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Lecture 02f
DC Gain

Lecture is on YouTube

The YouTube video entitled ‘DC Gain’ that covers this lecture is located at https://youtu.be/sgTt7v4-LYfE.

Outline

-DC gain

DC Gain

Definition: DC Gain - The DC Gain of a system is defined as the ratio of the steady state output to the input when the input is a step function.

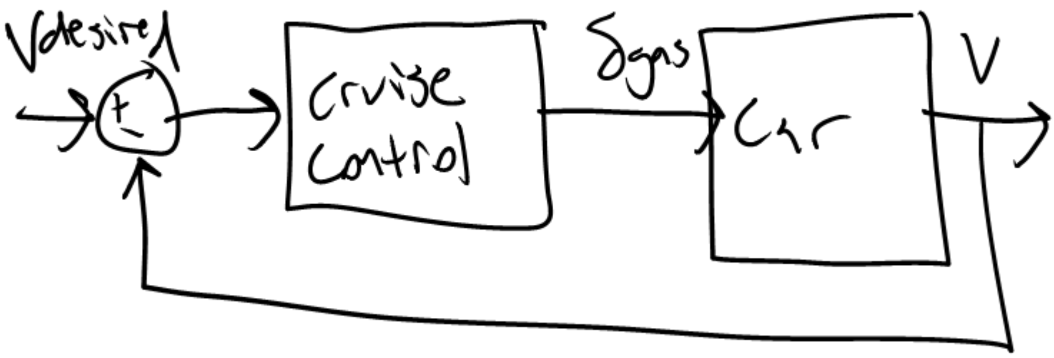
For example, if the input is step of magnitude A, the DC gain is given by

$$\text{DC gain} = \frac{\lim_{t\to\infty} y(t)}{A} = \frac{y_{ss}}{A} \qquad \textbf{\textit{(Eq.3)}}$$

where $\quad A$ = magnitude of step function

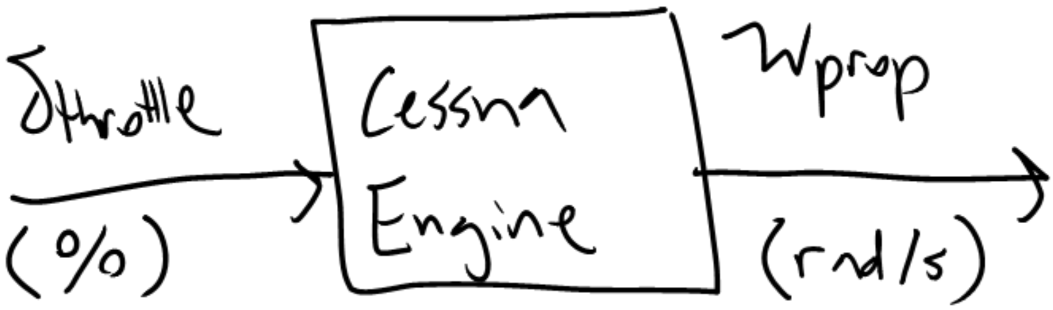
The DC gain is useful because if tells you information about the system at steady state.

In some scenarios, you would like to ensure that the DC gain is unity. For example, consider the block diagram of a car with a cruise control equipped.



The closed loop transfer function is defined as $T(s) = \frac{V(s)}{V_{desired}(s)}$. Obviously, we want the DC gain to be unity because this implies that at steady state, the total car/controller system will reach the desired velocity.

However, consider a case of a Cessna propulsion system



In this case the transfer function of $\frac{\omega_{\text{prop}}(s)}{\delta_{\text{throttle}}(s)}$ does not need to have a DC gain of unity. In fact, we hope that it has a much larger DC gain (for example, when the input is 50% full throttle, we hope the angular velocity of the motor is much larger than 50 rad/s)

We can see how the DC gain of a system is related to its transfer function.

Let us find the steady state response of the system, $y(t)$, in response to a for a step input, $r(t) = A * \mathbf{1}(t)$.

$y_{ss} = \lim_{t\to\infty} y(t)$ Note: Assume $y(t)$ is bounded so we can apply FVT

$= \lim_{s\to 0} s\, Y(s)$ recall: $Y(s) = G(s)\, R(s)$

$= \lim_{s\to 0} s\, G(s)\, R(s)$ recall: $R(s) = A/s$

$= \lim_{s\to 0} s\, G(s)\, \frac{A}{s}$

$y_{ss} = G(0)\, A$

So the DC gain of the system is given by

$$\text{DC gain } = \frac{y_{ss}}{A}$$

$$= \frac{G(0)\, A}{A}$$

$\text{DC gain } = G(0)$ ***(Eq.4)***

So the DC gain of the system is simply the transfer function of the system with $s = 0$.

