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Lecture 02c
Complex Variables and Functions

Lecture is on YouTube

The YouTube video entitled ‘Complex Numbers, Complex Variables, and Complex Functions’ that covers this lecture is located at https://youtu.be/WEYX-wa9csU.

Outline

-Complex Variables

- -Euler’s Formula
- -Complex algebra

Complex Numbers

A complex number can be represented as

$$z = x + y\,j \qquad \textbf{(Eq.1)}$$

where $j = \sqrt{-1}$ (imaginary number)

$i = \sqrt{-1}$ is another popular choice to represent the imaginary number.

Coding Warning:

Matlab uses ‘i’ and ‘j’ to represent complex numbers.

```
>> myComplexNumber = 2+3*i

myComplexNumber =

   2.0000 + 3.0000i
```

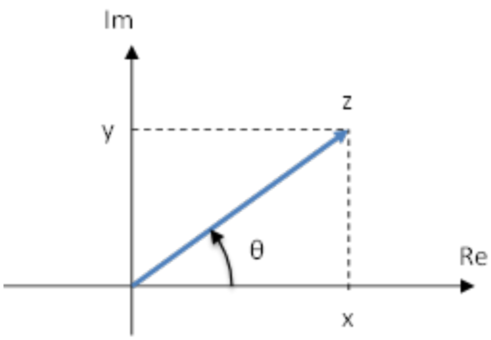
Therefore, do not use ‘i’ and ‘j’ as loop variables. An example of inadvisable code is shown below

```
for i=1:3
    b = 2*i;
    i
end

myComplexNumber = 2+3*i
```

This evaluates to '11', not '2+3i'

We can represent z in the complex plane (real/imaginary axis)



We can consider z to be either a point in the complex plane or a vector to the point, both will be useful later.

Math Joke: Imaginary Number
Have you ever dialed a wrong phone number and gotten the message, "I'm sorry, you've dialed an imaginary number, please rotate your phone 90 degrees and try again"?

We can define the magnitude and angle of z as

$$|\, z \,| = \sqrt{x^2 + y^2} \qquad \text{(magnitude of } z\text{)}$$

$\theta = \text{atan2}(y, x)$ (angle of z) (be sure to use the proper syntax with the 4 quadrant inverse tangent)

WARNING: when computing inverse tangents, be sure to use a 4 quadrant inverse tangent. You also need to be careful when using software as different systems use different order of input arguments. For example Mathematica requires that the first input argument be the x component and the second input argument is the y component. Matlab has the order of inputs reversed.

Table 1: Order of input parameters for inverse tangent

	function name	**1 st input**	**2 nd input**
Mathematica	ArcTan	x	y
Matlab	atan2	y	x

Euler's Formula

From the above figure, we can write

$$x = |z| \cos(\theta)$$
$$y = |z| \sin(\theta)$$

Substituting into the definition in Eq.1, we can write,

$$z = x + y\, j$$

$$= |z| \cos(\theta) + |z| \sin(\theta)\, j$$

$$z = |z| (\cos(\theta) + \sin(\theta)\, j) \qquad \textbf{\textit{(Eq.2)}}$$

Recall that cos and sin can be written as an infinite series

$$\cos(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!} = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots$$

$$\sin(\theta) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!} = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots$$

Let us investigate the quantity $\cos(\theta) + \sin(\theta)\, j$

$$\cos(\theta) + \sin(\theta)\, j = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\right) + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \ldots\right) j$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \ldots\right) + \left(\theta\, j - \frac{\theta^3}{3!}\, j + \frac{\theta^5}{5!}\, j - \frac{\theta^7}{7!}\, j + \ldots\right) \qquad \text{recall: } j^2 = -1,\ j^3 = -j,\ j^4 = 1$$

