Christopher Lum
lum@uw.edu

Lecture06d
Fourier Series

Lecture is on YouTube

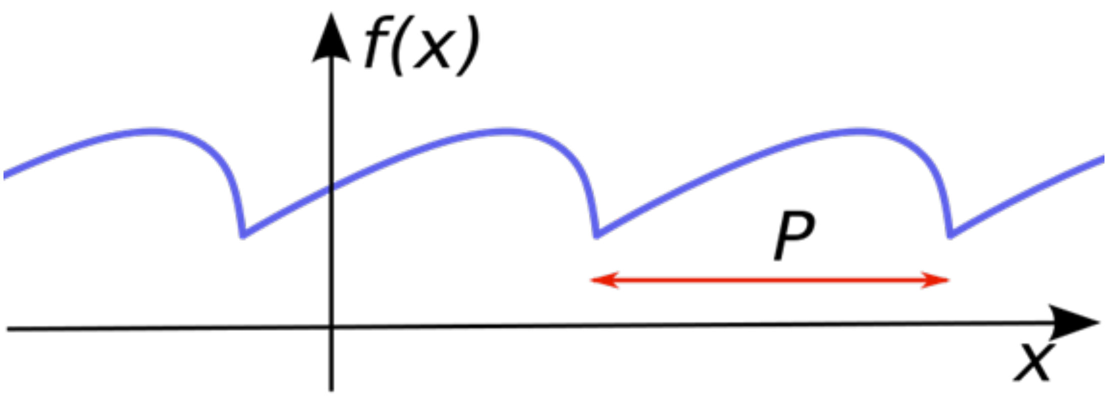
The YouTube video entitled ‘Fourier Series’ that covers this lecture is located at https://youtu.be/7GXbPYzW5JA

Periodic Functions

A function $f(x)$ is called **periodic** if it is defined for all real x and if there is some positive number P such that

$$f(x+P) = f(x) \quad \forall\, x \qquad \textit{(Eq.1)}$$

where $\quad P$ = period of the function



We immediately note that since the function repeats with period P, then

$$f(x+n\,P) = f(x) \qquad n \in Z \qquad (n \text{ is an integer}) \qquad \textit{(Eq.1.A)}$$

Furthermore, if $f(x)$ and $g(x)$ have period p, then the following function also has period p

$$h(x) = a\,f(x) + b\,g(x) \qquad (\text{has period } p) \qquad \textit{(Eq.1.B)}$$

where a, b = constants
$f(x)$, $g(x)$ have period p

Example

Consider the function

$$f(x) = 2\cos\left(\frac{2\pi}{P}\,x + 75\,\frac{\pi}{180}\right)$$

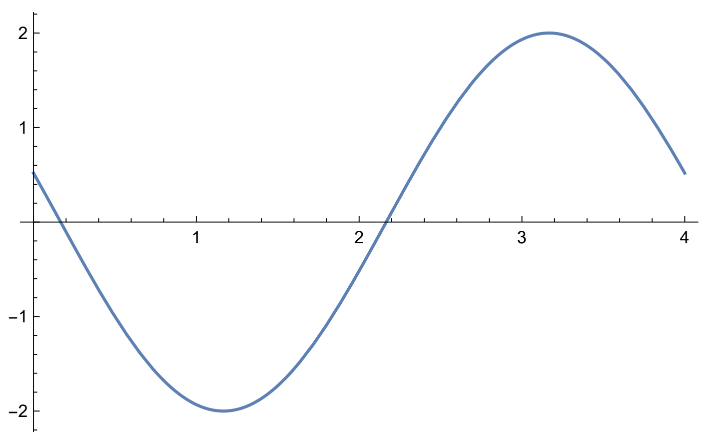
In[1]:=
```
P = 4;

(*Verify that f(x) is periodic*)
f[x] == f[x + P] // Simplify

(*Plot function*)
f[x_] = 2 Cos[2 π/P x + 75 π/180];
Plot[f[x], {x, 0, P}]

Print["Verify that f(x+n*P)=f(x) for n integer"]
Simplify[f[x + n P] == f[x], Element[n, Integers]]
```

Out[2]= `f[x] == f[4 + x]`

Out[4]=


`Verify that f(x+n*P)=f(x) for n integer`

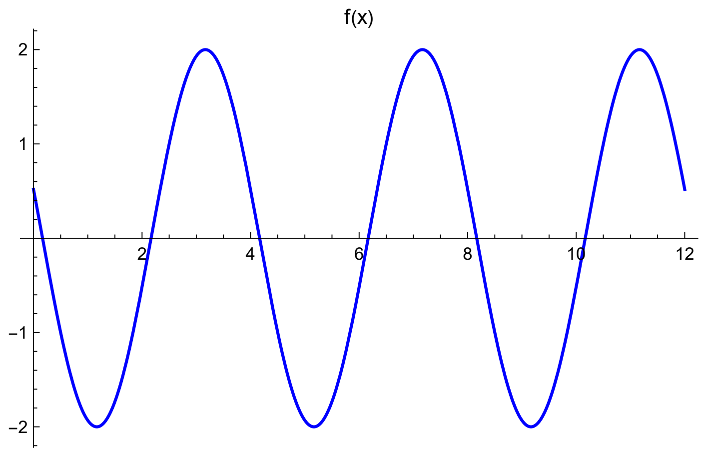
Out[6]= `True`

We can verify that $h(x) = a\,f(x) + b\,g(x)$ has period p.

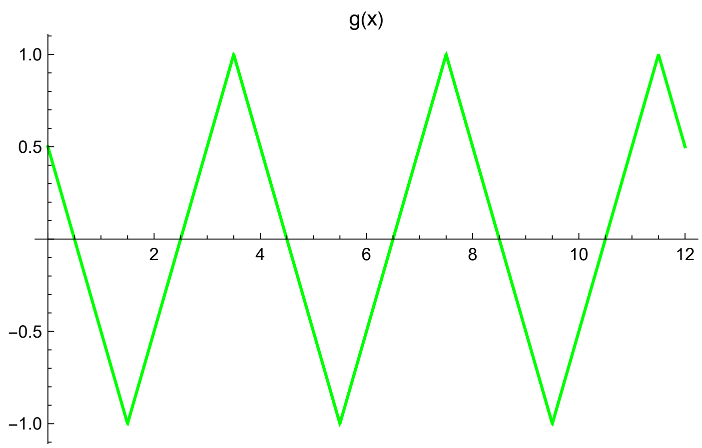
In[7]:= $g[x\_] = \text{TriangleWave}\left[\frac{x + 1.5}{P}\right];$

```
Plot[f[x], {x, 0, 3 P}, PlotStyle → {Blue}, PlotLabel → "f(x)"]
Plot[g[x], {x, 0, 3 P}, PlotStyle → {Green}, PlotLabel → "g(x)"]
Plot[-3 f[x] + 8 g[x], {x, 0, 3 P}, PlotStyle → {Red}, PlotLabel → "-3*f(x)+8*g(x)"]
```

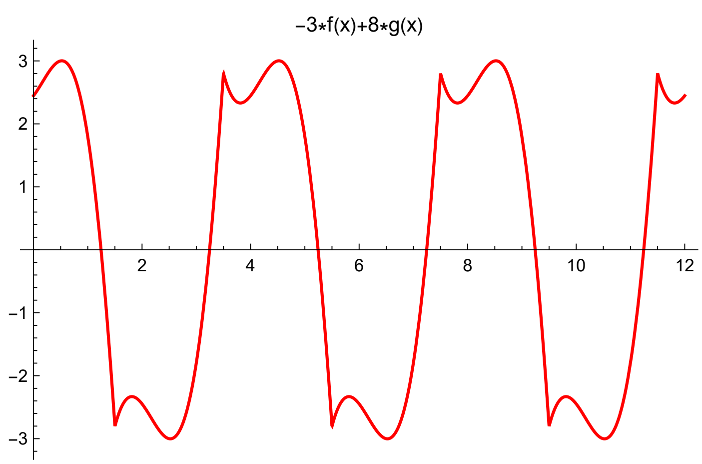
Out[8]=



Out[9]=



Out[10]=



```
In[]:= Clear[g, f, P]
```

Trigonometric Series

A **trigonometric series** is a series of the form

$$a_0 + (a_1 \cos(x) + b_1 \sin(x)) + (a_2 \cos(2\,x) + b_2 \sin(2\,x)) + \ldots$$

Or alternatively expressed as

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\,x) + b_n \sin(n\,x)) \qquad \textbf{\textit{(Eq.4)}}$$

where $\quad a_i, b_i$ = real constants

Fourier Series

The Fourier series is a type of trigonometric series with specific values of a_i, b_i coefficients

Let us assume that we have a function $f(x)$ with period of $p = 2\,\pi$. We would now like to find coefficients a_i and b_i from Eq.4 in the previous section such that

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\,x) + b_n \sin(n\,x)) \qquad \textbf{\textit{(Eq.1)}}$$

Note that in some cases, it may not be possible to compute the series to make the right hand side equal to the original function (the left hand side). As such, some texts will write this as

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\,x) + b_n \sin(n\,x))$$

Where the ~ is used because it is not always true that the sum of the Fourier series is equal to $f(x)$ (it can fail to converge or converge to something different from $f(x)$ but these differences are rarely significant in engineering).

