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Lecture 04j
Laplace Operator, Divergence, Curl

Lecture is on YouTube

The YouTube video entitled ‘TBD’ that covers this lecture is located at TBD

Outline

-The Laplace Operator (AKA Laplacian)
-Divergence of a Vector Field
-Curl of a Vector Field

The Laplace Operator (AKA Laplacian)

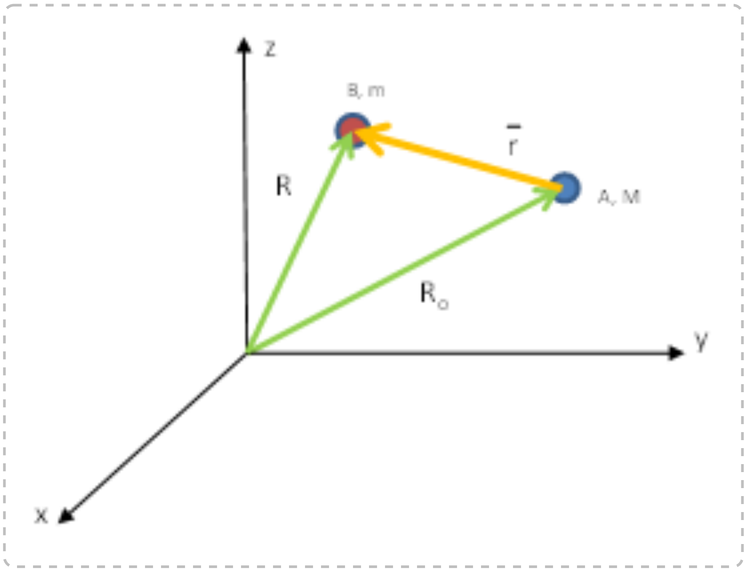
$\nabla^2 f = \Delta f$ is often referred to as the Laplacian of f (read “nabla squared” or “delta” f and sometimes referred to as the Laplace operator).

$$\nabla \cdot \nabla = \nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad \text{(Laplace operator)}$$

Recall: ∇ = “nabla” or “del”
Note 1: Do not confuse this with the Laplacian matrix from graph theory which is a different construct.
Note 2: To further confuse notation, in the context of optimization and control theory, sometimes $\nabla^2 f$ is used to refer to the Hessian of the function f. We will investigate this later during our discussion of Taylor series and again when we investigate optimization.

Example: Gravitational field. Laplace’s equation.

Let a particle A with mass M be fixed at a point $R_o = (\,x_o \quad y_o \quad z_o\,)$ and let a particle B of mass m be free to take up various positions $R = (\,x \quad y \quad z\,)$ in space (see diagram below).



We can define the vector $\bar{r}$ as the vector from R_o to R as

$$\bar{r} = (x - x_o)\,\hat{i} + (y - y_o)\,\hat{j} + (z - z_o)\,\hat{k} \qquad \textbf{\textit{(Eq.1)}}$$

The magnitude of the attractive force from B towards A is given as

$$|\,\bar{p}\,| = \frac{c}{|\bar{r}|^2} \qquad \textbf{\textit{(Eq.2)}}$$

where $c = G\,M\,m$

G = gravitational constant = $6.674 \times 10^{-11}\,N\,(m/\text{kg})^2$

We recognize $|\,\bar{r}\,|$ = distance between A and B = $\left((x - x_o)^2 + (y - y_o)^2 + (z - z_o)^2\right)^{1/2}$

The vector of the attractive force from B towards A has magnitude given by Eq.1 but direction of $-\bar{r}/\,|\,\bar{r}\,|$. So we have

$$\bar{p} = \frac{c}{|\bar{r}|^2}\left(\frac{-\bar{r}}{|\bar{r}|}\right)$$

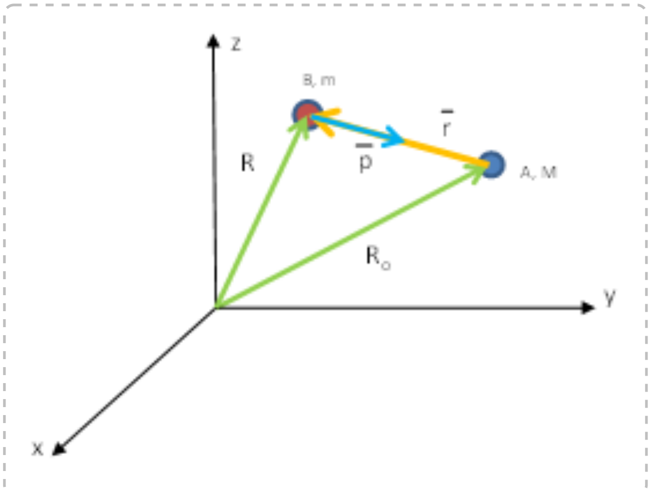
$$\bar{p} = -\frac{c}{|\bar{r}|^3}\,\bar{r} \qquad \textbf{\textit{(Eq.3)}}$$

In[9]:=
```
p[x_, y_, z_] = -c/normr^3 * {x - xo, y - yo, z - zo} /. {normr → ((x - xo)^2 + (y - yo)^2 + (z - zo)^2)^(1/2)};
p[x, y, z] // MatrixForm
```

Out[10]//MatrixForm=

$$\begin{pmatrix} -\frac{c\,(x-xo)}{\left((x-xo)^2+(y-yo)^2+(z-zo)^2\right)^{3/2}} \\ -\frac{c\,(y-yo)}{\left((x-xo)^2+(y-yo)^2+(z-zo)^2\right)^{3/2}} \\ -\frac{c\,(z-zo)}{\left((x-xo)^2+(y-yo)^2+(z-zo)^2\right)^{3/2}} \end{pmatrix}$$

