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Lecture01g
Eigenvalues and Eigenvectors

The YouTube video entitled ‘Eigenvalues and Eigenvectors’ that covers this lecture is located at https://youtu.be/PqaoKr1IlTg.

Outline

-Introduction
Computing Eigenvalues and Eigenvectors
- -Computing Eigenvalues and Eigenvectors (2x2 with Real Eigenvalues)
- -Computing Eigenvalues and Eigenvectors (2x2 with Complex Eigenvalues)
- -Computing Eigenvalues and Eigenvectors (3x3 with Real Eigenvalues)

Introduction

Consider simple multiplication of

$$b = A\,x$$

where A, x, b are all scalars

In this case, if we consider x to be the input, then A simply multiplies/scales the input x to produce the output, y. In other words, the output is simply a scaled version of the input. While this result is trivial for scalars, what if we instead consider

$$\overline{b} = A\,\overline{x}$$

where A = square matrix, $n-\text{by}-n$
$\overline{x}$, $\overline{b}$ are $n-\text{by}-1$ vectors

In this case, we know that the matrix A will transform the vector $\overline{x}$ to produce a new vector $\overline{b}$. In all likelihood, the output vector $\overline{b}$ will not be a simple scaled version of the input vector $\overline{x}$. The question is, are there certain values of the input $\overline{x}$ that will lead to the output $\overline{b}$ being a scaled version of the input? This is the crux of the eigenvalue/eigenvector problem.

Matrix eigenvalue/eigenvector problems consider the equation

$$A\,\overline{x} = \lambda\,\overline{x} \qquad \textbf{\textit{(Eq.1)}}$$

where A = square matrix, n – by – n
λ = scalar (eigenvalue)
$\overline{x}$ = vector, n – by – 1 (eigenvector)

This seemingly simple relationship has an incredible amount of application to various engineering problems.

Computing Eigenvalues and Eigenvectors

Consider a general matrix equation

$$A\,\overline{x} = \overline{b}$$

We can look at this as the matrix A operating on the vector $\overline{x}$ to produce a new vector, $\overline{b}$.

Consider the situation where

$$A = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix}$$

We can arbitrarily choose a vector $\overline{x} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$. In this case, we can easily compute $\overline{b}$

In[ ]:=
```
A = (-5  2
      2 -2);
x = (5
     1);

(*Compute b*)
b = A.x;

Print["b"]
b // MatrixForm
```

b

Out[ ]//MatrixForm=

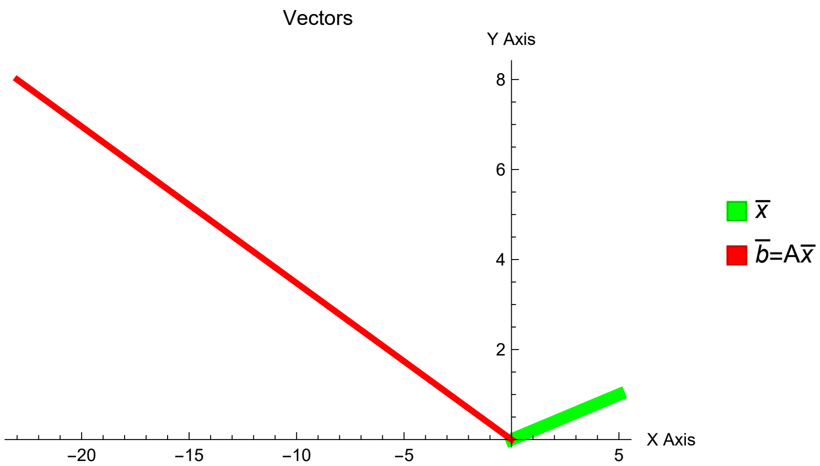
$$\begin{pmatrix} -23 \\ 8 \end{pmatrix}$$

We can plot $\overline{x}$ and $\overline{b}$

Plotting in Matlab is assigned during hw01 (Matlab03 - 2D Plotting in Matlab)

