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# Lecture01c
# Introduction to Matrices

**Lecture is on YouTube**

The YouTube video entitled ‘Introduction to Matrices’ that covers this lecture is located at https://youtu.be/N5R8IZ0R3PI.

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## Matrices

A matrix is a 2D array of numbers.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

Note that the book uses a bold face to denote a matrix. This is done for emphasis and most literature simply uses capital letters to denote matrices. We will follow the nomenclature of using capital letters to denote matrices.

Elements of a matrix are indexed/referenced using a row/column notation. In other words

$A_{ij}$ = element in the $i^{\text{th}}$ row, $j^{\text{th}}$ column

The dimension of the matrix is the number of rows-by-number of columns

If a matrix only has a single row or a single column, it is called a vector. Typically, vectors are denoted using lower case letters and a bold or over bar to denote that it is a vector.

$\overline{v}$ = vector

Often, it is useful to think of a matrix as being comprised on a series or rows or columns

$A_{N \times M}$ = a matrix with $N$ rows and $M$ columns

$$A = \begin{pmatrix} - & \overline{r}_1 & - \\ - & \overline{r}_2 & - \\ - & \ldots & - \\ - & \overline{r}_n & - \end{pmatrix} \qquad \text{or} \qquad A = \begin{pmatrix} | & | & | & | \\ \overline{c}_1 & \overline{c}_2 & \ldots & \overline{c}_M \\ | & | & | & | \end{pmatrix}$$

**Example**

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 3 & 5 \end{pmatrix}$$

$A$ is a $2 \times 3$ matrix.

$A_{2,3} = 5$

$A$ is comprised of 2 rows or 3 column vectors

$$A = \begin{pmatrix} - & \overline{r}_1 & - \\ - & \overline{r}_2 & - \end{pmatrix}$$

where $\overline{r}_1 = \begin{pmatrix} 1 & 3 & -1 \end{pmatrix}$
$\overline{r}_2 = \begin{pmatrix} 0 & 3 & 5 \end{pmatrix}$

or

$$A = \begin{pmatrix} | & | & | \\ \overline{c}_1 & \overline{c}_2 & \overline{c}_3 \\ | & | & | \end{pmatrix}$$

where $\overline{c}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\overline{c}_2 = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ $\overline{c}_3 = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$

# Matrix Addition/Subtraction

We can add and subtract matrices easily. This requires that both matrices be the same size/dimension.

Matrix addition/subtraction operates on a matrix $A$ (of dimensions $m-\text{by}-n$) with a matrix $B$ (of dimensions ($m-\text{by}-n$) to produce a matrix $C$ (which will have dimensions $m-\text{by}-n$).

$$C_{\text{ij}} = A_{\text{ij}} \pm B_{\text{ij}}$$

**Example**

$$A = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 3 & 5 \end{pmatrix} \qquad B = \begin{pmatrix} 2 & 4 & 2 \\ -4 & 1 & 1 \end{pmatrix}$$

$$A + B = \begin{pmatrix} 3 & 7 & 1 \\ -4 & 4 & 6 \end{pmatrix}$$

$$A - B = \begin{pmatrix} -1 & -1 & -3 \\ 4 & 2 & 4 \end{pmatrix}$$

**Rules for Matrix Addition**

$A + B = B + A$

$(A + B) + C = A + (B + C)$

$A + 0 = A$ (zero matrix)

$A + (-A) = 0$ (zero matrix)

# Matrix Multiplication

We should note that matrix multiplication can mean different things in different contexts. In general, there are two types of matrix multiplication that we will typically utilize

1. Standard matrix multiplication
2. Matrix element-wise multiplication
3. Matrix Scalar Multiplication

Let us examine each of these cases in examples.

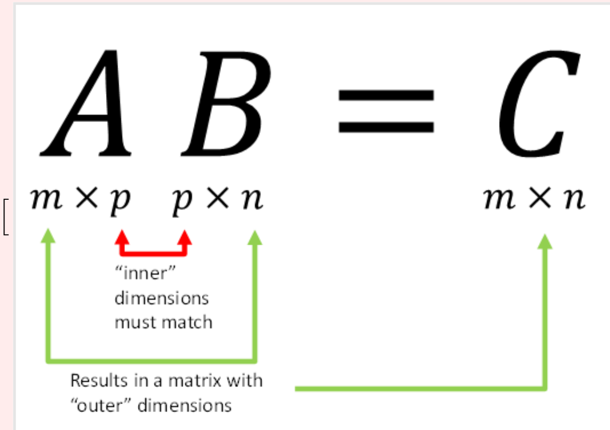
## 1. Standard Matrix Multiplication

Standard matrix multiplication multiplies a matrix $A$ (of dimensions $m - \text{by} - p$) with a matrix $B$ (of dimensions ($p - \text{by} - n$) to produce a matrix $C$ (which will have dimensions $m - \text{by} - n$).

