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Lecture 02g
Time Domain Analysis: Performance Metrics for a First Order System

Lecture is on YouTube

The YouTube video entitled ‘Time Domain Analysis: Performance Metrics for a First Order System’ that covers this lecture is located at https://youtu.be/5FmXwsrHmpA.

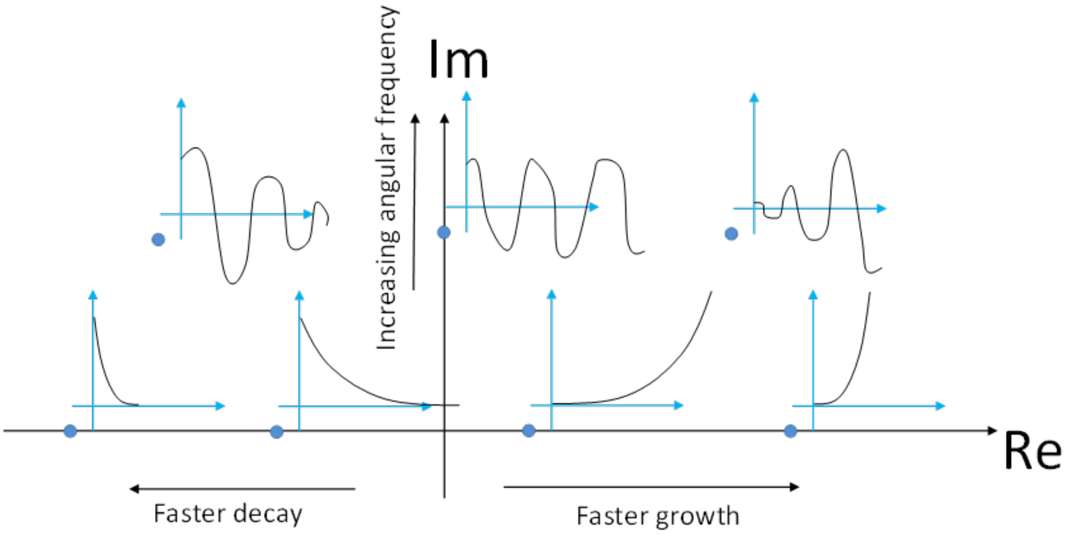
Outline

-Time Domain Analysis
-First Order Systems Step Response
    -Steady State Error
    -Settling Time
    -Rise Time
-First Order System Ramp Response
    -Steady State Error

Time Domain Analysis

Many physical processes can be modeled as dynamic systems (mechanical, electrical, thermal, etc.). In many cases, we would like to understand and predict the behavior of a dynamic system without needing to explicitly solve for the response as a function of time. Predicting the behavior of the system as a function of time is known as time domain analysis. Note that that we use time as the independent variable in time domain analysis. We will later contrast this with frequency domain analysis where we will predict the behavior of the system as input frequency changes.

If you have some familiarity with ordinary differential equations, you know that the response of the system was characterized by the poles of the system



So we have already done some time domain analysis on these systems where we can predict the response of the system without having to compute the response

$p = x + y\,i$ (pole of system)

more negative $x \Longleftrightarrow$ faster decay
more positive $x \Longleftrightarrow$ faster growth
larger $|\,y\,| \Longleftrightarrow$ faster oscillations

Transient-Response and Steady State

In general the response of the system is made up of transient and steady state behavior

In general, we can use the definitions of

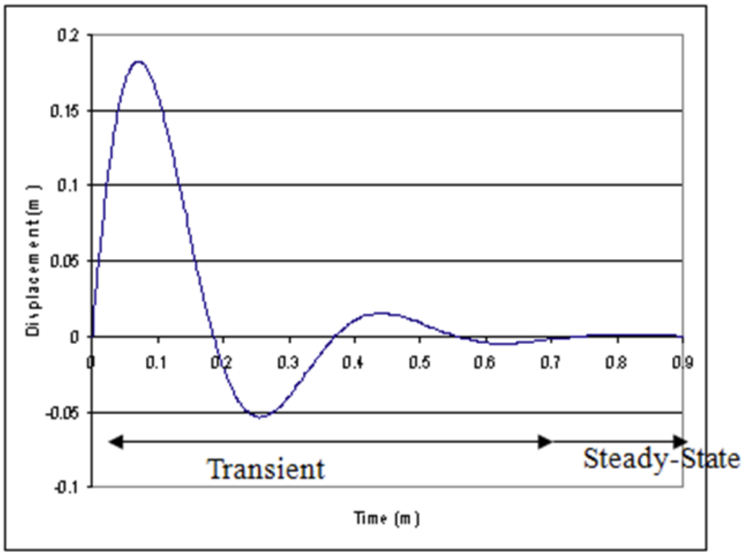
Transient Response: process of going from the initial to the final state
Steady State: how the system responds as $t \to \infty$.

