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Lecture 04f
Arc Length (AKA Length of a Curve)

The YouTube video entitled ‘Arc Length (AKA Length of a Curve)’ that covers this lecture is located at https://youtu.be/FoiuvPkFppg.

Outline

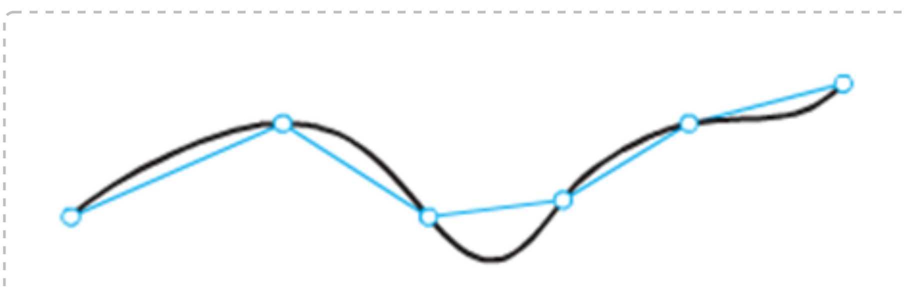
-Length of a Curve

Length of a Curve

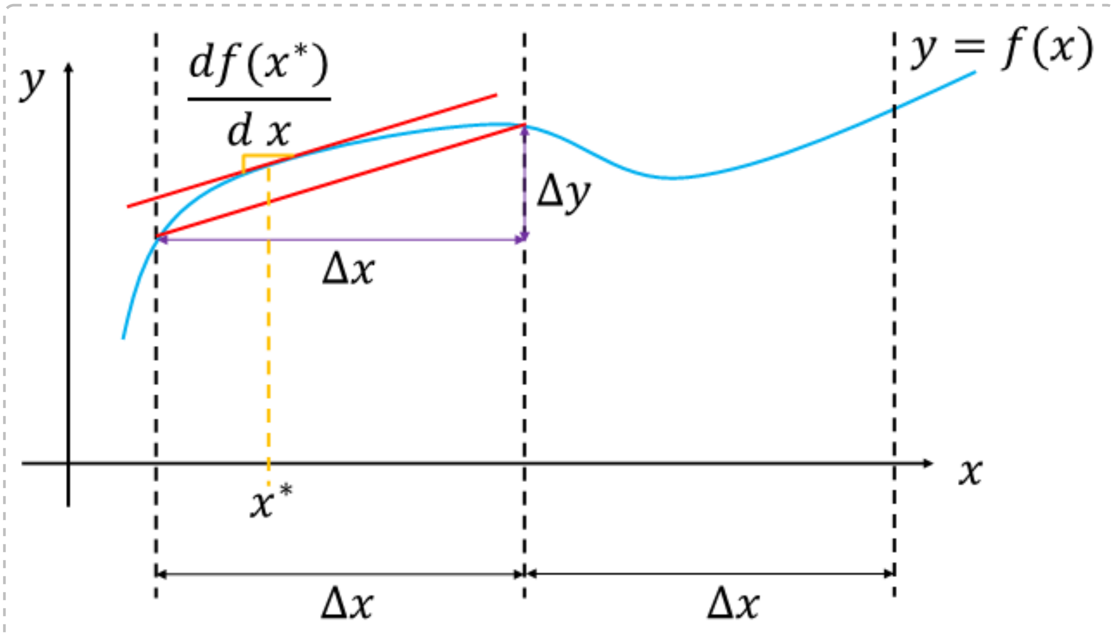
We may want to compute the length of a curve. We first do this using a standard, Cartesian approach and then investigate how to do this for a curve that has a parametric representation.

Standard Approach

Consider the geometry shown below



We can look closer at a few of these segments.