Graphically



Now, suppose that we define a scalar function as follows

$$f(x, y, z) = \frac{c}{|\bar{r}|}$$

In[11]:=
```
f[x_, y_, z_] = c/normr /. {normr → ((x - xo)^2 + (y - yo)^2 + (z - zo)^2)^(1/2)};
f[x, y, z] // MatrixForm
```

Out[12]//MatrixForm=

$$\frac{c}{\sqrt{(x-xo)^2+(y-yo)^2+(z-zo)^2}}$$

So this scalar function is given as

$$f(x, y, z) = \frac{c}{\left((x-x_o)^2+(y-y_o)^2+(z-z_o)^2\right)^{1/2}}$$

Side Note

The function f may seem somewhat arbitrary but we can see its significance if we compute the gradient of f using

$$\nabla f = \begin{pmatrix} \partial f / \partial x \\ \partial f / \partial y \\ \partial f / \partial z \end{pmatrix}$$

In[13]:=
```
delf[x_, y_, z_] = ( D[f[x, y, z], x]
                     D[f[x, y, z], y]
                     D[f[x, y, z], z] );
delf[x, y, z] // MatrixForm
```

Out[14]//MatrixForm=

$$\begin{pmatrix} -\frac{c\,(x-xo)}{\left((x-xo)^2+(y-yo)^2+(z-zo)^2\right)^{3/2}} \\ -\frac{c\,(y-yo)}{\left((x-xo)^2+(y-yo)^2+(z-zo)^2\right)^{3/2}} \\ -\frac{c\,(z-zo)}{\left((x-xo)^2+(y-yo)^2+(z-zo)^2\right)^{3/2}} \end{pmatrix}$$

We see that $\nabla f = \overline{p}$ and therefore, f is a potential of $\overline{p}$

In[15]:=
```
delf[x, y, z] == p[x, y, z]
```

Out[15]= True

End Side Note

The function f is interesting in that it satisfies Laplace’s equation which is given as

$$\nabla^2 f = \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

In[40]:=
```
Print["Individual Terms"]
d2fx2 = D[f[x, y, z], {x, 2}] // Simplify
d2fy2 = D[f[x, y, z], {y, 2}] // Simplify
d2fz2 = D[f[x, y, z], {z, 2}] // Simplify

Print["Combining terms to compute ∇²f = Δf"]
Δf = d2fx2 + d2fy2 + d2fz2 // Simplify
```

Individual Terms

Out[41]= $-\frac{c\left(-2x^2+4\,x\,xo-2\,xo^2+y^2-2\,y\,yo+yo^2+z^2-2\,z\,zo+zo^2\right)}{\left((x-xo)^2+(y-yo)^2+(z-zo)^2\right)^{5/2}}$

Out[42]= $-\frac{c\left(x^2-2\,x\,xo+xo^2-2y^2+4\,y\,yo-2\,yo^2+z^2-2\,z\,zo+zo^2\right)}{\left((x-xo)^2+(y-yo)^2+(z-zo)^2\right)^{5/2}}$

Out[43]= $-\frac{c\left(x^2-2\,x\,xo+xo^2+y^2-2\,y\,yo+yo^2-2z^2+4\,z\,zo-2\,zo^2\right)}{\left((x-xo)^2+(y-yo)^2+(z-zo)^2\right)^{5/2}}$

Combining terms to compute $\nabla^2 f = \Delta f$

Out[45]= 0

Divergence of a Vector Field

We now focus on the divergence of a vector field. Let

$$\overline{v}(x, y, z) = \begin{pmatrix} v_1(x, y, z) \\ v_2(x, y, z) \\ v_3(x, y, z) \end{pmatrix}$$

be a differentiable vector function

The divergence of $\overline{v}$ (sometime also referred to as the divergence of the vector field defined by $\overline{v}$) is defined as

$$\text{div}\,\overline{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \qquad \textbf{\textit{(Eq.1)}}$$

Another common notation for the divergence of $\overline{v}$ is $\nabla \cdot \overline{v}$

$$\text{div}\,\overline{v} = \nabla \cdot \overline{v} \qquad \text{(alternate notation)}$$

Note that $\nabla \cdot \overline{v}$ means scalar divergence of $\overline{v}$ whereas ∇v means the gradient of the scalar function v. The one little dot product symbol makes a lot of difference but also note that these are two inherently different operators. The divergence operates on a vector function and the gradient operates on a scalar function.

Let us examine how the divergence is related to the Laplacian of f. Suppose that $f(x, y, z)$ is a twice differentiable scalar function. We can easily compute the gradient of f

$$\nabla f = \begin{pmatrix} \partial f / \partial x \\ \partial f / \partial y \\ \partial f / \partial z \end{pmatrix}$$

We can compute the divergence of ∇f using Eq.1

$$\text{div}\,\nabla f = \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}\right] + \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial y}\right] + \frac{\partial}{\partial z}\left[\frac{\partial f}{\partial z}\right]$$

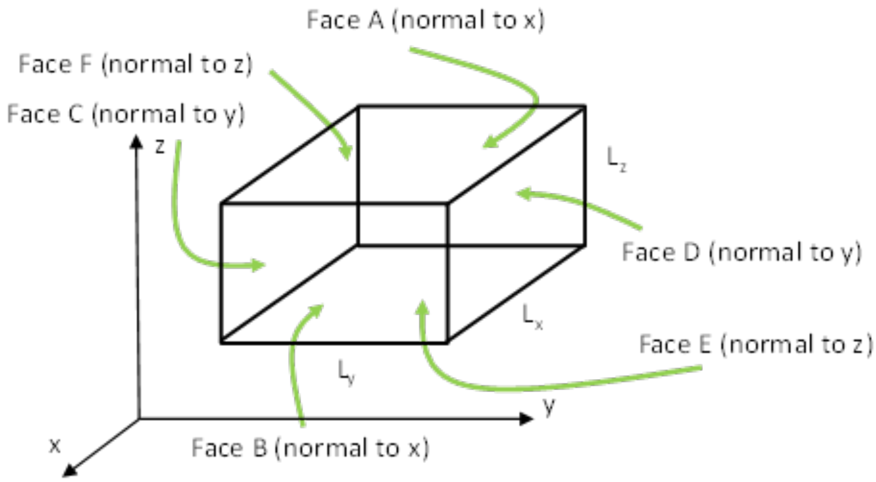
$$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \qquad \text{recall: } \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\text{div}\,\nabla f = \nabla^2 f \qquad \textbf{\textit{(Eq.3)}}$$

So we see that the divergence of the gradient of f is equal to the Laplacian of f.

