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Lecture 04C
Angle of Attack/Sideslip and the Stability/Wind Axes

Lecture is on YouTube

The YouTube video entitled ‘Angle of Attack/Sideslip and the Stability/Wind Axes’ that covers this lecture is located at https://youtu.be/4kaK569ug9Q

Outline

-Introduction
-Aerodynamic Angles α, β and Stability and Wind Axis
-Transformation from V_a, α, β to u, v, w and vice versa
-Aerodynamic Forces and Moments

Introduction

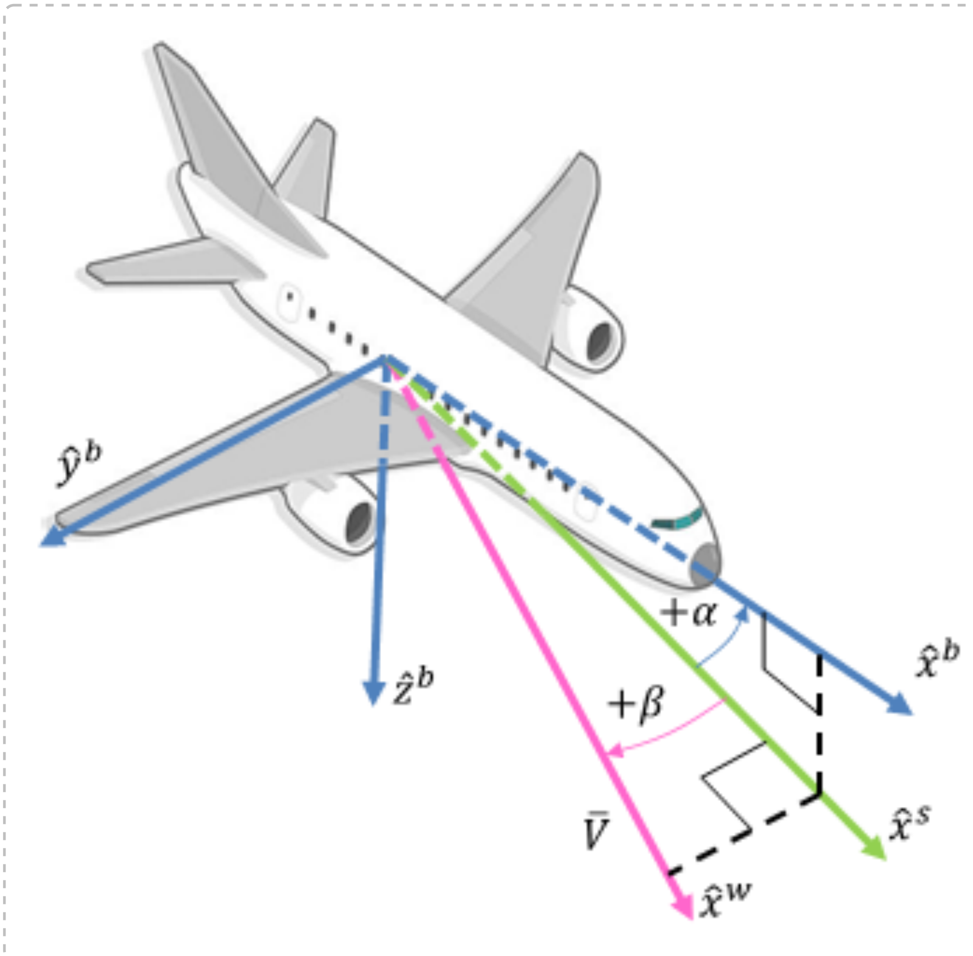
We have already defined the body frame, F_b, as attached to the aircraft. The wind frame, F_w, is aligned with the local wind. We will see that an intermediate frame called the stability frame, F_s, will be useful for defining aerodynamic angles as well as for wind tunnel testing. The stability frame is also used to study perturbations from steady-state flight.

F_b = body frame
F_s = stability frame
F_w = wind frame

Two very important aerodynamic angles are α (angle of attack) and β (angle of sideslip)

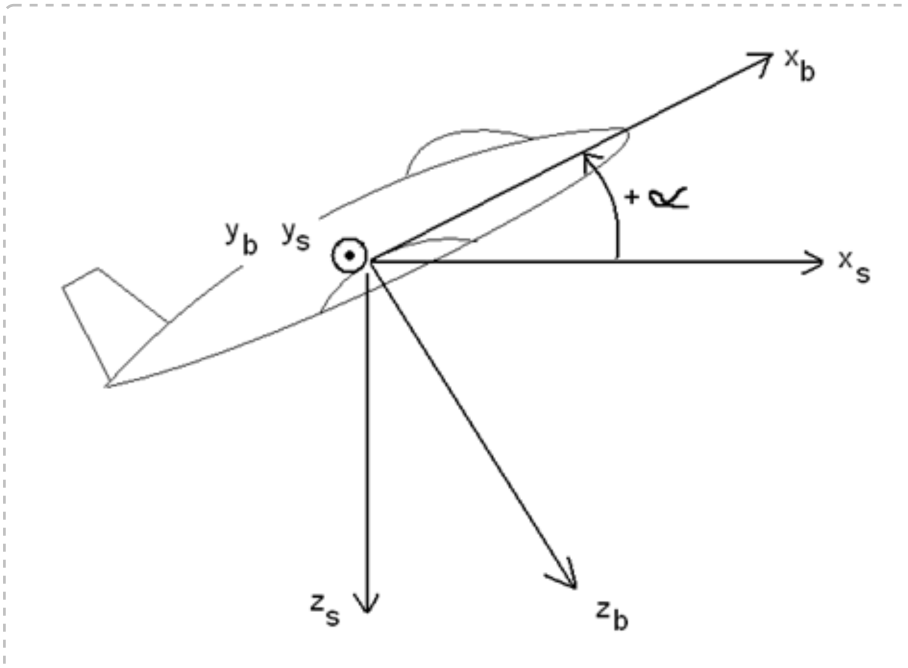
α defined positive when airstream hits under side of aircraft
β defined positive when airstream hits right side of aircraft

