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Lecture 05b
Potential Functions, Fundamental Theorem of Calculus Applied to Line Integrals, & Path Independence

The YouTube video entitled ‘Potential Functions, Fundamental Theorem of Calculus Applied to Line Integrals, & Path Independence’ that covers this lecture is located at https://youtu.be/7zYJYaftPu8 .

Outline

-Potential Functions
-Fundamental Theorem of Calculus Applied to Line Integrals
-Path Independence of Line Integrals
  -The Gradient Theorem
  -Exactness and Independence of Path

Introduction

In the previous video entitled ‘Line Integrals’ at https://youtu.be/0sIsoJYmVVM, we discussed the general concept of a line integral and how to compute it. We now investigate some additional details on this concept. Before diving into these discussions, we first discuss the concept of a potential function.

Potential Functions

Prerequisites:
-Video entitled ‘Scalar Functions, Vector Functions, and Vector Derivatives’ at https://youtu.be/haJVEtLN6-k
-Video entitled ‘Gradient of a Function and the Directional Derivative’ at https://youtu.be/obeu4B8mXuw

Some vector functions, $\overline{v}(x, y, z)$ have the advantage that they can be obtained as the gradient of a scalar function $f(x, y, z)$. In this case, $f(x, y, z)$ is called a potential function of $\overline{v}(x, y, z)$.

Example: Simple Potential Function

Consider the vector function

$$\overline{v}(x, y, z) = \begin{pmatrix} e^{2z} + 3y^2 \\ 6xy \\ 2e^{2z}x \end{pmatrix}$$

Now consider the scalar function

$$f(x, y, z) = 3xy^2 + e^{2z}x + \alpha$$

where α =constant

In[156]:=
```
f[x_, y_, z_] = 3 x y^2 + Exp[2 z] x + α;
```

We can compute the gradient of this scalar function

In[157]:=
```
Grad[f[x, y, z], {x, y, z}] // MatrixForm
```

Out[157]//MatrixForm=

$$\begin{pmatrix} e^{2z} + 3y^2 \\ 6xy \\ 2e^{2z}x \end{pmatrix}$$

We notice that in this case

$$\nabla f(x, y, z) = \overline{v}(x, y, z)$$

so f is the potential of $\overline{v}$.

Fundamental Theorem of Calculus Applied to Line Integrals

Recall that in general, the fundamental theorem of calculus states that

$$\int_a^b f(x)\,dx = g(x)\,|_{x=b} - g(x)\,|_{x=a}$$

where $f(x) = \frac{dg(x)}{dx} = g'(x)$

In other words, the process of computing this definite integral is as follows:

1. Find a function $g(x)$ such that its derivative is $f(x)$
2. Evaluate the difference of $g(x)$ evaluated at $x = b$ and $g(x)$ evaluated at $x = a$

Alternatively, this is more often written as

$$\int_a^b f'(x)\,dx = g(b) - g(a) \qquad \textbf{\textit{(Eq.1)}}$$

For a line integral, we have a similar result. Recall that the general form of the line integral of a vector field can be written as

$$\int_C \overline{F}\cdot d\,\overline{r} \qquad \textbf{\textit{(Eq.2)}}$$

Comparing this with Eq.1, we see that perhaps the equivalent question is when can we consider $\overline{F}$ to be the "derivative" of another function? It turns out that this occurs when $\overline{F}$ is the gradient of a scalar function.

$$\overline{F} = \nabla f \qquad \textbf{\textit{(Eq.3)}}$$

Let us now investigate a general line integral (Eq.2) when Eq.3 is true. For simplicity, we look at this in 2 dimensions but the analysis can extend to arbitrary dimensions.

$$\int_C \overline{F}\cdot d\,\overline{r} = \int_a^b \overline{F}(\overline{r}(t))\cdot \overline{r}\,'(t)\,dt \qquad \text{recall: } \overline{r}(t) = \langle x(t),\ y(t)\rangle \Rightarrow \overline{r}\,'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle$$

$$= \int_a^b \overline{F}(\overline{r}(t))\cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle dt$$

