Christopher Lum
lum@uw.edu

Lecture 01g
Block Diagram Algebra

Lecture is on YouTube

The YouTube video entitled ‘Block Diagram Algebra’ that covers this lecture is located at https://y-outu.be/OE9Va_ky6yU.

Outline

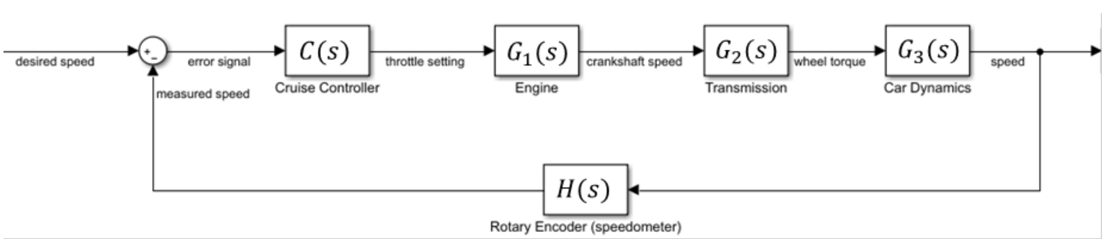
-Block Diagrams
-Block Diagrams Algebra
-Multi-Input, Multi-Output (MIMO) Transfer Functions

Block Diagrams

A block diagram of a dynamic system is a graphic representation of the functions performed by each component of the system and of the flow of signals within the system.

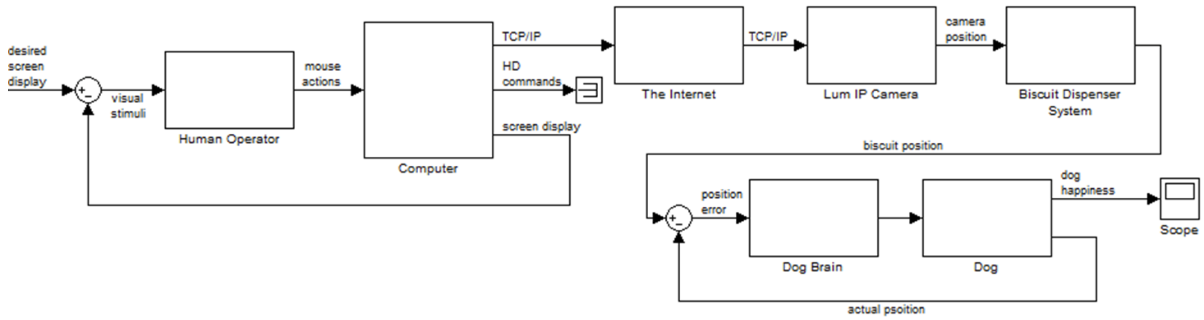
Example: Cruise Control

The classic example is a cruise control



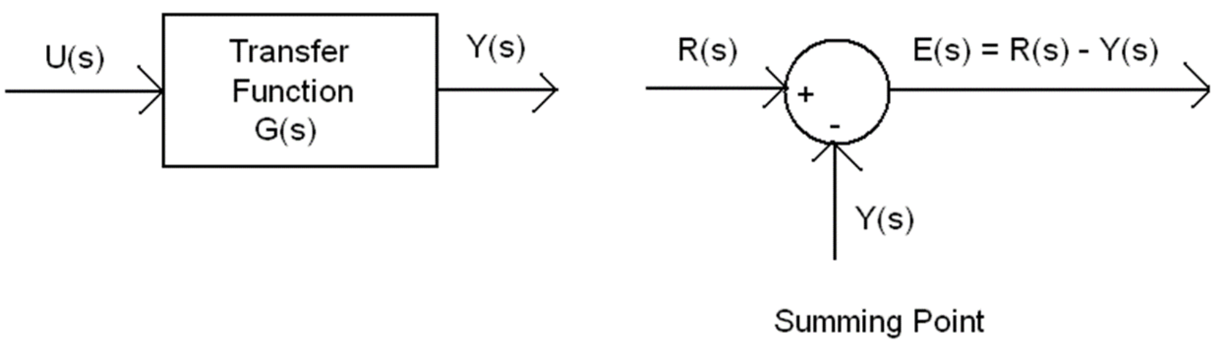
Example: Dog Feeder

Consider the remote doggie feeding system.



Alternatively, consider the physics engine for the Halo Warthog (YouTube Warthog jump video https://www.youtube.com/watch?v=nGQIQljaAc0). Grenade transfer function → vehicle physics transfer function

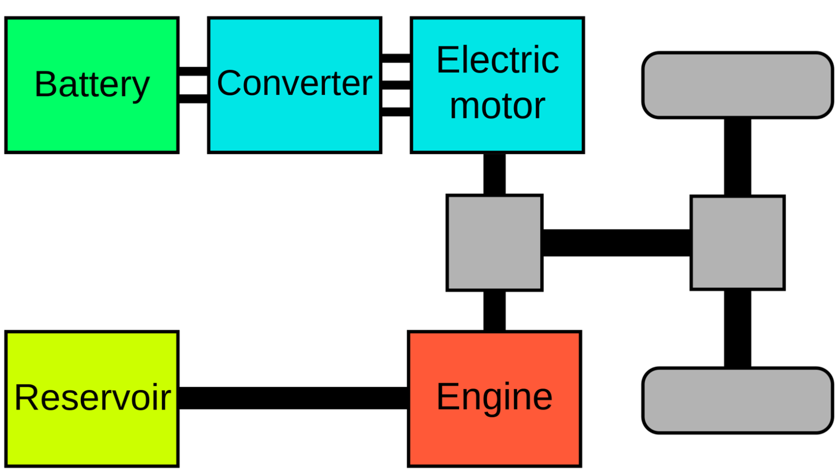
All system variables are linked to each other through blocks which represent the input/output relationship of the system in question.



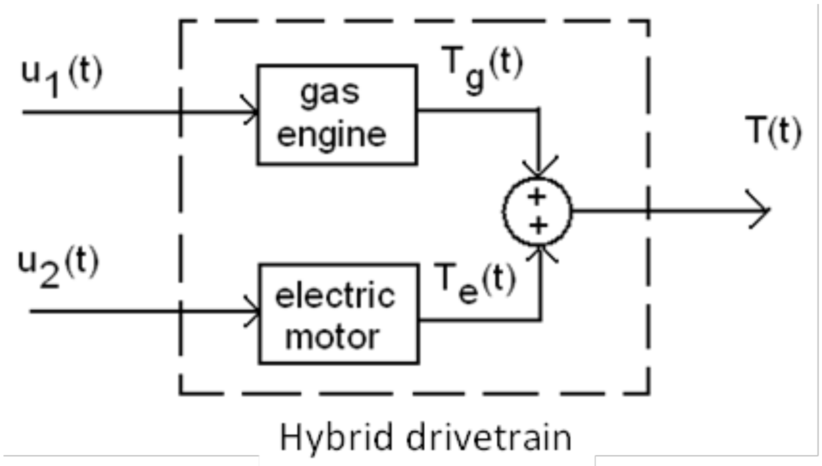
Note : For the summing point, you need to be sure that $R(s)$ and $Y(s)$ have the same units.