A graphic of these are shown below

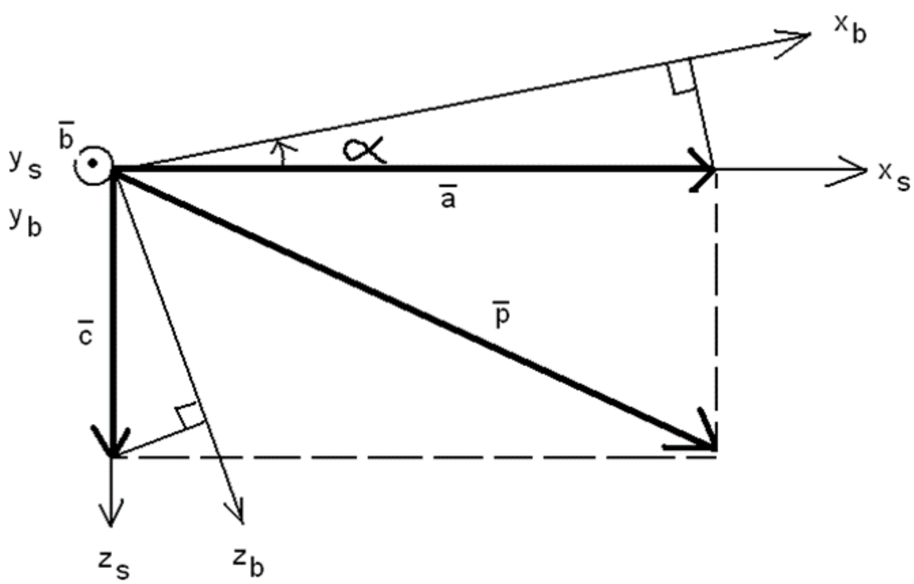


Aerodynamic Angles α, β and Stability and Wind Axis

We go from the stability axis to the body axis using a right handed rotation about the $y^s = y^b$ axis through the angle α



We can draw the two frames



So the vector $\overline{p}$ expressed in the stability frame is

$$\overline{p}^s = \overline{a}^s + \overline{b}^s + \overline{c}^s$$

$$= \begin{pmatrix} \gamma \\ 0 \\ 0 \end{pmatrix}^s + \begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix}^s + \begin{pmatrix} 0 \\ 0 \\ \eta \end{pmatrix}^s$$

$$\overline{p}^s = \begin{pmatrix} \gamma \\ \delta \\ \eta \end{pmatrix}^s$$

If instead, we would like to express vector $\overline{p}$ in the body frame, we can still write

$$\overline{p}^b = \overline{a}^b + \overline{b}^b + \overline{c}^b$$

So we can write

$$\overline{a}^b = \begin{pmatrix} \gamma\cos(\alpha) \\ 0 \\ \gamma\sin(\alpha) \end{pmatrix}^b \qquad \overline{b}^b = \begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix}^b \qquad \overline{c}^b = \begin{pmatrix} -\eta\sin(\alpha) \\ 0 \\ \eta\cos(\alpha) \end{pmatrix}^b$$

$$\overline{p}^b = \begin{pmatrix} \gamma\cos(\alpha) \\ 0 \\ \gamma\sin(\alpha) \end{pmatrix}^b + \begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix}^b + \begin{pmatrix} -\eta\sin(\alpha) \\ 0 \\ \eta\cos(\alpha) \end{pmatrix}^b$$

$$= \begin{pmatrix} \gamma\cos(\alpha) - \eta\sin(\alpha) \\ \delta \\ \gamma\sin(\alpha) + \eta\cos(\alpha) \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\alpha) & 0 & -\sin(\alpha) \\ 0 & 1 & 0 \\ \sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \\ \eta \end{pmatrix} \qquad \text{recall:} \quad \overline{p}^s = \begin{pmatrix} \gamma \\ \delta \\ \eta \end{pmatrix}$$

$$\overline{p}^b = C_{b/s}(\alpha)\,\overline{p}^s$$

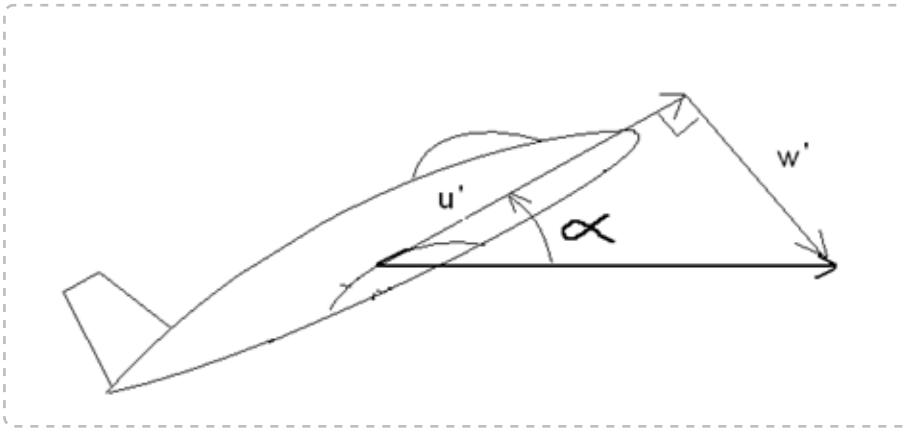
So the corresponding rotation matrix is

$$C_{b/s}(\alpha) = \begin{pmatrix} \cos(\alpha) & 0 & -\sin(\alpha) \\ 0 & 1 & 0 \\ \sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix} \qquad \text{(standard rotation about y-axis)}$$

