Deliberate model misspecification for weather radar signal Doppler spectrum moments estimation

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***Abstract*—Deliberate model misspecification can be a way to workaround over-complicated estimation problems. In weather radar signal estimation, the key parameters involved in most numerical weather prediction algorithms are the first two moments of the signal Doppler spectrum: the mean power, related to reflectivity coefficients and the mean Doppler frequency, related to the mean radial velocity. The spectral width of the Doppler spectrum is not always used (or even estimated) in operational systems which lets think that one could simply discard this parameter and apply potentially simpler estimators to obtain the desired estimates. This falls in the field of estimation theory under model misspecification in the case of an unconditional signal model. In this contribution, the misspecified Cramér-Rao bound for the problem at hand is evaluated to compare two estimators: the candidate mismatched unconditional maximum likelihood estimator and the pulse-pair estimator widely used in operational systems. Simulations show how the spectral width affects the considered estimators performance. The widely used pulse-pair estimator happens to remain the best option being both unbiased and with a fairly simple implementation, even though there is still room for improvement for the estimation of the mean Doppler frequency.**

***Index Terms*—weather radar, power spectrum, Doppler frequency, Cramér-Rao bound, model misspecification**

I. Introduction

Weather radar is at the center of most of the climate observation networks as it is able to detect rain drops over large areas. National meteorological agencies are using several weather radars along with rain gauges in order to monitor rain precipitations over inhabited regions. Weather radars operate in S-, C- and X-bands from 2 GHz to 12.5 GHz which is the ideal frequency range to observe rain drops without being too much attenuated.

Radars transmit short and powerful pulses that are partly absorbed and partly scattered in all directions by the raindrops contained in a so-called resolution volume. A small portion of these scattered pulses is received by the radar. They are then processed to create complex time series whose spectral features are closely related to physical properties of the scatterers within the resolution volume. For most conventional Doppler radars [1], the first two spectral moments are of interest: i) the zeroth moment, corresponding to the mean power, can be linked to rain rate through empirical Z-R relationships and ii) the first moment, or the mean Doppler frequency, gives an information on the mean radial motion of the drops with respect to the radar. The second moment, which is the standard deviation of the power spectrum (or spectral width) can be related to the velocity dispersion of the drops within the resolution volume. However, the spectral width is not always used and sometimes, it is not even estimated in operational systems.

In weather applications as well as in other Doppler-spread radar applications, spectrum moments estimation has been thoroughly studied and a large number of estimators have been proposed in the literature [2]. These methods can be separated into two categories: periodogram-based methods in which an estimate of the spectrum itself is obtained and its features are extracted based on the assumed shape [3], and the pulse-pair methods where variations of the signal autocorrelation are related to the spectrum moments [4]. Whatever the method used, this problem is an estimation problem that can be characterized using theoretical tools. In the case of the estimation of the first three spectrum moments, the Cramér-Rao bound (CRB) can be evaluated as a reference as it is done in [5], [6]. Efficient estimators that achieve this lower bound may not exist but unconditional maximum likelihood (UML) based estimators are known to be asymptotically efficient when the number of independent observations is large enough [7], [8].

Estimating properly all the unknown parameters of the Doppler spectrum can be cumbersome and the fact that the spectral width is not always used in operational systems can tempt one to overlook this parameter. But what is the price to ignore the spectral width? This question can be tackled under the angle of misspecified estimation: if one assumes that all the power is concentrated on a unique Doppler frequency, this results in a signal model misspecification. The estimator adapted to the assumed model turns out to be a misspecified estimator. Performance of such estimators was first studied in [9]–[11], and the mathematical framework to link such performance to the corresponding lower bounds was recently introduced in [12], [13]. In particular, it was shown that the mismatched maximum likelihood estimator is misspecified-unbiased and asymptotically efficient: it tends in probability to a so-called pseudo-true parameter which minimizes the Kullback-Leibler divergence between the true and the assumed probability density functions and its variance achieves the misspecified Cramér-Rao bound (MCRB). In [12], generalized Slepian-Bangs formulas were derived for the misspecified estimation problem.

