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Lecture 01d
Expressing Vectors in Different Frames Using Rotation Matrices

Lecture is on YouTube

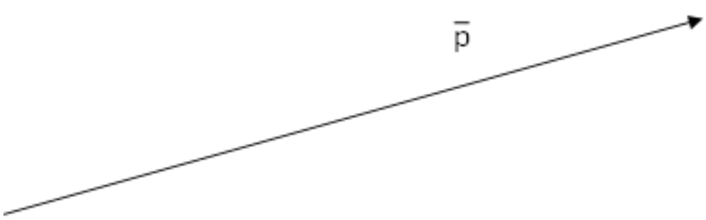
The YouTube video entitled 'Expressing Vectors in Different Frames Using Rotation Matrices' that covers this lecture is located at https://youtu.be/TODDZnOT3ro.

Outline

-Expressing Vectors In Different Frames (Coordinate Rotation)
-Vector Notation

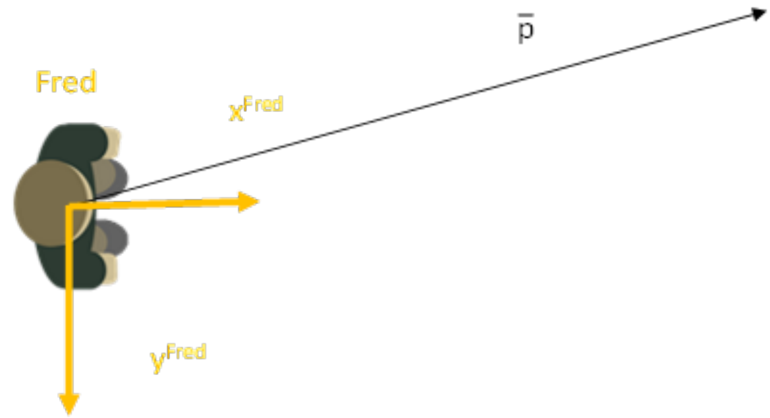
Expressing Vectors in Different Frames (Coordinate Rotation)

Previous expressions were just vector expression. It says nothing about what the components of the vector are. For example, assume that we have a vector $\overline{p}$



This vector exists independent of a reference frame. However, if we wish to express the components of the vector, we need to pick a frame. We can think of expressing this vector as describing its components. For example, suppose that Fred is standing at the base of the vector and looking directly to the right of the page. We then attach a reference frame to Fred as shown below. For the sake of argument, let us assume that the vector is approximately 2 units long. If you ask Fred to describe the vector (A.K.A. expressing the vector in his reference frame) he might say something like

"The vector $\overline{p}$ is 1.88 units along my x axis and -0.68 units along my y axis"

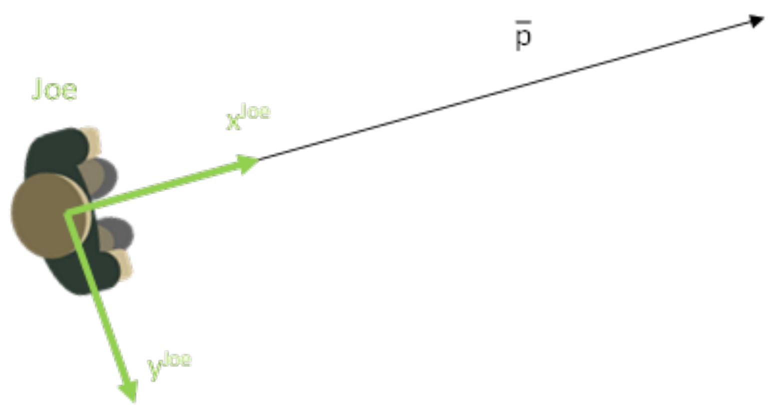


In other words, from Fred's perspective, we might write

$$\overline{p} = \begin{pmatrix} 1.88 \\ -0.68 \end{pmatrix} \qquad \text{(Fred's perspective)(notation is lacking as will be explained later)}$$

However, if another observer, Joe, is standing at the same location but facing in the same direction as the vector $\overline{p}$ and we ask him the same question, he might say

"The vector $\overline{p}$ is 2 units along my x axis and 0 units along my y axis"