Example: Motion of a compressible fluid. Meaning of the divergence.

Consider a control volume of a cube with faces aligned with the Cartesian coordinate system as shown below



Consider the velocity of the fluid to be

$$\overline{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = v_1\,\hat{i} + v_2\,\hat{j} + v_3\,\hat{k}$$

The fluid has a density of ρ. Using this, we define a function $\overline{u}$ to describe the mass flux of the fluid

$$\overline{u} = \rho\,\overline{v} \qquad \text{(mass flux)}$$

We can now calculate the change in mass in the control volume using the function $\overline{u}$ and considering the flux of matter across each of the 6 boundaries.

First, we consider the flux crossing Face A. This face has area of $L_y\,L_z$. Note that since this face is normal to x, v_2 and v_3 do not contribute because these are velocity components perpendicular to this face. Therefore, the total mass entering the control volume by crossing this face over the short time period dt is given as

$$\rho\,v_{1,A}\,L_y\,L_z\,\text{dt} \qquad \text{(mass entering via face A)}$$

where $\quad v_{1,A}$ = velocity of flow at face A

In a similar fashion, the mass leaving via face B is given by

$$\rho\,v_{1,B}\,L_y\,L_z\,\text{dt} \qquad \text{(mass leaving via face B)}$$

We can then formulate an expression for the mass leaving via these two faces as

$$\text{mass leaving} = \rho\,v_{1,B}\,L_y\,L_z\,\text{dt} - \rho\,v_{1,A}\,L_y\,L_z\,\text{dt}$$

$= (\rho v_{1,B} - \rho v_{1,A}) L_y L_x \,\mathrm{dt}$ note: $V = L_x L_y L_z \Rightarrow L_y L_z = V / L_x$

$= (\rho v_{1,B} - \rho v_{1,A}) \frac{V}{L_x} \,\mathrm{dt}$ recall: $\overline{u} = \rho \overline{v}$

$= (u_{1,B} - u_{1,A}) \frac{V}{L_x} \,\mathrm{dt}$

$= \mathrm{du}_1 \frac{V}{L_x} \,\mathrm{dt}$

where $\mathrm{du}_1 = u_{1,B} - u_{1,A}$
$u_{1,B}$ = mass flux at face B
$u_{1,A}$ = mass flux at face A
V = volume

Repeating this analysis for the other 4 faces yields the total mass leaving the control volume as

$$\text{total mass loss across control volume surfaces} = \mathrm{du}_1 \frac{V}{L_x} \,\mathrm{dt} + \mathrm{du}_2 \frac{V}{L_y} \,\mathrm{dt} + \mathrm{du}_3 \frac{V}{L_z} \,\mathrm{dt}$$

$$\text{total mass loss across control volume surfaces} = \left(\frac{\mathrm{du}_1}{L_x} + \frac{\mathrm{du}_2}{L_y} + \frac{\mathrm{du}_3}{L_z}\right) V \,\mathrm{dt} \qquad \textbf{\textit{(Eq.1)}}$$

where $\mathrm{du}_1 = u_{1,B} - u_{1,A}$
$\mathrm{du}_2 = u_{2,D} - u_{2,C}$
$\mathrm{du}_3 = u_{3,F} - u_{3,E}$

This total mass loss over time period dt is caused by the time rate of change of the density (the fluid is either expanding or contracting). Therefore, we can write

$$\text{total mass loss due to expansion/contraction} = -\frac{\partial \rho}{\partial t} V \,\mathrm{dt} \qquad \textbf{\textit{(Eq.2)}}$$

Setting Eq.1 equal to Eq.2 yields

$$\left(\frac{\mathrm{du}_1}{L_x} + \frac{\mathrm{du}_2}{L_y} + \frac{\mathrm{du}_3}{L_z}\right) V \,\mathrm{dt} = -\frac{\partial \rho}{\partial t} V \,\mathrm{dt}$$

$$\left(\frac{\mathrm{du}_1}{L_x} + \frac{\mathrm{du}_2}{L_y} + \frac{\mathrm{du}_3}{L_z}\right) = -\frac{\partial \rho}{\partial t}$$

As we let $L_x \to 0$, $L_y \to 0$, $L_z \to 0$, we see that this expression reduces to

$$\left(\frac{\mathrm{du}_1}{\mathrm{dx}} + \frac{\mathrm{du}_2}{\mathrm{dy}} + \frac{\mathrm{du}_3}{\mathrm{dz}}\right) = -\frac{\partial \rho}{\partial t} \qquad \text{recall: } \frac{\mathrm{du}_1}{\mathrm{dx}} + \frac{\mathrm{du}_2}{\mathrm{dy}} + \frac{\mathrm{du}_3}{\mathrm{dz}} = \mathrm{div}\,\overline{u}$$

$\text{div}\,\overline{u} = -\frac{\partial \rho}{\partial t}$ recall: $\overline{u} = \rho\,\overline{v}$

$\text{div}\,\rho\,\overline{v} = -\frac{\partial \rho}{\partial t}$

$\text{div}\,\rho\,\overline{v} + \frac{\partial \rho}{\partial t} = 0$ ***(Eq.5)***

This is sometimes referred to as the **conservation of mass** or **continuity equation** of a compressible fluid flow. It effectively states that rate of mass leaving a control volume must be equal to the time rate of change of mass within the volume.