$$= \left(1 + \frac{j^2\, \theta^2}{2!} + \frac{j^4\, \theta^4}{4!} + \frac{j^6\, \theta^6}{6!} + \ldots\right) + \left(\frac{\theta\, j}{1!} + \frac{j^3\, \theta^3}{3!} + \frac{j^5\, \theta^5}{5!} + \frac{J^7\, \theta^7}{7!} + \ldots\right)$$

$$= 1 + \frac{\theta\, j}{1!} + \frac{j^2\, \theta^2}{2!} + \frac{j^3\, \theta^3}{3!} + \frac{j^4\, \theta^4}{4!} + \frac{j^5\, \theta^5}{5!} + \frac{j^6\, \theta^6}{6!} + \frac{J^7\, \theta^7}{7!} + \ldots$$

$$= \sum_{n=0}^{\infty} \frac{(j\, \theta)^n}{n!} \qquad \text{recall: this is the definition of } e^{j\, \theta}$$

$$\cos(\theta) + \sin(\theta)\, j = e^{j\, \theta}$$

Substituting this into Eq .2 yields

$$z = |z|\, e^{j\, \theta} \qquad \textbf{\textit{(Eq.3)}}$$

This is known as **Euler's Formula**

Can use similar analysis to show that (in homework)

$$\cos(\theta) = \frac{e^{j\, \theta} + e^{-j\, \theta}}{2}$$

$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

We can write z in rectangular or polar form as

$z = x + y\,j$ (rectangular)

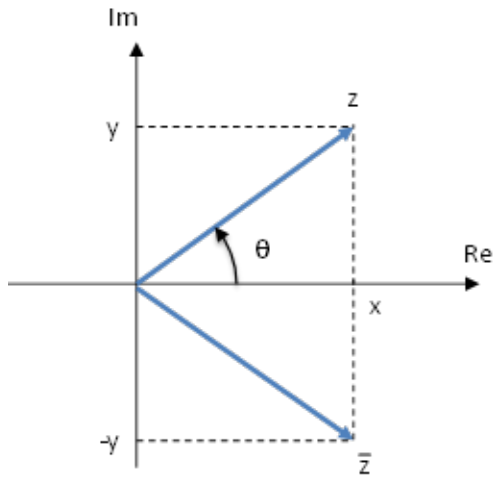
$z = |z|\,(\cos(\theta) + \sin(\theta)\,j)$ (rectangular)

$z = |z|\,\angle\theta$ (polar)

$z = |z|\,e^{j\theta}$ (polar)

The complex conjugate of $z = x + y\,j$ is defined as

$\bar{z} = x - y\,j$ (complex conjugate)



Using previous relationships

$\bar{z} = |z|\,\angle -\theta = |z|\,(\cos(\theta) - j\sin(\theta))$

Complex Algebra

Let us take two complex numbers, z and w

$z = x + y\,j = |z|\,\angle\theta = |z|\,e^{j\theta}$

$w = u + v\,j = |w|\,\angle\phi = |w|\,e^{j\phi}$

Equality: two complex numbers z and w are equal iff their real and imaginary parts are both equal

Addition/Subtraction: easily added in rectangular form

$z + w = (x + y\,j) + (u + v\,j)$

$= (x + u) + (y + v)\, j$

Multiplication: use FOIL rule

$$z\,w = (x + y\,j)\,(u + v\,j)$$

$$= x\,u + x\,v\,j + y\,u\,j + y\,v\,j^2 \qquad \text{recall: } j^2 = -1$$

$$= (x\,u - y\,v) + (x\,v + y\,u)\,j$$

Alternatively, we can use Euler's form

$$z\,w = |z|\,e^{j\,\theta}\,|w|\,e^{j\,\phi}$$

$$= |z||w|\,e^{j\,\theta + \mathrm{j}\phi}$$

$$z\,w = |z||w|\,e^{(\theta+\phi)\,j}$$

So we can write in polar form

$$z\,w = |z||w|\,\angle(\theta + \phi)$$

Using the multiplication rule, we can see why multiplying z with its complex conjugate is useful

