fix a cell whose coordinates will be $(0, 0)$, by definition. Then, we fix the directions, $North$, $South$, $East$ and $West$ and this requires to fix two cells: the one with coordinates $(1, 0)$ and the one with coordinates $(0, 1)$: the second choice amounts to define what means *clockwise*. These three choices are arbitrary as the Euclidean plane has no privileged point and as it has no intrinsic orientation.

Now, the hyperbolic plane has no privileged point and it is intrinsically non-oriented too. Basically, we do things in a very similar way with what is performed in the Euclidean case. First we fix a cell which will be the central one. This fixes the sectors around the central cell: each neighbour of the central cell is identified with a root of the Fibonacci tree which spans the considered sector. Next, fixing in a sector which cell will be the leftmost son of the root allows us to fix what means *clockwise* in this context, as the leftmost son of another sector is defined by the rotation which maps this sector to the first chosen one. This correspondence between the two processes allows us to say that defining the father of all the cells, except the central one, amounts to define a direction in the hyperbolic plane.

3 Gardens of Eden in the hyperbolic plane

Now, we turn to the examples which we announced in the introduction.

Theorem 3.1 *(Kari-Margenstern) − There is a cellular automaton A on the ternary heptagrid, or on the pentagrid, such that G_A is injective but G_A is not surjective. There is also a cellular automaton B on the ternary heptagrid, or on the pentagrid, such that G_B is surjective but G_B is not injective.*

Proof. First, consider the case of A.

We assume that the ternary heptagrid, or the pentagrid, has coordinates based on a central cell x_0 and the required number of sectors around it, each sector being spanned by a Fibonacci tree, as mentioned in Section 2.

Each cell x, with $x \neq x_0$, has a father which we denote by $f(x)$. For A, we consider that there are two states, 0 and 1 and that the transition function is defined by the following relations:

$$\eta_A(x_0, t{+}1) = \eta_A(x_0, t),$$
$$\eta_A(x, t{+}1) = \eta_A(f(x), t), \text{ when } x \neq x_0,$$

where $\eta_A(y, t)$ is the state of the cell y at time t under A.

It is not difficult to see that G_A is injective. Indeed, if c_1 and c_2 are two configurations with $c_1 \neq c_2$, there is a cell x such that $c_1(x) \neq c_2(x)$. If we consider a son y of x, we have that $G_A(c_1)(y) = c_1(x) \neq c_2(x) = G_A(c_2)(y)$. Now, G_A cannot be surjective as it cannot reach configurations in which two sons of the same node have different states.

Let us turn to the construction of B. We can give two variants of this construction.

In the first variant, we assume that we have a function σ_ℓ which, to each node, associates its leftmost son. If $x \neq x_0$, where x_0 again denotes the central cell, $\sigma_\ell(x)$