The entry in the $i^{\text{th}}$ row and $j^{\text{th}}$ column of $C$ is obtained by taking the inner/dot product of the entire $i^{\text{th}}$ row of $A$ with the entire $j^{\text{th}}$ column of $B$.

Note: Inner/dot product is discussed in 'Simple Vector Mechanics: Inner Product, Scalar/Vector Projection, and Cross Product' at https://youtu.be/fAZZJgm096w?si=B3ByzqhV46fGYtCM

A helpful pneumonic that is helpful to check/remember matrix multiplication dimensions is to write the dimension underneath the matrices. The "inner" dimensions much match and this produces a matrix based on the "outer" dimensions

Canvas[ CanvasFrame[ 

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**Example**

Consider the example below which illustrates matrix multiplication by hand

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \end{pmatrix} B = \begin{pmatrix} 2 & 4 \\ 6 & 8 \\ 10 & 12 \end{pmatrix}$$

$$C = A\,B$$

$$\begin{pmatrix} 1 & 3 & 5 \\ 7 & 9 & 11 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ 6 & 8 \\ 10 & 12 \end{pmatrix} = \begin{pmatrix} (1*2+3*6+5*10) & (1*4+3*8+5*12) \\ (7*2+9*6+11*10) & (7*4+9*8+11*12) \end{pmatrix}$$

$$= \begin{pmatrix} 70 & 88 \\ 178 & 232 \end{pmatrix}$$

## 2. Matrix Element-wise Multiplication

Matrix element-wise multiplication multiplies a matrix $A$ (of dimensions $m$ – by – $n$) with a matrix $B$ (of dimensions ($m$ – by – $n$)  to produce a matrix $C$ (which will have dimensions $m$ – by – $n$).

The entry in the $i^{\text{th}}$ row and $j^{\text{th}}$ column of $C$ is obtained by simply multiplying the entry in the $i^{\text{th}}$ row and $j^{\text{th}}$ column of $A$ with the entry in the $i^{\text{th}}$ row and $j^{\text{th}}$ column of $B$

$$C_{\text{ij}} = A_{\text{ij}} * B_{\text{ij}}$$

**Example**

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

$$C = A.*B$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} .* \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1*5 & 2*6 \\ 3*7 & 4*8 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 12 \\ 21 & 32 \end{pmatrix}$$

**Rules for Matrix Multiplication**

$$A\,(B\,C) = (A\,B)\,C$$

$$(A + B)\,C = A\,C + B\,C$$

$$C(A + B) = C\,A + C\,B$$

Note: In general, matrix multiplication is not commutative, $A\,B \neq B\,A$ in general.

## 3. Matrix Scalar Multiplication

The only time you can perform matrix multiplication with incompatible matrix dimensions is if one of the matrix is a $1 \times 1$ matrix (AKA a scalar). In this case, the scalar is simply multiplied to each element of the matrix.

$A = m \times n$ matrix
$k = 1 \times 1$ matrix (scalar)

$$C_{ij} = k\,A_{ij}$$

**Example**

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad k = 2$$

$$C = k\,A$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} 2 = \begin{pmatrix} 1*2 & 2*2 \\ 3*2 & 4*2 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 4 \\ 6 & 8 \end{pmatrix}$$

**Rules for Matrix Scalar Multiplication**

$$(k\,A)\,B = k(A\,B) = A(k\,B)$$

$$k(A + B) = k\,A + k\,B$$

$$(k+q)A = \text{kA} + qA$$

$$q(kA) = (qk)A$$

$$1A = A$$

where $k$, $q$, 1 are scalars

# Transpose

The operation of switching the rows and columns of a matrix is called a **transpose** operation. This is denoted by a $T$ superscript

$$B = A^T \qquad (B \text{ is the transpose of } A)$$

**Example: Transpose**

Consider the matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$$

It's transpose is given by

$$A^T = \begin{pmatrix} a & d \\ b & e \\ c & f \end{pmatrix}$$

**Rules for Matrix Transposes**

$$\left(A^T\right)^T = A$$

$$(A+B)^T = A^T + B^T$$

$$(kA)^T = kA^T$$

$$(AB)^T = B^T A^T$$

# Special Matrices

A **symmetric matrix** is one where the transpose equal itself

$$A^T = A \qquad \text{(symmetric matrix)}$$

A **skew-symmetric matrix** is one where the transpose equals the negative of the matrix

$A^T = -A$ (skew symmetric matrix)

A **diagonal matrix** is one where the only non-zero entries are on the main diagonal.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

A special type of diagonal matrix is the **identity matrix**, which has only 1's on the main diagonal. This is typically denotes as $I$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

An **upper triangular matrix** is one where the only non-zero entries are on the main diagonal or above.

$$A = \begin{pmatrix} 1 & 5 & 6 \\ 0 & 6 & 5 \\ 0 & 0 & 4 \end{pmatrix}$$

An **lower triangular matrix** is one where the only non-zero entries are on the main diagonal or below.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 6 & 0 \\ 4 & 3 & 4 \end{pmatrix}$$