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Lecture 03a
Homogeneous Linear Ordinary Differential Equations

Lecture is on YouTube

The YouTube video entitled 'Homogeneous Linear Ordinary Differential Equations' that covers this lecture is located at https://youtu.be/3Kox-3APznI.

Outline

-Homogeneous Linear Ordinary Differential Equations

Homogeneous Linear Ordinary Differential Equations

We are interested in solving linear, time invariant differential equations (often referred to as LTI systems). We now review the standard method of solving differential equations. Consider a second order, linear, homogeneous differential equation

$$a\,\ddot{x}(t) + b\,\dot{x}(t) + c\,x(t) = 0 \qquad \textbf{\textit{(Eq.1.1)}}$$

with $x(t_0) = x_0$
$\quad\;\; \dot{x}(t_0) = \dot{x}_0$

We can propose a solution of the form

$$x(t) = e^{r\,t} \qquad \textbf{\textit{(Eq.1.2)}}$$

where $\quad r$ is a parameter to be determined

```
x[t_] = Exp[r t];
```

We can substitute Eq.1.2 into Eq.1.1 to obtain

$$a\,\frac{d^2}{d\,t^2}\left[e^{r\,t}\right] + b\,\frac{d}{d\,t}\left[e^{r\,t}\right] + c\,e^{r\,t} = 0$$

$$a\,r^2\,e^{r\,t} + b\,r\,e^{r\,t} + c\,e^{r\,t} = 0$$

$$\left(a\,r^2\; + b\,r\; + c\right)e^{r\,t} = 0$$

```
Collect[a D[x[t], {t, 2}] + b D[x[t], t] + c x[t], Exp[-r t]]
```

$e^{r\,t}\,\left(c + b\,r + a\,r^2\right)$

Since $e^{r\,t} \neq 0\;\forall\,t$, we require that $a\,r^2 + b\,r + c = 0$. This is known as the characteristic equation and we must find values of r which make it zero. In this case, we seek the roots of the quadratic equation

$$r_{1,2} = \frac{-b\pm\sqrt{b^2-4\,a\,c}}{2\,a} \qquad \textbf{\textit{(Eq.1.3)}}$$

Since a, b, and c are real, there are three possibilities

1. Real distinct roots
2. Real, repeated roots
3. Complex conjugate roots

Case 1 : Real Distinct Roots

If we assume the roots are real and distinct, we denote them as r_1 and r_2 with $r_1 \neq r_2$.

Therefore, one solution is

$$x_1(t) = e^{r_1\,t} \qquad \textbf{\textit{(Eq.1.4)}}$$

```
x1[t_] = Exp[r1 t];
```

We can verify this is a solution

```
Collect[a D[x1[t], {t, 2}] + b D[x1[t], t] + c x1[t], Exp[r1 t]]
```

$e^{r1\,t}\,\left(c + b\,r1 + a\,r1^2\right)$

We see that the term in parenthesis is zero because r_1 is a root of the characteristic equation.

We notice that constant multiples of Eq.1.4 are also a solution

$$x_1(t) = c_1\,e^{r_1\,t} \qquad \textbf{\textit{(Eq.1.5)}}$$

```
x1[t_] = c1 Exp[r1 t];
```

We can verify this is a solution

$$a\,\frac{d^2}{d\,t^2}\left[c_1\,e^{r_1\,t}\right] + b\,\frac{d}{d\,t}\left[c_1\,e^{r_1\,t}\right] + c\,c_1\,e^{r_1\,t} = 0$$

$$a\,c_1\,{r_1}^2\,e^{r_1\,t} + b\,c_1\,r_1\,e^{r_1\,t} + c\,c_1\,e^{r_1\,t} = 0$$

$$c_1 e^{r_1 t}\left(a r_1^2 + b r_1 + c\right) = 0$$

Once again, the term in parenthesis is zero because r_1 is a root of the characteristic equation

```
Collect[a D[x1[t], {t, 2}] + b D[x1[t], t] + c x1[t], c1 Exp[r1 t]]
```

$$c1\, e^{r1\, t} \left(c + b\, r1 + a\, r1^2\right)$$

By similar logic, another solution is

$$x_2(t) = c_2 e^{r_2 t} \qquad \textbf{\textit{(Eq.1.6)}}$$

The combination of the two are also a solution. This is known as the homogeneous (AKA. general, complimentary) solution

$$x_{\text{homogeneous}}(t) = x_1(t) + x_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \qquad \textbf{\textit{(Eq.1.7)}}$$

