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Lecture 04 b
The Flat Earth Equations of Motion

Lecture is on YouTube

The YouTube video entitled 'The Flat Earth Equations of Motion' that covers this lecture is located at https://youtu.be/JhwYe7kOJPI .

Outline

-Translational Equations of Motion (Translational Velocity)
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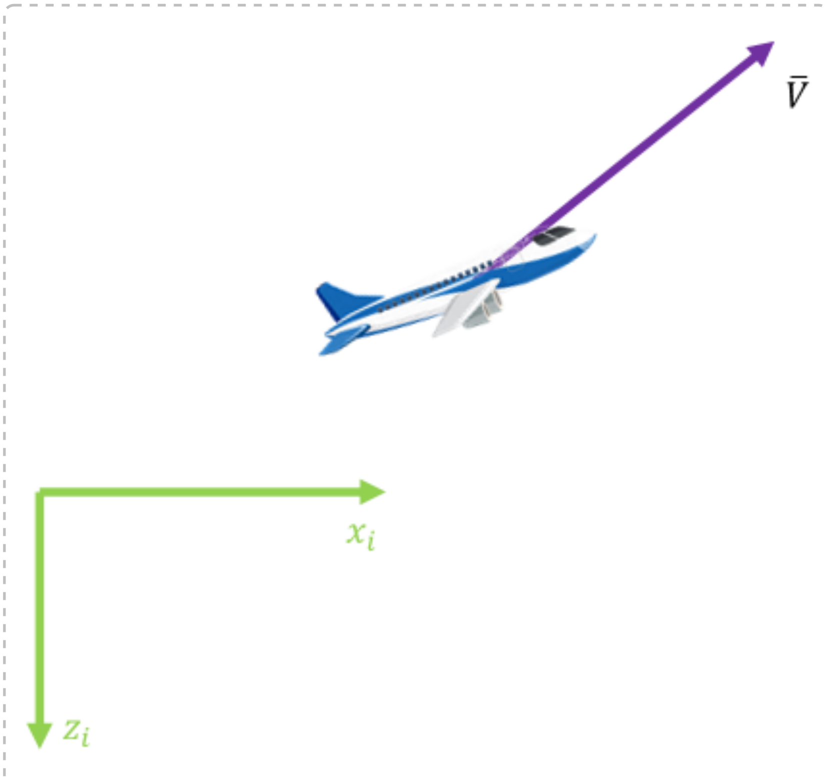
Translational Equations of Motion (Translational Velocity)

We now consider developing equations of motion for an object in an inertial reference frame.



For translation, we can make use of Newton's 2nd law to derive the equation of motion for translation. We can write

$$\bar{F} = \frac{d}{dt}\,|_i\left[m\bar{V}\right]$$

If we assume that the the mass is not changing appreciably (for example and electrically powered aircraft or glider), we can write

$$\bar{F} = m\ {}^i\dot{\bar{V}} \qquad \textbf{\textit{(Eq.1)}}$$

At this point, this is a valid equation of motion. If we want to express this in the inertial frame we can write

$$\bar{F}^i = m\ {}^i\dot{\bar{V}}^i$$

We can write this as

$${}^i\dot{\bar{V}}^i = \frac{1}{m}\bar{F}^i$$

So if the state is

$$\overline{V}^i = \begin{pmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{z}_i \end{pmatrix} = \frac{d}{dt}\Big|_i \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}$$

then this is a valid set of equation of motion.

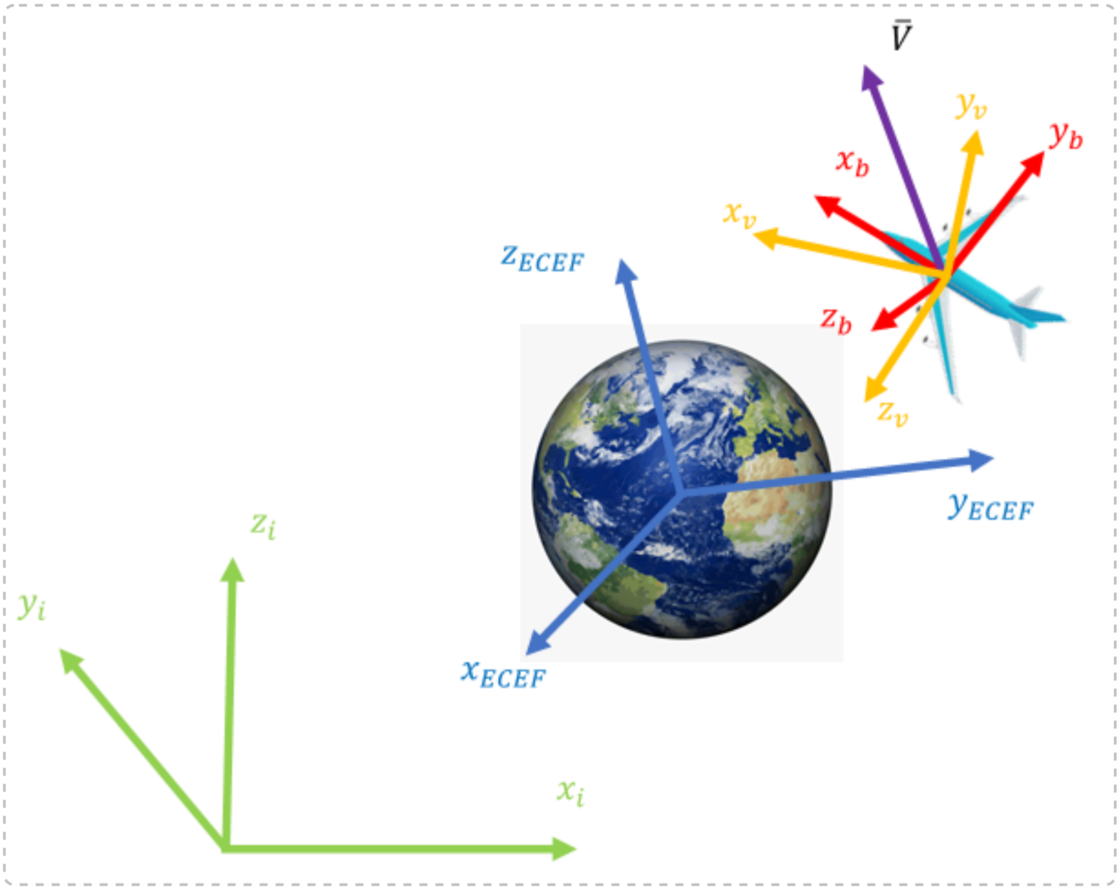
Note that with this formulation, we made the implicit assumption that