To start, let us try to find a_0. We can do this by integrating both sides from -π to π

$$\int_{-\pi}^{\pi} f(x)\, dx = \int_{-\pi}^{\pi} \left(a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\,x) + b_n \sin(n\,x))\right) dx$$

$$= \int_{-\pi}^{\pi} a_0\, dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (a_n \cos(n\,x) + b_n \sin(n\,x))\, dx$$

$$= 2\,\pi\, a_0 + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (a_n \cos(n\,x) + b_n \sin(n\,x))\, dx$$

Let us examine one of the integrals. For example, $\int_{-\pi}^{\pi} a_1 \cos(x)\, dx$.

In[11]:=
```
Integrate[a1 Cos[x], {x, -π, π}]
```

Out[11]= 0

We notice this is actually equal to 0. It turns out that the same is true if we look at any another term (below are 3 examples)

In[12]:=
```
Integrate[b3 Sin[3 x], {x, -π, π}]
Integrate[b4 Sin[4 x], {x, -π, π}]
Integrate[a40 Cos[40 x], {x, -π, π}]
```

Out[12]= 0

Out[13]= 0

Out[14]= 0

Therefore, we have $\int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (a_n \cos(n\,x) + b_n \sin(n\,x))\,dx = 0$. So the previous equation becomes

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\,\pi\,a_0$$

Solving for a_0, we obtain the first coefficient of the Fourier series

$$a_0 = \frac{1}{2\,\pi} \int_{-\pi}^{\pi} f(x)\,dx \qquad \textbf{\textit{(Eq.2)}}$$

We can now follow a similar procedure to find the remaining coefficients. Let us start with the a_n coefficients. We multiply Eq.1 by $\cos(m\,x)$ where m is any fixed positive integer.

$$\cos(m\,x)\,f(x) = \cos(m\,x)\,(a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\,x) + b_n \sin(n\,x)))$$

$$\cos(m\,x)\,f(x) = a_0 \cos(m\,x) + \sum_{n=1}^{\infty} (a_n \cos(n\,x) \cos(m\,x) + b_n \sin(n\,x) \cos(m\,x))$$

Integrating both sides from -π to π yields

$$\int_{-\pi}^{\pi} \cos(m\,x)\,f(x)\,dx = \int_{-\pi}^{\pi} \{a_0 \cos(m\,x) + \sum_{n=1}^{\infty} (a_n \cos(n\,x) \cos(m\,x) + b_n \sin(n\,x) \cos(m\,x))\}\,dx$$

$$= \int_{-\pi}^{\pi} a_0 \cos(m\,x)\,dx + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} (a_n \cos(n\,x) \cos(m\,x) + b_n \sin(n\,x) \cos(m\,x))\,dx$$

note: $\int_{-\pi}^{\pi} a_0 \cos(m\,x)\,dx = 0$

$$= \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n \cos(n\,x) \cos(m\,x) + b_n \sin(n\,x) \cos(m\,x))\,dx$$

$$= \sum_{n=1}^{\infty} \left\{a_n \int_{-\pi}^{\pi} \left(\cos(n\,x) \cos(m\,x)\,dx + b_n \int_{-\pi}^{\pi} \sin(n\,x) \cos(m\,x)\right)\,dx\right\}$$

At this point, we can make use of the trig identify to help evaluate these integrals

$$\cos(x) \cos(y) = \frac{1}{2}[\cos(x + y) + \cos(x - y)]$$

$$\sin(x)\cos(y) = \frac{1}{2}[\sin(x+y) + \sin(x-y)]$$

So applied to our integrals

$$\int_{-\pi}^{\pi} \cos(n\,x)\cos(m\,x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \cos((n+m)\,x)\,dx + \frac{1}{2}\int_{-\pi}^{\pi} \cos((n-m)\,x)\,dx \qquad \textbf{(Eq.A.1)}$$

$$\int_{-\pi}^{\pi} \sin(n\,x)\cos(m\,x)\,dx = \frac{1}{2}\int_{-\pi}^{\pi} \sin((n+m)\,x)\,dx + \frac{1}{2}\int_{-\pi}^{\pi} \sin((n-m)\,x)\,dx \qquad \textbf{(Eq.A.2)}$$

We note that since n, m are positive integers, each of these integrals on the right side become

$$\frac{1}{2}\int_{-\pi}^{\pi} \cos((n+m)\,x)\,dx = 0 \qquad \textbf{(Eq.A.3)}$$

$$\frac{1}{2}\int_{-\pi}^{\pi} \cos((n-m)\,x)\,dx = \begin{cases} 0 & n \neq m \\ \pi & n = m \end{cases} \qquad \textbf{(Eq.A.4)}$$

$$\frac{1}{2}\int_{-\pi}^{\pi} \sin((n+m)\,x)\,dx = 0 \qquad \textbf{(Eq.A.5)}$$

$$\frac{1}{2}\int_{-\pi}^{\pi} \sin((n-m)\,x)\,dx = 0 \qquad \textbf{(Eq.A.6)}$$

```
In[15]:= (*Spot check by choosing a value of m and n and show that we only have a non-
           zero result if m=n*)
         m = 3;
         n = 3;

         (*Evaluate integrals*)
         1/2 Integrate[Cos[(n + m) x], {x, -π, π}]
         1/2 Integrate[Cos[(n - m) x], {x, -π, π}]
         1/2 Integrate[Sin[(n + m) x], {x, -π, π}]
         1/2 Integrate[Sin[(n - m) x], {x, -π, π}]

Out[17]= 0

Out[18]= π

Out[19]= 0

Out[20]= 0
```

So we see that each of these terms are 0 except for Eq.A.4 which is non-zero only when $n = m$. In this case, the integral evaluates to π. Comparing with the result with what we had several lines previously, we see that this term multiplies the a_m coefficient so this reduces to

$$\int_{-\pi}^{\pi} \cos(m\,x)\,f(x)\,dx = a_m\,\pi$$

Solving for a_m yields

$$a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(m\,x)\, f(x)\, dx \qquad m = 1, 2, \ldots$$

or using n instead of m notation

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(n\,x)\, f(x)\, dx \qquad n = 1, 2, \ldots$$

We find the b_n coefficients in a similar manner (by multiplying by $\sin(m\,x)$) to obtain

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(n\,x)\, f(x)\, dx \qquad n = 1, 2, \ldots$$

So the Euler formulas for the Fourier coefficients are given by

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx \qquad \textbf{\textit{(Eq.6.0)}}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(n\,x)\, f(x)\, dx \qquad n = 1, 2, \ldots \qquad \textbf{\textit{(Eq.6.a)}}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(n\,x)\, f(x)\, dx \qquad n = 1, 2, \ldots \qquad \textbf{\textit{(Eq.6.b)}}$$

Which are applied to the generic trigonometric series

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\,x) + b_n \sin(n\,x))$$

We notice that this is an infinite series. Often it becomes useful to talk of partial sums which are finite approximations of the infinite series. These are often denoted as

$$S_N f = S_N = a_0 + \sum_{n=1}^{N} (a_n \cos(n\,x) + b_n \sin(n\,x)) \qquad \textbf{\textit{(Eq.8)}}$$

Example 1: Square Wave

Consider the periodic function $f(x)$. This function is described over one period of $x \in [-\pi, \pi]$ as

