possible to locate cells and which plays a key role in the proofs of the theorem of Section 3.

The important property indicated by Fig. 2 is that in the case of the hyperbolic plane, the penta- and the heptagrid are generated by a **tree**. The tree structure is underlined by the parts of the tiling which are detached and placed around the central tile. In the case of the pentagrid, five such regions, each one spanned by the same tree, are placed around the central tile, in a rotation symmetric way. In the case of the ternary heptagrid, we have seven regions. A remarkable property is that the generating tree is the same for the pentagrid and for the ternary heptagrid, see [11]. This tree is called **Fibonacci tree** as the number of its cells on a level k is f_{2k+1}, where $\{f_k\}_{k\in\mathbb{N}}$ is the Fibonacci sequence where $f_0 = f_1 = 1$, see [16]. From this, coordinates can be computed to locate the cells of a cellular automaton, see [7,11].

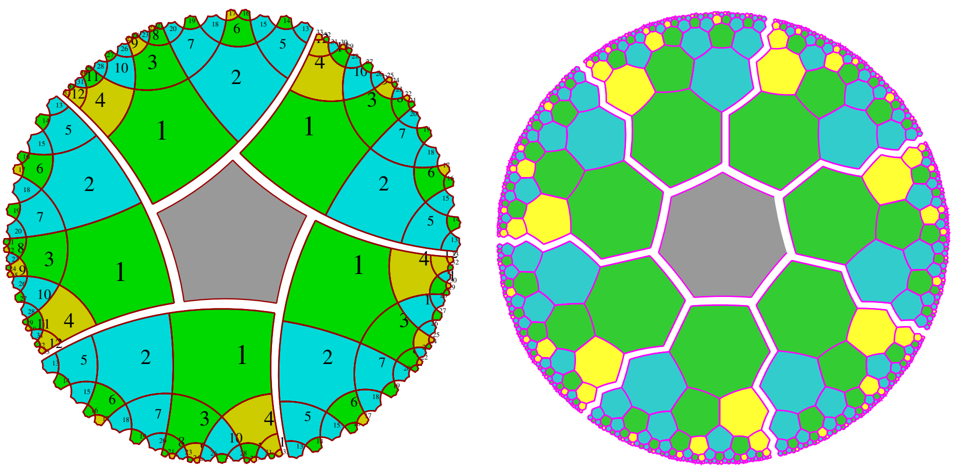

Fig. 2. On the left: the pentagrid, on the right: the underlying tree which spans the tiling.

As a consequence, except the cell which is placed at the central tile, we call it the **central cell**, each cell of the cellular automaton, has a father: its father as a node of the tree in the region it falls in.

This is an important point which will be used in Section 3.

The existence of a father for all cells, except the central one, plays the role of a direction in the hyperbolic plane, in the same way as the four traditional directions play a key role in the Euclidean plane.

In most formal presentations of cellular automata in the Euclidean plane − people usually say CA in the plane − the set of cells is identified with $\mathbb{Z}$ 2. This identification is so evident that it requires some effort to realize that it connects two different things and how it performs the connection.

In the Euclidean case, the above identification consists in three steps. First, we fix a cell whose coordinates will be $(0, 0)$, by definition. Then, we fix the directions, $North$, $South$, $East$ and $West$ and this requires to fix two cells: the one with coordinates $(1, 0)$ and the one with coordinates $(0, 1)$: the second choice amounts to define what means *clockwise*. These three choices are arbitrary as the Euclidean plane has no privileged point and as it has no intrinsic orientation.

Now, the hyperbolic plane has no privileged point and it is intrinsically non-oriented too. Basically, we do things in a very similar way with what is performed in the Euclidean case. First we fix a cell which will be the central one. This fixes the sectors around the central cell: each neighbour of the central cell is identified with a root of the Fibonacci tree which spans the considered sector. Next, fixing in a sector which cell will be the leftmost son of the root allows us to fix what means *clockwise* in this context, as the leftmost son of another sector is defined by the rotation which maps this sector to the first chosen one. This correspondence between the two processes allows us to say that defining the father of all the cells, except the central one, amounts to define a direction in the hyperbolic plane.