As a design engineer, it is important to be able to predict how your system will response during start-up (transient response) as well as when it has reached steady state. We need to consider trade-offs

System runs in steady state for the majority of the time
System must be able to reach steady state without failing

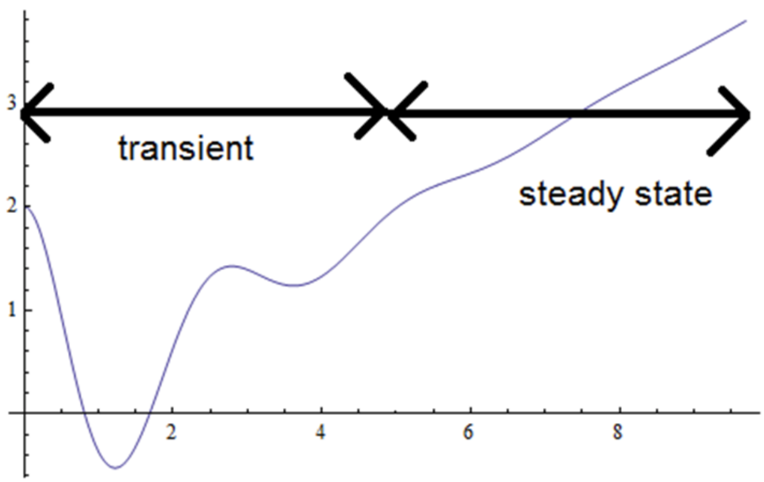
Talk about the example of a student trying to bungee jump off a second story balcony but did not count on transient behavior.

Example 1: Response due to purely initial conditions

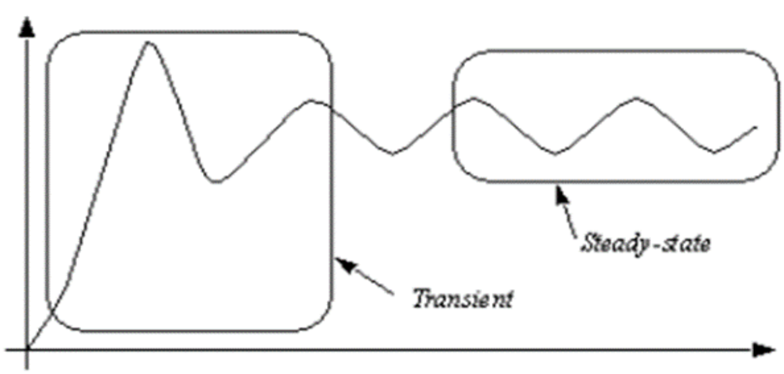


Example 2: Input a ramp with non-zero initial conditions

Recall the homework where we input a ramp input with nonzero initial conditions (hw02, p4, part b)



Example 3: Input a sin wave with non-zero initial conditions



Note that in this case, there is initial behavior of the signal, however at steady state, the system responds as a sin wave of the same magnitude and shifted frequency.

Consider a general linear, differential equation

$$x(t)^{(n)} + a_1\, x(t)^{(n-1)} + a_2\, x(t)^{(n-2)} + \ldots + a_{n-1}\, \dot{x}(t) + a_n\, x(t) = r(t)$$

where $\quad r(t)$ = input forcing function

Recall that the solution is comprised of two parts

$$x(t) = x_{\text{homogeneous}}(t) + x_{\text{particular}}(t)$$

Recall that we obtained the homogeneous solution by neglecting the forcing function

$$x(t)^{(n)} + a_1\, x(t)^{(n-1)} + a_2\, x(t)^{(n-2)} + \ldots + a_{n-1}\, \dot{x}(t) + a_n\, x(t) = 0$$

$x_{\text{homogneous}}(t) = x(t)$

Therefore, the solution of $x_{\text{homogeneous}}(t)$ is only the response to initial conditions. Provided the system is stable, these effects should die away as time goes to infinity. Therefore, $x_{\text{homogeneous}}(t)$ mostly influences the transient behavior of the system.

In a similar fashion, $x_{\text{particular}}(t)$ will mostly influence the steady state behavior of the system (if we shake a system with a sin wave, it will most likely go to a sin wave in steady state).

We can informally define the concept of steady state behavior as

$$x_{ss}(t) = \lim_{t\to\infty} x(t) \qquad \text{(steady state behavior)}$$

where this limit is taken in the sense that we seek the function that $x(t)$ becomes once transient effects die away.

