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Lecture 09b
Introduction to Full State Feedback Control

Lecture is on YouTube

The YouTube video entitled ‘Introduction to Full State Feedback Control’ that covers this lecture is located at https://youtu.be/1zIIcYfp5QA.

Outline

-Introduction
-Full State Feedback
    -Pole Placement
-Closing Thoughts

Introduction

In this lecture, we explore an incredibly powerful control architecture known as full state feedback (FSFB). This is an introduction to FSFB and future lectures will investigate the topic in more detail.

Full State Feedback

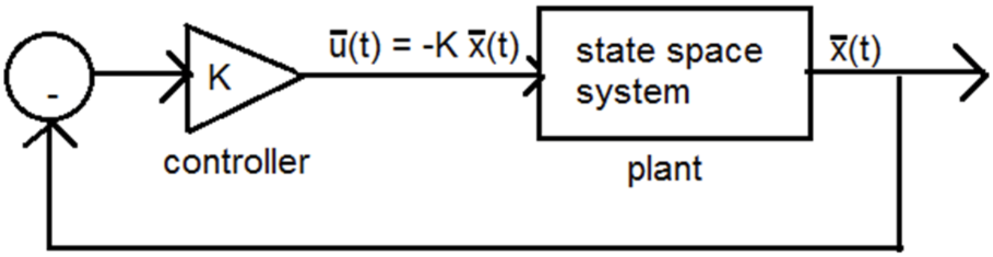
Consider a state space representation of a system

$$\dot{\overline{x}}(t) = A\,\overline{x}(t) + B\,\overline{u}(t) \qquad \textbf{\textit{(Eq.1)}}$$

We can choose a control law of the form

$$\overline{u}(t) = -K\,\overline{x}(t) \qquad \textbf{\textit{(Eq.2)}}$$

This can be implemented as shown below



Note that in this case, we assume that the system output, $\overline{y}(t)$, is the full state of the system. In other words

$$\overline{y}(t) = \overline{x}(t)$$

Note that this type of architecture is known as a regulator because there is no input for a desired control signal. Instead, this controller's primary function is to regulate the system state back to zero from a non-zero initial conditions $\left(\overline{x}(t) \to \overline{0}\right)$.

With this control law, the closed loop system becomes (plugging in Eq.2 into Eq.1)

$$\dot{\overline{x}}(t) = A\,\overline{x}(t) + B\,(-K\,\overline{x}(t))$$

$$= (A - B\,K)\,\overline{x}(t)$$

$$\dot{\overline{x}}(t) = A_{cl}\,\overline{x}(t) \qquad \textbf{\textit{(Eq.3)}}$$

where $\quad A_{cl} = A - B\,K$

We see that this control law has the ability to change the closed A matrix of the system. If the system is controllable, the eigenvalues of A_{cl} can be moved to any arbitrary locations. ***<go to lecture on controllability>***

Pole Placement

We can now compute the K matrix required to place the closed loop poles at the desired location. This process is known as pole placement.

Example 1: Controllable System

Consider the following system

$$\dot{\overline{x}}(t) = \begin{pmatrix} 0 & 3 \\ 2 & 4 \end{pmatrix} \overline{x}(t) + \begin{pmatrix} -2 \\ 1 \end{pmatrix} \overline{u}(t)$$

```
A = (0 3; 2 4); B = (-2; 1);
```

$$\mathbf{A} = \begin{pmatrix} 0 & 3 \\ 2 & 4 \end{pmatrix}; \ \mathbf{B} = \begin{pmatrix} -2 \\ 1 \end{pmatrix};$$

We can quickly verify that this is controllable

```
(*Check controllability*)
Pc = Transpose[Join[Transpose[B], Transpose[A.B]]];
Pc // MatrixForm
MatrixRank[Pc]
```

$$\begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}$$

2

We can compute A_{cl}, noting that K is a 1×2 matrix

$$K = \begin{pmatrix} k_1 & k_2 \end{pmatrix}$$

```
K = ( k1  k2 );
Acl = A - B.K;
Acl // MatrixForm
```

$$\begin{pmatrix} 2\,\text{k1} & 3+2\,\text{k2} \\ 2-\text{k1} & 4-\text{k2} \end{pmatrix}$$

So we have

$$A_{\text{cl}} = \begin{pmatrix} 2\,k_1 & 3+2\,k_2 \\ 2-k_1 & 4-k_2 \end{pmatrix}$$