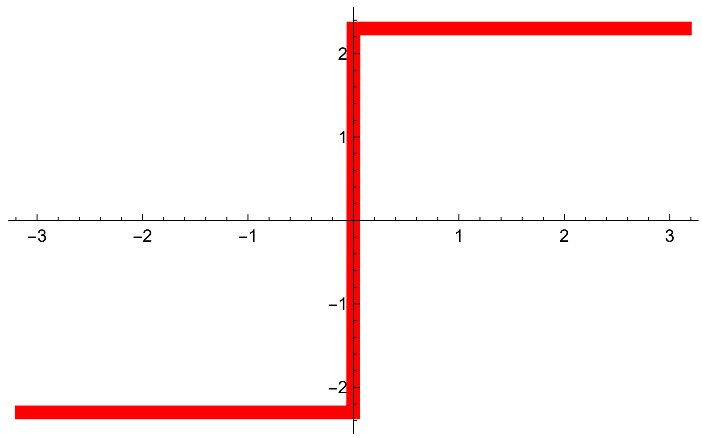
$$f(x) = \begin{cases} -k & \text{if } x \in [-\pi, 0] \\ k & \text{if } x \in [0, \pi] \end{cases}$$

At this point, it is useful to define the function f in Mathematica. However, we notice that we do not know what k is and this is a piecewise function. Therefore, it may be useful to use an 'If' statement to define the function. It may also be useful to use the 'set delay' operation in Mathematica. When defining the function, we use a ':=' instead of '=' when defining the function. This asks Mathematica to delay evaluating the right side until it is needed.

In[21]:=
```
f[x_] := If[
   x ≤ 0,
   -k,
   k
  ]

(*Choose a value of k for this plot*)
kGiven = 2.3;
Plot[f[x] /. {k → kGiven}, {x, -π, π}, PlotStyle → {Red, Thickness[0.02]}]
```

Out[23]=



We can find the coefficients by applying Eq.6.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{2\pi} \left(\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx \right)$$

$$= \frac{1}{2\pi} \left(\int_{-\pi}^{0} -k\,dx + \int_{0}^{\pi} k\,dx \right)$$

$$a_0 = 0$$

In[24]:=
```
Integrate[-k, {x, -π, 0}] + Integrate[k, {x, 0, π}]
```

Out[24]= `0`

We can now find the coefficients a_n

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(n\,x)\, f(x)\,dx \qquad n = 1,\, 2,\, \ldots$$

$$= \frac{1}{\pi} \left(\int_{-\pi}^{0} \cos(n\,x)\, f(x)\,dx + \int_{0}^{\pi} \cos(n\,x)\, f(x)\,dx \right)$$

$$= \frac{1}{\pi} \left(\int_{-\pi}^{0} \cos(n\,x)\,(-k)\,dx + \int_{0}^{\pi} \cos(n\,x)\, k\,dx \right)$$

$$= \frac{1}{\pi} \left(-k \int_{-\pi}^{0} \cos(n\,x)\,dx + k \int_{0}^{\pi} \cos(n\,x)\,dx \right)$$

$$= \frac{1}{\pi}\left(\frac{-k\sin(n\,x)}{n}\Big|_{x=-\pi}^{x=0} + \frac{k\sin(n\,x)}{n}\Big|_{x=0}^{x=\pi}\right) \qquad \text{note: } \sin(n\,\pi) = 0$$

$$a_n = 0 \qquad n = 1,\ 2,\ \ldots.$$

We can now find the coefficients b_n

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(n\,x)\,f(x)\,dx \qquad n = 1,\ 2,\ \ldots$$

using a similar procedure...

$$= \frac{1}{\pi}\left(\frac{k\cos(n\,x)}{n}\Big|_{x=-\pi}^{x=0} - \frac{k\cos(n\,x)}{n}\Big|_{x=0}^{x=\pi}\right)$$

$$= \frac{k}{n\,\pi}\left(\cos(n\,x)\Big|_{x=-\pi}^{x=0} - \cos(n\,x)\Big|_{x=0}^{x=\pi}\right)$$

$$= \frac{k}{n\,\pi}\left(\cos(0) - \cos(-n\,\pi) - (\cos(n\,\pi) - \cos(0))\right) \qquad \text{note: } \cos(0) = 1$$

$$= \frac{k}{n\,\pi}\left(1 - \cos(-n\,\pi) - \cos(n\,\pi) + 1\right)$$

$$= \frac{k}{n\,\pi}\left(2 - \cos(-n\,\pi) - \cos(n\,\pi)\right) \qquad \text{recall: } \cos(-\theta) = \cos(\theta)$$

$$= \frac{k}{n\,\pi}\left(2 - 2\cos(n\,\pi)\right)$$

$$= \frac{2\,k}{n\,\pi}\left(1 - \cos(n\,\pi)\right) \qquad \text{note: } \cos(n\,\pi) = \begin{cases} -1 & \text{for } n \text{ odd} \\ +1 & \text{for } n \text{ even} \end{cases}$$

So we see that $b_n = 0$ if n is even. Therefore, we investigate the more interesting case when n is odd

$$= \frac{2\,k}{n\,\pi}\left(1 - (-1)\right)$$

$$= \frac{4\,k}{n\,\pi} \qquad \text{(assuming } n \text{ is odd)}$$

So the general form we have for the b_n coefficients is

$$b_n = \begin{cases} \frac{4\,k}{n\,\pi} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

In[25]:=
```
a0 = 0;
a[n_] = 0;
b[n_] := If[
   OddQ[n],
   4 k/(n π) (*n is odd*),
   0(*n is even*)
  ];
```

So the various partial sums (Eq.8) are given as

In[28]:=
```
(*Note: upper limit of sum is M because N is a reserved Mathematica symbol*)
S[M_] := a0 + Sum[a[m] Cos[m x] + b[m] Sin[m x], {m, 1, M}]

(*Create a table*)
t1 = Table[i, {i, 1, 10}];
t2 = Table[S[N], {N, 1, 10}];
gridData = Transpose[{t1, t2}];
gridDataWithHeaders = Prepend[gridData, {"N", "S_N" }];
Grid[gridDataWithHeaders, Frame → All, Alignment → Left]
```

Out[33]=

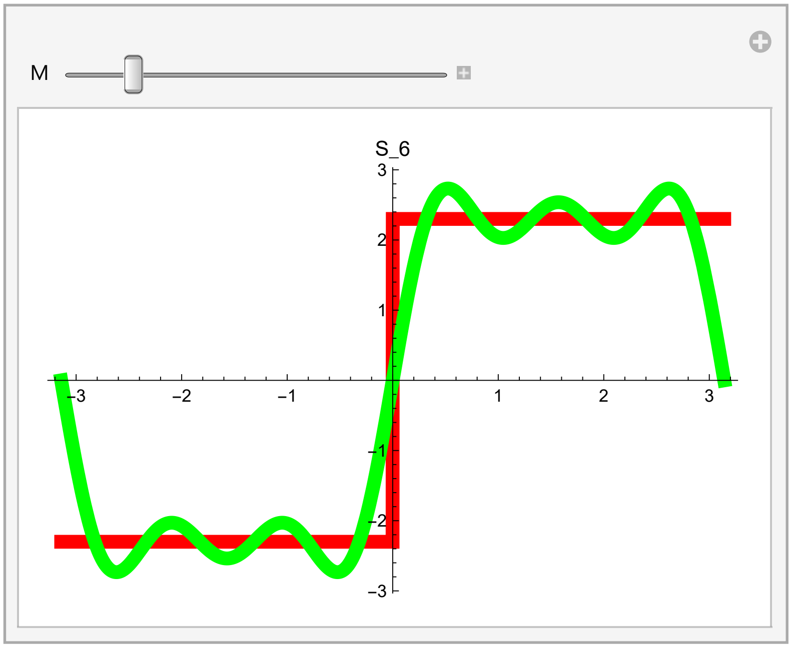
N	S_N
1	$\frac{4\,k\,\mathrm{Sin}[x]}{\pi}$
2	$\frac{4\,k\,\mathrm{Sin}[x]}{\pi}$
3	$\frac{4\,k\,\mathrm{Sin}[x]}{\pi}+\frac{4\,k\,\mathrm{Sin}[3\,x]}{3\,\pi}$
4	$\frac{4\,k\,\mathrm{Sin}[x]}{\pi}+\frac{4\,k\,\mathrm{Sin}[3\,x]}{3\,\pi}$
5	$\frac{4\,k\,\mathrm{Sin}[x]}{\pi}+\frac{4\,k\,\mathrm{Sin}[3\,x]}{3\,\pi}+\frac{4\,k\,\mathrm{Sin}[5\,x]}{5\,\pi}$
6	$\frac{4\,k\,\mathrm{Sin}[x]}{\pi}+\frac{4\,k\,\mathrm{Sin}[3\,x]}{3\,\pi}+\frac{4\,k\,\mathrm{Sin}[5\,x]}{5\,\pi}$
7	$\frac{4\,k\,\mathrm{Sin}[x]}{\pi}+\frac{4\,k\,\mathrm{Sin}[3\,x]}{3\,\pi}+\frac{4\,k\,\mathrm{Sin}[5\,x]}{5\,\pi}+\frac{4\,k\,\mathrm{Sin}[7\,x]}{7\,\pi}$
8	$\frac{4\,k\,\mathrm{Sin}[x]}{\pi}+\frac{4\,k\,\mathrm{Sin}[3\,x]}{3\,\pi}+\frac{4\,k\,\mathrm{Sin}[5\,x]}{5\,\pi}+\frac{4\,k\,\mathrm{Sin}[7\,x]}{7\,\pi}$
9	$\frac{4\,k\,\mathrm{Sin}[x]}{\pi}+\frac{4\,k\,\mathrm{Sin}[3\,x]}{3\,\pi}+\frac{4\,k\,\mathrm{Sin}[5\,x]}{5\,\pi}+\frac{4\,k\,\mathrm{Sin}[7\,x]}{7\,\pi}+\frac{4\,k\,\mathrm{Sin}[9\,x]}{9\,\pi}$
10	$\frac{4\,k\,\mathrm{Sin}[x]}{\pi}+\frac{4\,k\,\mathrm{Sin}[3\,x]}{3\,\pi}+\frac{4\,k\,\mathrm{Sin}[5\,x]}{5\,\pi}+\frac{4\,k\,\mathrm{Sin}[7\,x]}{7\,\pi}+\frac{4\,k\,\mathrm{Sin}[9\,x]}{9\,\pi}$