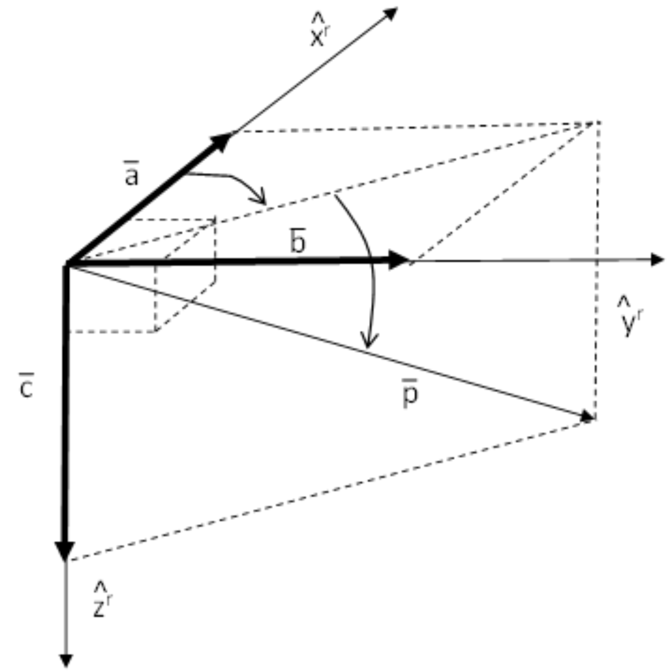


In other words, from Joe's perspective, we might write

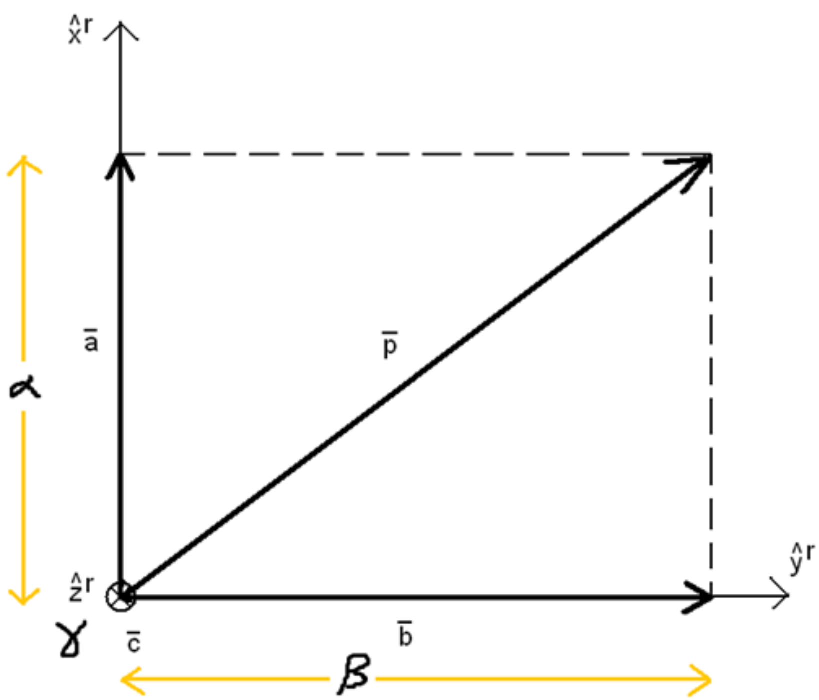
$$\overline{p} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \qquad \text{(Joe's perspective)} \qquad \text{(notation is lacking as will be explained later)}$$

So we see that our current notation is lacking because clearly $\begin{pmatrix} 1.88 \\ -0.68 \end{pmatrix} \neq \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. We need to be specific about what frame of reference the vector is expressed in. We also need mathematics to translate a vector expressed in one frame to be expressed in another frame.

For example we might choose Frame F_r to express the vector in.



To make drawing easier, let us just draw the diagram in the $\hat{x}^r$, $\hat{y}^r$ plane



