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Lecture 05a
Bode Plots of Complex Transfer Functions

Lecture is on YouTube

The YouTube video entitled ‘Bode Plots of Complex Transfer Functions’ that covers this lecture is located at https://youtu.be/cBMgRWOzLnw.

Outline

-Combining Components in Bode Plots
-Example Transfer Function
    -Component 1: Constant Gain
    -Component 2: Real Zero
    -Component 3: Pole at Origin
    -Component 4: Real Pole
-Teaser for Loop Shaping

Combining Components in Bode Plots

Recall that the final result is that the steady state output of the system (in response to an input of the form $u(t) = A\sin(\omega t)$) is a sin wave of the same frequency, ω, whose magnitude is multiplied by $|G(j\omega)|$ and whose phase is shifted by $\angle G(j\omega)$.

Now let us consider a system which is composed of multiple components.

$$G(s) = \frac{(s+z_1)}{(s+p_1)}$$

$$= \frac{z_1(s/z_1+1)}{p_1(s/p_1+1)}$$

$$G(s) = \frac{K(s/z_1+1)}{(s/p_1+1)}$$

where $\quad K = z_1/p_1$

We see that this transfer function is made up of 3 components

1. Constant gain
2. Single real zero
3. Single real pole

We can now compute $G(j\,\omega)$

$$G(j\,\omega) = \frac{K(j\,\omega/z_1+1)}{(j\,\omega/p_1+1)}$$

This is simply a complex number, so we can compute the magnitude

$$|\,G(j\,\omega)\,| = \left|\,\frac{K(j\,\omega/z_1+1)}{(j\,\omega/p_1+1)}\,\right|$$

We can consider the numerator and denominator separate complex numbers. Since we are only concerned with the magnitude of the complex number, we can write

$$|\,G(j\,\omega)\,| = \left|\,K\,\right| * \left|\,j\,\omega/z_1 + 1\,\right| * \left|\,\frac{1}{j\,\omega/p_1+1}\,\right|$$

Because we want to compute the magnitude in dB, we now can compute $20\log_{10}|\,G(j\,\omega)\,|$ for the magnitude plot.

$$20\log_{10}|\,G(j\,\omega)\,| = 20\log_{10}\left(\left|\,K\,\right| * \left|\,j\,\omega/z_1 + 1\,\right| * \left|\,\frac{1}{j\,\omega/p_1+1}\,\right|\right)$$

$$= 20\log_{10}(\,|\,K\,|\,) + 20\log_{10}(\,|\,j\,\omega/z_1 + 1\,|\,) + 20\log_{10}\left(\left|\,\frac{1}{j\,\omega/p_1+1}\,\right|\right)$$

$$20\log_{10}|\,G(j\,\omega)\,| = \text{magDB1} + \text{magDB2} + \text{magDB3}$$

where magDB1 = magnitude in dB contribution from component 1 (constant gain)
magDB2 = magnitude in dB contribution from component 2 (single real zero)
magDB3 = magnitude in dB contribution from component 3 (single real pole)

So we see that these components can be added together linearly.

For the phase plot, we can recall from our previous lectures on complex numbers that the angles of a quotient and product of two complex numbers are the difference and sum, respectively.

$$\angle\frac{u}{v} = \angle u - \angle v$$

$$\angle u\,v = \angle u + \angle v$$

So we can write the phase as

$$\angle G(j\,\omega) = \angle \frac{K(j\,\omega/z_1+1)}{(j\,\omega/p_1+1)}$$

$$= \angle K(j\,\omega/z_1+1)\left(\frac{1}{j\,\omega/p_1+1}\right)$$

$$= \angle K + \angle(j\,\omega/z_1+1) + \angle\left(\frac{1}{(j\,\omega/p_1+1)}\right)$$

$$\angle G(j\,\omega) = \text{angle1} + \text{angle2} + \text{angle3}$$

where angle1 = angle contribution from component 1 (constant gain)
angle2 = angle contribution from component 2 (single real zero)
angle3 = angle contribution from component 3 (single real pole)

So once again, we see that these add together linearly.

