Christopher Lum
lum@uw.edu

Lecture 01f
Vector Derivatives (the Equation of Coriolis) and the Angular Velocity Vector

The YouTube video entitled ‘Vector Derivatives (the Equation of Coriolis) and the Angular Velocity Vector’ that covers this lecture is located at https://youtu.be/-OyRCgv-hPs.

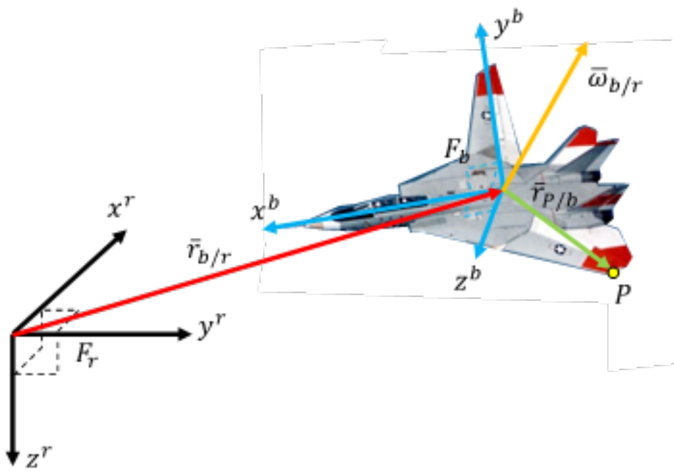
Outline

-Equation of Coriolis
-Angular Rotation Vector

Equation of Coriolis

Consider the example with two bodies with coordinate frames attached to each.

F_r = reference frame (assumed inertial, AKA non-rotating)
F_b = body frame (assumed to be rotating w.r.t. F_r)
$\overline{\omega}_{b/r}$ = vector describing how F_b is rotating w.r.t. F_r
$\overline{r}_{b/r}$ = position vector of origin of frame F_b w.r.t. origin of F_r (in other words, from F_r to F_b)
$\overline{r}_{P/b}$ = position vector of point P w.r.t. origin of F_b (in other words, from F_b to P)



Let us begin by considering the simple scenario where the following is true:

$\overline{r}_{P/b}$ is fixed in frame F_b (not moving w.r.t F_b)

F_b is rotating with respect to frame F_r (denoted $\overline{\omega}_{b/r}$)

An example which fits this scenario is where $\overline{r}_{P/b}$ is vector from CM of the aircraft to the left wingtip. So F_b is local aircraft body frame and F_r is earth frame (assumed inertial for this discussion)

In this setting, one could ask the question, how does the vector $\overline{r}_{P/b}$ change with time? In other words what is $\frac{d}{dt}\overline{r}_{P/b}$?

Can define derivative of a vector the same way as the derivative of a scalar

$$\frac{d}{dt}\overline{r}_{P/b} = \lim_{\delta t \to 0}\frac{\overline{r}_{P/b}(t+\delta t)-\overline{r}_{P/b}(t)}{\delta t}$$

If the observer is a pilot in aircraft (ie in F_b), then derivative is zero (the vector does not appear to be changing). However, if observer is in F_r then the derivative is non-zero because it looks like vector is changing from this observer's perspective.

This discrepancy highlights that the previous notation is lacking. We need to specify from what frame the derivative is taken from. To do so, we will use a left superscript on the vector to denote the perspective from which the derivative is taken.

$$\frac{d}{dt}\Big|_y \overline{V} = {}^y\dot{\overline{V}} \qquad \text{(how } \overline{V} \text{ is changing from the perspective of an observer in } F_y\text{)}$$

Coming back to the example of our aircraft wingtip, we see that

$$\frac{d}{dt}\Big|_b \overline{r}_{P/b} = {}^b\dot{\overline{r}}_{P/b} = 0 \qquad \text{(derivative of } \overline{r}_{P/b} \text{ w.r.t. } F_b\text{)}$$

$$\frac{d}{dt}\Big|_r \overline{r}_{P/b} = {}^r\dot{\overline{r}}_{P/b} \neq 0 \qquad \text{(derivative of } \overline{r}_{P/b} \text{ w.r.t. } F_r\text{)}$$

In many scenarios, we would like to obtain the derivative w.r.t. a non-rotating, inertial frame, F_r. In this case, we should derive an expression for ${}^r\dot{\overline{r}}_{P/b}$.

