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Lecture 04a
Using a Homogeneous Transformation Matrix to Combine Rotation and Translation

Lecture is on YouTube

The YouTube video entitled ‘Using a Homogeneous Transformation Matrix to Combine Rotation and Translation’ that covers this lecture is located at https://youtu.be/LftL6dA6tzE.

Outline

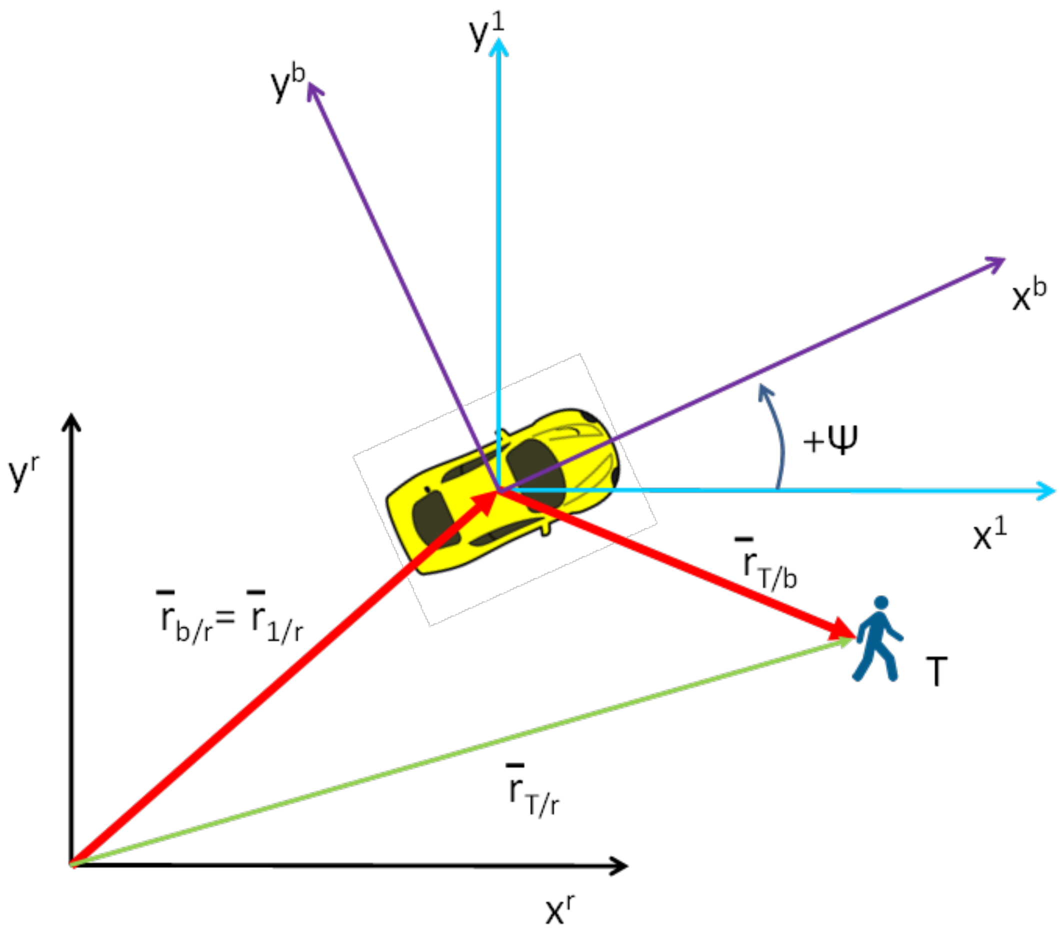
-Combining Rotation and Translation
-Composite Rotation/Translations
-Extension to 3D

Combining Rotation and Translation

In the previous example, we saw that the rotation matrix allows us to take a vector that is expressed in one frame and express it in another frame. Notice that this is correct only if the origin of the two frames are coincident.

Consider a case where the origins of the two frames are not coincident. For example, we have a base of operations aligned with an East/North/Up frame and from this base, we deploy a planar ground robot (the yellow car). This ground robot has a camera that it can use to locate target, T (the pedestrian). The relevant geometry is shown below. In this figure we define the following frames.

F_r = East/North/Up (ENU) frame
F_b = body frame attached to ground robot (origin at vehicle CM and aligned with body)
F_1 = intermediate vehicle carried frame (origin at vehicle CM but aligned with F_r) (blue)



In this scenario, the ground robot will be able to directly measure $\bar{r}_{T/b}{}^{b}$ with its camera attached to its local body frame.

