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Lecture 03e
State Space Representation

Lecture is on YouTube

The YouTube video entitled 'State Space Representation of Differential Equations' that covers this lecture is located at https://youtu.be/pXvAh1IOO4U.

Outline

-State Space Representation of a Dynamic System

  -Non-linear example: MBL

  -Linear example: two hydraulic tanks

-State Space Representation of an n^{th} Order Differential Equation

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State Space Representation of Ordinary Differential Equations

A state space representation is system of ordinary differential equations written in matrix/vector form. It contains a set of input, output, and state variables that are related by a set of first order differential equations. The inputs, outputs, and states are expressed as vectors. This provides a convenient and compact way to model and analyze the systems with multiple inputs and outputs.

The general form of a state space representation is

$$\dot{\bar{x}}(t) = \bar{f}(\bar{x}(t), \bar{u}(t))$$
$$\bar{y}(t) = \bar{g}(\bar{x}(t), \bar{u}(t))$$

where $\bar{x}(t) = n \times 1$ state vector (captures state of system, completely defines system)

$\bar{u}(t) = m \times 1$ control (or input) vector

$\bar{y}(t) = p \times 1$ output vector

$\bar{f} = n \times 1$ system of equations

$\bar{g} = p \times 1$ system of equations

If the differential equations are linear, they can be expressed in matrix form. The general form of a **linear** state space representation is

$\dot{\overline{x}}(t) = A\,\overline{x}(t) + B\,\overline{u}(t)$ ***(Eq.1)***

$\overline{y}(t) = C\,\overline{x}(t) + D\,\overline{u}(t)$ ***(Eq.2)***

where $A = n \times n$ state matrix
$B = n \times m$ control matrix
$C = p \times n$ output matrix
$D = p \times m$ feedthrough matrix

Example: Non-Linear State Space Representation - Magnetic Ball Levitator

<Show movie or demonstration of system.>

A close up of the magnet is shown below showing positive deflections

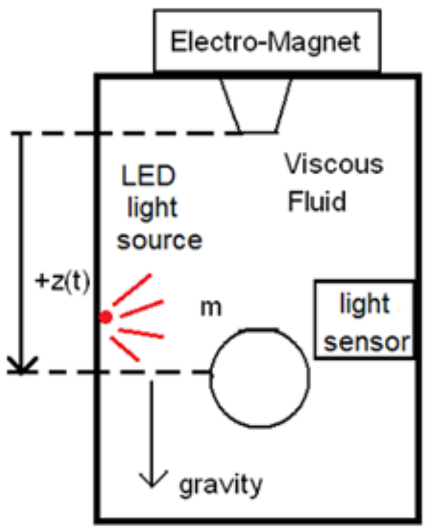


Figure 1: Test Apparatus

The system is actuated by changing the current through the electromagnet at the top. The LED light source emits light which is then observed by the light sensor which outputs a voltage related to the amount of light that it sees.

Note that we define the position, $z(t)$, as the distance between the tip of the magnet and the center of mass of the ball. A positive value of $z(t)$ implies that the center of mass of the ball is below the magnet tip.

We also note that gravity and aerodynamic drag has the potential to act on the ball.

The appropriate state and control vector for this system is

$$\overline{x}(t) = \begin{pmatrix} z(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \text{distance } b/w \text{ magnet tip and } C.M. \text{ of ball (positive down)} \\ \text{velocity of } C.M. \text{ of ball (positive down)} \end{pmatrix}$$

(Eq.1)

$$\overline{u}(t) = (V_m(t)) = (u_1(t)) = (\text{voltage on the magnet})$$

We can draw a free body diagram of the system at a positive displacement $z(t)$ with a positive velocity $\dot{z}(t)$

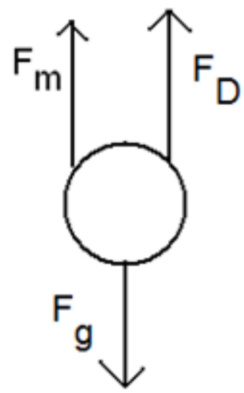


Figure 2: Free body diagram

The differential equation which governs the ball's motion is

$$\sum F = m\,\ddot{z}(t)$$

$$F_g - F_m - F_D = m\,\ddot{z}(t)$$

We now need to identify each term as a function of the states and controls.