If this vector is expressed in frame F_r, we can write

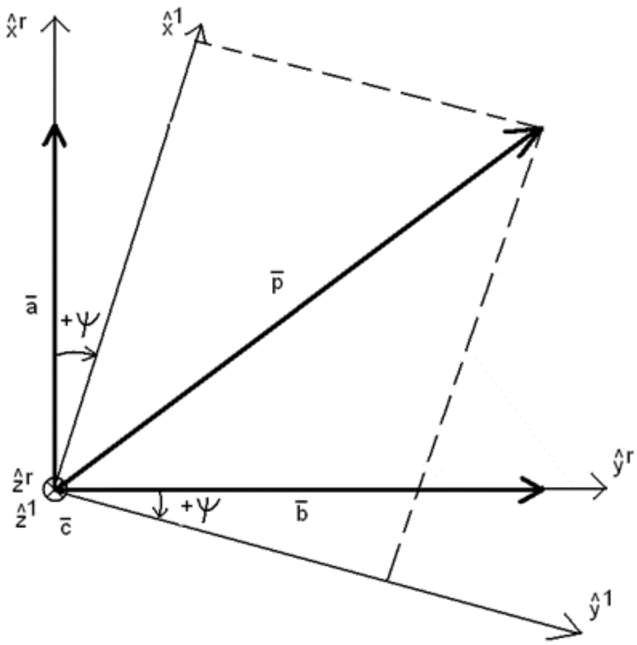
$$\bar{p}^r = \bar{a}^r + \bar{b}^r + \bar{c}^r$$

$$= \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix}^r + \begin{pmatrix} 0 \\ \beta \\ 0 \end{pmatrix}^r + \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix}^r$$

$$\overline{p}^r = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}^r$$

These components describe how to express the vector $\overline{p}$ in frame F_r. For example, to reach the end of vector $\overline{p}$, you travel a distance α in the $\hat{x}^r$ direction, a distance β in the $\hat{y}^r$, and a distance γ in the $\hat{z}^r$ direction.

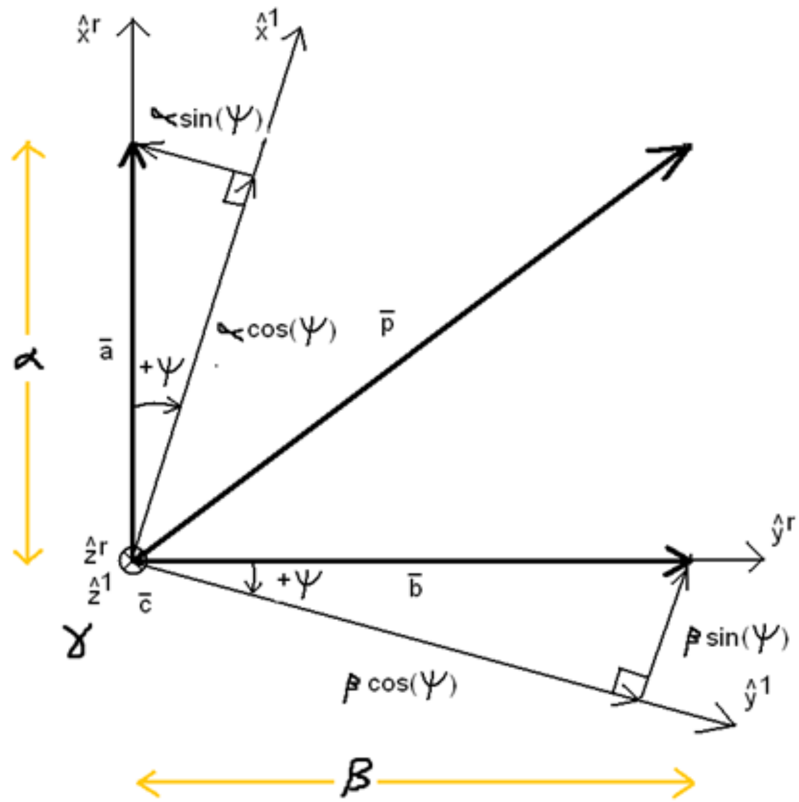
What if $\overline{p}$ is to be expressed in another frame, F_1 which is rotated about the $\hat{z}^r$ axis through an angle ψ (positive right hand rotation)



In this situation, the same vector $\overline{p}$ expressed in F_1 is no longer has $\hat{x}^1$, $\hat{y}^1$, $\hat{z}^1$ components given by α, β, and γ, respectively. However, we can still write that the vector is the sum of the other vectors, all expressed in the F_1 frame.

$$\overline{p}^1 = \overline{a}^1 + \overline{b}^1 + \overline{c}^1$$

The question now becomes, what are these vectors $\overline{a}^1$, $\overline{b}^1$, and $\overline{c}^1$? We can draw the picture as shown below.



