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Lecture01e
Elementary Row Operations, Row Echelon Form, and Reduced Row Echelon Form

Lecture is on YouTube

The YouTube video entitled ‘Elementary Row Operations, Row Echelon Form, and Reduced Row Echelon Form’ that covers this lecture is located at https://youtu.be/AxgzzJposVo

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Elementary Row Operations

There are 3 elementary row operations that can be performed on matrices

1. Row Switching

A row within a matrix can be switch with another row.

$$\bar{r}_i \longleftrightarrow \bar{r}_j$$

2. Row Multiplication

An entire row can be multiplied by a non-zero constant. This is also referred to as scaling a row.

$$k\,\bar{r}_i \to \bar{r}_i \qquad k \neq 0$$

3. Row Addition

A row can be replaced by the sum of that row and a multiple of another row.

$$\bar{r}_i + k\,\bar{r}_j \to \bar{r}_i \qquad i \neq j$$

Typically, the these operations are used to transform a matrix into an upper triangular matrix.

These operations are used to reduce a matrix to row echelon and reduced-row echelon form. We will later see that this is part of Gaussian elimination and closely related Gauss-Jordan elimination, respectively.

While these 3 operations can be done manually to a matrix, they can also be performed by generating an elementary matrix, E, and applying this to the left side of the matrix in question.

Relation to Systems of Linear Equations

Some insight into these elementary row operations can be obtained by considering a system of linear equations.

$$x_1 + 2\,x_2 - 2\,x_3 = 4$$
$$2\,x_1 + 3\,x_2 + x_3 = 3$$
$$3\,x_1 - x_3 = 2$$

In order to serve as a point of reference, we can solve this system of equations

In[◦]:=
```
Solve[{x1 + 2 x2 - 2 x3 == 4, 2 x1 + 3 x2 + x3 == 3, 3 x1 - x3 == 2}, {x1, x2, x3}]
```

Out[◦]= $\left\{\left\{x1 \to \frac{2}{5},\ x2 \to 1,\ x3 \to -\frac{4}{5}\right\}\right\}$

We will check this result once we perform some elementary row operations to verify that the operations do not change the nature/solution of the problem.

We can write this in matrix form as

$$\begin{pmatrix} 1 & 2 & -2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix} \overline{x} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$

$$A\,\overline{x} = \overline{b}$$

where $\quad A = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix} \qquad \overline{b} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$

In[◦]:= $\mathbf{A} = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 3 & 1 \\ 3 & 0 & -1 \end{pmatrix};$

$\mathbf{b} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$

Out[◦]= {{4}, {3}, {2}}

The equivalent augmented matrix is

$$\tilde{A} = \begin{pmatrix} A & \overline{b} \end{pmatrix} = \begin{pmatrix} 1 & 2 & -2 & 4 \\ 2 & 3 & 1 & 3 \\ 3 & 0 & -1 & 2 \end{pmatrix}$$