In this contribution, the question of the effect of purposefully omitting the spectral width for weather radar signals is tackled. The problem is formulated as a misspecified estimation problem with unconditional signal models. Helped with a numerical evaluation of both the MCRB and the CRB of the true signal model, a comparison between the mismatched UML estimator and the widely used pulse-pair estimator is proposed.

II. Weather signal models

The radar transmits pulses of duration T_p, with a given pulse repetition time T_s. This signal illuminates distributed rain drops and a small part of it is back-scattered towards the radar. This back-scattered signal is first pulse-by-pulse processed through matched filtering to get a complex time series sampled at T_s of the so-called weather signal $x(t)$.

A. True signal model

Given the potential large number of rain drops in the resolution volume, the power spectrum of the received signal is an average of all these contributors. Depending on the weather conditions in the considered volume, the rain drops will not have the same global motion with respect to the radar. Consequently, the received power may not be concentrated at a unique Doppler frequency but rather spread around a mean frequency with a certain distribution. Despite a number of counter examples, the power spectrum of weather signals is often assumed to have a Gaussian shape [14] defined by its first three moments the mean power P_0, the mean Doppler frequency f_0 and the spectral width σ_0. The power spectrum can then be written as in [6]:

$$P_x(f) = \frac{P_0}{\sqrt{2\pi\sigma_0^2}} \exp\left(-\frac{(f-f_0)^2}{2\sigma_0^2}\right) + \sigma_n^2, \qquad (1)$$

where σ_n^2 is the noise power, assumed white. From (1) one can evaluate the auto-correlation function of the signal by taking the inverse Fourier transform:

$$A(\tau) = P_0 \exp\left(-2\pi^2\sigma_0^2\tau^2\right) \exp\left(j2\pi f_0 \tau\right) + \sigma_n^2 \delta(\tau), \qquad (2)$$

where $\delta(\tau)$ is the Dirac function.

Consider the reception K independent recording data $\mathbf{y}_k$, $k \in (1, K)$ where each observation k consists of a N-by-1 vector sampled at a sampling period T_s. Dropping the index k for the sake of clarity, the received signal can be represented as a vector:

$$\mathbf{y} = \mathbf{x} + \mathbf{w}, \qquad (3)$$

where $\mathbf{y} = (\ldots, y(nT_s), \ldots)^T$, $\mathbf{x} = (\ldots, x(nT_s), \ldots)^T$ and $\mathbf{w} = (\ldots, w(nT_s), \ldots)^T$. The vector $\mathbf{w}$ represents the additive white complex Gaussian noise of power σ_n^2 and the vector $\mathbf{x}$ represents the signal of interest. Both $\mathbf{x}$ and $\mathbf{w}$ are stochastic processes assumed Gaussian centered so that :

$$\mathbf{w} \sim \mathcal{CN}\left(\mathbf{0}, \sigma_n^2 \mathbf{I}_N\right), \qquad (4)$$
$$\mathbf{x} \sim \mathcal{CN}\left(\mathbf{0}, \mathbf{B}_t(P_0, f_0, \sigma_0^2)\right), \qquad (5)$$

where

$$\mathbf{B}_t(P_0, f_0, \sigma_0^2) = P_0 \mathbf{D}(f_0) \mathbf{C}(\sigma_0^2) \mathbf{D}(f_0)^H, \qquad (6)$$
$$\mathbf{D}(f_0) = \mathrm{diag}\left(\ldots, \exp\left(j2\pi f_0 (n-1) T_s\right), \ldots\right), \qquad (7)$$
$$\left[\mathbf{C}(\sigma_0^2)\right]_{k,l} = \exp\left(-2\pi^2\sigma_0^2 (k-l)^2 T_s^2\right). \qquad (8)$$

From (4) and (5), the observed data $\mathbf{y}$ is a Gaussian-centered process with asymptotic covariance matrix $\mathbf{R}(\boldsymbol{\theta}) = \mathbf{B}_t(P_0, f_0, \sigma_0^2) + \sigma_n^2 \mathbf{I}_N$ where $\boldsymbol{\theta} = \left(P_0, f_0, \sigma_0^2, \sigma_n^2\right)^T$ is the vector of unknown parameters to be estimated.