We can check that this is a solution

$$a \frac{d^2}{dt^2}\left[c_1 e^{r_1 t} + c_2 e^{r_2 t}\right] + b \frac{d}{dt}\left[c_1 e^{r_1 t} + c_2 e^{r_2 t}\right] + c\left(c_1 e^{r_1 t} + c_2 e^{r_2 t}\right) = 0$$

$$a\left(c_1 r_1^2 e^{r_1 t} + c_2 r_2^2 e^{r_2 t}\right) + b\left(c_1 r_1 e^{r_1 t} + c_2 r_2 e^{r_2 t}\right) + c\left(c_1 e^{r_1 t} + c_2 e^{r_2 t}\right) = 0$$

$$a c_1 r_1^2 e^{r_1 t} + a c_2 r_2^2 e^{r_2 t} + b c_1 r_1 e^{r_1 t} + b c_2 r_2 e^{r_2 t} + c c_1 e^{r_1 t} + c c_2 e^{r_2 t} = 0$$

$$\left(a c_1 r_1^2 e^{r_1 t} + b c_1 r_1 e^{r_1 t} + c c_1 e^{r_1 t}\right) + \left(a c_2 r_2^2 e^{r_2 t} + b c_2 r_2 e^{r_2 t} + c c_2 e^{r_2 t}\right) = 0$$

$$c_1 e^{r_1 t}\left(a r_1^2 + b r_1 + c\right) + c_2 e^{r_2 t}\left(a r_2^2 + b r_2 + c\right) = 0$$

Once again, the terms in parenthesis will be zero since r_1 and r_2 are roots of the characteristic equation.

```
xhomogeneous[t_] = c1 Exp[r1 t] + c2 Exp[r2 t];
```

```
Collect[a D[xhomogeneous[t], {t, 2}] + b D[xhomogeneous[t], t] + c xhomogeneous[t],
 {c1 Exp[r1 t], c2 Exp[r2 t]}]
```

$$c1\, e^{r1\, t} \left(c + b\, r1 + a\, r1^2\right) + c2\, e^{r2\, t} \left(c + b\, r2 + a\, r2^2\right)$$

We can then use initial conditions to solve for specific values of c_1 and c_2. For example, if we have the initial conditions of

$$x(t_0) = x_0$$
$$\dot{x}(t_0) = \dot{x}_0$$

We can apply the first initial condition to obtain an equation

```
eq1 = xhomogeneous[t0]
```

$c1\, e^{r1\, t0} + c2\, e^{r2\, t0}$

We can then apply the second initial condition to obtain the second equation

```
eq2 = D[xhomogeneous[t], t] /. {t → t0}
```

$c1\, e^{r1\, t0}\, r1 + c2\, e^{r2\, t0}\, r2$

Solving simultaneously yields

```
Solve[{eq1 == x0, eq2 == xdot0}, {c1, c2}] // Simplify
```

$$\left\{\left\{c1 \to \frac{e^{-r1\, t0}\, (-r2\, x0 + xdot0)}{r1 - r2},\ c2 \to \frac{e^{-r2\, t0}\, (-r1\, x0 + xdot0)}{-r1 + r2}\right\}\right\}$$

So we have

$$c_1 = \frac{\dot{x}_0 - r_2\, x_0}{r_1 - r_2}\, e^{-r_1\, t_0}$$

$$c_2 = \frac{\dot{x}_0 - r_1\, x_0}{r_2 - r_1}\, e^{-r_2\, t_0}$$

Summary

If we have a differential equation of the form

$$a\, \ddot{x}(t) + b\, \dot{x}(t) + c\, x(t) = 0 \qquad \textbf{\textit{(Eq.1.1)}}$$

with $x(t_0) = x_0$

$\dot{x}(t_0) = \dot{x}_0$

The roots of the characteristic equation are given by

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4\, a\, c}}{2\, a} \qquad \textbf{\textit{(Eq.1.3)}}$$