(Eq.1)

```
In[ ]:= Cbs[α_] = {{Cos[α], 0, -Sin[α]}, {0, 1, 0}, {Sin[α], 0, Cos[α]}};
```

We can relate α to the velocity



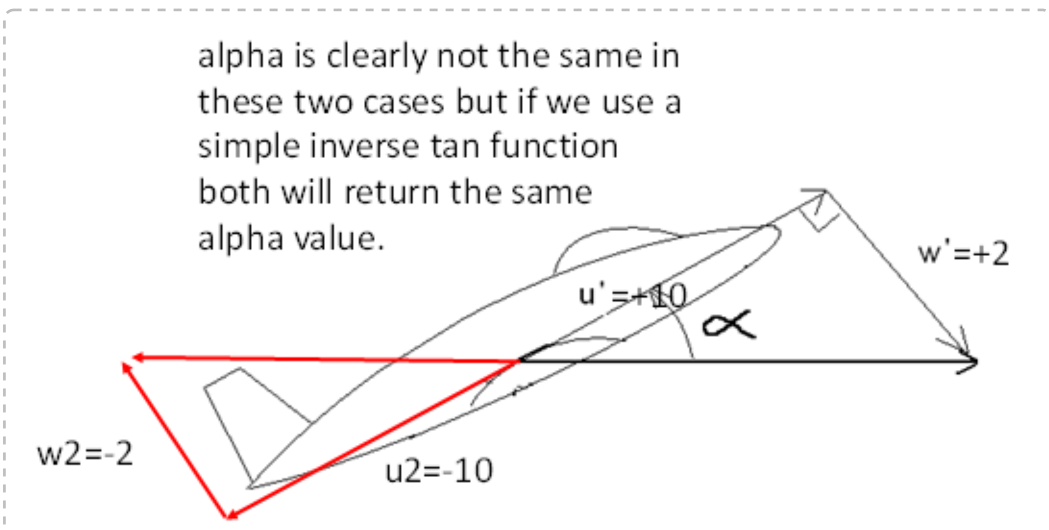
We also see that α is related to u and w through

$$\text{atan2}(y, x) = \text{atan2}(w, u) \qquad \textbf{(Eq.2)}$$

Again, we use the 4-quadrant inverse tangent to account for possible large angles of attack (https://-youtu.be/UWrkh_N1bfE)

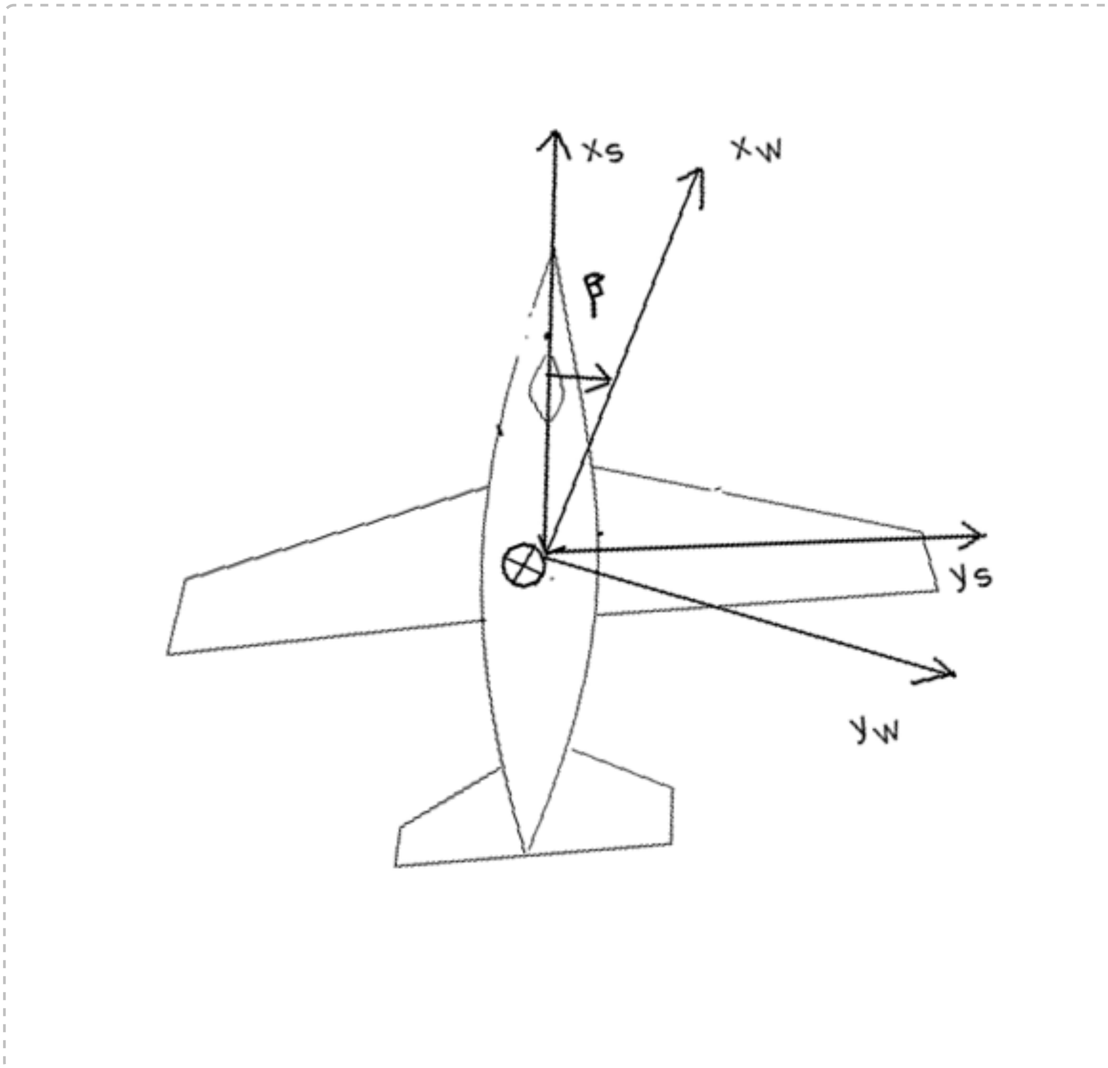
However for reasonable operating envelopes, we do not need to use the 4 quadrant inverse tangent and simply write