B. Assumed signal model

In operational conditions, it may be difficult to implement a real time consistent estimator of the vector $\boldsymbol{\theta}$. An alternative is to assume a simpler signal model for which an efficient estimator exists with an affordable computational burden. In the case of weather signals, a way to simplify the problem is to assume the Doppler spectrum of the signal to be concentrated around a single frequency f_0. The Doppler spectrum width σ_0 would be set to zero and the signal model for $\mathbf{x}$ would reduce to the following:

$$\mathbf{x} \sim \mathcal{CN}\left(\mathbf{0}, \mathbf{B}_a(P_0, f_0)\right), \qquad (9)$$

where

$$\mathbf{B}_a(P_0, f_0) = P_0 \mathbf{d}(f_0) \mathbf{d}(f_0)^H, \qquad (10)$$
$$\mathbf{d}(f_0) = \left(\ldots, \exp\left(j2\pi f_0 (n-1) T_s\right), \ldots\right). \qquad (11)$$

From (4) and (9), the observed data $\mathbf{y}$ is still a Gaussian-centered process with asymptotic covariance matrix $\mathbf{R}(\boldsymbol{\mu}) = \mathbf{B}_a(P_0, f_0) + \sigma_n^2 \mathbf{I}_N$ where $\boldsymbol{\mu} = \left(P_0, f_0, \sigma_n^2\right)^T$ is the vector of unknown parameters to be estimated in this simplified case.

C. Assumptions

In this study two important assumptions were made in order to simplify the problem: i) the spectrum aliasing was avoided by taking a small enough sampling period T_s compared to the considered spectrum global width and ii) the signal is assumed to be uniformly sampled. One must keep in mind that in most cases, both assumptions do not hold in operational weather radars: due to hardware limitations, the sampling frequency cannot be as large as desired. A consequence is that the estimation of the mean Doppler frequency is ambiguous which is worked around by applying non-uniform sampling [15].

III. Considered estimators

A. Mismatched unconditional maximum likelihood estimator

Given the assumed signal model (9), the UML estimator consists of minimizing the following negative log-likelihood function:

$$L(\boldsymbol{\mu}) = \ln\left(|\mathbf{R}(\boldsymbol{\mu})|\right) + \mathrm{Tr}\left\{\mathbf{R}(\boldsymbol{\mu})^{-1} \widehat{\mathbf{R}}_{\mathbf{y}}\right\}, \qquad (12)$$

where $\widehat{\mathbf{R}}_{\mathbf{y}} = \frac{1}{K}\sum_{k=1}^{K} \mathbf{y}_k \mathbf{y}_k^H$ is the sample covariance matrix. Given the expression of the asymptotic covariance matrix

$\mathbf{R}(\boldsymbol{\mu})$ in (10), it is possible to express the minimization of (12) a follows:

$$\widehat{\boldsymbol{\mu}} = \arg\min_{\boldsymbol{\mu}} L(\boldsymbol{\mu})$$
$$\Leftrightarrow \begin{cases} \widehat{f_0} = \arg\max_f \sum_{k=1}^{K} |r_k(f)|^2, \\ \widehat{P_0} = \frac{1}{KN(N-1)} \sum_{k=1}^{K} \left(|r_k(\widehat{f_0})|^2 - \|\mathbf{y}_k\|^2 \right), \\ \widehat{\sigma_n^2} = \frac{1}{K(N-1)} \sum_{k=1}^{K} \left(\|\mathbf{y}_k\|^2 - \frac{1}{N} |r_k(\widehat{f_0})|^2 \right), \end{cases} \quad (13)$$

where $r_k(f) = \mathbf{y}_k^H \mathbf{d}(f)$. See Appendix A for details.

