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Lecture 04d
Parameterizing Curves

The YouTube video entitled 'Parameterizing Curves' that covers this lecture is located at https://y-outu.be/MPcfaNIREN0 .

Outline

-Parametric Equations
-Examples

Parametric Equations

Consider a standard function

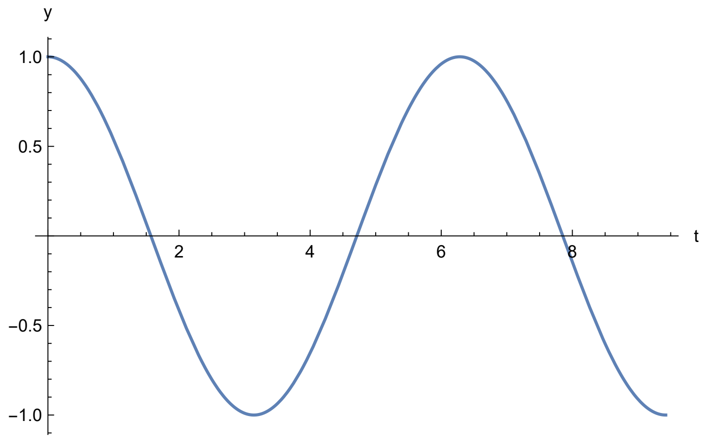
$$y = f(t)$$

In this case, the independent variable is t and the function f assigns a value y for every value of t.

Example

$$y = \cos(t)$$

In[1]:=
```
Plot[Cos[t], {t, 0, 3 π},
 AxesLabel → {"t", "y"}]
```

Out[1]=


In this sense, we see that the curve is a 1 dimensional object in the sense that one only requires a single piece of information (the value of the independent variable) to describe any location on the curve.

In the above example, the curve is “constrained” to the plane. However it is possible to describe curves in $\mathcal{R}^2$, $\mathcal{R}^3$ or even higher dimensions.

This is known as a parametric representation of a curve. A parametric representation is effectively a vector function that prescribes a 3D point of the curve at each value of the independent variable (*t* in this case).

For example, using Cartesian coordinates, we can write

$$\bar{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

where t = scalar parameter of the representation (AKA an independent variable)
$x(t)$ = function that assigns x values for each value of t
$y(t)$ = function that assigns y values for each value of t
$z(t)$ = function that assigns z values for each value of t

Examples

Example 2: 2D Ellipse

Consider a 2D curve parameterized by

$$\bar{r}(\theta) = \begin{pmatrix} a\cos(\theta) \\ b\sin(\theta) \end{pmatrix}$$

with $\theta \in [0, 2\pi]$

The parameterization, $\bar{r}$, describes locations on the curve at various points of θ (the independent variable). For example, one could make a table of the location at discrete values of θ.

$$\begin{pmatrix} \boldsymbol{\theta} & \boldsymbol{x = a\cos(\theta)} & \boldsymbol{y = b\sin(\theta)} \\ 0 & a & 0 \\ 1\pi/4 & \frac{a}{\sqrt{2}} & \frac{b}{\sqrt{2}} \\ 2\pi/4 & 0 & b \\ 3\pi/4 & -\frac{a}{\sqrt{2}} & \frac{b}{\sqrt{2}} \\ 4\pi/4 & -a & 0 \\ 5\pi/4 & -\frac{a}{\sqrt{2}} & -\frac{b}{\sqrt{2}} \\ 6\pi/4 & 0 & -b \\ 7\pi/4 & \frac{a}{\sqrt{2}} & -\frac{b}{\sqrt{2}} \\ 8\pi/4 & a & 0 \\ \ldots & \text{repeats} & \ldots \end{pmatrix}$$

Instead of computing this table manually as we did above, we can use the 'Table' and 'Grid' functions in Mathematica to create a similar table.