We consider the length of the red secant line which is an approximation of the length of the curve between x_1 and x_2. From the Pythagorean Theorem we have

$$H_i = \sqrt{\Delta x^2 + \Delta y^2} \qquad \textbf{(Eq.1.1)}$$

We note that by the Mean Value Theorem, there is some point $x^* \in [x_i, x_{i+1}]$ where the slope of the function f is equal to the slope of the secant line. In other words, there is a point where $\frac{d f(x^*)}{d x} = \frac{\Delta y}{\Delta x}$. Therefore

$$\Delta y = \frac{d f(x^*)}{dx} \Delta x \qquad \textbf{(Eq.1.2)}$$

Substituting this into the Eq.1.1 yields

$$H_i = \sqrt{\Delta x^2 + \Delta y^2}$$

$$= \sqrt{\Delta x^2 + \left(\frac{d f(x^*)}{dx} \Delta x\right)^2}$$

$$H_i = \sqrt{1 + \left(\frac{d f(x^*)}{dx}\right)^2} \Delta x \qquad \textbf{(Eq.1.3)}$$

So the length of the total curve between $x = a$ and $x = b$ can be approximated by summing the hypotenuse calculations

$$s \approx \sum_i H_i$$

$$s \approx \sum_i \sqrt{1 + \left(\frac{d f(x^*)}{\text{dx}}\right)^2}\ \Delta\text{x}$$

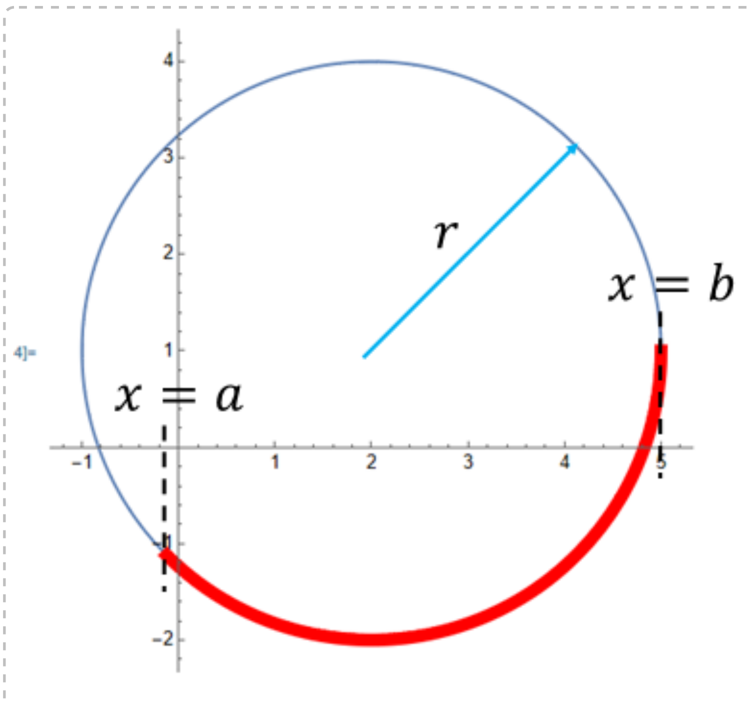
If we take the limit as $\Delta \text{x} \to 0$ this expression goes from an approximation to an equality

$$s = \lim_{\Delta\text{x}\to 0} \sum_i \sqrt{1 + \left(\frac{d f(x^*)}{\text{dx}}\right)^2}\ \Delta\text{x}$$

$$s = \int_a^b \sqrt{1 + \left(\frac{d f(x)}{\text{dx}}\right)^2}\ dx \qquad \textbf{\textit{(Eq.1.4)}}$$

Example: Partial Circle

Consider the picture below



Consider the red curve described by the function

$$y = f(x) = -\sqrt{r^2 - (x - x_o)^2} + y_o$$

```
In[◦]:= f[x_] = -√(r² - (x - xo)²) + yo;
```

We note that this is the equation of a circle centered at (x_o, y_o) with a radius r.

Let $x_o = 2$

$y_o = 1$

$r = 3$

In[ ]:=
```
xoGiven = 2;
yoGiven = 1;
rGiven = 3;
replaceString = {xo → xoGiven, yo → yoGiven, r → rGiven};
```

We chose to compute the length of the curve between

$$a = x_o - r\frac{1}{\sqrt{2}}$$

$$b = r + x_o$$

In[ ]:=
```
a = xo - r 1/Sqrt[2] /. replaceString
b = r + xo /. replaceString

a // N
b // N
```

