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Lecture 02i
Time to Double for a First and Second Order System

The YouTube video entitled ‘Time to Double for a First and Second Order System’ that covers this lecture is located at https://youtu.be/k-mli8-04RQ

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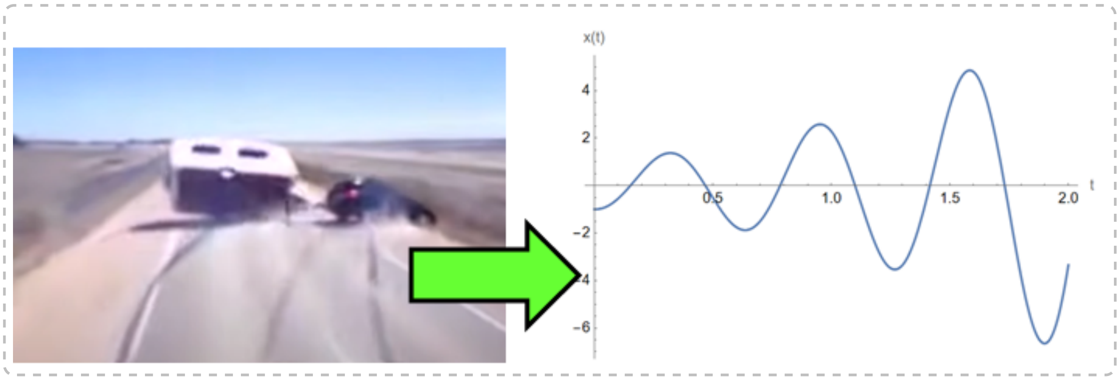
Introduction

In the previous lecture entitled ‘Deriving Percent Overshoot, Settling Time, and Other Performance Metrics of a 2nd Order Dynamic System’ at https://youtu.be/QWCLthgJEbc we derived performance metrics of how an underdamped, stable, 2nd order system responds to a step input.

In this lecture, we perform a similar analysis but instead look at performance metrics that are applicable to an unstable 1st and 2nd order system and look at how this responds to non-zero initial conditions (as opposed to response to an external input).

This arises in situations where we want to understand and quantify how unstable a system is. For example, consider the unstable system of an inverted pendulum falling down (see https://youtu.be/dOo_6OdXY6g?si=qd7rno81DeRs71g3&t=267)

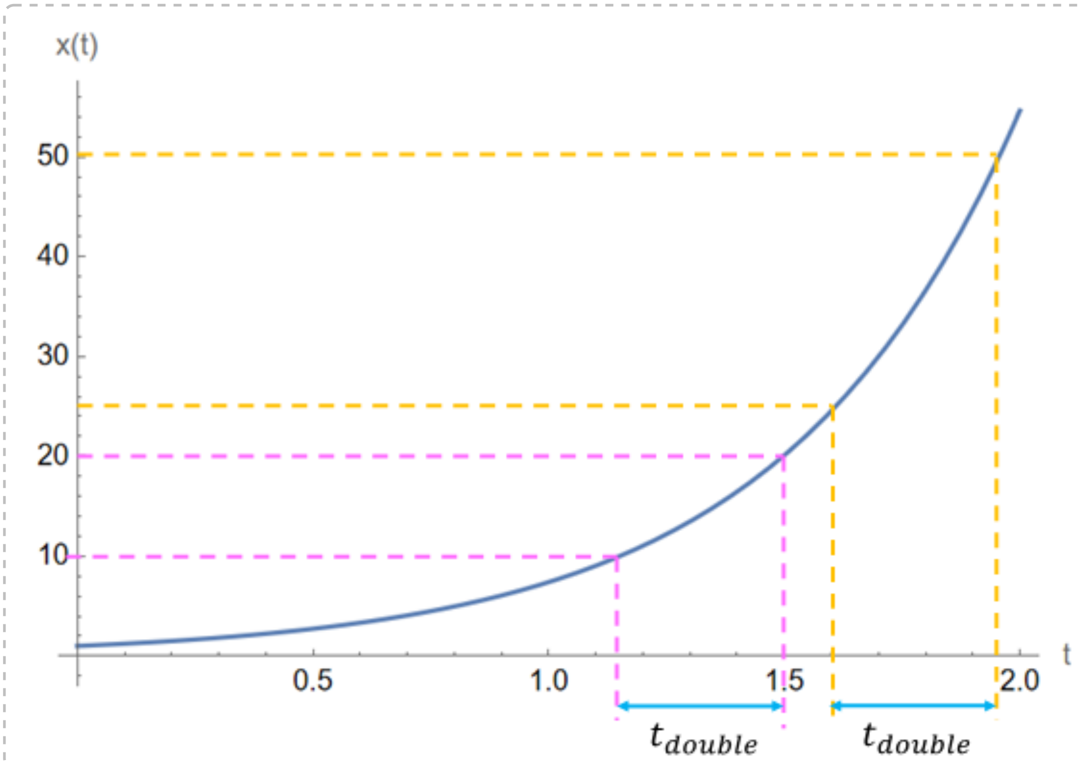
A system can also oscillate and become unstable (see https://youtu.be/nd-hUX8memY?si=e91Jlt-lJXdhQhKD)