As expected, the consecutive S_N values are the same (ie $S_1 = S_2$, $S_3 = S_4$, etc.)

We can plot these partial summations to see how the S_N approximates $f(x)$

In[34]:=
```
Manipulate[
 Show[
  (*Plot original function f(x)*)
  Plot[f[x] /. {k → kGiven}, {x, -π, π}, PlotStyle → {Red, Thickness[0.02]}],
  Plot[S[M] /. {k → kGiven}, {x, -π, π}, PlotStyle → {Green, Thickness[0.02]}],
  PlotLabel → StringJoin["S_", ToString[M]], PlotRange → All
 ]
 , {M, 1, 35, 1}]
```

Out[34]=



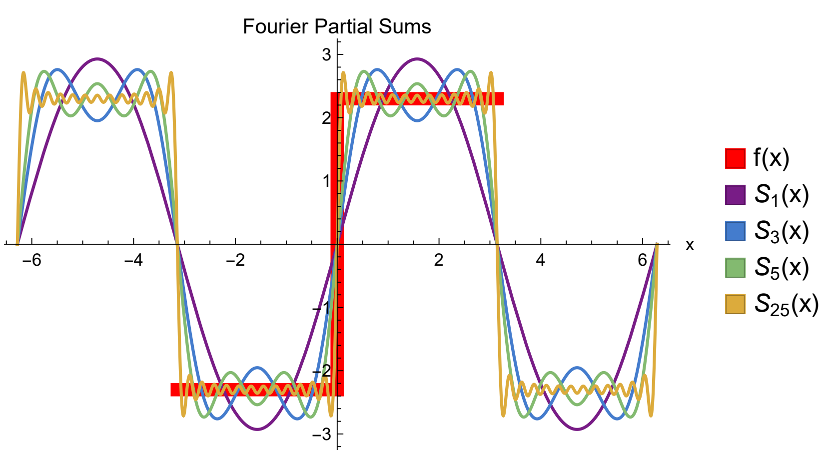
In[35]:=
```
xMax = 2 π;
Legended[
 Show[
  (*Plot original function f(x)*)
  Plot[f[x] /. {k → kGiven}, {x, -π, π}, PlotStyle → {Red, Thickness[0.02]}],

  (*Plot the partial sums*)
  Plot[S[1] /. {k → kGiven}, {x, -xMax, xMax}, PlotStyle → {ColorData["Rainbow"][0.0]}],
  Plot[S[3] /. {k → kGiven}, {x, -xMax, xMax}, PlotStyle → {ColorData["Rainbow"][0.25]}],
  Plot[S[5] /. {k → kGiven}, {x, -xMax, xMax}, PlotStyle → {ColorData["Rainbow"][0.50]}],
  Plot[S[25] /. {k → kGiven}, {x, -xMax, xMax}, PlotStyle → {ColorData["Rainbow"][0.75]}],

  (*Plot Options*)
  PlotLabel → "Fourier Partial Sums",
  AxesLabel → {"x"},
  PlotRange → All
 ],

 (*Add the legend information*)
 SwatchLegend[{Red,
   ColorData["Rainbow"][0.0],
   ColorData["Rainbow"][0.25],
   ColorData["Rainbow"][0.5],
   ColorData["Rainbow"][0.75]},
  {"f(x)", "S1(x)", "S3(x)", "S5(x)", "S25(x)"}]
]
```

Out[36]=



As can be seen, as N increases, the partial sum becomes a better and better approximation of the function

In[ ]:=
```
Clear[f, a0, a, b, S, t1, t2, gridData, gridDataWithHeaders, kGiven]
```

Modifications to the Fourier Series

Note that these techniques will be necessary for analyzing PDEs (ie the 1D wave equation, see YouTube video entitled 'Solving the 1D Wave Equation' at https://youtu.be/lMRnTd8yLeY) so it is recommended that you remember this section as we will return to use these results soon.

This section contains three topics:

1. Transition from period 2π to any period $2L$ or a period of P.

2. Simplifications of even and odd functions.

3. Expansion of f given for $x \in [0, L]$ using half range expansions.

Arbitrary Period

If the function has a period of $2L$ instead of 2π, we can simply re-derive the expressions for this new period. In this case, the Fourier series is given as

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L} x\right) + b_n \sin\left(\frac{n\pi}{L} x\right)\right) \qquad \textbf{(Eq.5)}$$

where $$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\, dx \qquad \textbf{(Eq.6.0)}$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi}{L} x\right) dx \qquad n = 1, 2, \ldots \qquad \textbf{(Eq.6.a)}$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L} x\right) dx \qquad n = 1, 2, \ldots \qquad \textbf{(Eq.6.b)}$$

Or alternatively if the function has a period of $P = 2L$ this becomes

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi}{L} x\right) + b_n \sin\left(\frac{n\pi}{L} x\right)\right)$$

where $$a_0 = \frac{1}{P} \int_{-P/2}^{P/2} f(x)\, dx$$

$$a_n = \frac{2}{P} \int_{-P/2}^{P/2} f(x) \cos\left(\frac{2n\pi}{P} x\right) dx \qquad n = 1, 2, \ldots$$

$$b_n = \frac{2}{P} \int_{-P/2}^{P/2} f(x) \sin\left(\frac{2n\pi}{P} x\right) dx \qquad n = 1, 2, \ldots$$

