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# Lecture06c
# Stokes' Theorem

Lecture is on YouTube

The YouTube video entitled 'Stokes' Theorem' that covers this lecture is located at https://youtu.be/40UUPvrHN-c

## Introduction

Stokes' Theorem (AKA Kelvin-Stokes Theorem or the Curl Theorem) relates the integral of the curl of a vector field over a surface to the line integral of the vector field around the boundary of the surface.

https://en.wikipedia.org/wiki/Stokes%27_theorem

**Prerequisites:**
-Video entitled 'Line Integrals' located at https://youtu.be/0sIsoJYmVVM
-Video entitled 'The Laplace Operator, Divergence, and Curl' at https://youtu.be/KOlVHPShCOk).
-Video entitled 'Surface Integrals of Scalar and Vector Fields/Functions' at https://youtu.be/34Xfij-7gcI

## Stokes' Theorem

Stokes' Theorem generalizes Green's theorem in the plane. Recall that the curl of a vector function $\overline{F}$ is given as

$$\text{curl}\,\overline{F} = \nabla \times \overline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix} \qquad \textbf{(Eq.1)}$$

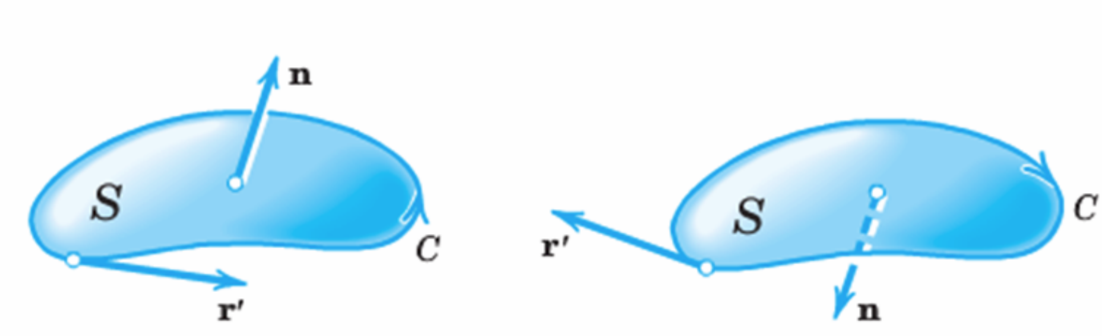
**Theorem 1: Stokes' Theorem: Transformation Between Surface and Line Integrals**
Let $S$ be a piecewise smooth oriented surface in space and let the boundary of $S$ be a piecewise smooth simple closed curve $C$. Let $\overline{F}(x, y, z)$ be a continuous vector function that has continuous first partial derivatives in a domain in space containing $S$. Then

$$\iint_S (\text{curl}\,\overline{F}) \cdot d\overline{A} = \oint_C \overline{F} \cdot d\overline{r}$$

Or alternatively expressed as

$$\iint_S (\text{curl}\,\overline{F}) \cdot \overline{n}\, dA = \oint_C \overline{F} \cdot \overline{r}\,'(s)\, ds \qquad \textbf{(Eq.2)}$$

Here $\overline{n}$ is a unit normal vector of $S$ and, depending on $\overline{n}$, the integration around $C$ is taken in the sense shown below. Note that the direction of integration can be determined that if you walk around $C$, the vector from your feet to your head will be generally pointing in the direction of $\overline{n}$ (effectively the right hand rule with the curling of your fingers in the direction of $C$ and your thumb pointing in the direction of $\overline{n}$).




Furthermore, $\overline{r}\,'(s) = d\overline{r}/ds$ is the unit tangent vector to $C$ and $s$ is the arc length along $C$.