The differential equation therefore has a homogeneous solution of

$$x_{\text{homogeneous}}(t) = c_1\, e^{r_1\, t} + c_2\, e^{r_2\, t} \qquad \textbf{\textit{(Eq.1.2)}}$$

where $c_1 = \frac{\dot{x}_0 - r_2\, x_0}{r_1 - r_2}\, e^{-r_1\, t_0}$

$c_2 = \frac{\dot{x}_0 - r_1\, x_0}{r_2 - r_1}\, e^{-r_2\, t_0}$

Example

$$\ddot{x}(t) + 5\, \dot{x}(t) + 6\, x(t) = 0$$

with $x(0) = 2$
$\dot{x}(0) = 3$

We can first find the homogeneous solution. We can look at the characteristic equation

$$r^2 + 5r + 6 = 0$$

Solving for the roots of the characteristic equation yields

```
Solve[r^2 + 5 r + 6 == 0, r]
{{r → -3}, {r → -2}}
```

So the roots are $r_1 = -2$, $r_2 = -3$. Since these are distinct, real roots, we can assume a solution of the form

$$x_{\text{homogeneous}}(t) = c_1 e^{-2t} + c_2 e^{-3t}$$

```
xhomogeneous[t_] = c1 Exp[-2 t] + c2 Exp[-3 t]
c2 e^(-3 t) + c1 e^(-2 t)
```

We can now use the initial conditions to find the coefficients. We require that

$x(0) = 2$
$\dot{x}(0) = 3$

Using the position initial condition

$$x_{\text{homogeneous}}(0) = 2$$

$$c_1 e^{-2\times(0)} + c_2 e^{-3\times(0)} = 2$$

$$c_1 + c_2 = 2$$

Using the velocity initial condition

$$\dot{x}_{\text{homogeneous}}(0) = 3$$

$$\frac{d}{dt}\left[c_1 e^{-2t} + c_2 e^{-3t}\right] \Big|_{t=0} = 3$$

$$-2 c_1 e^{-2\times(0)} - 3 c_2 e^{-3\times(0)} = 3$$

$$-2 c_1 - 3 c_2 = 3$$

Solving these two equation simultaneously yields

$$c_1 = 9$$

$c_2 = -7$

```
xDothomogneous[t_] = D[xhomogeneous[t], t];
temp = Solve[{xhomogeneous[0] == 2, xDothomogneous[0] == 3}, {c1, c2}];
c1 = c1 /. temp[[1]]
c2 = c2 /. temp[[1]]
```

9

-7

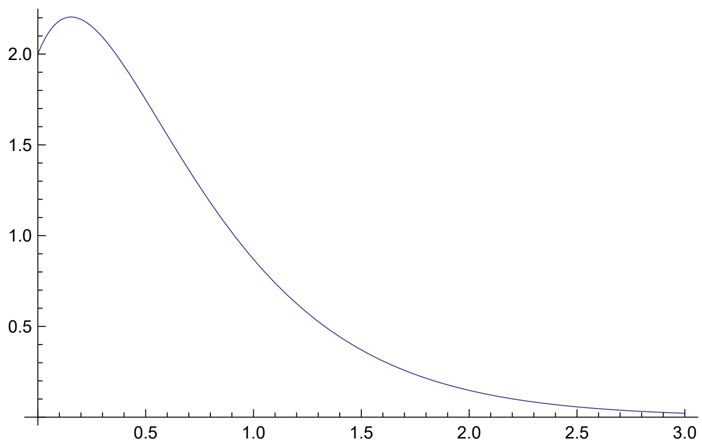
So the homogeneous solution is

$$x_{\text{homogeneous}}(t) = 9\,e^{-2\,t} - 7\,e^{-3\,t}$$

```
D[xhomogeneous[t], {t, 2}] + 5 D[xhomogeneous[t], t] + 6 xhomogeneous[t] // FullSimplify
```