Time to Double for a First Order System

We can investigate the time it takes for the system response to double. This is similar to the unstable pendulum case (the silo falling down) mentioned previously .

This time to double is shown below



Plant Model

Consider a first order differential equation of the form

$$\dot{x}(t) + p\,x(t) = 0 \qquad \textbf{(Eq.1)}$$

with $x(0) = x_0$

We can compute the response due to initial conditions using the Laplace method (see 'The Inverse Laplace Transform' at https://youtu.be/wZkrU1lPObM). We start by taking the Laplace transform of Eq.1

$$L\left[\dot{x}(t) + p\,x(t)\right] = 0$$

$$s\,X(s) - x(0) + p\,X(s) = 0$$

$$(s + p)\,X(s) = x(0)$$

$$X(s) = \frac{x(0)}{s+p} \qquad \textbf{(Eq.2)}$$

In[1]:=
```
xf[t_] = InverseLaplaceTransform[x0/(s + p), s, t]
```

Out[1]= $e^{-p t}$ x0

We can verify this satisfies the original ODE

In[2]:=
```
D[xf[t], t] + p xf[t] == 0
xf[0] == x0
```

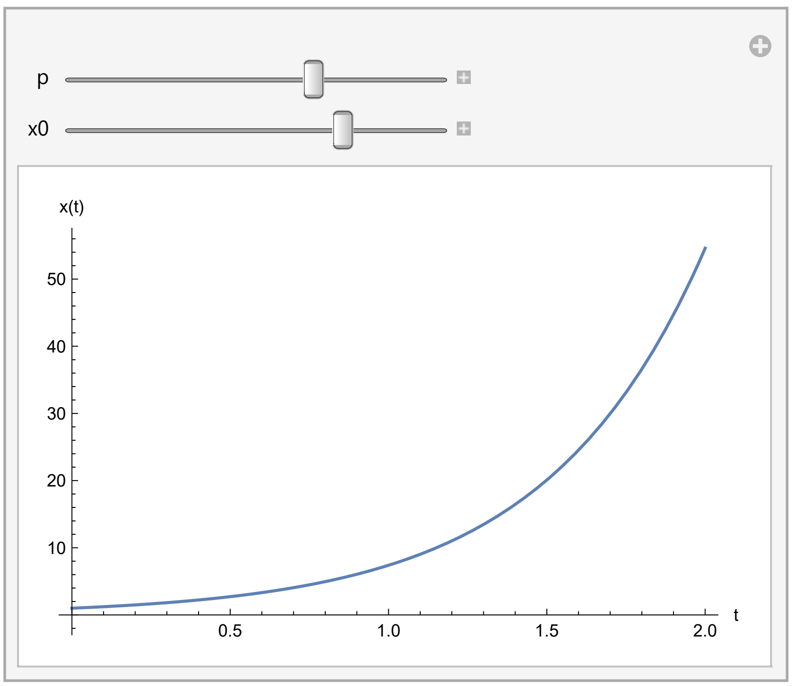
Out[2]= True

Out[3]= True

We see that this is an unstable (AKA growing) response for $p < 0$ and any $x(0)$.