In[ ]:=
```
Atilde = Transpose[Join[Transpose[A], Transpose[b]]];
Atilde // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & -2 & 4 \\ 2 & 3 & 1 & 3 \\ 3 & 0 & -1 & 2 \end{pmatrix}$$

$$\begin{matrix} x_1 + 2x_2 - 2x_3 = 4 \\ 2x_1 + 3x_2 + x_3 = 3 \\ 3x_1 - x_3 = 2 \end{matrix} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 2 & 3 & 1 & 3 \\ 3 & 0 & -1 & 2 \end{array}\right)$$

We now consider the implications of performing the 3 elementary row operations on this system of equations

1. Row Switching

We see that by switching rows of the *A* matrix is equivalent to simply switching the order of the original equations and therefore does not change the nature of the system.

Example: Switch row 1 and 3

The operation of $\bar{r}_1 \longleftrightarrow \bar{r}_3$ simply changes the order of equation 1 and equation 3.

$$\begin{matrix} x_1 + 2x_2 - 2x_3 = 4 \\ 2x_1 + 3x_2 + x_3 = 3 \\ 3x_1 - x_3 = 2 \end{matrix} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 2 & 3 & 1 & 3 \\ 3 & 0 & -1 & 2 \end{array}\right)$$

$$\Big\downarrow \; \bar{r}_1 \leftrightarrow \bar{r}_3$$

$$\begin{matrix} 3x_1 - x_3 = 2 \\ 2x_1 + 3x_2 + x_3 = 3 \\ x_1 + 2x_2 - 2x_3 = 4 \end{matrix} \Leftrightarrow \left(\begin{array}{ccc|c} 3 & 0 & -1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & 2 & -2 & 4 \end{array}\right)$$

2. Row Multiplication

We see that multiplying a row of the *A* matrix is equivalent to simply multiplying a single original equations and therefore does not change the nature of the system.

Example: Scale row 2 by *k*

The operation of $k\,\bar{r}_2 \to \bar{r}_2$ simply multiplies equation 2 by k.

$$k(2\,x_1 + 3\,x_2 + x_3) = 3\,k$$

$$2\,k\,x_1 + 3\,k\,x_2 + k\,x_3 = 3\,k$$

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 4 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ 3x_1 - x_3 &= 2 \end{aligned} \quad \Leftrightarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 2 & 3 & 1 & 3 \\ 3 & 0 & -1 & 2 \end{array}\right)$$

$$k\bar{r}_2 \leftrightarrow \bar{r}_2$$

$$\begin{aligned} x_1 + 2x_2 - 2x_3 &= 4 \\ 2k\,x_1 + 3kx_2 + kx_3 &= 3k \\ 3x_1 - x_3 &= 2 \end{aligned} \quad \Leftrightarrow \quad \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 2k & 3k & k & 3k \\ 3 & 0 & -1 & 2 \end{array}\right)$$

3. Row Addition

Example: Replace row 2 with sum of row 2 and *k* times row 1

For row addition, let us consider replacing row 2 with the sum of row 2 with a multiple of row 1

$$\bar{r}_2 + k\,\bar{r}_1 \to \bar{r}_2$$

Consider the original equation for row 2

$$2\,x_1 + 3\,x_2 + x_3 = 3 \qquad \textbf{(Eq.3.1)}$$

We can add a constant to both sides

$$2\,x_1 + 3\,x_2 + x_3 + \alpha = 3 + \alpha \qquad \textbf{(Eq.3.2)}$$

Now consider scaling row 1 by k

$$k(x_1 + 2\,x_2 - 2\,x_3) = 4\,k$$

$$k\,x_1 + 2\,k\,x_2 - 2\,k\,x_3 = 4\,k \qquad \textbf{(Eq.3.3)}$$

If we let $\alpha = 4\,k$ we can write Eq.3.2 as

$$2\,x_1 + 3\,x_2 + x_3 + 4\,k = 3 + 4\,k$$

Recall from Eq.3.3 that $k\, x_1 + 2\, k\, x_2 - 2\, k\, x_3 = 4\, k$ so we can replace the $4\, k$ term on the left with $k\, x_1 + 2\, k\, x_2 - 2\, k\, x_3$

$$2\, x_1 + 3\, x_2 + x_3 + k\, x_1 + 2\, k\, x_2 - 2\, k\, x_3 = 3 + 4\, k$$

$$(2 + k)\, x_1 + (3 + 2\, k)\, x_2 + (1 - 2\, k)\, x_3 = 3 + 4\, k$$

$$\begin{matrix} x_1 + 2x_2 - 2x_3 = 4 \\ 2x_1 + 3x_2 + x_3 = 3 \\ 3x_1 - x_3 = 2 \end{matrix} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 2 & 3 & 1 & 3 \\ 3 & 0 & -1 & 2 \end{array}\right)$$

$$\Big\downarrow \; \bar{r}_2 + k\bar{r}_1 \rightarrow \bar{r}_2$$

$$\begin{matrix} x_1 + 2x_2 - 2x_3 = 4 \\ (2+k)x_1 + (3+2k)x_2 + (1-2k)x_3 = 3 + 4\,k \\ 3x_1 - x_3 = 2 \end{matrix} \Leftrightarrow \left(\begin{array}{ccc|c} 1 & 2 & -2 & 4 \\ 2+k & 3+2k & 1-2k & 3+4k \\ 3 & 0 & -1 & 2 \end{array}\right)$$

We see that replacing a row with the sum of that row and a multiple of another row is simply manipulating a single original equation and therefore does not change the nature of the system.

In summary, all 3 elementary row operations are simply manipulating the original system of equations and therefore does not change the fundamental nature of the system of equations. This means the solution of a system remains unchanged and we can therefore analyze either the original set of equations (the original augmented matrix) or the system of equations after it has been operated on by the elementary row operations (a modified augmented matrix). This is the foundation of the Gaussian Elimination method and leads to a discussion on row echelon form.

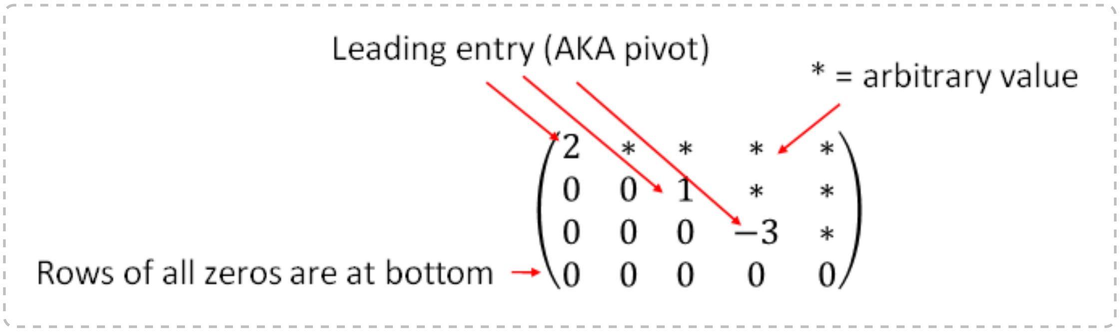
In later lectures, we will see that elementary row operations preserve the row space of the matrix and rank of the matrix.

Row Echelon Form

A matrix is in **row echelon form** when we perform elementary row operations and obtain a matrix that has the following characteristics:

1. All rows consisting of only zeros are at the bottom.
2. The leading entry (the left-most nonzero entry) of every nonzero row is to the right of the leading entry of every row above.

The corresponds to a roughly upper triangular matrix. An example of a matrix in row echelon form is shown below



We transform a matrix to row echelon form through the use of elementary row operations (AKA Gaussian Elimination).

The general idea for the algorithm is:

1. If necessary, switch rows so there is a non-zero value in the top left entry.
2. Start at leftmost column and perform elementary row operations to reduce all entries below the previous pivot to zero.
3. Move over to the right by 1 column and repeat.

Example: Row Echelon Form

Consider the example augmented matrix above. To keep track of iterations, we refer to this as the initial matrix, A_0

$$A_0 = \begin{pmatrix} 1 & 2 & -2 & 4 \\ 2 & 3 & 1 & 3 \\ 3 & 0 & -1 & 2 \end{pmatrix}$$

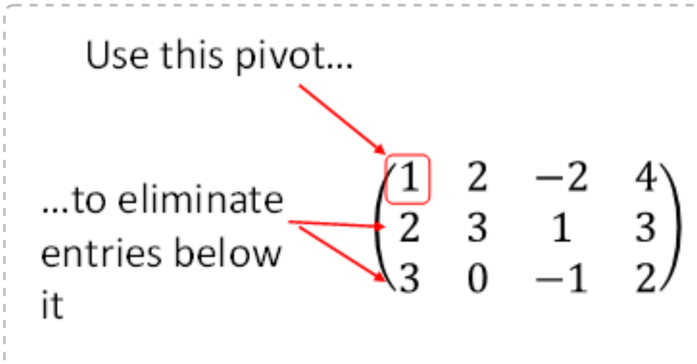
In[ ]:=
```
A0 = Atilde;
```

We will also perform the same operations on a 3x3 identify matrix in order to obtain the matrix equivalent of these elementary operations

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Pivot Column 1

Since there is a non-zero value in the top-left entry, we use this entry as the pivot and start eliminating entries below it



Step 1: $\bar{r}_3 + (-3)\,\bar{r}_1 \to \bar{r}_3$

While it would be standard practice to first manipulate row 2 instead of row 3, we first manipulate row 3 to illustrate the flexibility of the process and allow us the opportunity to exercise more of the operations.

$$A_1 = \begin{pmatrix} 1 & 2 & -2 & 4 \\ 2 & 3 & 1 & 3 \\ 3 + (-3*1) & 0 + (-3*2) & -1 + (-3*-2) & 2 + (-3*4) \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 1 & 2 & -2 & 4 \\ 2 & 3 & 1 & 3 \\ 0 & -6 & 5 & -10 \end{pmatrix}$$

Performing the same operation on E yields

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 + (-3*1) & 0 + (-3*0) & 1 + (-3*0) \end{pmatrix}$$

$$E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}$$

In[ ]:=
```
E1 = {{1, 0, 0}, {0, 1, 0}, {-3, 0, 1}};
```

We can now verify that

$$A_1 = E_1\,A_0$$

In[ ]:=
```
A1 = E1.A0;
A1 // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & -2 & 4 \\ 2 & 3 & 1 & 3 \\ 0 & -6 & 5 & -10 \end{pmatrix}$$