1st Order Systems

Consider a general, 1st order differential equation.

$$a\, \dot{y}(t) + b\, y(t) = c\, r(t) \qquad \textbf{\textit{(Eq.A)}}$$

We can compute the transfer function

$$(a\, s\, + b)\, Y(s) = c\, R(s)$$

$$G(s) = \frac{Y(s)}{R(s)} = \frac{c}{a\, s + b} \qquad \textbf{\textit{(Eq.B)}}$$

```
G[s_] = c/(a s + b);
```

So the DC gain is given by

$$\text{DC gain } = G(0) = \tfrac{c}{b}$$

```
G[0]
```

$\frac{c}{b}$

Steady State Error in Response to Step

So the steady state error in response to a step function is given by

$$e_{ss} = (1 - G(0))\,A$$

$$= \left(1 - \tfrac{c}{b}\right) A$$

```
(1 - G[0]) A
```

$A\left(1 - \frac{c}{b}\right)$

Settling Time (T_s)

Settling time, T_s, is defined as

T_s =time required for system to reach (and remain) within a specified bound, δ, of the final value.

We can compute the settling time of a first order system in response to a step function for the system defined in Eq.B $\left(G(s) = \frac{c}{a\,s+b}\right)$

```
yStep[t_] = InverseLaplaceTransform[G[s] A/s, s, t] // Simplify
```

$$\frac{A\,c\left(1 - e^{-\frac{b\,t}{a}}\right)}{b}$$

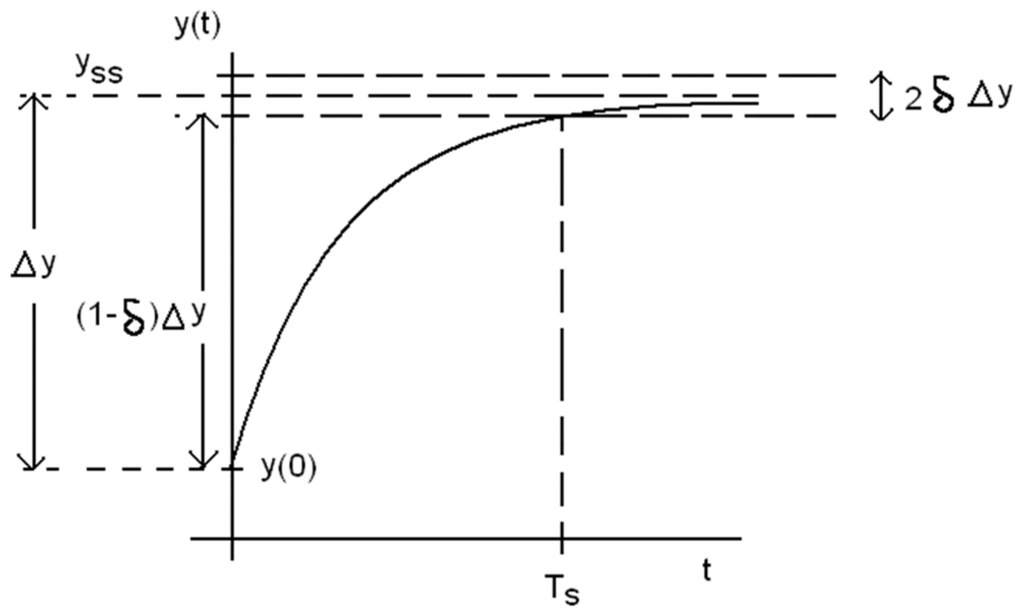
So we have

$$y(t) = \tfrac{A\,c}{b}\left(1 - e^{-\frac{b}{a}t}\right) \qquad \text{(response of 1st order system to a step function)}$$

Assuming that b and a are such that $b/a > 0$, the steady state value of the function is

$$y_{ss} = \tfrac{A\,c}{b}$$

We say that the system has settled once $y(t)$ is within δ of the final value. In other words



So we see that

$$y(T_s) = y(0) + (1-\delta)\,\Delta y \qquad \text{note: } \Delta y = y_{ss} - y(0)$$

$$= y(0) + (1-\delta)\,(y_{ss} - y(0))$$

$$= y(0) + y_{ss} - y(0) - \delta\, y_{ss} + \delta\, y(0)$$

$$= y_{ss} - \delta\, y_{ss} + \delta\, y(0)$$

$$= (1-\delta)\, y_{ss} + \delta\, y(0)$$

Assuming that the system starts at zero ($y(0) = 0$)

$$y(T_s) = (1-\delta)\, y_{ss} \qquad \text{recall: } y(t) = y(t) = \frac{A\,c}{b}\left(1 - e^{-\frac{b}{a}t}\right) \text{ and } y_{ss} = A\,\frac{c}{b}$$

$$\frac{A\,c}{b}\left(1 - e^{-\frac{b}{a}T_s}\right) = (1-\delta)\,A\,\frac{c}{b}$$

$$1 - e^{-\frac{b}{a}T_s} = 1 - \delta$$

$$e^{-\frac{b}{a}T_s} = \delta$$

$$T_s = -\frac{a}{b}\ln(\delta)$$

We will investigate in a later example how the initial conditions do not change the calculation of settling time.