We would like to know the location of the pedestrian with respect to the base, expressed in the ENU frame. In other words, we would like to compute $\bar{r}_{T/r}{}^{r}$.

Consider the intermediate step of calculating $\bar{r}_{T/b}{}^{r}$. We can start by computing $\bar{r}_{T/b}{}^{1}$. This is easily given by

$$\bar{r}_{T/b}{}^{1} = C_{1/b}(\psi)\,\bar{r}_{T/b}{}^{b} \qquad \textbf{(Eq.1)}$$

Recall $C_{b/1}(\psi) = \begin{pmatrix} \cos(\psi) & \sin(\psi) \\ -\sin(\psi) & \cos(\psi) \end{pmatrix} \Rightarrow \quad C_{1/b}(\psi) = \begin{pmatrix} \cos(\psi) & -\sin(\psi) \\ \sin(\psi) & \cos(\psi) \end{pmatrix}$

Note: Be careful with the definition of the rotation matrix (in other words, keep track of the ordering of the subscripts)

Since F_1 and F_r have the same orientation (they are not rotated with respect to each other) and only are different by a constant translation, we can write

$$\bar{r}_{T/r}{}^{r} = \bar{r}_{1/r}{}^{r} + \bar{r}_{T/b}{}^{1}$$

$$= \bar{r}_{1/r}{}^{r} + C_{1/b}(\psi)\,\bar{r}_{T/b}{}^{b} \qquad \text{note: } \bar{r}_{1/r}{}^{r} = \bar{r}_{b/r}{}^{r}$$

$$\bar{r}_{T/r}{}^{r} = \bar{r}_{b/r}{}^{r} + C_{1/b}(\psi)\,\bar{r}_{T/b}{}^{b} \qquad \textbf{\textit{(Eq.2)}}$$

If we write this in matrix form, we can write this as

$$\left(\bar{r}_{T/r}{}^{r}\right) = \begin{pmatrix} C_{1/b}(\psi) & \bar{r}_{b/r}{}^{r} \end{pmatrix} \begin{pmatrix} \bar{r}_{T/b}{}^{b} \\ 1 \end{pmatrix}$$

Note that we add an extra row at the bottom which appears to simply read “1 = 1”. While this seems extraneous, we will see this is useful when we want to perform successive operations.

$$\begin{pmatrix} \bar{r}_{T/r}{}^{r} \\ 1_{1\,\mathrm{x}1} \end{pmatrix} = \begin{pmatrix} C_{1/b}(\psi) & \bar{r}_{b/r}{}^{r} \\ 0_{1\,\mathrm{x}2} & 1_{1\,\mathrm{x}1} \end{pmatrix} \begin{pmatrix} \bar{r}_{T/b}{}^{b} \\ 1_{1\,\mathrm{x}1} \end{pmatrix} \qquad \textbf{\textit{(Eq.3)}}$$

Note that we introduce subscripts temporarily on the 1’s and 0’s to help illustrate the dimension of these objects.

We can call the augmented vector the pose of the system (dropping the subscript 1x1 notation)

$$\bar{p}_{T/r}{}^{r} = \begin{pmatrix} \bar{r}_{T/r}{}^{r} \\ 1 \end{pmatrix}$$

$$\bar{p}_{T/b}{}^{b} = \begin{pmatrix} \bar{r}_{T/b}{}^{b} \\ 1 \end{pmatrix}$$

We can write the previous expression as

$$\bar{p}_{T/r}{}^{r} = \xi_{r/b}\,\bar{p}_{T/b}{}^{b}$$

where $\xi_{r/b} = \begin{pmatrix} C_{1/b}(\psi) & \bar{r}_{b/r}{}^{r} \\ 0_{1\,\mathrm{x}2} & 1_{1\,\mathrm{x}1} \end{pmatrix}$ recall: F_1 is aligned with F_r but centered at F_b