This observation that the magnitude in dB and the phase add linearly leads to a procedure for sketching bode plots.

Procedure

Start with a transfer function of the form

$$G(s) = \frac{b_m\,s^m + b_{m-1}\,s^{m-1} + \ldots + b_1\,s + b_0}{a_n\,s^n + a_{n-1}\,s^{n-1} + \ldots + a_1\,s + a_0}$$

1. Factor transfer function to identify zeros and poles

$$G(s) = \alpha\,\frac{(s+z_1)\,(s+z_2)\ldots(s+z_m)}{(s+p_1)\,(s+p_2)\ldots(s+p_n)} \qquad (m \text{ zeros and } n \text{ poles})$$

2. Write transfer function in standard bode plot form (grouping complex and real poles/zeros)

$$G(s) = K\left\{\left[(s/z_1+1)\,(s/z_2+1)\ldots\left(s/z_p+1\right)\left(\frac{s^2}{\omega_{n,z1}{}^2} + \frac{2\,\zeta_{z1}\,s}{\omega_{n,z1}} + 1\right)\left(\frac{s^2}{\omega_{n,z2}{}^2} + \frac{2\,\zeta_{z2}\,s}{\omega_{n,z2}} + 1\right)\ldots\left(\frac{s^2}{\omega_{n,zq}{}^2} + \frac{2\,\zeta_{zq}\,s}{\omega_{n,zq}} + 1\right)\right]\Big/\left[(s/p_1+1)\,(s/p_2+1)\ldots(s/p_r+1)\left(\frac{s^2}{\omega_{n,p1}{}^2} + \frac{2\,\zeta_{p1}\,s}{\omega_{n,p1}} + 1\right)\left(\frac{s^2}{\omega_{n,p2}{}^2} + \frac{2\,\zeta_{p2}\,s}{\omega_{n,p2}} + 1\right)\ldots\left(\frac{s^2}{\omega_{n,pq}{}^2} + \frac{2\,\zeta_{pq}\,s}{\omega_{n,pq}} + 1\right)\right]\right\}$$

where $K = \frac{\alpha\,z_1\,z_2\ldots z_p\,\omega_{n,z1}{}^2\,\omega_{n,z2}{}^2\ldots\omega_{n,zq}{}^2}{p_1\,p_2\ldots p_r\,\omega_{n,p1}{}^2\,\omega_{n,p2}{}^2\ldots\omega_{n,pq}{}^2}$

3. Sketch the Bode plot contribution for each individual component.
4. Add all components together linearly to obtain the total bode plot.

Example Transfer Function

Consider the transfer function of

$$G(s) = \frac{50\,s+5}{s^2+30\,s}$$

Step 1: Factor to identify poles and zeros

$$G(s) = \frac{50\,(s+1/10)}{s(s+30)}$$

Step 2: Write transfer function in standard bode plot form

$$G(s) = 50\,\frac{\frac{1}{10}\times(10\,s+1)}{30\,s(s/30+1)}$$

$$G(s) = \frac{1}{6}\,\frac{(10\,s+1)}{s(s/30+1)}$$

```
1/6 (10 s + 1)/(s (s/30 + 1)) == (50 s + 5)/(s^2 + 30 s) // Simplify
```

$$\frac{1}{6}\,\frac{(10\,s+1)}{s\left(\frac{s}{30}+1\right)} == \frac{50\,s+5}{s^2+30\,s} \;\text{// Simplify}$$

```
True
```