$$\overline{V} = \frac{d}{dt}\Big|_i \overline{r}$$

The problem is that in this formulation, the velocity vector derivative is taken from the perspective of the inertial frame and is expressed in the inertial frame. In our situation, it will be more desirable to formulate the velocity vector derivative from the perspective of the body frame and expressed in the body frame. In other words, we want

$$\overline{V}^b = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

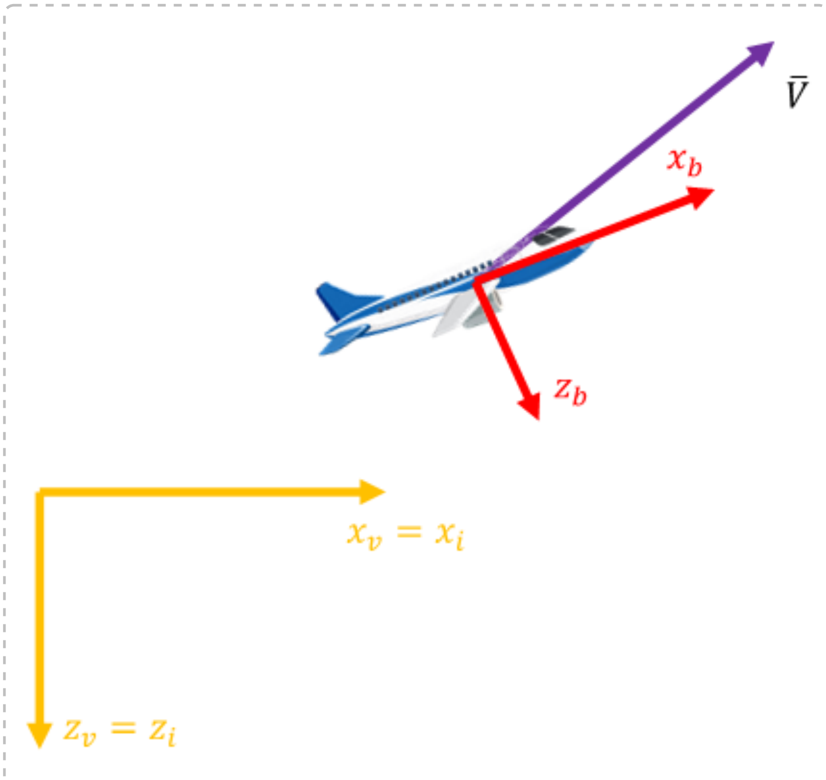
Furthermore, we note that the various frames are somewhat complicated as shown below



Note that in this general case is complicated as often we want to write the dynamics w.r.t. to the vehicle carried NED frame (F_v) frame which is rotating w.r.t. the inertial frame (F_i).

To simplify, let us assume that F_v is an inertial frame (in other words, we assume earth is not rotating and the vehicle is not moving fast enough such that F_v is not rotating).

For the remainder of the discussion, we assume that $F_v = F_i$ so we will use the subscripts and superscripts of v and i interchangeably and equivalently.



From Eq.1, we have $\overline{F} = m\ {}^i\dot{\overline{V}}$. From The Equation of Coriolis we know that we can rewrite ${}^i\dot{\overline{V}}$ as

$$ {}^i\dot{\overline{V}} = {}^b\dot{\overline{V}} + \overline{\omega}_{b/i} \times \overline{V} \qquad \textbf{\textit{(Eq.2)}}$$

Substituting Eq.2 into Eq.1 yields

$$\overline{F} = m\left({}^b\dot{\overline{V}} + \overline{\omega}_{b/i} \times \overline{V}\right)$$

Rearranging, we have

$${}^b\dot{\overline{V}} = \frac{1}{m}\overline{F} - \overline{\omega}_{b/i} \times \overline{V}$$

If we express this in F_b (the body frame), we have

$$ {}^{b}\dot{\overline{V}}^{b} = \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix}^{b} = \frac{1}{m}\overline{F}^{b} - \overline{\omega}_{b/i}{}^{b} \times \overline{V}^{b} \qquad \textbf{(Eq.3)} $$

Rotational Equations of Motion (Angular Velocity)

If the body can rotate in addition to translate, we need to write equations of motion for the angular velocity as well.