3 Gardens of Eden in the hyperbolic plane

Now, we turn to the examples which we announced in the introduction.

Theorem 3.1 *(Kari-Margenstern) − There is a cellular automaton A on the ternary heptagrid, or on the pentagrid, such that G_A is injective but G_A is not surjective. There is also a cellular automaton B on the ternary heptagrid, or on the pentagrid, such that G_B is surjective but G_B is not injective.*

Proof. First, consider the case of A.

We assume that the ternary heptagrid, or the pentagrid, has coordinates based on a central cell x_0 and the required number of sectors around it, each sector being spanned by a Fibonacci tree, as mentioned in Section 2.

Each cell x, with $x \neq x_0$, has a father which we denote by $f(x)$. For A, we consider that there are two states, 0 and 1 and that the transition function is defined by the following relations:

$$\eta_A(x_0, t{+}1) = \eta_A(x_0, t),$$
$$\eta_A(x, t{+}1) = \eta_A(f(x), t), \text{ when } x \neq x_0,$$

where $\eta_A(y, t)$ is the state of the cell y at time t under A.

It is not difficult to see that G_A is injective. Indeed, if c_1 and c_2 are two configurations with $c_1 \neq c_2$, there is a cell x such that $c_1(x) \neq c_2(x)$. If we consider a son y of x, we have that $G_A(c_1)(y) = c_1(x) \neq c_2(x) = G_A(c_2)(y)$. Now, G_A cannot be surjective as it cannot reach configurations in which two sons of the same node have different states.

Let us turn to the construction of B. We can give two variants of this construction.

In the first variant, we assume that we have a function σ_ℓ which, to each node, associates its leftmost son. If $x \neq x_0$, where x_0 again denotes the central cell, $\sigma_\ell(x)$

is known by the cell from $f(x)$. For x_0, as the sectors spanned by a Fibonacci tree are numbered, we define $\sigma_\ell(x_0)$ as the root of the tree which received the smallest number.

Now, we define B as follows:

$$\eta_B(x, t{+}1) = \mathrm{xor}(\eta_B(x, t), \eta_B(\sigma_\ell(x), t)),$$

where $\eta_B(x, t)$ is the state of the cell x at time t under B.

It is not very difficult to see that G_B is not injective. If we define c_0 by assigning the state 0 to all cells and c_1 by assigning the state 1 to each cell, it is not difficult to see that $G_B(c_0) = G_B(c_1) = c_0$.

Now, let us check that G_B is surjective. Indeed, fix a configuration c_1 and we have to define a configuration c_0 such that $G_B(c_0) = c_1$.

Consider x_0. Define $c_0(x_0) = 0$. Then, applying the definition, we have that $c_1(x_0) = \mathrm{xor}(0, c_0(\sigma_\ell(x_0))) = c_0(\sigma_\ell(x_0))$. And so, this defines c_0 at $\sigma_\ell(x_0)$. Define $c_0(x) = 0$ for all the other sons of x_0 than $\sigma_\ell(x_0)$.

By induction, assume that we have defined the level $n{+}1$ and that the surjectivity holds for all cells up to the level n, this level being included. From what we have seen, this is the case for $n = 0$.

On the level $n{+}2$, define all white nodes y by $c_0(y) = 0$. Now, consider a node x of the level $n{+}1$. As $c_0(x) = a$ is defined, we have, by definition, $G_B(c_0)(x) = \mathrm{xor}(a, c_0(\sigma_\ell(x)))$. This always defines c_0 at $\sigma_\ell(x)$. Indeed, $c_0(\sigma_\ell(x)) = c_1(x)$ if $a = 0$ and $c_0(\sigma_\ell(x)) = 1{-}c_1(x)$ if $a = 1$. And so, considering all nodes x of the level $n{+}1$, this defines c_0 for all black nodes of the level $n{+}2$. Moreover, now $G_B(c_0)(x) = c_1(x)$ for all cells x of the level $n{+}1$ too And the definition of c_0 at the level $n{+}2$ is complete.