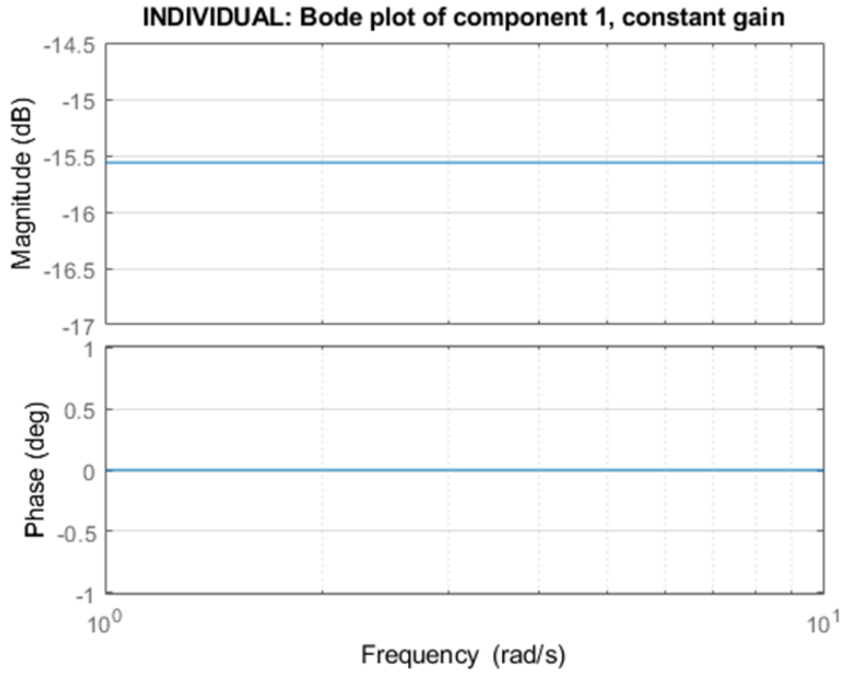
Step 3. Sketch the Bode plot contribution for each individual component

We see that there are 4 components

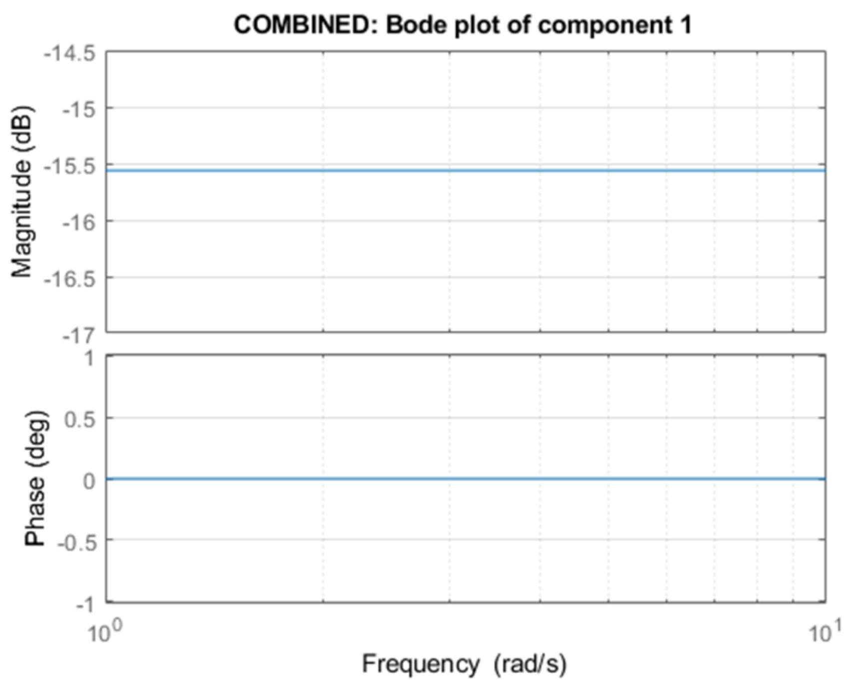
Component 1: Static gain of value 1/6
Component 2: real zero at $s = -0.1$
Component 3: pole at the origin
Component 4: real pole at $s = -30$