In[ ]:=
```
(*Plot both x and b*)
xPt1 = {0, 0};
xPt2 = {x[[1, 1]], x[[2, 1]]};

bPt1 = {0, 0};
bPt2 = {b[[1, 1]], b[[2, 1]]};

Legended[
 Show[
  (*Vector 1*)
  ListLinePlot[{xPt1, xPt2},
   PlotStyle → {Green, Thickness[0.02]}],

  (*Vector 2*)
  ListLinePlot[{bPt1, bPt2},
   PlotStyle → {Red, Thickness[0.01]}],

  (*Plot Options*)
  PlotLabel → "Vectors",
  AxesLabel → {"X Axis", "Y Axis"},
  PlotRange → All
 ],

 (*Add legend information*)
 SwatchLegend[{Green, Red}, {"x̄", "b̄=Ax̄"}]
]
```

Out[ ]=



As expected, the matrix A transforms the vector $\overline{x}$ into the vector $\overline{b}$.

Suppose we choose a different vector $\overline{x} = \begin{pmatrix} -4 \\ 2 \end{pmatrix}$. In this case, we can easily compute $\overline{b}$

In[ ]:=
```
x2 = (-4
       2);

(*Compute b*)
b2 = A.x2;

Print["b2"]
b2 // MatrixForm
```

b2

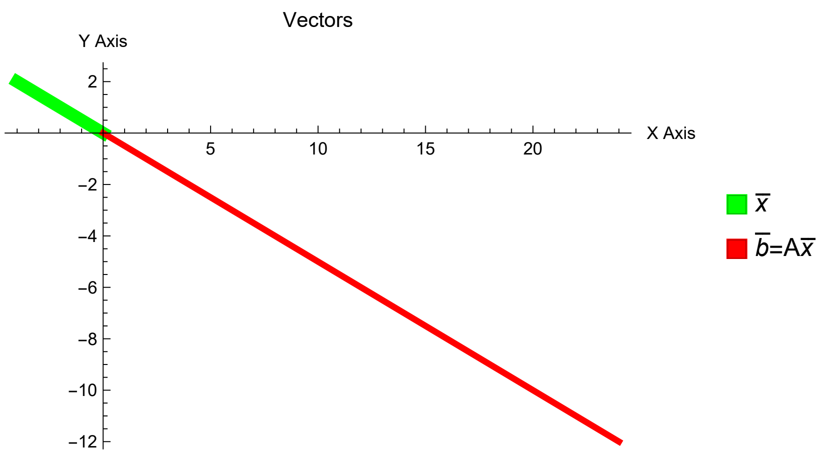
Out[ ]//MatrixForm=

$$\begin{pmatrix} 24 \\ -12 \end{pmatrix}$$

We can plot $\overline{x}$ and $\overline{b}$

In[•]:=
```
(*Plot both x and b*)
xPt1 = {0, 0};
xPt2 = {x2[[1, 1]], x2[[2, 1]]};

bPt1 = {0, 0};
bPt2 = {b2[[1, 1]], b2[[2, 1]]};

Legended[
 Show[
  (*Vector 1*)
  ListLinePlot[{xPt1, xPt2},
   PlotStyle → {Green, Thickness[0.02]}],

  (*Vector 2*)
  ListLinePlot[{bPt1, bPt2},
   PlotStyle → {Red, Thickness[0.01]}],

  (*Plot Options*)
  PlotLabel → "Vectors",
  AxesLabel → {"X Axis", "Y Axis"},
  PlotRange → All
 ],

 (*Add legend information*)
 SwatchLegend[{Green, Red}, {"x̄", "b̄=Ax̄"}]
]
```

Out[•]=



In this situation, the interesting result is that $\overline{b} = -6\,\overline{x}$. In other words, the output is a scaled version of the input.

In[•]:=
```
b2 == -6 x2
```

Out[•]= True

The value of $\lambda = -6$ is referred to as the eigenvalue and the vector $\overline{x} = (\,-4 \quad 2\,)^T$ is a corresponding eigenvector.