Transfer Function Example

Consider a hybrid car with a parallel hybrid system (https://en.wikipedia.org/wiki/Hybrid_vehicle_drivetrain)



The torque provided by the engine is the sum of the torque from the gas engine and the torque from the electric motor.



where $u_1(t)$ =control signal for gas engine (0=off, 1=full on)
$u_2(t)$ =control signal for electric motor (0=off, 1=full on)
$T_g(t)$ =torque from gas engine
$T_e(t)$ =torque from electric motor
$T(t)$ =total torque from hybrid engine

The gas engine is determined to have dynamics of

$$\dot{T}_g(t) + a_1 T_g(t) = a_2 u_1(t)$$

where $a_1 = 1$
$a_2 = 20$

```
a1 = 1;
a2 = 20;
```

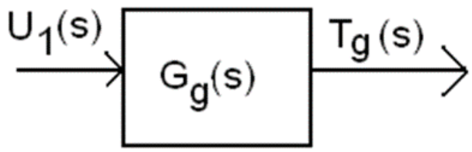
The transfer function is given as

$$s T_g(s) - T_g(t=0) + a_1 T_g(s) = a_2 U_1(s)$$ note: assume no initial conditions $T_g(t=0) = 0$

$$G_g(s) = \frac{T_g(s)}{U_1(s)} = \frac{a_2}{s+a_1}$$

```
Gg[s_] = a2/(s + a1);
```

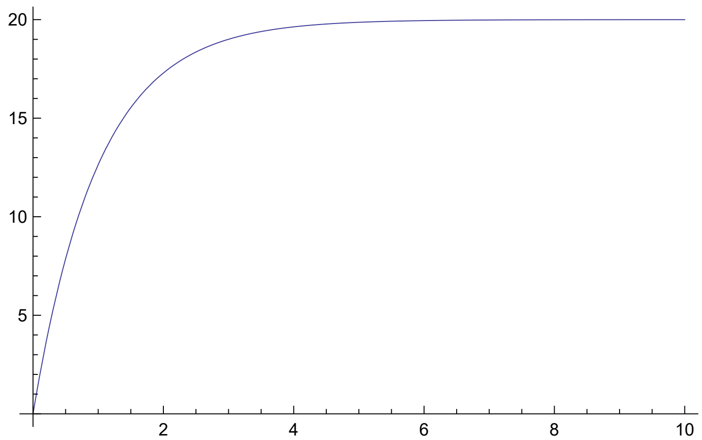
So the block diagram representation of this system is now



We can investigate the response of this system to a unit step function

```
TgStep[t_] = InverseLaplaceTransform[Gg[s] 1/s, s, t]
Plot[TgStep[t], {t, 0, 10}, PlotRange → All]
```

$20 \times \left(1 - e^{-t}\right)$



The electric motor has a similar model of

$$\dot{T}_e(t) + b_1\, T_e(t) = b_2\, u_2(t)$$

where $b_1 = 4$
$b_2 = 12$

```
b1 = 4;
b2 = 12;
```

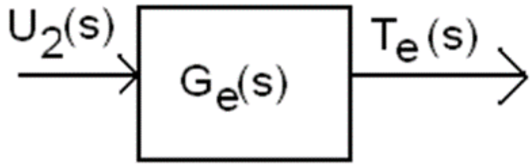
The transfer function is given as

$$s\, T_g(s) - T_g(t = 0) + a_1\, T_g(s) = a_2\, U_1(s) \qquad \text{note: assume } T_g(t = 0) = 0$$

$$G_e(s) = \frac{T_e(s)}{U_2(s)} = \frac{b_2}{s + b_1}$$

```
Ge[s_] = b2/(s + b1);
```

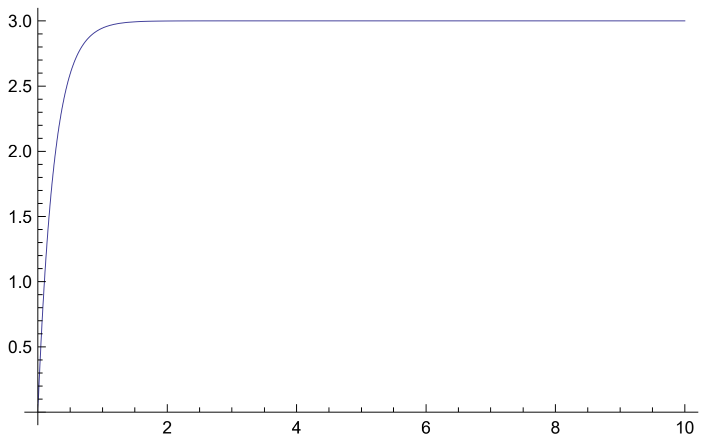
So the block diagram representation of this system is now



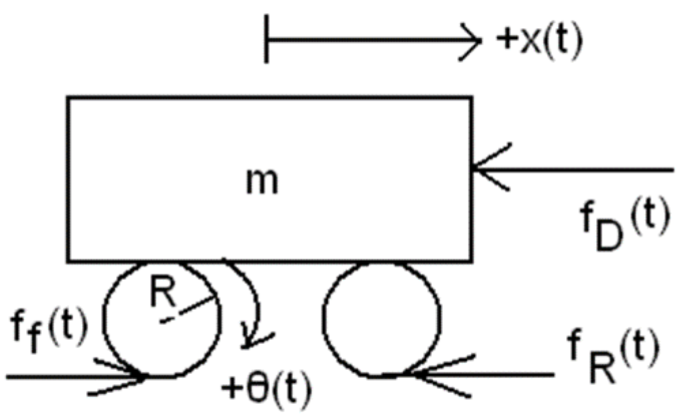
We can investigate the response of this system to a unit step function

```
TeStep[t_] = InverseLaplaceTransform[Ge[s] 1/s, s, t]
Plot[TeStep[t], {t, 0, 10}, PlotRange → All]
```

$$12 \times \left(\frac{1}{4} - \frac{e^{-4t}}{4}\right)$$

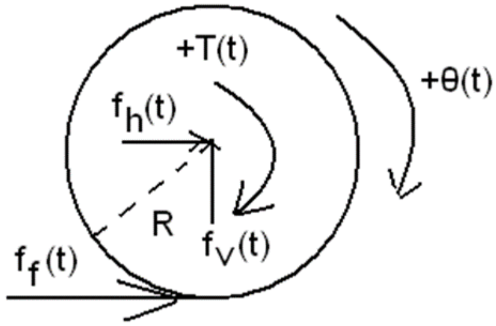


Let us now consider a model of a car



where $f_f(t)$ =propulsive force from friction between road and wheel
$f_R(t)$ =rolling resistance = $\mu_r\, \dot{\theta}(t)$
$f_D(t)$ =aerodynamic drag resistance = $c\, \dot{x}$

We can look at the wheel (assuming negligible viscous friction)



Newton's law of rotation about the rotation point is

$$\sum M = J\,\ddot{\theta}(t)$$

$$T(t) - f_f(t)\,R = J\,\ddot{\theta}(t)$$

$$T(t) - J\,\ddot{\theta}(t) = f_f(t)\,R$$

$$f_f(t) = \frac{1}{R}\,T(t) - \frac{J}{R}\,\ddot{\theta}(t)$$ note: $x(t) = R\,\theta(t) \Rightarrow \ddot{x}(t) = R\,\ddot{\theta}(t) \Rightarrow \ddot{\theta}(t) = \frac{1}{R}\,\ddot{x}(t)$

$$f_f(t) = \frac{1}{R}\,T(t) - \frac{J}{R^2}\,\ddot{x}(t)$$ ***(Eq.A)***

Newton's law of translation for the car is now

$$\sum F = m\,\dot{V}(t)$$

$$f_f(t) - f_D(t) - f_R(t) = m\,\dot{V}(t)$$

$$\frac{1}{R}\,T(t) - \frac{J}{R^2}\,\ddot{x}(t) - c\,V(t) - \mu_r\,\dot{\theta}(t) = m\,\dot{V}(t)$$ note: $V(t) = R\,\dot{\theta}(t) \Rightarrow \dot{\theta}(t) = \frac{1}{R}\,V(t)$