We applied Newton's second law of translation to obtain 3 equations so now let us apply Newton's second law of rotation to obtain another 3.

$$ \overline{M} = \frac{d}{dt}\Big|_{i} \overline{H} \qquad \textbf{(Eq.4)} $$

where $\overline{H} = I_b\,\overline{\omega}_{b/i}$

We need to represent inertia matrix in the body frame. This is one of the main reasons why we need to define the body frame so that we have a frame where the inertia does not change with respect to it.

So we have

$$ \overline{M} = \frac{d}{dt}\Big|_{i} \left[I_b\,\overline{\omega}_{b/i}\right] $$

This is slightly complicated because I_b is not a constant when taking a derivative with respect to F_i. This is because the as the aircraft moves, its is not necessarily aligned with the F_i axis all the time and therefore, from the perspective of F_i, the inertia of the aircraft about the F_i axes is changing constantly. Therefore, we employ the Coriolis theorem again to rewrite the right side of the equation

$$ \overline{M} = \frac{d}{dt}\Big|_{b} \left[I_b\,\overline{\omega}_{b/i}\right] + \overline{\omega}_{b/i} \times I_b\,\overline{\omega}_{b/i} $$

Here, I_b is constant when seen from F_b, so it can come outside the derivative operator.

$$ = I_b \frac{d}{dt}\Big|_{b} \left[\overline{\omega}_{b/i}\right] + \overline{\omega}_{b/i} \times I_b\,\overline{\omega}_{b/i} $$

$$ = I_b \; {}^{b}\dot{\overline{\omega}}_{b/i} + \overline{\omega}_{b/i} \times I_b\,\overline{\omega}_{b/i} $$

Rearranging yields

$$ {}^{b}\dot{\overline{\omega}}_{b/i} = I_b^{-1}\left(\overline{M} - \overline{\omega}_{b/i} \times I_b\,\overline{\omega}_{b/i}\right) $$

Once again, if we express this in F_b, we have

$$^{b}\dot{\overline{\omega}}_{b/i}{}^{b} = I_b^{-1}\left(\overline{M}^b - \overline{\omega}_{b/i}{}^{b} \times I_b\,\overline{\omega}_{b/i}{}^{b}\right) \qquad \textbf{\textit{(Eq.5)}}$$

Rotational Equations of Motion (Angular Position)

The equations of motion for angular position were detailed in the previous YouTube video entitled 'Computing Euler Angles: The Euler Kinematical Equations and Poisson's Kinematical Equations' (https://youtu.be/9GZjtfYOXao)

The resulting equations are

$$\dot{\Phi} = H(\Phi)\,\overline{\omega}_{b/v}{}^{b} \qquad \textbf{\textit{(Eq.6)}}$$

where $H(\Phi) = \begin{pmatrix} 1 & \tan(\theta)\sin(\phi) & \tan(\theta)\cos(\phi) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)/\cos(\theta) & \cos(\phi)/\cos(\theta) \end{pmatrix}$

$\Phi = \begin{pmatrix} \phi & \theta & \psi \end{pmatrix}^T$

Translational Equations of Motion (Translational Position)

The equations of motion for translational position were detailed in the previous YouTube video entitled 'The Navigation Equations: Computing Position North, East, and Down' (https://youtu.be/XQZV-YZ7asE).

The resulting equations are

$$\overline{V}^v = C_{v/b}(\Phi)\,\overline{V}^b$$

$$\begin{pmatrix} \dot{P}_N \\ \dot{P}_E \\ \dot{P}_D \end{pmatrix} = C_{v/b}(\Phi) \begin{pmatrix} u \\ v \\ w \end{pmatrix} \qquad \textbf{\textit{(Eq.7)}}$$

where $C_{v/b}(\Phi) = C_{b/2}(\phi)\,C_{2/1}(\theta)\,C_{1/v}(\psi)$

Complete Equations of Motion

Our goal is to develop a set of equations of the form

$$\dot{\overline{x}} = f(\overline{x}, \overline{u})$$

So at this point, we are in a position to develop the complete set of Flat-Earth equations of motion for the rigid body.

We first chose the state vector as

$$\overline{x} = \begin{pmatrix} u \\ v \\ w \\ p \\ q \\ r \\ \phi \\ \theta \\ \psi \\ P_N \\ P_E \\ P_D \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \\ x_{10} \\ x_{11} \\ x_{12} \end{pmatrix} = \begin{pmatrix} \text{velocity in body } x \text{ axis} \\ \text{velocity in body } y \text{ axis} \\ \text{velocity in body } z \text{ axis} \\ \text{angular rate about body } x \text{ axis} \\ \text{angular rate about body } y \text{ axis} \\ \text{angular rate about body } z \text{ axis} \\ \text{angular position 1} \\ \text{angular position 2} \\ \text{angular position 3} \\ \text{position north} \\ \text{position east} \\ \text{position down} \end{pmatrix} \qquad \textbf{\textit{(Eq.8)}}$$