Component 1: Constant Gain

Component 1's bode plot

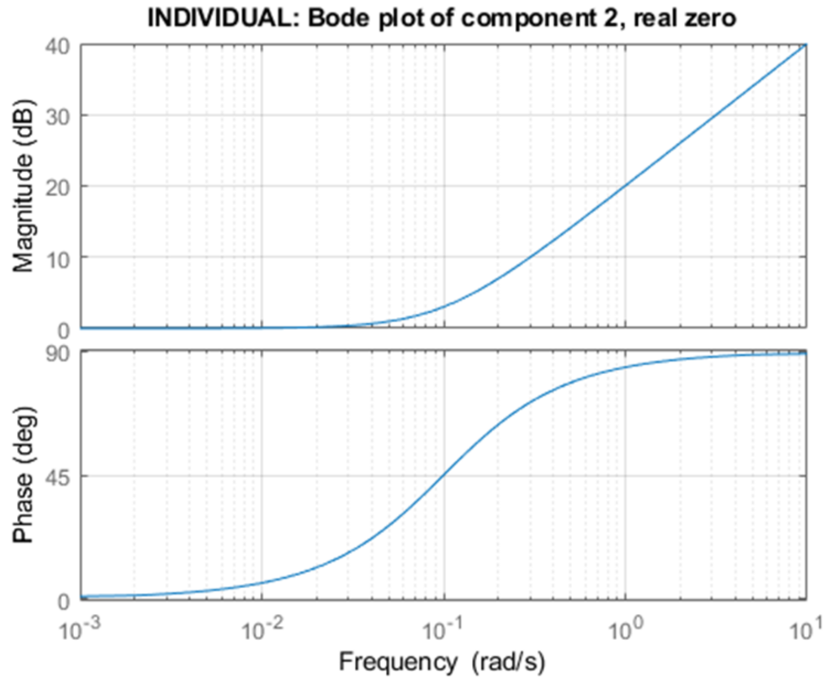


The combined plot is just component 1

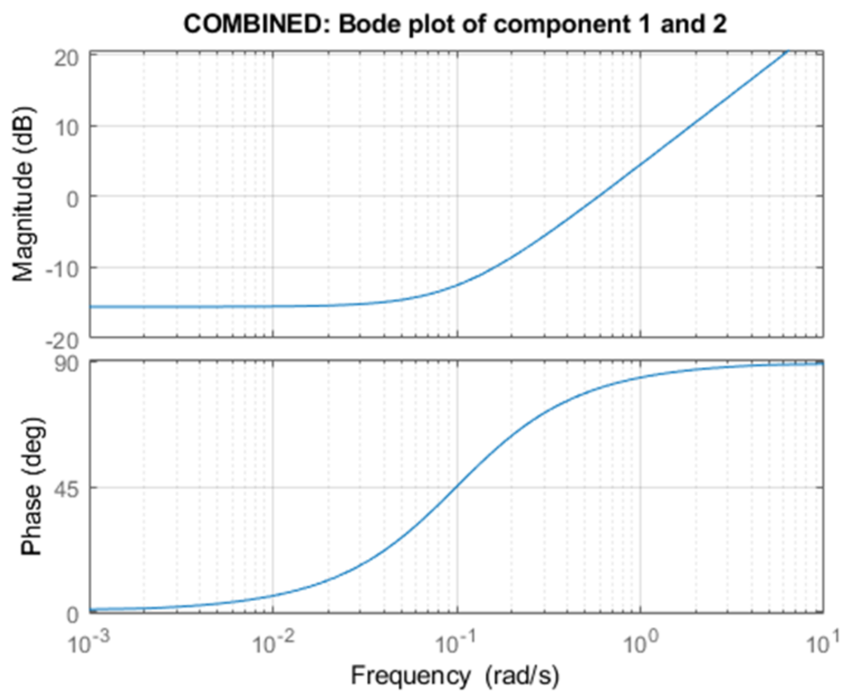


Component 2: Real Zero

Component 2’s bode plot

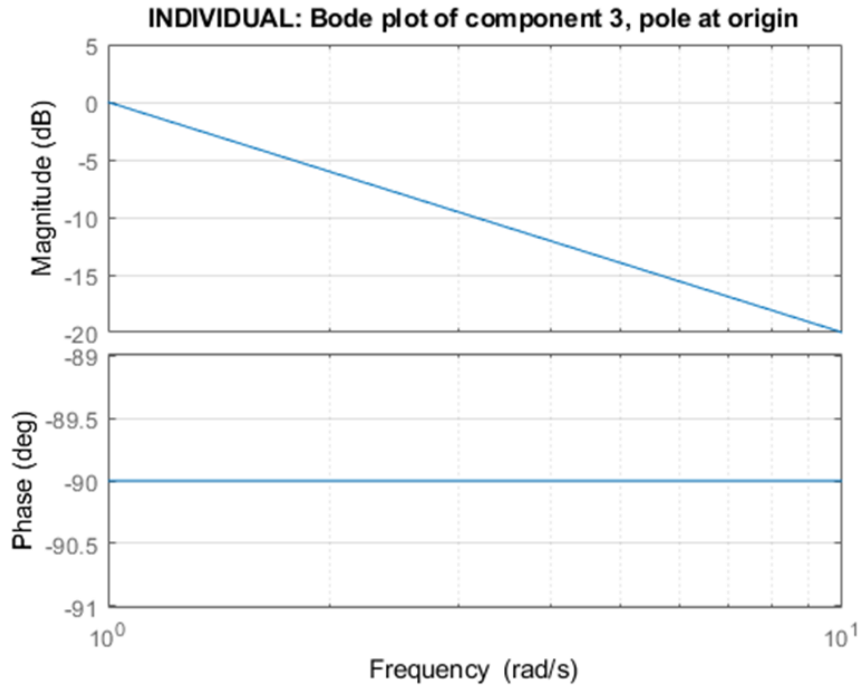


The combined plot is component 1 and 2 is shown below

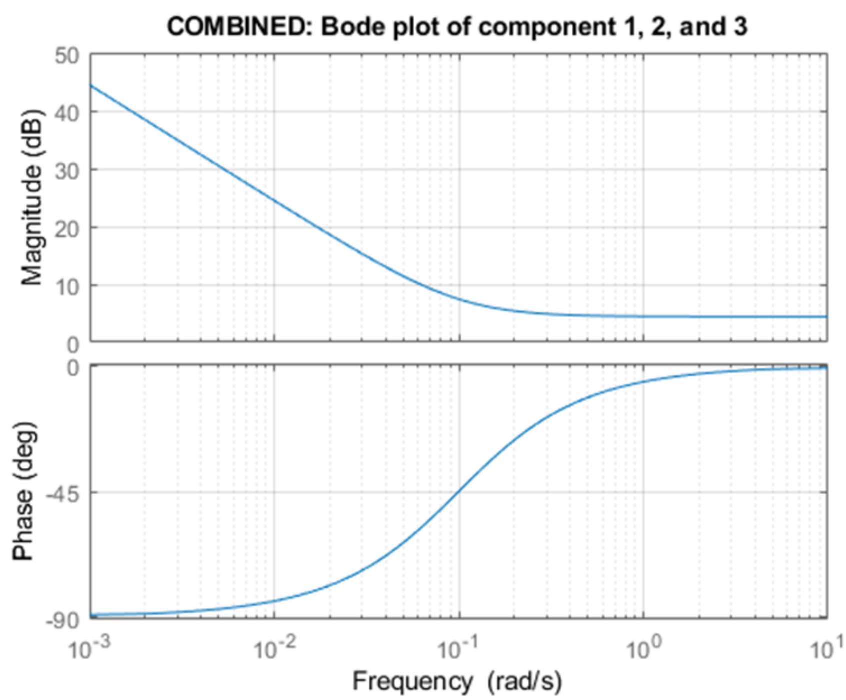