We can drop the subscript notation because both $\bar{p}_{T/r}{}^{r}$ and $\bar{p}_{T/b}{}^{b}$ are trying to describe the vector to the target, just from two different perspectives and this is captured by the superscript notation. In the end, we obtain

$$\bar{p}^{r} = \xi_{r/b}\,\bar{p}^{b} \qquad \textbf{\textit{(Eq.4)}}$$

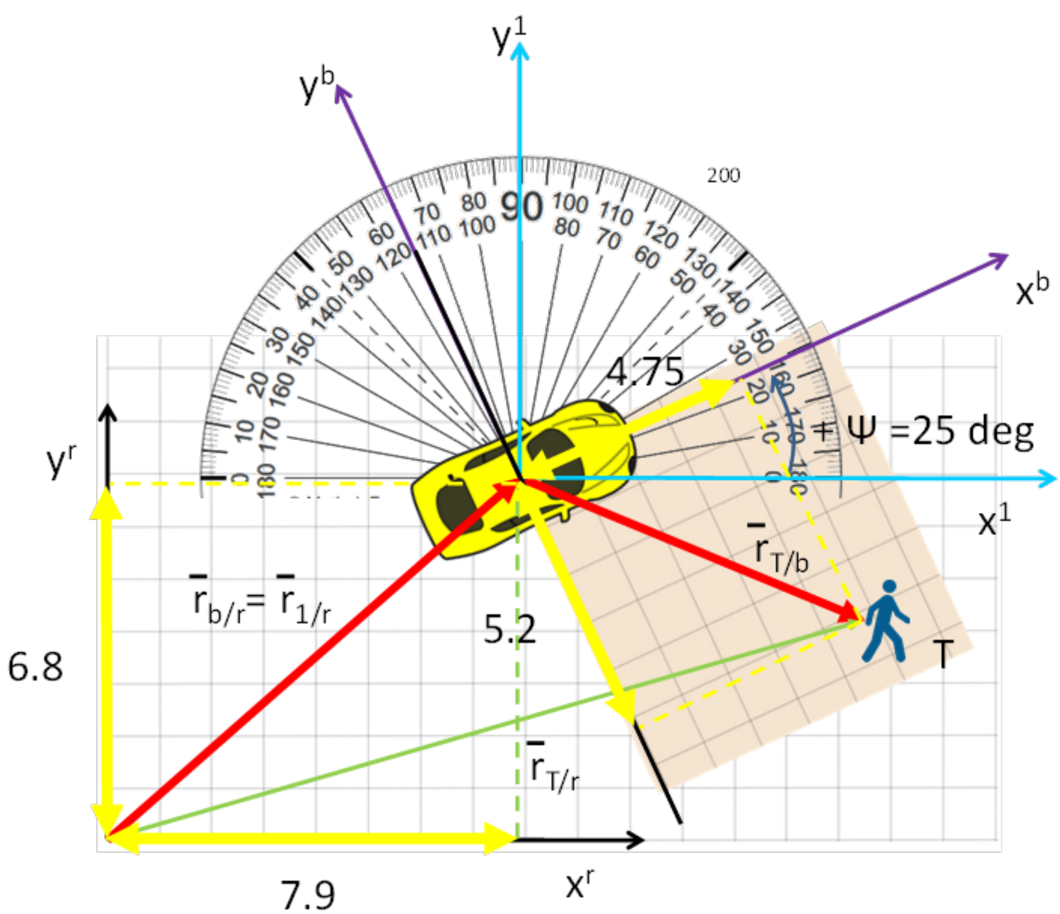
where $\quad \bar{p}^r = \begin{pmatrix} \bar{r}_{T/r}{}^r \\ 1 \end{pmatrix}$

$$\bar{p}^b = \begin{pmatrix} \bar{r}_{T/b}{}^b \\ 1 \end{pmatrix}$$

Eq.4 gives a way to relate the pose of the one system to that of another. This takes into account both the rotation and the translation between the two frames.

Example: Car

Consider the previously discussed car example. Note that we include accurate marking for distance and angle so we can check our solution for reasonableness.



So we see that $\bar{r}_{T/b}{}^b = \begin{pmatrix} 4.75 \\ -5.2 \end{pmatrix}$ so the pose is given as

$$\bar{p}_{T/b}{}^b = \bar{p}^b = \begin{pmatrix} \bar{r}_{T/b}{}^b \\ 1 \end{pmatrix} = \begin{pmatrix} 4.75 \\ -5.2 \\ 1 \end{pmatrix}$$

```
pb = (4.75
      -5.2
      1);
```

To build the matrix $\xi_{r/b}$ we see note that $\bar{r}_{b/r}{}^{r} = \begin{pmatrix} 7.9 \\ 6.8 \end{pmatrix}$ and $\psi = 25\,\frac{\pi}{180}$

$$\xi_{r/b} = \begin{pmatrix} C_{1/b}(\psi) & \bar{r}_{b/r}{}^{r} \\ 0_{1\times2} & 1_{1\times1} \end{pmatrix} = \begin{pmatrix} \cos\left(25\,\frac{\pi}{180}\right) & -\sin\left(25\,\frac{\pi}{180}\right) & 7.9 \\ \sin\left(25\,\frac{\pi}{180}\right) & \cos\left(25\,\frac{\pi}{180}\right) & 6.8 \\ 0 & 0 & 1 \end{pmatrix}$$

```
ξrb = (Cos[25 π/180]   -Sin[25 π/180]   7.9
       Sin[25 π/180]    Cos[25 π/180]   6.8
       0                0               1);
```