Out[ ]= $2 - \frac{3}{\sqrt{2}}$

Out[ ]= 5

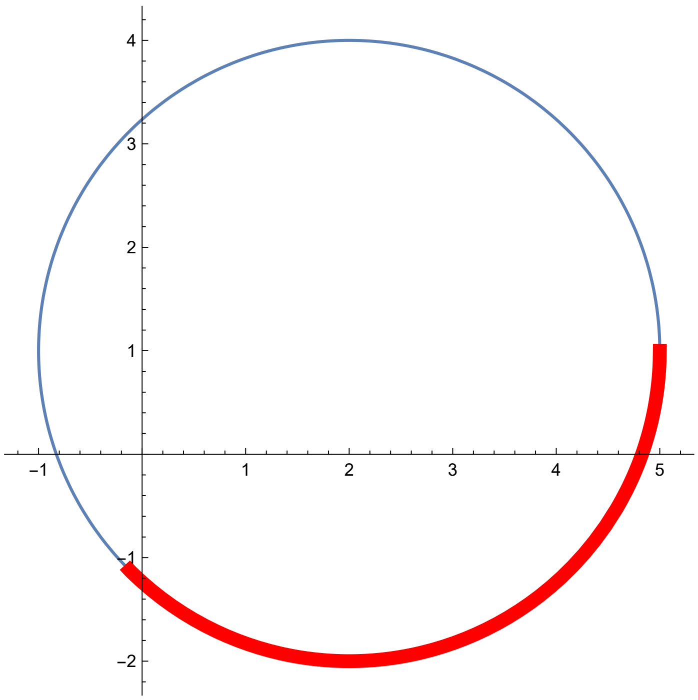
Out[ ]= -0.12132

Out[ ]= 5.

We can visualize the scenario

In[∘]:=
```
(*Plot an entire circle*)
p1 = ParametricPlot[{r Cos[θ] + xo, r Sin[θ] + yo} /. replaceString, {θ, 0, 2 π}];

(*Plot the curve in question*)
p2 = Plot[f[x] /. replaceString, {x, a, b},
   PlotStyle → {Red, Thickness[0.02]}, PlotRange → All];

Show[p1, p2]
```

Out[∘]=



To compute the length of the red curve, we apply Eq.1.4. This requires we compute the derivative of *f* w.r.t. *x*

In[∘]:=
```
dfdx[x_] = D[f[x], x]
```

Out[∘]=
$$\frac{x - xo}{\sqrt{r^2 - (x - xo)^2}}$$

$$s = \int_a^b \sqrt{1 + \left(\frac{d\,f(x)}{dx}\right)^2}\,dx$$

$$= \int_a^b \sqrt{1 + \left(\frac{x - x_o}{\sqrt{r^2 - (x - x_o)^2}}\right)^2}\,dx$$

$$= \int_{2-\frac{3}{\sqrt{2}}}^{5} \sqrt{1 + \left(\frac{x - 2}{\sqrt{3^2 - (x - 2)^2}}\right)^2}\,dx$$

In[]:=
```
s = Integrate[Sqrt[1 + dfdx[x]^2] /. replaceString, {x, a, b}]
```

Out[]= $\frac{9\pi}{4}$

We can confirm this result by computing the circumference of the circle and then realizing that the red curve is only $\frac{360-180-45}{360}$ of the entire circumference

In[]:=
```
sCheck = 2 π r ((360 - 180 - 45)/360) /. replaceString
s == sCheck
```

Out[]= $\frac{9\pi}{4}$

Out[]= True

So we see that this is the same.