Step 2: $\bar{r}_2 \longleftrightarrow \bar{r}_3$

We can switch row 2 and 3

$$A_2 = \begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & -6 & 5 & -10 \\ 2 & 3 & 1 & 3 \end{pmatrix}$$

Performing the same operation on E yields

$$E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

In[∘]:=
```
E2 = (1 0 0
      0 0 1
      0 1 0);
```

We can now verify that

$$A_2 = E_2 A_1$$

In[∘]:=
```
A2 = E2.A1;
A2 // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & -6 & 5 & -10 \\ 2 & 3 & 1 & 3 \end{pmatrix}$$

Step 3: $\bar{r}_3 + (-2)\,\bar{r}_1 \to \bar{r}_3$

$$A_3 = \begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & -6 & 5 & -10 \\ 2+(-2*1) & 3+(-2*2) & 1+(-2*-2) & 3+(-2*4) \end{pmatrix}$$

$$A_3 = \begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & -6 & 5 & -10 \\ 0 & -1 & 5 & -5 \end{pmatrix}$$

Performing the same operation on E yields

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0+(-2*1) & 0+(-2*0) & 1+(-2*0) \end{pmatrix}$$

$$E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

In[∘]:=
```
E3 = ( 1 0 0
       0 1 0
      -2 0 1);
```

We can now verify that

$$A_3 = E_3\,A_2$$

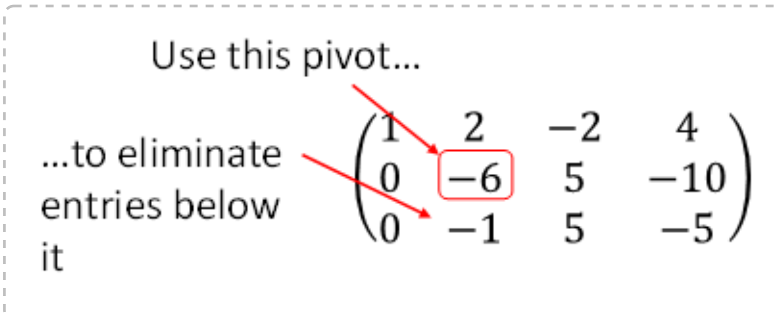
```
In[•]:= A3 = E3.A2;
        A3 // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & -6 & 5 & -10 \\ 0 & -1 & 5 & -5 \end{pmatrix}$$

Pivot Column 2

Now that all the entries below pivot 1 are zero, we move to the next column and repeat the process



Step 4: $-6\,\overline{r}_3 \to \overline{r}_3$

Again, for practice we can first scale the 3rd row (although this is not immediately necessary).

$$A_4 = \begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & -6 & 5 & -10 \\ (-6)*0 & (-6)*(-1) & (-6)*5 & (-6)*(-5) \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & -6 & 5 & -10 \\ 0 & 6 & -30 & 30 \end{pmatrix}$$

Performing the same operation on E yields

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (-6)*0 & (-6)*0 & (-6)*1 \end{pmatrix}$$

$$E_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

In[•]:= **E4 =** $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -6 \end{pmatrix}$**;**