In the case of a model misspecification, the UML estimates in (13) do not necessarily tend to the true value of $\boldsymbol{\mu}$: the estimator's output can be biased. If the power is distributed over a large range of frequencies, the UML estimator may underestimate the mean power and overestimate the noise power. An asymptotic property of the mismatched UML (i.e., when used in a misspecified configuration), is to be an efficient estimator of the so-called pseudo-true vector of parameters $\boldsymbol{\mu}_{pt}$ [13]: when the number of independent observations is large enough, its mean square error (MSE) tends to the MCRB. Besides, it can be shown that the pseudo-true vector, which is the one that minimizes the Kullback-Leibler divergence between the true and the assumed probability density functions [12], is also the solution of the mismatched UML (13). From this statement, it is possible to evaluate the pseudo-true estimates $\boldsymbol{\mu}_{pt}$ as the expectation of the UML's outputs when assuming the true signal model (5):

$$f_{pt} = E\{\widehat{f_0}\} = f_0, \quad P_{pt} = E\{\widehat{P_0}\} = P_0 - \delta P,$$
$$\sigma_{n,pt}^2 = E\{\widehat{\sigma_n^2}\} = \sigma_n^2 + N\delta P, \quad (14)$$

where $\delta P = P_0 \frac{N^2 - q_C}{N(N-1)}$ and $q_C = \sum_{k,l} \left[\mathbf{C}(\sigma_0^2)\right]_{k,l}$. See Appendix B for details.

B. Pulse-pair estimator

Estimators for both the mean power and the mean Doppler frequency of weather radar signals have been known and studied for decades now [1], [5]. The simplest ones relies on estimates of the signal auto-correlation function A (2):

$$\widehat{A}_k(nT_s) = \frac{1}{N - |n|} \sum_{i=0}^{N-1} [\mathbf{y}_k]_i^* [\mathbf{y}_k]_{i+n}. \quad (15)$$

From this estimate, the estimators for the mean power and the mean frequency are simply:

$$\widehat{P}_{PP} = \frac{1}{K} \sum_{k=1}^{K} \widehat{A}_k(0) \quad (16)$$

$$\widehat{f}_{PP} = \frac{1}{2\pi T_s} \frac{1}{K} \sum_{k=1}^{K} \arg\left(\widehat{A}_k(T_s) \right) \quad (17)$$

These estimators have the advantage to be very simple to implement, relying only on two points of the estimated auto-correlation function. The mean power estimator is expected to be slightly biased because of the noise but at large SNR, this bias should not be visible.

IV. Performance bounds

A. Misspecified Cramér-Rao bound

In [12, eq. (47)], a generalized Bangs formula, adapted to the problem at hand, was proposed. Given the true and assumed signal models, this formula can be written as follows

$$\mathbf{MCRB}(\boldsymbol{\mu}_{pt}) = \mathbf{G}(\boldsymbol{\mu}_{pt})^{-1} \mathbf{F}(\boldsymbol{\mu}_{pt}) \mathbf{G}(\boldsymbol{\mu}_{pt})^{-1}, \quad (18)$$

with

$$[\mathbf{F}(\boldsymbol{\mu})]_{k,l} = \mathrm{Tr}\left\{ \mathbf{R}^{-1} \frac{\partial \mathbf{R}}{\partial \mu_k} \mathbf{R}^{-1} \mathbf{B} \mathbf{R}^{-1} \frac{\partial \mathbf{R}}{\partial \mu_l} \mathbf{R}^{-1} \mathbf{B} \right\}, \quad (19)$$