$$\alpha = \text{atan}\left(\frac{w}{u}\right)$$

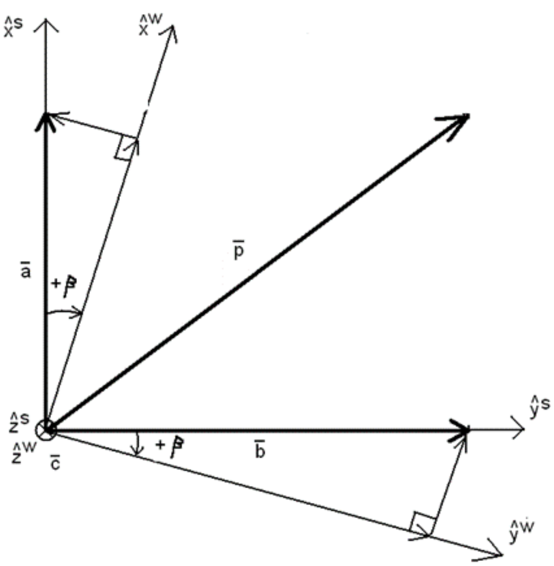


Note that in the Steven's and Lewis textbook, they use prime notation is used to denote velocity components relative to the atmosphere, as opposed to "inertial" components (account for wind, etc). (NOTE: typo on pg 74 of book, should have u' in equation for tan(α))

We go from the stability axis to the wind axis using a right handed rotation about the $z^s = z^W$ axis through the angle β



We can draw the two frames



So the vector $\overline{p}$ expressed in the stability frame is

$$\overline{p}^s = \overline{a}^s + \overline{b}^s + \overline{c}^s$$

$$= \begin{pmatrix} \gamma \\ 0 \\ 0 \end{pmatrix}^s + \begin{pmatrix} 0 \\ \delta \\ 0 \end{pmatrix}^s + \begin{pmatrix} 0 \\ 0 \\ \eta \end{pmatrix}^s$$

$$\overline{p}^s = \begin{pmatrix} \gamma \\ \delta \\ \eta \end{pmatrix}^s$$

If instead, we would like to express vector $\overline{p}$ in the wind frame, we can still write

$$\overline{p}^w = \overline{a}^w + \overline{b}^w + \overline{c}^w$$

So we can write

$$\overline{a}^w = \begin{pmatrix} \gamma \cos(\beta) \\ -\gamma \sin(\beta) \\ 0 \end{pmatrix}^w \qquad \overline{b}^w = \begin{pmatrix} \delta \sin(\beta) \\ \delta \cos(\beta) \\ 0 \end{pmatrix}^w \qquad \overline{c}^w = \begin{pmatrix} 0 \\ 0 \\ \eta \end{pmatrix}^w$$

$$\overline{p}^w = \begin{pmatrix} \gamma \cos(\beta) \\ -\gamma \sin(\beta) \\ 0 \end{pmatrix}^w + \begin{pmatrix} \delta \sin(\beta) \\ \delta \cos(\beta) \\ 0 \end{pmatrix}^w + \begin{pmatrix} 0 \\ 0 \\ \eta \end{pmatrix}^w$$

$$= \begin{pmatrix} \gamma \cos(\beta) + \delta \sin(\beta) \\ -\gamma \sin(\beta) + \delta \cos(\beta) \\ \eta \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\beta) & \sin(\beta) & 0 \\ -\sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \\ \eta \end{pmatrix} \qquad \text{recall:} \quad \overline{p}^s = \begin{pmatrix} \gamma \\ \delta \\ \eta \end{pmatrix}$$

$$\overline{p}^w = C_{w/s}(\beta)\, \overline{p}^s$$

So the corresponding rotation matrix is

$$C_{w/s}(\alpha) = \begin{pmatrix} \cos(\beta) & \sin(\beta) & 0 \\ -\sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \text{(standard rotation about z-axis)}$$

In[ ]:=
$$\text{Cws}[\alpha\_] = \begin{pmatrix} \text{Cos}[\beta] & \text{Sin}[\beta] & 0 \\ -\text{Sin}[\beta] & \text{Cos}[\beta] & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