In[•]:=
```
Clear[A, x, b, x2, b2, xPt1, xPt2, bPt1, bPt2]
```

Recall the eigenvalue/eigenvector equation is

$$A\,\overline{x} = \lambda\,\overline{x}$$

Of course, a solution of $\overline{x} = \overline{0}$ is a valid solution, but this is trivial and therefore of no interest to us. We seek non-zero values of λ and $\overline{x}$.

One method of finding eigenvalues is therefore to systematically and iteratively chose vectors $\overline{x}$ and examine when the output $\overline{b}$ is a scaled multiple of the input $\overline{x}$.

Show Matlab animation

From the numerical analysis, we obtain

2D case

$$\overline{v}_1 = \begin{pmatrix} -1.794 \\ 0.884 \end{pmatrix} \quad \lambda_1 = -6$$

$$\overline{v}_2 = \begin{pmatrix} 0.898 \\ 1.787 \end{pmatrix} \quad \lambda_2 = -1$$

3D case

$$\overline{v}_1 = \begin{pmatrix} -1.891 \\ 0 \\ 0.651 \end{pmatrix} \quad \lambda_1 = 1$$

Of course, this is not tractable for larger dimension problems so we should investigate how to analytically calculate eigenvalues and eigenvectors.

Computing Eigenvalues and Eigenvectors (2x2 with Real Eigenvalues)

We illustrate the process of finding an eigenvalue/eigenvector pair through an example

Example 1: Determination of Eigenvalues and Eigenvectors

Consider the following matrix

In[ ]:= $\mathbf{A} = \begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix};$

We first determine the eigenvalues. Recall that the eigenvalue λ and eigenvector $\overline{x}$ satisfy the following equation

$$A\,\overline{x} = \lambda\,\overline{x}$$

$$A\,\overline{x} - \lambda\,\overline{x} = \overline{0}$$

$$(A - \lambda\,I)\,\overline{x} = \overline{0}$$

As we stated previously, the solution of $\overline{x} = \overline{0}$ is valid, but of no use to us. Therefore, we need to find vectors $\overline{x}$ which are in the null space of the matrix $A - \lambda\,I$. Recall from basic linear algebra that the null space of $A - \lambda I$ is empty if the matrix is full rank. Therefore, we seek values λ which make the matrix $A - \lambda I$ singular. Recall that a matrix is singular if and only if its determinant is 0. Therefore, we seek values of λ which satisfy the equation

$$\det(A - \lambda I) = 0$$

In[ ]:=
```
charEq[λ_] = Det[A - λ IdentityMatrix[2]]
```

Out[ ]= $6 + 7\lambda + \lambda^2$

This is referred to as the **characteristic equation** or **characteristic polynomial** of the system.

We can easily solve this using the quadratic equation or any other technique you like (see Finding Roots of a Polynomial Using Matlab, Mathematica, and a TI-83 https://youtu.be/J8il5eB_VS8).

In[ ]:=
```
temp = Solve[charEq[λ] == 0, λ];
λ1 = λ /. temp[[1]]
λ2 = λ /. temp[[2]]
```

Out[ ]= -6

Out[ ]= -1

Eigenvector Corresponding to $\lambda_1 = -6$

We can now find the eigenvector corresponding to $\lambda_1 = -6$.

$$(A - \lambda_1 I)\,\overline{x} = \overline{0}$$

Recall that λ_1 was chosen specifically so that the matrix $(A - \lambda_1 I)$ was not full rank. Therefore, we do not expect there to be a unique solution, rather $\overline{x}$ can live anywhere within the null space of $A - \lambda_1 I$ and still be a solution. Writing this out yields

$$\left(\begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x_1 + 2x_2 = 0$$
$$2x_1 + 4x_2 = 0$$

From the first equation, we obtain $x_1 = -2x_2$.

From the second equation, we also obtain $x_1 = -2x_2$.

Therefore, we can arbitrarily choose either x_1 or x_2 to obtain the vector as

$$\overline{x} = \begin{pmatrix} -2x_2 \\ x_2 \end{pmatrix} = x_2\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

So an eigenvector is given by

$$\overline{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

In[ ]:=
```
v1 = (-2
       1)
```

Out[ ]=
```
{{-2}, {1}}
```

We can verify that it satisfies the eigenvalue equation.