In[2]:=
```
(*Create data*)
Nθ = 8;
t1 = Table[θ, {θ, 0, 2 π, 2 π/Nθ}];              (*list representing column 1*)
t2 = Table[a Cos[θ], {θ, 0, 2 π, 2 π/Nθ}];       (*list representing column 2*)
t3 = Table[b Sin[θ], {θ, 0, 2 π, 2 π/Nθ}];       (*list representing column 3*)

(*Now place them in a grid so it looks like a standard
 table (note the transpose so the orientation is vertical)*)
gridData = Transpose[{t1, t2, t3}];

(*Add headers to the table*)
gridDataWithHeaders = Prepend[gridData, {"θ", "x=a cos(θ)", "Y=b sin(θ)" }];

(*Display the table with a frame*)
Grid[gridDataWithHeaders, Frame → All]
```

Out[8]=

θ	x=a cos(θ)	Y=b sin(θ)
0	a	0
$\frac{\pi}{4}$	$\frac{a}{\sqrt{2}}$	$\frac{b}{\sqrt{2}}$
$\frac{\pi}{2}$	0	b
$\frac{3\pi}{4}$	$-\frac{a}{\sqrt{2}}$	$\frac{b}{\sqrt{2}}$
π	-a	0
$\frac{5\pi}{4}$	$-\frac{a}{\sqrt{2}}$	$-\frac{b}{\sqrt{2}}$
$\frac{3\pi}{2}$	0	-b
$\frac{7\pi}{4}$	$\frac{a}{\sqrt{2}}$	$-\frac{b}{\sqrt{2}}$
2π	a	0

We can substitute in values of *a* and *b* if so desired

In[9]:=
```
Grid[gridDataWithHeaders /. {a → 3, b → 1.2} // N, Frame → All]
```

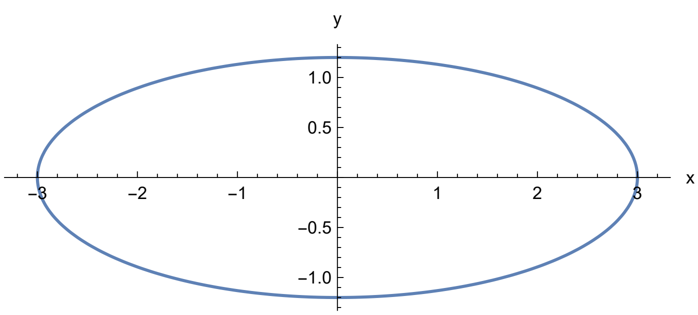
Out[9]=

θ	x=a cos(θ)	Y=b sin(θ)
0.	3.	0.
0.785398	2.12132	0.848528
1.5708	0.	1.2
2.35619	-2.12132	0.848528
3.14159	-3.	0.
3.92699	-2.12132	-0.848528
4.71239	0.	-1.2
5.49779	2.12132	-0.848528
6.28319	3.	0.

Mathematica provides the function ‘ParametricPlot’ to plot 2D parametric plots.

In[10]:=
```
a = 3;
b = 1.2;
ParametricPlot[{a Cos[θ], b Sin[θ]}, {θ, 0, 2 π},
 AxesLabel → {"x", "y"}]
Clear[a, b]
```

Out[12]=



Example 3: 3D Elliptical helix

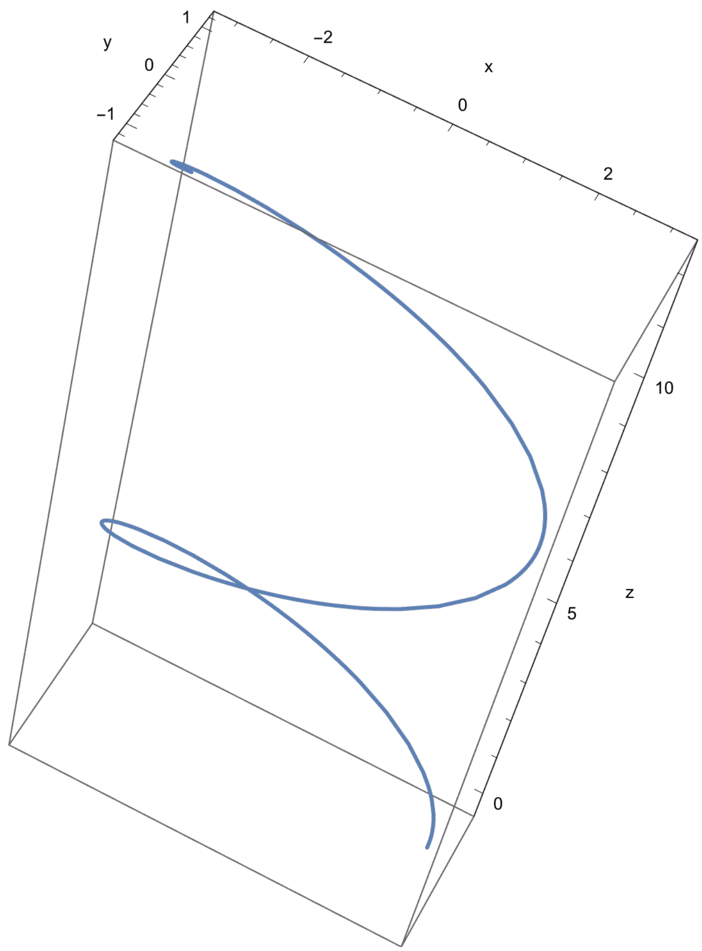
Consider extending our previous example to 3 dimensions with the parametrization