Let us derive this using two methods.

Method 1 (Use Rodrigues' Rotation Formula)

We want to know what $\overline{r}_{P/b}(t+\delta t)$ is. In other words, we want to know what the vector $\overline{r}_{P/b}$ looks like a small time in the future (after F_b has rotated through a small angle). We can apply Rodrigues' Rotation Formula (AKA Goldstein's Rotation of a Vector Formula) which is repeated here for convenience.

$$\overline{v} = (1-\cos(\mu)) < \overline{u}, \overline{n} > \overline{n} + \cos(\mu)\,\overline{u} - \sin(\mu)\,(\overline{n} \times \overline{u}) \qquad \textit{\textbf{(Eq.1.2-6)}}$$

where μ = left handed rotation about $\overline{n}$

$\overline{n}$ = unit vector describing axis of rotation

We first make a change of notation to be consistent with the book (see Stevens and Lewis pg. 9) and to change from a left handed to a right handed rotation.

$\overline{n} \Rightarrow \hat{s}$ (change notation to be consistent with book)

$\mu \Rightarrow -\phi$ (ϕ is a right handed rotation instead of left handed rotation)

$\overline{r}_{P/b} \Rightarrow \overline{a}$ (to avoid having to keep writing $\overline{r}_{P/b}$)

In the context of the rotation of a vector equation

$\overline{v} \Rightarrow \overline{a} + \delta\,\overline{a}$ (the location of $\overline{r}_{P/b}$ a short time later)

$\overline{u} \Rightarrow \overline{a}$ (original location of $\overline{r}_{P/b}$)

Substituting we obtain

$$\overline{a} + \delta\,\overline{a} = (1-\cos(-\phi)) < \overline{a}, \hat{s} > \hat{s} + \cos(-\phi)\,\overline{a} - \sin(-\phi)\left(\hat{s} \times \overline{a}\right)$$

$$= (1-\cos(\phi)) < \overline{a}, \hat{s} > \hat{s} + \cos(\phi)\,\overline{a} + \sin(\phi)\left(\hat{s} \times \overline{a}\right)$$

Since we are looking at small rotations ($\delta\,t$ is small) we can apply the small angle approximation ($\cos(\phi) \approx 1$, $\sin(\phi) \approx \phi$)

$$\approx \overline{a} + \phi\left(\hat{s} \times \overline{a}\right) \qquad \text{note } \phi = \dot{\phi}\,\delta\,t$$

$$\overline{a} + \delta\,\overline{a} \approx \overline{a} + \dot{\phi}\,\delta\,t\left(\hat{s} \times \overline{a}\right)$$

Subtracting $\overline{a}$ from both sides

$$\delta\,\overline{a} \approx \dot{\phi}\,\delta\,t\left(\hat{s}\times\overline{a}\right)$$

$$\frac{\delta\,\overline{a}}{\delta\,t} \approx \dot{\phi}\left(\hat{s}\times\overline{a}\right)$$

Taking the limit as $\delta\,t \to 0$ and remembering that this is from the perspective of an observer on F_r, we can write

$$^{r}\dot{\overline{a}} = \hat{s}\,\dot{\phi}\times\overline{a}$$

The instantaneous angular velocity vector is defined as

$$\overline{\omega}_{b/r} \equiv \hat{s}\,\dot{\phi} \text{ (angular velocity of } F_b \text{ with respect to } F_r\text{)}$$

Substituting this into the previous expression yields

$$^{r}\dot{\overline{a}} = \overline{\omega}_{b/r}\times\overline{a}$$

This is valid if $\overline{a}$ is of fixed "length" and the only movement is the rotation and translation of frames. However, if $\overline{a}$ is changing with respect to F_b (ie it is growing/shrinking or rotating with respect to F_b), this must be added to the right side of the equation. Using our example of the F-14, if the variable sweep wing mechanism was in action, then $^{b}\dot{\overline{a}} \neq 0$ and we need to add this to the right side of the equation.

$$^{r}\dot{\overline{a}} = {}^{b}\dot{\overline{a}} + \overline{\omega}_{b/r}\times\overline{a}$$

This is typically written using the vector $\overline{V}$ instead of $\overline{a}$ so you may see this as

$$^{r}\dot{\overline{V}} = {}^{b}\dot{\overline{V}} + \overline{\omega}_{b/r}\times\overline{V} \qquad \textit{\textbf{(Eq.1.2-7)}}$$

Eq.1.2-7 is sometimes referred to as the Equation of Coriolis (Blakelock 1965, pg. 296).