We can the compute the closed loop eigenvalues/poles of the system as

$$\det(\lambda I - A_{\text{cl}}) = 0$$

```
p[λ_] = Collect[Det[λ * IdentityMatrix[2] - Acl], {λ, λ^2}]
```

$-6 + 11\,\text{k1} - 4\,\text{k2} + (-4 - 2\,\text{k1} + \text{k2})\,\lambda + \lambda^2$

So we see that the closed loop characteristic equation of the system is given as

$$p(\lambda) = \det(\lambda I - A_{\text{cl}}) = \lambda^2 + (-4 - 2\,k_1 + k_2)\,\lambda + 11\,k_1 - 6 - 4\,k_2 \qquad \textbf{\textit{(Eq.4)}}$$

(closed loop char eq with full state feedback)

Suppose that we would like the closed loop poles to be at the locations

$$\lambda_{\text{cl},1} = -5 + 2\,i$$
$$\lambda_{\text{cl},2} = -5 - 2\,i$$

Therefore, the desired closed loop characteristic equation of the system should be

$$p_d(\lambda) = (\lambda + (-5 + 2\,i))\,(\lambda + (-5 - 2\,i)) \qquad \textbf{\textit{(Eq.5)}}$$

(desired closed loop char eq)

```
pd[λ_] = (λ + (5 + 2 I)) (λ + (5 - 2 I)) // Expand
```

$29 + 10\lambda + \lambda^2$

So we see that in order to place the poles at these desired locations, we need to choose gains k_1 and k_2 such that Eq.4 (the closed loop characteristic equation with full state feedback) matches Eq.5 (the desired closed loop characteristic equation). In other words, we require that

$$-4 - 2k_1 + k_2 = 10 \qquad \text{(coefficient of } \lambda\text{)}$$
$$11k_1 - 6 - 4k_2 = 29 \qquad \text{(constant)}$$

Solving these two equations simultaneously yields

```
temp = Solve[{Coefficient[p[λ], λ] == Coefficient[pd[λ], λ], p[0] == pd[0]}, {k1, k2}];

k1 = k1 /. temp[[1]]
k2 = k2 /. temp[[1]]
```

$\frac{91}{3}$

$\frac{224}{3}$

So we see that

$$k_1 = 91/3 \approx 30.33$$
$$k_2 = 224/3 \approx 74.67$$

We can verify that this yields the desired closed loop eigenvalues/poles

```
Eigenvalues[A - B.K]
```

$\{-5 + 2i, -5 - 2i\}$

Matlab provides a commands, ‘place’ and ‘acker’ to perform these operations (it is preferable to use ‘place’ rather than ‘acker’ for higher order systems due to instabilities in the Ackermann formula)

```
A = [0 3;
     2 4];

B = [-2;
     1];

pDesired = [-5+2*i -5-2*i];

[K] = place(A, B, pDesired)
```

```
Clear[k2, k1, temp, pd, p, Acl, K, Pc, B, A]
```

Example 2: Uncontrollable System

Consider the same system as in Example 1 except the B matrix is changed to be

$$B = \begin{pmatrix} \frac{1}{2} \times \left(-2 + \sqrt{10}\right) \\ 1 \end{pmatrix} = \begin{pmatrix} 0.581139 \\ 1 \end{pmatrix}$$

```
A = (0 3
     2 4); B = (1/2 × (-2 + √10)
                1);
```

Interestingly, this corresponds to an eigenvector of the *A* matrix and pursuant to our discussion of controllability (specifically the PBH Test), we know that this makes the system uncontrollable.

```
(*Check controllability*)
Pc = Transpose[Join[Transpose[B], Transpose[A.B]]];
Pc // MatrixForm
MatrixRank[Pc]
```

$$\begin{pmatrix} \frac{1}{2} \times \left(-2 + \sqrt{10}\right) & 3 \\ 1 & 2 + \sqrt{10} \end{pmatrix}$$

1

When we attempt to perform the same pole placement procedure we obtain

```
(*Compute Acl using FSFB*)
K = ( k1  k2 );
Acl = A - B.K;
Print["ACL"]
Acl // MatrixForm

(*Compute closed loop characteristic eq using FSFB*)
Print["p(λ)"]
p[λ_] = Collect[Det[λ * IdentityMatrix[2] - Acl], {λ, λ^2}]
```