$\ddot{x}(t) = \dot{V}(t)$

$$\frac{1}{R}\,T(t) - \frac{J}{R^2}\,\dot{V}(t) - c\,V(t) - \mu_r\,\frac{1}{R}\,V(t) = m\,\dot{V}(t)$$

$$\frac{1}{R}\,T(t) - \left(c + \mu_r\,\frac{1}{R}\right)V(t) = \left(m + \frac{J}{R^2}\right)\dot{V}(t)$$

We can find the transfer function between engine torque and velocity

$$\frac{1}{R}\,T(s) - \left(c + \mu_r\,\frac{1}{R}\right)V(s) = \left(m + \frac{J}{R^2}\right)(s\,V(s) - V(0))$$ note: assume $V(0) = 0$

$$\frac{1}{R}\,T(s) - \left(c + \mu_r\,\frac{1}{R}\right)V(s) = \left(m + \frac{J}{R^2}\right)s\,V(s)$$

$$\frac{1}{R}\,T(s) = \left(\left(m + \frac{J}{R^2}\right)s + \left(c + \mu_r\,\frac{1}{R}\right)\right)V(s)$$

$$G(s)=\frac{V(s)}{T(s)}=\frac{1/R}{\left(m+\frac{J}{R^2}\right)s+\left(c+\mu_r\,\frac{1}{R}\right)}$$

We can assume constants of

```
R = 1 / 4;
m = 50;
J = 3;
c = 2;
μr = 1;
```

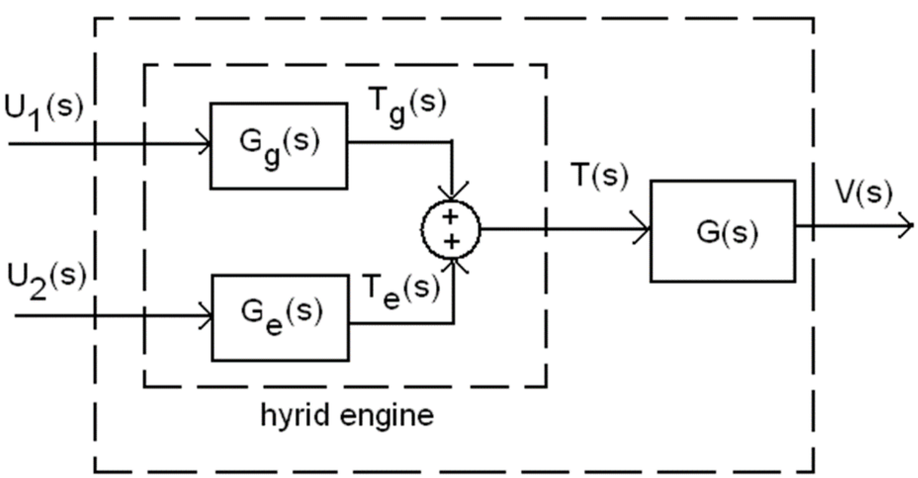
$$\texttt{G[s\_]} = \frac{1/R}{\left(m+\frac{J}{R^2}\right)s+\left(c+\mu r\,\frac{1}{R}\right)}$$

$$\frac{4}{6+98\,s}$$

So the block diagram representation of this system is now

Transfer function of car velocity to wheel torque (including rolling resistance and aerodynamic drag)

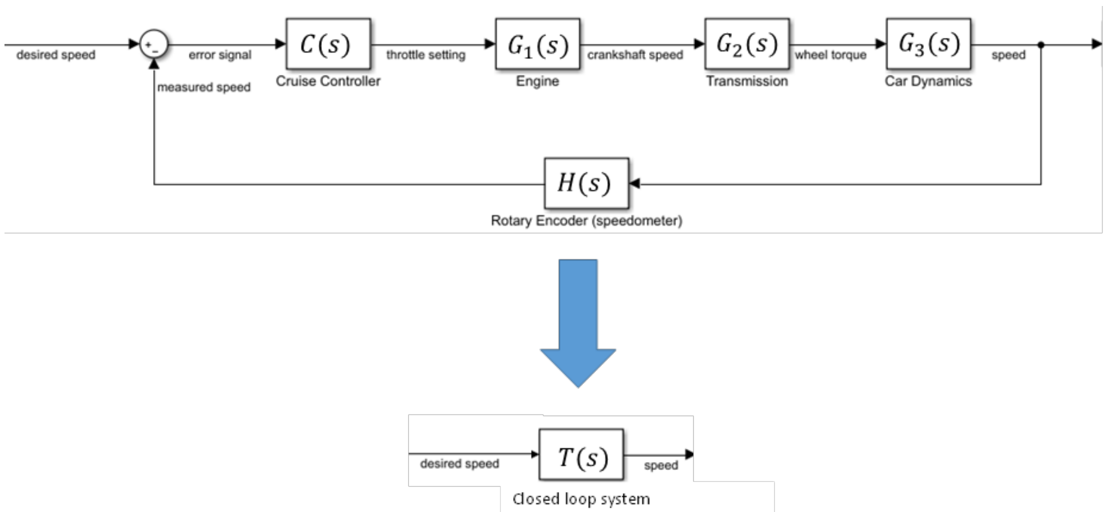
We can combine these systems together.



Block Diagram Algebra

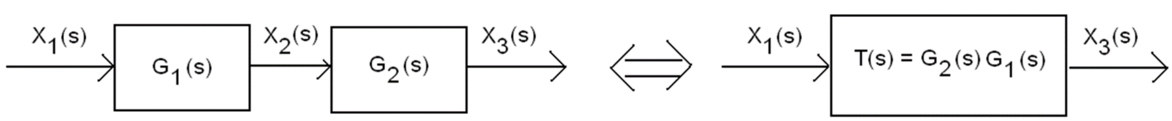
Pg. 81 of Dorf and Bishop. Also referred to as "Block Diagram Transformations"

The advantage of block diagrams is that we can abstract a complex system into a single block. The system can be made of up multiple, simpler blocks. For example, a car cruise control is made up of lots of different components, but we might only be interested in what happens to the velocity when you increase the speed setpoint.



Let's look at a series of simplifications that we can make to a block diagram

1. Combine in Series



Let' s look at a simple example, combining in series.

$$X_2(s) = G_1(s)\, X_1(s) \qquad \textbf{\textit{(Eq.1.1)}}$$

$$X_3(s) = G_2(s)\, X_2(s) \qquad \textbf{\textit{(Eq.1.2)}}$$

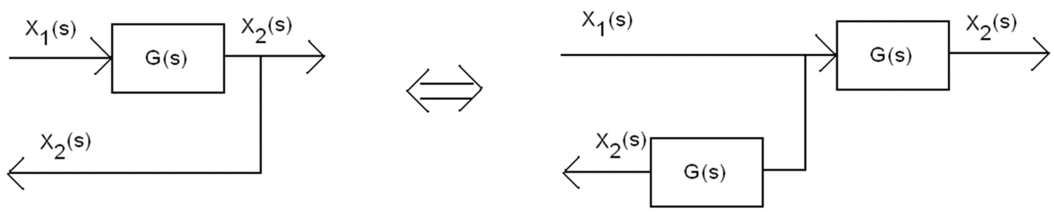
Substituting Eq.1.1 into Eq.1.2

$$X_3(s) = G_2(s)\, G_1(s)\, X_1(s)$$

$$X_3(s) = T(s)\, X_1(s) \qquad \textbf{\textit{(Eq.1.3)}}$$

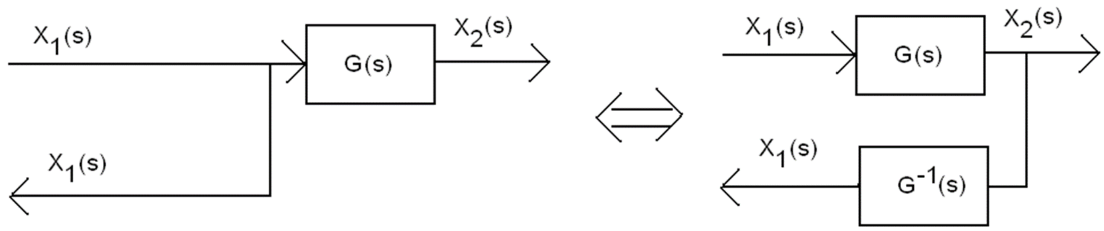
where $T(s) = G_2(s)\, G_1(s)$

2. Moving a Pickoff Point Ahead of a Block



From inspection, we see that $X_2(s) = G(s)\, X_1(s)$ and it is the same in both scenarios

