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# Lecture 01f
# Transfer Functions: Introduction and Implementation

Lecture is on YouTube

The YouTube video entitled ‘Transfer Functions: Introduction and Implementation’ that covers this lecture is located at https://youtu.be/Uh_-RZQIaEs.

## Outline

-Transfer Functions

- -Transfer Function Poles
- -Transfer Functions in Matlab and Simulink

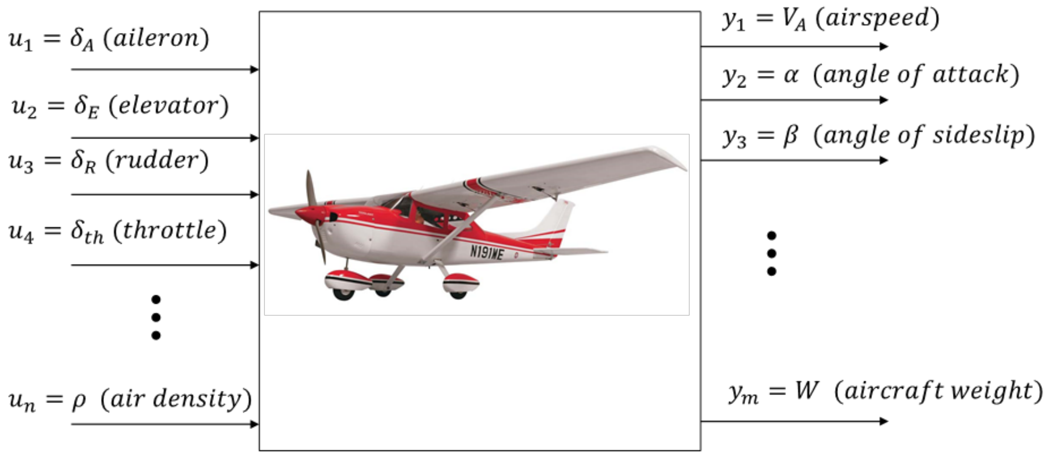
## Transfer Functions

So far in the class, we have only studied systems that response to initial conditions. The transfer function approach will help us study how a system responds to various inputs.

When speaking with control engineers casually, you may hear a transfer function referred to as a black box model that relates a specific input to a specific output.

**Example: Aircraft**

Consider and aircraft




Aircraft Model

For example, we could ask how does the elevator affect the angle of attack of the aircraft. In this case, we would be interested in the transfer function between the 2nd input and the 2nd output. We could conceptualize this as

$G_{22}$ = model that describes how the 2nd input affects the 2nd output (elevator to angle of attack)

Note that:
1. $G_{22}$ may be a complicated relationship that is likely dependent on the states or inputs of the system (how fast the aircraft is flying, the weight, the settings of the other control surfaces, etc).
2. The single input of aileron may likely affect other outputs besides the one that we are interested in (for example elevator will likely affect airspeed, $y_1$) as well as the angle of attack)

More precisely, a transfer function of a linear, time-invariant differential equation is defined as the ratio of the Laplace transform of the output (response function) to the Laplace transform of the input (driving or forcing function) under the assumption that all initial conditions are zero.

Consider the linear time-invariant system defined by the differential equation

$$a_n \frac{d^n y(t)}{d t^n} + a_{n-1} \frac{d^{n-1} y(t)}{d t^{n-1}} + \ldots + a_1 \frac{d y(t)}{d t} + a_0 y(t) = b_m \frac{d^m u(t)}{d t^m} + b_{m-1} \frac{d^{m-1} u(t)}{d t^{m-1}} + \ldots + b_1 \frac{d u(t)}{d t} + b_0 u(t)$$

***(Eq.1)***

where $y(t)$ is the system response (or output)
$u(t)$ is the input (or forcing function)

Assuming that all the initial conditions are zero ($y(0) = y^{(1)}(0) =. .. = y^{(n-1)}(0) = 0$ and $u(0) = u^{(1)}(0) =. .. = u^{(m-1)}(0) = 0$) the transfer function is defined as

$$G(s) = \frac{L[\text{output}]}{L[\text{input}]} \Big|_{\text{zero initial conditions}}$$

$$= \frac{Y(s)}{U(s)} \Big|_{\text{zero initial conditions}}$$

We can apply this to the general linear, ODE, shown in Eq.1.