In a steady state flow $\frac{\partial \rho}{\partial t} = 0$ and this reduces to

$\text{div}\,\rho\,\overline{v} = 0$

Furthermore, if the flow is incompressible, density is constant and this further reduces to

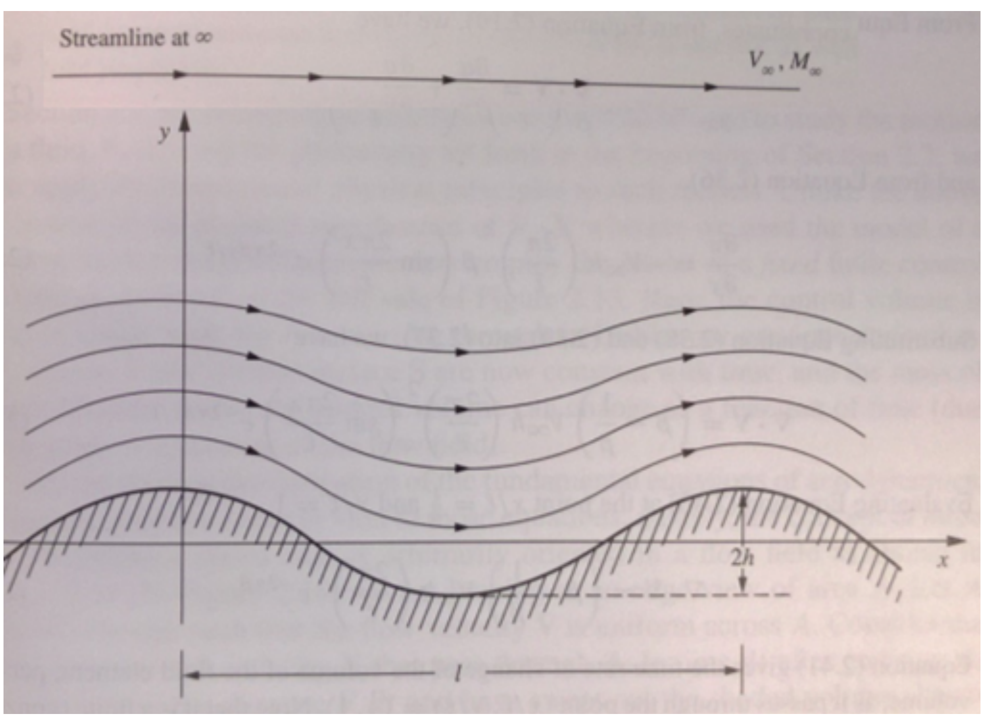
$\text{div}\;\overline{v} = 0$ (condition of incompressibility)

From this discussion, we see that divergence measures outflow minus inflow.

Example: Fluid Flow Along a Wavy Wall

It can be shown that if we have a vector field describing the velocity of a fluid flow, $\overline{v}$, then $\text{div}\,\overline{v} = \nabla \cdot \overline{v}$ is physically the time rate of change of the volume of a moving fluid element, per unit volume (see discussion during section 10.8). For now, we take this fact on faith and continue with our example.

The subsonic compressible flow over a cosine-shaped wavy wall is illustrated below.



The wavelength and amplitude of the wall are l and h, respectively. The streamlines exhibit the same qualitative shape as the wall, but with diminishing amplitude as distance above the wall increases.

Finally, as $y \to \infty$, the streamline becomes straight. Along this straight streamline, the freestream velocity and Mach number are V_∞ and M_∞, respectively. The velocity field in Cartesian coordinates is given by

$$u = V_\infty\left[1 + \frac{h}{\beta}\,\frac{2\pi}{l}\cos\left(\frac{2\pi}{l}\,x\right)e^{-2\pi\beta y/l}\right]$$

$$v = -V_\infty\, h\,\frac{2\pi}{l}\sin\left(\frac{2\pi}{l}\,x\right)e^{-2\pi\beta y/l}$$

where $\beta = \sqrt{1 - M_\infty^{\;2}}$

```
β = Sqrt[1 - M∞^2];
u = V∞ (1 + h/β (2 π)/l Cos[(2 π)/l x] Exp[-2 π β y / l]);
v = -V∞ h (2 π)/l Sin[(2 π)/l x] Exp[-2 π β y / l];
```

For example, if we consider the constants of

$l = 1\,m$
$h = 1/100 = 0.01\,m$
$V_\infty = 240\,m/s$
$M_\infty = 0.7$

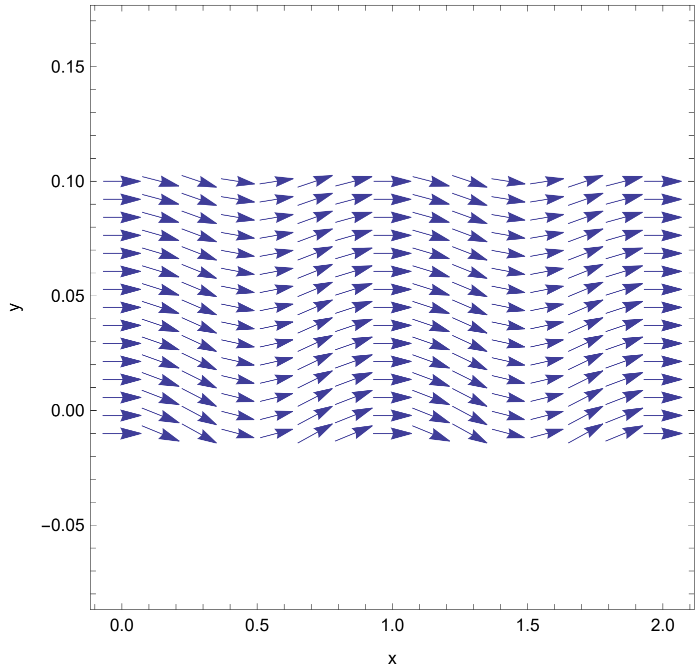
```
lGiven = 1;
hGiven = 1 / 100;
V∞Given = 240;
M∞Given = 7 / 10;
```

```
{u, v} /. {l → lGiven, h → hGiven, V∞ → V∞Given, M∞ → M∞Given} // MatrixForm
```

