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Lecture06a
Triple Integrals (AKA Volume Integrals)

Lecture is on YouTube

The YouTube video entitled ‘Triple Integrals (AKA Volume Integrals)’ that covers this lecture is located at https://youtu.be/jd-0thQnddY.

Introduction

We now turn our attention to triple integration (AKA Volume Integrals https://en.wikipedia.org/wiki/Volume_integral)

Triple Integrals

A **triple integral** is an integral of a function $f(x, y, z)$ taken over a closed bounded, three-dimensional region T in space. We subdivide T into boxes with planes parallel to the coordinate planes. Then we consider those boxes of the subdivision that lie entirely inside T, and number them from 1 to n. In each sub box we choose an arbitrary point (x_k, y_k, z_k) in box k. The volume of the box k we denote by ΔV_k. We now form the sum

$$J_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\,\Delta V_k$$

Which is imagined as the small volume weighted by the function at that point.

As these boxes get smaller and smaller towards 0, we have the triple integral

$$\text{Triple Integral} = \iiint_T f\, dV$$

$$\iiint_T f(x, y, z)\, dV = \iiint_T f(x, y, z)\, dx\, dy\, dz$$

Cartesian Coordinates

Example: Triple Integral of a Rectangle to Compute Total Mass

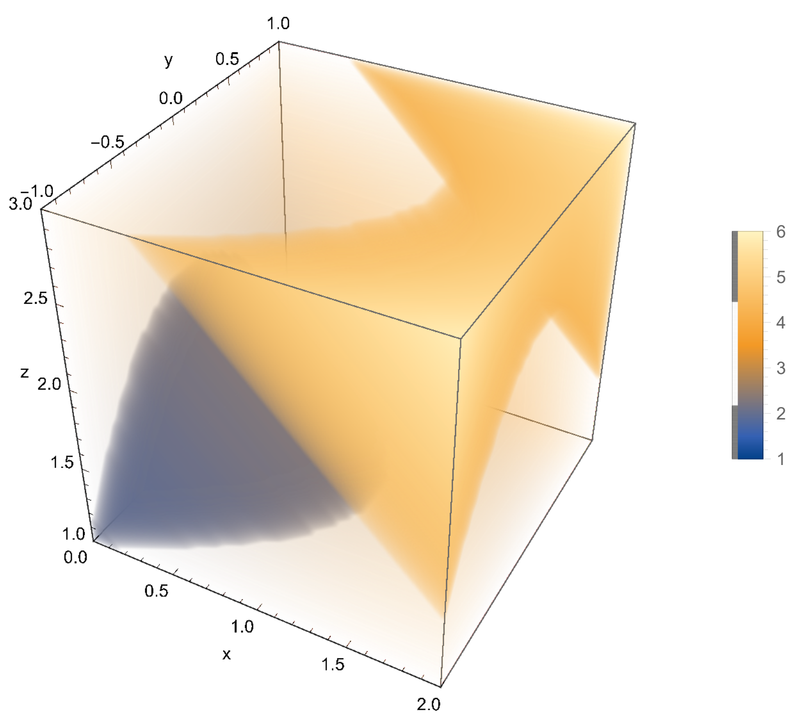
Consider the specific density in units of kg/m^3 of

$$\rho(x, y, z) = x + y^2 + z$$

In[1]:=
```
ρ[x_, y_, z_] = x + y^2 + z;

xMin = 0;
xMax = 2;
yMin = -1;
yMax = 1;
zMin = 1;
zMax = 3;
DensityPlot3D[ρ[x, y, z], {x, xMin, xMax}, {y, yMin, yMax}, {z, zMin, zMax},
 PlotLegends → Automatic,
 AxesLabel → {"x", "y", "z"}]
```

Out[8]=



We can compute the total mass using

$$\text{total mass} = \iiint_T \rho(x, y, z)\, dx\, dy\, dz$$

$$= \int_1^3 \int_{-1}^1 \int_0^2 x + y^2 + z\, dx\, dy\, dz$$

In[9]:=
```
term1 = Integrate[ρ[x, y, z], {x, xMin, xMax}]
term2 = Integrate[term1, {y, yMin, yMax}]
m = Integrate[term2, {z, zMin, zMax}]
```

Out[9]= $2 + 2\left(y^2 + z\right)$

Out[10]= $\frac{16}{3} + 4 z$

Out[11]= $\frac{80}{3}$

Cylindrical Coordinates

Note that many scenarios may lend themselves to evaluating the triple integral in cylindrical coordinates.