In[ ]:=
```
(A - λ1 IdentityMatrix[2]).v1 // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We can verify that any scalar multiple, α, of this satisfies the eigenvalue equation as well.

In[ ]:=
```
(A - λ1 IdentityMatrix[2]).(α * v1) // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We can compare this with the numerically computed eigenvalue we obtained earlier

$$\overline{v}_1 = \begin{pmatrix} -1.794 \\ 0.884 \end{pmatrix} \; \lambda_1 = -6$$

In[ ]:=
```
v1n = (-1.794
        0.884);
```

In[ ]:=
```
v1 / Norm[v1] // MatrixForm // N
v1n / Norm[v1n] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -0.894427 \\ 0.447214 \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} -0.897012 \\ 0.442006 \end{pmatrix}$$

So we see this is roughly equivalent and any discrepancies are easily attributed to numerical errors.

Eigenvector Corresponding to $\lambda_2 = -1$

We can now find the eigenvector corresponding to $\lambda_2 = -1$.

$$(A - \lambda_2 I)\,\overline{x} = \overline{0}$$

Writing this out yields

$$\left(\begin{pmatrix} -5 & 2 \\ 2 & -2 \end{pmatrix} + 1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-4\,x_1 + 2\,x_2 = 0$$

$$2\,x_1 - 1\,x_2 = 0$$

From the first equation, we obtain $x_1 = x_2/2$.

From the second equation, we also obtain $x_1 = x_2/2$.

Therefore, we can arbitrarily choose either x_1 or x_2 to obtain the vector as

$$\bar{x} = \begin{pmatrix} x_2/2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

So an eigenvector is given by

$$\bar{v}_2 = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

In[∘]:= $\mathbf{v2} = \begin{pmatrix} \mathbf{1/2} \\ \mathbf{1} \end{pmatrix};$

We can verify that it satisfies the eigenvalue equation.

In[∘]:= **(A - λ2 IdentityMatrix[2]).v2 // MatrixForm**

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We can compare this with the numerically computed eigenvalue we obtained earlier

$$\bar{v}_2 = \begin{pmatrix} 0.898 \\ 1.787 \end{pmatrix} \quad \lambda_2 = -1$$

In[∘]:= $\mathbf{v2n} = \begin{pmatrix} \mathbf{0.898} \\ \mathbf{1.787} \end{pmatrix};$

In[∘]:= **v2 / Norm[v2] // MatrixForm // N**
v2n / Norm[v2n] // MatrixForm

Out[∘]//MatrixForm=

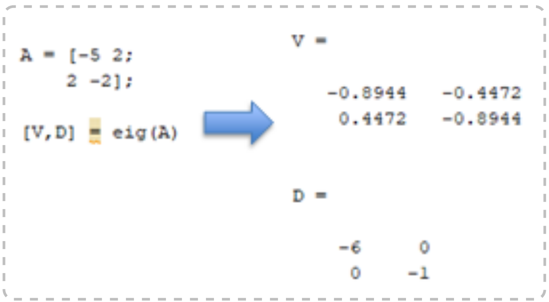
$$\begin{pmatrix} 0.447214 \\ 0.894427 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0.449013 \\ 0.893525 \end{pmatrix}$$

So we see this is roughly equivalent and any discrepancies are easily attributed to numerical errors.

We can use various numerical tools such as ‘eig’ in Matlab



In Matlab, note that the first returned object is the eigenvectors in column format and the second returned object is the eigenvalues in a diagonal matrix

We can also use 'Eigensystem' in Mathematica

In[ ]:=
```
{vals, vecs} = Eigensystem[A];
vals // MatrixForm
vecs // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -6 \\ -1 \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