In[4]:=
```
Manipulate[
 Plot[xf[t] /. {x0 → x0Given, p → pGiven}, {t, 0, 2},
  AxesLabel → {"t", "x(t)"}],
 {{pGiven, -2, "p"}, 0, -3},
 {{x0Given, 1, "x0"}, -2, 2}
]
```

Out[4]=



Time to Double

We can compute the time needed for the signal/response to double in magnitude.

We see that the relationship we see is

$$\frac{x(t+t_{\text{double}})}{x(t)} = 2 \qquad \textbf{(Eq.3)}$$

In[5]:=
```
LHS = xf[t + tdouble] / xf[t] // Simplify
```
$$\text{LHS} = \frac{\text{xf}[t + \text{tdouble}]}{\text{xf}[t]} \text{ // Simplify}$$

Out[5]= $e^{-p\,\text{tdouble}}$

So we have

$$e^{-p\, t_{\text{double}}} = 2$$

$$\ln\left(e^{-p\, t_{\text{double}}}\right) = \ln(2)$$

$$-p\, t_{\text{double}} = \ln(2)$$

$$t_{\text{double}} = \frac{\ln(2)}{-p} \quad p < 0 \qquad \textbf{\textit{(Eq.4)}}$$

The interesting thing to note is that this doubling time is not a function of the initial condition, $x(0)$, and it is not a function of t. This means that the signal will take the same amount of time to double regardless of the IC or when the time starts.

Again, note that we are considering $p < 0$ which corresponds to a pole in the right half plane (AKA an unstable pole).

Example

Consider the figure shown above which was generated using

$$x_0 = 1$$
$$p = -2$$

Using Eq.4 we have

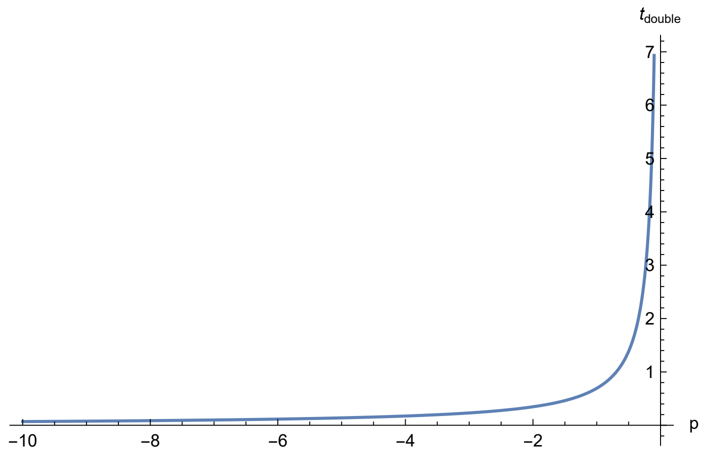
In[6]:=
```
tDouble = Log[2] / -p /. {p → -2} // N
```