Cylindrical to Cartesian

$x = r\cos(\theta)$

$y = r\sin(\theta)$

$z = z$

$$\iiint_T f(x, y, z)\, dV = \iiint_D f(r\cos(\theta), r\sin(\theta), z)\, r\, dr\, d\theta\, dz$$

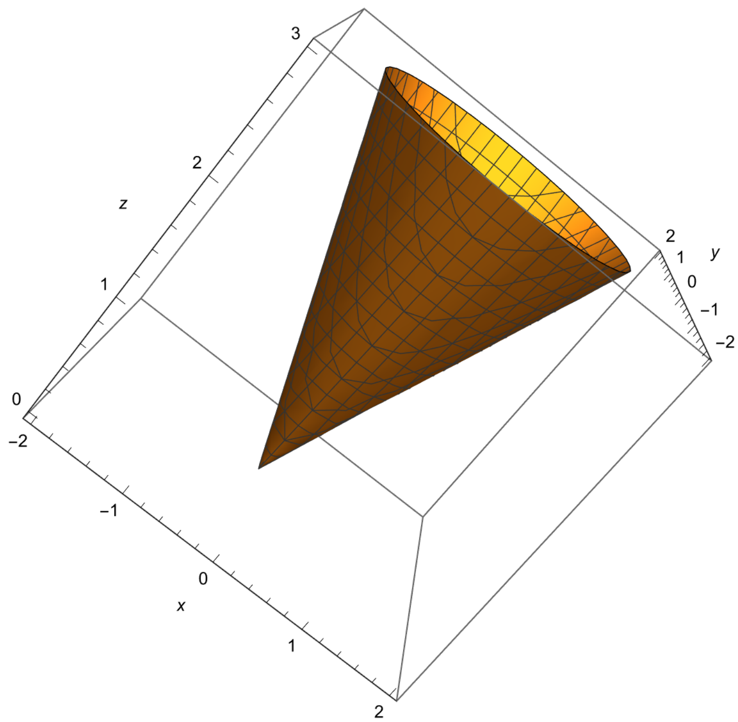
Example: Triple Integral of a Cone to Obtain Volume

Consider the cone with an outside shell defined by

$$z^2 = 4\left(x^2 + y^2\right)$$

In[12]:= `ContourPlot3D[4 (x^2 + y^2) == z^2, {x, -2, 2}, {y, -2, 2}, {z, 0, 3}, AxesLabel → {x, y, z}]`

Out[12]=



We note that

$r = 0$ when $z = 0$

$r = 3/2$ when $z = 3$

The volume of a cylinder with radius R can be calculated by integrating the constant function 1 over the volume

$$V = \iiint_T 1 \, dx \, dy \, dz$$

It is easier to change to cylindrical coordinates

$$= \iiint_D 1 \, r \, dr \, d\theta \, dz$$

$$= \iiint_D r \, dr \, d\theta \, dz$$

It is easier to first integrate over z

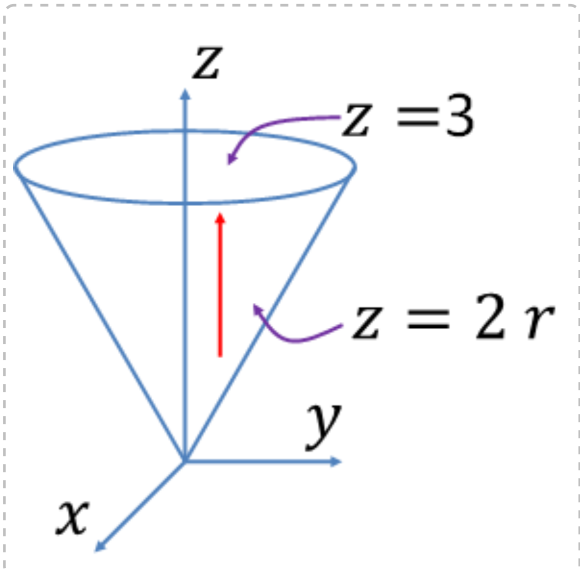
$$= \iiint_D r \, dz \, d\theta \, dr$$

We note that the radius and z are related through

$$\frac{3/2}{3} z = r$$

$$z = 2\,r$$

So if we integrate in the z direction we go from $z_{min} = 2\,r$ and $z_{max} = 3$



so we have

$$= \int_0^{3/2} \int_0^{2\,\pi} \int_{2\,r}^{3} r \, dz \, d\theta \, dr$$

$$= \int_0^{\pi} \int_0^{2\,\pi} (3 - 2\,r)\, r \, d\theta \, dr$$

$$= \int_0^{\pi} 2\,\pi (3 - 2\,r)\, r \, dr$$

$$V = \frac{9\,\pi}{4}$$

```
In[13]:= term1 = Integrate[r, {z, 2 r, 3}]
         term2 = Integrate[term1, {θ, 0, 2 π}]
         V = Integrate[term2, {r, 0, 3 / 2}]

Out[13]= (3 - 2 r) r

Out[14]= 2 π (3 - 2 r) r

Out[15]= 9 π
         ---
          4
```

We can check this with the familiar formula that gives the volume of a right cone as

$$V = \pi\, r^2 \, \frac{h}{3}$$

In[16]:=
```
Vcheck = π r^2 h/3 /. {r → 3 / 2, h → 3}
```

Out[16]= $\frac{9\pi}{4}$

As can be seen, these match.