3. Moving a Pickoff Point Behind a Block



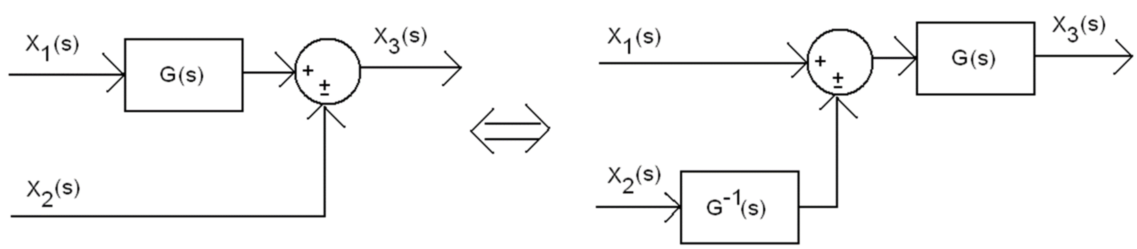
We need to be careful of this scenario. On the right hand diagram, we have

$$X_1(s) = G^{-1}(s)\, X_2(s)$$

$$= G^{-1}(s)\, G(s)\, X_1(s)$$ note: if the system is square and its inverse exists, then $G^{-1}(s)\, G(s) = I$

$$X_1(s) = X_1(s)$$

4. Moving Summing Point Ahead of a Block



We can verify the input/output relationship is the same. Let's look at the left diagram assuming the second sign on the sum block is a subtraction.

$$X_3(s) = G(s)\, X_1(s) - X_2(s)$$

Now for the right diagram

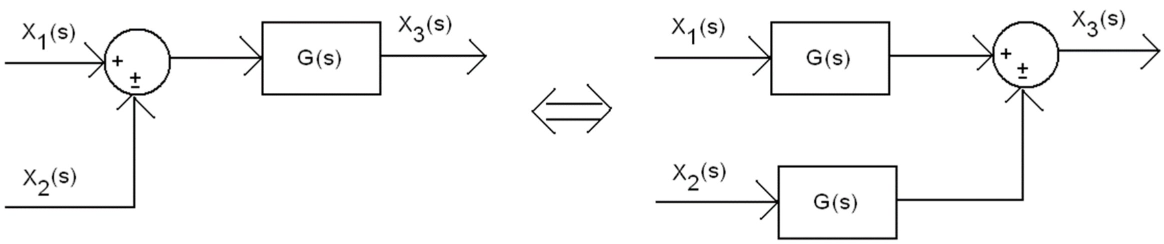
$$X_3(s) = G(s)\left(X_1(s) - G^{-1}(s)\, X_2(s)\right)$$

$$= G(s)\, X_1(s) - G(s)\, G^{-1}(s)\, X_2(s)$$ note: if the system is square and its inverse exists, then $G^{-1}(s)\, G(s) = I$

$$X_3(s) = G(s)\, X_1(s) - X_2(s)$$

So they are the same

5. Moving Summing Point Behind Block



We can verify the input/output relationship is the same. Let's look at the left diagram assuming the second sign on the sum block is a subtraction.

$$X_3(s) = G(s)\,(X_1(s) - X_2(s))$$

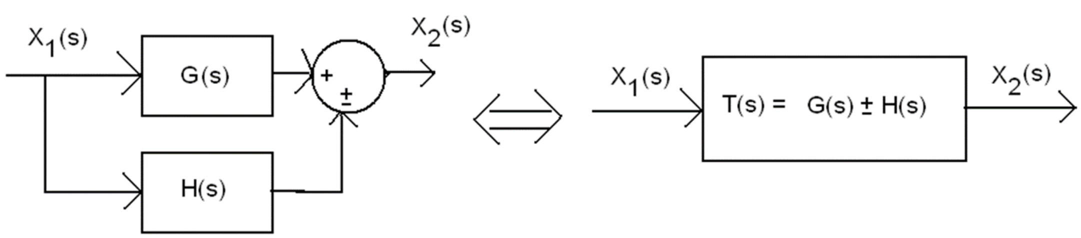
Now for the right diagram

$$X_3(s) = G(s)\, X_1(s) - G(s)\, X_2(s)$$

$$X_3(s) = G(s)\,(X_1(s) - X_2(s))$$

So they are the same

6. Eliminating a Feedforward Loop



We see that $X_2(s)$ is given by (assuming the sign is subtraction)

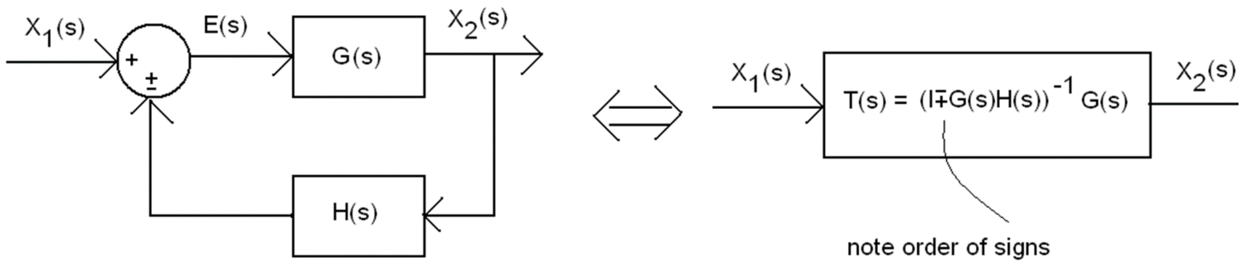
$$X_2(s) = G(s)\, X_1(s) - H(s)\, X_1(s)$$

$$= (G(s) - H(s))\, X_1(s)$$

$$X_2(s) = T(s)\, X_1(s)$$

where $\quad T(s) = G(s) - H(s)$

7. Eliminating a Feedback Loop



So we have (assuming the sign of the sum block is subtraction)