Rise Time (T_r)

The rise time is defined as the time it takes for the signal to reach a certain percentage of its steady state value. Since this system is first order, this is the same as the settling time. For example, if we calculate the rise time between 0 to 90%, we can simply use the equation for settling time with $\delta = 1 - 0.9 = 0.1$

T_r = time elapsed before signal first reaches specified percentage of final value

$$T_r = -\frac{a}{b} \ln(1 - \delta_r)$$

where δ_r = rise percentage (in range (0,1))

2. 1st Order System Response to a Ramp Function

Now let's investigate the response of a first order system to a ramp input

$$r(t) = \begin{cases} 0 & t < 0 \\ A\,t & t \geq 0 \end{cases}$$

Again, assuming the general, 1st order system, we compute the response to this ramp as

```
yRamp[t_] = Collect[Expand[InverseLaplaceTransform[G[s] A/s^2, s, t]], Exp[-t / τ]]
```

$$-\frac{a\,A\,c}{b^2} + \frac{a\,A\,c\,e^{-\frac{b\,t}{a}}}{b^2} + \frac{A\,c\,t}{b}$$

So we have

$$y(t) = \frac{A\,c}{b}\,t - \frac{A\,a\,c}{b^2} + \frac{A\,a\,c}{b^2}\,e^{-\frac{b}{a}\,t} \qquad \text{(response of 1st order system to a ramp)}$$

Assuming the constants a and b are such that $\frac{b}{a} > 0$, the exponential term will die away and at steady state we are left with

$$y_{\text{ss}}(t) = \frac{A\,c}{b}\,t - \frac{A\,a\,c}{b^2}$$

So we see that there is a steady state error since the output $y(t)$ does not approach the input function $r(t)$. Let's calculate the steady state error in this case

$$e_{\text{ss}} = r(t) - y_{\text{ss}}(t) \qquad \text{recall: } y_{\text{ss}}(t) = \frac{A\,c}{b}\,t - \frac{A\,a\,c}{b^2}$$

$$= A\,t - \left(A\,\frac{A\,c}{b}\,t - \frac{A\,a\,c}{b^2}\right)$$

```
Collect[A t - (A c)/b t - (A a c)/b^2, t]
```

$-\frac{a\,A\,c}{b^2} + \left(A - \frac{A\,c}{b}\right) t$

So we have

$$e_{ss} = \left(A - A\,\frac{c}{b}\right) t - \frac{a\,A\,c}{b^2} \qquad \text{(in response to a ramp)}$$

Note that the steady state error grows linearly with time.

It is interesting to note that if the $\frac{c}{b} = 1$, then the steady state error is constant. Note that DC gain is $\frac{c}{b}$ so another way to look at this is if the DC gain is unity, then the steady state error is constant.

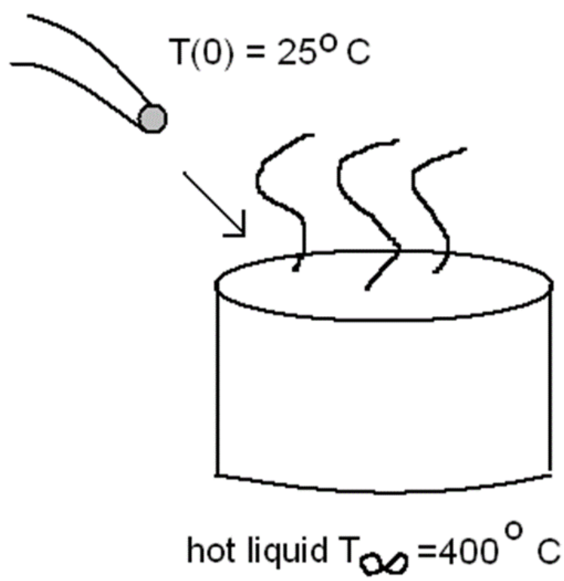
```
G[0]
```

$\frac{c}{b}$

Example: Thermocouple Response

<u>Situation 1: Step Input</u>

Consider a situation where we drop a thermocouple with an initial temperature is dropped into a pot of hot liquid.