In[ ]:=
```
vecs // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$$

In Mathematica, note that the first returned object is the eigenvalues as a list and the second returned object is the eigenvectors in row format

Computing Eigenvalues and Eigenvectors (2x2 with Complex Eigenvalues)

Consider another example of a matrix

In[ ]:= $A = \begin{pmatrix} -5 & 2 \\ -2 & -2 \end{pmatrix};$

We can repeat the process

In[ ]:=
```
Print["Characteristic Equation"]
charEq[λ_] = Det[A - λ IdentityMatrix[2]]
Print[]

Print["Solution of Characteristic Equation"]
temp = Solve[charEq[λ] == 0, λ];
λ1 = λ /. temp[[1]]
λ2 = λ /. temp[[2]]
Print[]
```

Characteristic Equation

Out[ ]= $14 + 7\lambda + \lambda^2$

```
Solution of Characteristic Equation
```

Out[•]= $\frac{1}{2} \times \left(-7 - i\sqrt{7}\right)$

Out[•]= $\frac{1}{2} \times \left(-7 + i\sqrt{7}\right)$

Eigenvector Corresponding to $\lambda_1 = -\frac{7}{2} - \frac{\sqrt{7}}{2}\, i$

We can now find the eigenvector corresponding to $\lambda_1 = -\frac{7}{2} - \frac{\sqrt{7}}{2}\, i$.

$$(A - \lambda_1 I)\,\overline{x} = \overline{0}$$

Recall that λ_1 was chosen specifically so that the matrix $(A - \lambda_1 I)$ was not full rank. Therefore, we do not expect there to be a unique solution, rather $\overline{x}$ can live anywhere within the null space of $A - \lambda_1 I$ and still be a solution. Writing this out yields

$$\left(\begin{pmatrix} -5 & 2 \\ -2 & -2 \end{pmatrix} - \left(-\frac{7}{2} - \frac{\sqrt{7}}{2}\, i\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} -5 & 2 \\ -2 & -2 \end{pmatrix} + \left(\frac{7}{2} + \frac{\sqrt{7}}{2}\, i\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{pmatrix} -5 & 2 \\ -2 & -2 \end{pmatrix} + \begin{pmatrix} \frac{7}{2} + \frac{\sqrt{7}}{2}\, i & 0 \\ 0 & \frac{7}{2} + \frac{\sqrt{7}}{2}\, i \end{pmatrix}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{7}{2} + \frac{\sqrt{7}}{2}\, i - 5 & 2 \\ -2 & \frac{7}{2} + \frac{\sqrt{7}}{2}\, i - 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{7}{2} + \frac{\sqrt{7}}{2}\, i - \frac{10}{2} & 2 \\ -2 & \frac{7}{2} + \frac{\sqrt{7}}{2}\, i - \frac{4}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \frac{-3}{2} + \frac{\sqrt{7}}{2}\, i & 2 \\ -2 & \frac{3}{2} + \frac{\sqrt{7}}{2}\, i \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left(\frac{-3}{2} + \frac{\sqrt{7}}{2}\, i\right) x_1 + 2\, x_2 = 0$$
$$-2\, x_1 + \left(\frac{3}{2} + \frac{\sqrt{7}}{2}\, i\right) x_2 = 0$$

In[ ]:=
```
A - λ1 IdentityMatrix[2] // Expand // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{3}{2} + \frac{i\sqrt{7}}{2} & 2 \\ -2 & \frac{3}{2} + \frac{i\sqrt{7}}{2} \end{pmatrix}$$

From the first equation, we can write

$$2\,x_2 = \left(\frac{3}{2} - \frac{\sqrt{7}}{2}\,i\right) x_1$$

$$x_2 = \frac{1}{2} \times \left(\frac{3}{2} - \frac{\sqrt{7}}{2}\,i\right) x_1$$

$$x_2 = \left(\frac{3}{4} - \frac{\sqrt{7}}{4}\,i\right) x_1$$

So we can write

$$\overline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 \\ \left(\frac{3}{4} - \frac{\sqrt{7}}{4}\,i\right) x_1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ \left(\frac{3}{4} - \frac{\sqrt{7}}{4}\,i\right) \end{pmatrix} x_1$$

So an eigenvector corresponding to λ_1 can be given as

$$\overline{v}_1 = \begin{pmatrix} 1 \\ \left(\frac{3}{4} - \frac{\sqrt{7}}{4}\,i\right) \end{pmatrix}$$

In[ ]:=
$$\textbf{v1} = \begin{pmatrix} 1 \\ \frac{3}{4} - \frac{\sqrt{7}}{4}\,\mathrm{I} \end{pmatrix};$$

We can verify that it satisfies the eigenvalue equation.