Component 3: Pole at Origin

Component 3's bode plot

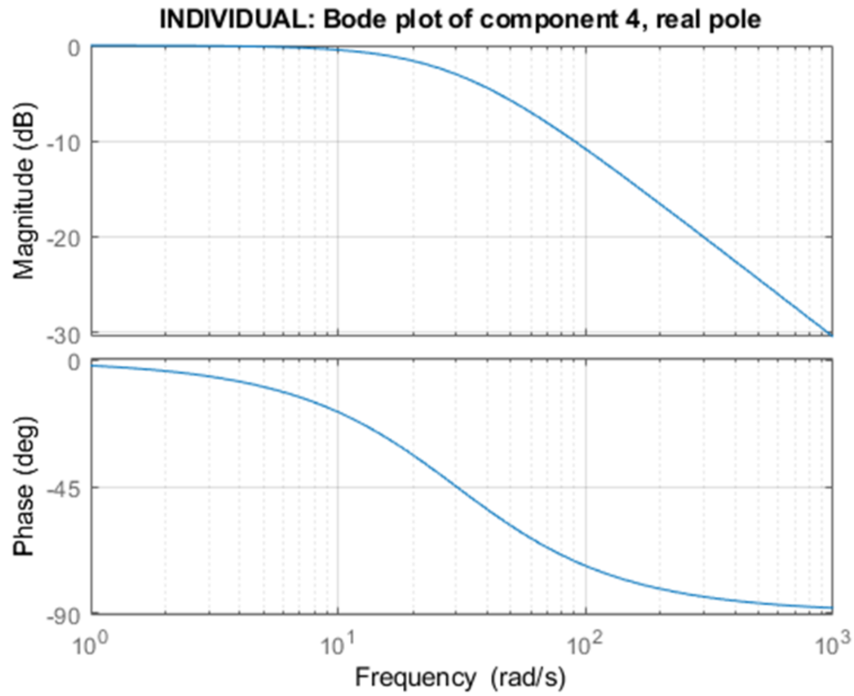


The combined plot is component 1, 2, and 3 is shown below

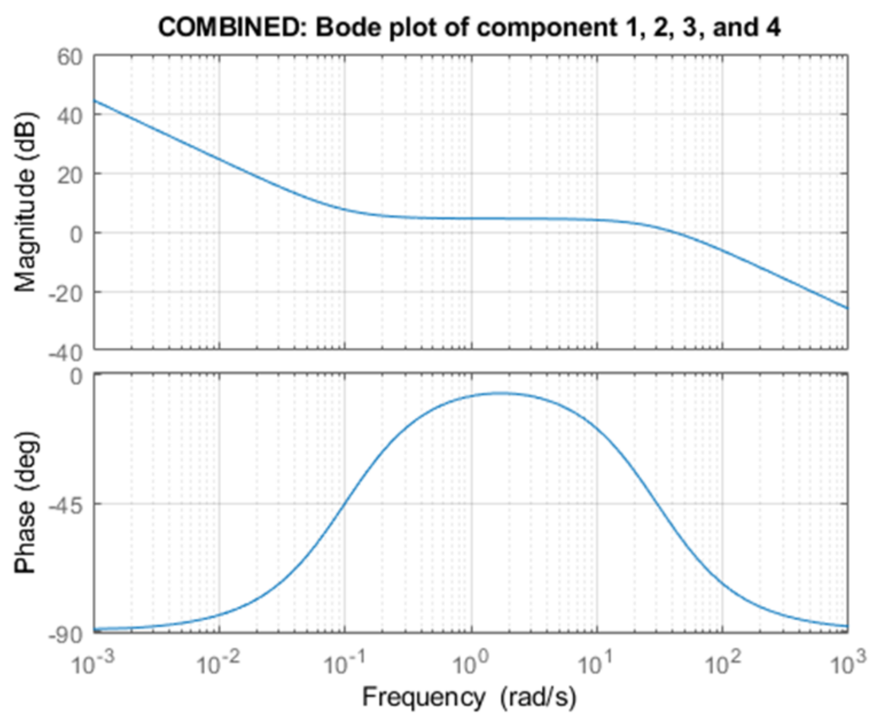


Component 4: Real Pole

Component 4's bode plot



The combined plot is component 1, 2, 3, and 4 is shown below



Teaser for Loop Shaping

The ability to linearly combine bode plots is extremely useful. For example, if the $G(s)$ that we considered in the previous example represented something like the position of an aircraft control surface respond to a command

$$G(s) = \frac{\delta_{E,\text{actual}}(s)}{\delta_{E.\text{commanded}}(s)} = \frac{50\,s+5}{s^2+30\,s}$$

Suppose that you need to operate this control surface at 100 rad/s (approx 16 Hz).