In order to do this, we will make use of the Laplace transform definition of

$$L\left[\frac{d^n}{d t^n} f(t)\right] = s^n F(s) - s^{n-1} f(0) - s^{n-2} \frac{d}{d t} f(0) - \ldots - \frac{d^{n-1}}{d t^{n-1}} f(0) \qquad \textbf{\textit{(Eq.2)}}$$

where $\frac{d^n}{d t^n} f(0) = \frac{d^n}{d t^n} [f(t)] \,|_{t=0}$

Applying this with zero initial conditions yields

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} \qquad \textbf{\textit{(Eq.3)}}$$

Comments on transfer functions

1. Transfer function is a mathematical model of that system to relate input to output

2. Transfer function is a property of the system itself and is not related to the input function
3. Transfer function includes necessary units but it does not provide any information concerning the physical structure of the system since it is only an input/output model. Therefore, some physically different systems can have identical transfer functions.
4. If the transfer function is known, the output to a given input can be predicted.
5. If the transfer function is unknown, it can be experimentally determined by introducing known inputs and studying the output of the system.

**Example 1 - Mass Spring Damper**

Consider the equation of motion of a mass spring damper with an arbitrary forcing function

$$m\,\ddot{y}(t) + c\,\dot{y}(t) + k\,y(t) = u(t)$$

Laplace transform

$$L\left[m\,\ddot{y}(t) + c\,\dot{y}(t) + k\,y(t)\right] = L[u(t)]$$

$$m\left(s^2\,Y(s) - s\,y(0) - \dot{y}(0)\right) + c(s\,Y(s) - y(0)) + k\,Y(s) = U(s)$$ note: assume initial conditions are zero

$$\left(m\,s^2 + c\,s + k\right)Y(s) = U(s)$$

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{m\,s^2 + c\,s + k} \qquad \textbf{\textit{(Eq.3)}}$$

Note that if the initial conditions are not zero, we cannot solve for the transfer function (cannot solve explicitly for the ratio of $Y(s)/U(s)$)

If we assume constants of

$$m = 1$$
$$c = 3$$
$$k = 5$$

The transfer function becomes

$$G(s) = \frac{1}{s^2 + 3\,s + 5}$$

```
G[s_] = 1/(m s^2 + c s + k) /. {m → 1, c → 3, k → 5}
```

$$\frac{1}{5 + 3\,s + s^2}$$

The reason the transfer function is useful is because we can now obtain the response of this system to an arbitrary input, $u(t)$. For example, if the input is a step of magnitude 1, we can compute the Laplace transform of this input

$$u(t) = 1$$

$$U(s) = L[u(t)] = \frac{1}{s}$$

So the response of the system due to this particular input is

$$Y(s) = G(s)\,U(s)$$

$$= \left(\frac{1}{s^2+3s+5}\right)\left(\frac{1}{s}\right)$$

```
Print["Y(s)"]
Y[s_] = G[s] 1/s

(*Use inverse Laplace transform to compute time domain response*)
Print["y(t)"]
y[t_] = InverseLaplaceTransform[Y[s], s, t] // Simplify
y[t] // N

(*Plot this*)
Plot[y[t], {t, 0, 10}, PlotRange → All]
```

