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Lecture 09a
Controllability of a Linear System: The Controllability Matrix and the PBH Test

The YouTube video entitled ‘Controllability of a Linear System: The Controllability Matrix and the PBH Test’ that covers this lecture is located at https://youtu.be/oQDi3Giv-DI.

Outline

-Introduction
-The Controllability Matrix
-The PBH Test

Introduction

Roughly speaking, a system is said to be controllable if you can move the system to arbitrary locations using certain admissible inputs/manipulations.

<Show movies, pictures, demos, etc. to introduce the concept of controllability>
WP_20140228_18_54_32_Pro.mp4 - different systems (dog and babies) react differently to different types of inputs (foods)

Gussie actuation using hand only, hand and voice, food incentives (illustrating different types of inputs to the system).

u_1 = hand motion
u_2 = voice
u_3 = treat

I want to transition him from standing → sitting → standing on 2 legs

pacific_rim_boat_sword.jpg - Pacific Rim is cool to some people but not to others. In other words, this type of input is able to influence some systems but not others)

The Controllability Matrix

The dictionary's definition of controllability is

"Ability to be controlled or managed"

The definition tends to vary depending on the application, framework, model, etc. A definition that is common with controls engineering is:

A system is said to be controllable if there exists a control $\overline{u}(t)$ that can transfer any initial state $\overline{x}(0)$ to any other desired location $\overline{x}(t)$.

We'll see that the ability to meet this definition depends on a combination of the system's dynamics as well the inputs the system.

While the aforementioned definition makes sense, it is somewhat impractical in the sense that it does not describe the control $\overline{u}(t)$ needed to take the system from $\overline{x}(0)$ to $\overline{x}(t)$.

If you have a linear system of

$$\dot{\overline{x}}(t) = A\,\overline{x}(t) + B\,\overline{u}(t) \quad \overline{x} \in \mathcal{R}^n \qquad \textbf{\textit{(Eq.1)}}$$

a more practical test/definition is that a system is controllable if and only if the matrix

$$P_c = \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix} \qquad \textbf{\textit{(Eq.2)}}$$

has full rank (there are at least n linearly independent rows columns, recall row and column rank are identical). In other words, rank(P_c) = n.

Matlab provides the function 'ctrb' to compute the controllability matrix, P_c. It also provides the function 'rank' that can be used to check the number of linearly independent rows or columns of a matrix.

Example 1: Controllable System

Consider a two state system

$$\dot{\overline{x}}(t) = \begin{pmatrix} 0 & 3 \\ 2 & 4 \end{pmatrix} \overline{x}(t) + \begin{pmatrix} -2 \\ 1 \end{pmatrix} \overline{u}(t)$$

$$\mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{3} \\ \mathbf{2} & \mathbf{4} \end{pmatrix}; \mathbf{B} = \begin{pmatrix} \mathbf{-2} \\ \mathbf{1} \end{pmatrix};$$

$$\left\{2 + \sqrt{10},\ 2 - \sqrt{10}\right\}$$

{5.16228, -1.16228}

Compute the controllability matrix

$$P_c = (B \quad AB)$$

```
col1 = B;
col2 = A.B;
Pc = Transpose[Join[Transpose[col1], Transpose[col2]]];
Pc // MatrixForm
```

$$\begin{pmatrix} -2 & 3 \\ 1 & 0 \end{pmatrix}$$

We can see that this is rank 2 (full rank) because the two columns are linearly independent, so the system is controllable. Note: Matlab has the 'rank' function to compute this.

```
MatrixRank[Pc]
```

2

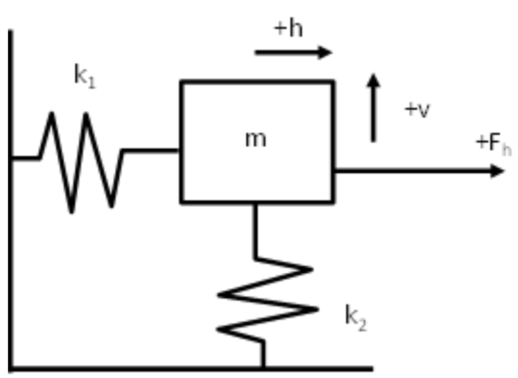
What is interesting about this result is that a single control input, $u_1(t)$, is able to move two states ($x_1(t)$ and $x_2(t)$) to arbitrary locations in the state space. However when you think about this, controlling more states than you have controls for isn't too much of a stretch of the imagination. For example consider your car. If we consider modeling it as a simple planar car with (see AA512 lecture on Planar Vehicles)

$$\overline{x}(t) = \begin{pmatrix} P_E \\ P_N \\ \theta \\ V \end{pmatrix} \qquad \overline{u} = \begin{pmatrix} \delta_{\text{gas}} \\ \delta_{\text{steering wheel}} \end{pmatrix}$$

You can chose appropriate inputs of gas and steering to reach any location in the state space from any initial condition. Sidenote: see Dubin's Path or Dubin's Vehicle (https://en.wikipedia.org/wiki/Dubins-_path).