$$\begin{aligned} [\mathbf{G}(\boldsymbol{\mu})]_{k,l} = {} & \mathrm{Tr}\left\{ \mathbf{R}^{-1} \frac{\partial^2 \mathbf{R}}{\partial \mu_k \partial \mu_l} \left(\mathbf{R}^{-1}\mathbf{B} - \mathbf{I}_N \right) \right\} \\ & - \mathrm{Tr}\left\{ \mathbf{R}^{-1} \frac{\partial \mathbf{R}}{\partial \mu_l} \mathbf{R}^{-1} \frac{\partial \mathbf{R}}{\partial \mu_k} \left(\mathbf{R}^{-1}\mathbf{B} - \mathbf{I}_N \right) \right\} \\ & - \mathrm{Tr}\left\{ \mathbf{R}^{-1} \frac{\partial \mathbf{R}}{\partial \mu_k} \mathbf{R}^{-1} \frac{\partial \mathbf{R}}{\partial \mu_l} \mathbf{R}^{-1} \mathbf{B} \right\}, \end{aligned} \quad (20)$$

where, for clarity sake, the parameters dependencies were omitted and, from (10), $\mathbf{R} = \mathbf{B}_a(P_0, f_0) + \sigma_n^2 \mathbf{I}_N$ and, from (6), $\mathbf{B} = \mathbf{B}_t(P_0, f_0, \sigma_0^2) + \sigma_n^2 \mathbf{I}_N$. Given the expression of $\mathbf{R}$, its inverse can be obtained as in (26) and its derivatives expressions are gathered in Appendix C.

B. Cramér-Rao bound for the true signal model

In order to compare the MCRB to the performance of an efficient estimator for the true data signal model (5), the CRB for the estimation of vector $\boldsymbol{\theta}$ are numerically evaluated based on the definition for the unconditional model [16, eq. (3.31)]. The Fisher information matrix, was already introduced in [6]:

$$\left[\mathrm{CRB}(\boldsymbol{\theta})^{-1}\right]_{k,l} = [\mathbf{I}(\boldsymbol{\theta})]_{k,l} = \mathrm{Tr}\left\{ \mathbf{B}^{-1} \frac{\partial \mathbf{B}}{\partial \theta_k} \mathbf{B}^{-1} \frac{\partial \mathbf{B}}{\partial \theta_l} \right\}. \quad (21)$$

V. Impact of spectrum bandwidth on power and velocity estimates

A. Methodology and simulation setup

In this section, the mismatched UML and the PP estimators are confronted to the MCRB and the CRB defined in Sec. IV. The UML estimator is known to be an asymptotically efficient estimator of the pseudo-true vector $\boldsymbol{\mu}_{pt}$. Consequently, its root MSE (RMSE) evaluated with respect to $\boldsymbol{\mu}_{pt}$ is expected to converge to the corresponding root MCRB (RMCRB).

A simulated signal is generated using (5) for different representative values of σ_0 [17]. As in [6], $K = 100$ independent observations of size $N = 8$ are considered with an SNR set to 20dB. This ratio is defined in linear as follows:

$$\mathrm{SNR}_{\text{lin.}} = \frac{P_0}{\sigma_n^2 \sqrt{2\pi\sigma_0^2}}. \quad (22)$$

The parameters true values used in the simulation are gathered in table I. Mean Doppler frequency f_0 and spectral width σ_0 are expressed in m/s by multiplying the frequency by $\lambda_c/2$ where λ_c is the carrier wavelength set to 5cm (C-band radar). The estimators RMSE is obtained from 2000 Monte Carlo runs. Because of the bias induced by the model misspecification, two RMSEs are computed: one with respect

TABLE I
SIMULATION SETTINGS.