<u>Gravity Force</u>

We can simply model the gravity force as a constant

$$F_g = m\,g$$

<u>Drag Force</u>

One popular model for drag is

$$F_D = C_D\,q\,S * \text{sign}\left(\dot{z}(t)\right)$$

However, this is too complicated, so let us simply consider linear drag of the form

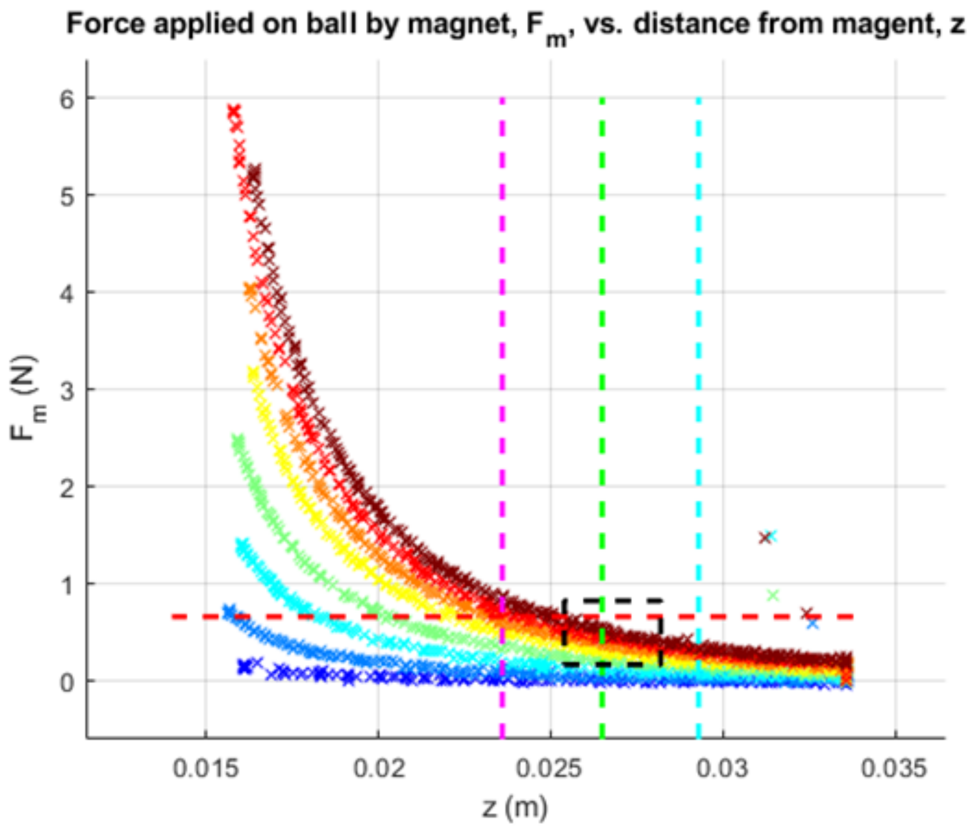
$$F_D = b\,\dot{z}(t) = b\,x_2(t)$$

<u>Magnet Force</u>

We know that the magnet force is a function of the voltage across the magnet and also the distance from the tip of the magnet. We can simply write this as a generic function

$$F_m = F_m(z(t), V_m(t)) = F_m(x_1(t), u_1(t))$$

This is likely a very complicated function



<u>Equations of Motion</u>

So our equation of motion becomes

$$F_g - F_m - F_D = m\,\ddot{z}(t)$$

$$m\,g - F_m(x_1(t), u_1(t)) - b\,x_2(t) = m\,\ddot{z}(t) \qquad \text{note: } \ddot{z}(t) = \dot{x}_2$$