Method 2 (Use Small Angle Analysis)

Let us start with the vector $\overline{V}$ and expressed this in frame F_b. If you need a refresher on expressing vectors in a specific frame, see the YouTube video entitled ‘Expressing Vectors in Different Frames Using Rotation Matrices’ (https://youtu.be/TODDZnOT3ro). We can write this as

$$\overline{V}^{b} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}^{b}$$

Let $\hat{x}_b$, $\hat{y}_b$, and $\hat{z}_b$ be unit vectors along F_b

$$\hat{x}^b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\hat{y}^b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\hat{z}^b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Note that u, v, and w are defined to be the components along the $\hat{x}^b$, $\hat{y}^b$, and $\hat{z}^b$ axes, respectively.

We can now ask what is $\frac{d}{dt}\big|_r \overline{V}^b$?

$$\frac{d}{dt}\Big|_r \overline{V}^b = \frac{d}{dt}\Big|_r \begin{pmatrix} u \\ v \\ w \end{pmatrix}^b$$

$$= \frac{d}{dt}\Big|_r \left[u\,\hat{x}^b + v\,\hat{y}^b + w\,\hat{z}^b\right]$$

$$= \frac{d}{dt}\Big|_r \left[u\,\hat{x}^b\right] + \frac{d}{dt}\Big|_r \left[v\,\hat{y}^b\right] + \frac{d}{dt}\Big|_r \left[w\,\hat{z}^b\right] \qquad \text{note: now use product rule}$$

$$= \frac{d}{dt}\Big|_r [u]\,\hat{x}^b + u\,\frac{d}{dt}\Big|_r [\hat{x}^b] + \frac{d}{dt}\Big|_r [v]\,\hat{y}^b + v\,\frac{d}{dt}\Big|_r [\hat{y}^b] + \frac{d}{dt}\Big|_r [w]\,\hat{z}^b + w\,\frac{d}{dt}\Big|_r [\hat{z}^b]$$

Scalar terms of $\frac{d}{dt}\big|_r [u]$, $\frac{d}{dt}\big|_r [v]$, and $\frac{d}{dt}\big|_r [w]$ are easy to deal with since they do not change based on perspective (they are scalars, not vectors). Therefore their derivatives are not a function of reference frames/perspective. We can write

$$\frac{d}{dt}\Big|_r [u] = \frac{d}{dt}[u] = \dot{u}$$

$$\frac{d}{dt}\Big|_r [v] = \frac{d}{dt}[v] = \dot{v}$$

$$\frac{d}{dt}\Big|_r [w] = \frac{d}{dt}[w] = \dot{w}$$

With the understanding that $\dot{u}$, $\dot{v}$, and $\dot{w}$ are derivatives of the components along F_b.

So the expression becomes

$$= \dot{u}\,\hat{x}^b + u \left.\frac{d}{dt}\right|_r [\hat{x}^b] + \dot{v}\,\hat{y}^b + v \left.\frac{d}{dt}\right|_r [\hat{y}^b] + \dot{w}\,\hat{z}^b + w \left.\frac{d}{dt}\right|_r [\hat{z}^b]$$

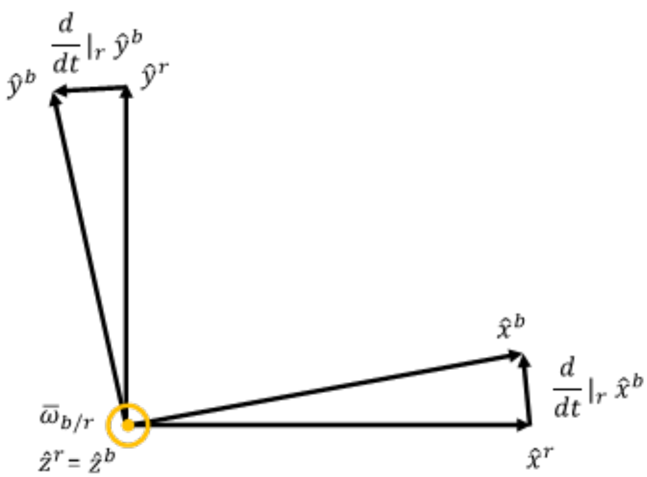
$$= \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix}^b + u \left.\frac{d}{dt}\right|_r [\hat{x}^b] + v \left.\frac{d}{dt}\right|_r [\hat{y}^b] + w \left.\frac{d}{dt}\right|_r [\hat{z}^b]$$