A_{CL}

$$\begin{pmatrix} -\frac{1}{2} \times \left(-2 + \sqrt{10}\right) k1 & 3 - \frac{1}{2} \times \left(-2 + \sqrt{10}\right) k2 \\ 2 - k1 & 4 - k2 \end{pmatrix}$$

p(λ)

$$-6 + 7\,k1 - 2\sqrt{10}\,k1 - 2\,k2 + \sqrt{10}\,k2 + \left(-4 - k1 + \sqrt{\frac{5}{2}}\,k1 + k2\right)\lambda + \lambda^2$$

We keep the same desired pole locations and therefore the same closed loop characteristic equation as before

$$p_d(\lambda) = \lambda^2 + 10\,\lambda + 29 \qquad \text{(desired closed loop char eq)}$$

```
pd[λ_] = λ^2 + 10 λ + 29
```

$29 + 10\,\lambda + \lambda^2$

So we see that in order to place the poles at these desired locations, we need to choose gains k_1 and k_2 such that Eq.4 (the closed loop characteristic equation with full state feedback) matches Eq.5 (the desired closed loop characteristic equation). In other words, we require that

$$-4 - k_1 + \sqrt{\frac{5}{2}}\ k_1 + k_2 = 10 \qquad \text{(coefficient of } \lambda\text{)}$$

$$-6 + 7\,k_1 - 2\,\sqrt{10}\ k_1 - 2\,k_2 + \sqrt{10}\ k_2 = 29 \qquad \text{(constant)}$$

Writing in matrix form yields

$$\begin{pmatrix} \sqrt{\frac{5}{2}} - 1 & 1 \\ 7 - 2\,\sqrt{10} & \sqrt{10} - 2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} + \begin{pmatrix} -4 \\ -6 \end{pmatrix} = \begin{pmatrix} 10 \\ 29 \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{\frac{5}{2}} - 1 & 1 \\ 7 - 2\,\sqrt{10} & \sqrt{10} - 2 \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 14 \\ 35 \end{pmatrix}$$

We can see that the matrix on the left is rank deficient

MatrixRank$\left[\begin{pmatrix} \sqrt{\frac{5}{2}} - 1 & 1 \\ 7 - 2\,\sqrt{10} & \sqrt{10} - 2 \end{pmatrix}\right]$

1

So we know there there are either infinitely many solution or no solution. We can now create the augmented matrix of and attempt to row reduce it

RowReduce$\left[\begin{pmatrix} \sqrt{\frac{5}{2}} - 1 & 1 & 14 \\ 7 - 2\,\sqrt{10} & \sqrt{10} - 2 & 35 \end{pmatrix}\right]$ **// MatrixForm**

$$\begin{pmatrix} 1 & \frac{2}{-2+\sqrt{10}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So we see that this is inconsistent so there are no solutions.

We can verify by attempting to solve these two equations simultaneously using ‘Solve’

```
temp = Solve[{Coefficient[p[λ], λ] == Coefficient[pd[λ], λ], p[0] == pd[0]}, {k1, k2}]
```

{}

```
Clear[temp, pd, p, Acl, K, Pc, B, A]
```

Example 3: Controllable System with Multiple Controls

Again, consider the same system the B matrix is changed have multiple control inputs

$$\dot{\overline{x}}(t) = \begin{pmatrix} 0 & 3 \\ 2 & 4 \end{pmatrix} \overline{x}(t) + \begin{pmatrix} -2 & 1 \\ 1 & 1 \end{pmatrix} \overline{u}(t)$$

```
A = (0 3
     2 4); B = (-2 1
                 1 1);
```

We can quickly verify that this is controllable

```
(*Check controllability*)
Pc = Transpose[Join[Transpose[B], Transpose[A.B]]];
Pc // MatrixForm
MatrixRank[Pc]
```

$$\begin{pmatrix} -2 & 1 & 3 & 3 \\ 1 & 1 & 0 & 6 \end{pmatrix}$$

2

We can compute A_{cl}, noting that K is now a 2×2 matrix

$$K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$$

```
K = (k11 k12
     k21 k22);
Acl = A - B.K;
Acl // MatrixForm
```

$$\begin{pmatrix} 2\,\text{k11} - \text{k21} & 3 + 2\,\text{k12} - \text{k22} \\ 2 - \text{k11} - \text{k21} & 4 - \text{k12} - \text{k22} \end{pmatrix}$$

We can the compute the closed loop eigenvalues/poles of the system as

$$\det(\lambda I - A_{\text{cl}}) = 0$$

```
p[λ_] = Collect[Det[λ * IdentityMatrix[2] - Acl], {λ, λ^2}]
```