$$\bar{r}(t) = \begin{pmatrix} a\cos(t) \\ b\sin(t) \\ c\,t \end{pmatrix}$$

Mathematica provides the function 'ParametricPlot3D' to plot 3D parametric plots.

In[14]:=
```
a = 3;
b = 1.2;
c = 1.2;
ParametricPlot3D[{a Cos[t], b Sin[t], c t}, {t, 0, 10},
 AxesLabel → {"x", "y", "z"}]
Clear[a, b, c]
```

Out[17]=



Example 3: Hypotrochoid

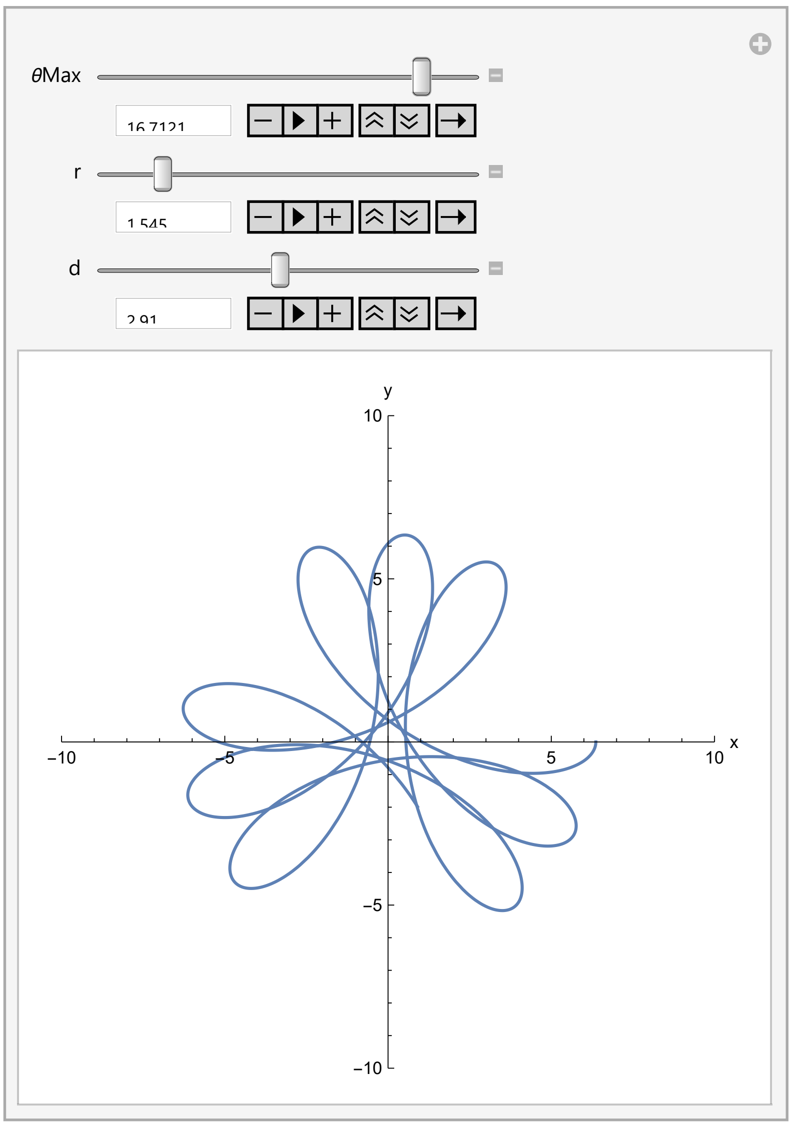
Another interesting parameterization is a hypotrochoid which is a curve traced by a point attached to a circle of radius r rolling around the inside of a fixed circle of radius R, where the point is at a distance d from the center of the interior circle.