Using vector notation, we can write (recall from a previous discussion that Φ is not a proper vector but rather a collection of 3 angles, each is relevant in a different frame).

$$\overline{V}^b = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\overline{\omega}_{b/i}{}^b = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$\Phi = \begin{pmatrix} \phi \\ \theta \\ \psi \end{pmatrix}$$

$$\overline{P}_{\text{NED}}{}^v = \begin{pmatrix} P_N \\ P_E \\ P_D \end{pmatrix}$$

So we have

$$\overline{x} = \begin{pmatrix} \overline{V}^b \\ \overline{\omega}_{b/i}{}^b \\ \Phi \\ \overline{P}_{\text{NED}}{}^v \end{pmatrix}$$

Translational Equations of Motion (Translational Velocity)

Recall Eq.3 was given as

$$ {}^{b}\dot{\overline{V}}^{b} = \frac{1}{m}\overline{F}^{b} - \overline{\omega}_{b/i}{}^{b} \times \overline{V}^{b} $$

$$ \begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \frac{1}{m}\overline{F}^{b} - \begin{pmatrix} p \\ q \\ r \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} $$

$$ \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \frac{1}{m}\overline{F}^{b} - \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} $$

Rotational Equations of Motion (Angular Velocity)

Recall Eq.5 was given as

$$ {}^{b}\dot{\overline{\omega}}_{b/i}{}^{b} = I_b{}^{-1}\left(\overline{M}^{b} - \overline{\omega}_{b/i}{}^{b} \times I_b\,\overline{\omega}_{b/i}{}^{b}\right) $$

$$ \begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} \end{pmatrix}^{b} = I_b{}^{-1}\left(\overline{M}^{b} - \begin{pmatrix} p \\ q \\ r \end{pmatrix} \times I_b \begin{pmatrix} p \\ q \\ r \end{pmatrix}\right) $$

$$ \begin{pmatrix} \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{pmatrix} = I_b{}^{-1}\left(\overline{M}^{b} - \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix} \times I_b \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix}\right) $$

Rotational Equations of Motion (Angular Position)

Recall Eq.6 was given as

$$ \dot{\Phi} = H(\Phi)\,\overline{\omega}_{b/v}{}^{b} $$

$$ \begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = H(\Phi) \begin{pmatrix} p \\ q \\ r \end{pmatrix} $$

$$\begin{pmatrix} \dot{x}_7 \\ \dot{x}_8 \\ \dot{x}_9 \end{pmatrix} = H(x_7, x_8, x_9) \begin{pmatrix} x_4 \\ x_5 \\ x_6 \end{pmatrix}$$

where $H(x_7, x_8, x_9) = \begin{pmatrix} 1 & \tan(x_8)\sin(x_7) & \tan(x_8)\cos(x_7) \\ 0 & \cos(x_7) & -\sin(x_7) \\ 0 & \sin(x_7)/\cos(x_8) & \cos(x_7)/\cos(x_8) \end{pmatrix}$

Translational Equations of Motion (Translational Position)

Recall Eq.7 was given as

$$\overline{V}^v = C_{v/b}(\Phi)\, \overline{V}^b$$

$$\begin{pmatrix} \dot{P}_N \\ \dot{P}_E \\ \dot{P}_D \end{pmatrix} = C_{v/b}(\Phi) \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\begin{pmatrix} \dot{x}_{10} \\ \dot{x}_{11} \\ \dot{x}_{12} \end{pmatrix} = C_{v/b}(x_7, x_8, x_9) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

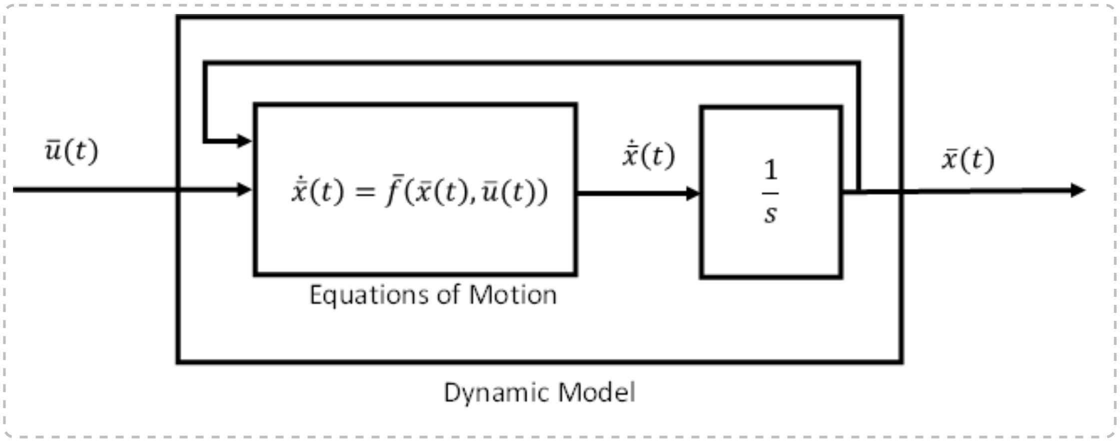
where $C_{v/b}(x_7, x_8, x_9) = C_{b/2}(x_7)\, C_{2/1}(x_8)\, C_{1/v}(x_9)$