Parametric Representation

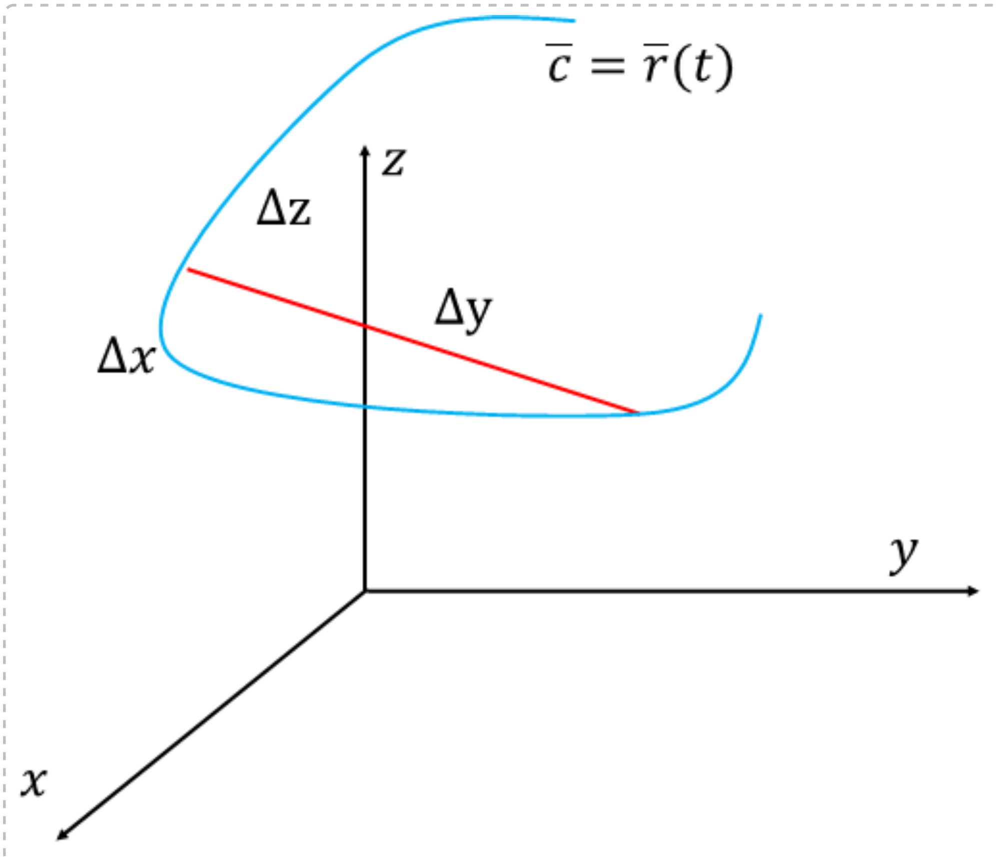
In the previous example, we noted that Δx was constant. We can generalize this approach to a parametric representation of a curve where Δx, Δy, and Δz are not constant and instead depend on where we are in the parameterization.

The parameterization is given as

$$x(t) = r_1(t)$$
$$y(t) = r_2(t)$$
$$z(t) = r_3(t)$$

Or in vector form

$$\overline{c}(t) = \overline{r}(t) = \begin{pmatrix} r_1(t) \\ r_2(t) \\ r_3(t) \end{pmatrix}$$



In 3 dimensions the secant line is still given curve between x_1 and x_2. From the Pythagorean Theorem we have

$$H_i = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2} \qquad \textbf{\textit{(Eq.2.1)}}$$

Again we note that by the Mean Value Theorem, there is some point t^*, t^{**}, and t^{***} where the slope of the function x,y,z parametrization is equal to the slope of the secant line, respectively. In other words, there is a point where

$$\frac{d\,r_1(t^*)}{d\,t} = \frac{\Delta x}{\Delta t}$$

$$\frac{d\,r_2(t^{**})}{d\,t} = \frac{\Delta y}{\Delta t}$$

$$\frac{d\,r_3(t^{***})}{d\,t} = \frac{\Delta z}{\Delta t}$$

Therefore

$$\Delta x = \frac{d\,r_1(t^*)}{d\,t}\,\Delta t$$

$$\Delta y = \frac{d\,r_2(t^{**})}{d\,t}\,\Delta t$$

$$\Delta z = \frac{d\,r_3(t^{***})}{d\,t}\,\Delta t \qquad \textbf{\textit{(Eq.2.2)}}$$

Substituting this into the Eq.2.1 yields

$$H_i = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$$

$$= \sqrt{\left(\frac{d\,r_1(t^*)}{d\,t}\,\Delta t\right)^2 + \left(\frac{d\,r_2(t^{**})}{d\,t}\,\Delta t\right)^2 + \left(\frac{d\,r_3(t^{**})}{d\,t}\,\Delta t\right)^2}$$

$$= \sqrt{\left(\frac{d\,r_1(t^*)}{d\,t}\right)^2 + \left(\frac{d\,r_2(t^{**})}{d\,t}\right)^2 + \left(\frac{d\,r_3(t^{***})}{d\,t}\right)^2}\,\Delta t$$

$$= \sqrt{\frac{d\,\overline{r}(\tilde{t})}{d\,t}.\frac{d\,\overline{r}(\tilde{t})}{d\,t}}\,\Delta t$$ where $\tilde{t}$ denotes that t^*, t^{**}, and t^{***} may be different points

$$H_i = \sqrt{\overline{r}'(\tilde{t}).\overline{r}'(\tilde{t})}\,\Delta t \qquad \textbf{\textit{(Eq.2.3)}}$$