We can now investigate the consequence of $\overline{F} = \nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle$. Substituting this into our result so far yields

$$= \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right) dt$$

At this point, we recall that f is a function of x and y and that x and y are functions of t (in other words: $f = f(x(t),\ y(t))$). So from the Chain Rule (see video entitled 'The Chain Rule' at https://youtu.be/tf-pLFQB-7sU), we see that $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$, which is exactly the expression in the integrand, so we have

$$= \int_a^b \frac{df}{dt}\,dt \qquad \text{recall from the fundamental theorem of calculus } \int_a^b \frac{df}{dt}\,dt = f(b) - f(a)$$

$$\int_C \overline{F}\cdot d\,\overline{r} = f(\overline{r}(b)) - f(\overline{r}(a)) = f(x(b),\ y(b)) - f(x(a),\ y(a)) \qquad (\text{if } \overline{F} = \nabla f) \qquad \textbf{\textit{(Eq.4)}}$$

As we discussed previously (also in Kreyszig section 9.7), the function f is called the **potential function** of $\overline{F}$.

If we take a second to think about the ramifications of Eq.4, we see that if $\overline{F} = \nabla f$, then the evaluation of the integral is simple as it is simply the difference in the potential function at the limits of integration. In other words, the integral is independent of the curve C.

We investigate this further in the next section. For now, let us look at several examples.

Example: Finding and Using a Potential Function

Note that for this example (and other notation in this context), the notation of $< a, b >$ means a vector $(a \quad b)^T$, not the inner product of a and b. Hopefully the seemingly ambiguous use of $< a, b >$ will be clear by examining a and b.

if a, b are scalar, then $< a, b >$ means a vector $\overline{c} = \begin{pmatrix} a \\ b \end{pmatrix}$

if a, b are vectors, then $< a, b >$ means dot product of a and b $(a \cdot b)$

Consider the vector field and curve given by

$$\overline{F}(x, y) = < F_1, F_2 > = < 3 + 2 x y, x^2 - 3 y^2 >$$

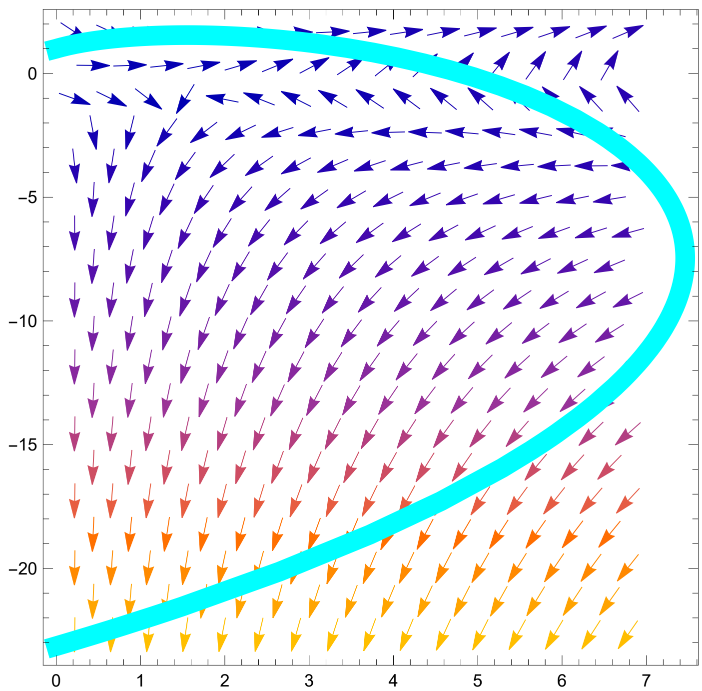
$$C : \overline{r} = < x(t), y(t) > = < r_1(t), r_2(t) > = < e^t \sin(t), e^t \cos(t) > \quad t \in [0, \pi]$$

In[158]:=
```
F1[x_, y_] = 3 + 2 x y;
F2[x_, y_] = x^2 - 3 y^2;
r1[t_] = Exp[t] Sin[t];
r2[t_] = Exp[t] Cos[t];

(*Plot the vector field and curve*)
plotF = VectorPlot[{F1[x, y], F2[x, y]}, {x, 0, 7}, {y, -23, 2}, VectorStyle → Blue];
plotC = ParametricPlot[{r1[t], r2[t]}, {t, 0, π},
   AxesLabel → {x, y},
   PlotStyle → {Cyan, Thickness[0.03]}];

Show[
 plotF,
 plotC,
 PlotRange → All
]
```

Out[164]=



We now wish to compute the line integral of $\int_C \overline{F}(\overline{r}) \cdot d\overline{r}$.