P_0	f_0 [m/s]	σ_0 [m/s]	σ_n^2
100	0	0.25 : 0.25 : 4	$P_0 / \left(\mathrm{SNR}_{\text{lin.}} \sqrt{2\pi\sigma_0^2}\right)$

to the true parameters values and one with respect to the pseudo-true parameters values, which are defined for the i-th element of the vector of unknown parameters $\boldsymbol{\mu}$ as

$$\mathrm{RMSE}([\boldsymbol{\mu}_0]_i) \triangleq E\left\{\left([\widehat{\boldsymbol{\mu}}]_i - [\boldsymbol{\mu}_0]_i\right)^2\right\}, \tag{23}$$

$$\mathrm{RMSE}([\boldsymbol{\mu}_{pt}]_i) \triangleq E\left\{\left([\widehat{\boldsymbol{\mu}}]_i - [\boldsymbol{\mu}_{pt}]_i\right)^2\right\}. \tag{24}$$

In addition, the theoretical bias induced by the model misspecification (14), defined as $\mathrm{bias}([\boldsymbol{\mu}]_i) = [\boldsymbol{\mu}_{pt}]_i - [\boldsymbol{\mu}_0]_i$ is also presented in the results. Finally, in order to evaluate what is lost when assuming a misspecified signal model, the root CRB (RCRB) for the true signal model is also displayed.

B. Results and discussions

This section presents the RMSEs defined previously against the RMCRB and RCRB for different values of the spectral width σ_0. For the mean power estimation in Fig. 1, one can see that the bias induced by the misspecification on the UML output can be important even for small value of the spectral width. At large SNR, the variance of the estimation of the mean power is known to be proportional to the mean power value itself [5], consequently when σ_0 is larger, the apparent mean power is smaller and a consequence is to see the MCRB and the CRB getting smaller. At a certain value of the spectral width, the MCRB gets actually smaller than the CRB. Using a mismatched UML would then yield an estimate of the mean power more concentrated than an efficient estimator of the true data model, although biased. The pulse-pair estimator is robust against the misspecification and does not suffer from any bias. Although it is not efficient, its RMSE is not too far from the true model RCRB which makes it a good estimator of the mean power.

Considering the mean Doppler frequency estimation in Fig. 2, since the misspecification leads to no bias, only one RMSE is displayed for the UML. For large values of spectral width, one can observe that the spectrum becomes too large for the chosen sampling rate T_s and aliasing starts affecting the overall performance of the UML (RMSE achieving better than the RMCRB). On the other hand, the pulse-pair estimator behaves better for most of the considered spectral width values, exception is for very small spectral width where the UML behaves better since the assumed model is not so far from the true model. However, compared to the RCRB, the pulse-pair estimator's RMSE remains moderately far (3dB to 5dB) which might be a motivation for future estimator improvements.

VI. CONCLUSION

In this study, the question of the toll to be paid in omitting the spectral width of weather radar signals was tackled. The problem has been formulated as a model misspecification which can be fully characterized by the MCRB. Simulations show that applying a mismatched UML may be relevant only for very small spectral width values. If the spectral width is too large, operational pulse-pair algorithm remains a very good estimator of the mean power and a rather good estimator of the mean Doppler frequency.

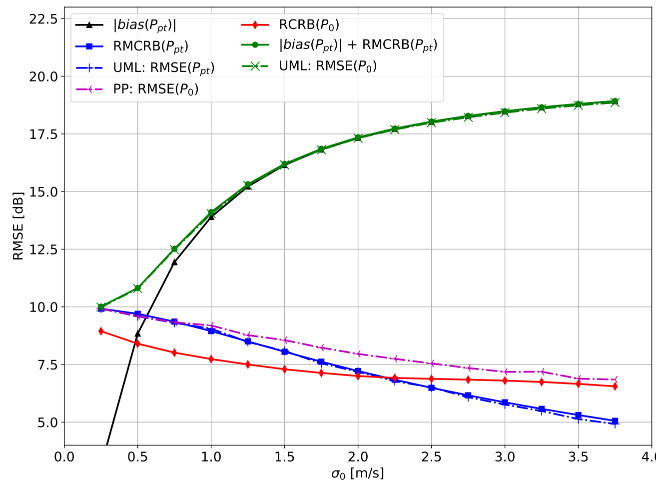

Fig. 1. Mismatched UML RMSE for the estimation of the mean power P_0. The use of a mismatched UML induces (i) a bias (black line) and (ii) a change of RMSE lower bound (blue line) where the matched UML was (i) asymptotically unbiased and (ii) with with its RMSE lower bounded by the RCRB (red line).