It may be useful to consider the reverse rotation (note the transpose difference between $C_{w/s}(\beta)$ and $C_{s/w}(\beta)$)

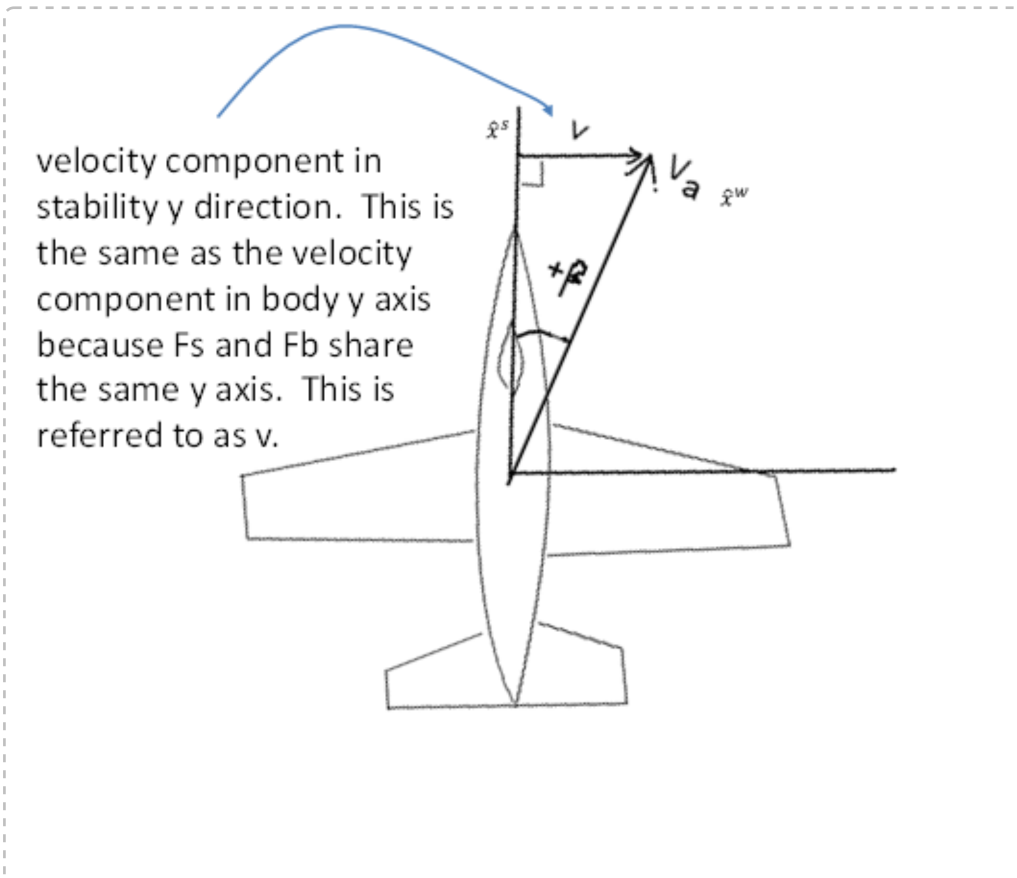
$$C_{s/w}(\beta) = \begin{pmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \textbf{(Eq.3)}$$

In[ ]:=
```
Csw[β_] = Transpose[Cws[β]];
Csw[β] // MatrixForm
```

Out[ ]//MatrixForm=
$$\begin{pmatrix} \text{Cos}[\beta] & -\text{Sin}[\beta] & 0 \\ \text{Sin}[\beta] & \text{Cos}[\beta] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can also relate β to aerodynamic velocities by drawing a triangle with the appropriate velocity components

$$\sin(\beta) = \frac{\text{velocity component in } \hat{y}^s}{V_a}$$



So we have

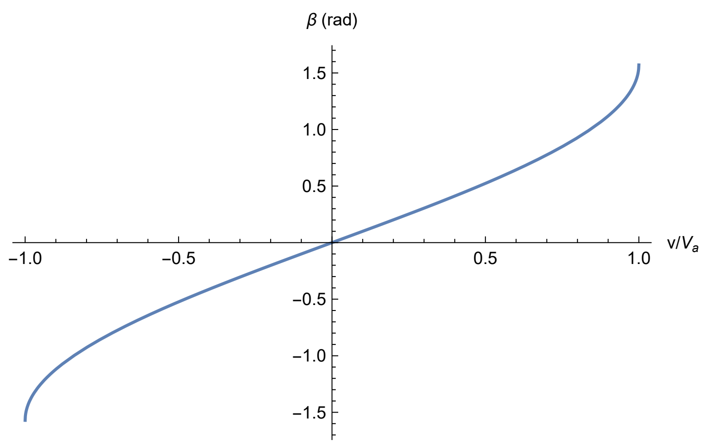
$$\sin(\beta) = \frac{v}{V_a}$$

$$\beta = \text{asin}\left(\frac{v}{V_a}\right) \qquad \textbf{\textit{(Eq.4)}}$$

Again note that we have simply written this using a standard inverse sin function, thereby assuming that $\beta \in [-\pi/2, \pi/2] = [-90, 90]$ deg

In[∘]:=
```
Plot[ArcSin[y], {y, -1, 1}, AxesLabel → {"v/Va", "β (rad)"}]
```

Out[∘]=



It may be tempting to consider a version of inverse sin that can support a larger range but since we are using a 4 quadrant inverse tangent to compute $\alpha \in [-\pi, \pi]$, then a range of $\beta \in [-\pi/2, \pi/2]$ is actually enough to completely describe all locations of the wind vector (similar to how we can describe any location on a sphere with lon $\in [-\pi, \pi]$ and lat $\in [-\pi/2, \pi/2]$.