$$z\,\overline{z} = (x + y\,j)\,(x - y\,j)$$

$$= x^2 - y^2\,j^2 \qquad \text{recall: } j^2 = -1$$

$$z\,\overline{z} = x^2 + y^2$$

Using the polar form

$$z\,\overline{z} = |z|\,e^{j\,\theta}\,|z|\,e^{-j\,\theta}$$

$$= |z|^2\,e^0 \qquad \text{recall: } |z| = \sqrt{x^2 + y^2}$$

$$z\,\overline{z} = x^2 + y^2$$

So we see that multiplication of a complex number with its complex conjugate results in a real number.

Division: this is somewhat messier

$\frac{z}{w} = \frac{x+y\,j}{u+v\,j}$ multiply by complex conjugate

$= \frac{(x+y\,j)(u-v\,j)}{(u+v\,j)(u-v\,j)}$

$= \frac{x\,u-x\,v\,j+y\,u\,j-y\,v\,j^2}{u^2-v^2\,j^2}$ recall: $j^2 = -1$

$= \frac{(x\,u+y\,v)+(y\,u-x\,v)\,j}{u^2+v^2}$

$\frac{z}{w} = \frac{x\,u+y\,v}{u^2+v^2} + \frac{y\,u-x\,v}{u^2+v^2}\,j$

Powers and Roots: We can look at z^n as z multiplied by itself n times, so

$$z^n = |z|^n \angle(n\,\theta)$$

The n^{th} root is equivalent to raising to the $1/n$ power

$$z^{1/n} = |z|^{1/n} \angle(\theta/n)$$

Example

Consider the complex numbers

$z = 3 - 2\,j$
$w = -4 + j$

Compute

Part a. $\bar{z}$
Part b. $z + w$
Part c. $z\,\bar{z}$
part d. z^5

```
z = 3 - 2 I;
w = -4 + I;
```

Part a. Complex conjugate

We simply have $\bar{z} = 3 + 2\,j$

Part b. Addition

We simply have

$z + w = (3 - 2\,j) + (-4 + j)$

$z + w = -1 - j$

```
z + w
```

```
-1 - i
```

Part c. Multiplication

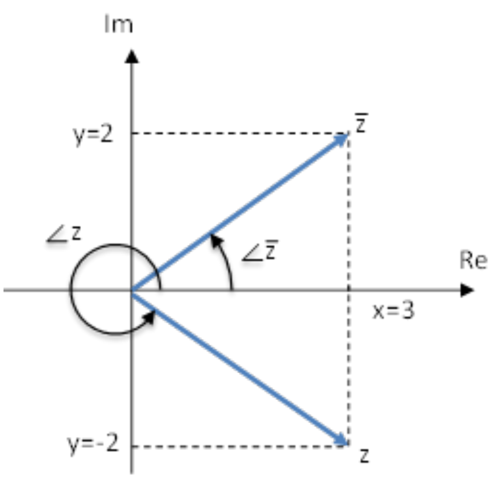
We can first find the complex conjugate of z, this is

$\bar{z} = 3 + 2\,j$

For practice, let us use Euler's formula. We first compute the magnitude and angle of both z and $\bar{z}$ recall

$|\,z\,| = \sqrt{x^2 + y^2}$ (magnitude of z)

$\angle z = \text{atan2}(y,\,x)$ (angle of z) (WARNING: be careful with inverse tangents)



So we have

```
x = Re[z];
y = Im[z];
magZ = √(x² + y²)
angleZ = ArcTan[x, y];
(*Recall that Mathematica expects the input arguments to be x then y*)
angleZ * 180/π // N
```

```
√13
-33.6901
```

So we have

$$| \ z \ | = \sqrt{13} = 3.605$$
$$\angle z = -0.588 \text{ rad} = -33.69^o$$

Similarly for $\bar{z}$

```
magZbar = √(x² + (-y)²)
angleZbar = ArcTan[x, -y];
angleZbar * 180/π // N
```