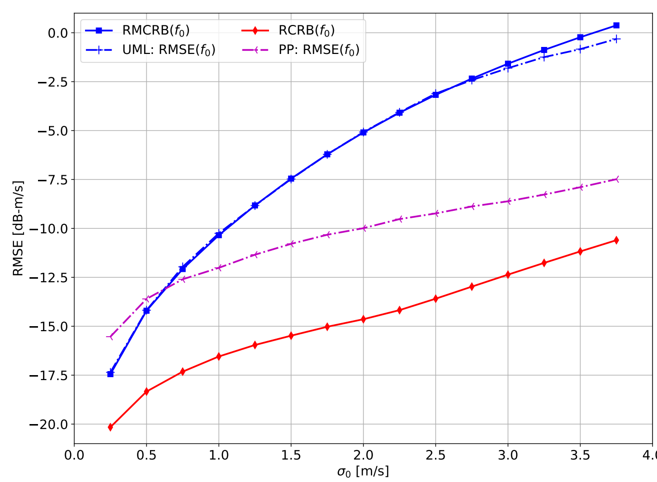

Fig. 2. Mismatched UML RMSE for the estimation of the mean Doppler frequency f_0, converted in m/s.

APPENDIX A
DETAILS ON THE UNCONDITIONAL MAXIMUM LIKELIHOOD

From its definition reminded here: $\mathbf{R} = P_0\mathbf{d}\mathbf{d}^H + \sigma_n^2\mathbf{I}_N$, the determinant of the assumed covariance matrix $\mathbf{R}$ can be expressed using [18, eq. (24)]:

$$|\mathbf{R}| = \left(\sigma_n^2\right)^N \left|\frac{P_0}{\sigma_n^2}\mathbf{d}\mathbf{d}^H + \mathbf{I}_N\right| = \left(\sigma_n^2\right)^{N-1}\left(P_0 N + \sigma_n^2\right), \tag{25}$$

and its inverse is obtained in a similar way [18, eq. (160)]:

$$\mathbf{R}^{-1} = \frac{1}{\sigma_n^2}\left(\mathbf{I}_N - \frac{P_0}{P_0 N + \sigma_n^2}\mathbf{d}\mathbf{d}^H\right). \tag{26}$$

Using (26), the trace term in (12) can be further developed.

$$\begin{aligned}\mathrm{Tr}\left\{\mathbf{R}^{-1}\widehat{\mathbf{R}}_{\mathbf{y}}\right\} &= \frac{1}{\sigma_n^2}\frac{1}{K}\sum_{k=1}^{K}\mathrm{Tr}\left\{\mathbf{y}_k\mathbf{y}_k^H\right\} \\ &\quad - \frac{P_0}{\sigma_n^2\left(P_0 N + \sigma_n^2\right)}\frac{1}{K}\sum_{k=1}^{K}\mathrm{Tr}\left\{\mathbf{d}\mathbf{d}^H\mathbf{y}_k\mathbf{y}_k^H\right\} \\ &= \frac{1}{K\sigma_n^2}\sum_{k=1}^{K}\|\mathbf{y}_k\|^2 - \frac{P_0}{K\sigma_n^2\left(P_0 N + \sigma_n^2\right)}\sum_{k=1}^{K}\left|\mathbf{y}_k^H\mathbf{d}\right|^2. \end{aligned}\tag{27}$$

Injecting (25) and (27) in (12) and zeroing the partial derivatives with respect to P_0 and σ_n^2, the results in (13) are obtained.

