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Lecture 08a
Laplace's Equation

Lecture is on YouTube

The YouTube video entitled 'Derivation and Solution of Laplace's Equation' that covers this lecture is located at https://youtu.be/GCESkCyZt4g.

Steady Two-Dimensional Heat Problems. Laplace's Equation

We now extend our discussion from one dimension to two dimensions and consider the 2D heat equation

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

where $c^2 = \frac{K}{\sigma \rho}$ = thermal diffusivity

u = temperature (K)

We first consider steady (aka time-independent) problems. In this situation $\partial u / \partial t = 0$ and the heat equation reduces to Laplace's equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad \textit{\textbf{(Eq.14)}}$$

Solving Laplace's equation involves solving this PDE in some region R of the xy plane and a given boundary condition on the boundary curve C of R. This is a **boundary value problem (BVP)** which can be further classified as follows.

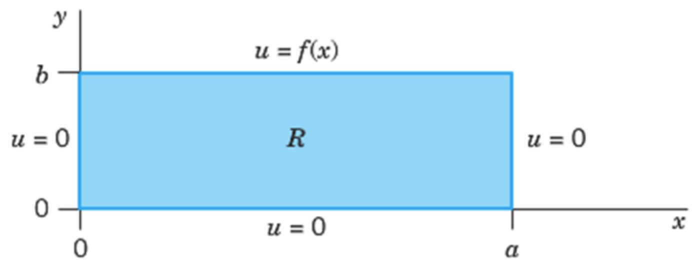
First BVP or **Dirichlet Problem** if u is prescribed on C ("**Dirichlet boundary condition**")

Second BVP or **Neumann Problem** if the normal derivative $u_n = \partial u / \partial n$ is prescribed on C ("**Neumann boundary condition**")

Third BVP, **Mixed BVP**, or **Robin Problem** if u is prescribed on a portion of C and u_n on the rest of C ("**Mixed boundary condition**")

Example: Dirichlet Problem on a Rectangle

For the remainder of this lecture, we consider a Dirichlet problem for Laplace's Equation (Eq.14) in a rectangle R, assuming that the temperature $u(x, y)$ equals a given function $f(x)$ on the upper side and 0 on the other three sides of the rectangle.



The procedure to solve this particular type of PDE with these specified boundary conditions (a type of Dirichlet problem) is as follows

Step 1. Apply method of separating variables (aka product method) to obtain two ordinary differential equations.

Step 2. Determine solutions of those two equations that satisfy the boundary conditions at the top.

Step 1. Two ODEs from the PDE

We solve this problem by separating variables. Substituting $u(x, y) = F(x)\, G(y)$ into Eq.14 yields

$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

$$= \frac{d^2 F(x)}{d x^2} G(y) + F(x) \frac{d^2 G(y)}{d y^2}$$

We note that $G(y) \neq 0$ and $F(x) \neq 0$ (or else we would obtain a trivial solution or $u(x, y) = 0$). So dividing by the quantity $F(x)\, G(y)$ yields (note that we can drop the ∂ notation in favor of the 'd' notation because these are only functions of one independent variable).

$$= \frac{d^2 F(x)}{d x^2} \frac{1}{F(x)} + \frac{1}{G(y)} \frac{d^2 G(y)}{d y^2}$$

$$\frac{d^2 F(x)}{d x^2} \frac{1}{F(x)} = -\frac{1}{G(y)} \frac{d^2 G(y)}{d y^2}$$

Once again, we see that we have isolated each side as functions of x and y, respectively and we can equate each side to a separation constant, $-k$, and obtain two 2nd order differential equations

$$\frac{d^2 F(x)}{d x^2} \frac{1}{F(x)} = -k \qquad \Rightarrow \qquad \frac{d^2 F(x)}{d x^2} + k\, F(x) = 0 \qquad \textbf{(Eq.A)}$$

$$-\frac{1}{G(y)} \frac{d^2 G(y)}{d y^2} = -k \qquad \Rightarrow \qquad \frac{d^2 G(y)}{d y^2} - k\, G(y) = 0 \qquad \textbf{(Eq.B)}$$

