boxed{1000}$ This is a characteristic of the expectation of an exponential distributuion.

(b) What is the probability a light bulb will still function after 2,000 hours of use?

From the hazard function, we know that

$$S(x) = e^{-\int_0^x \frac{1}{1000}du} = e^{\frac{-x}{1000}}$$

Solving the survival function at X = 2000, we have:

$$S(X = 2000) = exp(\frac{-2000}{1000} = e^{-2} \approx \boxed{0.135})$$

which means the probability a light bulb will survive over 2,000 hours of use is about 0.135.

4. The time in days to development of a tumor for rats exposed to a carcinogen follows a Weibull distribution with $\alpha = 2$ and $\lambda = 0.001$. (10 points)

(a) What is the probability a rat will be tumor free at 30 days?

Let X be the time to tumor development in days where $X \sim Weibull(\alpha = 2, \lambda = 0.001)$

The p.d.f. of the weibull distribution is

$$f(x) = \alpha\lambda e^{-\lambda x^{\alpha}} \times x^{\alpha-1} = \frac{2}{1000} e^{\frac{-x^2}{1000}} \times x^{2-1} = \frac{x}{500} e^{\frac{-x^2}{1000}}$$

From this p.d.f., the $S(t)$ or $P(X > t)$ can be found by integration from t to ∞:

$$S(t) = P(X > t) = \int_t^{\infty} \frac{x}{500} e^{\frac{-x^2}{1000}} dx = -e^{\frac{-x^2}{1000}} |_t^{\infty} = e^{\frac{-t^2}{1000}}$$

Setting \$ t = 30\$, we find:

$$P(X > 30) = e^{\frac{-30^2}{1000}} = e^{-0.9} \approx \boxed{0.407}$$

(b) What is the mean time to tumor? $(\Gamma(0.5) = \sqrt{\pi})$.

We know that for the Weibull distribution, the

$$E(X) = \Gamma(1 + \frac{1}{\alpha}) \times \lambda^{\frac{-1}{\alpha}}$$

so

$$E(X) = \Gamma(1 + \frac{1}{2}) \times (\frac{1}{1000})^{-\frac{1}{2}} = \Gamma(\frac{3}{2}) \times \sqrt{1000} = \frac{1}{2}\sqrt{\pi} \times \sqrt{1000} \approx \boxed{28.02} days$$

(c) Find the median time to tumor.

Let $S(x) = 0.5$ and solve for x to find the median time to tumor:

$$S(t) = 0.5 = e^{\frac{-t^2}{1000}}$$

$$log(0.5) = \frac{-t^2}{1000}$$

$$\sqrt{-1000 \times log(0.5)} = t \approx \boxed{26.33}$$