We can now verify that

$$A_4 = E_4\, A_3$$

In[ ]:=
```
A4 = E4.A3;
A4 // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & -6 & 5 & -10 \\ 0 & 6 & -30 & 30 \end{pmatrix}$$

Step 5: $\bar{r}_3 + 1\,\bar{r}_2 \rightarrow \bar{r}_3$

$$A_5 = \begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & -6 & 5 & -10 \\ 0 + (1*0) & 6 + (1*-6) & -30 + (1*5) & 30 + (1*-10) \end{pmatrix}$$

$$A_5 = \begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & -6 & 5 & -10 \\ 0 & 0 & -25 & 20 \end{pmatrix}$$

Performing the same operation on E yields

$$E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 + (1*0) & 0 + (1*1) & 1 + (1*0) \end{pmatrix}$$

$$E_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

In[ ]:=
$$\mathbf{E5} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix};$$

We can now verify that

$$A_5 = E_5\, A_4$$

In[ ]:=
```
A5 = E5.A4;
A5 // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & -6 & 5 & -10 \\ 0 & 0 & -25 & 20 \end{pmatrix}$$

At this point, the matrix satisfies the aforementioned conditions and is said to be in row echelon form. This process of performing elementary row operations to get the matrix in row echelon form is also known as **Gaussian Elimination**

In[•]:=
```
A5Check = E5.E4.E3.E2.E1.A0;
A5Check == A5
```

Out[•]=
```
True
```

It should be noted that the row echelon form is not unique (you can easily see this by arbitrarily scaling any of the rows of A_5). Furthermore, note that this corresponds to a system of linear equations of the form

$$x_1 + 2\,x_2 - 2\,x_3 = 4$$
$$-6\,x_2 + 5\,x_3 = -10$$
$$-25\,x_3 = 20$$

The triangular nature of this allows for **back-substitution** (ie starting with the bottom equation and successively back substituting results).

It is also worth mentioning that the process of manipulating a matrix into row echelon form is also referred to as **Gaussian Elimination**.

We can solve this modified system of equations to verify we obtain the same result as before.

In[•]:=
```
Solve[{x1 + 2 x2 - 2 x3 == 4, -6 x2 + 5 x3 == -10, -25 x3 == 20}, {x1, x2, x3}]
```

Out[•]= $\left\{\left\{x1 \to \frac{2}{5},\ x2 \to 1,\ x3 \to -\frac{4}{5}\right\}\right\}$

We can continue elementary row operations to place them matrix in reduced row echelon form.

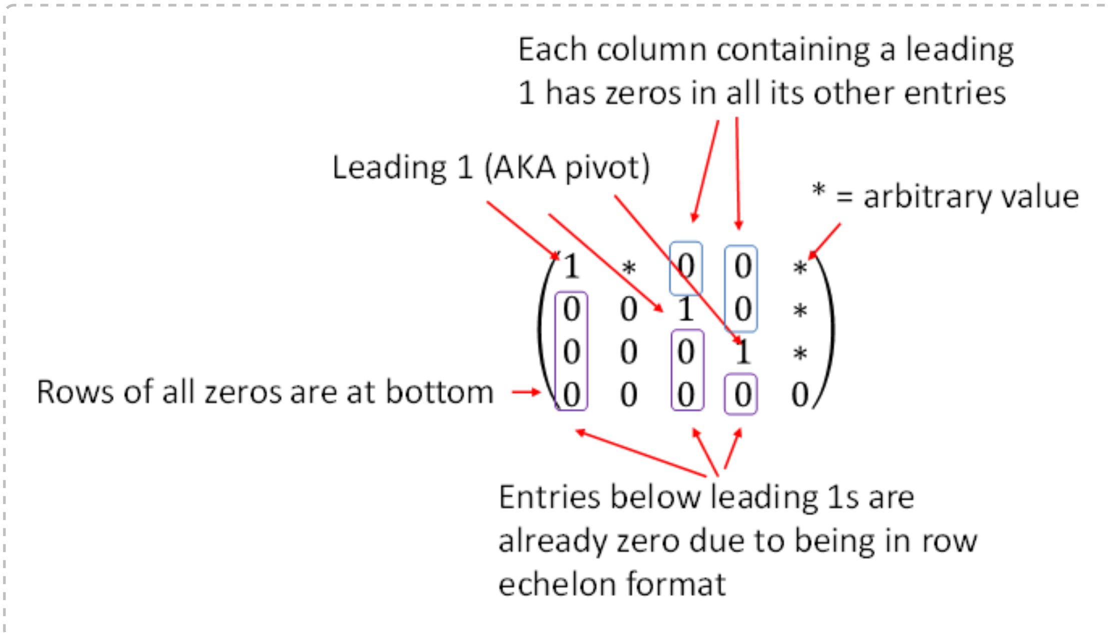
Reduced Row Echelon Form

We can continue elementary row operations to further manipulate the matrix into reduced row echelon format.

A matrix is in **reduced row echelon form** when we perform elementary row operations and obtain a matrix that has the following characteristics:

1. It is in row echelon form.
2. The leading entry in each nonzero row is a 1 (called a leading 1)
2. Each column containing a leading 1 has zeros in all its other entries

An example of a matrix in reduced row echelon form is shown below



We transform a matrix to reduced row echelon form through the continued use of elementary row operations.