0

```
Plot[xhomogeneous[t], {t, 0, 3}]
```



Let us look at solving this using the Laplace method. The differential equation is

$$\ddot{x}(t) + 5\,\dot{x}(t) + 6\,x(t) = 0$$

Taking the Laplace transform of both sides we obtain

$$L\left[\ddot{x}(t) + 5\,\dot{x}(t) + 6\,x(t)\right] = L[0] \qquad \text{note: } L[1] = \frac{1}{s}$$

$$L\left[\ddot{x}(t)\right] + 5\,L\left[\dot{x}(t)\right] + 6\,L[x(t)] = \frac{0}{s}$$

$$\left(s^2\,X(s) - s\,x(0) - \dot{x}(0)\right) + 5\,(s\,X(s) - x(0)) + 6\,X(s) = 0 \qquad \text{note: } x(0) = 2,\ \dot{x}(0) = 3$$

$$\left(s^2\,X(s) - 2\,s - 3\right) + 5\,(s\,X(s) - 2) + 6\,X(s) = 0$$

Collect terms

```
Collect[s^2 X - 2 s - 3 + 5 (s X - 2) + 6 X, X]
-13 - 2 s + (6 + 5 s + s^2) X
```

$$-13 - 2s + (6 + 5s + s^2)X(s) = 0$$

$$X(s) = \frac{13+2s}{s^2+5s+6}$$ note: characteristic equation shows up again!

Partial fraction expansion (recall that we know that the roots are distinct and real).

$$\frac{13+2s}{s^2+5s+6} = \frac{a_1}{(s+2)} + \frac{a_2}{(s+3)}$$

$$= \frac{a_1(s+3)+a_2(s+2)}{(s+2)(s+3)}$$

```
num[s_] = Collect[a1 (s + 3) + a2 (s + 2), s];
Solve[{Coefficient[num[s], s] == 2, num[0] == 13}, {a1, a2}]
{{a1 → 9, a2 → -7}}
```

So we have

$$X(s) = \frac{9}{s+2} - \frac{7}{s+3}$$

Performing the inverse Laplace transform yields

$$x(t) = L^{-1}\left[\frac{9}{s+2} - \frac{7}{s+3}\right]$$

$$= 9\,L^{-1}\left[\frac{1}{s+2}\right] - 7\,L^{-1}\left[\frac{1}{s+3}\right]$$

$$x(t) = 9\,e^{-2t} - 7\,e^{-3t}$$

So we arrive at the same solution

```
Clear[num, c2, c1, temp, xDothomogneous, xhomogeneous]
```

Case 2 : Real Repeated Roots

In the case where we have repeated roots we need to assume a different form of the solution. This is very similar to what we did when we were solving ODEs using the Laplace method with partial fraction expansion and needed to assume a different form of the expansion.

Example

$$\ddot{x}(t) + 4\,\dot{x}(t) + 4\,x(t) = 0$$

with $x(0) = 2$

$\dot{x}(0) = 3$

We can first find the homogeneous solution. We can look at the characteristic equation

$$r^2 + 4 r + 4 = 0$$

Solving for the roots of the characteristic equation yields

```
Solve[r^2 + 4 r + 4 == 0, r]
```

```
{{r → -2}, {r → -2}}
```

So the roots are $r_1 = -2$, $r_2 = -2$. Since these are repeated real roots, if we assume a solution of the same form as we did for the real, distinct roots, we will run into issues

$$x_{\text{incorrect}}(t) = c_1 e^{-2 t} + c_2 e^{-2 t}$$

```
xincorrect[t_] = c1 Exp[-2 t] + c2 Exp[-2 t]
```

$$\text{c1}\, e^{-2 t} + \text{c2}\, e^{-2 t}$$

We can show that this satisfies the differential equation

```
D[xincorrect[t], {t, 2}] + 4 D[xincorrect[t], t] + 4 xincorrect[t] == 0 // Simplify
```

```
True
```

However if we attempt to satisfy the initial conditions

$$x(0) = 2$$
$$\dot{x}(0) = 3$$

Using the position initial condition

$$x_{\text{incorrect}}(0) = 2$$

$$c_1 e^{-2\times(0)} + c_2 e^{-2\times(0)} = 2$$

$$c_1 + c_2 = 2$$

Using the velocity initial condition

$$\dot{x}_{\text{incorrect}}(0) = 3$$

$$\frac{d}{dt}\left[c_1 e^{-2 t} + c_2 e^{-2 t}\right] \Big|_{t=0} = 3$$

$$-2 c_1 e^{-2\times(0)} - 2 c_2 e^{-2\times(0)} = 3$$

$$-2 c_1 - 2 c_2 = 3$$

$$c_1 + c_2 = \frac{-3}{2}$$

We we see that we cannot solve simultaneously for the constants to satisfy the initial conditions in this scenario.

```
xDotincorrect[t_] = D[xincorrect[t], t];
temp = Solve[{xincorrect[0] == 2, xDotincorrect[0] == 3}, {c1, c2}]
```

```
{}
```