For this specific thermocouple, let us assume

$$k = 20\,\frac{W}{m\,K} \qquad \text{(material's conductivity)}$$
$$C = 400\,\frac{J}{\text{kg}\,K} \qquad \text{(specific heat)}$$
$$\rho = 8500\,\frac{\text{kg}}{m^3} \qquad \text{(density)}$$

$d = 3$ mm (diameter)
$T(0) = 25^\circ C = 298^\circ K$ (initial temperature)

```
kgiven = 20;
Cgiven = 400;
ρgiven = 8500;
dgiven = 3 * 10^-3;
T0given = 298;
```

For the hot liquid:

$h = 400 \frac{W}{m^2 K}$ (convective heat transfer coefficient)
$T_\infty = 400^\circ C = 673^\circ K$ (liquid temperature)

```
hgiven = 400;
Tinfinity = 673;
```

Let's assume that the thermocouple is a sphere. The volume and surface area are given by

$$V = \frac{4}{3} \pi \left(\frac{d}{2}\right)^3$$

$$A_s = 4 \pi \left(\frac{d}{2}\right)^2$$

```
Vgiven = 4/3 π (dgiven/2)^3;
Asgiven = 4 π (dgiven/2)^2;
Vgiven // N
Asgiven // N
```

1.41372×10^{-8}

0.0000282743

Let's first compute the characteristic length, L_c

$$L_c = \frac{V}{A_s}$$

$$= \frac{\frac{4}{3} \pi \left(\frac{d}{2}\right)^3}{4 \pi \left(\frac{d}{2}\right)^2}$$

$$= \frac{1}{3} \left(\frac{d}{2}\right)$$

$$L_c = 1 \times 10^{-3} \text{ m}$$

Let's first check B_i to see if lumped capacitance model is valid.

$$B_i = \frac{h\,L_c}{k}$$

$$= \frac{\left(400\,\frac{W}{m^2\,K}\right)\times\left(1\times10^{-3}\,m\right)}{\left(20\,\frac{W}{m\,K}\right)}$$

$$B_i = 0.02$$

Since $B_i < 0.1$, lumped capacitance model is valid. So the governing equation becomes

$$\frac{d\,E_s}{d\,t} = \dot{E}_{\text{in}} - \dot{E}_{\text{out}}$$

Since $T_\infty > T(0)$, the thermocouple is being heated, so $\dot{E}_{\text{out}} = 0$

$$\frac{d\,E_s}{d\,t} = \dot{E}_{\text{in}}$$

Furthermore, we assume that convection is the only mode of heat transfer into the thermocouple

$$\rho\,V\,C\,\frac{d\,T(t)}{d\,t} = h\,A_s(T_\infty(t) - T(t)) \quad \text{recall: } h = \text{convective heat transfer coefficient } \left(\frac{W}{m^2\,K}\right)$$

$$\frac{d\,T(t)}{d\,t} = \frac{h\,A_s}{\rho\,V\,C}\,(T_\infty(t) - T(t))$$

$$\frac{d\,T(t)}{d\,t} + \beta\,T(t) = \beta\,T_\infty(t)$$

where $\beta = \frac{h\,A_s}{\rho\,V\,C} = \frac{\left(400\,\frac{W}{m^2\,K}\right)\left(0.0000282743\,m^2\right)}{\left(8500\,\frac{\text{kg}}{m^3}\right)\times\left(1.41372\times10^{-8}\,m^3\right)\times\left(400\,\frac{J}{\text{kg}\,K}\right)} = \frac{4}{17}$ (units of $1/\text{sec}$)

$$\beta\textbf{given} = \frac{\textbf{hgiven Asgiven}}{\rho\textbf{given Vgiven Cgiven}}$$

$$\frac{4}{17}$$

Let compute the units of β

$$\frac{\left(\frac{W}{m^2\,K}\right)\left(m^2\right)}{\left(\frac{\text{kg}}{m^3}\right)\left(m^3\right)\left(\frac{J}{\text{kg}\,K}\right)} = \frac{\left(\frac{W}{K}\right)}{(\text{kg})\left(\frac{J}{\text{kg}\,K}\right)} \qquad \text{recall: } W = J/s$$

$$= \frac{\left(\frac{J}{s\,K}\right)}{\left(\frac{J}{K}\right)}$$

$$= \frac{K}{J}\;\frac{J}{s\,K}$$

$= 1/s$

So we see that the units of β are 1/s

We can find the response of the system using Laplace that includes initial conditions

$$s\,T(s) - T(0) + \beta\,T(s) = \beta\,T_\infty(s) \qquad \text{note:} \qquad T_\infty(s) = \frac{T_\infty}{s}$$

$$s\,T(s) - T(0) + \beta\,T(s) = \beta\,\frac{T_\infty}{s}$$

$$(s + \beta)\,T(s) = \beta\,\frac{T_\infty}{s} + T(0)$$

$$T(s) = \beta\,\frac{T_\infty}{(s+\beta)\,s} + \frac{T(0)}{(s+\beta)}$$

$$T(t) = T_\infty + (T(0) - T_\infty)\,e^{-\beta t} \qquad \textbf{\textit{(Eq.A.1)}}$$

```
TwithICs[t_] = Tinfty + (T0 - Tinfty) Exp[-β t];
```

We can also calculate the transfer function of the system.

$$s\,T(s) + \beta\,T(s) = \beta\,T_\infty(s)$$

$$G(s) = \frac{T(s)}{T_\infty(s)} = \frac{\beta}{s+\beta}$$

```
G[s_] = β/(s + β);
```

Let us calculate the DC gain of this system.

$$\text{DC gain} = G(0)$$

$$= \frac{\beta}{\beta}$$

$$\text{DC gain} = 1$$

So we expect no steady state error in response to a step input. Recall that the steady state error in response to a step function is

$$e_{ss} = (1 - G(0))\,A$$

$$= (1 - 1)\,A$$

$$e_{ss} = 0$$

This means that eventually, the system (thermocouple) will reach the temperature of the liquid.