From the bode plot above, we see that at 100 rad/s, the bode plot is at approximately -6.43 dB. Therefore, the response is severely attenuated ($|\,G(100\,j)\,| \approx 0.48$). This means that it is deflecting less than half of what is commanded.

```
Solve[20 Log10[A] == -6.43, A]
```

```
{{A → 0.47698}}
```

We need the bring the magnitude of the bode plot up by 6.43 dB at this frequency. We can achieve this with a pair of complex poles that have a resonant frequency of $\omega_r = 10\,\text{rad}/s$.

Recall from the lecture entitled ‘Resonant Frequency of a Dynamic System’ we have

$$\omega_r = \omega_n\,\sqrt{1-2\,\zeta^2} = 100\,\text{rad}/s \qquad \textbf{\textit{(Eq.1)}}$$

Furthermore, the magnitude at the resonant frequency was derived in the lecture entitled ‘Understanding and Sketching Individual Bode Plots Components’ as

$$20\log_{10}(\,|\,G(\omega_r\,j)\,|\,) = -10\log_{10}\left(-4\,\zeta^2\left(\zeta^2-1\right)\right) = 6.43\,\text{dB} \qquad \textbf{\textit{(Eq.2)}}$$

We first solve Eq.2 for the the damping ratio that will yield the appropriate magnitude

```
temp = Solve[-10 Log10[-4 ξ^2 (ξ^2 - 1)] == 6.43, ξ]
```

Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

```
{{ξ → -0.246055}, {ξ → 0.246055}, {ξ → -0.969256}, {ξ → 0.969256}}
```

Recall that there is only a magnitude peak if $\zeta \in \left[0,\,1\big/\sqrt{2}\right]$, so the only valid solution is $\zeta = 0.246$

```
ξd = ξ /. temp[[2]]
```

```
0.246055
```

We can now solve Eq.1 for the necessary ω_n that will place ω_r at 100 rad/s

```
temp = Solve[ωn (1 - 2 ξd^2)^(1/2) == 100, ωn];
ωnd = ωn /. temp[[1]]
```