$$X_2(s) = G(s)\, E(s)$$

$$= G(s)\, (X_1(s) - H(s)\, X_2(s))$$

$$X_2(s) = G(s)\, X_1(s) - G(s)\, H(s)\, X_2(s)$$

$$X_2(s) + G(s)\, H(s)\, X_2(s) = G(s)\, X_1(s)$$

$$(I + G(s)\, H(s))\, X_2(s) = G(s)\, X_1(s)$$

$$X_2(s) = (I + G(s)\, H(s))^{-1}\, G(s)\, X_1(s)$$

$$X_2(s) = T(s)\, X_1(s)$$

where $T(s) = (I + G(s)\, H(s))^{-1}\, G(s)$

In the case of a SISO system

$$T(s) = \frac{G(s)}{1+G(s)\, H(s)}$$

If the sign of the sum block was addition, we would have

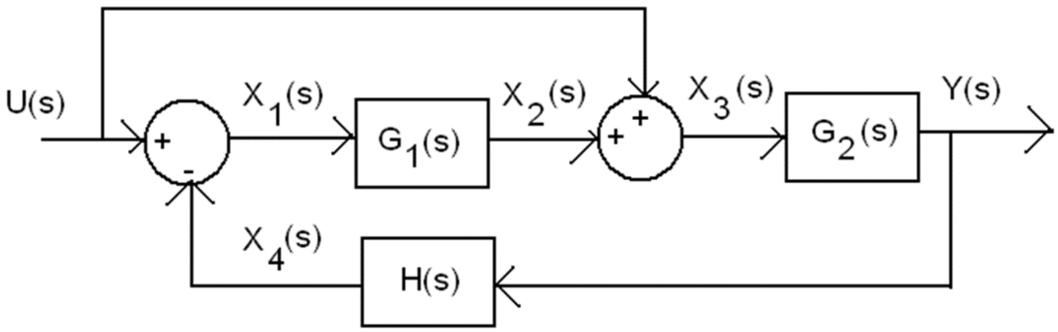
$$T(s) = \frac{G(s)}{1-G(s)\, H(s)}$$

Note that the order of signs is "opposite" from what the sum block has.

Block Diagram Algebra

Let's just look at an example to illustrate block diagram algebra.

Example: Block diagram simplifications



Questions : -If we know what the poles of each individual block are, what do the poles of the overall system look like?

-Is this overall system stable (ie do the signals go to 0)?

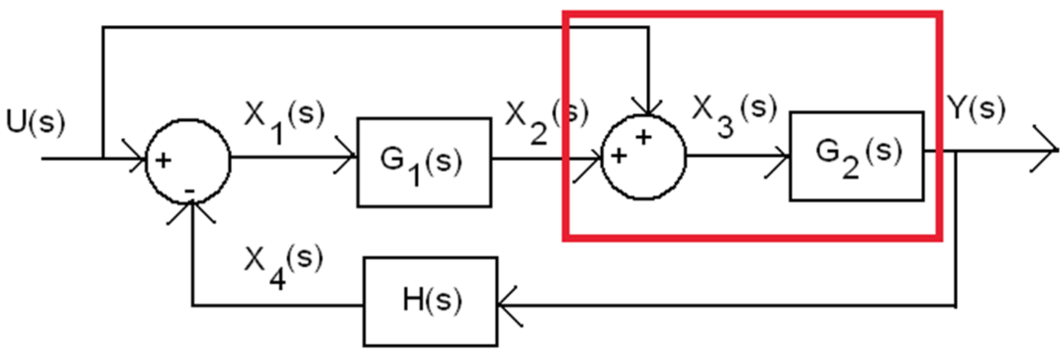
-If the system is stable, is it underdamped, overdamped, etc.?

-What kind of performance does the overall system have?

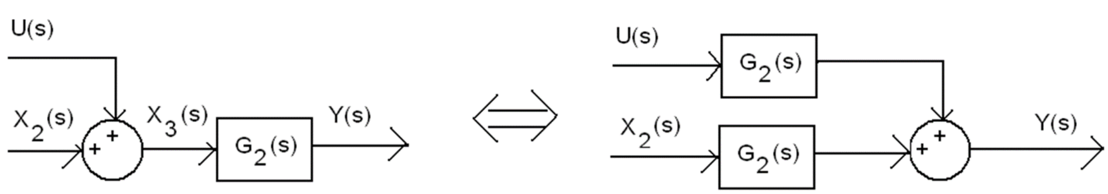
In order to answer these questions, we want to simplify it to a single block.

<u>Modification 1 : Move Sum Block</u>

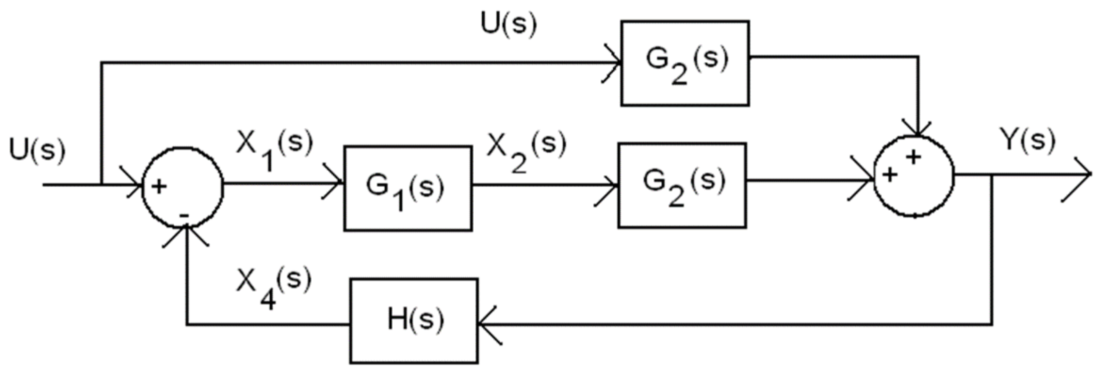
Let' s first move the sum block behind the $G_2(s)$ block.