We can calculate the settling time. We can use $\delta = 0.02$ (meaning the system is settled once it is within 2% of the final value).

Recall that previously we showed that for an equation of motion of the form $a\,\dot{y}(t) + b\,y(t) = c\,r(t)$ has expressions of

$$T_s = -\frac{a}{b}\ln(\delta)$$

$$T_r = -\frac{a}{b}\ln(1-\delta_r)$$

We can apply these results to our situation. Recall that our equation of motion is $\frac{d\,T(t)}{d\,t} + \beta\,T(t) = \beta\,T_\infty(t)$

$$T_s = -\frac{a}{b}\ln(\delta)$$

$$= -\frac{1}{\beta}\ln(\delta) \qquad \text{recall: } \beta = \frac{h\,A_s}{\rho\,V\,C}$$

$$= -\frac{\rho\,V\,C}{h\,A_s}\ln(\delta)$$

$$= -\frac{\left(8500\,\frac{\text{kg}}{m^3}\right)\times\left(1.41372\times10^{-8}\,m^3\right)\times\left(400\,\frac{J}{\text{kg}\,K}\right)}{\left(400\,\frac{W}{m^2\,K}\right)\left(0.0000282743\,m^2\right)}\ln(0.02)$$

```
Ts = -1/β Log[0.02] /. {β → βgiven}
```

```
16.6261
```

So the settling time is

$$T_s = 16.63 \text{ seconds}$$

How about the rise time to reach 90% of the final value?

$$T_r = -\frac{a}{b}\ln(1-\delta_r)$$

$$= -\frac{1}{\beta}\ln(1-\delta_r) \qquad \text{recall: } \beta = \frac{h\,A_s}{\rho\,V\,C}$$

$$= -\frac{\rho\,V\,C}{h\,A_s}\ln(1-\delta_r)$$

$$= -\frac{\left(8500\,\frac{\text{kg}}{m^3}\right)\times\left(1.41372\times10^{-8}\,m^3\right)\times\left(400\,\frac{J}{\text{kg}\,K}\right)}{\left(400\,\frac{W}{m^2\,K}\right)\left(0.0000282743\,m^2\right)}\ln(1-0.9)$$

```
Trise = -1/β Log[1 - 0.9] /. {β → βgiven}
```

```
9.78599
```

So the rise time is

$T_r = 9.786$ seconds

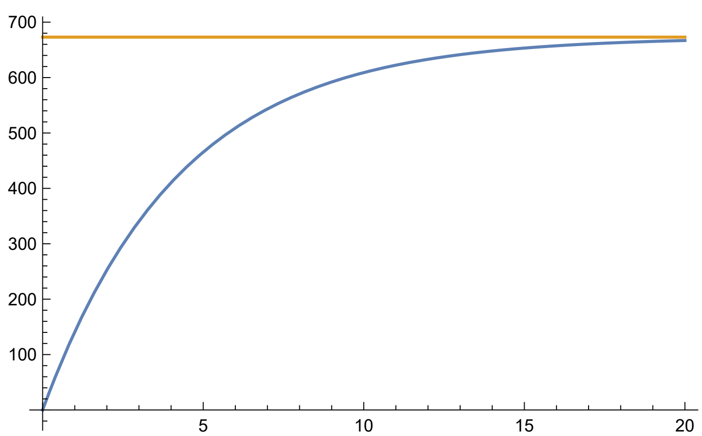
Recall that β had units of 1/seconds so the units of rise and settling time work out to units of seconds, which is what we expect for a settling time.

Let's verify these results. Note that earlier we defined settling time as the time to settle from zero initial conditions. If we would like to predict the behavior of the system using the transfer function approach, we have the temperature in the Laplace domain given by

$T(s) = G(s)\,\frac{673}{s}$

```
Agiven = 673;
T[t_] = InverseLaplaceTransform[G[s] Agiven/s, s, t]
Plot[{T[t] /. {β → βgiven}, Agiven}, {t, 0, 20}, PlotRange → All]
```