So the length of the total curve between $t = a$ and $t = b$ can be approximated by summing the hypotenuse calculations

$$s \approx \sum_i H_i$$

$$s \approx \sum_i \sqrt{\overline{r}'(\tilde{t}) \cdot \overline{r}'(\tilde{t})}\,\Delta t$$

If we take the limit as $\Delta t \to 0$ this expression goes from an approximation to an equality

$$s = \lim_{\Delta t \to 0} \sum_i \sqrt{\overline{r}'(t) \cdot \overline{r}'(t)}\,\Delta t$$

$$s = \int_a^b \sqrt{\overline{r}'(t) \cdot \overline{r}'(t)}\,dt \qquad \textbf{\textit{(Eq.2.4)}}$$

In practice, evaluating this integral may prove difficult although we will investigate some situations where it is feasible.

Example: Partial Circle (repeated)

We can use the same example as before by simply describing the curve using a parametric representation

$$\overline{r}(t) = \begin{pmatrix} r\cos(t) + x_o \\ r\sin(t) + y_o \\ 0 \end{pmatrix}$$

In[ ]:=
```
r[t_] = {{r Cos[t] + xo}, {r Sin[t] + yo}, {0}};
```

Computing $\bar{r}'(t) = \frac{d}{dt}[\bar{r}(t)]$

In[ ]:=
```
drdt[t_] = D[r[t], t];
drdt[t] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -r\,\mathrm{Sin}[t] \\ r\,\mathrm{Cos}[t] \\ 0 \end{pmatrix}$$

Computing the end points of the parametrization

In[ ]:=
```
a = (180 + 45)/360 * 2 π;
b = 2 π;
```

Inserting into Eq.2.4 yields

In[ ]:=
```
Integrand = (Sqrt[(Transpose[drdt[t]].drdt[t])])[[1, 1]] // Simplify
```

Out[ ]= $\sqrt{r^2}$

So this becomes

$$s = \int_a^b \sqrt{r^2}\, dt$$

In[ ]:=
```
sParam = Integrate[Integrand /. replaceString, {t, a, b}]
```

Out[ ]= $\frac{9\pi}{4}$

In[ ]:=
```
sParam == s
```

Out[ ]= True

In[ ]:=
```
Clear[a, b, s]
```

Alternatively, the length of the curve between a and an arbitrary stop point t can be given as

$$s(t) = \int_a^t \sqrt{\bar{r}'(\tau) \cdot \bar{r}'(\tau)}\, d\tau \qquad \textbf{\textit{(Eq.11)}}$$

where $s(t)$ = arc length of the curve from a to t

Note that τ is the variable of integration because t is the upper limit of integration.

Note that the earlier parametrization of the curve $C = \bar{r}(t)$ uses t as the parameter. Often, it is more convenient to parameterize the curve based on the arc length s, so we write $C = \bar{r}(s)$. We can investigate this in the next example.

Example 5: Circular helix. Circle. Arc length as parameter

Let us revisit the elliptical helix from the ‘Tangent to a Curve’ video except we will make it a circular helix where $a = b$ (the semi-major axis is equal to the semi-minor axis). Initially, this was parameterized using t as the independent parameter

$$\bar{r}(t) = \begin{pmatrix} a\cos(t) \\ a\sin(t) \\ c\,t \end{pmatrix}$$

In[∘]:=
```
r[t_] = ( a Cos[t]
          a Sin[t]
            c t    );
```

We can first compute the derivative of this function. Recall that this is effectively the tangent vector to curve at the point $P(t)$

In[∘]:=
```
drdt[t_] = D[r[t], t];
drdt[t] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} -\text{a Sin}[t] \\ \text{a Cos}[t] \\ \text{c} \end{pmatrix}$$

We can now compute the arc length from a starting point of 0 to an ending point of t using Eq.11.

$$s(t) = \int_0^t \sqrt{\bar{r}'(\tilde{t}) \cdot \bar{r}'(\tilde{t})}\, d\tilde{t}$$

$$= \int_0^t \sqrt{\begin{pmatrix} -a\sin(\tilde{t}) \\ a\cos(\tilde{t}) \\ c \end{pmatrix} \cdot \begin{pmatrix} -a\sin(\tilde{t}) \\ a\cos(\tilde{t}) \\ c \end{pmatrix}}\, d\tilde{t}$$

$$= \int_0^t \sqrt{a^2\cos(\tilde{t})^2 + a^2\sin(\tilde{t})^2 + c^2}\, d\tilde{t} \qquad \text{recall: } \cos(\tilde{t})^2 + \sin(\tilde{t})^2 = 1$$