Let us expand Eq.2. We can write $\overline{F}$ in components

$$\overline{F} = \langle F_1, F_2, F_3 \rangle$$

Furthermore, recall the following relationships

$$\overline{N} = \langle N_1, N_2, N_3 \rangle = \overline{r}_u \times \overline{r}_v$$
$$\overline{n}\, dA = \overline{N}\, du\, dv$$
$$\overline{r}\,'\, ds = \langle dx, dy, dz \rangle$$

So Eq.2 can be written in components as

$$\iint_R \left[ \left( \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) N_1 + \left( \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} \right) N_2 + \left( \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) N_3 \right] du\, dv = \oint_{\overline{C}} (F_1\, dx + F_2\, dy + F_3\, dz)$$

where $\quad R$ = region in the uv-plane corresponding to $S$ ($S$ represented by $\overline{r}(u, v)$)

## Example 1: Verification of Stoke's Theorem

Consider

$$\overline{F} = \langle y, z, x \rangle$$

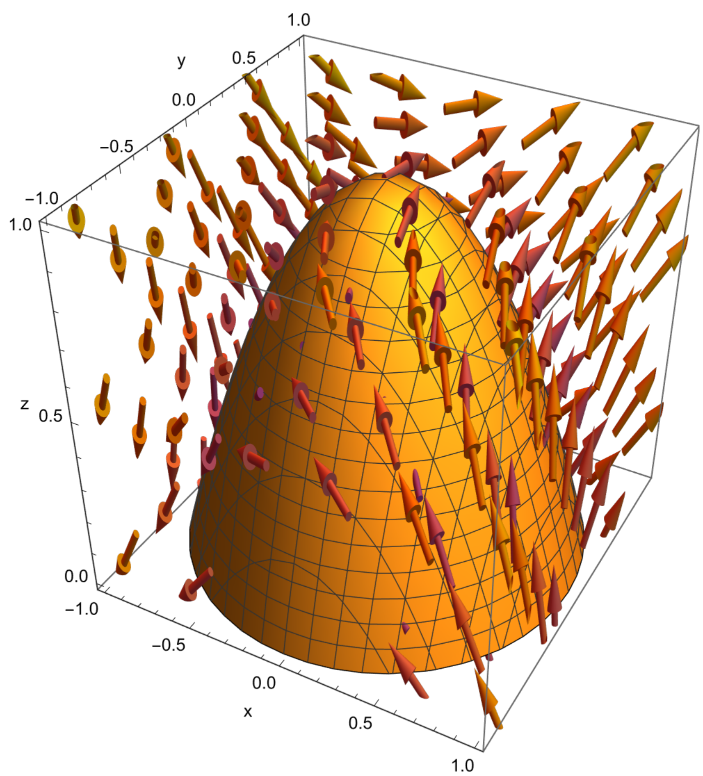
$S$ is the paraboloid defined by

$$S = \left\{ \langle x, y, z \rangle \mid z = f(x, y) = 1 - \left(x^2 + y^2\right), z \geq 0 \right\}$$

We can visualize the surface $S$ and the vector field $\overline{F}$

In[ ]:=
```
(*Define the surface*)
f[x_, y_] = 1 - (x^2 + y^2);

(*Define the vector field*)
F[x_, y_, z_] = {y, z, x};

Show[
 (*Plot surface S*)
 ContourPlot3D[f[x, y] == z, {x, -1, 1},
  {y, -1, 1}, {z, 0, 1}, AxesLabel → {"x", "y", "z"}],

 (*Plot the vector field*)
 VectorPlot3D[F[x, y, z], {x, -1, 1}, {y, -1, 1}, {z, 0, 1}]
]
```