State Space Representation

So our state space representation can be written in first order form as

$$\dot{\overline{x}} = \begin{pmatrix} \frac{1}{m}\overline{F}^b - \overline{\omega}_{b/e}{}^b \times \overline{V}^b \\ I_b{}^{-1}\left(\overline{M}^b - \overline{\omega}_{b/e}{}^b \times I_b\, \overline{\omega}_{b/e}{}^b\right) \\ H(\Phi)\, \overline{\omega}_{b/v}{}^b \\ C_{v/b}\,(\Phi)\, \overline{V}^b \end{pmatrix} \qquad \textbf{\textit{(Eq.9)}}$$

Note that these are not specific to an aircraft. They are valid for any rigid body in an inertial frame.



Note that these are not specific to an aircraft. They are valid for any rigid body in an inertial frame.

This is referred to as 'The Flat-Earth Equations of Motion' (see Steven's and Lewis 2nd Edition pg. 52).

Thoughts on EOMs

As mentioned earlier, Eq.9 is valid for all rigid bodies. The only things that are specific to a particular vehicle/model is

m
I_b
$\overline{F}^b$
$\overline{M}^b$

Control Vector

The control vector is also specific to a particular vehicle/model. For example, for a 2 engine aircraft model might have

$$\overline{u}(t) = \begin{pmatrix} \delta_A(t) \\ \delta_T(t) \\ \delta_R(t) \\ \delta_{\text{th1}}(t) \\ \delta_{\text{th2}}(t) \end{pmatrix} = \begin{pmatrix} \text{aileron} \\ \text{tail (elevator)} \\ \text{rudder} \\ \text{throttle 1} \\ \text{throttle 2} \end{pmatrix}$$

However, if we have a quadrotor we might have

$$\overline{u}(t) = \begin{pmatrix} \delta_{\text{th1}}(t) \\ \delta_{\text{th2}}(t) \\ \delta_{\text{th3}}(t) \\ \delta_{\text{th4}}(t) \end{pmatrix} = \begin{pmatrix} \text{throttle 1} \\ \text{throttle 2} \\ \text{throttle 3} \\ \text{throttle 4} \end{pmatrix}$$

External Forces

The major/complicated term that makes this model specific to a particular vehicle are the terms $\overline{F}^b$ and $\overline{M}^b$. The task is to now find expressions for $\overline{F}^b$ and $\overline{M}^b$ in terms of $\overline{x}$, $\overline{u}$, and constants.

Consider the forces, $\overline{F}^b$

$$\overline{F}^b = \overline{F}_g{}^b + \overline{F}_E{}^b + \overline{F}_A{}^b$$

where $\overline{F}_g{}^b$ =force due to gravity expressed in F_b

$\overline{F}_E{}^b$ =force due to engine/propulsion expressed in F_b

$\overline{F}_A{}^b$ =force due to aerodynamics expressed in F_b

What is $\overline{F}_g{}^b$? For our flat earth model we can write

$$\overline{F}_g{}^v = \begin{pmatrix} 0 \\ 0 \\ m\,g \end{pmatrix}$$

Note that if we assumed a ellipsoid model, then it is possible that there is a non-zero north and east components

So in this case we can write

$$\overline{F}_g{}^b = C_{b/v}\,\overline{F}_b{}^e$$

where

$$C_{b/v} = \begin{pmatrix} \cos(\theta)\cos(\psi) & \cos(\theta)\sin(\psi) & -\sin(\theta) \\ -\cos(\phi)\sin(\psi) + \sin(\phi)\sin(\theta)\cos(\psi) & \cos(\phi)\cos(\psi) + \sin(\phi)\sin(\theta)\sin(\psi) & \sin(\phi)\cos(\theta) \\ \sin(\phi)\sin(\psi) + \cos(\phi)\sin(\theta)\cos(\psi) & -\sin(\phi)\cos(\psi) + \cos(\phi)\sin(\theta)\sin(\psi) & \cos(\phi)\cos(\theta) \end{pmatrix}$$

So we have

$$\overline{F}_g{}^b = m\,g \begin{pmatrix} -\sin(x_8) \\ \sin(x_7)\cos(x_8) \\ \cos(x_7)\cos(x_8) \end{pmatrix}$$

What about $\overline{F}_E{}^b$? Here is an example of a model of the propulsive force