$$\dot{x}_2(t) = g - \frac{1}{m}\,F_m(x_1(t), u_1(t)) - \frac{b}{m}\,x_2(t)$$

So we can now write our state space representation by differentiating the state vector

$$\dot{\overline{x}}(t) = \frac{d}{d\,t}[\overline{x}]$$

$$= \frac{d}{d\,t}\left[\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right]$$

$$= \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} \qquad \text{recall: } x_1 = \text{postion so } \dot{x}_1 = \text{velocity} = x_2$$

$$= \begin{pmatrix} x_2 \\ g - \frac{1}{m} F_m(x_1(t), u_1(t)) - \frac{b}{m} x_2(t) \end{pmatrix}$$

$$\dot{\overline{x}}(t) = f(\overline{x}(t), \overline{u}(t))$$

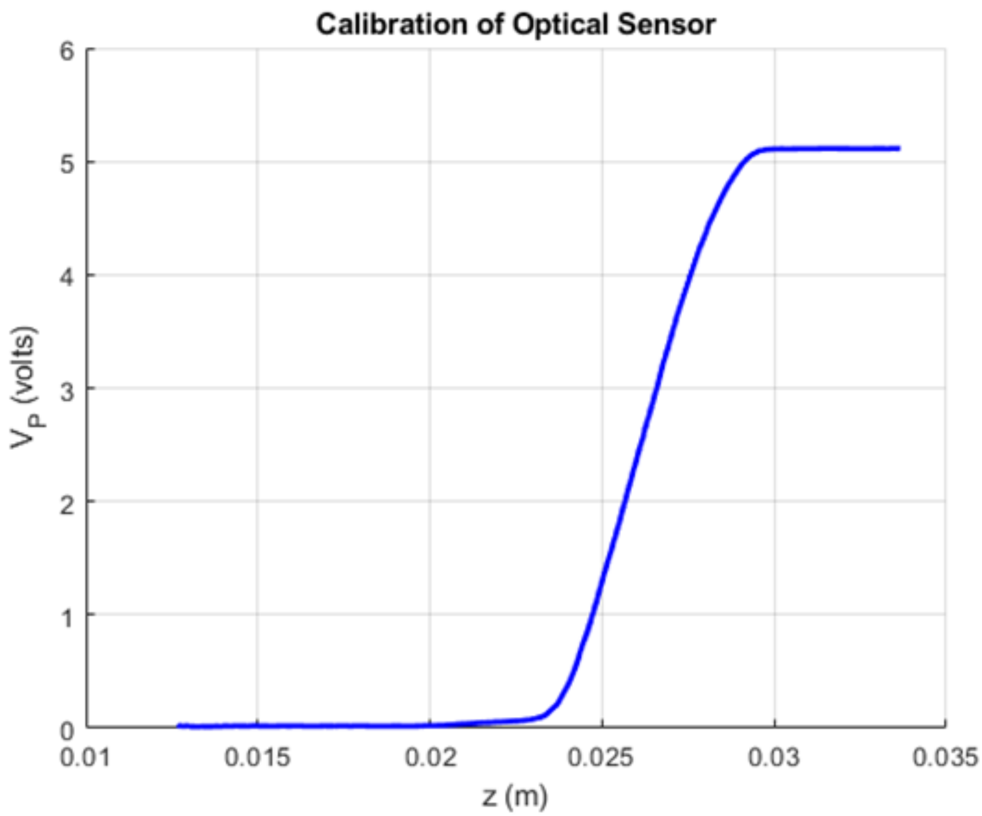
where $$f(\overline{x}(t), \overline{u}(t)) = \begin{pmatrix} x_2 \\ g - \frac{1}{m} F_m(x_1(t), u_1(t)) - \frac{b}{m} x_2(t) \end{pmatrix}$$

We see that this is a non-linear state equation because we cannot write it in matrix form.