$$\begin{pmatrix} 240 \times \left(1 + \dfrac{e^{-\frac{1}{5}\sqrt{51}\,\pi y}\,\pi\,\mathrm{Cos}[2\,\pi\,x]}{5\,\sqrt{51}}\right) \\ -\dfrac{24}{5}\,e^{-\frac{1}{5}\sqrt{51}\,\pi y}\,\pi\,\mathrm{Sin}[2\,\pi\,x] \end{pmatrix}$$

We can plot the vector field

```
VectorPlot[{u, v} /. {l → lGiven, h → hGiven, V∞ → V∞Given, M∞ → M∞Given},
 {x, 0, 2 lGiven}, {y, -hGiven, 10 * hGiven},
 FrameLabel → {"x", "y"}]
```



We can compute the time rate of change of the volume of a moving fluid element of fixed mass, per unit volume using

$$\operatorname{div} \overline{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

```
divV[x_, y_] = D[u, x] + D[v, y] // Simplify
```

$$-\frac{4\, e^{-\frac{2\sqrt{1-M\infty^2}\,\pi\, y}{l}}\, h\, M\infty^2\, \pi^2\, V\infty\, \mathrm{Sin}\left[\frac{2\,\pi\, x}{l}\right]}{l^2\,\sqrt{1-M\infty^2}}$$

We can now evaluate this at a point

```
xGiven = 1/4;
yGiven = 1;
divV[xGiven, yGiven] /. {l → lGiven, h → hGiven, V∞ → V∞Given, M∞ → M∞Given} // N
```

```
-0.731582
```

So we obtain

$$\nabla \cdot \overline{V} = -0.7315$$

The physical significant of this result is that, as the fluid element is passing through the point (1/4, 1) in the flow, it is experience a 73 percent rate of decrease of volume per second (the negative quantity denotes a decrease in volume). That is, the density of the fluid element is increasing. Hence, the point (1/4,1) is in a compression region of the flow, where the fluid element will experience an increase in density. Expansion regions are defined by values of x and y which yield positive $\nabla \cdot \overline{v}$.

As we move close to the wall, we see that this is behavior is exacerbated.

```
xGiven = 1/4;
yGiven = 0.95;
divV[xGiven, yGiven] /. {l → lGiven, h → hGiven, V∞ → V∞Given, M∞ → M∞Given} // N
```

```
-0.915585
```

```
Clear[yGiven, xGiven, divV, M∞Given, V∞Given, hGiven, lGiven, v, u, β]
```

Properties of the Divergence Operator

The most important property of the divergence operator is that it is linear. So for two vector fields $\overline{F}$ and $\overline{G}$ and two scalars a and b, we have

$$\text{div}\left(a\,\overline{F} + b\,\overline{G}\right) = a\,\text{div}(\overline{F}) + b\,\text{div}(\overline{G})$$

There is a product rule of the following type. If φ is a scalar valued function and $\overline{F}$ is a vector field, then

$$\text{div}\left(\varphi\,\overline{F}\right) = \text{grad}(\varphi)\cdot\overline{F} + \varphi\,\text{div}(\overline{F})$$

This can be written using alternative notation as shown below

$$\nabla\cdot\left(\varphi\,\overline{F}\right) = (\nabla\varphi)\cdot\overline{F} + \varphi\left(\nabla\cdot\overline{F}\right)$$

Curl of a Vector Field

Let x, y, z be right-handed Cartesian coordinates and let $\overline{v}$ be a differentiable vector function.

$$\overline{v}(x, y, z) = v_1\,\hat{i} + v_2\,\hat{j} + v_3\,\hat{k}$$

The curl of the vector function $\overline{v}$ (also referred to as the curl of the vector field defined by $\overline{v}$) is given by

$$\text{curl}\,\overline{v} = \nabla\times\overline{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= \begin{vmatrix} \partial/\partial y & \partial/\partial z \\ v_2 & v_3 \end{vmatrix}\hat{i} - \begin{vmatrix} \partial/\partial x & \partial/\partial z \\ v_1 & v_3 \end{vmatrix}\hat{j} + \begin{vmatrix} \partial/\partial x & \partial/\partial y \\ v_1 & v_2 \end{vmatrix}\hat{k}$$

$$\text{curl}\,\overline{v} = \nabla \times \overline{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k} \qquad \textbf{(Eq.1)}$$

Example 2: Rotation of a rigid body. Relation to curl (extended)

In a previous example we showed that the velocity field of a rigid body in rotation can be described by

$$\overline{v} = \overline{\omega} \times \overline{r}$$

```
v = Cross[{ω1, ω2, ω3}, {x, y, z}];
v // MatrixForm
```

$$\begin{pmatrix} z\,\omega 2 - y\,\omega 3 \\ -z\,\omega 1 + x\,\omega 3 \\ y\,\omega 1 - x\,\omega 2 \end{pmatrix}$$

Let us calculate curl $\overline{v}$

$$\text{curl}\,\overline{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

```
v1 = v[[1]];
v2 = v[[2]];
v3 = v[[3]];
curlV = {D[v3, y] - D[v2, z], D[v1, z] - D[v3, x], D[v2, x] - D[v1, y]};
curlV // MatrixForm
```

$$\begin{pmatrix} 2\,\omega 1 \\ 2\,\omega 2 \\ 2\,\omega 3 \end{pmatrix}$$

So we see that

$$\text{curl}\,\overline{v} = 2\,\overline{\omega}$$

Let us use Mathematica's 'Curl' function to check

```
Curl[v, {x, y, z}]
```

$\{2\,\omega 1,\ 2\,\omega 2,\ 2\,\omega 3\}$

```
Clear[v, v1, v2, v3]
```

We notice that curl $\overline{v}$ is exactly twice the angular velocity vector. This leads to a theorem.