Simplifications: Even and Odd Functions

Recall that a function is **even** if

$$g(-x) = g(x) \qquad \forall x \qquad \text{(even function)}$$

Similarly, a function is **odd** if

$$g(-x) = -g(x) \qquad \forall\, x \qquad \text{(odd function)}$$

For example, $\cos(x)$ is even and $\sin(t)$ is odd

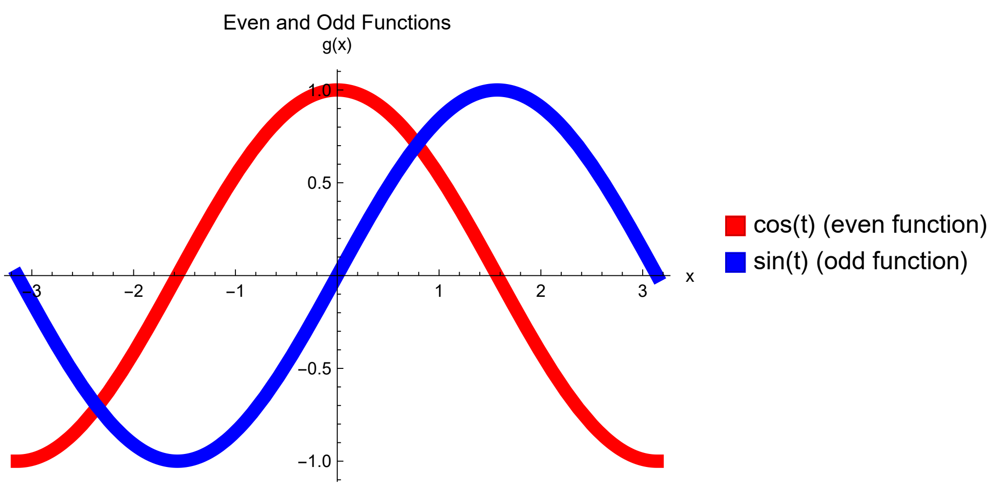
In[ ]:=
```
Legended[
 Show[
  (*Plot 1*)
  Plot[Cos[x], {x, -π, π}, PlotStyle → {Red, Thickness[0.02]}],

  (*Plot 2*)
  Plot[Sin[x], {x, -π, π}, PlotStyle → {Blue, Thickness[0.02]}],

  (*Plot Options*)
  PlotLabel → "Even and Odd Functions",
  AxesLabel → {"x", "g(x)"}
 ],

 (*Add the legend information*)
 SwatchLegend[{Red, Blue}, {"cos(t)  (even function)", "sin(t)  (odd function)"}]
]
```

Out[ ]=



Note that in general

$$\int_{-L}^{L} g(x)\,dx = \begin{cases} 2\int_0^L g(x)\,dx & \text{if } g(x) \text{ is even} \\ 0 & \text{if } g(x) \text{ is odd} \end{cases}$$

Theorem 1 - Fourier series of even and odd functions

With these simplifications, the Fourier series of an even function with period $2L$ is referred to as a **Fourier cosine series** and is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right) \qquad \text{(if } f \text{ is even)} \qquad \textbf{\textit{(Eq.5*)}}$$

where $a_0 = \frac{1}{L}\int_0^L f(x)\,dx$ ***(Eq.6*)***

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)dx \qquad n = 1, 2, \ldots$$

Similarly, the Fourier series of an odd function with period $2L$ is referred to as a **Fourier sine series** and is given by

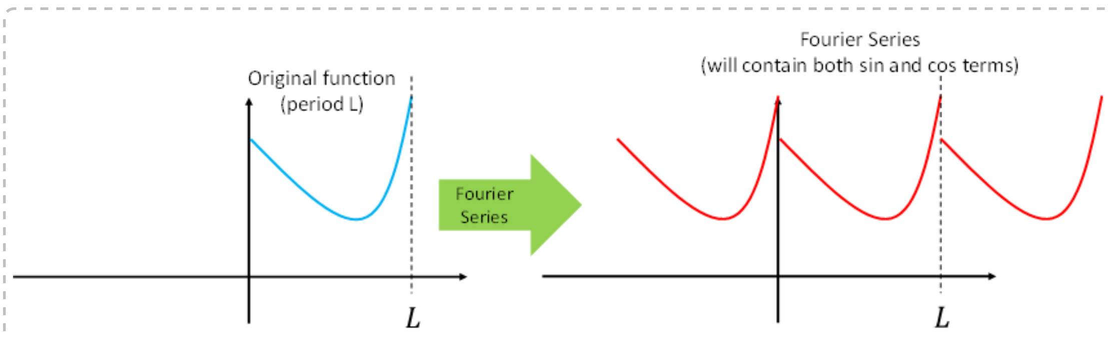
$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) \qquad \text{(if } f \text{ is odd)} \qquad \textbf{(Eq.5**)}$$

where $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right)dx$ ***(Eq.6**)***

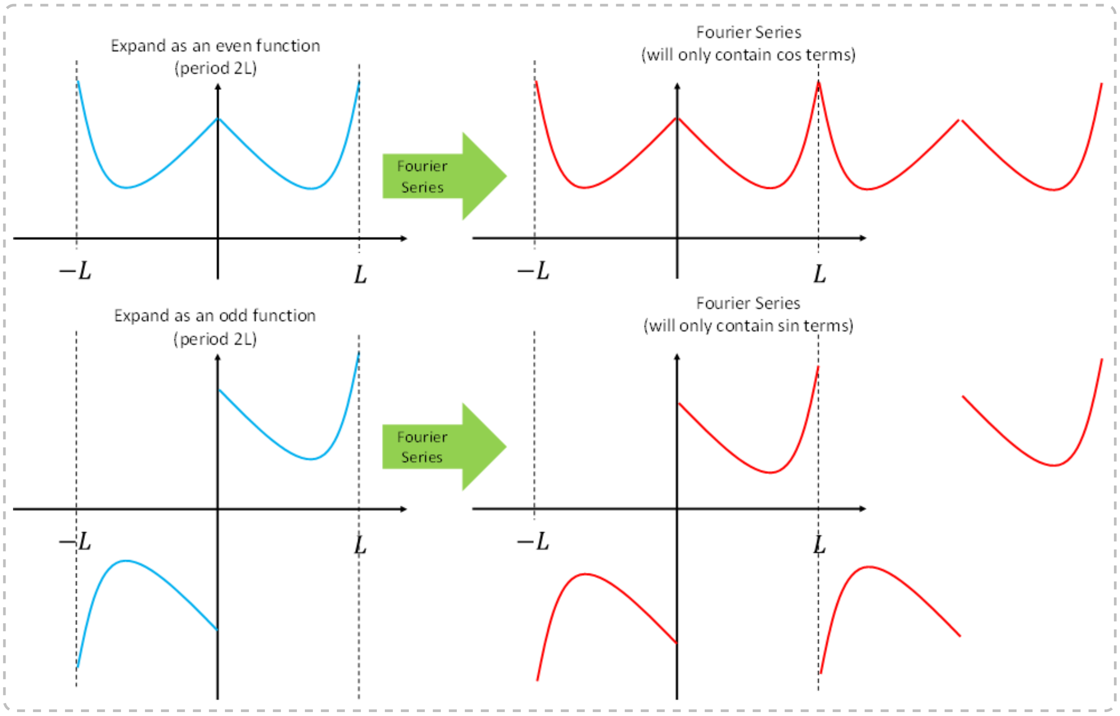
Note that in these cases, due to the symmetry, we only integrate from 0 to L even though the function is periodic with period $2L$

Half-Range Expansions

There are times when we have a function that is only defined over the period L ($x \in [0, L]$) (we will examine this later in PDEs for the deflection of a string, heat in a bar, etc.). We can immediately use the results from the discussion above to compute the Fourier Series for this function as shown below.



We note that in this case, it is very likely that the series will contain both cos and sin terms. We can simplify the Fourier Series by first expanding/extending the original function as an even or odd function to take advantage of the fact that even and odd functions only have cos and sin terms, respectively.



If the function is extended as an even function, the Fourier series can be written using only cos terms using Eq.5* as mentioned previously.

If the function is extended as an odd function, the Fourier series can be written using only sin terms using Eq.5** as mentioned previously.

In future lectures, we will see how these expansions are useful in solving partial differential equations.

Theorem 1 - Sum of Functions

The Fourier coefficients of a sum $f_1 + f_2$ are the sums of the corresponding Fourier coefficients of f_1 and f_2

The Fourier coefficients of $c\, f$ are c times the corresponding Fourier coefficients of f

Forced Oscillations

In previous undergraduate classes on system dynamics, you may have looked at the behavior of linear ordinary differential equation in response to sinusoidal inputs (ie bode plots). However, if the input is not a pure sinusoid, we can use Fourier analysis to look at the response. We will investigate this in a homework assignment.

Approximation by Trigonometric Polynomials

We see that the trigonometric series is an exact representation of a function

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\,x) + b_n \sin(n\,x))$$

The major disadvantage of this form is the fact that we require an infinite series.