So we can write

$$\bar{a}^1 = \begin{pmatrix} \alpha\cos(\psi) \\ -\alpha\sin(\psi) \\ 0 \end{pmatrix}^1 \qquad \bar{b}^1 = \begin{pmatrix} \beta\sin(\psi) \\ \beta\cos(\psi) \\ 0 \end{pmatrix}^1 \qquad \bar{c}^1 = \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix}^1$$

$$\bar{p}^1 = \begin{pmatrix} \alpha\cos(\psi) \\ -\alpha\sin(\psi) \\ 0 \end{pmatrix}^1 + \begin{pmatrix} \beta\sin(\psi) \\ \beta\cos(\psi) \\ 0 \end{pmatrix}^1 + \begin{pmatrix} 0 \\ 0 \\ \gamma \end{pmatrix}^1$$

$$= \begin{pmatrix} \alpha\cos(\psi) + \beta\sin(\psi) \\ -\alpha\sin(\psi) + \beta\cos(\psi) \\ \gamma \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \qquad \text{recall:} \quad \bar{p}^r = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

$$\bar{p}^1 = C_{1/r}(\psi)\,\bar{p}^r$$

where $\quad C_{1/r}(\psi) = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$

This is used to rotate a vector from one coordinate frame to another. This is known as a rotation matrix and sometimes as the direction cosines matrix.

There are lots of special properties of this matrix. For one, it is a unitary matrix (more specifically, it is an orthogonal matrix). Unitary can be specialized to orthogonal matrix if all elements are real.

Also, given that the physical interpretation of an eigenvalue is that it measures the amount that a matrix "stretches" a vector, from physical intuition we see that this rotation matrix should have all eigenvalues of magnitude 1 (because the rotation matrix does not stretch a vector, it only rotates it).

```
C1r[ψ] = ( Cos[ψ]   Sin[ψ]  0
          -Sin[ψ]   Cos[ψ]  0
             0        0     1 );
Eigenvalues[C1r[ψ]]
```

```
{1, Cos[ψ] - ⅈ Sin[ψ], Cos[ψ] + ⅈ Sin[ψ]}
```

So we see that all eigenvalues have magnitude 1.

We can use identical analysis to derive rotation matrices about the y and x axes thorough angles θ and ϕ, respectively. These are given as

$$C_{1/r}(\theta) = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \qquad \text{(rotation about } y \text{ axis)}$$

$$C_{1/r}(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix} \qquad \text{(rotation about } x \text{ axis)}$$

Unitary Matrix

We can show that this matrix is in fact unitary, meaning that $C_{1/r}(\psi)^{-1} = C_{1/r}(\psi)^T$. To do so, we compute the quantify $C_{1,r}(\psi)\, C_{1,r}(\psi)^T$

```
C1rC1rT = C1r[ψ].Transpose[C1r[ψ]];
C1rC1rT // MatrixForm
```

$$\begin{pmatrix} \text{Cos}[\psi]^2 + \text{Sin}[\psi]^2 & 0 & 0 \\ 0 & \text{Cos}[\psi]^2 + \text{Sin}[\psi]^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If we simplify this expression

```
C1rC1rT // Simplify // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can also compute $C_{1,r}(\psi)^T C_{1,r}(\psi)$

```
C1rTC1r = Transpose[C1r[ψ]].C1r[ψ];
% // Simplify // MatrixForm
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We also obtain the identify matrix showing that indeed, $C_{1/r}(\psi)^{-1} = C_{1/r}(\psi)^T$.

We can do the same for the rotation matrices about the y and x axes

$$A = \begin{pmatrix} \text{Cos}[\theta] & 0 & -\text{Sin}[\theta] \\ 0 & 1 & 0 \\ \text{Sin}[\theta] & 0 & \text{Cos}[\theta] \end{pmatrix};$$

```
A.Transpose[A] == IdentityMatrix[3] // Simplify
Transpose[A].A == IdentityMatrix[3] // Simplify
```

```
True
```

```
True
```

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \text{Cos}[\phi] & \text{Sin}[\phi] \\ 0 & -\text{Sin}[\phi] & \text{Cos}[\phi] \end{pmatrix};$$

```
A.Transpose[A] == IdentityMatrix[3] // Simplify
Transpose[A].A == IdentityMatrix[3] // Simplify
```