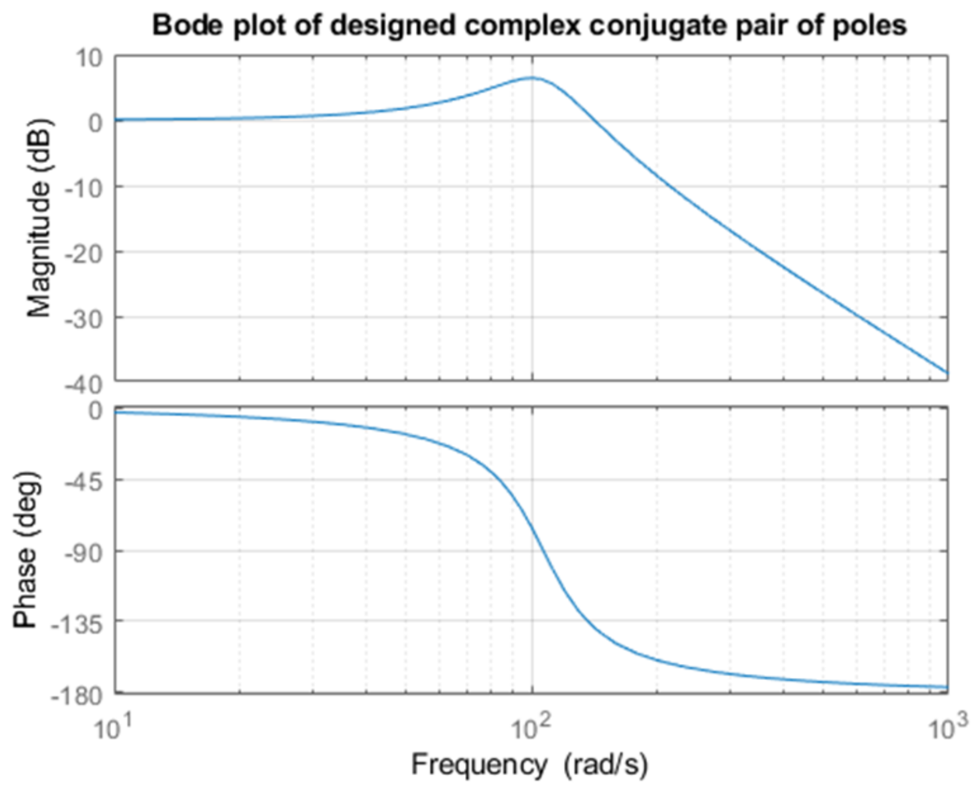
```
106.666
```

So we have

$$C(s) = \frac{1}{\frac{1}{\omega_n^2} s^2 + 2\frac{\zeta}{\omega_n} s + 1}$$

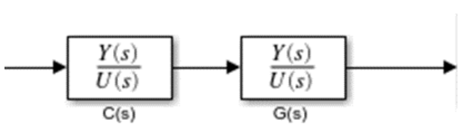
where $\omega_n = 106.66$
$\zeta = 0.246$

The bode plot of this new component is shown below

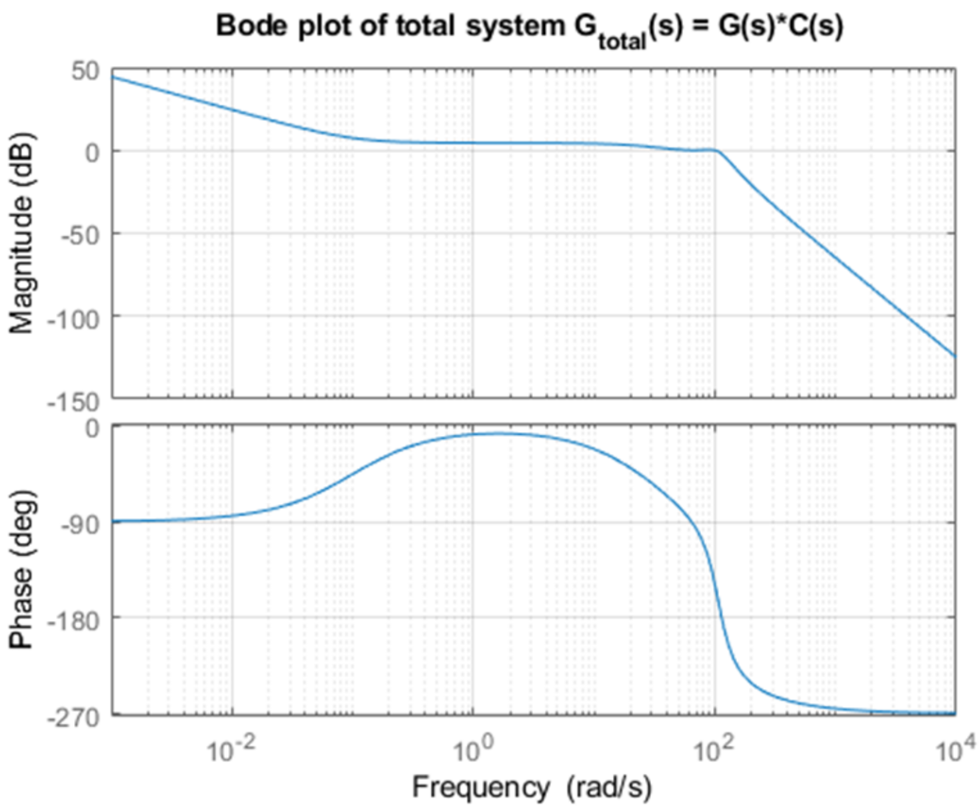


If we add this component in series before the plant model, we have

$$G_{\text{total}}(s) = G(s)\, C(s)$$



Computing the bode plot of the new, total system we obtain



We see that we now achieve the desired 0 dB magnitude at $\omega = 100$ rad/s.

We will explore this concept further when we discuss loop shaping, lead/lag compensators, and notch filtering.