We can isolate the pertinent blocks and signals and we see

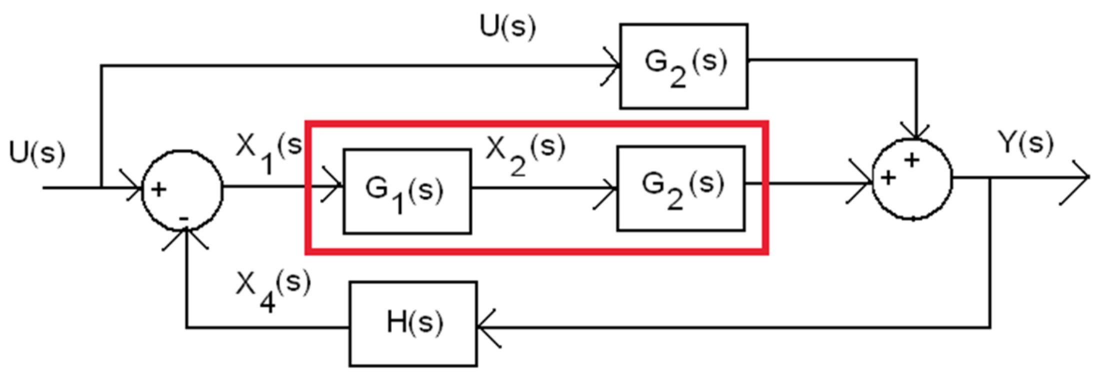


So we can redraw the diagram as

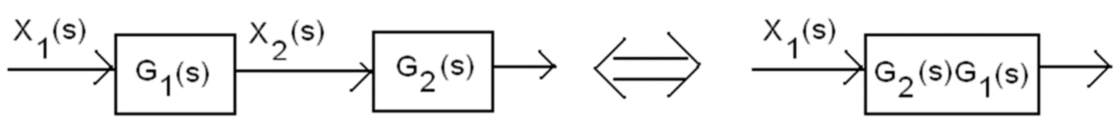


Modification 2 : Combine Blocks in Series

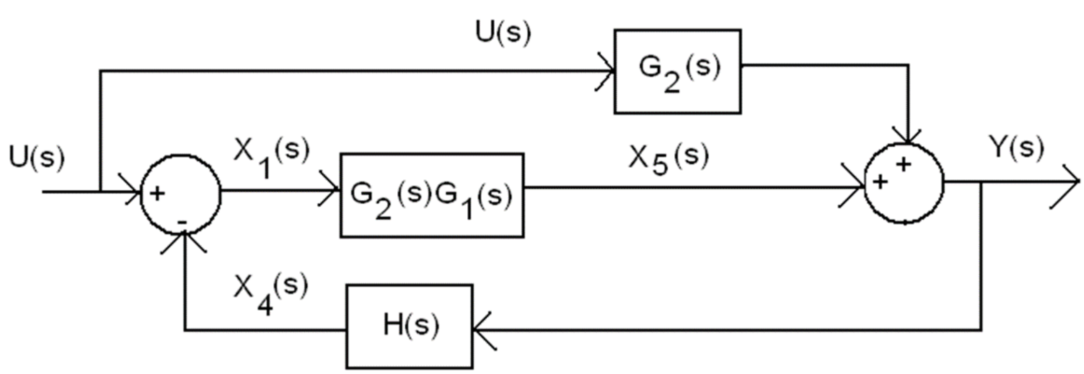
We can easily combine the blocks in series.



We can isolate the pertinent blocks and signals and we see

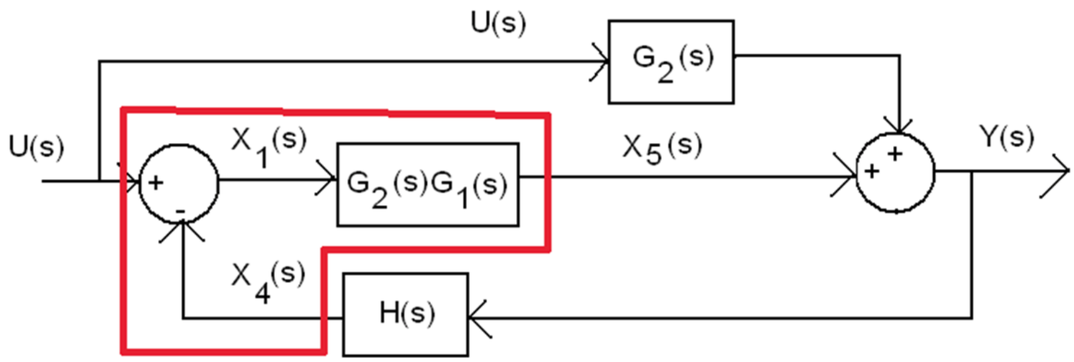


So we can redraw the diagram as

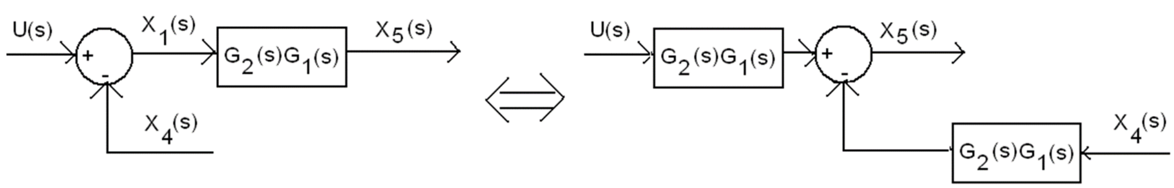


Modification 3 : Move Sum Behind Block

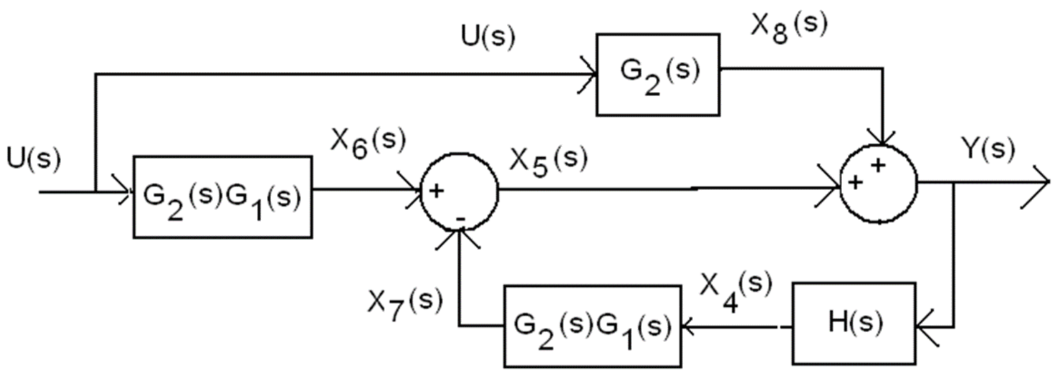
Let' s now move the first sum block behind the $G_2(s)\,G_1(s)$ block.



We can isolate the pertinent blocks and signals and we see

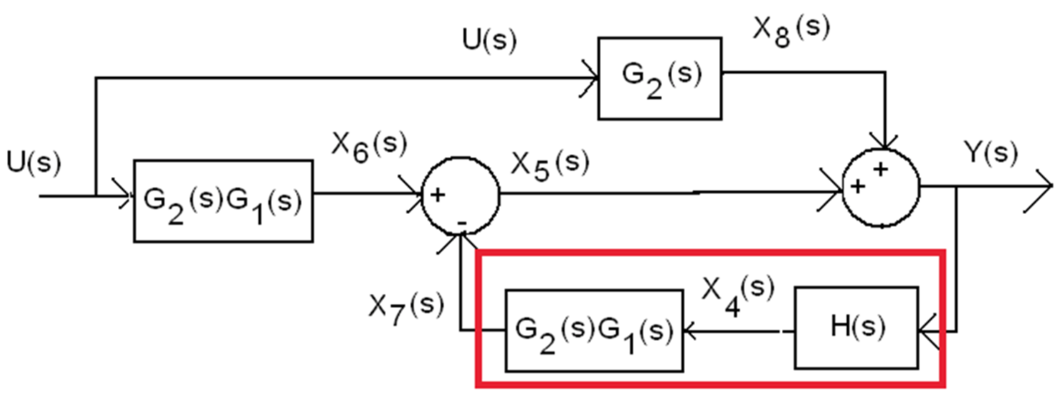


So we can redraw the diagram as

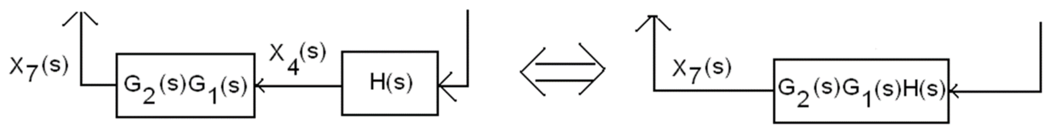


<u>Modification 4 : Combine Blocks in Series</u>

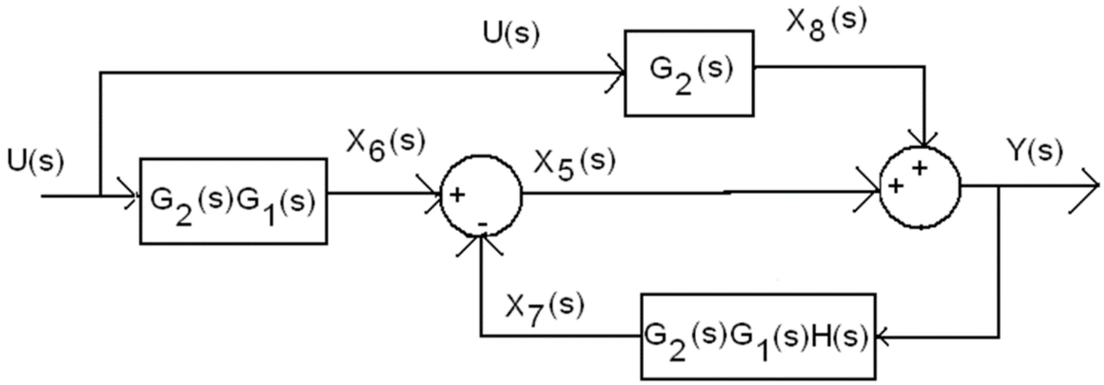
We can easily combine the blocks in series.