Appendix B
Pseudo-true parameter expression

The pseudo-true vector is obtained by taking the expectation of the UML estimates (13) when $\mathbf{y}$ is Gaussian process according to the true signal model (5). A first step is to evaluate the expectation of the terms that depend on $\mathbf{y}_k$:

$$\begin{aligned}E\left\{\left|\mathbf{y}_k^H\mathbf{d}\right|^2\right\} &= E\left\{\mathbf{d}^H\mathbf{y}_k\mathbf{y}_k^H\mathbf{d}\right\} \\ &= P_0\mathbf{d}^H\mathbf{D}\mathbf{C}\mathbf{D}^H\mathbf{d} + \sigma_n^2\mathbf{d}^H\mathbf{d} \\ &= P_0\left[1,\ldots,1\right]\mathbf{C}\left[1,\ldots,1\right]^T + \sigma_n^2 N \\ &= P_0\sum_{k,l}\left[\mathbf{C}\right]_{k,l} + \sigma_n^2 N = P_0 q_C + \sigma_n^2 N. \end{aligned}\tag{28}$$

$$\begin{aligned}E\left\{\|\mathbf{y}_k\|^2\right\} &= E\left\{\mathrm{Tr}\left\{\mathbf{y}_k\mathbf{y}_k^H\right\}\right\} \\ &= \mathrm{Tr}\left\{P_0\mathbf{D}\mathbf{C}\mathbf{D}^H + \sigma_n^2\mathbf{I}_N\right\} \\ &= P_0 N + \sigma_n^2 N. \end{aligned}\tag{29}$$

Then, using (28) and (29) in (13),

$$\begin{aligned}P_{pt} &= E\left\{\frac{1}{KN(N-1)}\sum_{k=1}^{K}\left(|\mathbf{y}_k^H\mathbf{d}|^2 - \|\mathbf{y}_k\|^2\right)\right\} \\ &= \frac{1}{N(N-1)}\left(P_0 q_C + \sigma_n^2 N - \left(P_0 N + \sigma_n^2 N\right)\right) \\ &= P_0\frac{q_C - N}{N(N-1)} = P_0 - P_0\frac{N^2 - q_C}{N(N-1)} = P_0 - \delta P. \end{aligned}\tag{30}$$

$$\begin{aligned}\sigma_{n,pt}^2 &= E\left\{\frac{1}{K(N-1)}\sum_{k=1}^{K}\left(\|\mathbf{y}_k\|^2 - \frac{|\mathbf{y}_k^H\mathbf{d}|^2}{N}\right)\right\} \\ &= \frac{1}{(N-1)}\left(P_0 N + \sigma_n^2 N - \frac{1}{N}\left(P_0 q_C + \sigma_n^2 N\right)\right) \\ &= \sigma_n^2 + P_0\frac{N^2 - q_C}{N-1} = \sigma_n^2 + N\delta P. \end{aligned}\tag{31}$$

Since the true spectrum has a symmetrical shape centered on the true mean Doppler frequency f_0 (1), the estimate that maximizes the power of the Fourier transform $|\mathbf{y}_k^H\mathbf{d}(f)|^2$ under the true signal model (5) is the same as under the assumed signal model (9). Consequently, $f_{pt} = f_0$.

Appendix C
Assumed covariance matrix derivatives

Using the definition of $\mathbf{d}$ in (11), the derivatives of $\mathbf{R} = P_0\mathbf{d}\mathbf{d}^H + \sigma_n^2\mathbf{I}_N$ can be directly obtained as follows:

$$\frac{\partial\mathbf{R}}{\partial P_0} = \mathbf{d}\mathbf{d}^H, \qquad \frac{\partial\mathbf{R}}{\partial\sigma_n^2} = \mathbf{I}_N, \tag{32}$$

$$\frac{\partial\mathbf{R}}{\partial f_0} = j2\pi T_s\left(\mathbf{U}\mathbf{d}\mathbf{d}^H - \mathbf{d}\mathbf{d}^H\mathbf{U}\right), \tag{33}$$

where $\mathbf{U} = \mathrm{diag}(\ldots, n, \ldots)$.

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