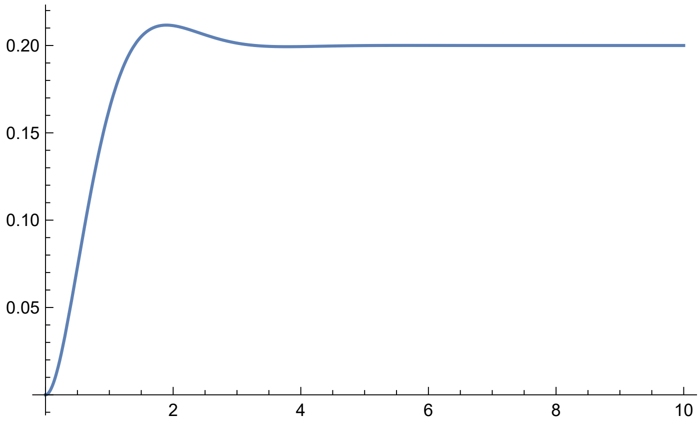
Y(s)

$$\frac{1}{s\left(5+3s+s^2\right)}$$

y(t)

$$\frac{1}{5} - \frac{e^{-3t/2}\left(\sqrt{11}\,\text{Cos}\left[\frac{\sqrt{11}\,t}{2}\right] + 3\,\text{Sin}\left[\frac{\sqrt{11}\,t}{2}\right]\right)}{5\sqrt{11}}$$

$$0.2 - 0.0603023 \times 2.71828^{-1.5t} \times (3.31662\,\text{Cos}[1.65831\,t] + 3.\,\text{Sin}[1.65831\,t])$$




In a  similar fashion, we could subject the system to an input of

$$u(t) = 5\sin(2\,t)$$

```
Usin[s_] = LaplaceTransform[5 Sin[2 t], t, s]
```

$$\frac{10}{4+s^2}$$

So we have

$U(s) = \frac{10}{s^2+4}$

So the response is

```
Print["Y(s)"]
Y[s_] = G[s] × Usin[s]

(*Use inverse Laplace transform to compute time domain response*)
Print["y(t)"]
y[t_] = InverseLaplaceTransform[Y[s], s, t] // Simplify
y[t] // N

(*Plot this*)
Plot[y[t], {t, 0, 10}, PlotRange → All]
```

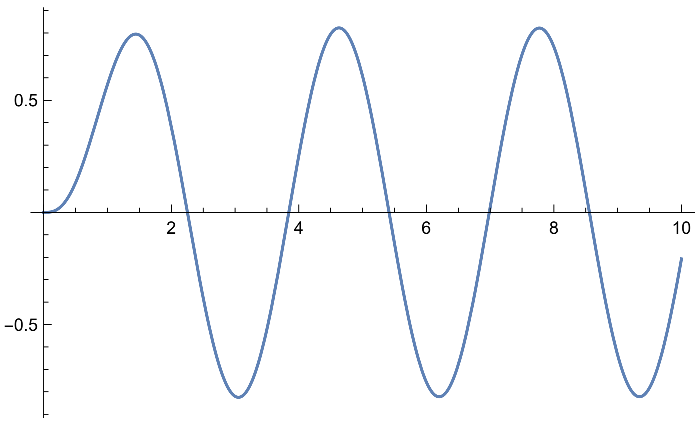
Y(s)

$$\frac{10}{(4+s^2)\times(5+3\,s+s^2)}$$

y(t)

$$\frac{5}{407}\times\left(11\times(-6\,\mathrm{Cos}[2\,t]+\mathrm{Sin}[2\,t])+2\,e^{-3\,t/2}\left(33\,\mathrm{Cos}\left[\frac{\sqrt{11}\,t}{2}\right]+7\,\sqrt{11}\,\mathrm{Sin}\left[\frac{\sqrt{11}\,t}{2}\right]\right)\right)$$

$$0.012285\times\left(2.\times 2.71828^{-1.5\,t}\times(33.\,\mathrm{Cos}[1.65831\,t]+23.2164\,\mathrm{Sin}[1.65831\,t])+11.\times(-6.\,\mathrm{Cos}[2.\,t]+\mathrm{Sin}[2.\,t])\right)$$