Equation 1: F(x)

We can first solve Eq. A, $\frac{d^2 F(x)}{d x^2} + k\, F(x) = 0$, which we recognize as a 2nd order, undamped system with general solution of

$$F(x) = A\cos\left(\sqrt{k}\; x\right) + B\sin\left(\sqrt{k}\; x\right)$$

```
F[x_] = A Cos[√k x] + B Sin[√k x];
```

```
(*Verify general solution satisfies ODE*)
D[F[x], {x, 2}] + k F[x] == 0 // Simplify
```

```
True
```

Note that the left boundary condition on our region requires that $F(0) = 0$

```
F[0]
```

```
A
```

So we see that $A = 0$ in order to satisfy this boundary condition

The right boundary condition on our region requires that $F(a) = 0$.

```
F[a] /. {A → 0}
```

```
B Sin[a √k]
```

$B = 0$ is a valid solution but it yields a trivial solution of $F(x) = 0$. So we see that in order to obtain a non-trivial solution we require that

$$a\sqrt{k} = n\,\pi \qquad n \in Z$$

$$k = \left(\frac{n\,\pi}{a}\right)^2 \qquad n \in Z \qquad \textbf{(Eq.C)}$$

So the corresponding non-trivial solution is given by

$$F(x) = F_n(x) = B\sin\left(\frac{n\,\pi}{a}\, x\right) \qquad n \in Z$$

```
Fn[x_] = B Sin[(n π)/a x];

(*Verify that this satisfies equation and boundary
 conditions as long as k satisfies Eq.C and n is an intger value*)
((D[Fn[x], {x, 2}] + k Fn[x]) /. {k → ((n π)/a)^2}) == 0 // Simplify
Fn[0] == 0
Simplify[Fn[a] == 0, Element[n, Integers]]
```

```
True

True

True
```

Equation 2:G(y)

Turning our attention to Eq.2 (with the understanding that $k = (n\pi/a)^2$ and $n = 1, 2, 3, \ldots$) we have

$$\frac{d^2 G(y)}{d y^2} - \left(\frac{n\pi}{a}\right)^2 G(y) = 0$$

This is slightly different than the previous cases we investigated (the 1D wave equation and the 1D heat equation had ODEs of the form $F''(x) + p^2 F(x) = 0$ which were undamped oscillators).

We recognize the current ODE as a 2nd order, unstable system (1 positive and 1 negative real root). Therefore, the general solution is

$$G(y) = G_n(y) = A_n e^{\frac{n\pi}{a} y} + B_n e^{-\frac{n\pi}{a} y}$$

```
Gn[y_] = An Exp[(n π)/a y] + Bn Exp[-(n π)/a y];
D[Gn[y], {y, 2}] - ((n π)/a)^2 Gn[y] == 0 // Simplify
```

```
True
```

Now the boundary condition on the bottom of the region states that $u(x, 0) = 0$. This implies that

$$G(0) = G_n(0) = 0$$

$$A_n + B_n = 0$$

$$A_n = -B_n$$

Substituting this back into the general solution for $G_n(y)$ yields

$$G(y) = G_n(y) = A_n\left(e^{\frac{n\pi}{a} y} - e^{-\frac{n\pi}{a} y}\right) \qquad \text{recall: } e^{a y} - e^{-a y} = 2\sinh(a y)$$

$$= 2\, A_n \sinh\left(\frac{n\,\pi}{a}\, y\right) \qquad \text{let } A_n{}^{*} = 2\, A_n$$

$$G(y) = G_n(y) = A_n{}^{*} \sinh\left(\frac{n\,\pi}{a}\, y\right)$$

```
temp = FullSimplify[Gn[y] /. {Bn → -An}] /. {2 An → AnStar};
Clear[Gn]
Gn[y_] = temp
```

```
AnStar Sinh[n π y / a]
```

We can verify that this satisfies the ODE and the boundary condition on the bottom

```
Print["Satisfies ODE"]
D[Gn[y], {y, 2}] - (n π / a)^2 Gn[y] == 0 // Simplify

Print["Satisfies Boundary Condition on Bottom"]
Gn[0] == 0
```

```
Satisfies ODE
True
Satisfies Boundary Condition on Bottom
True
```