$\sqrt{13}$

```
33.6901
```

So we have

$$| \ \bar{z} \ | = \sqrt{13} = 3.605$$
$$\angle \bar{z} = 0.588 \text{ rad} = 33.69^o$$

So from Euler's formula

$$z\,\bar{z} = \left| \ z \right| \left| \ \bar{z} \ \right| e^{(\angle z + \angle \bar{z})\,j}$$

$$= \sqrt{13} \ \sqrt{13} \ e^{(-0.588+0.588)\,j}$$

$$= \sqrt{13} \ \sqrt{13} \ e^{0\,j} \qquad \text{note } e^{0\,j} = 1$$

$$z\,\bar{z} = 13$$

Alternatively, we could have used the formula

$$z\,\bar{z} = x^2 + y^2$$

$$= 3^2 + 2^2$$

$$z\,\bar{z} = 13$$

Part d. Powers

$$z^5 = \ | \ z \ |^5 \angle (5\,\theta)$$

```
mag = magZ^5;
θ = 5 * angleZ;
mag // N
θ // N
```

609.338

-2.94001

So we have

$$z^5 = 609.3 \angle -2.94$$

$$= 609.3\, e^{-2.94\, j}$$

We can convert this to rectangular coordinates using

$$x = 609.3 \cos(-2.94)$$
$$y = 609.3 \sin(-2.94)$$

```
xcheck = mag Cos[θ] // Simplify
ycheck = mag Sin[θ] // Simplify
```

-597

-122

So we alternatively have

$$z^5 = -597 - 122\, j$$

```
z^5
```

-597 - 122 i

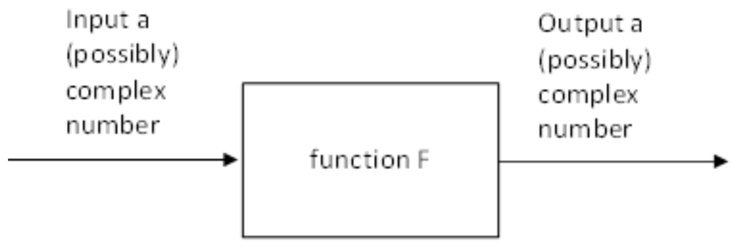
Complex Variables and Functions

If a complex number has a variable real, imaginary, or both parts, it is referred to as a complex variable. Typical notation is

$$s = \sigma + \omega\, j$$

A complex function $F(s)$ maps a complex number to another possibly complex number

$$F : C^1 \to C^1$$



This function has a real part and an imaginary part

$$F(s) = F_x + F_y\, j$$

where F_x and F_y are real.

Most function we will deal with in this class have the form

$$F(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

where $-z_1, -z_2, \ldots, -z_m$ are referred to as the zeros of the function
$-p_1, -p_2, \ldots, -p_n$ are referred to as the poles of the function

We can see why these are called poles and zeros. In the case of the zeros, $-z_1, \ldots, -z_m$, we see that if s takes on any of these values then the function will be zero. Similarly, in the case of the poles, $-p_1, \ldots, -p_n$, we see that if s takes on any of these values, then the function will be infinite (sticking straight up like a pole). Also note that a value cannot simultaneously be a pole and a zero otherwise the term will cancel in the numerator and denominator (pole/zero cancellation).

Example

$$G(s) = \frac{K(s+2)(s+10)}{s(s+1)(s+5)(s+15)^2}$$

note that at $s \to \infty$, $G(s) \to \frac{K}{s^3}$ and therefore $G(s) \to 0$ for large values of s

simple zeros at $s = -2$ and $s = -10$
simple (distinct) poles at $s = 0$, $s = -1$, $s = -5$
two repeated poles at $s = -15$
three zeros at infinity at $s = \infty$