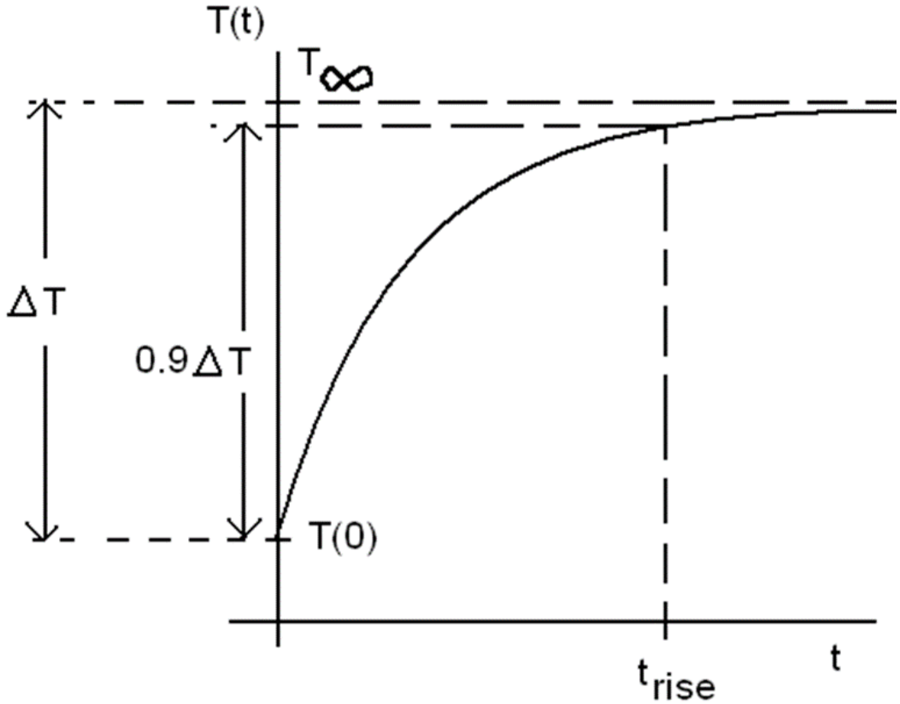
$673\left(\frac{1}{\beta} - \frac{e^{-t\beta}}{\beta}\right)\beta$



From the plot, we see that a rise time of $T_r = 9.786$ seconds and settling time of $T_s = 16.63$ seconds appear correct.

Note that with this previous analysis, the transfer function assumed that the initial condition was 0. However, in our example, we see that there is a non-zero initial condition. We can still conduct our same analysis with the non-zero initial condition, we just have to be careful of how we define the change in signal.

Recall that the time constant tells us how long it takes the system to reach ≈63% of the change in value. If the signal starts at 0, this is the same as 63% of the final value. A similar definition follows for rise and settling time. For example, a more general definition of rise time is the time required for the signal to change by 90%.



So we see we are really interested to see when the signal reaches $T(0) + 0.90\,\Delta T$. This corresponds to 90% <u>change</u> of the signal

$T(t_{\text{rise}}) = T(0) + 0.9\,\Delta T$ note: $\Delta T = T_\infty - T(0)$

$T(t_{\text{rise}}) = T(0) + 0.9\,(T_\infty - T(0))$ recall: $T(t) = T_\infty + (T(0) - T_\infty)\,e^{-\beta t}$

$$T_\infty + (T(0) - T_\infty)\,e^{-\beta t_{\text{rise}}} = T(0) + 0.9\,(T_\infty - T(0))$$

$$T_\infty + (T(0) - T_\infty)\,e^{-\beta t_{\text{rise}}} = 0.1\,T(0) + 0.9\,T_\infty$$

$$e^{-\beta t_{\text{rise}}} = \frac{0.1\,T(0) - 0.1\,T_\infty}{T(0) - T_\infty}$$

```
(1/10 T0 - 1/10 Tinfty)/(T0 - Tinfty) // Simplify
```

```
1/10
```

$$e^{-\beta t_{\text{rise}}} = \frac{1}{10}$$

$$-\beta\,t_{\text{rise}} = \ln\!\left(\frac{1}{10}\right)$$

$$\beta\, t_{\text{rise}} = \ln(10)$$

$$t_{\text{rise}} = \frac{\ln(10)}{\beta}$$

So we obtain

$$t_{\text{rise}} = T_r = \frac{\ln(10)}{\beta}$$

```
triseWithICs = Log[10]/β;
```

Notice that the initial and final temperature do not matter and we obtain the same rise time.

```
triseWithICs /. {β → βgiven} // N
Trise == triseWithICs /. {β → βgiven}
```

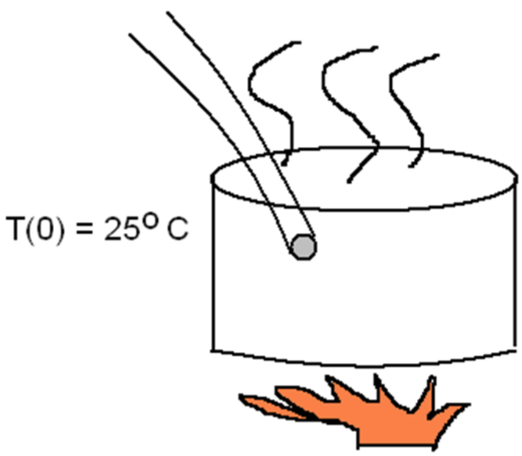
```
9.78599
```

```
True
```

So in summary, the initial condition does not affect the rise time in a linear system as long as the rise time is defined in terms of amount of change in signal.