We can isolate the pertinent blocks and signals and we see

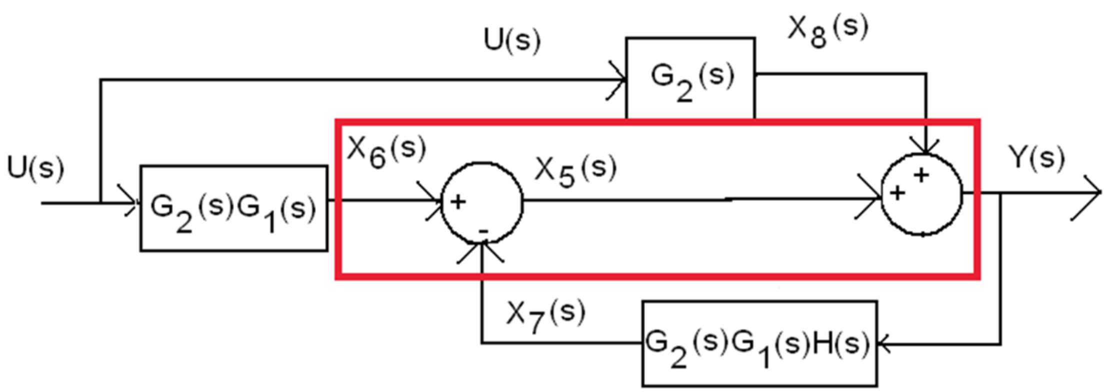


So we can redraw the diagram as



<u>Modification 5: Switch Order of Sum Blocks</u>

Let's take a closer look at the two summation blocks

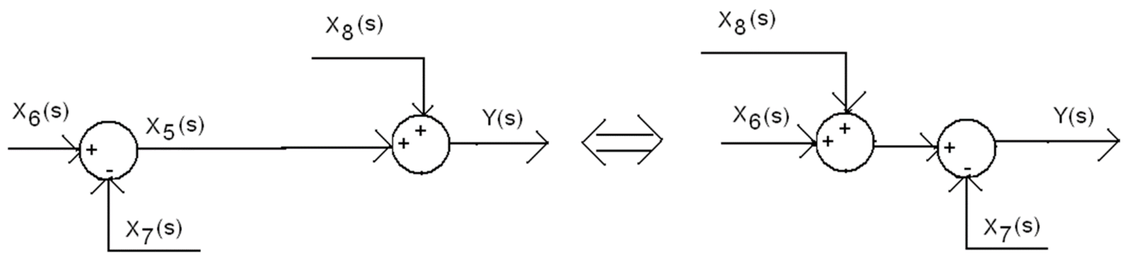


We note that

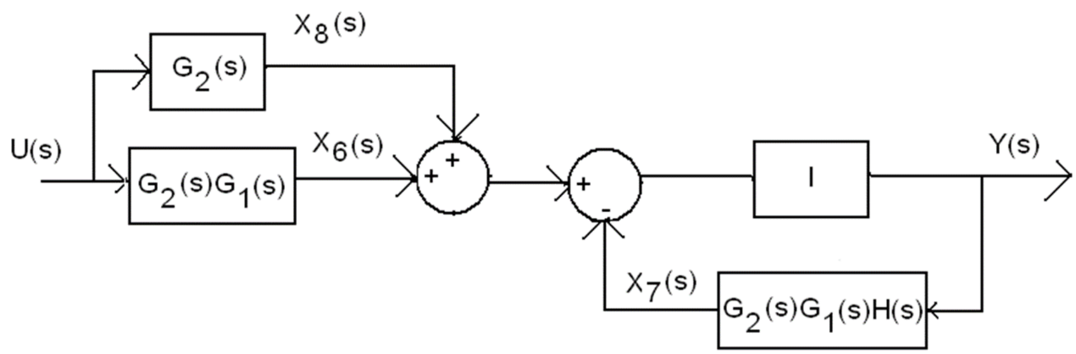
$$Y(s) = X_8(s) + X_5(s)$$

$$= X_8(s) + X_6(s) - X_7(s)$$

We can rearrange the order of the sum blocks so this is preserved. We can isolate the pertinent blocks and signals and we see

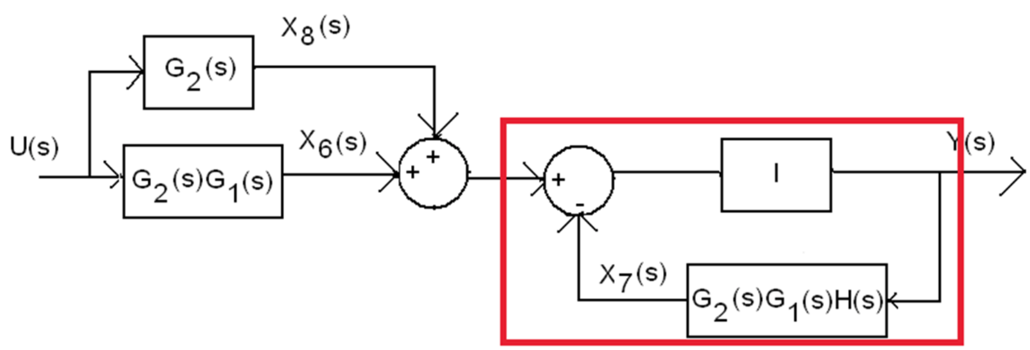


We can redraw the diagram as (inserting a identity gain on the $Y(s)$ signal



<u>Modification 6: Eliminate Feedback Loop</u>

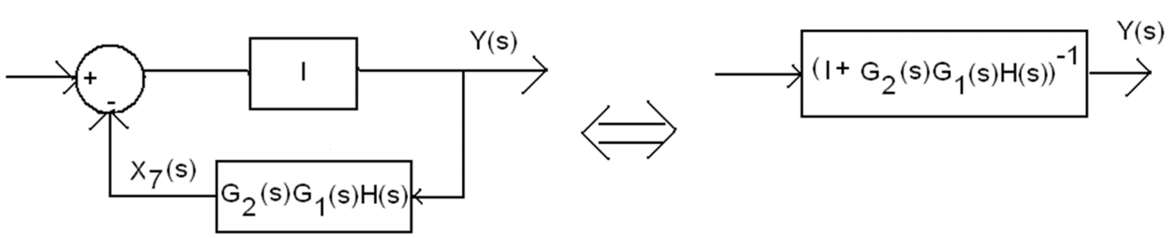
We can eliminate the feedback loop



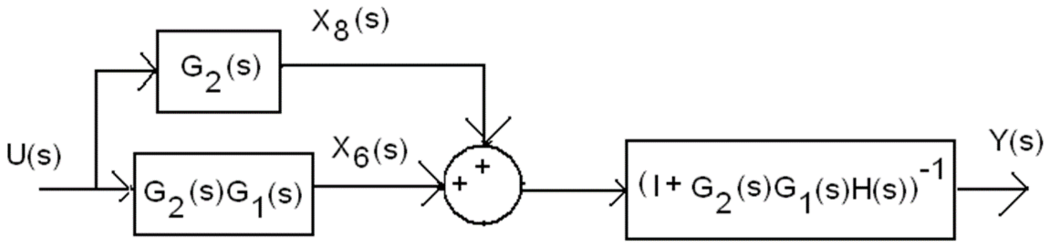
We noticing that the sign on the sum block is subtraction, so the feedback loop is given by