Recall from previous sections, we investigated partial sums which involved cutting off the infinite series at a maximum integer N.

$$f(x) \approx S_N = a_0 + \sum_{n=1}^{N} (a_n \cos(n\,x) + b_n \sin(n\,x)) \qquad \textbf{\textit{(Eq.1)}}$$

where $\quad a_n, b_n$ = Fourier coefficients

The natural questions is,

1. How accurate an approximation is the partial sum to the infinite sum (and therefore the function f)?

A related question is

2. Are there other coefficients, α_n and β_n which might yield a better approximation than the Fourier coefficients?

Let us denote a finite approximation of f as

$$F(x) = \alpha_0 + \sum_{n=1}^{N} (\alpha_n \cos(n\,x) + \beta_n \sin(n\,x)) \qquad \textbf{\textit{(Eq.2)}}$$

where $\quad N$ = degree of the approximation

Note: the use of α and β instead of a and b

In[ ]:=
```
(*Use M as the upper limit instead of N because N is a reserved Mathematica symbol*)
F[x_, M_] := α0 + Sum[α[m] Cos[m x] + β[m] Sin[m x], {m, 1, M}]
```

One measure for the error of the fit on the interval $x \in [-\pi, \pi]$ is the **total square error**, given as

$$E = \int_{-\pi}^{\pi} (f(x) - F(x))^2\,d\,x \qquad \textbf{\textit{(Eq.3)}}$$

So the problem becomes: N being fixed, find the coefficients in Eq.2 such that E is minimum.

Let us expand the square in Eq.3

$$E = \int_{-\pi}^{\pi} f(x)^2\,d\,x - 2\int_{-\pi}^{\pi} f(x)\,F(x)\,d\,x + \int_{-\pi}^{\pi} F(x)^2\,d\,x$$

Let us look at each integral individually, starting with the last integral.

$$\int_{-\pi}^{\pi} F(x)^2\,d\,x = \int_{-\pi}^{\pi} \left(\alpha_0 + \sum_{n=1}^{N} (\alpha_n \cos(n\,x) + \beta_n \sin(n\,x))\right)^2 d\,x$$

Perhaps we can gain some insight if we consider the case of $N = 2$

In[∘]:=
```
F[x, 2]^2 // Expand
```

Out[∘]=
```
α0^2 + 2 α0 Cos[x] α[1] + Cos[x]^2 α[1]^2 + 2 α0 Cos[2 x] α[2] + 2 Cos[x] Cos[2 x] α[1] × α[2] +
 Cos[2 x]^2 α[2]^2 + 2 α0 Sin[x] β[1] + 2 Cos[x] Sin[x] α[1] × β[1] + 2 Cos[2 x] Sin[x] α[2] × β[1] +
 Sin[x]^2 β[1]^2 + 2 α0 Sin[2 x] β[2] + 2 Cos[x] Sin[2 x] α[1] × β[2] +
 2 Cos[2 x] Sin[2 x] α[2] × β[2] + 2 Sin[x] Sin[2 x] β[1] × β[2] + Sin[2 x]^2 β[2]^2
```

We see that all the $\int_{-\pi}^{\pi} \cos(n\,x)^2\,dx = \int_{-\pi}^{\pi} \sin(n\,x)^2\,dx = \pi$ because $n \in \mathbb{Z}$.

In[∘]:=
```
Simplify[Integrate[Cos[n x]^2, {x, -π, π}], Element[n, Integers]]
Simplify[Integrate[Sin[n x]^2, {x, -π, π}], Element[n, Integers]]
```

Out[∘]= π

Out[∘]= π

We also see that all mixed terms
$\int_{-\pi}^{\pi} \cos(n\,x)\sin(m\,x)\,dx = \int_{-\pi}^{\pi} \cos(n\,x)\cos(m\,x)\,dx = \int_{-\pi}^{\pi} \sin(n\,x)\sin(m\,x)\,dx = 0$

In[∘]:=
```
Integrate[Cos[n x] Sin[m x], {x, -π, π}]
Simplify[Integrate[Cos[n x] Cos[m x], {x, -π, π}],
 {Element[m, Integers], Element[n, Integers]}]
Simplify[Integrate[Sin[n x] Sin[m x], {x, -π, π}],
 {Element[m, Integers], Element[n, Integers]}]
```

Out[∘]= 0

Out[∘]= π

Out[∘]= π

So the resulting integral is

$$\int_{-\pi}^{\pi} F(x)^2\,dx = \pi\left(2\,\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \beta_1^2 + \beta_2^2\right) \qquad \text{(for } N = 2\text{)}$$

In[∘]:=
```
Integrate[F[x, 2]^2, {x, -π, π}]
Integrate[F[x, 3]^2, {x, -π, π}]
Integrate[F[x, 5]^2, {x, -π, π}]
```

Out[∘]= π (2 α0^2 + α[1]^2 + α[2]^2 + β[1]^2 + β[2]^2)

Out[∘]= π (2 α0^2 + α[1]^2 + α[2]^2 + α[3]^2 + β[1]^2 + β[2]^2 + β[3]^2)

Out[∘]= π (2 α0^2 + α[1]^2 + α[2]^2 + α[3]^2 + α[4]^2 + α[5]^2 + β[1]^2 + β[2]^2 + β[3]^2 + β[4]^2 + β[5]^2)

So the general pattern is

$$\int_{-\pi}^{\pi} F(x)^2\,dx = \pi\left(2\,\alpha_0^2 + \alpha_1^2 + \alpha_2^2 \ldots + \alpha_N^2 + \beta_1^2 + \beta_2^2 + \ldots + \beta_N^2\right)$$

In a similar fashion, let us now examine the integral

$$\int_{-\pi}^{\pi} f(x)\, F(x)\, dx = \int_{-\pi}^{\pi} \left(a_0 + \sum_{n=1}^{N} (a_n \cos(n\,x) + b_n \sin(n\,x))\right) \left(\alpha_0 + \sum_{n=1}^{N} (\alpha_n \cos(n\,x) + \beta_n \sin(n\,x))\right) dx$$

```
In[ ]:= (*just use 5 terms in the Fourier series, this will illustrate what we need to show*)
f = a0 + Sum[a[p] Cos[p x] + b[p] Sin[p x], {p, 1, 5}];
Integrate[f F[x, 2], {x, -π, π}]
Out[ ]= π (2 a0 α0 + a[1] × α[1] + a[2] × α[2] + b[1] × β[1] + b[2] × β[2])
```

So we see that the only terms that are non-zero are the ones with matching coefficients in $F(x)$. So we have

$$\int_{-\pi}^{\pi} f(x)\, F(x)\, dx = \pi(2\, a_0\, \alpha_0 + a_1\, \alpha_1 + a_2\, \alpha_2 + \ldots + a_N\, \alpha_N + b_1\, \beta_1 + b_2\, \beta_2 + \ldots + b_N\, \beta_N$$

So, using the choice of α_n and β_n as coefficients to our approximation, the error between the approximation and the actual function can be written as

$$E = \int_{-\pi}^{\pi} f(x)^2\, dx - 2 \int_{-\pi}^{\pi} f(x)\, F(x)\, dx + \int_{-\pi}^{\pi} F(x)^2\, dx$$

$$E = \int_{-\pi}^{\pi} f(x)^2\, dx - 2\,\pi \left[2\,\alpha_0\, a_0 + \sum_{n=1}^{N} (\alpha_n\, a_n + \beta_n\, b_n)\right] + \pi\left[2\,\alpha_0{}^2 + \sum_{n=1}^{N} \left(\alpha_n{}^2 + \beta_n{}^2\right)\right] \qquad \textbf{\textit{(Eq.5)}}$$

Recall that Eq.5 describes the error between $F(x)$ and $f(x)$ if we choose α_n, β_n as the coefficients of the trigonometric series. What if we choose to use coefficients of the Fourier series? In other words

$\alpha_n = a_n$
$\beta_n = b_n$

In this case, the error becomes

$$E^* = \int_{-\pi}^{\pi} f(x)^2\, dx - 2\,\pi \left[2\, a_0{}^2 + \sum_{n=1}^{N} \left(a_n{}^2 + b_n{}^2\right)\right] + \pi\left[2\, a_0{}^2 + \sum_{n=1}^{N} \left(a_n{}^2 + b_n{}^2\right)\right]$$

$$E^* = \int_{-\pi}^{\pi} f(x)^2\, dx - \pi \left[2\, a_0{}^2 + \sum_{n=1}^{N} \left(a_n{}^2 + b_n{}^2\right)\right] \quad \text{(error we would obtain if we use Fourier coefficients)}$$
(Eq.6)