Spherical Coordinates

Note that many scenarios may lend themselves to evaluating the triple integral in spherical coordinates.

Spherical to Cartesian

$x = r \sin(\phi) \cos(\theta)$

$y = r \sin(\phi) \sin(\theta)$

$z = r \cos(\phi)$

$$\iiint_T f(x, y, z)\, dV = \iiint_D f(r \sin(\phi) \cos(\theta), r \sin(\phi) \sin(\theta), r \cos(\phi))\, r^2 \sin(\phi)\, dr\, d\theta\, d\phi$$

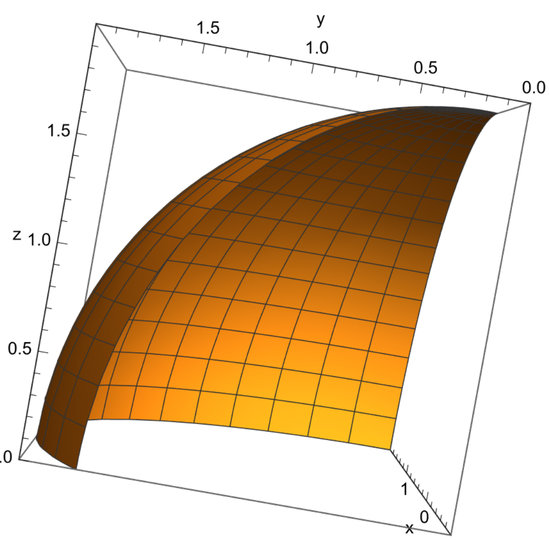
Example: Triple Integral of a Partial Sphere to Obtain Volume

Consider the partial sphere (1/6 of a full sphere) shown below

In[17]:=
```
polarAngleMax = π / 2;  (*We call this ϕ*)
azimuthAngleMax = 2 π / 3;   (*We call this θ*)

SphericalPlot3D[r /. {r → 2},
 {polarAngle, 0, polarAngleMax}, {azimuthAngle, 0, azimuthAngleMax},
 AxesLabel → {"x", "y", "z"}]
```

Out[19]=



The volume of a partial sphere with radius R can be calculated by integrating the constant function 1 over the partial sphere

$$V = \iiint_T 1 \, dx \, dy \, dz$$

It is easier to change to spherical coordinates

$$= \iiint_D 1 \, r^2 \sin(\phi) \, dr \, d\theta \, d\phi$$

$$= \iiint_D r^2 \sin(\phi) \, dr \, d\theta \, d\phi$$

Note that the sphere is described by $r \in [0, R]$, $\theta \in [0, 2\pi/3]$, $\phi \in [0, \pi/2]$

so we have

$$= \int_0^{\pi/2} \int_0^{2\pi/3} \int_0^R r^2 \sin(\phi) \, dr \, d\theta \, d\phi$$

$$= \int_0^{\pi/2} \int_0^{2\pi/3} \frac{R^3 \sin(\phi)}{3} \, d\theta \, d\phi$$

$$= \int_0^{\pi} \frac{2\pi R^3 \sin(\phi)}{9} \, d\phi$$

$$V = \frac{2\pi R^3}{9}$$

```
In[20]:= term1 = Integrate[r^2 Sin[ϕ], {r, 0, R}]
         term2 = Integrate[term1, {θ, 0, 2 π / 3}]
         V = Integrate[term2, {ϕ, 0, π / 2}]
```

Out[20]= $\frac{1}{3} R^3 \text{Sin}[\phi]$

Out[21]= $\frac{2}{9} \pi R^3 \text{Sin}[\phi]$

Out[22]= $\frac{2 \pi R^3}{9}$

We can compare this against the known formula of a sphere of $V_t = \frac{4}{3} \pi R^3$and dividing this result by 6 because we only have 1/6 of a full sphere.

```
In[23]:= Vcheck = 4/3 π R^3 / 6
```

Out[23]= $\frac{2 \pi R^3}{9}$

As can be seen, these match.