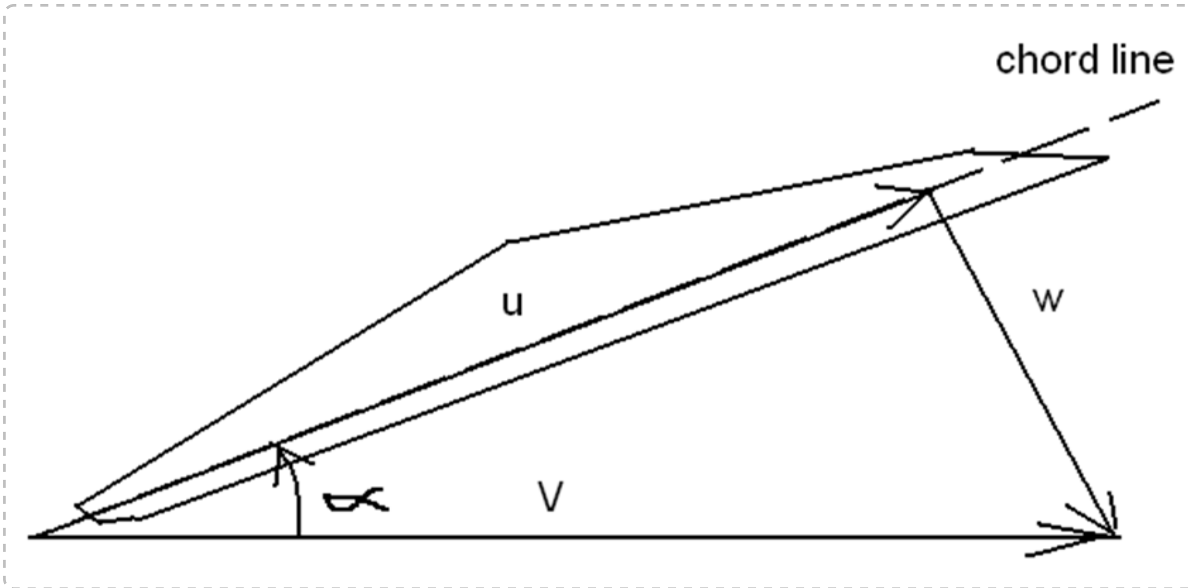
$$\overline{F}_E{}^b = \begin{pmatrix} \frac{m\,g}{2}\,(\delta_{\text{th1}} + \delta_{\text{th2}}) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{m\,g}{2}\,(u_4 + u_5) \\ 0 \\ 0 \end{pmatrix} \qquad u_4,\, u_5 \in [0,\ 1]$$

so at max throttle, thrust to weight ratio is 1.

Lastly, we have $\overline{F}_A{}^b$, the aerodynamic forces. This is what makes our model really an aircraft. These are the most complicated to deal with.

Aerodynamic forces don't depend on Euler angles or orientation of aircraft. For example, aircraft in a loop, the Euler angles and orientation are constantly changing, but the angle of attack is constant. So the aerodynamic forces are most dependent on the aerodynamic angles α and β.

We now need to describe how V, α, δ_e, etc. are function of states or controls. For example, what is α in terms of the states?



$$\tan(\alpha) = \frac{w}{u}$$

$$\alpha = \text{atan2}(w, u)$$

$$\alpha = \text{atan2}(x_3, x_1)$$

So $\quad \dot{\alpha} = h(\dot{x}_3, \dot{x}_1)$

In a similar fashion, $V = (u^2 + v^2 + w^2)^{1/2} = (x_1{}^2 + x_2{}^2 + x_3{}^2)^{1/2}$. So substituting this yields

$$\dot{x}_3 = g(x_8, x_7) - \frac{1}{m} L\big(V(x_1, x_2, x_3), \alpha, u_1, u_5, x_5, h(\dot{x}_3, \dot{x}_1)\big)$$

So we have an implicit form (cannot explicitly solve for $\dot{x}_3$)

If we move to the other side

$$0 = g(x_8, x_7) - \frac{1}{m} L\big(V(x_1, x_2, x_3), \text{atan2}(x_3, x_1), u_1, u_5, x_5, h(\dot{x}_3, \dot{x}_1)\big) - \dot{x}_3$$

$$0 = F\left(\dot{\bar{x}}, \bar{x}, \bar{u}\right)$$

This is an implicit form of a nonlinear ODE (compared to explicit where $\dot{\bar{x}} = f(\bar{x}, \bar{u})$). We will deal with this later.

If we neglect some of the higher order effects (namely the effect of $\dot{\alpha}$ on L, we can most likely write this in an explicit form of

$$\dot{x}_3 = g(x_8, x_7) - \frac{1}{m} L(V(x_1, x_2, x_3), \text{atan2}(x_3, x_1), u_1, u_5)$$

$$\dot{x}_3 = f_3\left(x_1, x_2, x_3, x_{7,} x_8, u_1, u_5\right)$$

If we perform a similar operation on the $\dot{x}_1$ and $\dot{x}_2$ equations, we can massage this to a form of

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} f_1(\bar{x}, \bar{u}) \\ f_2(\bar{x}, \bar{u}) \\ f_3(\bar{x}, \bar{u}) \end{pmatrix}$$

We will discuss this more when we investigate wind tunnel testing.

External Moments

Consider the moments, $\overline{M}^b$

$$\overline{M}^b = \overline{M}_g{}^b + \overline{M}_E{}^b + \overline{M}_A{}^b$$

If we choose the moments to be about the CoG of the vehicle, then we do not need to worry about gravity (so $\overline{M}_g{}^b = \bar{0}$). As such $\overline{M}^b$ is completely comprised of aerodynamic moments and moments due to propulsion.

$$\overline{M}^b = \overline{M}_E{}^b + \overline{M}_A{}^b$$

Invertibility of Inertia Matrix

TO MOVE: In this situation, the inertia matrix is most convenient to be attached to the body axis. This would be how the aircraft body resists rotation about its body axis. This could easily be measured in a lab and should be constant.