So the eigenfunctions of the system are

$$u_n(x,\, y) = F_n(x)\, G_n(y)$$

$$= \left(B \sin\left(\frac{n\,\pi}{a}\, x\right)\right) \left(A_n{}^{*} \sinh\left(\frac{n\,\pi}{a}\, y\right)\right)$$

We note that assuming $B = 1$ is valid because this constant could be rolled into the $A_n{}^{*}$ constant

$$u_n(x,\, y) = A_n{}^{*} \sin\left(\frac{n\,\pi}{a}\, x\right) \sinh\left(\frac{n\,\pi}{a}\, y\right) \qquad \textbf{\textit{(Eq.16)}}$$

```
(*Define eigenfunctions*)
Print["Eigenfunctions"]
un[x_, y_] = Fn[x] × Gn[y] /. {B → 1}
Print[" "]

(*Verify this satisfies original PDE*)
Print["Satisfies original PDE"]
D[un[x, y], {x, 2}] + D[un[x, y], {y, 2}] == 0 // Simplify
Print[" "]

(*Verify that this satisfies BC on bottom*)
Print["Satisfies boundary condition on bottom"]
un[x, 0] == 0 // Simplify
Print[" "]

(*Verify that this satisfies BC on left side*)
Print["Satisfies boundary condition on left side"]
un[0, y] == 0 // Simplify
Print[" "]

(*Verify that this satisfies BC on right side*)
Print["Satisfies boundary condition on right side"]
Simplify[un[a, y] == 0, Element[n, Integers]]
Print[" "]
```

Eigenfunctions

$\text{AnStar} \, \text{Sin}\left[\frac{n \pi x}{a}\right] \text{Sinh}\left[\frac{n \pi y}{a}\right]$

Satisfies original PDE

True

Satisfies boundary condition on bottom

True

Satisfies boundary condition on right side

True

Satisfies boundary condition on left side

True

Step 2. Satisfying the Boundary Conditions

Boundary Condition on Top: Fourier Series

To satisfy the boundary condition at the top of the region, we consider the infinite series

$$u(x,\ y) = \sum_{n=1}^{\infty} u_n(x,\ y)$$

$$= \sum_{n=1}^{\infty} A_n{}^* \sin\left(\frac{n\pi}{a}\ x\right) \sinh\left(\frac{n\pi}{a}\ y\right)$$

At $y = b$, we would like this to equal $f(x)$

$$f(x) = \sum_{n=1}^{\infty} A_n{}^* \sin\left(\frac{n\pi}{a}\ x\right) \sinh\left(\frac{n\pi}{a}\ b\right)$$

$$= \sum_{n=1}^{\infty} \left[A_n{}^* \sinh\left(\frac{n\pi}{a}\ b\right)\right] \sin\left(\frac{n\pi}{a}\ x\right)$$

Comparing this with Eq.5** in section 11.2, we see that the term in brackets must be the coefficients, b_n, of the Fourier odd expansion of $f(x)$. So we have

$$b_n = A_n{}^* \sinh\left(\frac{n\pi}{a}\ b\right) = \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi}{a}\ x\right) dx$$

Solving for $A_n{}^*$, and substituting this back into our general solution yields the general solution to the entire problem

$$u(x,\ y) = \sum_{n=1}^{\infty} A_n{}^* \sin\left(\frac{n\pi}{a}\ x\right) \sinh\left(\frac{n\pi}{a}\ y\right) \qquad \textbf{\textit{(Eq.17)}}$$

where $$A_n{}^* = \frac{2}{a \sinh\left(\frac{n\pi}{a}\ b\right)} \int_0^a f(x) \sin\left(\frac{n\pi}{a}\ x\right) dx \qquad \textbf{\textit{(Eq.18)}}$$

Note that this solution is valid for our scenario of bottom and sides of the region constrained to 0 and the top of the region constrained to $f(x)$.

We investigate an example of this in the homework.

Heat Equation: Modeling Very Long Bars. Solution by Fourier Integrals and Transforms

Come back to this if we have time.