<u>Option 1: Compute Line Integral By Hand</u>

We can compute the line integral by hand

$$\int_C \overline{F}(\overline{r}) \cdot d\overline{r} = \int_a^b \overline{F}(\overline{r}(t)) \cdot \overline{r}'(t)\, dt$$

We can first compute $\overline{F}(\overline{r}(t))$

In[165]:=
```
F1r[t_] = F1[x, y] /. {x → r1[t], y → r2[t]}
F2r[t_] = F2[x, y] /. {x → r1[t], y → r2[t]}
```

Out[165]= $3 + 2\, e^{2t} \mathrm{Cos}[t]\, \mathrm{Sin}[t]$

Out[166]= $-3\, e^{2t} \mathrm{Cos}[t]^2 + e^{2t} \mathrm{Sin}[t]^2$

We can now compute $\bar{r}'(t)$

In[167]:=
```
r1Prime[t_] = D[r1[t], t]
r2Prime[t_] = D[r2[t], t]
```

Out[167]= $e^{t} \mathrm{Cos}[t] + e^{t} \mathrm{Sin}[t]$

Out[168]= $e^{t} \mathrm{Cos}[t] - e^{t} \mathrm{Sin}[t]$

We can now compute the dot product of $\bar{F}(\bar{r}(t)) \cdot \bar{r}'(t)$

In[169]:=
```
integrand[t_] = F1r[t] * r1Prime[t] + F2r[t] * r2Prime[t] // Expand
```

Out[169]= $3\, e^{t} \mathrm{Cos}[t] - 3\, e^{3t} \mathrm{Cos}[t]^3 + 3\, e^{t} \mathrm{Sin}[t] + 5\, e^{3t} \mathrm{Cos}[t]^2 \mathrm{Sin}[t] + 3\, e^{3t} \mathrm{Cos}[t]\, \mathrm{Sin}[t]^2 - e^{3t} \mathrm{Sin}[t]^3$

We can now integrate with respect to t

In[170]:=
```
g[t_] = Integrate[integrand[t], t]
```

Out[170]= $\frac{1}{2} \left(-e^{3t} \mathrm{Cos}[t] - e^{3t} \mathrm{Cos}[3t] + 6\, e^{t} \mathrm{Sin}[t]\right)$

Evaluating at $t = \pi$ and subtracting the value of the function evaluated at $t = 0$ yields the final result

In[171]:=
```
g[π] - g[0]
```

Out[171]= $1 + e^{3\pi}$

<u>Option 2: Find Potential Function</u>

An alternative method involves finding the potential function f such that $\nabla f = \bar{F}$. We see that we require

$$\nabla f = \bar{F}$$

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} 3 + 2\,x\,y \\ x^2 - 3\,y^2 \end{pmatrix}$$

$$\frac{\partial f}{\partial x} = 3 + 2\,x\,y \qquad \textbf{\textit{(Eq.A)}}$$

$$\frac{\partial f}{\partial y} = x^2 - 3\,y^2 \qquad \textbf{\textit{(Eq.B)}}$$

<u>Method A: Integrate both equations</u>

Let us look at Eq.A, we can integrate with respect to x to obtain

$$\int \frac{\partial f}{\partial x}\, dx = \int 3 + 2\,x\,y\, dx$$

$$f = 3x + x^2 y + g(y) \qquad \textbf{(Eq.C)}$$

Note that the constant of integration is actually only a constant with respect to x (it can be a function of y).

Now consider Eq.B and integrate with respect to y to obtain

$$\int \frac{\partial f}{\partial y} \, dy = \int x^2 - 3y^2 \, dy$$

$$f = x^2 y - y^3 + h(x) \qquad \textbf{(Eq.D)}$$

Comparing the two (Eq.C and Eq.D)

$$3x + x^2 y + g(y) = x^2 y - y^3 + h(x)$$

$$3x + g(y) = -y^3 + h(x)$$

From inspection, we see that if we choose $g(y) = -y^3$ and $h(x) = 3x$, we should have a consistent solution. Substituting these into Eq.C and Eq.D, we obtain

$$f = 3x + x^2 y - y^3 \qquad \textbf{(Eq.E)}$$

$$f = x^2 y - y^3 + 3x$$

Which are both the same solution. We can verify that the gradient of this potential is $\overline{F}$