We now turn out attention to the output equation, we see that the output of this system is the voltage from the light sensor, so we can write

$$\overline{y}(t) = (V_{\text{light,out}}(t)) = (y_1(t)) = (\text{voltage from light sensor})$$

Again, this is likely a complicated function that we need to obtain experimentally.



We assume that this is a function of the ball's position, for example, one model of this might be a sigmoid function

```
zo = 0.025;
f[z_] = 5.1 (1/(1 + Exp[-α (z - zo)]) /. {α → 1000})
Plot[f[x], {x, 0.015, 0.035}, PlotRange → All]
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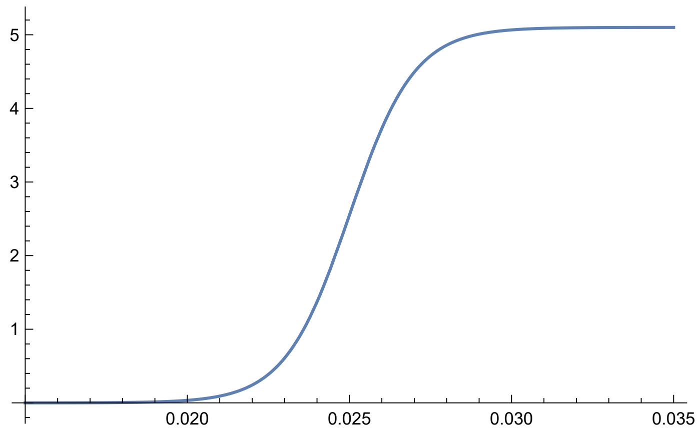
$$\frac{5.1}{1 + e^{-1000 \times (-0.025+z)}}$$



This models the fact that as the ball gets too close to the magnet, it blocks all the light and therefore the sensor outputs 0 volts. As the ball moves away, it start letting in more light and therefore the sensor outputs its full range of 2 volts.

We can write this in the state space form

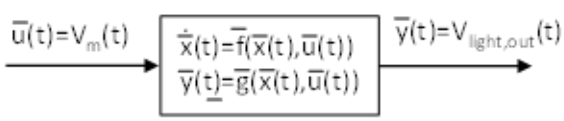
$$\overline{y}(t) = g(\overline{x}(t), \overline{u}(t))$$

where $\quad g(\overline{x}(t), \overline{u}(t)) = \frac{1}{1+e^{-1000\,(x_1-0.025)}}$

As we see, this is a non-linear output equation as well.

We have now characterized and modeled the MBL system as a non-linear state space system.

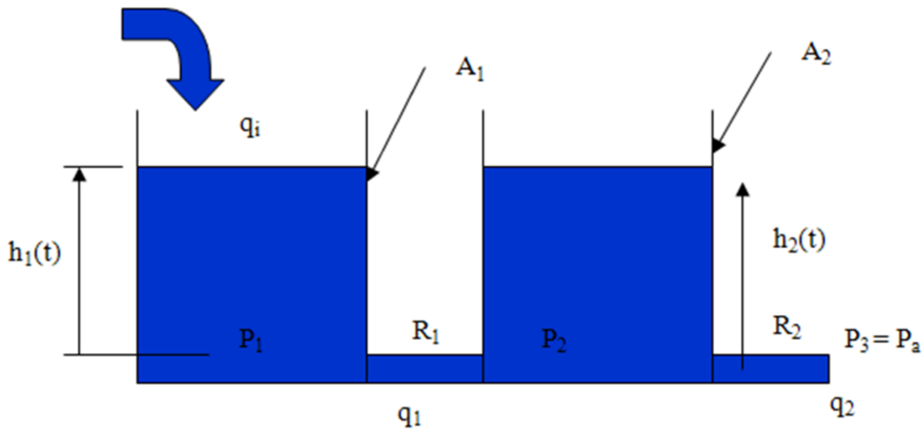
From a block diagram perspective, , we can visualize this as



State Space Representation
Of Magnetic Ball Levitator

Example: Linear State Space Representation - Two Tank Hydraulic System

Consider a two tank hydraulic system as shown below



The differential equations which describe this system are given as

$$\frac{d h_1(t)}{d t} = \frac{1}{A_1} q_i - \frac{\rho g}{A_1 R_1} h_1(t) + \frac{\rho g}{A_1 R_1} h_2(t) \qquad \textbf{\textit{(Eq.1)}}$$

$$\frac{d h_2(t)}{d t} = \frac{\rho g}{A_2 R_1} h_1(t) - \frac{\rho g}{A_2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) h_2(t) \qquad \textbf{\textit{(Eq.2)}}$$

where $h_k(t)$ =height of fluid in tank k (m)