Example 2: Uncontrollable System

Consider a system shown below



We can write the state vector and control vector as

$$\overline{x}(t) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} h(t) \\ \dot{h}(t) \\ v(t) \\ \dot{v}(t) \end{pmatrix} \qquad \overline{u}(t) = (u_1(t)) = (F_h(t))$$

We can write the state space representation of this system as

$$\dot{\overline{x}}(t) = \begin{pmatrix} x_2(t) \\ \frac{1}{m} F_h(t) - \frac{k_1}{m} x_1(t) \\ x_4(t) \\ -\frac{k_2}{m} x_3(t) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{-k_1}{m} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-k_2}{m} & 0 \end{pmatrix} \overline{x}(t) + \begin{pmatrix} 0 \\ \frac{1}{m} \\ 0 \\ 0 \end{pmatrix} u(t)$$

```
A = {{0, 1, 0, 0}, {-k1 / m, 0, 0, 0}, {0, 0, 0, 1}, {0, 0, -k2 / m, 0}}; B = {{0}, {1 / m}, {0}, {0}};
```

We can compute the controllability matrix of this system

$$P_c = (\, B \quad A B \quad A^2 B \quad A^3 B \,)$$

```
col1 = B;
col2 = A.B;
col3 = A.A.B;
col4 = A.A.A.B;

(*Horizontally concatenate the matrices*)
Pc =
  Transpose[Join[Transpose[col1], Transpose[col2], Transpose[col3], Transpose[col4]]];
Pc // MatrixForm
```

$$\begin{pmatrix} 0 & \frac{1}{m} & 0 & -\frac{k1}{m^2} \\ \frac{1}{m} & 0 & -\frac{k1}{m^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We can now compute the rank of the controllability matrix

```
MatrixRank[Pc]
```

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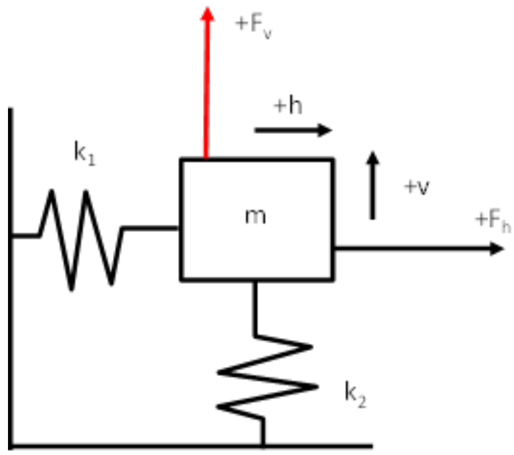
We see that since this matrix is not full rank, the system is not fully controllable. This makes sense considering that the input force cannot influence the vertical dynamics of the system.

Example 3: Make an Uncontrollable System Controllable

However, if we add a second actuator/control to the system as shown below in red.



We see the control vector is augmented

$$\overline{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} F_h(t) \\ F_v(t) \end{pmatrix}$$

And the B matrix becomes

$$B = \begin{pmatrix} 0 & 0 \\ \frac{1}{m} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m} \end{pmatrix}$$

```
B = ( 0     0   )
    ( 1 / m 0   )
    ( 0     0   )
    ( 0   1 / m );
```

If we recompute the controllability matrix, P_c, and check is rank we now obtain

```
(*Compute Pc*)
col1 = B;
col2 = A.B;
col3 = A.A.B;
col4 = A.A.A.B;

(*Horizontally concatenate the matrices*)
Pc =
  Transpose[Join[Transpose[col1], Transpose[col2], Transpose[col3], Transpose[col4]]];
Pc // MatrixForm

(*Check rank*)
MatrixRank[Pc]
```

$$\begin{pmatrix} 0 & 0 & \frac{1}{m} & 0 & 0 & 0 & -\frac{k1}{m^2} & 0 \\ \frac{1}{m} & 0 & 0 & 0 & -\frac{k1}{m^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{m} & 0 & 0 & 0 & -\frac{k2}{m^2} \\ 0 & \frac{1}{m} & 0 & 0 & 0 & -\frac{k2}{m^2} & 0 & 0 \end{pmatrix}$$