In[172]:=
```
f = 3 x + x^2 y - y^3;
D[f, x] == 3 + 2 x y // Simplify
D[f, y] == x^2 - 3 y^2 // Simplify
```

Out[173]= `True`

Out[174]= `True`

<u>Method B: Integrate one equation w.r.t *x* then differentiate result w.r.t. *y*</u>

Another method is to first integrate Eq.A w.r.t. x. We would again obtain Eq.C (repeated here for convenience)

$$f = 3x + x^2 y + g(y)$$

We can now differentiate this w.r.t. y to obtain

$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial}{\partial y}[g(y)]$$

We would like this to equal Eq.B

$$x^2 + \frac{\partial}{\partial y}[g(y)] = x^2 - 3\,y^2$$

$$\frac{\partial}{\partial y}[g(y)] = -3\,y^2$$

At this point, we need to find a function of only y such that when we differentiate it w.r.t. y, we obtain $-3\,y^2$. From inspection, we see that $g(y) = -y^3$ satisfies this constraint. Substituting this into Eq.C, we obtain

$$f = 3\,x + x^2\,y - y^3$$

which is the same as Eq.E that we found using method A.

At this point, we have found f such that $\nabla f = \overline{F}$, we can apply the fundamental rule of calculus for line integrals (Eq.4) to obtain

$$\int_C \overline{F}\cdot d\,\overline{r} = f(\overline{r}(b)) - f(\overline{r}(a)) = f(x(b),\ y(b)) - f(x(a),\ y(a)) \qquad (\text{if } \overline{F} = \nabla f)$$

Recall that $\overline{r}(t)$ is given by

$$\overline{r}(t) = \begin{pmatrix} e^t \sin(t) \\ e^t \cos(t) \end{pmatrix}$$

So $\overline{r}(a) = \overline{r}(0)$ and $\overline{r}(b) = \overline{r}(\pi)$ are given as

```
In[175]:= (r1[t]
           r2[t]) /. {t → 0} // MatrixForm
          (r1[t]
           r2[t]) /. {t → π} // MatrixForm
```

Out[175]//MatrixForm=

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Out[176]//MatrixForm=

$$\begin{pmatrix} 0 \\ -e^{\pi} \end{pmatrix}$$

So the potential function evaluated at these points are given by

$$f(\overline{r}(b)) = 3\times(0) + (0)^2\,(-e^{\pi}) - (-e^{\pi})^3 = e^{3\,\pi}$$

$$f(\overline{r}(a)) = 3\times(0) + (0)^2\,(1) - (1)^3 = -1$$

So we obtain the final result of

$$\int_C \overline{F}\cdot d\,\overline{r} = e^{3\,\pi} + 1$$

which is the same result we obtained previously

In[177]:=
```
(f /. {x → r1[π], y → r2[π]}) - (f /. {x → r1[0], y → r2[0]})
```

Out[177]= $1 + e^{3\pi}$

To further reinforce the idea that the result of the line integral is independent of the path, we can consider an alternate path, C_{alt}, that has the same start and end point and manually compute the line integral and verify we still obtain this same result.

$$C_{\text{alt}} : \bar{r} = < x(s), y(s) > = < r_1(s), r_2(s) > = < 0, (-e^{\pi})\, t + 1 > \qquad s \in [0, 1]$$

In[178]:=
```
r1alt[s_] = 0;
r2alt[s_] = - (1 + Exp[π]) s + 1;

(*verify parameterization has same end points*)
r1alt[s] == r1[t] /. {s → 0, t → 0}
r2alt[s] == r2[t] /. {s → 0, t → 0}

r1alt[s] == r1[t] /. {s → 1, t → π}
r2alt[s] == r2[t] /. {s → 1, t → π}

(*Plot this alternate parameterization and overlay with previous picture*)
plotCalt = ParametricPlot[{r1alt[s], r2alt[s]}, {s, 0, 1},
   AxesLabel → {x, y},
   PlotStyle → {Red, Thickness[0.03]}];

Show[
 plotF,
 plotC,
 plotCalt,
 PlotRange → All
]
```