A_k =cross sectional area of tube k (m^2)
q_i =volumetric input flow rate (m^3/s)
ρ =density of fluid (kg/m^3)
R_k =resistance of pipe k $(N s/m^5)$
g =gravitational acceleration (m/s^2)

We can write this in matrix form

$$\begin{pmatrix} \dot{h}_1(t) \\ \dot{h}_2(t) \end{pmatrix} = \begin{pmatrix} -\frac{\rho g}{A_1 R_1} & \frac{\rho g}{A_1 R_1} \\ \frac{\rho g}{A_2 R_1} & -\frac{\rho g}{A_2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \end{pmatrix} \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{A_1} \\ 0 \end{pmatrix} q_i(t)$$

Let us choose a state and control vector of

$$\overline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix} \qquad \text{(state vector)}$$

$$\overline{u}(t) = (u_1(t)) = (q_i(t)) \qquad \text{(control vector)}$$

This system can be rewritten as

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -\frac{\rho g}{A_1 R_1} & \frac{\rho g}{A_1 R_1} \\ \frac{\rho g}{A_2 R_1} & -\frac{\rho g}{A_2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} \frac{1}{A_1} \\ 0 \end{pmatrix} (u_1(t))$$

$$\dot{\bar{x}}(t) = A\,\bar{x}(t) + B\,\bar{u}(t)$$

where $$A = \begin{pmatrix} -\frac{\rho g}{A_1 R_1} & \frac{\rho g}{A_1 R_1} \\ \frac{\rho g}{A_2 R_1} & -\frac{\rho g}{A_2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{1}{A_1} \\ 0 \end{pmatrix}$$

Suppose that we have two sensors to measure the height of both tanks.

$$\bar{y}(t) = \begin{pmatrix} h_1(t) \\ h_2(t) \end{pmatrix}$$

$$= \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} (u_1(t))$$

$$\bar{y}(t) = C\,\bar{x}(t) + D\,\bar{u}(t)$$

where $$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

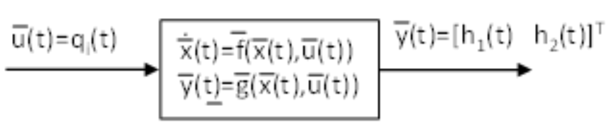
$$D = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

So together, the state space representation of the system is

$$\dot{\bar{x}}(t) = A\,\bar{x}(t) + B\,\bar{u}(t) \qquad \text{(state equation)}$$
$$\bar{y}(t) = C\,\bar{x}(t) + D\,\bar{u}(t) \qquad \text{(output equation)}$$

with $\bar{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} h_1(0) \\ h_2(0) \end{pmatrix}$

