is known by the cell from $f(x)$. For $x_0$, as the sectors spanned by a Fibonacci tree are numbered, we define $\sigma_\ell(x_0)$ as the root of the tree which received the smallest number.

Now, we define $B$ as follows:

$$\eta_B(x, t{+}1) = \mathrm{xor}(\eta_B(x, t), \eta_B(\sigma_\ell(x), t)),$$

where $\eta_B(x, t)$ is the state of the cell $x$ at time $t$ under $B$.

It is not very difficult to see that $G_B$ is not injective. If we define $c_0$ by assigning the state 0 to all cells and $c_1$ by assigning the state 1 to each cell, it is not difficult to see that $G_B(c_0) = G_B(c_1) = c_0$.

Now, let us check that $G_B$ is surjective. Indeed, fix a configuration $c_1$ and we have to define a configuration $c_0$ such that $G_B(c_0) = c_1$.

Consider $x_0$. Define $c_0(x_0) = 0$. Then, applying the definition, we have that $c_1(x_0) = \mathrm{xor}(0, c_0(\sigma_\ell(x_0))) = c_0(\sigma_\ell(x_0))$. And so, this defines $c_0$ at $\sigma_\ell(x_0)$. Define $c_0(x) = 0$ for all the other sons of $x_0$ than $\sigma_\ell(x_0)$.

By induction, assume that we have defined the level $n{+}1$ and that the surjectivity holds for all cells up to the level $n$, this level being included. From what we have seen, this is the case for $n = 0$.

On the level $n{+}2$, define all white nodes $y$ by $c_0(y) = 0$. Now, consider a node $x$ of the level $n{+}1$. As $c_0(x) = a$ is defined, we have, by definition, $G_B(c_0)(x) = \mathrm{xor}(a, c_0(\sigma_\ell(x)))$. This always defines $c_0$ at $\sigma_\ell(x)$. Indeed, $c_0(\sigma_\ell(x)) = c_1(x)$ if $a = 0$ and $c_0(\sigma_\ell(x)) = 1{-}c_1(x)$ if $a = 1$. And so, considering all nodes $x$ of the level $n{+}1$, this defines $c_0$ for all black nodes of the level $n{+}2$. Moreover, now $G_B(c_0)(x) = c_1(x)$ for all cells $x$ of the level $n{+}1$ too And the definition of $c_0$ at the level $n{+}2$ is complete.

And so, by induction, we proved that $G_B$ is surjective.

The second variant requires to know the sons of a cell. This is easy to define from $f(x)$ for any cell $x$ with $x \neq x_0$. For $x_0$, we consider that all its neighbours are its sons. Now, we define $G_B$ as follows:

$$\eta_B(x, t{+}1) = \min\{\mathrm{xor}(\eta_B(x, t), \eta_B(y, t)) \mid y \in S_x\},$$

where $S_x$ is the set of the sons of $x$.

The argument is the same as in the first variant.

Note that, in the proof of the surjectivity, we can easily see that $B$ cannot be injective, as long as the state of many cells can be fixed arbitrarily. □

## 4 The case of rotation invariant cellular automata

It was proved in [15] that an analog of Hedlund's for cellular automata hold for the hyperbolic plane provided that an additional property is satisfied by the automaton, namely that the set of its rules is **rotation invariant**.

Intuitively, this means that if the neighbourhood of a cell is changed by a rotation of the neighbourhood around the cell, then the new state of the cell is the same as