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Lecture03h
Computing the Matrix Exponential Using the Laplace Method

The YouTube video entitled ‘Computing the Matrix Exponential Using the Laplace Method’ that covers this lecture is located at https://youtu.be/belZC9XGBtM .

Outline

-Computing the Matrix Exponential Using the Laplace Method

Computing the Matrix Exponential Using the Laplace Method

The state space system is given as

$$\dot{\overline{x}}(t) = A\,\overline{x}(t) + B\,\overline{u}(t) \qquad \textbf{\textit{(Eq.1)}}$$

As discussed previously, the solution to this is given by

$$\overline{x}(t) = e^{A(t-t_o)}\,\overline{x}(t_o) + \int_{t_o}^{t} e^{A(t-\tau)}\,B\,\overline{u}(\tau)\,d\tau \qquad \textbf{\textit{(Eq.2)}}$$

Let us consider the autonomous case (no inputs and $\overline{u}(t) = \overline{0}$). In this case, the governing state space equation (Eq.1) becomes

$$\dot{\overline{x}}(t) = A\,\overline{x}(t)$$

$$\frac{d}{dt}[\overline{x}(t)] = A\,\overline{x}(t) \qquad \textbf{\textit{(Eq.3)}}$$

Recall that the solution to this is given by Eq.2 and in the autonomous/homogeneous case with $t_o = 0$ becomes

$$\overline{x}(t) = e^{A\,t}\,\overline{x}(0) = \phi(t)\,\overline{x}(0) \qquad \textbf{\textit{(Eq.4)}}$$

where $\quad \phi(t) = e^{A\,t}$

Substituting this into Eq.3 for $\overline{x}(t)$ yields

$$\frac{d}{dt}[\phi(t)\,\overline{x}(0)] = A\,\phi(t)\,\overline{x}(0)$$

$$\frac{d}{dt}[\phi(t)]\,\overline{x}(0) = A\,\phi(t)\,\overline{x}(0)$$

This would obviously be true if $\overline{x}(0) = 0$ but since this must be true for all $\overline{x}(0)$, we have

$$\frac{d}{dt}\,\phi(t) = A\,\phi(t) \qquad \textbf{\textit{(Eq.5)}}$$

We see that Eq.5 is a differential equation. We need an initial condition ($\phi(0)$) to fully define the differential equation.

If we again recall that the state is given by $\overline{x}(t) = \phi(t)\,\overline{x}(0)$, we can note that this expression at $t = 0$ can be written as

$$\overline{x}(0) = \phi(0)\,\overline{x}(0)$$

Obviously this implies that

$$\phi(0) = I \qquad \textbf{\textit{(Eq.6)}}$$

which becomes the initial condition for Eq.5. As such, the fully defined ODE is given by

$$\frac{d}{dt}\,\phi(t) = A\,\phi(t) \qquad \textbf{\textit{(Eq.7)}}$$

with $\phi(0) = I$

Taking the Laplace transform (see YouTube video entitled 'The Laplace Transform' https://youtu.be/q0nX8uIFZ_k) of Eq.7 yields

$$s\,\Phi(s) - \phi(0) = A\,\Phi(s) \qquad \text{recall: } \phi(0) = I$$

$$s\,\Phi(s) - I = A\,\Phi(s)$$

$$s\,\Phi(s) - A\,\Phi(s) = I$$

$$(s\,I - A\,)\,\Phi(s) = I$$

$$\Phi(s) = (s\,I - A\,)^{-1} \qquad \textbf{\textit{(Eq.8)}}$$

Eq.8 shows how to compute the transition matrix in the Laplace domain. Once we have computed the transition matrix in the Laplace domain, we can perform an inverse Laplace transform to obtain the transition matrix in the time domain.

Example: 2x2 matrix

Consider the following linear system

$$\dot{\overline{x}}(t) = A\,\overline{x}(t)$$

where $\quad A = \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix}$

We can compute the transition matrix in the Laplace domain using Eq.6. Let us first compute some intermediate terms

$$s\,I - A = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} s+2 & -1 \\ 1 & s-2 \end{pmatrix}$$

We can now calculate the determinant

$$\det(s\,I - A) = (s+2)\,(s-2) + 1$$

$$= s^2 - 3$$

We can now calculate the inverse

$$(s\,I - A)^{-1} = \frac{1}{\det(A)} \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

$$= \frac{1}{s^2-3} \begin{pmatrix} s-2 & 1 \\ -1 & s+2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{s-2}{s^2-3} & \frac{1}{s^2-3} \\ \frac{-1}{s^2-3} & \frac{s+2}{s^2-3} \end{pmatrix}$$

```
In[]:= A = ( -2  1
             -1  2 );
       Φ[s_] = Inverse[s * IdentityMatrix[2] - A];
       Φ[s] // MatrixForm
```