$$= {}^b\dot{\bar{V}}^b + u \left.\frac{d}{dt}\right|_r [\hat{x}^b] + v \left.\frac{d}{dt}\right|_r [\hat{y}^b] + w \left.\frac{d}{dt}\right|_r [\hat{z}^b]$$

Now we need to look at terms $\left.\frac{d}{dt}\right|_r [\hat{x}^b]$, $\left.\frac{d}{dt}\right|_r [\hat{y}^b]$, and $\left.\frac{d}{dt}\right|_r [\hat{z}^b]$.

Consider a 2D example where $\overline{\omega}_{b/r}$ is aligned with the $\hat{z}^b = \hat{z}^r$ axis.

$$\overline{\omega}_{b/r}{}^b = \begin{pmatrix} 0 \\ 0 \\ \omega_z \end{pmatrix}$$



So in the instantaneous limit, we have

$$\left.\frac{d}{dt}\right|_r [\hat{x}^b] = \begin{pmatrix} 0 \\ \omega_z \, |\, \hat{x}^b \,| \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \omega_z \\ 0 \end{pmatrix}$$

$$\left.\frac{d}{dt}\right|_r [\hat{y}^b] = \begin{pmatrix} -\omega_z \, |\, \hat{y}^b \,| \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\omega_z \\ 0 \\ 0 \end{pmatrix}$$

$$\frac{d}{dt}\Big|_r [\hat{z}^b] = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Note that this can be written using a cross product.

$$\frac{d}{dt}\Big|_r [\hat{x}^b] = \overline{\omega}_{b/r} \times \hat{x}^b = \overline{\omega}_{b/r} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \textbf{(Eq.1)}$$

$$\frac{d}{dt}\Big|_r [\hat{y}^b] = \overline{\omega}_{b/r} \times \hat{y}^b = \overline{\omega}_{b/r} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad \textbf{(Eq.2)}$$

$$\frac{d}{dt}\Big|_r [\hat{z}^b] = \overline{\omega}_{b/r} \times \hat{z}^b = \overline{\omega}_{b/r} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \textbf{(Eq.3)}$$

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ωbr = {0, 0, ωz};
xb = {1, 0, 0};
yb = {0, 1, 0};
zb = {0, 0, 1};
dxbdtr = Cross[ωbr, xb]
dybdtr = Cross[ωbr, yb]
dzbdtr = Cross[ωbr, zb]
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{0, ωz, 0}

{-ωz, 0, 0}

{0, 0, 0}
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Eq.1, Eq.2, and Eq.3 generalize to the scenario where $\overline{\omega}_{b/r}$ is a a full vector. Therefore we can write

$$\frac{d}{dt}\Big|_r \overline{V}^b = {}^b\dot{\overline{V}}^b + u\left(\overline{\omega}_{b/r} \times \hat{x}^b\right) + v\left(\overline{\omega}_{b/r} \times \hat{y}^b\right) + w\left(\overline{\omega}_{b/r} \times \hat{z}^b\right) \qquad \text{recall:} \qquad \alpha(\overline{u} \times \overline{v}) = \alpha\,\overline{u} \times \overline{v} = \overline{u} \times \alpha\,\overline{v}$$

$$= {}^b\dot{\overline{V}}^b + \overline{\omega}_{b/r} \times u\,\hat{x}^b + \overline{\omega}_{b/r} \times v\,\hat{y}^b + \overline{\omega}_{b/r} \times w\,\hat{z}^b \qquad \text{recall:} \qquad (\overline{u} \times \overline{v}) + (\overline{u} \times \overline{w}) = \overline{u} \times (\overline{v} + \overline{w})$$

$$= {}^b\dot{\overline{V}}^b + \overline{\omega}_{b/r} \times \left(u\,\hat{x}^b + v\,\hat{y}^b + w\,\hat{z}^b\right)$$

$$= {}^b\dot{\overline{V}}^b + \overline{\omega}_{b/r} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix}^b$$

$${}^r\dot{\overline{V}}^b = {}^b\dot{\overline{V}}^b + \overline{\omega}_{b/r} \times \overline{V}^b$$

We can drop the notation denoting which frame we express this in (F_b) to again arrive at the Equation of Coriolis.

$$^{r}\dot{\overline{V}} = {}^{b}\dot{\overline{V}} + \overline{\omega}_{b/r} \times \overline{V}$$

Angular Rotation Vector ($\overline{\omega}_{b/a}$)

Let us apply the Equation of Coriolis to find some properties of the angular rotation vector.