In[ ]:=
```
(A - λ1 IdentityMatrix[2]).v1 // Simplify // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We can verify that any scalar multiple, α, of this satisfies the eigenvalue equation as well.

In[ ]:=
```
(A - λ1 IdentityMatrix[2]).(α * v1) // Simplify // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Eigenvector Corresponding to $\lambda_2 = -\frac{7}{2} + \frac{\sqrt{7}}{2}\,i$

We can now find the eigenvector corresponding to $\lambda_2 = -\frac{7}{2} + \frac{\sqrt{7}}{2}\, i$.

$$(A - \lambda_2 I)\,\overline{x} = \overline{0}$$

In[∘]:= `A - λ2 IdentityMatrix[2] // Expand // MatrixForm`

Out[∘]//MatrixForm=

$$\begin{pmatrix} -\frac{3}{2} - \frac{i\sqrt{7}}{2} & 2 \\ -2 & \frac{3}{2} - \frac{i\sqrt{7}}{2} \end{pmatrix}$$

Writing this out yields

$$\left(\frac{-3}{2} - \frac{\sqrt{7}}{2}\, i\right) x_1 + 2\, x_2 = 0$$
$$-2\, x_1 + \left(\frac{3}{2} - \frac{\sqrt{7}}{2}\, i\right) x_2 = 0$$

For variety, we can inspect the 2nd equation

$$-2\, x_1 + \left(\frac{3}{2} - \frac{\sqrt{7}}{2}\, i\right) x_2 = 0$$

$$2\, x_1 = \left(\frac{3}{2} - \frac{\sqrt{7}}{2}\, i\right) x_2$$

$$x_1 = \left(\frac{3}{4} - \frac{\sqrt{7}}{4}\, i\right) x_2$$

Therefore, we can obtain the vector as

$$\overline{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{3}{4} - \frac{\sqrt{7}}{4}\, i\right) x_2 \\ x_2 \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{3}{4} - \frac{\sqrt{7}}{4}\, i\right) \\ 1 \end{pmatrix} x_2$$

So an eigenvector is given by

$$\overline{v}_2 = \begin{pmatrix} \left(\frac{3}{4} - \frac{\sqrt{7}}{4}\, i\right) \\ 1 \end{pmatrix}$$

In[∘]:= **v2** = $\begin{pmatrix} \frac{3}{4} - \frac{\sqrt{7}}{4}\, \text{I} \\ 1 \end{pmatrix}$;

We can verify that it satisfies the eigenvalue equation.

In[ ]:=
```
(A - λ2 IdentityMatrix[2]).v2 // Expand // MatrixForm
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We can verify

In[ ]:=
```
temp = Eigenvectors[A];
v1check = temp[[1]]
v1