Theorem 1: Rotating Body and Curl

The curl of the velocity field of a rotating rigid body has the direction of the axis of the rotation, and its magnitude equals twice the angular speed of the rotation.

Intuitive Interpretation of Curl

Suppose the vector field describes the velocity field of a fluid flow and a small ball is located within the fluid or gas. If there is sufficient friction between the ball's surface and the fluid, the fluid flowing past it will make the ball rotate rotate. The rotation axis (oriented according to the right hand rule) points in the direction of the curl of the field at the center of the ball, and the angular speed of the rotation is half the magnitude of the curl at this point.

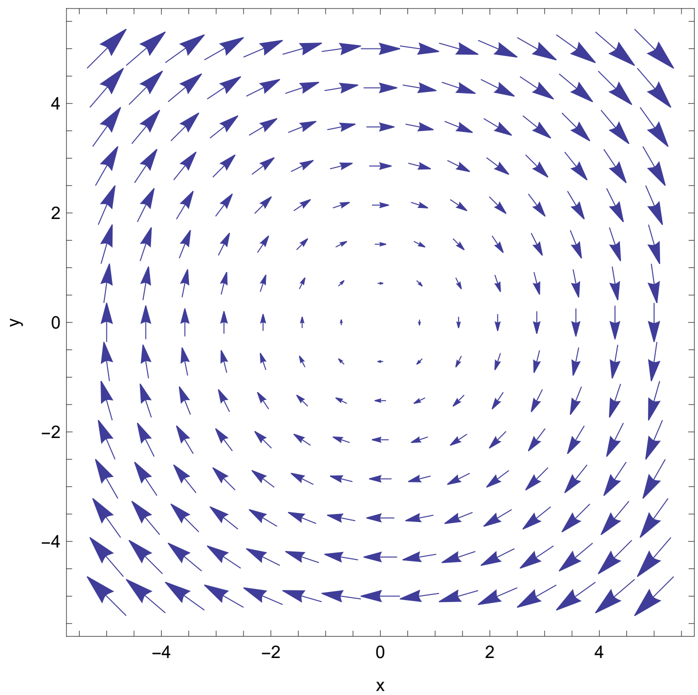
Example: Simple Vector Field

Consider the vector field

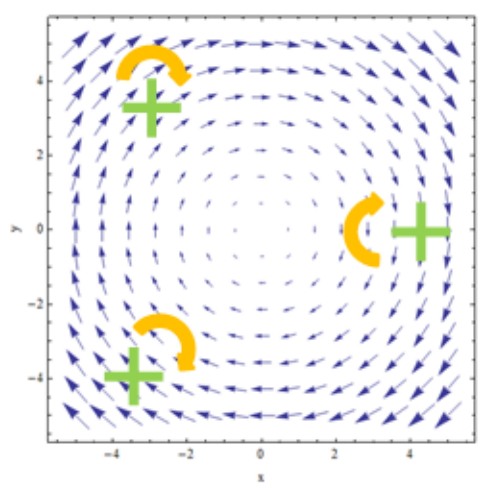
$$v(x, y, z) = y\,\hat{i} - x\,\hat{j} + 0\,\hat{k} = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix}$$

```
v[x_, y_, z_] = (y
                 -x
                 0);
```

```
VectorPlot[{v[x, y, z][[1, 1]], v[x, y, z][[2, 1]]}, {x, -5, 5}, {y, -5, 5},
 FrameLabel → {"x", "y"}]
```



By simple inspection, we see that if we place a ball or paddlewheel anywhere in the flow, it should have a tendency to turn clockwise (or into the page).



The question is, will all of these wheels spin at the same speed? To answer this, we can compute the curl of the vector field

$$\text{curl}\,\overline{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

```
Curl[{v[x, y, z][[1, 1]], v[x, y, z][[2, 1]], v[x, y, z][[3, 1]]}, {x, y, z}]
```

```
{0, 0, -2}
```

So we obtain

$$\text{curl}\,\overline{v} = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$$

which basically shows that the flow has the same amount of rotation at any point and as expected, this generates clock-wise behavior (according to the right hand rule).

```
Clear[v]
```