Show $\overline{\omega}_{a/b} = -\overline{\omega}_{b/a}$

The Equation of Coriolis is given as

$$^{a}\dot{\overline{V}} = {}^{b}\dot{\overline{V}} + \overline{\omega}_{b/a} \times \overline{V} \qquad \textbf{\textit{(Eq.1)}}$$

But if we apply this from the frame b's perspective, we have

$$^{b}\dot{\overline{V}} = {}^{a}\dot{\overline{V}} + \overline{\omega}_{a/b} \times \overline{V} \qquad \textbf{\textit{(Eq.2)}}$$

Substituting Eq.2 into Eq.1

$$^{a}\dot{\overline{V}} = {}^{a}\dot{\overline{V}} + \overline{\omega}_{a/b} \times \overline{V} + \overline{\omega}_{b/a} \times \overline{V}$$

$$0 = \overline{\omega}_{a/b} \times \overline{V} + \overline{\omega}_{b/a} \times \overline{V} \qquad \text{(note: } \overline{x} \times \overline{y} = -(\overline{y} \times \overline{x})\text{)}$$

$$0 = -\left(\overline{V} \times \overline{\omega}_{a/b}\right) - \left(\overline{V} \times \overline{\omega}_{b/a}\right)$$

$$0 = \left(\overline{V} \times \overline{\omega}_{a/b}\right) + \left(\overline{V} \times \overline{\omega}_{b/a}\right)$$

$$0 = \overline{V} \times \left(\overline{\omega}_{a/b} + \overline{\omega}_{b/a}\right)$$

Since this must be true for all $\overline{V}$, we have

$$\overline{\omega}_{a/b} + \overline{\omega}_{b/a} = 0$$

$$\overline{\omega}_{a/b} = -\overline{\omega}_{b/a} \qquad \textbf{\textit{(Eq.3)}}$$

This yields the expected result that the angular velocity of frame a w.r.t. frame b is the opposite of frame b w.r.t. frame a.

Show $^a\dot{\overline{\omega}}_{b/a} = {}^b\dot{\overline{\omega}}_{b/a}$

Recall that the Equation of Coriolis relates how any vector $\overline{V}$ changes from the perspective of a different frame

$$^a\dot{\overline{V}} = {}^b\dot{\overline{V}} + \overline{\omega}_{b/a} \times \overline{V}$$

Substituting $\overline{V} = \overline{\omega}_{b/a}$ yields

$$^a\dot{\overline{\omega}}_{b/a} = {}^b\dot{\overline{\omega}}_{b/a} + \overline{\omega}_{b/a} \times \overline{\omega}_{b/a}$$

We showed previously that $\overline{\omega}_{b/a} = -\overline{\omega}_{a/b}$ so we see that $\overline{\omega}_{b/a}$ and $\overline{\omega}_{a/b}$ point in exactly opposite directions, therefore $\overline{\omega}_{b/a} \times \overline{\omega}_{b/a} = \overline{0}$

$$^a\dot{\overline{\omega}}_{b/a} = {}^b\dot{\overline{\omega}}_{b/a} \qquad \textbf{\textit{(Eq.4)}}$$

So in summary, some properties of the angular rotation vector

(i) It is a unique vector that relates the derivatives of a vector taken in two different frames.
(ii) $\overline{\omega}_{a/b} = -\overline{\omega}_{b/a}$ (relative motion condition)
(iii) $\overline{\omega}_{c/a} = \overline{\omega}_{c/b} + \overline{\omega}_{b/a}$ (additive over multiple frames, not true of angular acceleration)
(iv) $^a\dot{\overline{\omega}}_{b/a} = {}^b\dot{\overline{\omega}}_{b/a}$ (derivative is the same in either frame)