$$= \int_0^t \sqrt{a^2 + c^2}\, d\tilde{t}$$

$$s(t) = \sqrt{a^2 + c^2}\; t$$

In[∘]:=
```
s[t_] = Integrate[Sqrt[(Transpose[drdt[t]].drdt[t])[[1, 1]]], t]
```

Out[∘]= $\sqrt{a^2 + c^2}\; t$

We can check the validity of this result. For example, if $c = 0$, we see that the curve is actually a simple circle with radius of a. The circumference of the circle is given by $2\pi a$. Looking at the original parametrization of the curve, we see that we should complete a full circle when $t \in [0,\ 2\pi]$. So we see that $s(2\pi)$ should equal 2π a

In[∘]:=
```
Simplify[s[2 π] /. {c → 0}, a > 0]
```

Out[∘]= $2\, a\, \pi$

The relationship between *s* and *t* is given by our solution of $s = \left(a^2 + c^2\right)^{1/2} t$ so solving for *t* yields how the parameter *t* is related to the length along the curve, *s*.

$$t = \frac{s}{\left(a^2+c^2\right)^{1/2}}$$

So we can now parameterize based on *s* instead of *t* by substituting this relationship into our original parametrization to obtain an alternative parametrization of the curve that uses the arc length, *s*, as the independent parameter

$$\bar{r}^*(s) = \bar{r}(t)\ \Big|_{t=\frac{s}{\left(a^2+c^2\right)^{1/2}}} = \bar{r}\left(\frac{s}{\left(a^2+c^2\right)^{1/2}}\right)$$

In[∘]:=
```
rstar[s_] = r[s/(a^2 + c^2)^(1/2)];

rstar[s] // MatrixForm
```

Out[∘]//MatrixForm=

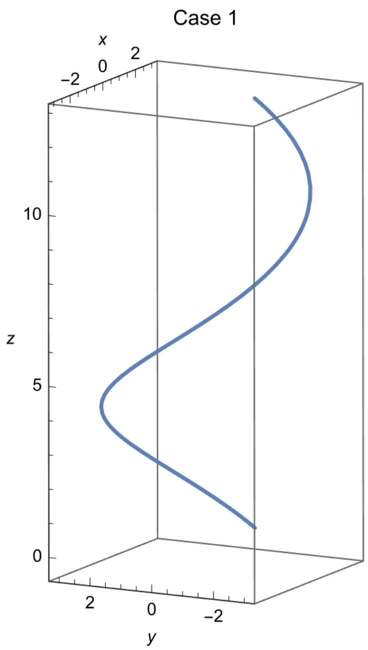
$$\begin{pmatrix} a \operatorname{Cos}\left[\frac{s}{\sqrt{a^2+c^2}}\right] \\ a \operatorname{Sin}\left[\frac{s}{\sqrt{a^2+c^2}}\right] \\ \frac{c\, s}{\sqrt{a^2+c^2}} \end{pmatrix}$$

We can now investigate the differences between these two parameterizations. If $c \neq 0$, we can compare

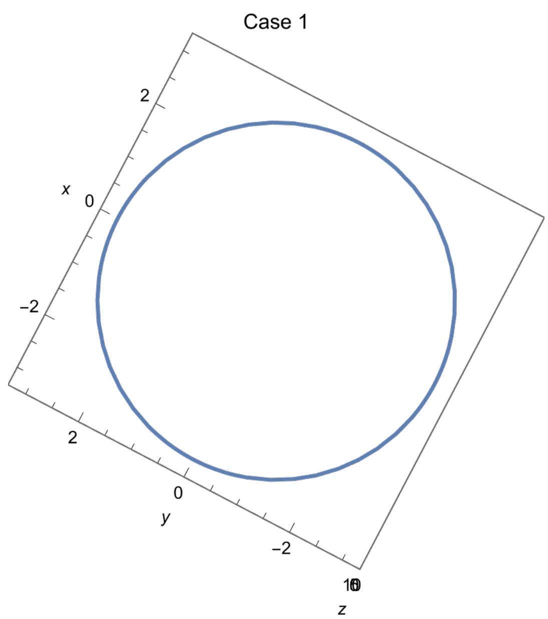
Case 1: $r(t)$ with $t \in [0, 2\pi]$ $\Rightarrow$ should generate 1 full spiral
Case 2: $r^*(s)$ with $s \in [0, 2\pi a]$ $\Rightarrow$ will not generate a 1 full spiral