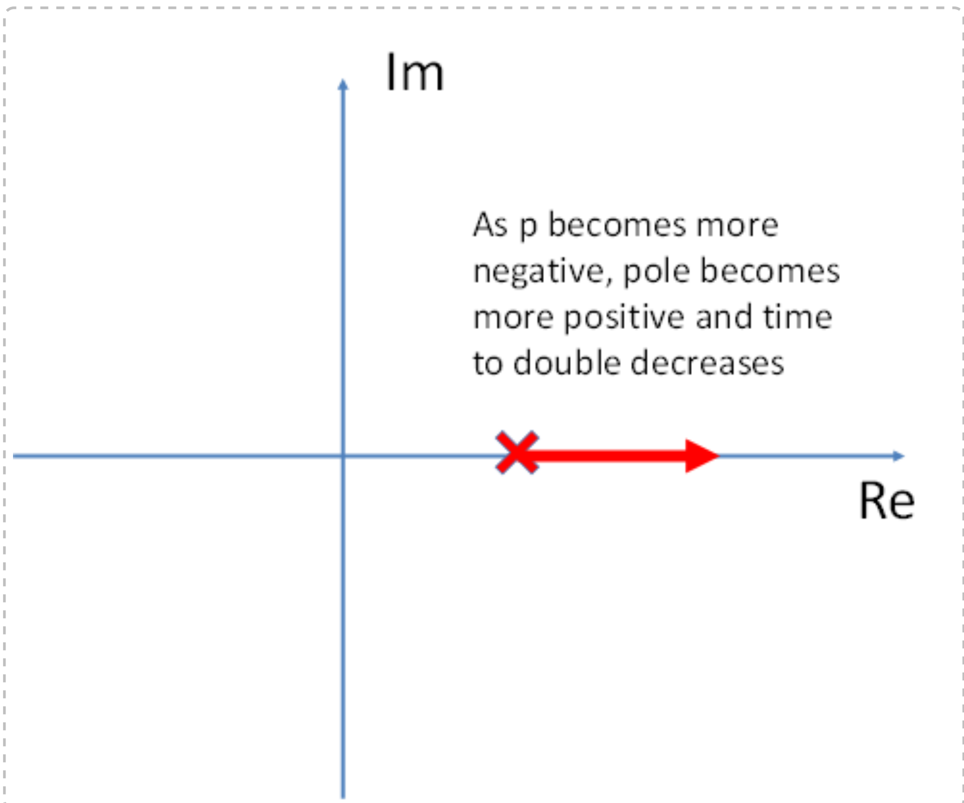
Out[6]= 0.346574

As p becomes more negative, the pole becomes more positive and the time to double decreases

In[7]:=
```
Plot[Log[2]/-p, {p, -1/10, -10},
 AxesLabel → {"p", "t_double"}, PlotRange → All]
```

Out[7]=





Time to Double for a Second Order System

We can investigate the time it takes for an unstable second order system response to double. This is similar to the oscillating car example shown previously.

Plant Model

Consider a first order differential equation of the form (see 'Standard 2nd Order ODEs: Natural Frequency and Damping Ratio' at https://youtu.be/eJMf9CYHr6c?si=2cdNEHlzgWD0-AOf)

$$\ddot{x}(t) + 2\,\zeta\,\omega_n\,\dot{x}(t) + \omega_n^{\,2}\,x(t) = 0 \qquad \textbf{\textit{(Eq.5)}}$$

with $x(0) = x_0$

$\dot{x}(0) = \dot{x}_0$

We can compute the response due to initial conditions using the Laplace method. We start by taking the Laplace transform of Eq.5

$$L\left[\ddot{x}(t) + 2\,\zeta\,\omega_n\,\dot{x}(t) + \omega_n^{\,2}\,x(t)\right] = 0$$

$$s^2\,X(s) - s\,x(0) - \dot{x}(0) + 2\,\zeta\,\omega_n\,(s\,X(s) - x(0)) + \omega_n^{\,2}\,X(s) = 0$$

$$s^2\,X(s) - s\,x(0) - \dot{x}(0) + 2\,\zeta\,\omega_n\,s\,X(s) - 2\,\zeta\,\omega_n\,x(0) + \omega_n^{\,2}\,X(s) = 0$$

$$\left(s^2 + 2\,\zeta\,\omega_n\,s + \omega_n^{\,2}\right)X(s) - s\,x(0) - \dot{x}(0) - 2\,\zeta\,\omega_n\,x(0) = 0$$

$$X(s) = \frac{s\,x(0) + \dot{x}(0) + 2\,\zeta\,\omega_n\,x(0)}{s^2 + 2\,\zeta\,\omega_n\,s + \omega_n^{\,2}} \qquad \textbf{\textit{(Eq.6)}}$$