The general idea for the algorithm is:

1. Place the matrix in row echelon form.
2. Scale each row so each pivot equals 1 (we can now refer to this as a leading 1)
3. Use each leading 1 to eliminate the nonzero entries above it (entries below should already be 0 as it is in row echelon form)

Example: Reduced Row Echelon Form

We can continue to use the same example as above (recall that we already placed it in row echelon form) as

$$A_5 = \begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & -6 & 5 & -10 \\ 0 & 0 & -25 & 20 \end{pmatrix}$$

We can scale each row so each pivot equals 1

Step 6: $\left(-\frac{1}{6}\right)\bar{r}_2 \to \bar{r}_2$

$$A_6 = \begin{pmatrix} 1 & 2 & -2 & 4 \\ \left(-\frac{1}{6}\right)*0 & \left(-\frac{1}{6}\right)*-6 & \left(-\frac{1}{6}\right)*5 & \left(-\frac{1}{6}\right)*-10 \\ 0 & 0 & -25 & 20 \end{pmatrix}$$

$$A_6 = \begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & 1 & -5/6 & 5/3 \\ 0 & 0 & -25 & 20 \end{pmatrix}$$

Performing the same operation on E yields

$$E_6 = \begin{pmatrix} 1 & 0 & 0 \\ \left(-\frac{1}{6}\right)*0 & \left(-\frac{1}{6}\right)*1 & \left(-\frac{1}{6}\right)*0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/6 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[ ]:=
```
E6 = {{1, 0, 0}, {0, -1/6, 0}, {0, 0, 1}};
```

We can now verify that

$$A_6 = E_6\, A_5$$

In[ ]:=
```
A6 = E6.A5;
A6 // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & 1 & -\frac{5}{6} & \frac{5}{3} \\ 0 & 0 & -25 & 20 \end{pmatrix}$$

Step 7: $\left(-\frac{1}{25}\right)\bar{r}_3 \to \bar{r}_3$

$$A_7 = \begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & 1 & -5/6 & 5/3 \\ \left(-\frac{1}{25}\right)*0 & \left(-\frac{1}{25}\right)*0 & \left(-\frac{1}{25}\right)*-25 & \left(-\frac{1}{25}\right)*20 \end{pmatrix}$$

$$A_7 = \begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & 1 & -5/6 & 5/3 \\ 0 & 0 & 1 & -4/5 \end{pmatrix}$$

Performing the same operation on E yields

$$E_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \left(-\frac{1}{25}\right)*0 & \left(-\frac{1}{25}\right)*0 & \left(-\frac{1}{25}\right)*1 \end{pmatrix}$$

$$E_7 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/25 \end{pmatrix}$$

In[ ]:=
```
E7 = {{1, 0, 0}, {0, 1, 0}, {0, 0, -1/25}};
```

We can now verify that

$$A_7 = E_7\,A_6$$

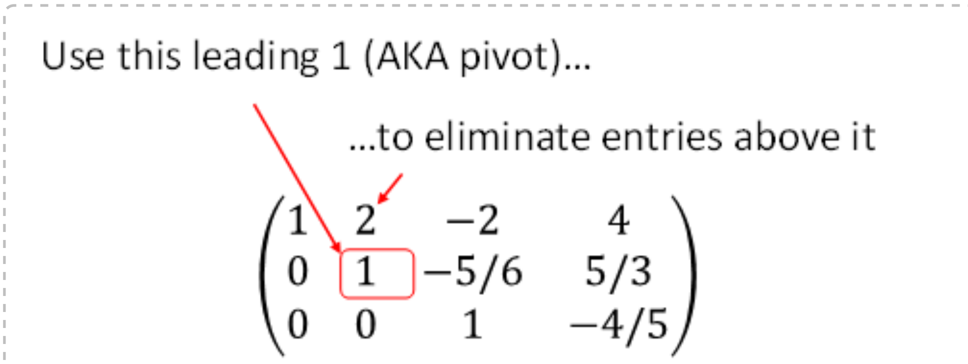
In[∘]:=
```
A7 = E7.A6;
A7 // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & -2 & 4 \\ 0 & 1 & -\frac{5}{6} & \frac{5}{3} \\ 0 & 0 & 1 & -\frac{4}{5} \end{pmatrix}$$

Pivot Column 2

We can now the leading 1 in column 2 to eliminate non-zero entries above it (the entries below it are already zero as the matrix is in row echelon format).



Step 8: $\bar{r}_1 + (-2)\,\bar{r}_2 \to \bar{r}_1$

$$A_8 = \begin{pmatrix} 1+(-2)*0 & 2+(-2)*1 & -2+(-2)*(-5/6) & 4+(-2)*(5/3) \\ 0 & 1 & -5/6 & 5/3 \\ 0 & 0 & 1 & -4/5 \end{pmatrix}$$

$$A_8 = \begin{pmatrix} 1 & 0 & -1/3 & 2/3 \\ 0 & 1 & -5/6 & 5/3 \\ 0 & 0 & 1 & -4/5 \end{pmatrix}$$

Performing the same operation on E yields

$$E_8 = \begin{pmatrix} 1+(-2)*0 & 0+(-2)*1 & 0+(-2)*0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_8 = \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[∘]:=
```
E8 = (1  -2  0
      0   1  0
      0   0  1);
```