And so, by induction, we proved that G_B is surjective.

The second variant requires to know the sons of a cell. This is easy to define from $f(x)$ for any cell x with $x \neq x_0$. For x_0, we consider that all its neighbours are its sons. Now, we define G_B as follows:

$$\eta_B(x, t{+}1) = \min\{\mathrm{xor}(\eta_B(x, t), \eta_B(y, t)) \,|\, y \in S_x\},$$

where S_x is the set of the sons of x.

The argument is the same as in the first variant.

Note that, in the proof of the surjectivity, we can easily see that B cannot be injective, as long as the state of many cells can be fixed arbitrarily. □

4 The case of rotation invariant cellular automata

It was proved in [15] that an analog of Hedlund's for cellular automata hold for the hyperbolic plane provided that an additional property is satisfied by the automaton, namely that the set of its rules is **rotation invariant**.

Intuitively, this means that if the neighbourhood of a cell is changed by a rotation of the neighbourhood around the cell, then the new state of the cell is the same as what it was defined before the change. This informal definition can be made more precise, by fixing a way to number the neighbours of a cell or, which is equivalent, to number the sides of the polygon which supports the automaton. Let us say that for the central cell, side 1 is fixed once and for all. The other sides are numbered increasingly while counter-clockwise turning around the cell starting from side 1. For the other cells, fix side 1 to be the side shared by the cell with its father and, similarly number the other sides by counter-clockwise turning around the cell. Then, a rule of the automaton can be displayed in the following format, see [12,19,18] :

$$\eta_0, \eta_1, ..., \eta_\alpha \rightarrow \eta_0^1,$$

where η_0 is the current state of the cell, η_i, with $i \in \{1..\alpha\}$, where $\alpha = 5$ or $\alpha = 7$, is the current state of neighbour i and η_0^1 is the new state of the cell.

We say that a cellular automaton on the pentagrid or on the heptagrid is rotation invariant if and only if for each rule as above, the rule

$$\eta_0, \eta_{\pi(1)}, ..., \eta_{\pi(\alpha)} \rightarrow \eta_0^1,$$

also belongs to the set of rules for any circular permutation π on $\{1..\alpha\}$.

In what follows, $s(x, t)$ denotes the state at x and at time t. We have the following two properties :

Theorem 4.1 *(Margenstern-Kari) There is a rotation invariant cellular automaton on the pentagrid or on the heptagrid which is surjective but not injective.*

Theorem 4.2 *(Margenstern-Kari) There is a rotation invariant cellular automaton on the pentagrid or on the heptagrid which is surjective but not injective even on finite configurations.*

Proof of Theorem 4.1. It will be enough that $s(x, t)$ takes its values in $\{0, 1\}$. We define the transition rules in such a way that:

$$s(x, t+1) = \begin{cases} 0 & \text{if } \#\{s(y,t) = 0\} \text{ is even} \\ 1 & \text{if } \#\{s(y,t) = 1\} \text{ is even} \end{cases},$$

where, in this formula, y denotes a neighbour of x which is not x and $\#\{P(s(y, t))\}$ is the number of y's around x such that $s(y, t)$ satisfy P. This is clearly a rotation invariant function.

Consider a configuration c and again, denote the central cell by x_0. It is not difficult to see that $c_0(x_0)$ can be 0 or 1 indifferently. Once it is fixed, we can fix the level 1 around x_0, setting two cells at the value $c(x_0)$ and the others at $1-c(x_0)$. Then, by induction, on each level, we have that the number of neighbours to be fixed is 3 or 2. In fact, the number is 4 for the first tile which we consider at level 1. For the others it is 3, the last one being excepted, for which it is 2. Now, having 2 or 3 cells at our disposal is enough to fix the parity of the required value to be odd. If the number of the required value is odd it is enough to give this value to one new cell and the other value to the others. If it is even, we have the choice: even to give the required value to an even number of cells, 0 or 2, we have the choice, and the other value to the other cells.

This proves that the global function is surjective. Moreover, the just above