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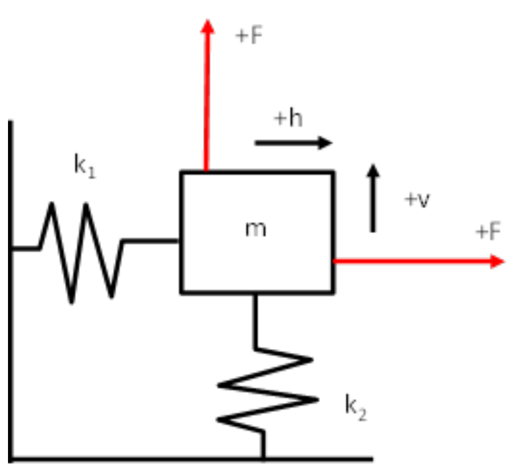
We see that system is now controllable. This illustrates that in some situations, you can make a system controllable by simply adding more actuators/inputs.

Example 4: System Is Controllable Using Single Input

What if we consider only using a single control. For example what if the actuator was mounted at a 45 degree angle on the block. This effectively yields equal vertical and horizontal control forces as shown below



In this case the *B* matrix becomes

$$B = \begin{pmatrix} 0 \\ \frac{1}{m} \\ 0 \\ \frac{1}{m} \end{pmatrix}$$

Choosing constants of

$$k_1 = 1$$
$$k_2 = 4$$
$$m = 1$$

In this case we have

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -4 & 0 \end{pmatrix} B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

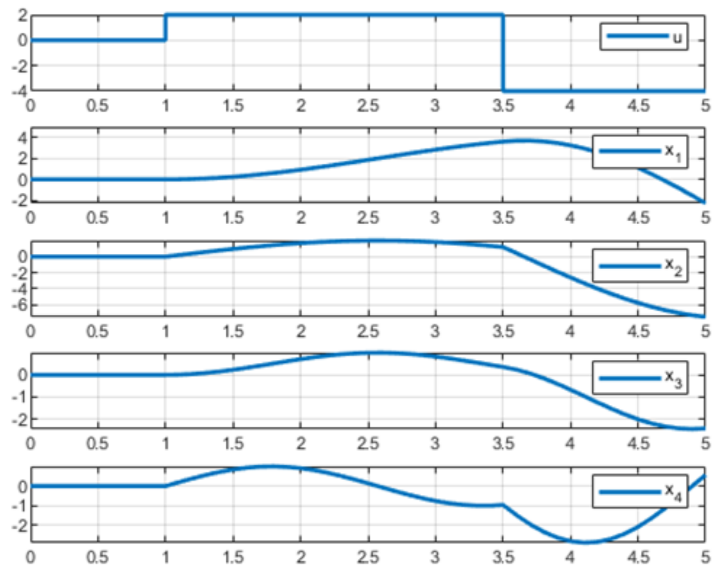
And the controllability matrix is

$$P_c = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -4 \\ 1 & 0 & -4 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \text{rank}(P_c) = 4$$

This might seem counterintuitive but since the spring constants are different, by appropriately choosing the control input signal, you can have the system reach arbitrary locations in the state space. For example, consider starting at the origin and we desire the system to go to

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} -2.27 \\ -7.5 \\ -2.41 \\ 0.57 \end{pmatrix}$$

Below is a simulation showing this is achievable



Example 5: Symmetry Makes System Uncontrollable with Single Input

The previous system was still controllable due to the differences in k_1 and k_2 yielding an asymmetric system that one can conceivably exploit in order to move to state to arbitrary location. However, if both spring constants are identical,

$$k_1 = k_2 = 1/3$$
$$m = 1$$

the system in Example 4 becomes

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1/3 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

And the controllability matrix is

$$P_c = \begin{pmatrix} 0 & 1 & 0 & -1/3 \\ 1 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -1/3 \\ 1 & 0 & -1/3 & 0 \end{pmatrix} \qquad \Rightarrow \qquad \text{rank}(P_c) = 2$$

We see that this is uncontrollable again, which makes sense as the is no way to independently actuate horizontal and vertical states.

The PBH Test

Note that much of the following discussion is based on Steve Brunton's YouTube video at http-s://www.youtube.com/watch?v=0XJHgLrcPeA.