Question, is I_b invertible? What does it look like? If the aircraft was symmetrical across all three major planes (ie football, brick, etc.), we would have

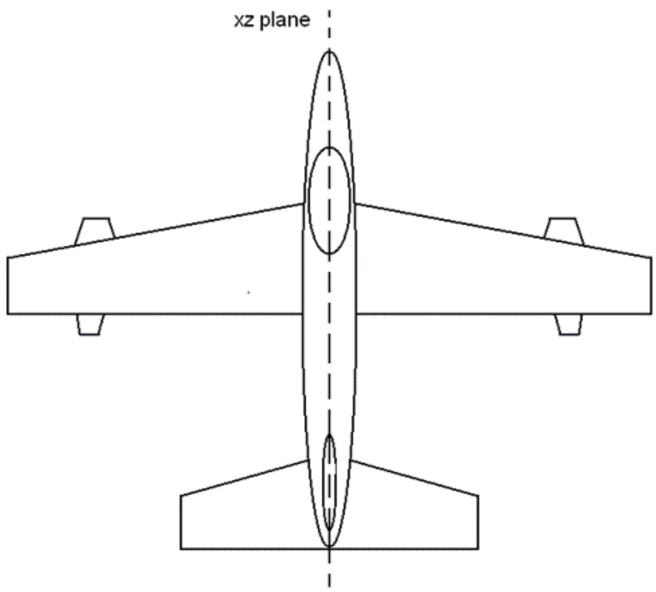
$$I_b = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

In this case, the inverse of the matrix is simple

$$I_b^{-1} = \begin{pmatrix} 1/I_{xx} & 0 & 0 \\ 0 & 1/I_{yy} & 0 \\ 0 & 0 & 1/I_{zz} \end{pmatrix}$$

If we assume that aircraft has one plane of symmetric (xz plane), everything else is not symmetric. In this case, inertia matrix becomes

$$I_b = \begin{pmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{pmatrix}$$



The notation on the cross terms implies that I_{xz} is positive so these cross terms have negative numerical values. For example for the RCAM aircraft

$$I_b = m\begin{pmatrix} 40.07 & 0 & -2.0923 \\ 0 & 64 & 0 \\ -2.0923 & 0 & 99.92 \end{pmatrix}$$

The inverse is given by

$$I_b^{-1} = \Gamma\begin{pmatrix} I_{zz} & 0 & I_{xz} \\ 0 & \Gamma/I_{yy} & 0 \\ I_{xz} & 0 & I_{xx} \end{pmatrix}$$

where $\quad \Gamma = I_{xx}\, I_{zz} - I_{xz}^2$

```
Ib = {{Ixx, 0, -Ixz}, {0, Iyy, 0}, {-Ixz, 0, Izz}};
Inverse[Ib] // Simplify // MatrixForm
```

$$\begin{pmatrix} \frac{\text{Izz}}{-\text{Ixz}^2+\text{Ixx Izz}} & 0 & \frac{\text{Ixz}}{-\text{Ixz}^2+\text{Ixx Izz}} \\ 0 & \frac{1}{\text{Iyy}} & 0 \\ \frac{\text{Ixz}}{-\text{Ixz}^2+\text{Ixx Izz}} & 0 & \frac{\text{Ixx}}{-\text{Ixz}^2+\text{Ixx Izz}} \end{pmatrix}$$

Cross Coupling from Inertia Matrix

Recall that the rotational equation of motion was given by

$${}^{b}\dot{\overline{\omega}}_{b/i}{}^{b} = I_b^{-1}\left(\overline{M}^b - \overline{\omega}_{b/i}{}^{b} \times I_b\,\overline{\omega}_{b/i}{}^{b}\right)$$

We can write this as

$${}^{b}\dot{\overline{\omega}}_{b/i}{}^{b} = I_b^{-1}\,\overline{M}^b - I_b^{-1}\left(\overline{\omega}_{b/i}{}^{b} \times I_b\,\overline{\omega}_{b/i}{}^{b}\right)$$

Let us take a closer look at the $I_b^{-1}\left(\overline{\omega}_{b/i}{}^{b} \times I_b\,\overline{\omega}_{b/i}{}^{b}\right)$ term (which we can interpret as an angular acceleration due to rotation)

$$I_b^{-1}\left(\overline{\omega}_{b/i}{}^{b} \times I_b\,\overline{\omega}_{b/i}{}^{b}\right) = I_b^{-1}\left(\begin{pmatrix} p \\ q \\ r \end{pmatrix} \times \begin{pmatrix} I_{xx} & 0 & -I_{xz} \\ 0 & I_{yy} & 0 \\ -I_{xz} & 0 & I_{zz} \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix}\right)$$

$$= I_b^{-1}\left(\begin{pmatrix} p \\ q \\ r \end{pmatrix} \times \begin{pmatrix} I_{xx}\,p - I_{xz}\,r \\ I_{yy}\,q \\ I_{zz}\,p - I_{xz}\,r \end{pmatrix}\right)$$

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SetDirectory["D:\lum\mathematica\libraries"];
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ResetDirectory[];
Off[General::"spell1", General::"spell"]
```

$$\omega = \begin{pmatrix} p \\ q \\ r \end{pmatrix};$$