In[8]:=
```
x[t_] = InverseLaplaceTransform[(s x0 + xDot0 + 2 ζ ωn x0)/(s^2 + 2 ζ ωn s + ωn^2), s, t]
```

Out[8]=

$$\frac{1}{2\sqrt{(-1+\zeta^2)\,\omega n^2}}\Big(-e^{t\left(-\zeta\,\omega n-\sqrt{(-1+\zeta^2)\,\omega n^2}\right)}\,xDot0 + e^{t\left(-\zeta\,\omega n+\sqrt{(-1+\zeta^2)\,\omega n^2}\right)}\,xDot0 - e^{t\left(-\zeta\,\omega n-\sqrt{(-1+\zeta^2)\,\omega n^2}\right)}\,x0\,\zeta\,\omega n + e^{t\left(-\zeta\,\omega n+\sqrt{(-1+\zeta^2)\,\omega n^2}\right)}\,x0\,\zeta\,\omega n + e^{t\left(-\zeta\,\omega n-\sqrt{(-1+\zeta^2)\,\omega n^2}\right)}\,x0\,\sqrt{(-1+\zeta^2)\,\omega n^2} + e^{t\left(-\zeta\,\omega n+\sqrt{(-1+\zeta^2)\,\omega n^2}\right)}\,x0\,\sqrt{(-1+\zeta^2)\,\omega n^2}\Big)$$

We can verify this satisfies the original ODE

In[9]:=
```
D[x[t], {t, 2}] + 2 ζ ωn D[x[t], t] + ωn^2 x[t] == 0 // Simplify
x[0] == x0
(D[x[t], t] /. {t → 0}) == xDot0 // Simplify
```

Out[9]= True

Out[10]= True

Out[11]= True

Eq.6 may be more useful if we can write this in terms of cos and sin

In[12]:= **x[t] // Expand**

Out[12]= $\frac{1}{2} e^{t\left(-\zeta\,\omega n-\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)} x0+\frac{1}{2} e^{t\left(-\zeta\,\omega n+\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)} x0-\frac{e^{t\left(-\zeta\,\omega n-\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)} xDot0\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n^2}+\frac{e^{t\left(-\zeta\,\omega n+\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)} xDot0\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n^2}-\frac{e^{t\left(-\zeta\,\omega n-\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)} x0\,\zeta\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n}+\frac{e^{t\left(-\zeta\,\omega n+\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)} x0\,\zeta\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n}$

We can write this as

$$x(t)=\frac{1}{2} e^{t\left(-\zeta\,\omega n-\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)} x0+\frac{1}{2} e^{t\left(-\zeta\,\omega n+\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)} x0-\frac{e^{t\left(-\zeta\,\omega n-\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)} xDot0\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n^2}+\frac{e^{t\left(-\zeta\,\omega n+\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)} xDot0\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n^2}-\frac{e^{t\left(-\zeta\,\omega n-\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)} x0\,\zeta\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n}+\frac{e^{t\left(-\zeta\,\omega n+\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)} x0\,\zeta\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n}$$

$$=\frac{x0}{2} e^{t\left(-\zeta\,\omega n-\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)}+\frac{x0}{2} e^{t\left(-\zeta\,\omega n+\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)}-\frac{xDot0\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n^2} e^{t\left(-\zeta\,\omega n-\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)}+\frac{xDot0\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n^2} e^{t\left(-\zeta\,\omega n+\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)}-\frac{x0\,\zeta\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n} e^{t\left(-\zeta\,\omega n-\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)}+\frac{x0\,\zeta\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n} e^{t\left(-\zeta\,\omega n+\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)}$$

$$=\left(\frac{x0}{2} e^{t\left(-\zeta\,\omega n-\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)}-\frac{xDot0\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n^2} e^{t\left(-\zeta\,\omega n-\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)}-\frac{x0\,\zeta\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n} e^{t\left(-\zeta\,\omega n-\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)}\right)+\left(\frac{x0}{2} e^{t\left(-\zeta\,\omega n+\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)}+\frac{xDot0\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n^2} e^{t\left(-\zeta\,\omega n+\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)}+\frac{x0\,\zeta\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n} e^{t\left(-\zeta\,\omega n+\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)}\right)$$