In[∘]:=
```
a = 3;
c = 2;

(*Case 1*)
case1 = ParametricPlot3D[{r[t][[1, 1]], r[t][[2, 1]], r[t][[3, 1]]}, {t, 0, 2 π},
  AxesLabel → {x, y, z},
  PlotLabel → "Case 1"]

Show[case1,
 ViewPoint → {0, 0, ∞}]
```

Out[∘]=



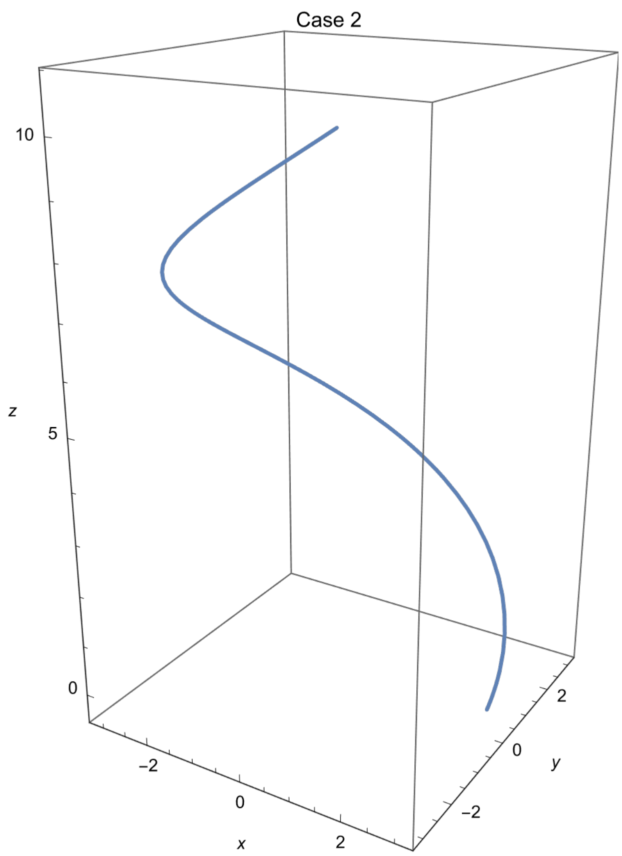
Out[∘]=



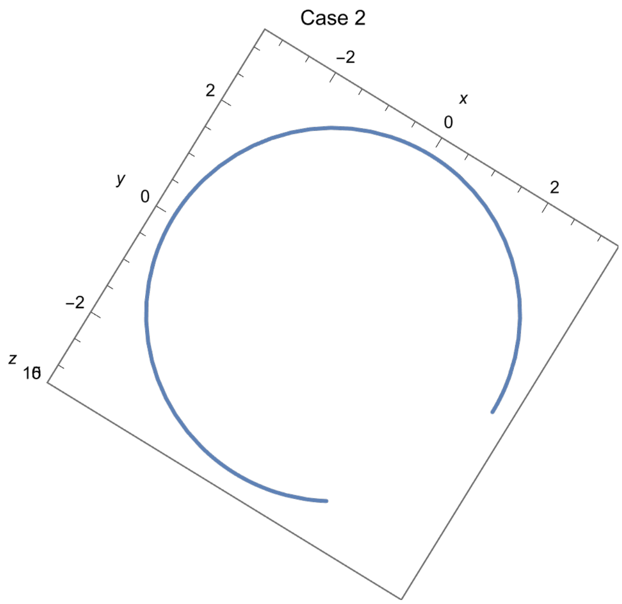
In[ ]:=

```
(*Case 2*)
case2 =
 ParametricPlot3D[{rstar[s][[1, 1]], rstar[s][[2, 1]], rstar[s][[3, 1]]}, {s, 0, 2 π a},
  AxesLabel → {x, y, z},
  PlotLabel → "Case 2"]

Show[case2,
 ViewPoint → {0, 0, ∞}]
```

Out[ ]=



Out[∘]=



For case 2, if $c \neq 0$, then $s \in [0, \ 2\,\pi\,a]$ will not obtain a full circle since the arc length required to complete a full "circle" is increased

In[∘]:=
```
Clear[c, a, rstar, s, drdt, r, case1, case2]
```