Out[ ]=




Let us check that Stokes' Theorem holds for this example

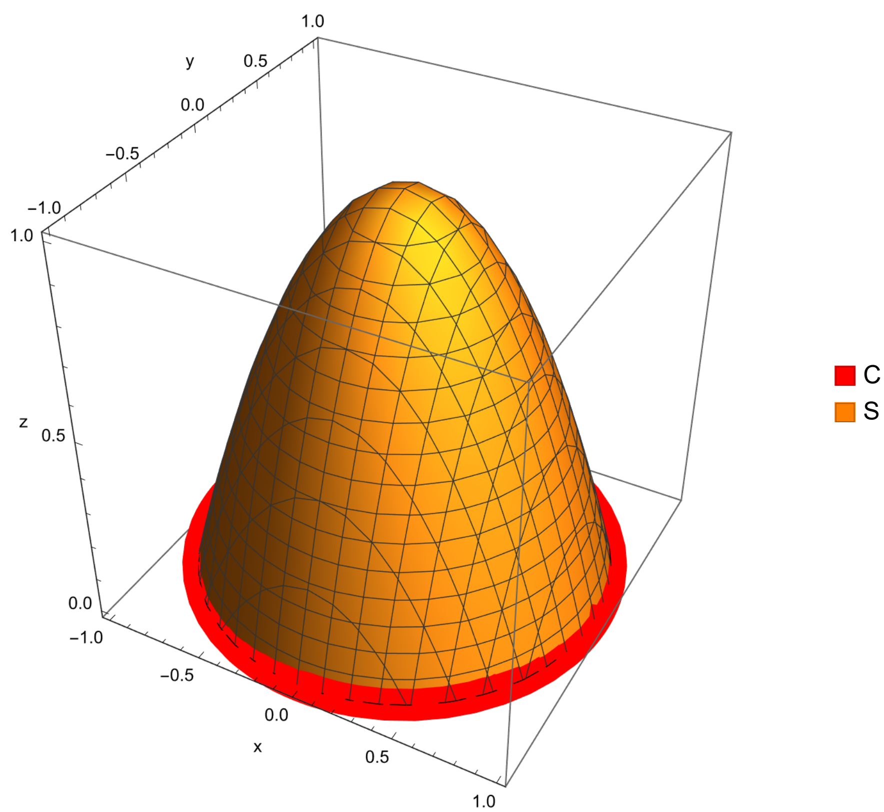
**Compute Right Side:** Compute $\oint_C \overline{F} \cdot d\overline{r} = \oint_a^b \overline{F} \cdot \overline{r}'(s)\, ds$

We choose the curve $C$ as

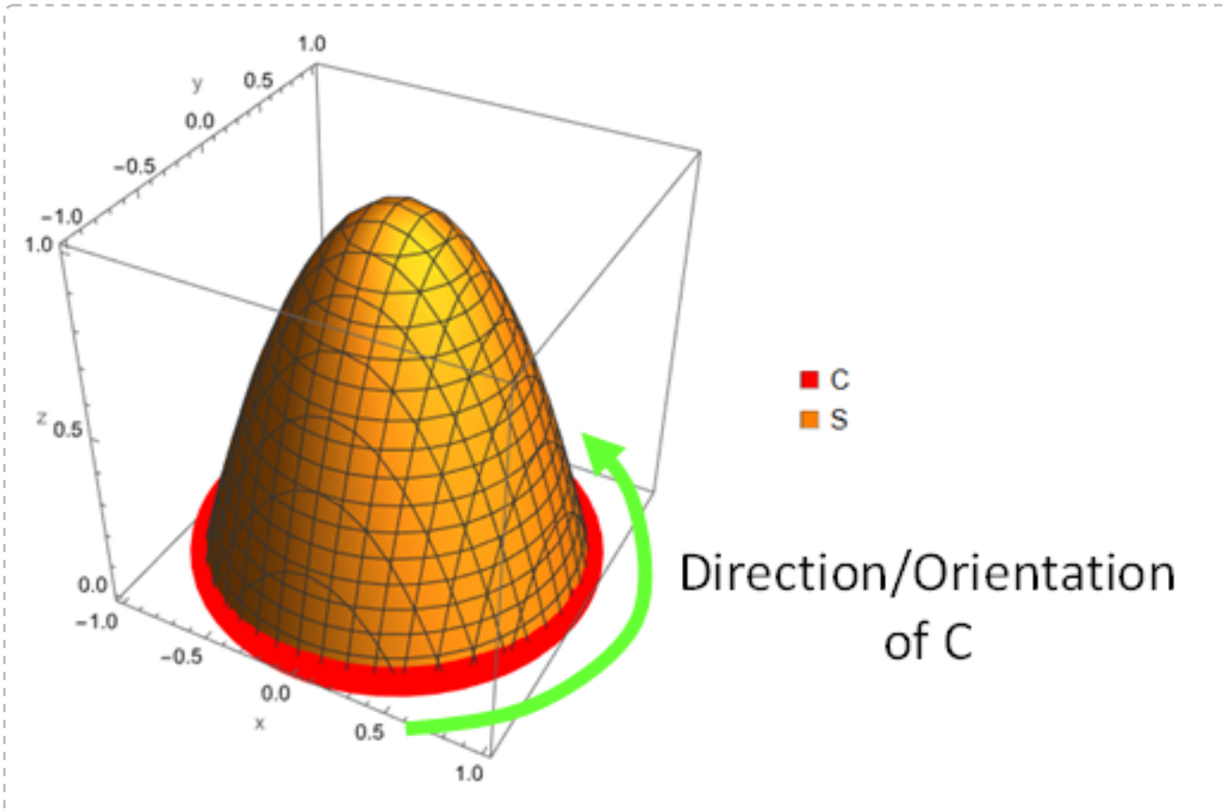
$$C = \overline{r}(s) = \langle \cos(s), \sin(s), 0 \rangle \quad s \in [0, 2\pi]$$