Now computing the difference between the two error terms yields

$$E - E^* = A - 2\,\pi \left[2\,\alpha_0\, a_0 + \sum_{n=1}^{N} (\alpha_n\, a_n + \beta_n\, b_n)\right] + \pi\left[2\,\alpha_0{}^2 + \sum_{n=1}^{N} \left(\alpha_n{}^2 + \beta_n{}^2\right)\right] - \left\{A - \pi \left[2\, a_0{}^2 + \sum_{n=1}^{N} \left(a_n{}^2 + b_n{}^2\right)\right]\right\}$$

where $\quad A = \int_{-\pi}^{\pi} f(x)^2\, dx$

$$= A - 2\,\pi \left[2\,\alpha_0\, a_0 + \sum_{n=1}^{N} (\alpha_n\, a_n + \beta_n\, b_n)\right] + \pi\left[2\,\alpha_0{}^2 + \sum_{n=1}^{N} \left(\alpha_n{}^2 + \beta_n{}^2\right)\right] - A + \pi \left[2\, a_0{}^2 + \sum_{n=1}^{N} \left(a_n{}^2 + b_n{}^2\right)\right]$$

$$= -2\,\pi \left[2\,\alpha_0\, a_0 + \sum_{n=1}^{N} (\alpha_n\, a_n + \beta_n\, b_n)\right] + \pi\left[2\,\alpha_0{}^2 + \sum_{n=1}^{N} \left(\alpha_n{}^2 + \beta_n{}^2\right)\right] + \pi \left[2\, a_0{}^2 + \sum_{n=1}^{N} \left(a_n{}^2 + b_n{}^2\right)\right]$$

$$= \pi\left\{-4\,\alpha_0\,a_0 - 2\sum_{n=1}^{N}(\alpha_n\,a_n + \beta_n\,b_n) + 2\,\alpha_0^{\,2} + \sum_{n=1}^{N}\left(\alpha_n^{\,2} + \beta_n^{\,2}\right) + 2\,a_0^{\,2} + \sum_{n=1}^{N}\left(a_n^{\,2} + b_n^{\,2}\right)\right\}$$

$$= \pi\left\{-4\,\alpha_0\,a_0 + 2\,\alpha_0^{\,2} + 2\,a_0^{\,2} - \sum_{n=1}^{N}2\,(\alpha_n\,a_n + \beta_n\,b_n) + \sum_{n=1}^{N}\left(\alpha_n^{\,2} + \beta_n^{\,2}\right) + \sum_{n=1}^{N}\left(a_n^{\,2} + b_n^{\,2}\right)\right\}$$

$$= \pi\left\{-4\,\alpha_0\,a_0 + 2\,\alpha_0^{\,2} + 2\,a_0^{\,2} + \sum_{n=1}^{N}\left(-2\,\alpha_n\,a_n - 2\,\beta_n\,b_n + \alpha_n^{\,2} + \beta_n^{\,2} + a_n^{\,2} + b_n^{\,2}\right)\right\}$$

$$= \pi\left\{2\left(a_0^{\,2} - 2\,\alpha_0\,a_0 + \alpha_0^{\,2}\right) + \sum_{n=1}^{N}\left(a_n^{\,2} - 2\,\alpha_n\,a_n + \alpha_n^{\,2} + b_n^{\,2} - 2\,\beta_n\,b_n + \beta_n^{\,2}\right)\right\}$$ note:

$(\alpha_0 - a_0)^2 = a_0^{\,2} - 2\,\alpha_0\,a_0 + \alpha_0^{\,2}$

$(\alpha_n - a_n)^2 = a_n^{\,2} - 2\,\alpha_n\,a_n + a_n^{\,2}$

$(\beta_n - b_n)^2 = b_n^{\,2} - 2\,\beta_n\,b_n + b_n^{\,2}$

$$E - E^* = \pi\left\{2\,(\alpha_0 - a_0)^2 + \sum_{n=1}^{N}\left((\alpha_n - a_n)^2 + (\beta_n - b_n)^2\right)\right\}$$

We see that the entire right side of this expression is the sum of squares, so it must be positive. Therefore

$$E - E^* \geq 0 \iff E \geq E^*$$

Recall that E was error if we chose any arbitrary constants α_n, β_n as coefficients of the trigonometric series and E^* was the error if we chose the Fourier coefficients. This shows that $E = E^*$ if and only if $\alpha_n = a_n$ and $\beta_n = b_n$. This leads to the following theorem

Theorem 1: Minimum square error

If we define the N^{th} partial sum approximation of $f(x)$ with period $2\,\pi$ as

$$F(x) = \alpha_0 + \sum_{n=1}^{N}(\alpha_n\cos(n\,x) + \beta_n\sin(n\,x)) \qquad \textbf{\textit{(Eq.2)}}$$

where N = degree of the approximation

And denote the total square error as

$$E = \int_{-\pi}^{\pi}(f(x) - F(x))^2\,dx$$

Then E is minimized on the domain $x \in [-\pi, \pi]$ if and only if the coefficients of F are the Fourier coefficients of f.

```
In[•]:= Clear[f, F]
```

Fourier Integral (OPTIONAL)

From Fourier Series to Fourier Integral

Some of the previous sections dealt with Fourier series and involved treatment of periodic functions. We now look at treatment of non-periodic functions. Recall that any periodic function $f_L(x)$ with period

$2L$ can be represented by a Fourier series

$$f_L(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(\omega_n x) + b_n \sin(\omega_n x)) \qquad \textbf{(Eq.1)}$$

where $\quad \omega_n = n\pi/L$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(v)\,dv$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(v)\cos(\omega_n v)\,dv \qquad n = 1, 2, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(v)\sin(\omega_n v)\,dv \qquad n = 1, 2, \ldots$$

We now ask, what if $L \to \infty$

With some analysis as outlined in the text and provided some conditions on $f_{L\to\infty} = f$ hold, we can write this as (note: the variable of integration is ω, not x)

$$f_{L\to\infty}(x) = f(x) = \int_0^{\infty} [A(\omega)\cos(\omega x) + B(\omega)\sin(\omega x)]\,d\omega \qquad \textbf{(Eq.5)}$$

where $\quad A(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos(\omega v)\,dv$

$$B(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin(\omega v)\,dv$$

This representation of f is known as the **Fourier integral.**

A concept that will be useful in this derivation (and also useful later) is the idea of **absolutely integrable** on the x-axis; that is, the following limits exist (and are finite)

$$\lim_{a\to-\infty}\int_a^0 \left| f(x) \right| dx + \lim_{b\to\infty}\int_0^b \left| f(x) \right| dx \qquad \text{(written as } \int_{-\infty}^{\infty} | f(x) |\,dx\text{)}$$

Example: Single pulse, sine integral

Consider the function

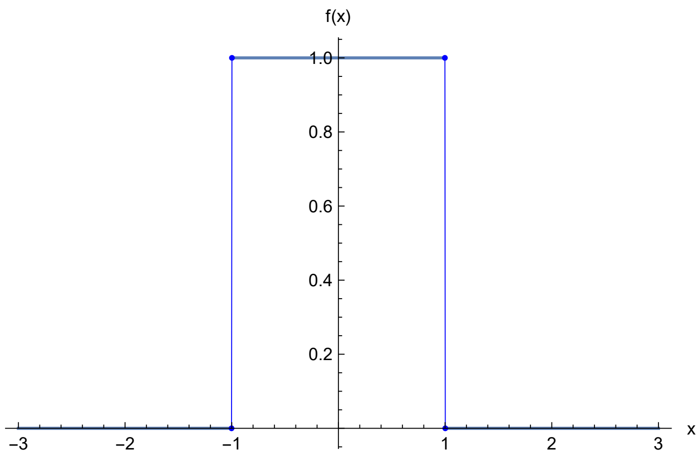
$$f(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$$

In[ ]:=
```
f[x_] := If[
   Abs[x] < 1,
   1,
   0
  ]

Plot[f[x], {x, -3, 3},
 AxesLabel → {"x", "f(x)"},

 (*Connect locations where there are discontinuities/jumps*)
 ExclusionsStyle → {Blue, Blue}]
```