Graphically, we can visualize this as



State Space Representation
Of Two Tank Hydraulic System

State Space Representation of an n^{th} Order ODE

This method is particularly useful for reducing an n^{th} order differential equation to 1st order vector representation. This procedure will help write either linear or nonlinear ODEs in state space format.

The general procedure is as follows:

1. Solve differential equation for highest order derivative term (ie $\frac{d^n z(t)}{dt^n}$)
2. Define state vector as $\overline{x}(t) = \begin{pmatrix} z(t) & \frac{dz(t)}{dt} & \frac{d^2 z(t)}{dt} & \ldots & \frac{d^{n-1} z(t)}{dt^{n-1}} \end{pmatrix}^T$
3. Define control vector as $\overline{u}(t) = (\, u_1(t) \quad u_2(t) \quad \ldots \quad u_m(t)\,)^T$
4. Write state equation
    a. $\dot{\overline{x}}(t) = \overline{f}(\overline{x}(t), \overline{g}(t))$ (if nonlinear)
    b. $\dot{\overline{x}}(t) = A\,\overline{x}(t) + B\,\overline{u}(t)$ (if linear)
5. Write output equation
    a. $\overline{y}(t) = \overline{g}(\overline{x}(t), \overline{u}(t))$ (if nonlinear)
    b. $\overline{y}(t) = C\,\overline{x}(t) + D\,\overline{u}(t)$ (if linear)

<u>Example: 4^{th} Order Linear Differential Equation</u>

Consider the linear differential equation of the form.

$$\frac{d^4 z(t)}{dt^4} - 3\frac{d^3 z(t)}{dt^3} + 2\frac{d^2 z(t)}{dt^2} + 6\frac{dz(t)}{dt} + 4z(t) = 3p(t) + 5q(t)$$

where $p(t), q(t)$ = external inputs or forcing functions

Step 1: Solve for highest order term

$$\frac{d^4 z(t)}{dt^4} = 3\frac{d^3 z(t)}{dt^3} - 2\frac{d^2 z(t)}{dt^2} - 6\frac{dz(t)}{dt} - 4z(t) + 3p(t) + 5q(t)$$

Step 2: We can define the state vector as

$$\overline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = \begin{pmatrix} z(t) \\ \frac{dz(t)}{dt} \\ \frac{d^2 z(t)}{dt^2} \\ \frac{d^3 z(t)}{dt^3} \end{pmatrix}$$

Step 3: We can define the control vector as

$$\overline{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} p(t) \\ q(t) \end{pmatrix}$$

Step 4: We can write the state equation