In[◦]:=
```
Legended[
 Show[
  (*Plot surface S*)
  ContourPlot3D[f[x, y] == z, {x, -1, 1},
   {y, -1, 1}, {z, 0, 1}, AxesLabel → {"x", "y", "z"}],

  (*Plot C*)
  ParametricPlot3D[{Cos[s], Sin[s], 0}, {s, 0, 2 π}, PlotStyle → {Red, Thickness[0.04]}]
 ],

 (*Add legend information*)
 SwatchLegend[{Red, Orange}, {"C", "S"}]
]
```

Out[◦]=




Note that this choice of parameterization of $C$ defines a direction/orientation of $C$ (as $s$ goes from 0 tot $2\pi$). This orientation is shown below and will matter in a latter step when we compute a surface normal for Stokes' Theorem




So

$\bar{r}'(s) = <-\sin(s), \cos(s), 0>$

And

$\bar{F}(\bar{r}(s)) = <y, z, x>$ recall: $x = \cos(s)$, $y = \sin(s)$, $z = 0$

$\bar{F}(\bar{r}(s)) = <\sin(s), 0, \cos(s)>$

So we have

$$\oint_C \bar{F} \cdot d\bar{r} = \oint_a^b \bar{F}(\bar{r}(s)) \cdot \bar{r}'(s)\, ds$$

$$= \oint_0^{2\pi} <\sin(s), 0, \cos(s)> \cdot <-\sin(s), \cos(s), 0>\, ds$$

$$= \oint_0^{2\pi} (\sin(s))(-\sin(s)) + 0 + 0\, ds$$

$$= -\oint_0^{2\pi} \sin^2(s)\, ds$$

$$\oint_C \bar{F} \cdot d\bar{r} = -\pi$$

In[]:= 
```
-Integrate[Sin[s]^2, {s, 0, 2 π}]
```
Out[]= $-\pi$

**Compute Left Side:** Compute $\iint_S (\text{curl}\,\overline{F}) \cdot \overline{n}\, dA$

We can compute curl $\overline{F}$

In[]:= 
```
CurlF = Curl[{y, z, x}, {x, y, z}]
```
Out[]= {-1, -1, -1}

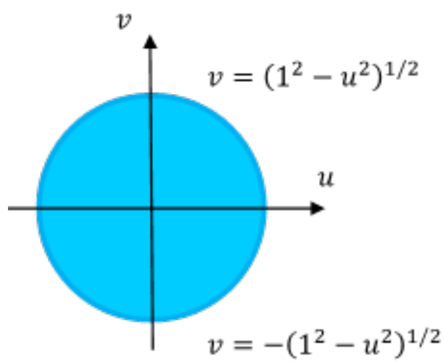
So we have

$$\text{curl}\,\overline{F} = <-1, -1, -1>$$

We can parameterize the surface $S$ as

$$S = \overline{r}(u, v) = <u, v, 1 - (u^2 + v^2)>$$

It is interesting to note that $u$ and $v$ for this parametrization is not a simple box. Instead, $u$ and $v$ take values along a circle with radius $r$ in the uv – plane