Out[ ]=



Let us now compute the Fourier integral of this function. We can first compute $A(\omega)$ and $B(\omega)$

$$A(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos(\omega v)\, dv$$

$$= \frac{1}{\pi} \int_{-1}^{1} \cos(\omega v)\, dv$$

$$= \frac{1}{\pi \omega} \sin(\omega v) \;\Big|_{-1}^{+1}$$

$$A(\omega) = \frac{2 \sin(\omega)}{\pi \omega}$$

Similarly,

$$B(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \sin(\omega v)\, dv$$

$$= \frac{1}{\pi} \int_{-1}^{1} \sin(\omega v)\, dv$$

$$= \frac{-1}{\pi \omega} \cos(\omega v) \;\Big|_{-1}^{+1}$$

$$B(\omega) = 0$$

In[ ]:=
```
A[ω_] = Integrate[1/π Cos[ω v], {v, -1, 1}]
B[ω_] = Integrate[1/π Sin[ω v], {v, -1, 1}]
```

Out[ ]= $\frac{2\,\mathrm{Sin}[\omega]}{\pi\,\omega}$

Out[ ]= 0

So the Fourier integral of f is given by Eq.5

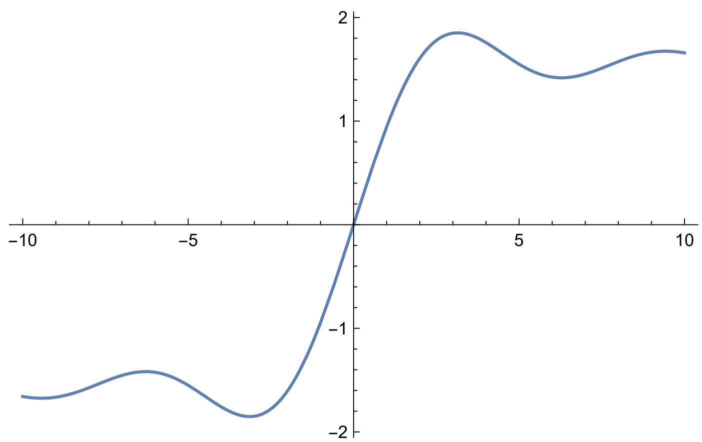
$$f(x) = \int_0^\infty [A(\omega)\cos(\omega\, x) + B(\omega)\sin(\omega\, x)]\, d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin(\omega)\cos(\omega\, x)}{\omega}\, d\omega$$

At this point, we cannot evaluate the integral although we can note that the sin-integral has a similar form

$$\mathrm{SinIntegral}(z) = \mathrm{Si}(z) = \int_0^z \frac{\sin(t)}{t}\, dt$$

In[ ]:=
```
Plot[SinIntegral[x], {x, -10, 10}]
```

Out[ ]=


In the case of a Fourier series, we looked at partial sums by approximating the sum $\sum_{n=1}^{\infty}$ with $\sum_{n=1}^{N}$. In a similar fashion, with the Fourier integral, we approximate the integral in Eq.5 $\int_0^\infty$ with $\int_0^a$ where a is a finite number. Therefore, an approximation of f is

$$f_a(x) = \frac{2}{\pi}\int_0^a \frac{\sin(\omega)\cos(\omega\, x)}{\omega}\, d\omega$$

Using Mathematica to evaluate this integral yields

In[ ]:=
```
fa[x_] = 2/π Integrate[Sin[ω] Cos[ω x]/ω, {ω, 0, a}]
```

Out[ ]=
$$\frac{\text{SinIntegral}[a\,(1+x)] + \text{SinIntegral}[a - a\,x]}{\pi}$$

We can now look at various approximations of the function

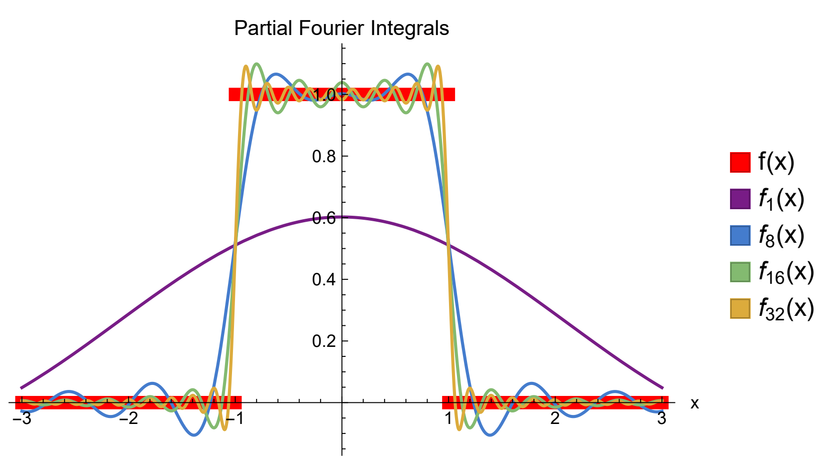
In[ ]:=
```
Legended[
 Show[
  (*Plot original function f(x)*)
  Plot[f[x], {x, -3, 3}, PlotStyle → {Red, Thickness[0.02]}],

  (*Plot the partial Fourier integrals*)
  Plot[fa[x] /. {a → 1}, {x, -3, 3}, PlotStyle → {ColorData["Rainbow"][0.0]}],
  Plot[fa[x] /. {a → 8}, {x, -3, 3}, PlotStyle → {ColorData["Rainbow"][0.25]}],
  Plot[fa[x] /. {a → 16}, {x, -3, 3}, PlotStyle → {ColorData["Rainbow"][0.50]}],
  Plot[fa[x] /. {a → 32}, {x, -3, 3}, PlotStyle → {ColorData["Rainbow"][0.75]}],

  (*Plot Options*)
  PlotLabel → "Partial Fourier Integrals",
  AxesLabel → {"x"},
  PlotRange → All
 ],

 (*Add the legend information*)
 SwatchLegend[{
   Red,
   ColorData["Rainbow"][0.0],
   ColorData["Rainbow"][0.25],
   ColorData["Rainbow"][0.50],
   ColorData["Rainbow"][0.75]},
  {"f(x)",
   "f1(x)",
   "f8(x)",
   "f16(x)",
   "f32(x)"}]
]
```

Out[ ]=



As expected, as *a* increases, the approximation becomes better and better. Also note the Gibb's Phenomenon at the discontinuities.

Fourier Cosine Integral and Fourier Sine Integral

For even or odd functions, the Fourier integral can be simplified.

If f is even, this reduces to the so called **Fourier cosine integral**.

$$f(x) = \int_0^\infty A(\omega)\cos(\omega x)\,d\omega \qquad \text{(if } f \text{ is even)} \qquad \textbf{\textit{(Eq.11)}}$$

where $\quad A(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\cos(\omega v)\,dv$

If f is odd this reduces to the so called **Fourier sine integral**.

$$f(x) = \int_0^\infty B(\omega)\sin(\omega x)\,d\omega \qquad \text{(if } f \text{ is odd)} \qquad \textbf{\textit{(Eq.13)}}$$

where $\quad B(\omega) = \frac{2}{\pi}\int_0^\infty f(v)\sin(\omega v)\,dv$

Evaluation of Integrals

In a roundabout way, the Fourier integral can be used to help evaluate integrals.

For example, consider the function

$$f(x) = e^{-k x} \qquad (x > 0,\ k > 0)$$

If we compute the Fourier cosine integral of this, we obtain (derivation in text)

$$e^{-k x} = \frac{2k}{\pi}\int_0^\infty \frac{\cos(\omega x)}{k^2+\omega^2}\,d\omega \qquad (x > 0,\ k > 0)$$

Therefore, if you ever need to evaluate the integral of $\int_0^\infty \frac{\cos(\omega x)}{k^2+\omega^2}\,d\omega$, we see that it is given by

$$\int_0^\infty \frac{\cos(\omega x)}{k^2+\omega^2}\,d\omega = \frac{\pi}{2k}e^{-k x} \qquad (x > 0,\ k > 0)$$

In a similar fashion, if we use the Fourier sine integral of this, we can show that

$$\int_0^\infty \frac{\omega\sin(\omega x)}{k^2+\omega^2}\,d\omega = \frac{\pi}{2}e^{-k x} \qquad (x > 0,\ k > 0)$$