With the rotation matrices $C_{b/s}(\alpha)$ and $C_{s/w}(\beta)$ established, we can generate the full rotation matrix to go from wind to body axis as

$$C_{b/w}(\alpha, \beta) = C_{b/s}(\alpha)\, C_{s/w}(\beta)$$

$$= \begin{pmatrix} \cos(\alpha) & 0 & -\sin(\alpha) \\ 0 & 1 & 0 \\ \sin(\alpha) & 0 & \cos(\alpha) \end{pmatrix} \begin{pmatrix} \cos(\beta) & -\sin(\beta) & 0 \\ \sin(\beta) & \cos(\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[∘]:=
```
Cbw[α_, β_] = Cbs[α].Csw[β];
Cbw[α, β] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\alpha]\ \text{Cos}[\beta] & -\text{Cos}[\alpha]\ \text{Sin}[\beta] & -\text{Sin}[\alpha] \\ \text{Sin}[\beta] & \text{Cos}[\beta] & 0 \\ \text{Cos}[\beta]\ \text{Sin}[\alpha] & -\text{Sin}[\alpha]\ \text{Sin}[\beta] & \text{Cos}[\alpha] \end{pmatrix}$$

Or we can rotate in the opposite direction (for example to verify against Matlab's implementation https://www.mathworks.com/help/aeroblks/directioncosinematrixbodytowind.html)

In[ ]:=
```
Cwb[α_, β_] = Transpose[Cbw[α, β]];
Cwb[α, β] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\alpha]\,\text{Cos}[\beta] & \text{Sin}[\beta] & \text{Cos}[\beta]\,\text{Sin}[\alpha] \\ -\text{Cos}[\alpha]\,\text{Sin}[\beta] & \text{Cos}[\beta] & -\text{Sin}[\alpha]\,\text{Sin}[\beta] \\ -\text{Sin}[\alpha] & 0 & \text{Cos}[\alpha] \end{pmatrix}$$

Transformation from V_a, α, β to u, v, w and vice versa

With $C_{b/w}(\alpha, \beta)$ established, we can derive a relationship between the set u, v, w and the set V_a, α, β by recognizing that $\overline{V}^w = (\, V_a \quad 0 \quad 0\,)^T$ and then rotating this to the body axis using

$$\overline{V}^b = C_{b/w}(\alpha, \beta)\,\overline{V}^w$$

In[ ]:=
$$\textbf{Vb = Cbw}[\alpha, \beta].\begin{pmatrix} \textbf{Va} \\ 0 \\ 0 \end{pmatrix};$$
```
Vb // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Va}\,\text{Cos}[\alpha]\,\text{Cos}[\beta] \\ \text{Va}\,\text{Sin}[\beta] \\ \text{Va}\,\text{Cos}[\beta]\,\text{Sin}[\alpha] \end{pmatrix}$$

So we see that the airspeed, V_a, and the aerodynamic angles are related to the body velocities (u, v, w) through

$$u = V_a \cos(\alpha) \cos(\beta) \qquad \textbf{(Eq.5.1)}$$
$$v = V_a \sin(\beta) \qquad \textbf{(Eq.5.2)}$$
$$w = V_a \sin(\alpha) \cos(\beta) \qquad \textbf{(Eq.5.3)}$$

The inverse transformation can be obtained by brute force solving Eq.5.1 - 3 for V_a, α, β