Example: More Complex Vector Field

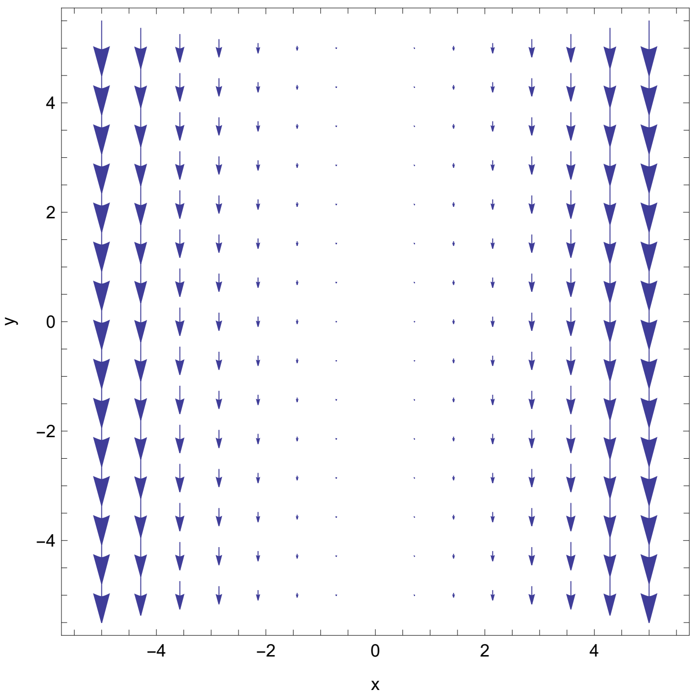
Consider the vector field

$$v(x, y, z) = 0\,\hat{i} - x^2\,\hat{j} + 0\,\hat{k} = \begin{pmatrix} 0 \\ -x^2 \\ 0 \end{pmatrix}$$

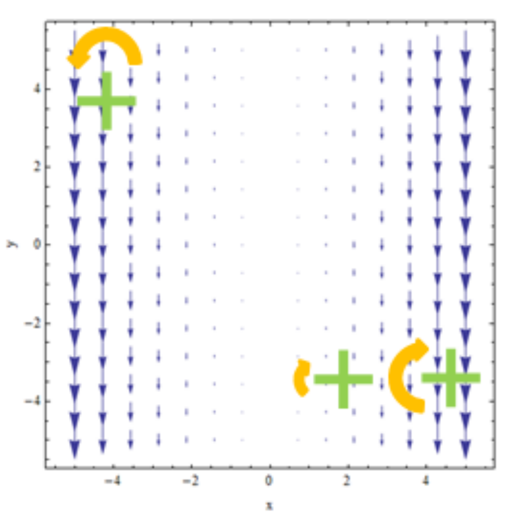
```
v[x_, y_, z_] = (0; -x^2; 0);
```

$$v[x\_, y\_, z\_] = \begin{pmatrix} 0 \\ -x^2 \\ 0 \end{pmatrix};$$

```
VectorPlot[{v[x, y, z]⟦1, 1⟧, v[x, y, z]⟦2, 1⟧}, {x, -5, 5}, {y, -5, 5},
 FrameLabel → {"x", "y"}]
```



By simple inspection, we see that if we place a ball or paddlewheel anywhere in the flow, it should have a tendency to turn but which direction is dependent on the location



The question is, will all of these wheels spin at the same speed? To answer this, we can compute the curl of the vector field.

$$\text{curl}\,\overline{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

```
Curl[{v[x, y, z][[1, 1]], v[x, y, z][[2, 1]], v[x, y, z][[3, 1]]}, {x, y, z}]
```

```
{0, 0, -2 x}
```

So we obtain

$$\text{curl}\,\overline{v} = \begin{pmatrix} 0 \\ 0 \\ -2\,x \end{pmatrix}$$

which basically shows that the rotation is a function of the x position.

```
Clear[v]
```

Relationship to Potential Functions

We note that for any twice continuously differentiable scalar function f

$$\text{curl}\,(\text{grad}\,f) = \overline{0} \qquad \textbf{\textit{(Eq.3)}}$$

We can verify this by direction calculation

$$\text{curl}(\text{grad}\,f) = \text{curl}\left(\frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j} + \frac{\partial f}{\partial z}\,\hat{k}\right)$$

$$= \left(\frac{\partial}{\partial y}\frac{\partial f}{\partial z} - \frac{\partial}{\partial z}\frac{\partial f}{\partial y}\right)\hat{i} + \left(\frac{\partial}{\partial z}\frac{\partial f}{\partial x} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z}\right)\hat{j} + \left(\frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x}\right)\hat{k}$$ recall: by symmetry of second derivatives $\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right) = \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}\right)$

$$= 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k}$$

$$\text{curl}\,(\text{grad}\,f) = \overline{0}$$

Let us consider some implications of this. For example, suppose we have a vector field g. We do not know if this is irrotational or not because it is entirely feasible that curl(g) $\neq \overline{0}$. However, what if g is the gradient of a potential function, f?

$$g = \text{grad}\,f$$

In this case, applying the above discussion yields the fact that

$$\text{curl}(g) = \text{curl}(\text{grad}\,f) = 0$$

Hence if a vector function is the gradient of a scalar function, its curl is the zero vector. As we saw in the previous example, the curl characterizes the rotation in a field. Combining this, we say that gradient fields describing a motion are irrotational. In other words, any vector field that is obtained as the gradient of a potential function is irrotational.

Summary of Operators

It may be useful to summarize all of these functions and operators

Dot Product

$$< \overline{u}, \overline{v} > = \overline{u} \cdot \overline{v} = |\overline{u}||\overline{v}| \cos(\theta) = \sum_{i=1}^{n} u_i v_i$$

Input: two vectors
Output: scalar

Scalar Projection

$$\text{comp}_{\overline{v}} \overline{u} = |\overline{u}| \cos(\theta) = \frac{<\overline{u}, \overline{v}>}{|\overline{v}|}$$

Input: two vectors
Output: scalar

Vector Projection

$$\text{proj}_{\overline{v}} \overline{u} = \frac{|\overline{u}| \cos(\theta)}{|\overline{v}|} \overline{v} = \frac{<\overline{u}, \overline{v}>}{|\overline{v}|^2} \overline{v}$$

Input: two vectors
Output: vector

Cross Product

$$\overline{u} \times \overline{v} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

Input: two vectors
Output: vector

Gradient

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

Input: scalar $f(x, y, z)$
Output: vector

Directional Derivative

$$D_{\overline{b}} f = \frac{\overline{b}}{|\overline{b}|} \cdot \nabla f$$

Input: scalar $f(x, y, z)$ and a vector $\overline{b}$

Output: scalar

Laplacian

$$\nabla^2 f = \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Input: scalar $f(x, y, z)$
Output: scalar

Divergence

$$\text{div}\, \overline{v} = \nabla \cdot \overline{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Input: vector
Output: scalar

Curl

$$\text{curl}\, \overline{v} = \nabla \times \overline{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

Input: vector
Output: vector

A3.4: Grad, Div, Curl, ∇^2 in Curvilinear Coordinates

First present lecture04c_cylindrical_spherical_coordinates.nb

Note: A helpful Wikipedia link for some of these identities
http://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates
WARNING: The Wikipedia article may use different definitions of θ and ϕ.