We can now verify that

$$A_8 = E_8\, A_7$$

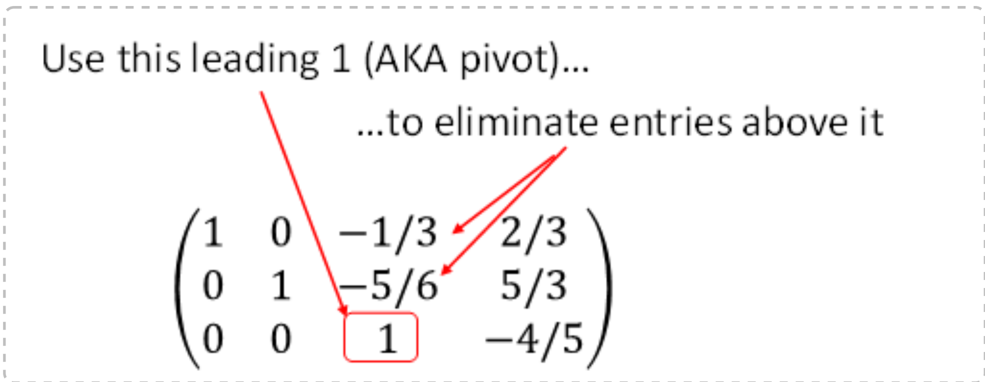
In[ ]:=
```
A8 = E8.A7;
A8 // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & -\frac{5}{6} & \frac{5}{3} \\ 0 & 0 & 1 & -\frac{4}{5} \end{pmatrix}$$

Pivot Column 3

We can now the leading 1 in column 3 to eliminate non-zero entries above it (the entries below it are already zero as the matrix is in row echelon format).



Step 9: $\bar{r}_2 + \left(\frac{5}{6}\right)\bar{r}_3 \to \bar{r}_2$

$$A_9 = \begin{pmatrix} 1 & 0 & -1/3 & 2/3 \\ 0 + \left(\frac{5}{6} * 0\right) & 1 + \left(\frac{5}{6} * 0\right) & -5/6 + \left(\frac{5}{6} * 1\right) & 5/3 + \left(\frac{5}{6} * \frac{-4}{5}\right) \\ 0 & 0 & 1 & -4/5 \end{pmatrix}$$

$$A_9 = \begin{pmatrix} 1 & 0 & -1/3 & 2/3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -4/5 \end{pmatrix}$$

Performing the same operation on E yields

$$E_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 + \left(\frac{5}{6} * 0\right) & 1 + \left(\frac{5}{6} * 0\right) & 0 + \left(\frac{5}{6} * 1\right) \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 5/6 \\ 0 & 0 & 1 \end{pmatrix}$$

In[ ]:=
```
E9 = (1 0 0
      0 1 5/6
      0 0 1);
```

We can now verify that

$$A_9 = E_9\, A_8$$

In[ ]:=
```
A9 = E9.A8;
A9 // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{4}{5} \end{pmatrix}$$

Step 10: $\bar{r}_1 + \left(\frac{1}{3}\right)\bar{r}_3 \to \bar{r}_1$

$$A_{10} = \begin{pmatrix} 1+\left(\frac{1}{3}*0\right) & 0+\left(\frac{1}{3}*0\right) & -1/3+\left(\frac{1}{3}*1\right) & 2/3+\left(\frac{1}{3}*-\frac{4}{5}\right) \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -4/5 \end{pmatrix}$$

$$A_{10} = \begin{pmatrix} 1 & 0 & 0 & 2/5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -4/5 \end{pmatrix}$$

Performing the same operation on E yields

$$E_{10} = \begin{pmatrix} 1+\left(\frac{1}{3}*0\right) & 0+\left(\frac{1}{3}*0\right) & 0+\left(\frac{1}{3}*1\right) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_{10} = \begin{pmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In[ ]:=
```
E10 = (1 0 1/3
       0 1 0
       0 0 1);
```