Situation 2: Ramp Input

Let's consider a situation where we heat up a pot with a thermocouple in it. The thermocouple has an initial temperature of $T(0) = T_0 = 25 + 273 = 298\,K$ and the pot is being heated at a rate of 5 deg/s.




```
Agiven = 5;
```

Recall that we showed that in response to a ramp

$$e_{ss} = \left(A - A\frac{c}{b}\right)t - \frac{a\,A\,c}{b^2} \qquad \text{(in response to a ramp)}$$

So in our case $A = 5$, $a = 1$, $b = \beta = 4/17$, $c = \beta = 4/17$, so we have

```
essRamp = ((A - A c/b) t - (a A c)/b^2) /. {A → Agiven, a → 1, b → βgiven, c → βgiven}
essRamp // N
```

$-\frac{85}{4}$

-21.25

So we see that at steady state, we expect the thermocouple temperature to be 21.25 degrees cooler than the water actually is.

We can confirm this by direct calculation. Recall the governing equation was

$$\frac{d\,T(t)}{d\,t} + \beta\,T(t) = \beta\,T_\infty(t)$$

where $\beta = \frac{h\,A_s}{\rho\,V\,C}$

If the liquid temperature is rising linearly at rate A, we have

$$\frac{d\,T(t)}{d\,t} + \beta\,T(t) = \beta\,A\,t$$

$$(s\,T(s) - T(0)) + \beta\,T(s) = \frac{\beta\,A}{s^2}$$

$$T(s) = \frac{\frac{\beta\,A}{s^2} + T(0)}{s+\beta}$$

```
T[t_] = InverseLaplaceTransform[(β A/s^2 + T0)/(s + β), s, t]
```

$A\,t - \frac{A}{\beta} + \frac{e^{-t\,\beta}\,(A + T0\,\beta)}{\beta}$

So we see that the steady state behavior is

$$T_{ss}(t) = A\,t - \frac{A}{\beta}$$

So we see that there is a steady state error since the output $T(t)$ does not approach the input function $T_\infty(t)$. Let's calculate the steady state error in this case

$$e_{ss} = T_\infty(t) - T_{ss}(t) \qquad \text{recall: } T_{ss}(t) = A\,t - \frac{A}{\beta}$$

$$= A\,t - \left(A\,t - \frac{A}{\beta}\right)$$

$$e_{ss} = \frac{A}{\beta} = \frac{5\,\text{deg}/s}{4/17}$$

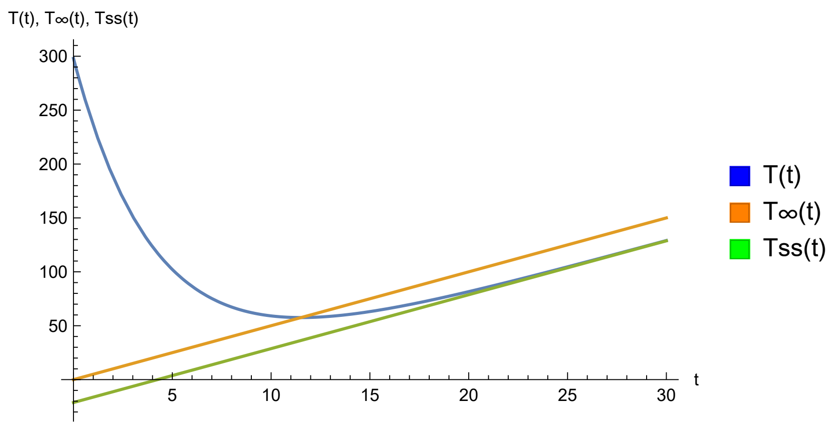
```
essCheck = Agiven / βgiven // N
```

```
21.25
```

```
T0given = 25 + 273;

Legended[
 Plot[{T[t] /. {A → Agiven, β → βgiven, T0 → T0given}, Agiven t, Agiven t - Agiven / βgiven},
  {t, 0, 30}, PlotRange → All, AxesLabel → {"t", "T(t), T∞(t), Tss(t)"}],

 (*Add legend information*)
 SwatchLegend[{Blue, Orange, Green}, {"T(t)", "T∞(t)", "Tss(t)"}]
]
```



So we see that the thermocouple will not be able to track the temperature of the pot and will constantly be lagging behind by 21.25^o.