```
crossTerms[p_, q_, r_] = Inverse[Ib].(CrossProduct[ω, Ib.ω]) // Simplify;
crossTerms[p, q, r] // MatrixForm
```

$$\begin{pmatrix} \frac{q\left(\text{Ixx Ixz } p+\text{Ixz }(-\text{Iyy}+\text{Izz})\, p-\text{Ixz}^2\, r+(\text{Iyy}-\text{Izz})\text{ Izz } r\right)}{\text{Ixz}^2-\text{Ixx Izz}} \\ \frac{(\text{Ixx}-\text{Izz})\, p\, r+\text{Ixz}\left(p^2-r^2\right)}{\text{Iyy}} \\ \frac{q\left(\text{Ixx}^2\, p-\text{Ixx }(\text{Iyy } p+\text{Ixz } r)+\text{Ixz }(\text{Ixz } p+(\text{Iyy}-\text{Izz})\, r)\right)}{\text{Ixz}^2-\text{Ixx Izz}} \end{pmatrix}$$

What if there is only one non-zero term of p, q, or r and no moments? For example, if

$$p \neq 0$$
$$q = 0$$
$$r = 0$$
$$\overline{M} = \overline{0}$$

```
crossTerms[p, 0, 0] // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{\text{Ixz } p^2}{\text{Iyy}} \\ 0 \end{pmatrix}$$

This is interesting because the rotational equation becomes

$${}^b\dot{\bar{\omega}}_{b/e}{}^b = \begin{pmatrix} 0 \\ \frac{I_{xz}\, p^2}{I_{yy}} \\ 0 \end{pmatrix}$$

So we see that $\dot{q} \neq 0$. So a non-zero roll rate will result in an acceleration in pitch rate.

A similar situation arises from a non-zero yaw rate.

```
crossTerms[0, 0, r] // MatrixForm
```

$$\begin{pmatrix} 0 \\ -\frac{\text{Ixz } r^2}{\text{Iyy}} \\ 0 \end{pmatrix}$$

However, if we look at a non-zero pitch rate

```
crossTerms[0, q, 0] // MatrixForm
```

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We see that due to the symmetry in the plane, a pure pitch rate does not yield angular accelerations in other axes.

This illustrates the cross coupling effect of the inertia matrix.

We can look at combinations when there are two non-zero terms.

```
crossTerms[p, q, 0] // MatrixForm
```

$$\begin{pmatrix} \frac{(\text{Ixx Ixz p}+\text{Ixz }(-\text{Iyy}+\text{Izz})\text{ p})\text{ q}}{\text{Ixz}^2-\text{Ixx Izz}} \\ \frac{\text{Ixz p}^2}{\text{Iyy}} \\ \frac{(\text{Ixx}^2\text{ p}+\text{Ixz}^2\text{ p}-\text{Ixx Iyy p})\text{ q}}{\text{Ixz}^2-\text{Ixx Izz}} \end{pmatrix}$$

```
crossTerms[p, 0, r] // MatrixForm
```

$$\begin{pmatrix} 0 \\ \frac{(\text{Ixx}-\text{Izz})\text{ p r}+\text{Ixz }(\text{p}^2-\text{r}^2)}{\text{Iyy}} \\ 0 \end{pmatrix}$$

```
crossTerms[0, q, r] // MatrixForm
```

$$\begin{pmatrix} \frac{\text{q }(-\text{Ixz}^2\text{ r}+(\text{Iyy}-\text{Izz})\text{ Izz r})}{\text{Ixz}^2-\text{Ixx Izz}} \\ -\frac{\text{Ixz r}^2}{\text{Iyy}} \\ \frac{\text{q }(-\text{Ixx Ixz r}+\text{Ixz }(\text{Iyy}-\text{Izz})\text{ r})}{\text{Ixz}^2-\text{Ixx Izz}} \end{pmatrix}$$

What about if there was a diagonal I_b matrix?

$$\text{IbD} = \begin{pmatrix} \text{Ixx} & 0 & 0 \\ 0 & \text{Iyy} & 0 \\ 0 & 0 & \text{Izz} \end{pmatrix};$$

```
crossTermsD[p_, q_, r_] = Inverse[IbD].(CrossProduct[ω, IbD.ω]) // Simplify;
crossTermsD[p, q, r] // MatrixForm
```

$$\begin{pmatrix} \frac{(-\text{Iyy}+\text{Izz})\text{ q r}}{\text{Ixx}} \\ \frac{(\text{Ixx}-\text{Izz})\text{ p r}}{\text{Iyy}} \\ \frac{(-\text{Ixx}+\text{Iyy})\text{ p q}}{\text{Izz}} \end{pmatrix}$$

```
crossTermsD[p, 0, 0]
```

{{0}, {0}, {0}}

```
crossTermsD[0, q, 0]
```

{{0}, {0}, {0}}

```
crossTermsD[0, 0, r]
{{0}, {0}, {0}}
```

Position States

Also notice that we only have 9 equations. In homework 2, we discovered that we should expect 12 states and thus (12 equations) for the aircraft model because it was a 6 degree of freedom system.

We do not require 3 position equations because the model behaves the same regardless of location.
This would not be true if we modeled gravity or thrust as a function of altitude as this would make the dynamics dependent on position.
Furthermore, we do not need the $\dot{\psi}$ equation. This is because the aircraft model performs the same regardless of heading. Note that we need ϕ and θ because of gravity is a function of these states.