Out[]//MatrixForm=

$$\begin{pmatrix} \frac{-2+s}{-3+s^2} & \frac{1}{-3+s^2} \\ -\frac{1}{-3+s^2} & \frac{2+s}{-3+s^2} \end{pmatrix}$$

We now have the transition matrix in the Laplace domain

$$\Phi(s) = \begin{pmatrix} \frac{s-2}{s^2-3} & \frac{1}{s^2-3} \\ \frac{-1}{s^2-3} & \frac{s+2}{s^2-3} \end{pmatrix}$$

We now have to use the Inverse Laplace transform in order to obtain the transition matrix in time domain (see YouTube video entitled 'Partial Fraction Expansion/Decomposition' https://youtu.be/vlCd-CAEtRag and 'The Inverse Laplace Transform' https://youtu.be/wZkrU1lPObM).

Term a_{11}

We have the term

$$\frac{s-2}{s^2-3}$$

We need to try and calculate the partial fraction decomposition of this term. We see that the denominator is a perfect square and we can write $s^2 - 3 = \left(s + \sqrt{3}\right)\left(s - \sqrt{3}\right)$. Therefore

$$\frac{s-2}{s^2-3} = \frac{s-2}{\left(s+\sqrt{3}\right)\left(s-\sqrt{3}\right)} \qquad \textbf{\textit{(Eq.6)}}$$

We would like to break these apart to the form

$$\frac{\alpha}{s-\sqrt{3}} + \frac{\beta}{s+\sqrt{3}} = \frac{s-2}{\left(s+\sqrt{3}\right)\left(s-\sqrt{3}\right)}$$

Therefore, we need to solve for coefficients α and β which make this equal to Eq.6. Obtaining a common denominator for the previous equation yields

$$\frac{s-2}{\left(s+\sqrt{3}\right)\left(s-\sqrt{3}\right)} = \frac{\alpha\left(s+\sqrt{3}\right)}{\left(s-\sqrt{3}\right)\left(s+\sqrt{3}\right)} + \frac{\beta\left(s-\sqrt{3}\right)}{\left(s+\sqrt{3}\right)\left(s-\sqrt{3}\right)}$$

$$\frac{s-2}{\left(s+\sqrt{3}\right)\left(s-\sqrt{3}\right)} = \frac{\alpha\left(s+\sqrt{3}\right)+\beta\left(s-\sqrt{3}\right)}{\left(s-\sqrt{3}\right)\left(s+\sqrt{3}\right)}$$

$$\frac{s-2}{\left(s+\sqrt{3}\right)\left(s-\sqrt{3}\right)} = \frac{\alpha s+\alpha\sqrt{3}+\beta s-\beta\sqrt{3}}{\left(s-\sqrt{3}\right)\left(s+\sqrt{3}\right)}$$

$$\frac{s-2}{\left(s+\sqrt{3}\right)\left(s-\sqrt{3}\right)} = \frac{(\alpha+\beta)\, s+(\alpha-\beta)\sqrt{3}}{\left(s-\sqrt{3}\right)\left(s+\sqrt{3}\right)}$$

Therefore, by comparing coefficients, we obtain two equations

$$1 = \alpha + \beta$$

$$-2 = (\alpha - \beta)\sqrt{3}$$

We can solve these two equations.

```
Solve[{α + β == 1, (α - β) √3 == -2}, {α, β}]
```

$$\left\{\left\{\alpha \to \frac{-2+\sqrt{3}}{2\sqrt{3}}, \beta \to \frac{1}{2} + \frac{1}{\sqrt{3}}\right\}\right\}$$

From this, we see that we can write this expression as

$$\frac{s-2}{s^2-3} = \frac{\frac{1}{2}-\frac{1}{\sqrt{3}}}{s-\sqrt{3}} + \frac{\frac{1}{2}+\frac{1}{\sqrt{3}}}{s+\sqrt{3}}$$

$$= \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)\frac{1}{s-\sqrt{3}} + \left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)\frac{1}{s+\sqrt{3}}$$

We can now look up the Inverse Laplace Transform for each of these terms in a simple table.

$$L^{-1}\left(\frac{s-2}{s^2-3}\right) = \left(\frac{1}{2} - \frac{1}{\sqrt{3}}\right)e^{\sqrt{3}\,t} + \left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)e^{-\sqrt{3}\,t}$$

We can verify this using Mathematica

```
a11 = InverseLaplaceTransform[(s - 2)/(s^2 - 3), s, t] // Expand
```

$$\frac{1}{2}e^{-\sqrt{3}\,t} + \frac{e^{-\sqrt{3}\,t}}{\sqrt{3}} + \frac{e^{\sqrt{3}\,t}}{2} - \frac{e^{\sqrt{3}\,t}}{\sqrt{3}}$$

As can be seen, this is correct.