Applying Eq.4 yields

$$\bar{p}^{r} = \xi_{r/b}\,\bar{p}^{b}$$

$$\begin{pmatrix} \bar{r}_{T/r}{}^{r} \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\left(25\,\frac{\pi}{180}\right) & -\sin\left(25\,\frac{\pi}{180}\right) & 7.9 \\ \sin\left(25\,\frac{\pi}{180}\right) & \cos\left(25\,\frac{\pi}{180}\right) & 6.8 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4.75 \\ -5.2 \\ 1 \end{pmatrix}$$

```
pr = ξrb.pb;
pr // MatrixForm
```

$$\begin{pmatrix} 14.4026 \\ 4.09464 \\ 1. \end{pmatrix}$$

So we see that the position of the target with respect to the ENU frame expressed in the ENU frame is given by

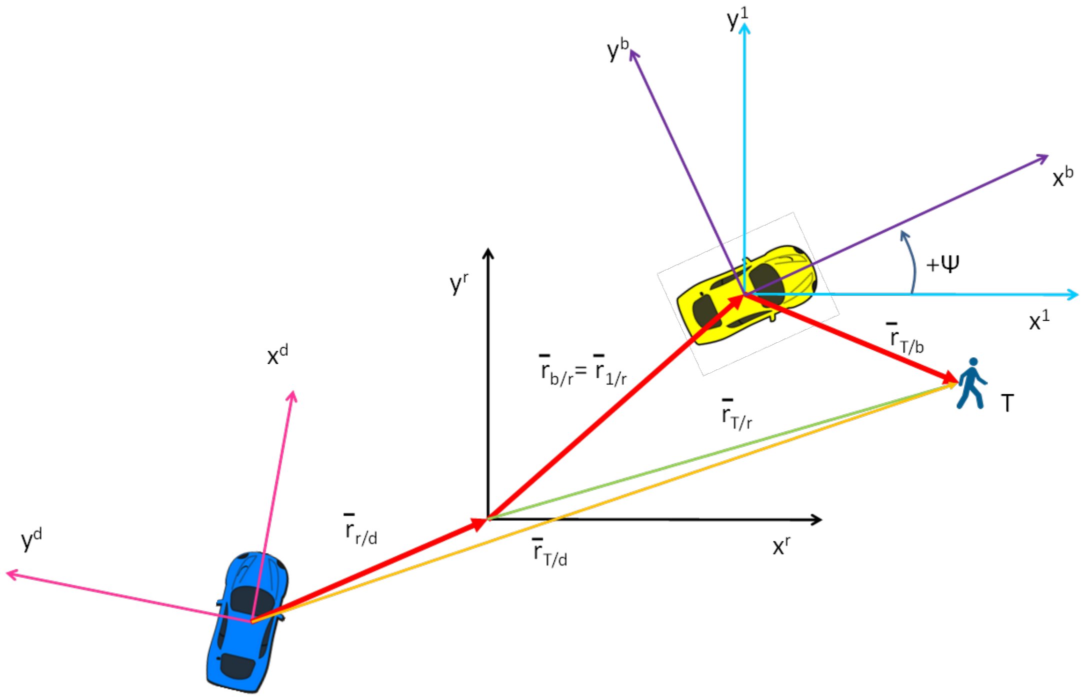
$$\bar{r}_{T/r}{}^{r} = \begin{pmatrix} 14.4 \\ 4.1 \end{pmatrix}$$

This agrees with manual measurements made on the figure above.