$$\dot{\overline{x}}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ \dot{x}_4(t) \end{pmatrix} = \begin{pmatrix} \frac{d\,z(t)}{d\,t} \\ \frac{d^2\,z(t)}{d\,t^2} \\ \frac{d^3\,z(t)}{d\,t^3} \\ \frac{d^4\,z(t)}{d\,t^4} \end{pmatrix} \quad \text{recall: } \frac{d\,z(t)}{d\,t} = x_2(t),\ \frac{d^2\,z(t)}{d\,t^2} = x_3(t),\ \frac{d^3\,z(t)}{d\,t^3} = x_4(t)$$

$$= \begin{pmatrix} x_2(t) \\ x_3(t) \\ x_4(t) \\ 3\,\frac{d^3\,z(t)}{d\,t^3} - 2\,\frac{d^2\,z(t)}{d\,t^2} - 6\,\frac{d\,z(t)}{d\,t} - 4\,z(t) + 3\,p(t) + 5\,q(t) \end{pmatrix}$$

$$= \begin{pmatrix} x_2(t) \\ x_3(t) \\ x_4(t) \\ 3\,x_4(t) - 2\,x_3(t) - 6\,x_2(t) - 4\,x_1(t) + 3\,u_1(t) + 5\,u_2(t) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -6 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 3 & 5 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

$$\dot{\overline{x}}(t) = A\,\overline{x}(t) + B\,\overline{u}(t)$$

where $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -4 & -6 & -2 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 3 & 5 \end{pmatrix}$

Step 5: We can write the output equation

In this example, let us assume that the outputs of this system are $z(t)$, $\ddot{z}(t) + 3\,\dddot{z}(t)$, and $4\,q(t)$

$$\overline{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix}$$

$$= \begin{pmatrix} z(t) \\ \frac{d^2\,z(t)}{d\,t^2} + 3\,\frac{d^3\,z(t)}{d\,t^3} \\ 4\,q(t) \end{pmatrix}$$

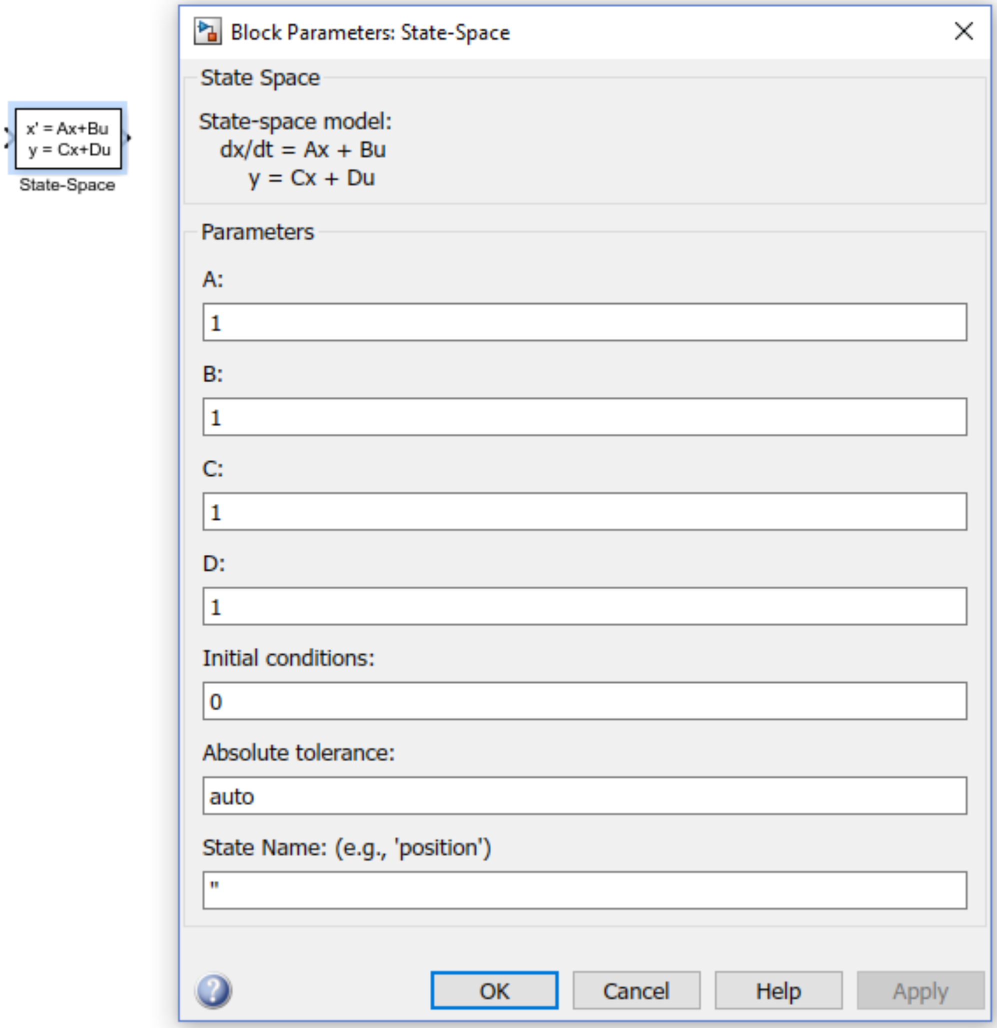
$$= \begin{pmatrix} x_1(t) \\ x_3(t) + 3\, x_4(t) \\ 4\, u_2(t) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$$

$$\overline{y}(t) = C\, \overline{x}(t) + D\, \overline{u}(t)$$

where $\quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad D = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 4 \end{pmatrix}$

Simulink has a "State-Space" block which implements this exact system of equations (under Continuous > State-Space)



Of course in this situation, the initial condition is really a vector of initial conditions

$$\overline{x}_0 = \overline{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{pmatrix}$$