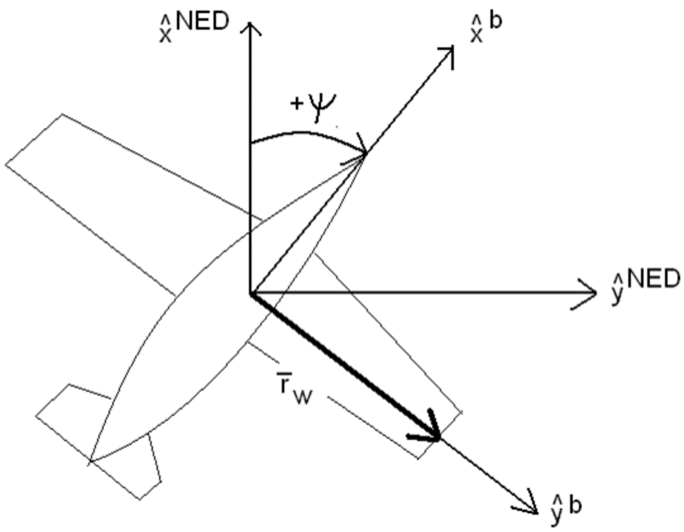
```
True
```

```
True
```

Example

Consider an example where we have an aircraft as shown below. We attach two frames to it

F_b = body frame. Origin at CM of AC. x is positive out the nose, y is positive out the right wingtip
F_{NED} = North, East down frame. Origin at CM of AC. x is pointing north, y is pointing east.



The vector from the center of mass to the wingtip is denoted $\bar{r}_w$. This exists regardless of frames but it is most easily expressed in the body frame

$$\bar{r}_w{}^b = \begin{pmatrix} 0 \\ 1.76 \\ 0.12 \end{pmatrix}^b$$

From last time

$$C_{b/\text{NED}}(\psi) = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So we can write

$$C_{b/\text{NED}}(\psi)\,\bar{r}_w{}^{\text{NED}} = \bar{r}_w{}^b$$

$$\bar{r}_w{}^{\text{NED}} = C_{b/\text{NED}}{}^{-1}(\psi)\,\bar{r}_w{}^b$$

We notice that we have $C_{b/\text{NED}}{}^{-1}(\psi) = C_{\text{NED}/b}(\psi)$. So using our notation, the inverse is the same as flipping the indices.

```
Cinv = Inverse[( Cos[ψ]   Sin[ψ]  0
                 -Sin[ψ]  Cos[ψ]  0
                 0        0       1 )] // Simplify;
```

```
Cinv // MatrixForm
```

$$\begin{pmatrix} \text{Cos}[\psi] & -\text{Sin}[\psi] & 0 \\ \text{Sin}[\psi] & \text{Cos}[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So for an example rotation of 45^o

```
Cinv.(0, 1.76, 0.12) /. {ψ → π / 4}
```

$$\mathbf{Cinv}.\begin{pmatrix} 0 \\ 1.76 \\ 0.12 \end{pmatrix} /. \{\psi \to \pi/4\}$$

```
{{-1.24451}, {1.24451}, {0.12}}
```

So we have

$$\overline{r}_w^{NED} = \begin{pmatrix} -1.244 \\ 1.244 \\ 0.12 \end{pmatrix}^{NED}$$

So only the x and y components change.

Vector Notation

-Vectors will be in boldface or with a bar on top.

-A right subscript will be used to designate two points for a position vector, and a point and a frame for a velocity or acceleration vector. A "/" in a subscript will mean "with respect to"

-A left superscript will specify the frame in which a derivative is taken, and the dot notation will indicate a derivative.

-A right superscript on a vector will specify a coordinate system. It will therefore denote an array of the components of that vector in the specified system.

Examples

$\overline{p}_{A/B}$ ≡ position vector of point A with respect to point B (vector ending at A and starting from B)
$\overline{v}_{A/i}$ ≡ velocity of point A in frame i (F_i)
$\overline{v}_{A/i}^{\,c} \equiv (\overline{v}_{A/i})^c \equiv \overline{v}_{A/i}$ expressed in coordinate system c

For rotation matrices

$C_{1/r}(\psi)$ = rotation from frame F_r to F_1 though an angle ψ

Some new notation will be

${}^{b}\dot{\overline{v}}_{A/i}$ ≡ derivative of $\overline{v}_{A/i}$ taken with respect to frame b
${}^{b}\dot{\overline{v}}_{A/i}^{\,c}$ ≡derivative of $\overline{v}_{A/i}$ with respect to frame b with components expressed in coordinate system c

Rotations About X and Y Axes

The same theory can be applied to rotations about the x and y axes. We will investigate this later when we explore Euler angles. For completeness, they are shown below (you will derive these in a homework assignment)

$$C_{2/1}(\theta) = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix} \qquad \text{(y-axis rotation through } \theta\text{)}$$

$$C_{b/2}(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix} \qquad \text{(x-axis rotation through } \phi\text{)}$$