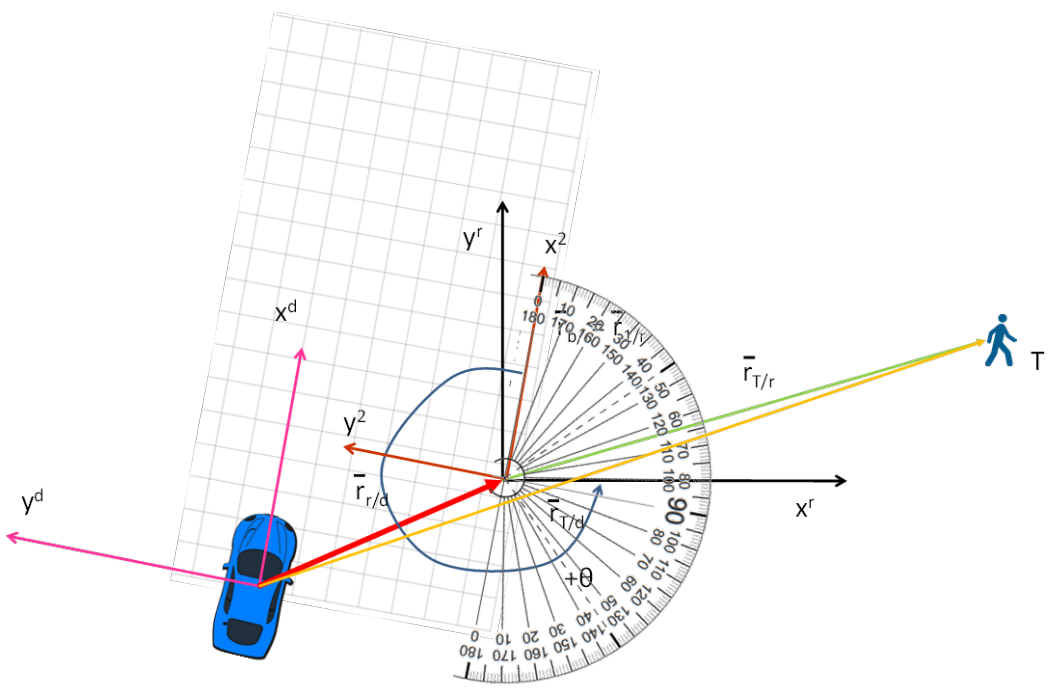
Composite Rotation/Translations

Eq.4 also shows why it is useful to add the extra bottom row of “1 = 1” as this allows us to operate on the pose of the system and generate another pose vector, thereby allowing us to successively apply another ξ matrix to rotate/translate to another pose. This can be done successively as many times as necessary

Returning to the previous example, suppose that another vehicle, that is not equipped with the ability to detect pedestrians (we designate this as the “dumb” vehicle). Frame *d* is related to the *r* frame as shown below. We now wish to compute $\bar{r}_{T/d}{}^{d}$ (position of the pedestrian w.r.t to the dumb vehicle, expressed in F_d)



At this point, it may be useful to “declutter” the diagram and note that since we already computed $\bar{r}_{T/r}{}^{r}$, we can simply leave the orange vector (we no longer need the *b* frame, nor any of these intermediate frames).



So we see that

$$\theta = -80 \frac{\pi}{180}$$

$$\overline{r}_{r/d}{}^{d} = \begin{pmatrix} 4.5 \\ -6.5 \end{pmatrix}$$

So we can calculate the pose using

$$\overline{p}^{d} = \xi_{d/r}\, \overline{p}^{r} \qquad \textbf{\textit{(Eq.5)}}$$

where $\overline{p}^{d} = \begin{pmatrix} \overline{r}_{T/d}{}^{d} \\ 1 \end{pmatrix}$

$$\overline{p}^{r} = \begin{pmatrix} \overline{r}_{T/r}{}^{r} \\ 1 \end{pmatrix} \qquad \text{recall: from previous calculation } \overline{r}_{T/r}{}^{r} = \begin{pmatrix} 14.4 \\ 4.1 \end{pmatrix}$$

$$\xi_{d/r} = \begin{pmatrix} C_{2/r}(\theta) & \overline{r}_{r/d}{}^{d} \\ 0_{1\,x2} & 1_{1\,x1} \end{pmatrix} = \begin{pmatrix} \cos\left(-80 \frac{\pi}{180}\right) & -\sin\left(-80 \frac{\pi}{180}\right) & 4.5 \\ \sin\left(-80 \frac{\pi}{180}\right) & \cos\left(-80 \frac{\pi}{180}\right) & -6.5 \\ 0 & 0 & 1 \end{pmatrix}$$

```
pr = {{14.4}, {4.1}, {1}};
ξdr = {{Cos[-80 π/180], -Sin[-80 π/180], 4.5}, {Sin[-80 π/180], Cos[-80 π/180], -6.5}, {0, 0, 1}};
```