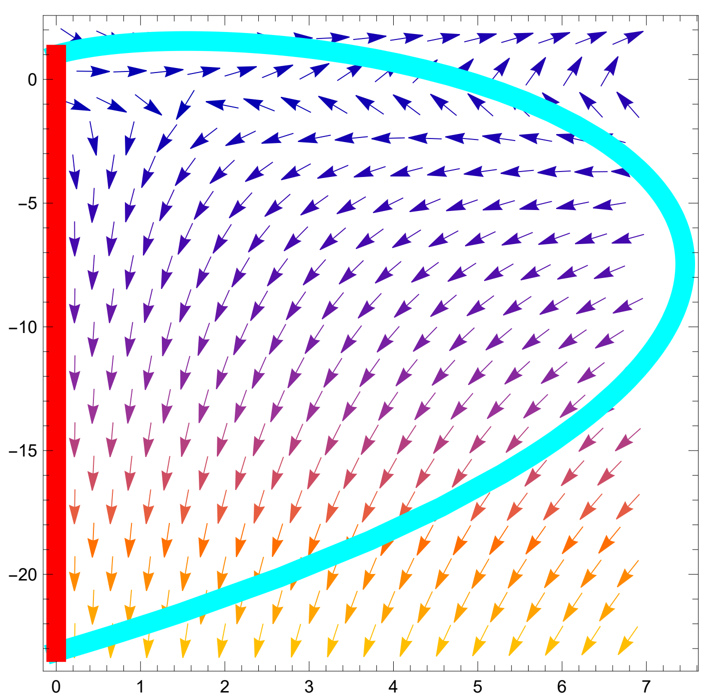
Out[180]= True

Out[181]= True

Out[182]= True

Out[183]= True

Out[185]=



We can compute the line integral by hand

$$\int_C \overline{F}(\overline{r}_{\text{alt}}) \cdot d\,\overline{r}_{\text{alt}} = \int_a^b \overline{F}(\overline{r}_{\text{alt}}(s)) \cdot \overline{r}_{\text{alt}}{}'(s)\, ds$$

We can first compute $\overline{F}(\overline{r}_{\text{alt}}(s))$

```
In[186]:= F1ralt[s_] = F1[x, y] /. {x → r1alt[t], y → r2alt[s]}
          F2ralt[s_] = F2[x, y] /. {x → r1alt[t], y → r2alt[s]}
```

Out[186]= 3

Out[187]= $-3\,(1 + (-1 - e^{\pi})\,s)^2$

We can now compute $\overline{r}_{\text{alt}}{}'(s)$

```
In[188]:= r1altPrime[s_] = D[r1alt[s], s]
          r2altPrime[s_] = D[r2alt[s], s]
```

Out[188]= 0

Out[189]= $-1 - e^{\pi}$

We can now compute the dot product of $\overline{F}(\overline{r}_{\text{alt}}(s)) \cdot \overline{r}_{\text{alt}}{}'(s)$

```
In[190]:= integrandalt[s_] = F1ralt[s] * r1altPrime[s] + F2ralt[s] * r2altPrime[s] // Expand
```

Out[190]= $3 + 3\,e^{\pi} - 6\,s - 12\,e^{\pi}\,s - 6\,e^{2\pi}\,s + 3\,s^2 + 9\,e^{\pi}\,s^2 + 9\,e^{2\pi}\,s^2 + 3\,e^{3\pi}\,s^2$

We can now integrate with respect to s

```
In[191]:= galt[s_] = Integrate[integrandalt[s], s]
```

Out[191]= $3\,s + 3\,e^{\pi}\,s - 3\,s^2 - 6\,e^{\pi}\,s^2 - 3\,e^{2\pi}\,s^2 + s^3 + 3\,e^{\pi}\,s^3 + 3\,e^{2\pi}\,s^3 + e^{3\pi}\,s^3$

Evaluating at $s = 1$ and subtracting the value of the function evaluated at $s = 0$ yields the final result

```
In[192]:= galt[1] - galt[0]
```

Out[192]= $1 + e^{3\pi}$

Which is the same result we obtained previously.

```
In[193]:= Clear[F1, F2, r1, r2, F1r, F2r, r1Prime, r2Prime, integrand, g, f, r1alt, r2alt,
            F1ralt, F2ralt, r1altPrime, r2altPrime, integrandalt, galt, plotF, plotC, plotCalt]
```

While this method is obviously very powerful, the natural question that follows is “when is it not possible to find the potential function f”?