v2check = temp[[2]]
v2
```
Out[ ]= $\left\{-\frac{1}{4} i \left(3 i + \sqrt{7}\right), 1\right\}$

Out[ ]= $\left\{\{1\}, \left\{\frac{3}{4} - \frac{i\sqrt{7}}{4}\right\}\right\}$

Out[ ]= $\left\{\frac{1}{4} i \left(-3 i + \sqrt{7}\right), 1\right\}$

Out[ ]= $\left\{\left\{\frac{3}{4} - \frac{i\sqrt{7}}{4}\right\}, \{1\}\right\}$

In[ ]:=
```
v2check // Expand
```
Out[ ]= $\left\{\frac{3}{4} + \frac{i\sqrt{7}}{4}, 1\right\}$

In[ ]:=
```
v2 // Expand
```
Out[ ]= $\left\{\left\{\frac{3}{4} - \frac{i\sqrt{7}}{4}\right\}, \{1\}\right\}$

In[ ]:=
```
v1 // Expand
```
Out[ ]= $\left\{\{1\}, \left\{\frac{3}{4} - \frac{i\sqrt{7}}{4}\right\}\right\}$

In[ ]:=
```
{vals, vecs} = Eigensystem[A]
```
Out[ ]= $\left\{\left\{\frac{1}{2} \times \left(-7 + i\sqrt{7}\right), \frac{1}{2} \times \left(-7 - i\sqrt{7}\right)\right\}, \left\{\left\{-\frac{1}{4} i \left(3 i + \sqrt{7}\right), 1\right\}, \left\{\frac{1}{4} i \left(-3 i + \sqrt{7}\right), 1\right\}\right\}\right\}$

In[ ]:=
```
vals // Expand // MatrixForm
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{7}{2} + \frac{i\sqrt{7}}{2} \\ -\frac{7}{2} - \frac{i\sqrt{7}}{2} \end{pmatrix}$$

In[ ]:=
```
vecs // Expand // MatrixForm
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{3}{4} - \frac{i\sqrt{7}}{4} & 1 \\ \frac{3}{4} + \frac{i\sqrt{7}}{4} & 1 \end{pmatrix}$$

In[]:=
```
λ1 // Expand
```
Out[]= $-\frac{7}{2}-\frac{i\sqrt{7}}{2}$

Computing Eigenvalues and Eigenvectors (3x3 with Real Eigenvalues)

For practice, we can illustrate how this process extends to a 3x3 matrix. Consider the following *A* matrix

In[]:=
$$A = \begin{pmatrix} -2 & -2 & -9 \\ -1 & 1 & -3 \\ 1 & 1 & 4 \end{pmatrix};$$

We can easily compute the eigenvalues

In[]:=
```
Print["Characteristic Equation"]
charEq[λ_] = Det[A - λ IdentityMatrix[3]]
Print[]

Print["Solution of Characteristic Equation"]
temp = Solve[charEq[λ] == 0, λ];
λ1 = λ /. temp[[1]]
λ2 = λ /. temp[[2]]
λ3 = λ /. temp[[3]]
Print[]
```

Characteristic Equation

Out[]= $2-4\lambda+3\lambda^2-\lambda^3$

Solution of Characteristic Equation

Out[]= 1

Out[]= $1-i$

Out[]= $1+i$

Eigenvector Corresponding to $\lambda_1 = 1$

We can now find the eigenvector corresponding to $\lambda_1 = -1$.

$$(A - \lambda_1 I)\,\overline{x} = \overline{0}$$

In[]:=
```
Atilde = A - λ1 IdentityMatrix[3];
Atilde // MatrixForm
```

Out[]//MatrixForm=
$$\begin{pmatrix} -3 & -2 & -9 \\ -1 & 0 & -3 \\ 1 & 1 & 3 \end{pmatrix}$$

$$\begin{pmatrix} -3 & -2 & -9 \\ -1 & 0 & -3 \\ 1 & 1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We can row reduce the matrix to obtain

In[∘]:=
```
RowReduce[Atilde] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So we have

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From the second equation, we obtain $x_2 = 0$.

From the first equation we obtain

$$x_1 + 3\, x_3 = 0$$

$$x_1 = -3\, x_3$$

Therefore, we can write

$$\overline{x} = \begin{pmatrix} -3\, x_3 \\ 0 \\ x_3 \end{pmatrix} = x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

So an eigenvector is given by

$$\overline{v}_1 = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$$

In[∘]:=
```
v1 = ( -3
        0
        1 );
```

We can verify that it satisfies the eigenvalue equation.

In[∘]:=
```
(A - λ1 IdentityMatrix[3]).v1 // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

We can verify that any scalar multiple, α, of this satisfies the eigenvalue equation as well.

In[ ]:=
```
(A - λ1 IdentityMatrix[3]).(α * v1) // MatrixForm
```
Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

8.2: Some Applications of Eigenvalue Problems

We will investigate several applications in the coming weeks when we study ordinary differential equations and partial differential equations.