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Lecture 05c
Equations of Motion for a Planar Vehicle

Lecture is on YouTube

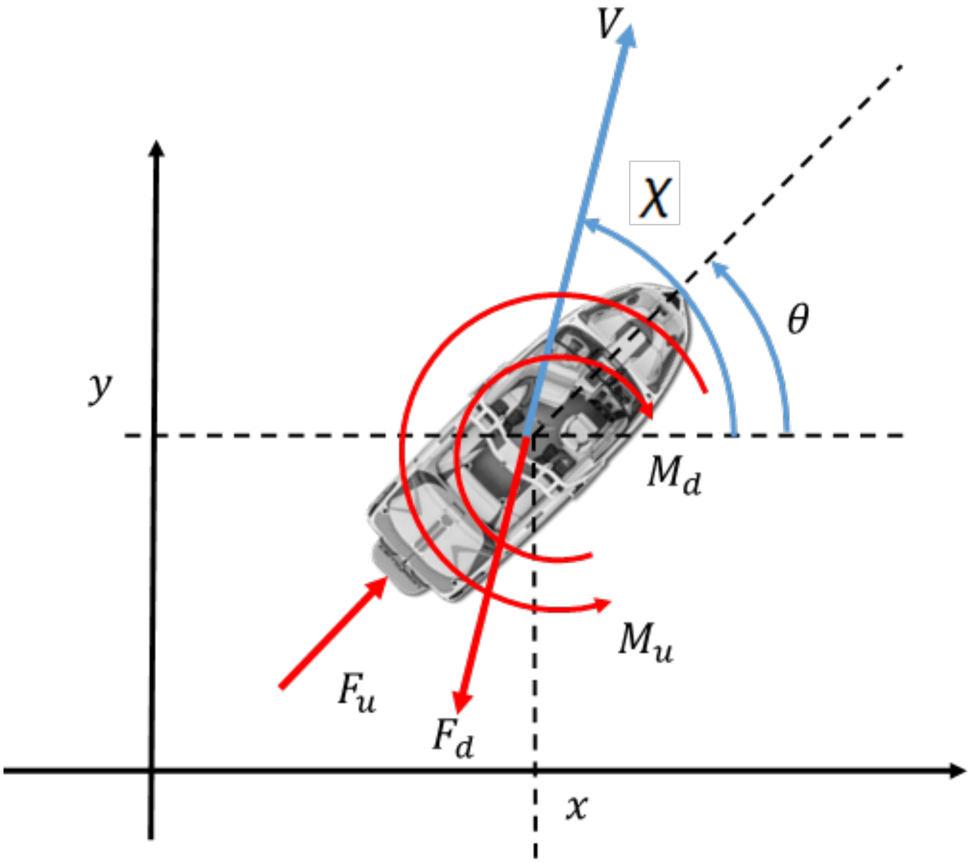
The YouTube video entitled ‘Equations of Motion for a Planar Vehicle’ that covers this lecture is located at https://youtu.be/kbGal6xKLB4.

Outline

-Planar Vehicle Model

Planar Vehicle Model

This represents a simple, planar vehicle as shown below. This is suitable for modeling a vehicle that can spin in place such as a boat with rotational thrusters or a multi-rotor. Note that there is viscous damping that retards both the translational and rotational motion of the vehicle. In effect, this is basically like the vehicle in the video game ‘Asteroids’.



The state vector for this vehicle is given as

$$\bar{x} = \begin{pmatrix} x \\ y \\ \dot{x} \\ \dot{y} \\ \theta \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} x \text{ position} \\ y \text{ position} \\ x \text{ velocity} \\ y \text{ velocity} \\ \text{angular position} \\ \text{angular velocity} \end{pmatrix} \qquad \textbf{(Eq.1)}$$

We see that we can obtain some secondary parameters as

$$V = \left(\dot{x}^2 + \dot{y}^2\right)^{1/2} \qquad \text{(velocity of CG)} \qquad \textbf{(Eq.2)}$$

$$\chi = \text{atan2}(\dot{y}, \dot{x}) \qquad \text{(course angle of CG)} \qquad \textbf{(Eq.3)}$$

Note that we assume atan2 has input arguments in the order of (vertical, horizontal). For more information on atan2 see ‘The 4 Quadrant Inverse Tangent (atan2) and Other Inverse Trigonometric Functions’ (https://youtu.be/UWrkh_N1bfE)

The control vector for this vehicle is given as

$$\bar{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \text{fraction of axial thrust force} \\ \text{fraction of moment about CG} \end{pmatrix} \qquad \textbf{(Eq.4)}$$

where $u_i \in [-1, 1]$ (-1 = full negative force/moment +1 = full positive force/moment)

In this case, we relate the translational force and rotational moment to the control input as

$$F_u = F_{\max} u_1 \qquad \textbf{(Eq.5)}$$
$$M_u = M_{\max} u_2 \qquad \textbf{(Eq.6)}$$

where $F_{\max}$, $M_{\max}$ = maximum force/moment that can be applied

Writing Newton's 2nd law for translation yields

$$\overline{F} = m\,\overline{a}$$

$$\begin{pmatrix} F_x \\ F_y \end{pmatrix} = m \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$$

$$\begin{pmatrix} F_u \cos(\theta) - F_d \cos(\chi) \\ F_u \sin(\theta) - F_d \sin(\chi) \end{pmatrix} = m \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix}$$