To address this issue, we need to assume a different form of solution

$$x_{\text{homogeneous}}(t) = c_1\, e^{r\,t} + c_2\, t\, e^{r\,t}$$

```
xhomogeneous[t_] = c1 Exp[-2 t] + c2 t Exp[-2 t]
```

$c1\, e^{-2\,t} + c2\, e^{-2\,t}\, t$

We can show that this satisfies the differential equation

```
D[xhomogeneous[t], {t, 2}] + 4 D[xhomogeneous[t], t] + 4 xhomogeneous[t] == 0 // Simplify
```

We can now solve for the constants to satisfy the initial conditions

$$x(0) = 2$$
$$\dot{x}(0) = 3$$

Using the position initial condition

$$x_{\text{homogeneous}}(0) = 2$$

$$c_1\, e^{-2\times(0)} + c_2\,(0)\, e^{-2\times(0)} = 2$$

$$c_1 = 2$$

Using the velocity initial condition

$$\dot{x}_{\text{homogeneous}}(0) = 3$$

$$\frac{d}{d\,t}\left[c_1\, e^{-2\,t} + c_2\, t\, e^{-2\,t}\right] \Big|_{t=0} = 3$$

$$\left[-2\,c_1\, e^{-2\,t} - 2\,c_2\, t\, e^{-2\,t} + c_2\, e^{-2\,t}\right] \Big|_{t=0} = 3$$

$$-2\,c_1\, e^{-2\times(0)} - 2\,c_2\,(0)\, e^{-2\times(0)} + c_2\, e^{-2\times(0)} = 3$$

$$-2\,c_1 + c_2 = 3$$

Solving these two equation simultaneously yields

$c_1 = 2$
$c_2 = 7$

```
xDothomogneous[t_] = D[xhomogeneous[t], t];
temp = Solve[{xhomogeneous[0] == 2, xDothomogneous[0] == 3}, {c1, c2}];
c1 = c1 /. temp[[1]]
c2 = c2 /. temp[[1]]
```

```
2
```

```
7
```

So the homogeneous solution is

$$x_{\text{homogeneous}}(t) = 2\,e^{-2\,t} + 7\,t\,e^{-2\,t}$$

```
D[xhomogeneous[t], {t, 2}] + 4 D[xhomogeneous[t], t] + 4 xhomogeneous[t] == 0 // FullSimplify
xhomogeneous[0] == 2
D[xhomogeneous[t], t] /. {t → 0} == 3
```

```
0
```

```
Plot[xhomogeneous[t], {t, 0, 3}]
```

```
Clear[c2, c1, temp, xDothomogneous, xhomogeneous, xDotincorrect, xincorrect]
```

Case 3 : Complex Conjugate Roots

The last case to consider is if there are complex conjugate roots

Example

$$\ddot{x}(t) + 4\,\dot{x}(t) + 13\,x(t) = 0$$

with $x(0) = 2$
$\dot{x}(0) = 3$

We can first find the homogeneous solution. We can look at the characteristic equation

$$r^2 + 4\,r + 13 = 0$$

Solving for the roots of the characteristic equation yields

```
Solve[r^2 + 4 r + 13 == 0, r]
```

```
{{r → -2 - 3 i}, {r → -2 + 3 i}}
```

So the roots are $r_1 = -2 + 3\,i$, $r_2 = -2 - 3\,i$. Interestingly, these are actually distinct roots, but they are not real. Therefore, let us assume the same form of solution that we assumed for the distinct, real roots case

$$x_{\text{homogeneous}}(t) = c_1\, e^{-2\,t} + c_2\, e^{-3\,t}$$

```
xhomogeneous[t_] = c1 Exp[r1 t] + c2 Exp[r2 t] /. {r1 → -2 + 3 I, r2 → -2 - 3 I}
```

$c2\, e^{(-2-3\,i)\,t} + c1\, e^{(-2+3\,i)\,t}$

We note that this solves the differential equation

```
D[xhomogeneous[t], {t, 2}] + 4 D[xhomogeneous[t], t] + 13 xhomogeneous[t] == 0 // Simplify
```