```
In[ ]:= Simplify[Solve[{u == Vb[[1, 1]], v == Vb[[2, 1]], w == Vb[[3, 1]]}, {Va, α, β}]]
```

Out[ ]=

$$\left\{\left\{Va \to -\sqrt{u^2+v^2+w^2},\ \alpha \to \boxed{\text{ArcTan}\left[-\frac{u}{\sqrt{u^2+w^2}}, -\frac{w}{\sqrt{u^2+w^2}}\right] + 2\pi c_1 \text{ if } c_1 \in \mathbb{Z}},\right.\right.$$

$$\left.\beta \to \boxed{\text{ArcTan}\left[\frac{\sqrt{u^2+w^2}}{\sqrt{u^2+v^2+w^2}}, -\frac{v}{\sqrt{u^2+v^2+w^2}}\right] + 2\pi c_2 \text{ if } c_2 \in \mathbb{Z}}\right\},$$

$$\left\{Va \to -\sqrt{u^2+v^2+w^2},\ \alpha \to \boxed{\text{ArcTan}\left[\frac{u}{\sqrt{u^2+w^2}}, \frac{w}{\sqrt{u^2+w^2}}\right] + 2\pi c_1 \text{ if } c_1 \in \mathbb{Z}},\right.$$

$$\left.\beta \to \boxed{\text{ArcTan}\left[-\frac{\sqrt{u^2+w^2}}{\sqrt{u^2+v^2+w^2}}, -\frac{v}{\sqrt{u^2+v^2+w^2}}\right] + 2\pi c_2 \text{ if } c_2 \in \mathbb{Z}}\right\},$$

$$\left\{Va \to \sqrt{u^2+v^2+w^2},\ \alpha \to \boxed{\text{ArcTan}\left[-\frac{u}{\sqrt{u^2+w^2}}, -\frac{w}{\sqrt{u^2+w^2}}\right] + 2\pi c_1 \text{ if } c_1 \in \mathbb{Z}},\right.$$

$$\left.\beta \to \boxed{\text{ArcTan}\left[-\frac{\sqrt{u^2+w^2}}{\sqrt{u^2+v^2+w^2}}, \frac{v}{\sqrt{u^2+v^2+w^2}}\right] + 2\pi c_2 \text{ if } c_2 \in \mathbb{Z}}\right\},$$

$$\left\{Va \to \sqrt{u^2+v^2+w^2},\ \alpha \to \boxed{\text{ArcTan}\left[\frac{u}{\sqrt{u^2+w^2}}, \frac{w}{\sqrt{u^2+w^2}}\right] + 2\pi c_1 \text{ if } c_1 \in \mathbb{Z}},\right.$$

$$\left.\left.\beta \to \boxed{\text{ArcTan}\left[\frac{\sqrt{u^2+w^2}}{\sqrt{u^2+v^2+w^2}}, \frac{v}{\sqrt{u^2+v^2+w^2}}\right] + 2\pi c_2 \text{ if } c_2 \in \mathbb{Z}}\right\}\right\}$$

Again, these solutions account for all possible conditions/permutations of V_a, α, and β (for example negative V_a).

For example, if we examine solution 4, we directly see that

$$V_a = \left(u^2 + v^2 + w^2\right)^{1/2} \qquad \textbf{\textit{(Eq.6.1)}}$$

We can further examine the solution 4 expression for α and obtain

$\alpha = \text{ArcTan}[x, y] = \text{ArcTan}[A, O]$ where A = adjacent, O = opposite (recall Mathematica reports ArcTan in order of x,y)

$$= \text{ArcTan}\left[\frac{u}{\sqrt{u^2+w^2}}, \frac{w}{\sqrt{u^2+w^2}}\right]$$

so $A = \frac{u}{\sqrt{u^2+w^2}}$, $O = \frac{w}{\sqrt{u^2+w^2}}$

$\tan(\alpha) = O/A$

$= \frac{w}{\sqrt{u^2+w^2}} \Big/ \frac{u}{\sqrt{u^2+w^2}}$

$\tan(\alpha) = w/u$

Or more safely

$\alpha = \text{atan2}(y,\ x) = \text{atan2}(w,\ u)$

In a similar fashion, we can examine the solution 4 expression for β to obtain

$\beta = \text{ArcTan}[x,\ y] = \text{ArcTan}[A,\ O]$

$= \text{ArcTan}\left[\frac{\sqrt{u^2+w^2}}{\sqrt{u^2+v^2+w^2}},\ \frac{v}{\sqrt{u^2+v^2+w^2}}\right]$

so $A = \frac{\sqrt{u^2+w^2}}{\sqrt{u^2+v^2+w^2}}$, $O = \frac{v}{\sqrt{u^2+v^2+w^2}}$

$\tan(\beta) = O/A$

$= \frac{v}{\sqrt{u^2+v^2+w^2}} \Big/ \frac{\sqrt{u^2+w^2}}{\sqrt{u^2+v^2+w^2}}$

$\tan(\beta) = \frac{v}{\sqrt{u^2+w^2}}$

Or more safely

$\beta = \text{atan2}(y,\ x) = \text{atan2}\left(v,\ \sqrt{u^2 + w^2}\right)$

Alternatively, for β we can use Eq.2 that we derived previously or solve Eq.5.2 for β which both yield

$\beta = \text{asin}(v/V_a)$

So one possible set of inverse mappings is

$V_a = \left(u^2 + v^2 + w^2\right)^{1/2}$ ***(Eq.6.1)***

$\alpha = \text{atan2}(y,\ x) = \text{atan2}(w,\ u)$ ***(Eq.6.2)***

$\beta = \text{asin}(v/V_a)$ or $\beta = \text{atan2}(y,\ x) = \text{atan2}\left(v,\ \sqrt{u^2 + w^2}\right)$ ***(Eq.6.3)***

Keep these relationships in mind as they will be useful when we discuss similarity transformations in a future lecture.

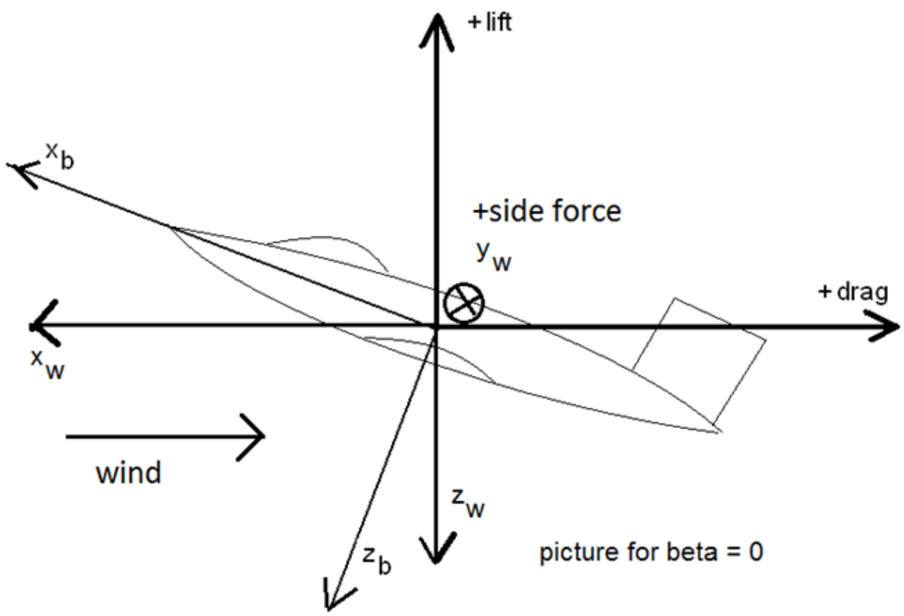