Applying Eq.5 yields

$$\overline{p}^{d} = \xi_{d/r}\, \overline{p}^{r}$$

$$\begin{pmatrix} \overline{r}_{T/d}{}^{d} \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\left(-80 \frac{\pi}{180}\right) & -\sin\left(-80 \frac{\pi}{180}\right) & 4.5 \\ \sin\left(-80 \frac{\pi}{180}\right) & \cos\left(-80 \frac{\pi}{180}\right) & -6.5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 14.4 \\ 4.1 \\ 1 \end{pmatrix}$$

```
pd = ξdr.pr;
pd // MatrixForm
```

$$\begin{pmatrix} 11.0382 \\ -19.9693 \\ 1. \end{pmatrix}$$

So we see that

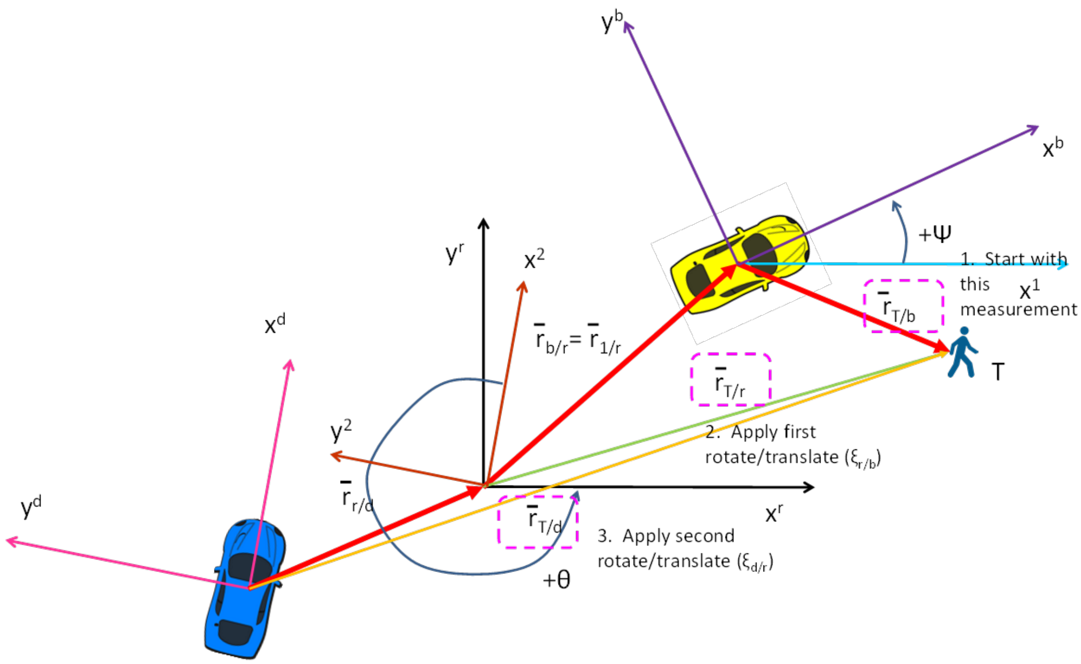
$$\bar{r}_{T/d}{}^{d} = \begin{pmatrix} 11.04 \\ -19.97 \end{pmatrix}$$

Again, this result agrees with physical measurements we make on the diagram.

So we see that we can chain these composite operations together

$$\bar{p}^{d} = \xi_{d/r}\,\bar{p}^{r} \qquad \text{recall: } \bar{p}^{r} = \xi_{r/b}\,\bar{p}^{b}$$

$$\bar{p}^{d} = \xi_{d/r}\,\xi_{r/b}\,\bar{p}^{b}$$



Extension to 3D

This can easily be extended to 3D, we merely need to augment the size of the vectors, rotation matrices, and padding zeros and ones. If we simply repeat Eq.2

$$\bar{r}_{T/r}{}^{r} = \bar{r}_{b/r}{}^{r} + C_{1/b}(\psi)\,\bar{r}_{T/b}{}^{b}$$

We note that vectors are now 3x1 and rotation matrix is 3x3 and can be a fully populated DCM

$$\begin{pmatrix} \bar{r}_{T/r}{}^{r} \\ 1_{1\times 1} \end{pmatrix} = \begin{pmatrix} C_{v/b}(\phi,\,\theta,\,\psi) & \bar{r}_{b/r}{}^{r} \\ 0_{1\times 3} & 1_{1\times 1} \end{pmatrix} \begin{pmatrix} \bar{r}_{T/b}{}^{b} \\ 1_{1\times 1} \end{pmatrix}$$