Term a_{12}

For this element, we have the term

$$\frac{1}{s^2-3}$$

We can perform a similar analysis and write

$$\frac{1}{s^2-3} = \frac{1}{\left(s+\sqrt{3}\right)\left(s-\sqrt{3}\right)}$$

Once again, we would like to break this into two parts

$$\frac{\alpha}{s-\sqrt{3}} + \frac{\beta}{s+\sqrt{3}} = \frac{1}{\left(s+\sqrt{3}\right)\left(s-\sqrt{3}\right)}$$

We can now solve for the coefficients.

$$\frac{1}{\left(s+\sqrt{3}\right)\left(s-\sqrt{3}\right)} = \frac{\alpha}{s-\sqrt{3}} + \frac{\beta}{s+\sqrt{3}}$$

$$\frac{1}{\left(s+\sqrt{3}\right)\left(s-\sqrt{3}\right)} = \frac{\alpha\left(s+\sqrt{3}\right)}{\left(s-\sqrt{3}\right)\left(s+\sqrt{3}\right)} + \frac{\beta\left(s-\sqrt{3}\right)}{\left(s+\sqrt{3}\right)\left(s-\sqrt{3}\right)}$$

$$\frac{1}{\left(s+\sqrt{3}\right)\left(s-\sqrt{3}\right)} = \frac{\alpha s+\alpha\sqrt{3}+\beta s-\beta\sqrt{3}}{\left(s-\sqrt{3}\right)\left(s+\sqrt{3}\right)}$$

$$\frac{1}{\left(s+\sqrt{3}\right)\left(s-\sqrt{3}\right)} = \frac{(\alpha+\beta)\, s+(\alpha-\beta)\sqrt{3}}{\left(s-\sqrt{3}\right)\left(s+\sqrt{3}\right)}$$

By comparing coefficients, we see that

$$0 = \alpha + \beta$$

$$1 = (\alpha - \beta)\sqrt{3}$$

We can solve these two equations.

```
temp = Solve[{α + β == 0, (α - β) √3 == 1}, {α, β}] // Simplify
```

$$\left\{\left\{\alpha \to \frac{1}{2\sqrt{3}},\ \beta \to -\frac{1}{2\sqrt{3}}\right\}\right\}$$

From this we see that we can write

$$\frac{1}{s^2-3} = \frac{\frac{1}{2\sqrt{3}}}{s-\sqrt{3}} + \frac{-\frac{1}{2\sqrt{3}}}{s+\sqrt{3}}$$

$$= \left(\frac{1}{2\sqrt{3}}\right)\frac{1}{s-\sqrt{3}} - \left(\frac{1}{2\sqrt{3}}\right)\frac{1}{s+\sqrt{3}}$$

We can now easily take the inverse Laplace Transform to obtain.

$$L^{-1}\left(\frac{1}{s^2-3}\right) = \left(\frac{1}{2\sqrt{3}}\right)e^{\sqrt{3}\,t} - \left(\frac{1}{2\sqrt{3}}\right)e^{-\sqrt{3}\,t}$$

We can check this with Mathematica

```
a12 = InverseLaplaceTransform[1/(s^2 - 3), s, t] // Expand
```

$$-\frac{e^{-\sqrt{3}\,t}}{2\sqrt{3}}+\frac{e^{\sqrt{3}\,t}}{2\sqrt{3}}$$

As we can see, this is correct.

Term a_{21}

As we can see,

$$a_{21}=-a_{12}$$

Therefore, we can easily find the Laplace Transform

$$L^{-1}\left(\frac{-1}{s^2-3}\right)=-\left(\frac{1}{2\sqrt{3}}\right)e^{\sqrt{3}\,t}+\left(\frac{1}{2\sqrt{3}}\right)e^{-\sqrt{3}\,t}$$

```
a21 = InverseLaplaceTransform[-1/(s^2 - 3), s, t] // Expand
```

$$\frac{e^{-\sqrt{3}\,t}}{2\sqrt{3}}-\frac{e^{\sqrt{3}\,t}}{2\sqrt{3}}$$

Term a_{22}

We have the term

$$\frac{s+2}{s^2-3}$$

As can be seen, we can once again perform the same analysis to obtain,

$$\frac{s+2}{\left(s+\sqrt{3}\right)\left(s-\sqrt{3}\right)}=\frac{(\alpha+\beta)\,s+(\alpha-\beta)\,\sqrt{3}}{\left(s-\sqrt{3}\right)\left(s+\sqrt{3}\right)}$$