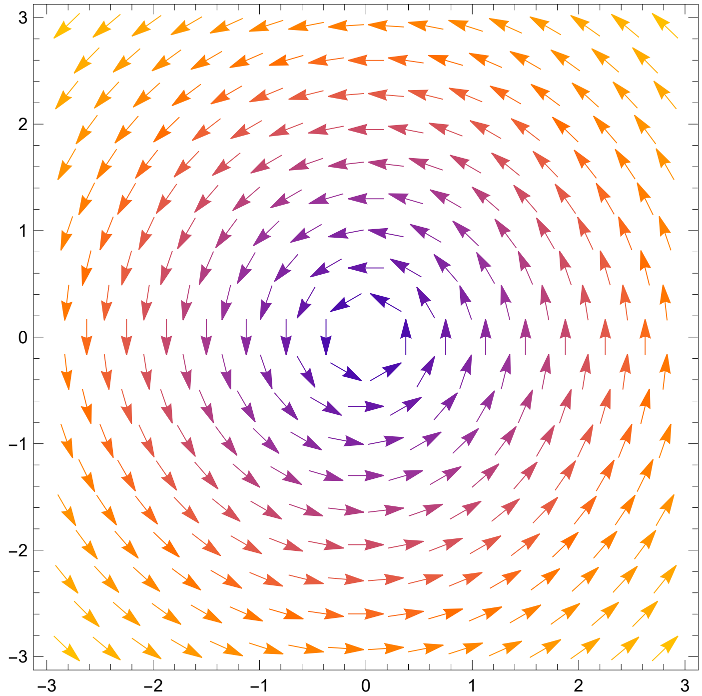
Example: Unable to Find a Potential Function

Consider the vector field

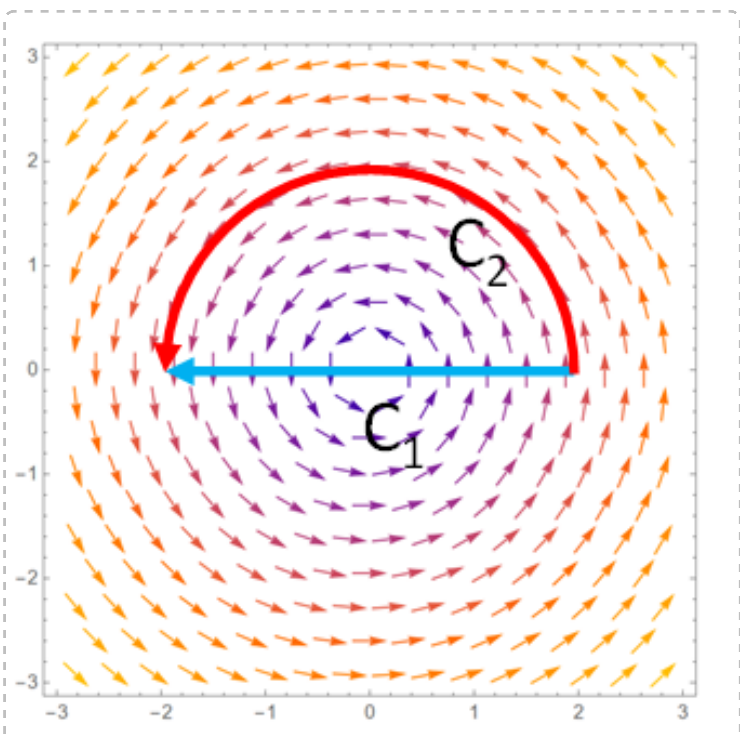
$\overline{F}(x, y) = < -y, x >$ note: this notation of $< a, b >$ means a vector $(\,a \quad b\,)^T$, not the inner product of a and b

In[194]:=
```
VectorPlot[{-y, x}, {x, -3, 3}, {y, -3, 3}]
```

Out[194]=



From physical intuition, we see that the work done by the field depends on the path. For example, consider the path C_1 as shown below. In this case, the direction of travel is normal to the field at all points so the total work is 0. However the path C_2 which has the same start and end point moves in the same direction as the field and as such we expect the work to be positive.