$$T(s) = (I + H(s)\, G_2(s)\, G_1(s))^{-1}$$

We can isolate the pertinent blocks and signals and we see

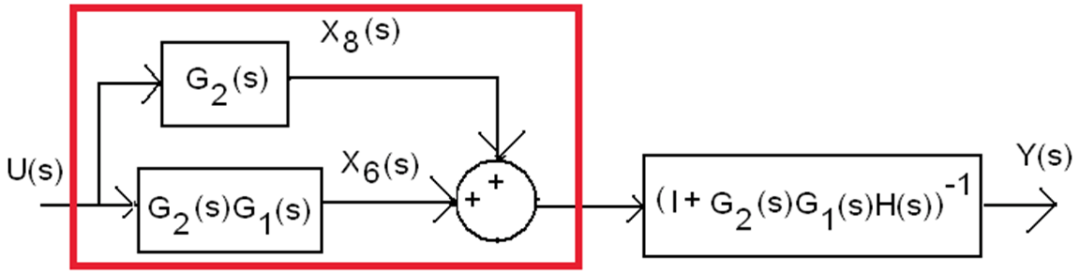


We can redraw the diagram as

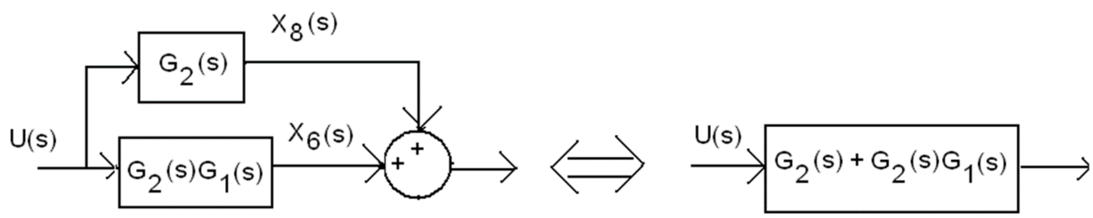


Modification 7: Eliminate Feedforward Loop

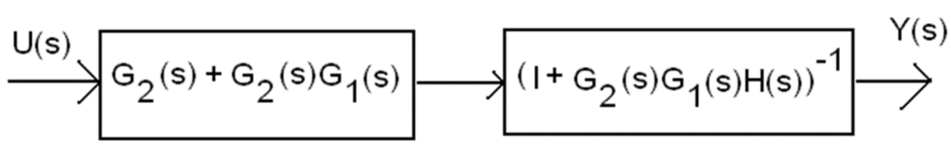
We can easily eliminate the feedfoward loop.



We can isolate the pertinent blocks and signals and we see to obtain

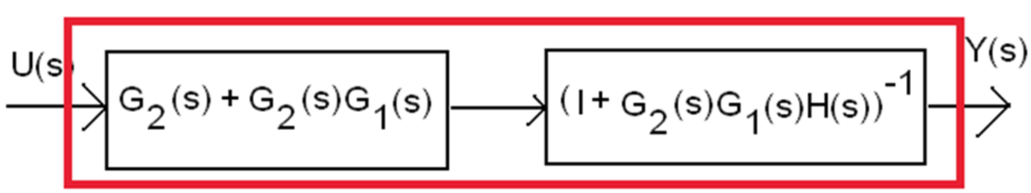


We can redraw the diagram as

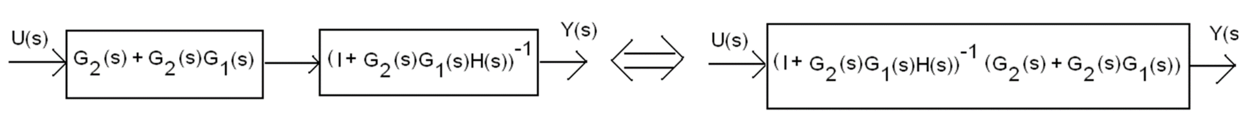


Modification 8: Combine Blocks in Series

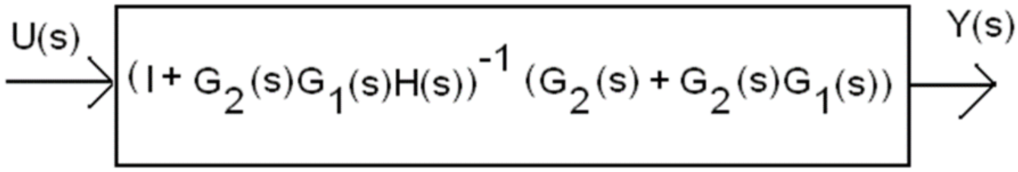
We combine the blocks in series.



We can isolate the pertinent blocks and signals and we see to obtain



So the final result is as shown below



So the final closed loop transfer function is given by

$$T(s) = (I + G_2(s)\,G_1(s)\,H(s))^{-1}\,(G_2(s) + G_2(s)\,G_1(s))$$

We can check this with Matlab using a specific example

$$G_1(s) = \frac{3\,(s+2)\,(s-3)}{(s+2)\,(s+3)\,(s+4)}$$

$$G_2(s) = \frac{-2\,(s+1)\,(s+3)\,(s+17)}{(s+12)^2\left(s^2+2\,s+5\right)}$$

$$H(s) = \frac{3\left(s^2+2\,s+5\right)}{(s+1)\,(s+2)\,(s+4)\,(s+6)}$$

$$\mathbf{G1} = \frac{3\,(s+2)\,(s-3)}{(s+2)\,(s+3)\,(s+4)};$$

$$\mathbf{G2} = \frac{-2\,(s+1)\,(s+3)\,(s+17)}{(s+12)^2\left(s^2+2\,s+5\right)};$$

$$\mathbf{H} = \frac{3\left(s^2+2\,s+5\right)}{(s+1)\,(s+2)\,(s+4)\,(s+6)};$$

We need to expand the numerator and denominator of each transfer function so we can easily input them into Simulink.

```
G1num = Expand[Numerator[G1]]
G1den = Expand[Denominator[G1]]
```

$-9 + 3\,s$

$12 + 7\,s + s^2$

```
G2num = Expand[Numerator[G2]]
G2den = Expand[Denominator[G2]]
```

$-102 - 142\,s - 42\,s^2 - 2\,s^3$

$720 + 408\,s + 197\,s^2 + 26\,s^3 + s^4$

```
Hnum = Expand[Numerator[H]]
Hden = Expand[Denominator[H]]
```

$15 + 6\,s + 3\,s^2$

$48 + 92\,s + 56\,s^2 + 13\,s^3 + s^4$

```
temp = Simplify[(1 + G2 G1 H)^-1 (G2 + G2 G1)];
num = Numerator[temp];
den = Denominator[temp];
T = Expand[num] / Expand[den]
```

$$\left(-4896 - 25\,992\,s - 40\,136\,s^2 - 25\,766\,s^3 - 7808\,s^4 - 1160\,s^5 - 80\,s^6 - 2\,s^7\right) \Big/ \left(142\,830 + 240\,192\,s + 195\,780\,s^2 + 97\,888\,s^3 + 31\,370\,s^4 + 6176\,s^5 + 705\,s^6 + 42\,s^7 + s^8\right)$$

Using Simulink, we see they are the same (See Matlab example files)