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \frac{1}{m} \begin{pmatrix} F_u \cos(\theta) - F_d \cos(\chi) \\ F_u \sin(\theta) - F_d \sin(\chi) \end{pmatrix} \qquad \textbf{(Eq.7)}$$

Similarly, Newton's 2nd law for rotation yields

$$M = I_b\,\ddot{\theta}$$

$$M_u - M_d = I_b\,\ddot{\theta}$$

$$\ddot{\theta} = \frac{1}{I_b}(M_u - M_d) \qquad \textbf{(Eq.8)}$$

Modeling Drag

The drag force can be modeling using non-linear drag as follows

$$F_d = \frac{1}{2}\rho V^2 C_D S = c_T V^2 \qquad \textbf{(Eq.9)}$$

where $c_T = \frac{1}{2}\rho C_D S$ (units of $N\frac{s^2}{m^2}$)

The drag moment can be modeled using linear drag as follows

$$M_d = c_R\,\dot{\theta} = c_R\,x_6 \qquad \textbf{(Eq.10)}$$

where $\quad c_R$ = coefficient of viscous rotational damping (units of $N\,m\,s$)

We can now write a state space representation of the system as

$$\dot{\underline{x}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \ddot{x} \\ \ddot{y} \\ \dot{\theta} \\ \ddot{\theta} \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ \frac{1}{m}\left(F_u \cos(x_5) - F_d \cos(\chi)\right) \\ \frac{1}{m}\left(F_u \sin(x_5) - F_d \sin(\chi)\right) \\ x_6 \\ \frac{1}{I_b}\left(M_u - M_d\right) \end{pmatrix} \qquad \textbf{(Eq.9)}$$

where $\quad V = \left(x_3{}^2 + x_4{}^2\right)^{1/2}$

$\chi = \text{atan2}(x_4, x_3)$

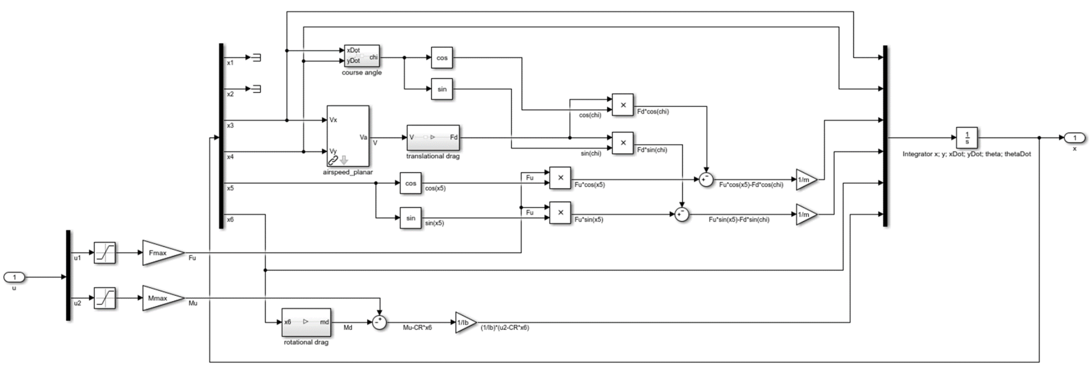
$F_d = c_T V^2$

$M_d = c_R x_6$

$F_u = F_{\max} u_1$

$M_u = M_{\max} u_2$

We can implement this as a Simulink model as shown below.



Things to note:

-We saturate the inputs to be in the range of [-1,1].

-We compute intermediate signals such as V and χ

-The state $x_5 = \theta$ is not restricted to the range of $[0,2\pi)$. This may or may not cause issues (we will explore this later)

Example: Boat

Consider an example of a boat with rotational thrusters. The appropriate parameters are as follows:

Geometry

$m = 10$ kg

$I_b = 5$ kg m^2

Propulsion

$F_{\max} = 10\,N$

$M_{\max} = 0.5\,N\,m$

Environment

$C_T = 0.025\,N\,s^2 / m^2$

$C_R = 0.75\,N\,m\,s$

With an initial condition of

$$\overline{x}(0) = \begin{pmatrix} 200 \\ 200 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

We can subject the system to various types of inputs to test the behavior of the system (go to Simulink model for demonstrations)

Future Usages of the Planar Vehicle

We will use the planar vehicle model in future videos
-Trimming a Simulink Model Using the Linear Analysis Tool (https://youtu.be/kypswO4RLkk)
-Linearizing a Simulink Model Using the Linear Analysis Tool and 'linmod' (https://-youtu.be/M6FQfLmir0I)

We have other videos/lectures that discuss building more complicated models
-A Nonlinear, 6 DOF Dynamic Model of an Aircraft: the Research Civil Aircraft Model (RCAM) (https://-youtu.be/bFFAL9lI2IQ)
-Building a Matlab/Simulink Model of an Aircraft: the Research Civil Aircraft Model (RCAM) (https://-youtu.be/YzZI1V2mJw8)