Comparing coefficients yields

$$\alpha+\beta=1$$
$$(\alpha-\beta)\,\sqrt{3}=2$$

```
temp = Solve[{α + β == 1, (α - β) √3 == 2}, {α, β}] // Simplify
```

$$\left\{\left\{\alpha\to\frac{1}{2}+\frac{1}{\sqrt{3}},\ \beta\to\frac{1}{2}-\frac{1}{\sqrt{3}}\right\}\right\}$$

From this we see that

$$\frac{s+2}{s^2-3} = \frac{\frac{1}{2}+\frac{1}{\sqrt{3}}}{s-\sqrt{3}} + \frac{\frac{1}{2}-\frac{1}{\sqrt{3}}}{s+\sqrt{3}}$$

$$= \left(\frac{1}{2}+\frac{1}{\sqrt{3}}\right)\frac{1}{s-\sqrt{3}} + \left(\frac{1}{2}-\frac{1}{\sqrt{3}}\right)\frac{1}{s+\sqrt{3}}$$

So the inverse Laplace transform is given by

$$L^{-1}\left(\frac{s+2}{s^2-3}\right) = \left(\frac{1}{2}+\frac{1}{\sqrt{3}}\right) e^{\sqrt{3}\,t} + \left(\frac{1}{2}-\frac{1}{\sqrt{3}}\right) e^{-\sqrt{3}\,t}$$

Once again, we can check this with Mathematica

```
a22 = InverseLaplaceTransform[(s + 2)/(s^2 - 3), s, t] // Expand
```

$$\frac{1}{2} e^{-\sqrt{3}\,t} - \frac{e^{-\sqrt{3}\,t}}{\sqrt{3}} + \frac{e^{\sqrt{3}\,t}}{2} + \frac{e^{\sqrt{3}\,t}}{\sqrt{3}}$$

As can be seen, this is correct

We now have the complete transition matrix in the time domain, which is given by

```
ϕlaplace[t_] = {{a11, a12}, {a21, a22}};
ϕlaplace[t] // MatrixForm
```

$$\begin{pmatrix} \frac{1}{2} e^{-\sqrt{3}\,t} + \frac{e^{-\sqrt{3}\,t}}{\sqrt{3}} + \frac{e^{\sqrt{3}\,t}}{2} - \frac{e^{\sqrt{3}\,t}}{\sqrt{3}} & -\frac{e^{-\sqrt{3}\,t}}{2\sqrt{3}} + \frac{e^{\sqrt{3}\,t}}{2\sqrt{3}} \\ \frac{e^{-\sqrt{3}\,t}}{2\sqrt{3}} - \frac{e^{\sqrt{3}\,t}}{2\sqrt{3}} & \frac{1}{2} e^{-\sqrt{3}\,t} - \frac{e^{-\sqrt{3}\,t}}{\sqrt{3}} + \frac{e^{\sqrt{3}\,t}}{2} + \frac{e^{\sqrt{3}\,t}}{\sqrt{3}} \end{pmatrix}$$

Alternatively, this can be written as

$$\phi(t) = \begin{pmatrix} \left(\frac{1}{2}-\frac{1}{\sqrt{3}}\right)e^{\sqrt{3}\,t} + \left(\frac{1}{2}+\frac{1}{\sqrt{3}}\right)e^{-\sqrt{3}\,t} & \left(\frac{1}{2\sqrt{3}}\right)e^{\sqrt{3}\,t} - \left(\frac{1}{2\sqrt{3}}\right)e^{-\sqrt{3}\,t} \\ -\left(\frac{1}{2\sqrt{3}}\right)e^{\sqrt{3}\,t} + \left(\frac{1}{2\sqrt{3}}\right)e^{-\sqrt{3}\,t} & \left(\frac{1}{2}+\frac{1}{\sqrt{3}}\right)e^{\sqrt{3}\,t} + \left(\frac{1}{2}-\frac{1}{\sqrt{3}}\right)e^{-\sqrt{3}\,t} \end{pmatrix}$$

We can check this with Mathematica

```
ϕlaplaceCheck[t_] = InverseLaplaceTransform[ϕ[s], s, t];
ϕlaplace[t] == ϕlaplaceCheck[t] // Simplify
```

```
True
```

We can check this again using the MatrixExp function

```
ϕlaplace[t] == MatrixExp[A t] // Simplify
```

```
True
```

```
Clear[ϕlaplaceCheck, a22, a21, a12, a11, temp, ϕ, ϕlaplace]
```