The following definitions use notation that is consistent with the textbook.

We can transform the gradient, divergence, and curl operators to work in cylindrical and spherical coordinates (see text for derivations)

Gradient

For the gradient, assuming that we have a scalar function in cylindrical coordinates, $f(r, \theta, z)$, the gradient can be computed using

$$\text{grad}\, f = \nabla f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{\partial f}{\partial z}\hat{e}_z \qquad \text{(cylindrical coordinates)}$$

Similarly, if we have a scalar function in spherical coordinates, $f(r, \theta, \phi)$, the gradient can be computed using

$$\text{grad}\, f = \nabla f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r\sin(\phi)}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{e}_\phi \qquad \text{(spherical coordinates)}$$

Divergence

For divergence, assuming we have a vector function in cylindrical coordinates, $\overline{F} = F_r\,\hat{e}_r + F_\theta\,\hat{e}_\theta + F_z\,\hat{e}_z$, the divergence can be computed using

$$\text{div}\,\overline{F} = \nabla \cdot \overline{F} = \frac{1}{r}\frac{\partial}{\partial r}[r\,F_r] + \frac{1}{r}\frac{\partial}{\partial \theta}[F_\theta] + \frac{\partial}{\partial z}[F_z] \qquad \text{(cylindrical coordinates)}$$

Similarly, if we have a vector function in spherical coordinates, $\overline{F} = F_r\,\hat{e}_r + F_\theta\,\hat{e}_\theta + F_\phi\,\hat{e}_\phi$, the divergence can be computed using

$$\text{div}\,\overline{F} = \nabla \cdot \overline{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\,F_r\right] + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \theta}[F_\theta] + \frac{1}{r\sin(\phi)}\frac{\partial}{\partial \phi}\left[\sin(\phi)\,F_\phi\right] \quad \text{(spherical coordinates)}$$

Laplacian

Recall that in Cartesian coordinates, the Laplacian is given as $\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

So for the Laplace operator assuming that we have a scalar function in cylindrical coordinates, $f(r,\ \theta,\ z)$, the Laplacian can be computed using

$$\nabla^2 f = \Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial f}{\partial r}\right] + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

Alternatively, if we notice that $\frac{\partial}{\partial r}\left[r\frac{\partial f}{\partial r}\right] = \frac{\partial f}{\partial r} + r\frac{\partial^2 f}{\partial r^2}$, we can write this as

$$\nabla^2 f = \Delta f = \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \qquad \text{(cylindrical coordinates)}$$

Similarly, if we have a scalar function in spherical coordinates, $f(r,\ \theta,\ \phi)$, the Laplacian can be computed using

$$\nabla^2 f = \Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin^2(\phi)}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2\sin(\phi)}\frac{\partial}{\partial \phi}\left(\sin(\phi)\frac{\partial f}{\partial \phi}\right)$$

Alternatively, by performing some of the differentiations using the product rule, we can alternatively write this as

$$\nabla^2 f = \Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2\sin^2(\phi)}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\cot(\phi)}{r^2}\frac{\partial f}{\partial \phi} \qquad \text{(spherical coordinates)}$$

Curl

For curl $\overline{F} = \nabla \times \overline{F}$, assuming we have a vector function in cylindrical coordinates, $\overline{F} = F_r\,\hat{e}_r + F_\theta\,\hat{e}_\theta + F_z\,\hat{e}_z$, the curl can be computed using

$$\nabla \times \overline{F} = \frac{1}{r} \begin{vmatrix} \hat{e}_r & r\,\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & r\,F_\theta & F_z \end{vmatrix}$$

$$= \left(\frac{1}{r}\frac{\partial F_z}{\partial \theta} - \frac{\partial F_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\hat{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}[r\,F_\theta] - \frac{\partial F_r}{\partial \theta}\right)\hat{e}_z \quad \text{(cylindrical coordinates)}$$

Similarly, if we have a vector function in spherical coordinates, $\overline{F} = F_r\,\hat{e}_r + F_\theta\,\hat{e}_\theta + F_\phi\,\hat{e}_\phi$, the curl can be computed using

$$\nabla \times \overline{F} = \frac{1}{r^2 \sin(\phi)} \begin{vmatrix} \hat{e}_r & r\,\hat{e}_\phi & r\sin(\phi)\,\hat{e}_\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial \theta} \\ F_r & r\,F_\phi & r\sin(\phi)\,F_\theta \end{vmatrix}$$

$$= \frac{1}{r\sin(\phi)}\left(\frac{\partial}{\partial\phi}[F_\theta \sin(\phi)] - \frac{\partial F_\phi}{\partial \theta}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\,F_\phi - \frac{\partial F_r}{\partial \phi}\right)\right)\hat{e}_\theta + \frac{1}{r}\left(\frac{1}{\sin(\phi)}\frac{\partial F_r}{\partial \theta} - \frac{\partial}{\partial r}[r\,F_\theta]\right)\hat{e}_\phi \quad \text{(spherical coordinates)}$$