A related test for controllability is the Popov-Belevitch-Hautus test (or PBH Test)

Vasile Popov (1973) - Hyperstability of Control Systems
Vitold Belevitch (1968) - Classical Network Theory
Malo Hautus - https://www.win.tue.nl/~wscomalo/ (updated 2004)

Hautus Lemma - https://en.wikipedia.org/wiki/Hautus_lemma

The PBH test is a little more cumbersome than the rank of the controllability matrix test but we'll see that it gives more information and insight into the controllability of a system.

The PBH test states

A, B is controllable if and only if $\text{rank}(\,[(A - \lambda I\;)\quad B]\,) = n \qquad \forall\, \lambda \in C$ ***(Eq.3)***

We notice that the first n columns of the matrix are given by $A - \lambda I$. This is the familiar formula for eigenvalues/eigenvectors. Therefore, we notice that $\text{rank}(A - \lambda I) = n$ unless λ is an eigenvalue of A. Therefore, we only need to perform the PBH check at eigenvalues. We can revise Eq.3 to

A, B is controllable if and only if $\text{rank}(\,[(A - \lambda_i I\;)\quad B]\,) = n \qquad \forall\, \lambda_i \in \text{eig}(A)$ ***(Eq.4)***

So we see that at most we need to check n values of λ_i.

In addition, we see that when λ_i is an eigenvalue of A, then $A - \lambda_i I$ is rank deficient.

Case 1: $A - \lambda_i I$ is rank deficient by 1

Let us start by assuming that $\text{rank}(A - \lambda_i I) = n - 1$. In other words, it is rank deficient by only 1.

In order for Eq.4 to be satisfied, since we know that $\text{rank}(A - \lambda_i I) = n - 1$, we only need matrix B to "add rank" of 1. In other words, we need it to be linearly independent from $A - \lambda_i I$. In other words we simply need

$B \notin \text{range}(A - \lambda_i I)$ ***(Eq.5)***

This is for a single eigenvalue but this needs to be true for all eigenvalues so we see that we simply have the requirement that

$B \notin \text{range}(A - \lambda_1 I) \cup \text{range}(A - \lambda_2 I) \cup \ldots \cup \text{range}(A - \lambda_n I)$ ***(Eq.6)***

Eq.6 is useful because it allows us to see the structure or "shape" of B needed for controllability.

Example 6: PBH Test

Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 0 \\ 0 & 2 & 4 \end{pmatrix}$$

This has eigenvalues and eigenvectors of

$$\lambda_1 = 0 \qquad \overline{v}_1 = (\ 0.4082 \quad -0.8165 \quad 0.4082\ )^T$$

$$\lambda_2 = 1 \qquad \overline{v}_2 = (\ 0.0000 \quad 0.8321 \quad -0.5547\ )^T$$

$$\lambda_3 = 5 \qquad \overline{v}_3 = (\ 0.6667 \quad 0.3333 \quad 0.6667\ )^T$$

Go to Matlab demo to illustrate various spaces

So we see than an example of an uncontrollable B is

$$B_{\text{uncontrollable}} = -0.8\ \text{range}(A - \lambda_1 I)\,(\,:\,,\,1) + 1.2\ \text{range}(A - \lambda_1 I)\,(\,:\,,\,2) = \begin{pmatrix} -0.2674 \\ 0.9886 \\ -1.0155 \end{pmatrix}$$

An example of a controllable B is one that simply is not in any of the planes that represent range$(A - \lambda_1 I)$, range$(A - \lambda_2 I)$, or range$(A - \lambda_3 I)$.

$$B_{\text{controllable}} = \begin{pmatrix} 0.1 \\ -0.25 \\ 1 \end{pmatrix}$$

We see that in many cases, you actually need to be somewhat unlucky to have a B that does not satisfy Eq.6 since each term range$(A - \lambda_i I)$ is generally a lower dimensional shape.

We can expand on this if we recall that the Rank-Nullity Theorem which states that for an arbitrary matrix Z

$$\text{rank}(Z) + \text{nullity}(Z) = n \qquad \textbf{\textit{(Eq.7)}}$$

where $\quad$ nullity(Z) = dimension of nullspace of Z

In our case, we are interested in the matrix $Z = A - \lambda_i I$

$$\text{rank}(A - \lambda_i I) + \text{nullity}(A - \lambda_i I) = n \qquad \textbf{\textit{(Eq.8)}}$$

Recall the definition of an eigenvector $\overline{v}_i$ corresponding to eigenvalue λ_i is

$$\overline{v}_i \in \text{null}(A - \lambda_i I) \qquad \textbf{\textit{(Eq.9)}}$$

where $\quad \overline{v}_i$ = eigenvector of A corresponding to eigenvalue λ_i

So we know that

$\overline{v}_i \notin \text{range}(A - \lambda_i I)$

So $\overline{v}_i$ should satisfy Eq.4 for a single λ_i. However we note that this same eigenvector will be in the range of other combinations. In other words.