We can attempt to find a potential function f such that $\nabla f = \overline{F}$.

$$\nabla f = \overline{F}$$

$$\begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

$$\frac{\partial f}{\partial x} = -y \qquad \textbf{(Eq.A)}$$

$$\frac{\partial f}{\partial y} = x \qquad \textbf{(Eq.B)}$$

<u>Method A: Integrate both equations</u>

Let's look at Eq.A, we can integrate with respect to x to obtain

$$\int \frac{\partial f}{\partial x} \, dx = \int -y \, dx$$

$$f = -x\,y + g(y) \qquad \textbf{(Eq.C)}$$

Note that the constant of integration is actually only a constant with respect to x (it can be a function of y).

Now consider Eq.B and integrate with respect to y to obtain

$$\int \frac{\partial f}{\partial y} \, dy = \int x \, dy$$

$$f = x\,y + h(x) \qquad \textbf{(Eq.D)}$$

Comparing the two

$$-x\,y + g(y) = x\,y + h(x)$$

$$g(y) = 2\,x\,y + h(x)$$

At this point, we encounter an inconsistency. There is no function $g(y)$ which is only a function of y which will equal $2\,x\,y + h(x)$ (note: we cannot choose $h(x) = -2\,x\,y$ because $h(x)$ must be only a function of x).

Therefore, we determine that we cannot find a potential of this vector field.

 We would reach this same conclusion if we use Method B

<u>Method B: Integrate one equation w.r.t *x* then differentiate result w.r.t. *y*</u>

Another method is to Integrate Eq.A w.r.t. x and we would again obtain Eq.C (repeated here for convenience)

$$f = -x\,y + g(y)$$

We can now differentiate this w.r.t. y to obtain

$$\frac{\partial f}{\partial y} = -x + \frac{\partial}{\partial y}[g(y)]$$

We would like this to equal Eq.B

$$-x + \frac{\partial}{\partial y}[g(y)] = x$$

$$\frac{\partial}{\partial y}[g(y)] = 2\,x$$

At this point, we need to find a function of only y such that when we differentiate it w.r.t. y, we obtain $2\,x$. Once again, this is an inconsistency because there is no function of only y which contains x.

Path Independence of Line Integrals

The Gradient Theorem

As we mentioned previously, in general, the line integral $\int_C \overline{F} \cdot d\,\overline{r}$ will take different values depending on the path C even if two paths, C_1 and C_2 have the same end points. However we saw that if $\overline{F}$ is the gradient of a potential function f, then the line integral is independent of path. We can now state this formally.

Consider the line integral of the form

$$\int_C \overline{F}(\overline{r}) \cdot d\overline{r} = \int_C (F_1\,dx + F_2\,dy + F_3\,dz) \qquad \textbf{(Eq.1)}$$

Theorem 1 : Path Independence

A line integral with continuous F_1, F_2, and F_3 in a domain D in space is independent of path in D if and only if $\overline{F}$ is the gradient of some function f in D.

$$\overline{F} = \text{grad}\,f \qquad \textbf{(Eq.2)}$$

<u>Proof:</u> $\{\overline{F} = \text{grad}\,f\} \Rightarrow \{\int_C \overline{F}(\overline{r}) \cdot d\overline{r}$ is independent of the path in a domain D in space$\}$

We first show that if $\overline{F} = \text{grad}\,f$, then the line integral is independent of the path in a domain D in space.

Let C be any path in D from any point A to any point B, given by

$\overline{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k} \qquad t \in [a, b] \qquad$ (note $t = a$ corresponds to point A and $t = b$ corresponds to point B)

Since we assume that

$$\overline{F} = \text{grad}\, f = \frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j} + \frac{\partial f}{\partial z}\,\hat{k}$$

So the line integral

$$\int_C \overline{F} \cdot d\overline{r} = \int_A^B \left(\frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j} + \frac{\partial f}{\partial z}\,\hat{k}\right) \cdot \left(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}\right) \qquad \text{(note: we integrate from } \overline{r} = A \text{ to } \overline{r} = B\text{)}$$

$$= \int_A^B \left(\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz\right)$$

$$= \int_a^b \frac{\mathrm{df}}{\mathrm{dt}}\,dt \qquad \text{recall: } \int_a^b f'(t)\,dt = f(b) - f(a)$$

$$= f(x(t), y(t), z(t))\,|_{t=b} - f(x(t), y(t), z(t))\,|_{t=a}$$

$$\int_C \overline{F} \cdot d\,\overline{r} = f(\overline{r}(b)) - f(\overline{r}(a))$$

So this shows that the value of the line integral is simply the difference of the values of f at the two endpoints of C and is therefore independent of the path C

Proof: $\left\{\int_C \overline{F}(\overline{r}) \cdot d\,\overline{r}\right.$ is independent of the path in a domain D in space$\} \Rightarrow \left\{\overline{F} = \text{grad}\, f\right\}$

This portion of the proof is left as an exercise to the reader.