$-6 + 11\,\text{k11} - 4\,\text{k12} - \text{k21} + 3\,\text{k12}\,\text{k21} + 2\,\text{k22} - 3\,\text{k11}\,\text{k22} + (-4 - 2\,\text{k11} + \text{k12} + \text{k21} + \text{k22})\,\lambda + \lambda^2$

We keep the same desired pole locations and therefore the same closed loop characteristic equation as before

$$p_d(\lambda) = \lambda^2 + 10\,\lambda + 29 \qquad \text{(desired closed loop char eq)}$$

```
pd[λ_] = λ^2 + 10 λ + 29
```

$29 + 10\,\lambda + \lambda^2$

So we see that in order to place the poles at these desired locations, we need to choose gains k_1 and k_2 such that Eq.4 (the closed loop characteristic equation with full state feedback) matches Eq.5 (the desired closed loop characteristic equation). In other words, we require that

$$-4 - 2\,k_{11} + k_{12} + k_{21} + k_{22} = 10 \qquad \text{(coefficient of } \lambda\text{)}$$
$$-6 + 11\,k_{11} - 4\,k_{12} - k_{21} + 3\,k_{12}\,k_{21} + 2\,k_{22} - 3\,k_{11}\,k_{22} = 29 \qquad \text{(constant)}$$

So we see that we actually have 2 equations and 4 unknowns. Furthermore, since the system is controllable, we have infinite solutions. These extra degrees of freedom allow for more design decisions. For example, we can solve k_{21} and k_{22} in terms of k_{11} and k_{12}

```
temp = Solve[{Coefficient[p[λ], λ] == Coefficient[pd[λ], λ], p[0] == pd[0]}, {k21, k22}];
k21 = k21 /. temp[[1]]
k22 = k22 /. temp[[1]]
```

$$-\frac{-7 - 27\,\text{k11} - 6\,\text{k11}^2 - 6\,\text{k12} + 3\,\text{k11}\,\text{k12}}{3 \times (-1 + \text{k11} + \text{k12})}$$

$$-\frac{49 - 9\,\text{k11} - 39\,\text{k12} - 6\,\text{k11}\,\text{k12} + 3\,\text{k12}^2}{3 \times (-1 + \text{k11} + \text{k12})}$$

Now suppose we wanted to ensure that u_1 was small in comparison to u_2. In this case, we could specify that k_{11} and k_{12} are small, say 1/10 and then compute the required k_{21} and k_{22} to ensure that

```
k11 = 1 / 10;
k12 = 1 / 10;
k21 = k21
k22 = k22
```

$$-\frac{1033}{240}$$

$$\frac{4417}{240}$$

Numerically we could also write

```
k21 // N
k22 // N
```

-4.30417

18.4042

So we see that

```
K // MatrixForm // N
```

$$\begin{pmatrix} 0.1 & 0.1 \\ -4.30417 & 18.4042 \end{pmatrix}$$

```
Eigenvalues[A - B.K]
```

$\{-5 + 2\,i, -5 - 2\,i\}$

Again, note that we could do this in Matlab

```
K =

   -1.6667    1.3333
    1.6667    7.6667
```

However note that Matlab does not allow us to specify how to use the extra degrees of freedom. It simply returns a K matrix that is guarantee to place the eigenvalues at the desired location. This sets the stage for some of our discussion on Linear Quadratic Regulators, or LQR, which will give us a way to specify how to tune the *K* matrix to get the behavior we desire.

```
Clear[k11, k12, k21, k22, temp, pd, p, Acl, K, Pc, B, A]
```

Closing Thoughts

FSFB is effectively proportional control since the control law is simply directly proportional to the current state. There is no need for knowledge of past state nor a need to try and predict where the future state of the system will be.

Comparison to Root Locus: In previous discussions, we looked at using root locus to place the closed loop poles but we saw that the pole locations were constrained to locus lines. However with FSFB, we can arbitrarily move the closed loop poles anywhere.

If you think about it, THIS MEANS YOU CAN CHANGE ANY CONTROLLABLE SYSTEM TO BEHAVE LIKE ANY OTHER SYSTEM. <Talk about transforming a general aviation aircraft to act like a fighter or an experimental prototype. Can also talk about transforming yourself into a dog/person). Obviously this is a ridiculous example and sets the stage for the discussion on practical implementation issues which might illustrate why FSFB controller are not as common as you might think.