Aerodynamic Forces and Moments

Why are these frames useful? This is due to the typical definition of lift, drag, and side force.

Drag is typically defined as the component of aerodynamic force parallel to the relative wind
Lift is typically defined as the component of aerodynamic force perpendicular to the relative wind
Side force typically defined as the component of aerodynamic force perpendicular to the relative wind

In other words

$$\overline{F}_A^{\,w} = \begin{pmatrix} -D \\ \text{SF} \\ -L \end{pmatrix}^w$$



We now need to determine how these aerodynamic forces are a function of the states. Typically, we go to a wind tunnel or other analysis to obtain these functions.

The aerodynamic forces and moments are a function of many things. Using lift as an example, one model might be

$$L = L_{\text{static}} + L_{\text{unsteady}}$$

where $\quad L_{\text{static}}$ = lift due to steady aerodynamics

L_{unsteady} = lift due to unsteady aerodynamics

In a wind tunnel, the model is pitched and yawed to a certain orientation and data is taken once things have settled down. By definition, this is static forces.

$$L_{\text{static}} = L_{\text{static}}(\alpha,\ \beta,\ V,\ h,\ \delta_e,\ \delta_a,\ \delta_r,\ \delta_{\text{th}},\ \rho_{\text{air}},\ \mu_\infty\ ...) \qquad \text{(static lift)}$$

where ρ_{air} = air density (altitude dependence)
μ_∞ = fluid viscosity

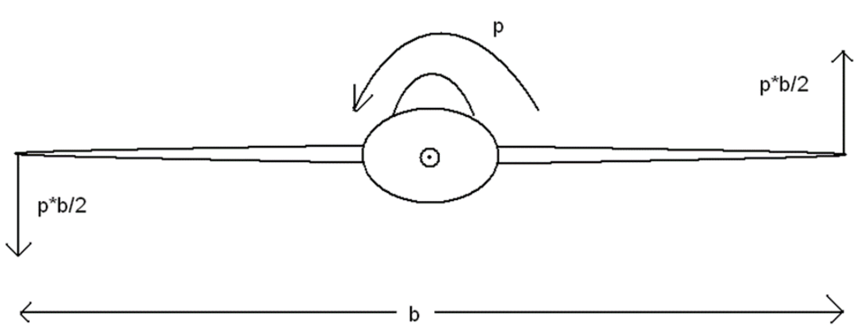
This is already a very complicated function. Can obtain this data from a wind tunnel. May need to be implemented with a look up table.

What about non-static forces? These can be broken into two categories

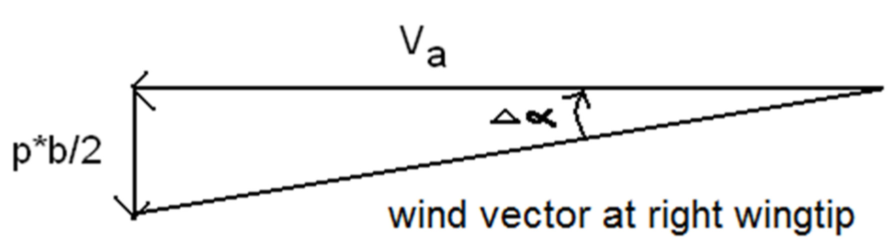
1. Aerodynamic forces or moments that can be modeled as linearly proportional to the angular rate that produced them
2. Unsteady aerodynamics

Case 1: Aerodynamic Derivatives

For example, roll damping. Consider aircraft with constant roll rate *p*



So this roll rate increases the angle of attack on the right wing and decreases the angle of attack on the left wing.



So we have

$$\tan(\Delta\,\alpha) = \frac{p\,b/2}{V_a}$$

If small relative to airspeed we have

$$\Delta\alpha \approx \frac{p\,b}{2\,V_a}$$

This increase in angle of attack tends to increase the lift force on the right wingtip.

Note that this generates a restoring moment which counters the roll motion, so this is called roll damping

$C_{l,p}$ = rolling moment coefficient due to roll rate

$$C_{l,p} = \alpha\,p$$

where α =roll damping coefficient (typically negative)

A similar analysis can be carried out for the other axes.

Case 2: Unsteady Aerodynamics

For example, dynamic lift. This is where rapidly pitched to a high angle of attack. Wing doesn't stall because it takes time for the separation to occur so there is a temporary increase in lift force.

These effects are much more difficult to quantify and model.

So total force and moment is given by combination of static and dynamic forces/moments

$$L = L_{\text{static}}(\alpha,\ \beta,\ V,\ h,\ \delta_e, \delta_a, \delta_r, \delta_{\text{th}},\ \ldots) + L_{\text{linear}}(p, q, r, V,\ \ldots) + L_{\text{unsteady}}\left(\dot{\theta}, \dot{\phi}, \dot{\psi},\ \ldots\right)$$

The dynamic effects are somewhat hard to come by and for most flight regimes, these are negligible. For our approximations, we just use

$$L = L_{\text{static}}(\alpha,\ \beta,\ M,\ h,\ \delta_e, \delta_a, \delta_r, \delta_{\text{th}},\ \ldots)$$

This is still a complex function, how do we get data for this function? Go to the wind tunnel.