End proof

This is also known as the 'Gradient Theorem'

The Gradient Theorem

Let $\overline{F} = \nabla f$. Then

$$\int_C \overline{F} \cdot d\,\overline{r} = f(\overline{r}(b)) - f(\overline{r}(a))$$

where b =corresponds to the end point of C
a =corresponds to the start point of C

Additional References: http://en.wikipedia.org/wiki/Gradient_theorem

Exactness and Independence of Path

If the vector field is a gradient of a scalar potential function, the vector field is called an exact vector field.

$$\overline{F} = \nabla f \qquad \xRightarrow{\text{(definition of exact)}} \overline{F} \text{ is an exact vector field}$$

If the line integral is independent of path, the vector field is called conservative vector field.

$$\int_C \overline{F} \cdot d\overline{r} \text{ only depends on end points} \xRightarrow{\text{(definition of conservative)}} \overline{F} \text{ is a conservative vector field}$$

In practice, these definitions are usually blurred and for the most part, exact and conservative are the same thing. However, to be clear, we see that

$$\overline{F} \text{ is exact} \Rightarrow \overline{F} \text{ is conservative}$$

We can make some interesting observations if we have an exact/conservative vector field

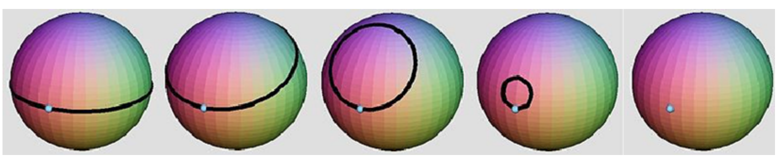
1) $\overline{F} = \nabla f$ (definition of an exact vector field)

2) $\overline{F}$ is conservative $\Longleftrightarrow$ $\int_C \overline{F} \cdot d\overline{r} = f(\overline{r}(b)) - f(\overline{r}(a))$ (independence of path)

3) $\overline{F}$ is conservative $\Longleftrightarrow$ $\oint_C \overline{F} \cdot d\overline{r} = 0$ (line integral over any closed path is 0)

Another useful concept is that of a domain D being **simply connected**. A domain D is called is called simply connected if every closed curve in D can be continuously shrunk to any point in D without leaving D.

The surface of the sphere is simply connected because every loop can be contracted to a point.



Informally, a object is simply connect if it consists of one piece and does not have any "holes".

Recall in our video entitled 'The Laplace Operator, Divergence, and Curl' at 26:45 (see https://youtu.be/KOlVHPShCOk?si=hH8tXJ8sF9kVeZdy&t=1605) we showed that if $\overline{F} = \nabla f$ (AKA $\overline{F}$ is conservative) then curl (grad f) = $\overline{0}$ and we refer to $\overline{F}$ as irrotational.

$$\overline{F} \text{ is conservative} \Rightarrow \overline{F} \text{ is irrotational (AKA curl}(\overline{F}) = \overline{0})$$

If the domain D is simply connected then the reverse implication is true as well

$$\overline{F} \text{ is irrotational } \left(\text{AKA curl}(\overline{F}) = \overline{0}\right) \Rightarrow \overline{F} \text{ is conservative} \qquad \text{(if } D \text{ is simply connected)}$$

Or stated more formally

Theorem 3 (Criterion for exactness and independence of path)

Let F_1, F_2, F_3 in the line integral

$$\int_C \overline{F}(\overline{r}) \cdot d\overline{r} = \int_C (F_1\, dx + F_2\, dy + F_3\, dz)$$

be continuous and have continuous first partial derivatives in a domain *D* in space. Then:

(a) If the line integral is independent of path in *D* - and thus the differential form under the integral sign is exact- then in *D*

$$\text{curl}\,\overline{F} = \overline{0}$$

In components

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}, \quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y} \qquad \textbf{(Eq.6')}$$

(b) If Eq.6' holds in *D* and *D* is simply connected, then the line integral is independent of path in *D*