## Transfer Function Poles

Recall that a transfer function is a property of the system and not of the input

$G(s) = \frac{Y(s)}{U(s)} = \frac{\text{num}(s)}{\text{den}(s)}$

Therefore, we know that the output of the system is given by

$Y(s) = G(s)\,U(s)$

We would like to understand how the poles of the transfer function relate to the output of the system. Consider an example transfer function with two real poles

$$G(s) = \frac{K(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}$$

where $K$ = constant
$z_i$ = zeros of system
$p_i$ = poles of system

We now consider the output of the system. We know that the output of the system (in the Laplace domain) is simply obtained by multiplying the transfer function with the Laplace transform of the input

$$Y(s) = G(s)\,U(s)$$

$$Y(s) = \frac{K(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}\,U(s) \qquad \textbf{(Eq.1.1)}$$

From Eq.1.1, we see that the poles of $Y(s)$ will be a combination of the poles of $G(s)$ (ie $p_1$ and $p_2$) and the poles of $U(s)$. In other words, no matter what the poles of $U(s)$ are, it cannot change the poles of $G(s)$ and therefore the poles of $G(s)$ will appear in the poles of $Y(s)$.

Let us now consider two cases for the input $U(s)$

Case 1: $U(s)$ is bounded (ie $u(t) = 3$, $u(t) = \sin(2\,t)$, etc.)
Case 2: $U(s)$ is unbounded (ie $u(t) = t$, $u(t) = -e^{2\,t}\cos(t)$, etc.)

**Case 1: *U*(*s*) is Bounded**

Consider the case where $u(t)$ is bounded. That is to say $\lim_{t\to\infty} u(t) \neq \infty$. Let us consider the case of a simple sinusoidal input.

$$u(t) = \cos(t)$$

```
LaplaceTransform[Cos[t], t, s]
```

$$\frac{s}{1+s^2}$$

So we see that as expected, $U(s)$ has poles at $s = \pm i$

We can find the output $Y(s)$ using Eq.1.1

$$Y(s) = \left(\frac{K(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}\right)\left(\frac{s}{s^2+1}\right)$$

We see that $Y(s)$ has poles at $s = -p_1,\ -p_2,\ +i,$ and $-i$. We showed (in a previous homework assignment), that the resulting partial fraction expansion of this expression has the form

$$Y(s) = \frac{a_1}{s+p_1} + \frac{a_2}{s+p_2} + \frac{a_3}{s+i} + \frac{a_4}{s-i}$$

Once again, the important thing to notice here is that the poles of $G(s)$ are not affected by the poles of $U(s)$

Therefore, the resulting signal in the time domain is given by

$$y(t) = a_1 e^{-p_1 t} + a_2 e^{-p_2 t} + \left(a_3 e^{-i t} + a_4 e^{i t}\right)$$

Using Euler's theorem, we can show that the terms in parenthesis (containing the imaginary terms) can be written in terms of trigonometric sin or cos

```
ExpToTrig[a3 Exp[-I t] + a4 Exp[I t]] // Simplify
(a3 + a4) Cos[t] - i (a3 - a4) Sin[t]
```

We also know that if the original transfer function had no imaginary constant, $a_3$ and $a_4$ should be values such that $a_3 - a_4 = 0$ (so there will be no imaginary terms in $y(t)$ ) so we are left with

$$y(t) = a_1 e^{-p_1 t} + a_2 e^{-p_2 t} + (a_3 + a_4) \cos(t) \qquad \textbf{\textit{(Eq.1.2)}}$$

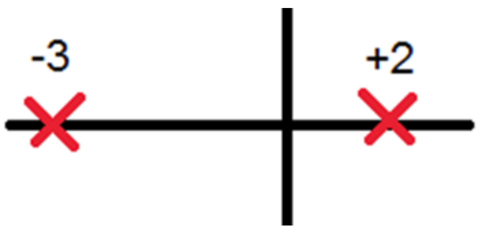
We now ask the question, what are the conditions which lead to $y(t)$ becoming unbounded (ie another word for unstable)? By inspecting Eq.1.2, we see that the only way that $\lim_{t\to\infty} y(t) = \pm\infty$ is if <u>either</u> $p_1$ or $p_2$ are negative.