Now that we have confidence that we setup the limits of integration correctly, let’s consider a function f that is not simply 1. Consider the same density function in the first example

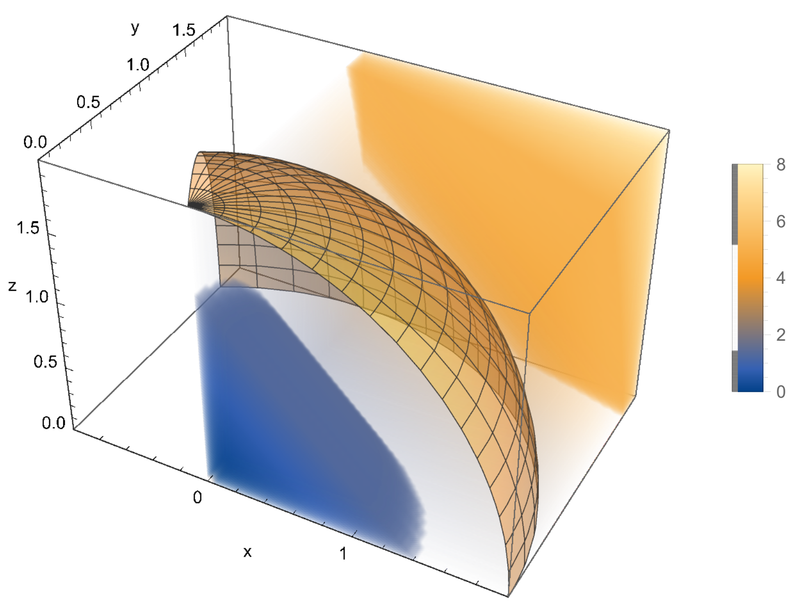
$$f(x, y, z) = \rho(x, y, z) = x + y^2 + z$$

In[24]:=
```
plot1 = SphericalPlot3D[r /. {r → 2},
   {polarAngle, 0, polarAngleMax}, {azimuthAngle, 0, azimuthAngleMax},
   PlotStyle → Opacity[0.4]];

xMin = 0;
xMax = 2;
yMin = 0;
yMax = 2;
zMin = 0;
zMax = 2;
plot2 = DensityPlot3D[ρ[x, y, z], {x, xMin, xMax}, {y, yMin, yMax}, {z, zMin, zMax},
   PlotLegends → Automatic];

Show[plot1, plot2,
 AxesLabel → {"x", "y", "z"}]
```

Out[32]=



So the integral becomes

$$\text{total mass} = \int_0^{\pi/2} \int_0^{2\pi/3} \int_0^R f(r\sin(\phi)\cos(\theta), r\sin(\phi)\sin(\theta), r\cos(\phi))\, r^2 \sin(\phi)\, dr\, d\theta\, d\phi$$

$$x = r\sin(\phi)\cos(\theta)$$

$$y = r\sin(\phi)\sin(\theta)$$

$$z = r\cos(\phi)$$

So the integrand becomes

In[33]:=
```
integrand = ρ[x, y, z] * r^2 Sin[ϕ] /. {x → r Sin[ϕ] Cos[θ], y → r Sin[ϕ] Sin[θ], z → r Cos[ϕ]}
```

Out[33]= $r^2 \operatorname{Sin}[\phi] \left(r \operatorname{Cos}[\phi] + r \operatorname{Cos}[\theta] \operatorname{Sin}[\phi] + r^2 \operatorname{Sin}[\theta]^2 \operatorname{Sin}[\phi]^2\right)$

So we have

total mass = $\int_0^{\pi/2}\int_0^{2\pi/3}\int_0^{R} r^2 \operatorname{Sin}[\phi]\left(r \operatorname{Cos}[\phi] + r \operatorname{Cos}[\theta] \operatorname{Sin}[\phi] + r^2 \operatorname{Sin}[\theta]^2 \operatorname{Sin}[\phi]^2\right) dr\, d\theta\, d\phi$ consider $R = 2$

In[34]:=
```
term1 = Integrate[integrand, {r, 0, 2}]
term2 = Integrate[term1, {θ, 0, 2π / 3}]
totalMass = Integrate[term2, {ϕ, 0, π / 2}]
totalMass // N
```

Out[34]= $4 \operatorname{Cos}[\phi] \operatorname{Sin}[\phi] + 4 \operatorname{Cos}[\theta] \operatorname{Sin}[\phi]^2 + \frac{32}{5} \operatorname{Sin}[\theta]^2 \operatorname{Sin}[\phi]^3$

Out[35]= $\frac{8}{3} \pi \operatorname{Cos}[\phi] \operatorname{Sin}[\phi] + 2\sqrt{3} \operatorname{Sin}[\phi]^2 + \frac{4}{15} \times \left(3\sqrt{3} + 8\pi\right) \operatorname{Sin}[\phi]^3$

Out[36]= $\frac{1}{90} \times \left(48\sqrt{3} + \left(248 + 45\sqrt{3}\right)\pi\right)$

Out[37]= 12.3013