We can now verify that

$$A_{10} = E_{10}\, A_9$$

In[∘]:=
```
A10 = E10.A9;
A10 // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & \frac{2}{5} \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -\frac{4}{5} \end{pmatrix}$$

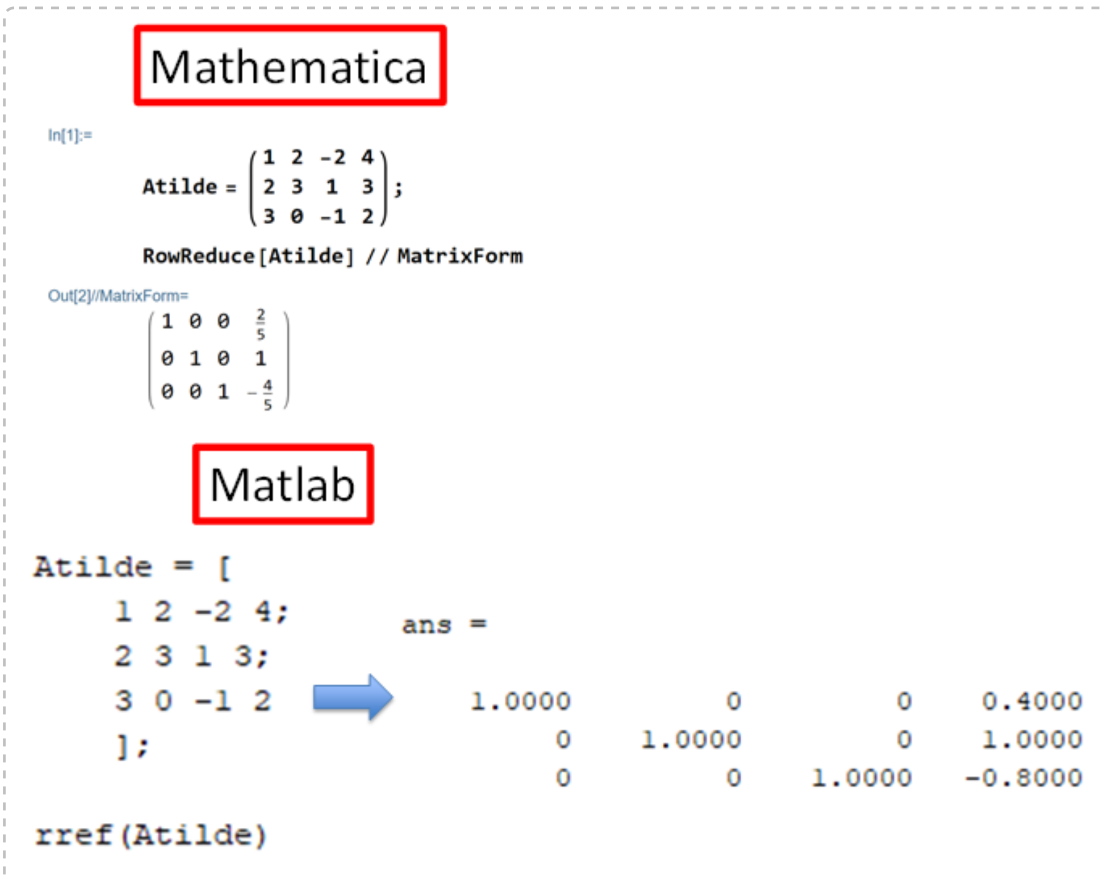
We have now placed them matrix in reduced row echelon form. Unlike row echelon form, the reduced row echelon form is unique.

We can verify using Mathematica's 'RowReduce' function (this performs operation to place the matrix in reduced row echelon form)

Matlab has the equivalent function 'rref' to perform this operation

In[∘]:=
```
RowReduce[Atilde] == A10
```

Out[∘]= True

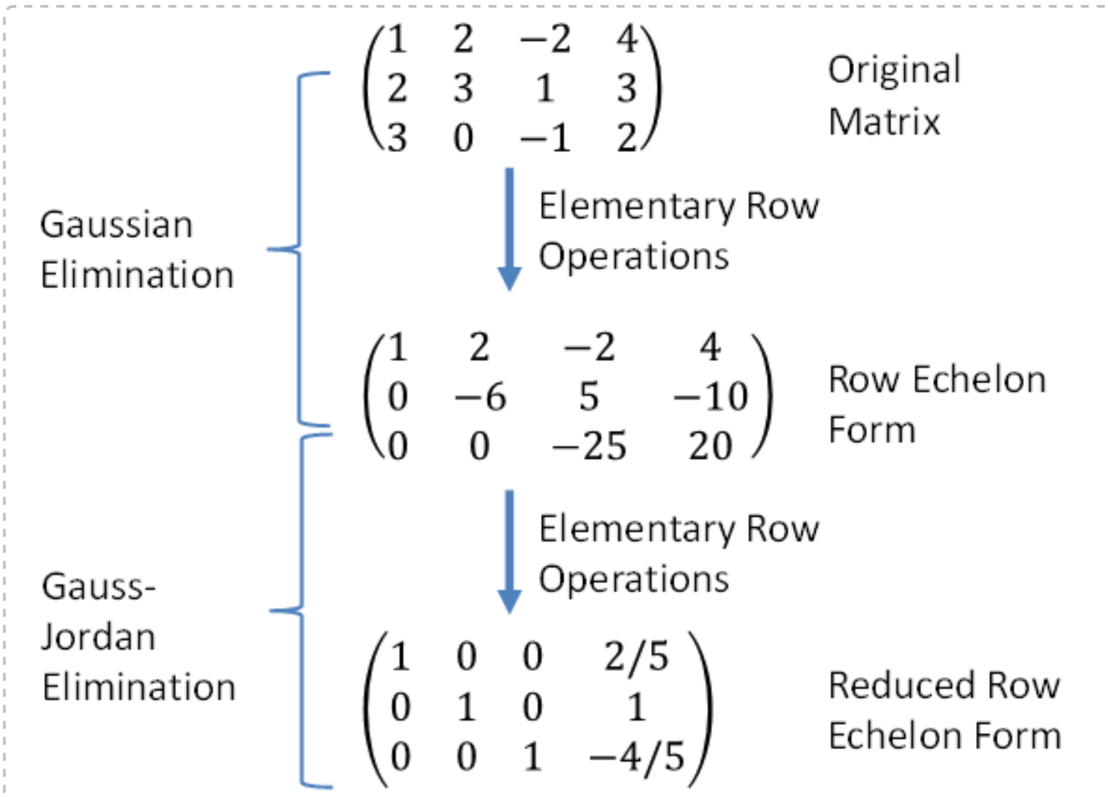


This format is useful as it allows us to solve the original system of equations. We can directly see that A_{10} can be rewritten as

$x_1 = 2/5$
$x_2 = 1$
$x_3 = -4/5$

This is the same solution we obtained previously.

This additional process to place the matrix in reduced row echelon format is sometimes referred to as **Gauss-Jordan Elimination**.



Row echelon form and reduced row echelon form will be useful in future discussions on systems of linear equations, row/column space, and other linear algebra discussions.

Calculating the Inverse of the Matrix

In addition, we note that the left portion of A_{10} is an identify matrix. Furthermore, the sequence of $E_{10}\, E_9 \ldots E_1$ captures the sequence of elementary row operations performed. In the example above we applied this sequence to $\tilde{A}$ but if we instead apply this to just A we can write

$$I = E_{10}\, E_9\, E_8\, E_7\, E_6\, E_5\, E_4\, E_3\, E_2\, E_1\, A$$

In[ ]:=
```
E10.E9.E8.E7.E6.E5.E4.E3.E2.E1.A // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