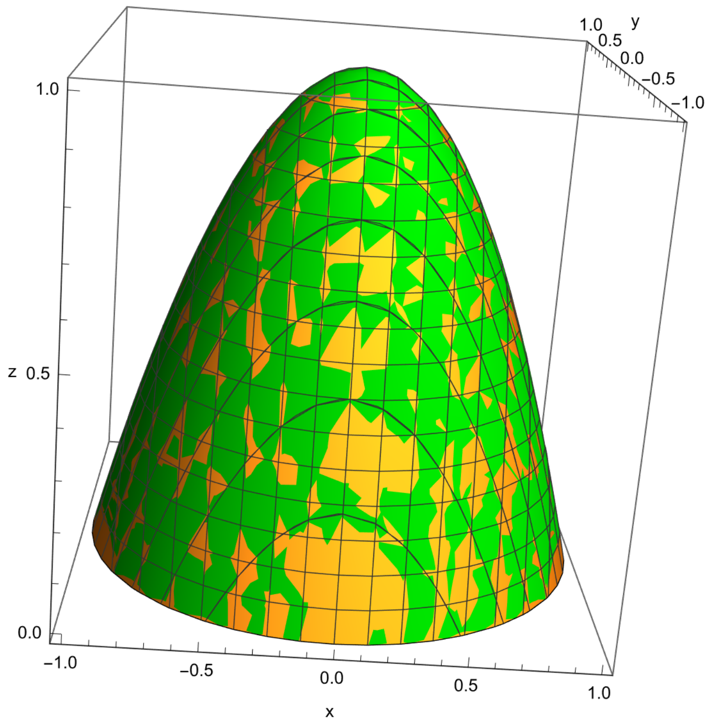



In[]:= 
```
r[u_, v_] = {u, v, 1 - (u^2 + v^2)};
```

We can verify that this parametric representation is the same by plotting the two on top of each other

In[ ]:=
```
Show[
 (*Plot surface S*)
 ContourPlot3D[f[x, y] == z, {x, -1, 1},
  {y, -1, 1}, {z, 0, 1}, AxesLabel → {"x", "y", "z"}],

 (*Plot surface S using parameterized representation*)
 ParametricPlot3D[{u, v, 1 - (u^2 + v^2)},
  {u, -1, 1}, {v, -(1^1 - u^2)^(1/2), (1^1 - u^2)^(1/2)}, PlotStyle → {Green}]
]
```

Out[ ]=




We can compute a surface normal using

$$\overline{N} = \overline{r}_u \times \overline{r}_v$$

$$= \frac{\partial \overline{r}(u,v)}{\partial u} \times \frac{\partial \overline{r}(u,v)}{\partial v} \qquad \text{recall: } \overline{r}(u, v) = < u, v, 1 - (u^2 + v^2) >$$

$$= < 1, 0, -2\,u > \times < 0, 1, -2\,v >$$

$$\overline{N} = < 2\,u, 2\,v, 1 >$$

In[ ]:=
```
NVector[u_, v_] = Cross[D[r[u, v], u], D[r[u, v], v]]
```
Out[ ]=
```
{2 u, 2 v, 1}
```

At this point, we check the orientation of this to ensure that $\overline{N}$ is consistent with the orientation of $C$. A quick spot check at $u = v = 0$ (the top of the surface) shows that $\overline{N}(0, 0) = <0, 0, 1>$ which is indeed pointing in the "upwards" direction which is consistent with the orientation of $C$

We now need to compute the term $\overline{n}\, dA$. Recall from our video entitled 'Parameterizing Surfaces and Computing Surface Normal Vectors' at https://youtu.be/a3_c4c9PYNg?t=1617 (timestamp 26:58) (or see Kreyseig Section 10.6 Eq.3*), that

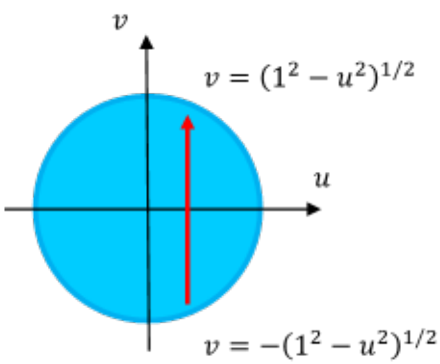
$$\overline{n}\, dA = \overline{n}\, |\, \overline{N}\, |\, du\, dv = \overline{N}\, du\, dv \qquad \textbf{\textit{(Eq.3*)}}$$

where $\overline{N} = \overline{r}_u \times \overline{r}_v$

So finally, we have

$$\iint_S (\mathrm{curl}\, \overline{F}) \cdot \overline{n}\, dA = \iint_R (\mathrm{curl}\, \overline{F}(\overline{r}(u, v))) \cdot \overline{N}(u, v)\, du\, dv$$

Recall from our ideo entitled 'Double Integrals' at https://youtu.be/C-yfMjxxsz0, we can treat $R$ as a type I region