True

We can now apply initial conditions to solve for c_1 and c_2

```
xDothomogneous[t_] = D[xhomogeneous[t], t];
temp = Solve[{xhomogeneous[0] == 2, xDothomogneous[0] == 3}, {c1, c2}];
c1 = c1 /. temp[[1]]
c2 = c2 /. temp[[1]]
```

$1 - \frac{7\,i}{6}$

$1 + \frac{7\,i}{6}$

So the proposed solution is

```
xhomogeneous[t]
```

$\left(1 + \frac{7\,i}{6}\right) e^{(-2-3\,i)\,t} + \left(1 - \frac{7\,i}{6}\right) e^{(-2+3\,i)\,t}$

Again, we note that this solves the initial conditions

```
xhomogeneous[0]
xDothomogneous[0]
```

2

3

We note that we can actually write this using cos and sin via Euler's Formula $e^{i\,\theta} = \cos(\theta)\ +\ \sin(\theta)\,i =$ (https://www.youtube.com/watch?v=WEYX-wa9csU&t=641s) and eliminate imaginary components (we leave this as an exercise in homework).

```
solution1[t_] = ExpToTrig[xhomogeneous[t]] // FullSimplify
```

$\frac{1}{3}\, e^{-2\,t}\,(6\,\text{Cos}[3\,t] + 7\,\text{Sin}[3\,t])$

For those that are familiar with ODEs, we know that imaginary roots lead to oscillatory solutions. Therefore, another form of the solution to assume is

$$x_{\text{homogeneous}}(t) = e^{\sigma\,t}(c_1 \cos(\omega\,t) + c_2 \sin(\omega\,t))$$

where $\quad r = \sigma + \omega\,i$

```
Clear[c1, c2]
xhomogeneous[t_] = Exp[-2 t] (c1 Cos[3 t] + c2 Sin[3 t])
```

e^{-2t} (c1 Cos[3 t] + c2 Sin[3 t])

We can verify that this satisfies the differential equation

```
D[xhomogeneous[t], {t, 2}] + 4 D[xhomogeneous[t], t] + 13 xhomogeneous[t] == 0 // Simplify
```

True

We can now apply initial conditions to solve for c_1 and c_2

```
xDothomogeneous[t_] = D[xhomogeneous[t], t];
temp = Solve[{xhomogeneous[0] == 2, xDothomogneous[0] == 3}, {c1, c2}];
c1 = c1 /. temp[[1]]
c2 = c2 /. temp[[1]]
```

2

$\frac{7}{3}$

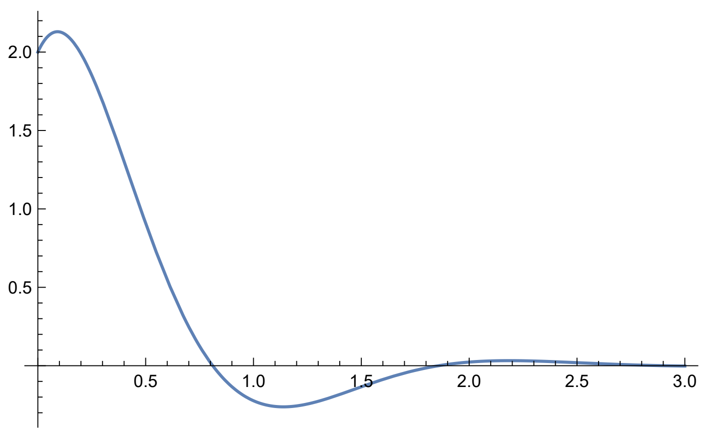
```
xhomogeneous[t]
```

$e^{-2t}\left(2\,\text{Cos}[3\,t] + \frac{7}{3}\,\text{Sin}[3\,t]\right)$

So the homogeneous solution is

$$x_{\text{homogeneous}}(t) = e^{-2t}\left(2\cos(3\,t) + \tfrac{7}{3}\sin(3\,t)\right)$$

```
Plot[xhomogeneous[t], {t, 0, 3}]
```



This is the same as the solution we obtained earlier using complex numbers

```
xhomogeneous[t] == solution1[t] // Simplify
```

True

In the next lecture, we examine non-homogeneous ordinary differential equations.