$$x(\theta) = (R - r)\cos(\theta) + d\cos\left(\tfrac{R-r}{r}\,\theta\right)$$

$$y(\theta) = (R - r)\sin(\theta) - d\sin\left(\tfrac{R-r}{r}\,\theta\right)$$

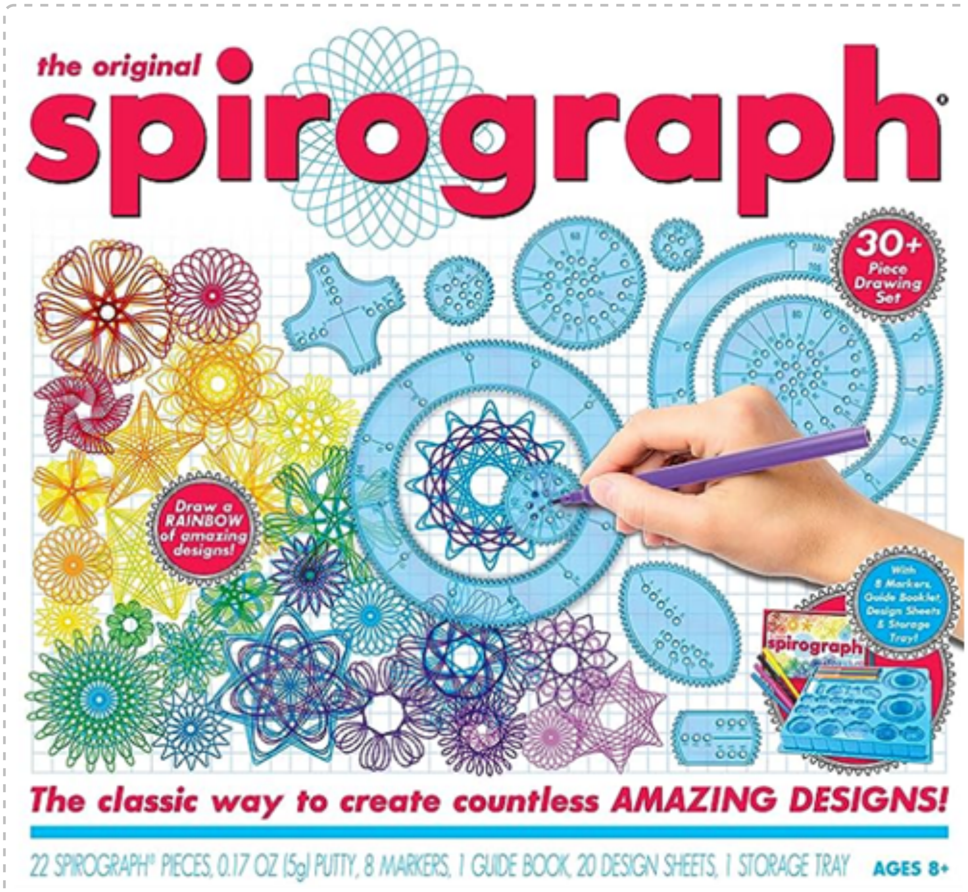
In[22]:=
```
(*A good curve is generated using R=5, r=3, d=5*)
R = 5;

Manipulate[
 ParametricPlot[
  {(R - r) Cos[θ] + d Cos[(R - r)/r θ], (R - r) Sin[θ] - d Sin[(R - r)/r θ]}, {θ, 0, θMax},
  AxesLabel → {"x", "y"},
  PlotRange → {{-2 R, 2 R}, {-2 R, 2 R}}], {θMax, 0.1, 6 π}, {r, 1, R}, {d, 1, R}
]
```

Out[23]=



In[21]:=
```
Clear[R]
```

This has been commercialized by British engineer Denys Fisher in 1965.



We will later expand this idea of parameterizing a curve into parameterizing an entire surface (see YouTube video entitled ‘Parameterizing Surfaces and Computing Surface Normal Vectors’ that covers this lecture is located at https://youtu.be/a3_c4c9PYNg)