$\overline{v}_1 \notin \text{range}(A - \lambda_1 I)$ but $\overline{v}_1 \in \text{range}(A - \lambda_2 I)$ and $\overline{v}_1 \in \text{range}(A - \lambda_3 I)$ ***(Eq.10a)***
$\overline{v}_2 \notin \text{range}(A - \lambda_2 I)$ but $\overline{v}_2 \in \text{range}(A - \lambda_1 I)$ and $\overline{v}_2 \in \text{range}(A - \lambda_3 I)$ ***(Eq.10b)***
$\overline{v}_3 \notin \text{range}(A - \lambda_3 I)$ but $\overline{v}_3 \in \text{range}(A - \lambda_1 I)$ and $\overline{v}_3 \in \text{range}(A - \lambda_2 I)$ ***(Eq.10c)***

So a single eigenvector $\overline{v}_i$ by itself does not satisfy Eq.6. As such, B cannot be a single eigenvector by itself. However if we combine this with the observation that in general you need to be fairly unlucky to chose a B that does not satisfy Eq.6 then in most cases, $B = \overline{v}_1 + \overline{v}_2 + \overline{v}_3$ should work and yield a controllable system (show Matlab demo).

Case 2: $A - \lambda_i I$ is rank deficient by more than 1

We can now expand our discussion to situations where $\text{rank}(A - \lambda_i I) < n - 1$. In other words, it is rank deficient by more than 1.

Consider the index p that yields the most "loss of rank"

$$\text{rank}\left(A - \lambda_p I\right) = m \qquad \textbf{\textit{(Eq.11)}}$$

Again by considering Eq.4, we see that B now needs to "make up" more than a single rank It needs to "make up" at least a rank of $n - m$. This means that it needs at least $n - m$ linear independent columns from $\text{range}\left(A - \lambda_p I\right)$. This means that B can no longer be a single vector, it must be a matrix that satisfies.

i. $\dim(B) \geq n - m$ ***(Eq.12)***
ii. $\text{range}(B)$ is linearly independent from $\text{range}\left(A - \lambda_p I\right)$ ***(Eq.13)***

Again, this is for a single eigenvalue, we will require that B satisfies Eq.12 and Eq.13 for all eigenvalues.

Physically, condition i means that we need at least $n - m$ control inputs to the system.

The natural extension of this question is when is $\text{rank}(A - \lambda_i I) < n - 1$ (when do we lose more than 1 rank)?

Again, by recalling the rank-nullity theorem, we see that this occurs when

$$\text{nullity}(A - \lambda_i I) > 1$$

We see that this occurs when the eigenvector space spans more than 1 dimension and we know that this occurs when there is a multiplicity of eigenvalues.

If there is a multiplicity of eigenvalues in the A matrix of degree m, the system requires at least m individual control inputs to be controllable ***(Eq.14)***

Example 7: Revising Example 5

Again, consider the symmetrical spring mass system we examined in Example 5. Recall with our choice of $k_1 = k_2 = 1/3$ and $m = 1$ we have

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1/3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1/3 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

The eigenvalues of this are

$$\lambda_1 = 0.866$$
$$\lambda_2 = -0.5\,i$$
$$\lambda_3 = 0.866$$
$$\lambda_4 = -0.5\,i$$

So for example, if we apply Eq.4 to λ_1, we have

$$A, B \text{ is controllable if and only if } \operatorname{rank}([(A - \lambda_1 I) \quad B]) = n$$

So we see that since $\operatorname{rank}(A - \lambda_1 I) = 2$, then not only does $\operatorname{range}(B)$ need to have at least 2 dimensions, it also requires both of which are linearly independent from $\operatorname{range}(A - \lambda_1 I)$ so the overall $\operatorname{rank}([(A - \lambda_1 I) \quad B]) = 4$. This means that in this case, we require at least 2 independent control signals which makes sense given the physical system. You will need to repeat the check with $\lambda_{2,4}$ to ensure that B is also linearly independent from $\operatorname{range}(A - \lambda_2 I)$.

Additional Resources

-Steve Brunton's video on Controllability: https : // www.youtube.com/watch ? v = u5Sv7YKAkt4.

-Steve Brunton's video on PBH Test: https://www.youtube.com/watch?v=0XJHgLrcPeA