The DC gain is related to the steady state error in response to a step function? If the input is a step function, then we know

$\text{DC gain} = \frac{y_{ss}}{A}$ note: $\text{DC gain} = G(0)$

$y_{ss} = G(0)\, A$

The steady error in response to a step function is the difference between the step value and the system's steady state value

$e_{ss} = A - y_{ss}$ note: $y_{ss} = G(0)\,A$

$e_{ss} = A - G(0)\,A$

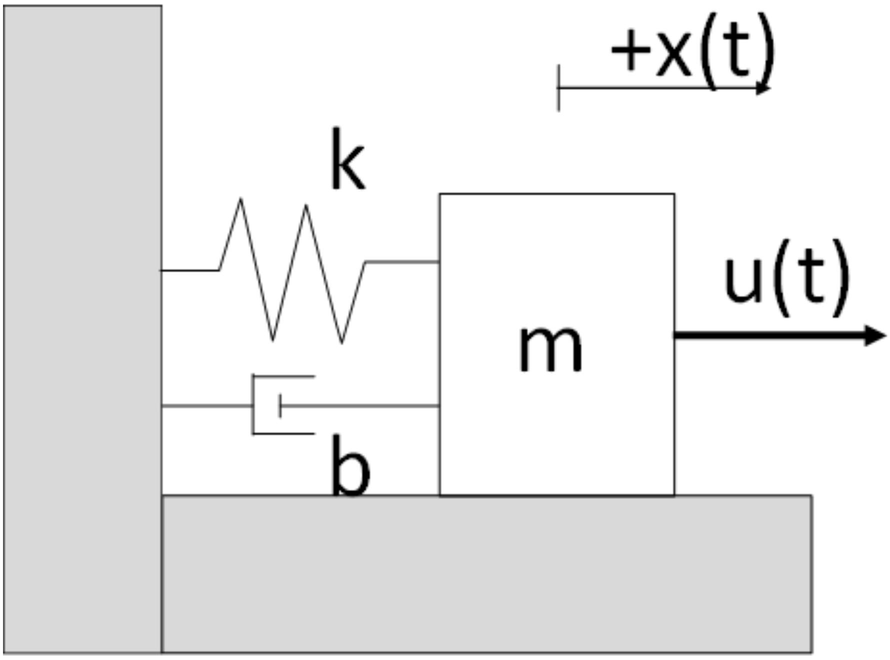
$e_{ss} = (1 - G(0))\,A$ (steady state error in response to a step of magnitude A)
(Eq.5)

In other words, we see that if the system does not have a DC gain of unity, there will be some steady state error in response to a step function.

Note that this result is valid for <u>any</u> transfer function. Nothing in the previous DC gain analysis was restricted to a 1st, 2nd, or nth order system.

Example: Spring Mass Damper

Recall the mass spring damper system that we studied in our lecture on Laplace transforms



The transfer function of this system can be shown to be

$$G(s) = \frac{X(s)}{U(s)} = \frac{1/m}{s^2 + \frac{b}{m}\,s + \frac{k}{m}}$$

To make analysis simpler, let us chose constants of

$m = 2$
$b = 1/10$
$k = 5$

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G[s_] = (1 / m) / (s^2 + b/m s + k/m) /. {m → 2, b → 1 / 10, k → 5} // Simplify
```

$$\frac{10}{50 + s + 20\,s^2}$$

The response of the system to a step input of magnitude A is given as

$$X(s) = G(s)\,\frac{A}{s}$$

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XStep[s_] = G[s] A/s
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$$\frac{10\,A}{s\,(50 + s + 20\,s^2)}$$

Converting back to the time domain yields

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xStep[t_] = InverseLaplaceTransform[XStep[s], s, t]
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$$10\,A\left(\frac{1}{50} - \frac{e^{-t/40}\left(\sqrt{3999}\,\mathrm{Cos}\left[\frac{\sqrt{3999}\,t}{40}\right] + \mathrm{Sin}\left[\frac{\sqrt{3999}\,t}{40}\right]\right)}{50\,\sqrt{3999}}\right)$$

If we take the limit as time goes to infinity we have

```
Limit[xStep[t], t → ∞]
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$$\frac{A}{5}$$

So we see that

$$x_{ss} = \frac{A}{5}$$

Recall that the input to the system was a step of magnitude A so we have

$$\text{DC gain} = \frac{x_{ss}}{A} = \frac{A/5}{A} = \frac{1}{5}$$

We can verify this with

$$\text{DC gain} = G(0)$$

```
G[0]
```

$$\frac{1}{5}$$

So the system attenuates the input by a factor of 5. If we input a step of magnitude $A = 5$ then eventually this should reach a steady state value of 1.

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Plot[xStep[t] /. {A → 5}, {t, 0, 200}, PlotRange → All]
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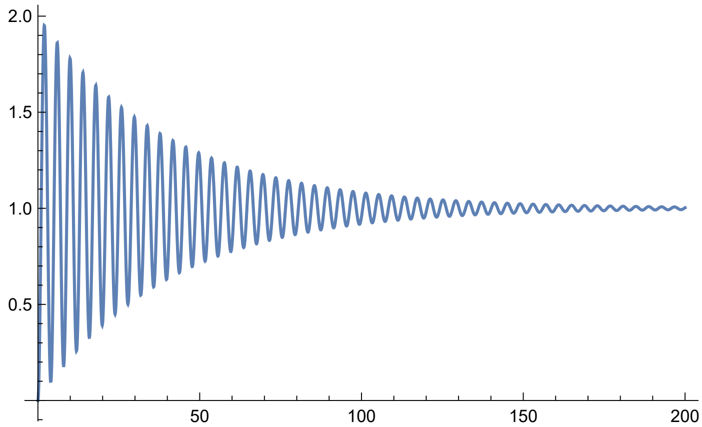

The steady state error is then given by

$$e_{ss} = (1 - G(0))\,A$$

$$= \left(1 - \frac{1}{5}\right) \times 5$$

$$e_{ss} = 4$$

Again, the word “error” may be misleading because there is nothing wrong with the input not equalling the output in this scenario.