For example, if $p_1 = -2$ and $p_2 = 3$, then $G(s)$ would have the form of

$G(s) = \frac{K(s+z_1)(s+z_2)}{(s-2)(s+3)}$ $\qquad$ $G(s)$ is unstable in the sense that $y(t) \to \pm\infty$ for any type of input

The poles of this unstable transfer function are at

$$s = +2, -3$$




In this case, $y(t)$ becomes

$$y(t) = a_1 e^{2t} + a_2 e^{-3t} + (a_3 + a_4) \cos(t)$$

And as time becomes large, this reduces to

$$y(t) = a_1 e^{2t} + (a_3 + a_4) \cos(t)$$

So we see that the pole at +2 in the original transfer function was what caused the system to become unstable (the output becomes unbounded)

**Case 2: *U*(*s*) is Unbounded**

Consider the case where $u(t)$ is unbounded. That is to say $\lim_{t\to\infty} u(t) = \pm\infty$. Let us consider the case of a simple exponential input

$$u(t) = e^{2t}$$

```
LaplaceTransform[Exp[2 t], t, s]
```

$$\frac{1}{-2+s}$$

So we see that as expected, $U(s)$ has a single pole at $s = +2$

We can find the output $Y(s)$ using Eq.1.1

$$Y(s) = \left(\frac{K(s+z_1)(s+z_2)}{(s+p_1)(s+p_2)}\right)\left(\frac{1}{s+2}\right)$$

We see that $Y(s)$ has poles at $s = -p_1, -p_2, +2$. We showed (in a previous homework assignment), that the resulting partial fraction expansion of this expression has the form

$$Y(s) = \frac{a_1}{s+p_1} + \frac{a_2}{s+p_2} + \frac{a_3}{s+2}$$

Once again, the important thing to notice here is that the poles of $G(s)$ are not affected by the poles of $U(s)$

Therefore, the resulting signal in the time domain is given by

$$y(t) = a_1 e^{-p_1 t} + a_2 e^{-p_2 t} + a_3 e^{2t} \qquad \textbf{\textit{(Eq.1.3)}}$$

We now ask the question, what are the conditions which lead to $y(t)$ becoming unbounded (ie another word for unstable)? By inspecting Eq.1.3, we see $\lim_{t\to\infty} y(t) = \pm\infty$ regardless of the values that $p_1$ or $p_2$ take. However, the important thing to note in this situation is that the reason $y(t) \to \pm\infty$ is because the pole associated with the input (the pole of $U(s)$ at +2) had a positive real part.

Physically, this makes sense. If you consider the system to be a mechanical system, then typically the input signal $u(t)$ is a force or a torque. If the input force or torque becomes increasingly large without bound, then the resulting output of the system (ie the position or velocity) will also become unbounded regardless of the type of system.

A homework assignment will more thoroughly investigate how the poles of the transfer function are related to the stability of the system.

## Transfer Functions in Matlab and Simulink

Recall that a transfer function has the form of

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

To create a transfer function in Matlab, we define a row vector of just the coefficients of $s$ in descending order

G_num = [ $b_m$ $b_{m-1}$ ... $b_1$ $b_0$ ];
G_den = [ $a_n$ $a_{n-1}$ ... $a_1$ $a_0$ ];

$G$ = tf(G_num, G_den);

For example, recall the mass/spring/damper example that has a transfer function of

$$G(s) = \frac{1}{m s^2 + c s + k}$$

We can see the response of the system using the ‘step’ command in Matlab

step(G)

However, what if the input is not a simple input like a step function or what if we want to include this as part of a more complicated model?  This is easier to do in Simulink.  Simulink is a

-numerical solver for dynamic systems
-uses a graphical interface to connect input/outputs from subsystems together
-instead of writing functions, can use blocks (graphical representations of subsystems)

We can implement this model graphically in Simulink using the ‘Transfer Fcn’ block (see Matlab files)

A future lecture will expand on using transfer functions and other dynamic system representations in Simulink (see ‘Ordinary Differential Equations and Dynamic Systems in Simulink’ - https://youtu.be/Cvu2zWk3gYw).

Now that we understand how to model a single system as a transfer function, we can investigate how to combine multiple systems together in a block diagram.