So we see that in the plane we have

$$R = \{(u, v)\ |\ -1 \le u \le 1,\ g(u) \le v \le h(u)\} \qquad \text{(Type I)}$$

where $g(u) = -\sqrt{1^2 - u^2}$

$h(u) = \sqrt{1^2 - u^2}$

So the double integral becomes

$$\iint_R \left(\mathrm{curl}\, \overline{F}(\overline{r}(u, v)) \cdot \overline{N}(u, v)\right) du\, dv = \int_{-1}^{1} \int_{-\sqrt{1^2-u^2}}^{\sqrt{1^2-u^2}} <-1, -1, -1> \cdot <2u, 2v, 1>\, dv\, du$$

$$= \int_{-1}^{1} \int_{-\sqrt{1^2-u^2}}^{\sqrt{1^2-u^2}} -1 - 2u - 2v\, dv\, du$$

$$= \int_{-1}^{1} -2\sqrt{1-u^2} - 4u\sqrt{1-u^2}\, du$$

$$\int\int_R \left(\text{curl}\,\overline{F}(\overline{r}(u,v)) \cdot \overline{N}(u,v)\right) du\, dv = -\pi$$

```
In[•]:= t1 = Integrate[CurlF.NVector[u, v], {v, -Sqrt[1^2 - u^2], Sqrt[1^2 - u^2]}]
        t2 = Integrate[t1, {u, -1, 1}]
```

Out[•]= $2 \times (-1 - 2u)\sqrt{1-u^2} - 2 \times \left(\frac{1}{2} \times (1-u^2) + \frac{1}{2} \times (-1+u^2)\right)$

Out[•]= $-\pi$

So we have the same answer, showing that Stokes theorem holds for this example.

# Green's Theorem in the Plane as a Special Case of Stokes' Theorem

Recall our video entitled 'Green's Theorem: Relating Closed Line Integrals to Double Integrals' at https://youtu.be/p7PSZW9NhLU.

If we consider a two dimensional situation, then $\overline{F} = < F_1, F_2, 0 >$. Since we are in the xy-plane, the unit vector normal to $S$ is simply $\overline{n} = \hat{k}$. So we have

$$\left(\text{curl}\,\overline{F}\right) \cdot \overline{n} = \left(\text{curl}\,\overline{F}\right) \cdot \hat{k} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$$

So Stokes' Theorem in this situation becomes

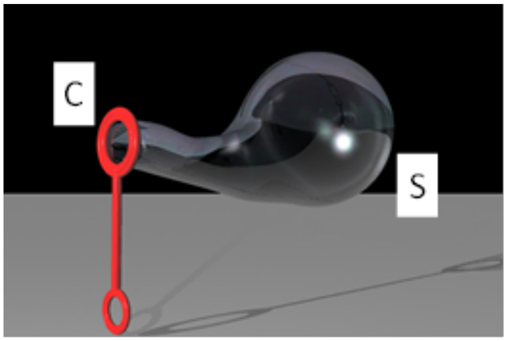
$$\int\int_S \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA = \oint_C (F_1\, dx + F_2\, dy)$$

which is Green's Theorem in the plane. This shows that Green's Theorem in the plane is a specialized case of Stokes' Theorem.

# Discussion of Stokes' Theorem

There are many times when it becomes convenient to use Stokes' theorem. For example, it can be used to simplify computing a surface integral.

Suppose that you are tasked with computing the surface integral of $\overline{F}$ over the surface of a bubble as shown below (in other words compute $\int\int_S \overline{F} \cdot d\overline{s}$ where $S$ is the surface of the bubble)




As it stands, there is no shortcut and we must directly compute this. However, what if the question was instead to compute the surface integral of curl $\overline{F}$ over the surface of the bubble? In other words, the situation requires the computation of $\int\int_S \text{curl}\,\overline{F}\cdot d\overline{s}$. If we did not know about Stokes' Theorem, we may try to directly compute this, which would require a parametrization of the bubble's complex surface, $S$. However, with Stokes' Theorem, this becomes

$$\int\int_S \text{curl}\,\overline{F}\cdot d\overline{s} = \oint_C \overline{F}\cdot d\overline{r}$$

We see that the line integral is significantly easier to compute as we can choose $C$ as the perfect circle of the bubble wand. This is perhaps why this concept of applying Stokes' theorem is sometimes referred to as the bubble property.