In other words, the sequence $E_{10}\, E_9\, E_8\, E_7\, E_6\, E_5\, E_4\, E_3\, E_2\, E_1$ is actually the inverse of A

$$A^{-1} = E_{10}\, E_9\, E_8\, E_7\, E_6\, E_5\, E_4\, E_3\, E_2\, E_1$$

In[ ]:=
```
Ainv = E10.E9.E8.E7.E6.E5.E4.E3.E2.E1;
Ainv // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\frac{3}{25} & \frac{2}{25} & \frac{8}{25} \\ \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{9}{25} & \frac{6}{25} & -\frac{1}{25} \end{pmatrix}$$

In[ ]:=
```
Ainv.A // MatrixForm
A.Ainv // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

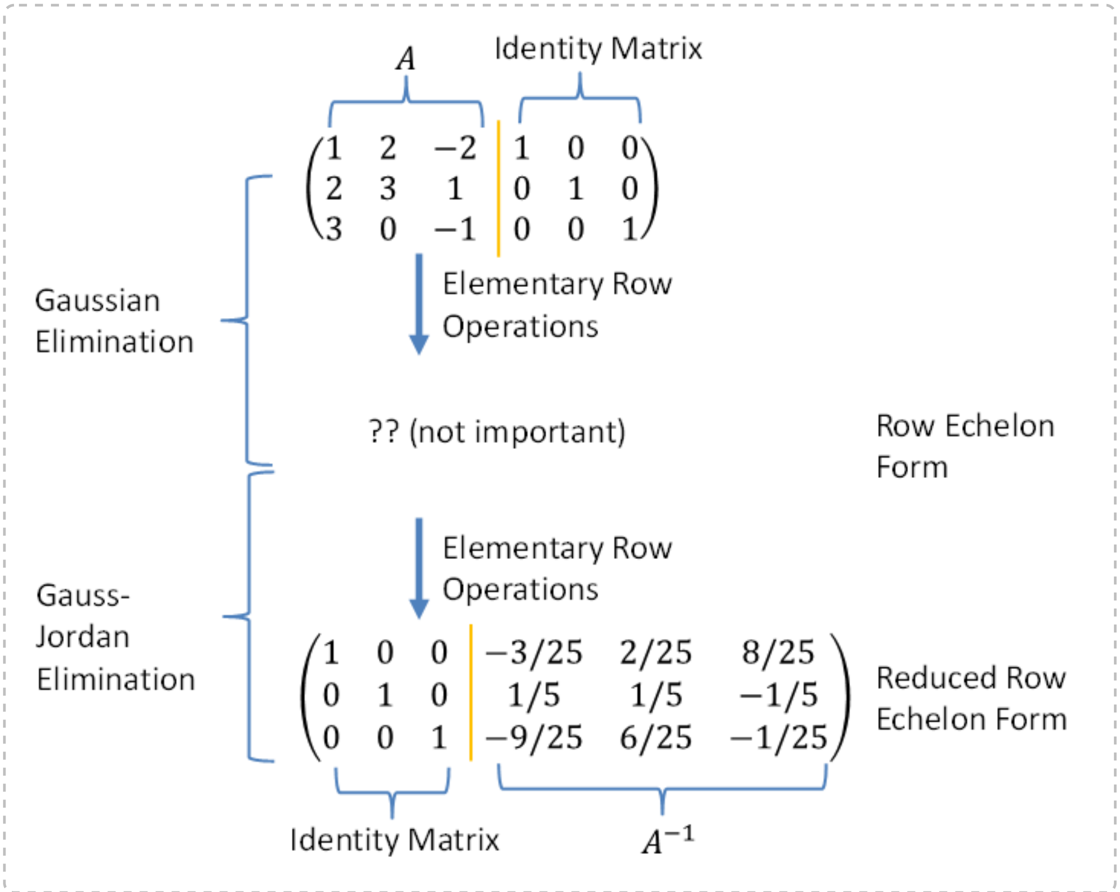
Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This is referred to as the Gaussian elimination method to compute a matrix inverse.

The method that is commonly taught in university linear algebra classes is as follows:

1. Augment the original matrix with an identify matrix on the right.
2. Place augmented matrix in reduced row echelon form.
3. If there is an identify matrix on the left, then the inverse of the original matrix is now on the right.



As we now see, this is simply the same procedure as above (namely Gaussian elimination followed by Gauss-Jordan elimination)

```
In[]:= (*Step 1: Augment matrix with identify matrix on right*)
Aaug = Transpose[Join[Transpose[A], IdentityMatrix[3]]];

Print["Augmented matrix (A|I)"]
Aaug // MatrixForm

(*Step 2: Perform Gaussian elimination followed by Gauss-
  Jordan elimination on augmented matrix*)
result = E10.E9.E8.E7.E6.E5.E4.E3.E2.E1.Aaug;

Print["Result in redueced row echelon form"]
result // MatrixForm

(*Step 3: Extract A^-1 from right*)
Ainv2 = result[[1 ;; 3, 4 ;; 6]];

Print["A^-1"]
Ainv2 // MatrixForm
```

Augmented matrix (A|I)

Out[]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & -2 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 3 & 0 & -1 & 0 & 0 & 1 \end{pmatrix}$$

Result in redueced row echelon form

Out[]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & -\frac{3}{25} & \frac{2}{25} & \frac{8}{25} \\ 0 & 1 & 0 & \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & -\frac{9}{25} & \frac{6}{25} & -\frac{1}{25} \end{pmatrix}$$

A^{-1}

Out[]//MatrixForm=

$$\begin{pmatrix} -\frac{3}{25} & \frac{2}{25} & \frac{8}{25} \\ \frac{1}{5} & \frac{1}{5} & -\frac{1}{5} \\ -\frac{9}{25} & \frac{6}{25} & -\frac{1}{25} \end{pmatrix}$$

```
In[]:= Ainv2 == Inverse[A]
```

Out[]= True