$$=\left(\frac{x0}{2}-\frac{xDot0\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n^2}-\frac{x0\,\zeta\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n}\right) e^{t\left(-\zeta\,\omega n-\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)}+\left(\frac{x0}{2}+\frac{xDot0\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n^2}+\frac{x0\,\zeta\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n}\right) e^{t\left(-\zeta\,\omega n+\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)}$$

$$=a\,e^{t\left(-\zeta\,\omega n-\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)}+b\,e^{t\left(-\zeta\,\omega n+\sqrt{\left(-1+\zeta^2\right)\omega n^2}\right)} \qquad \textbf{(Eq.7a)}$$

where $a=\left(\frac{x0}{2}-\frac{xDot0\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n^2}-\frac{x0\,\zeta\sqrt{\left(-1+\zeta^2\right)\omega n^2}}{2\times\left(-1+\zeta^2\right)\omega n}\right)$

$$b = \left(\frac{x0}{2} + \frac{xDot0 \sqrt{\left(-1+\zeta^2\right) \omega n^2}}{2 \times \left(-1+\zeta^2\right) \omega n^2} + \frac{x0\, \zeta \sqrt{\left(-1+\zeta^2\right) \omega n^2}}{2 \times \left(-1+\zeta^2\right) \omega n} \right)$$

In[13]:=
```
a = (x0/2 - (xDot0 Sqrt[(-1 + ζ^2) ωn^2])/(2 × (-1 + ζ^2) ωn^2) - (x0 ζ Sqrt[(-1 + ζ^2) ωn^2])/(2 × (-1 + ζ^2) ωn));
b = (x0/2 + (xDot0 Sqrt[(-1 + ζ^2) ωn^2])/(2 × (-1 + ζ^2) ωn^2) + (x0 ζ Sqrt[(-1 + ζ^2) ωn^2])/(2 × (-1 + ζ^2) ωn));
```

We can verify that this is the same equation

In[15]:=
```
xCheck[t_] = a Exp[t (-ζ ωn - Sqrt[(-1 + ζ^2) ωn^2])] + b Exp[t (-ζ ωn + Sqrt[(-1 + ζ^2) ωn^2])];
xCheck[t] == x[t] // Simplify
```

Out[16]= True

We can perform further manipulations

$$x(t) = a\, e^{t\left(-\zeta\, \omega n - \sqrt{\left(-1+\zeta^2\right) \omega n^2}\right)} + b\, e^{t\left(-\zeta\, \omega n + \sqrt{\left(-1+\zeta^2\right) \omega n^2}\right)} \qquad \text{let } \sigma = \zeta\, \omega n,\ \omega = \sqrt{\left(-1+\zeta^2\right) \omega n^2}$$

$$= a\, e^{t\,(-\sigma-\omega)} + b\, e^{t\,(-\sigma+\omega)}$$

$$= a\, e^{(-\sigma\, t-\omega\, t)} + b\, e^{(-\sigma\, t+\omega\, t)}$$

$$= a\, e^{(-\sigma\, t)}\, e^{(-\omega\, t)} + b\, e^{(-\sigma\, t)}\, e^{(\omega\, t)}$$

$$= e^{(-\sigma\, t)}\left(a\, e^{(-\omega\, t)} + b\, e^{(\omega\, t)}\right) \qquad \textbf{\textit{(Eq.7b)}}$$

In[17]:=
```
xCheck2[t_] = Exp[-σ t] (a Exp[-ω t] + b Exp[ω t]) /. {σ → ζ ωn, ω → Sqrt[(-1 + ζ^2) ωn^2]};
xCheck2[t] == x[t] // Simplify
```

Out[18]= True

We can perform further manipulations

$$x(t) = e^{(-\sigma\, t)}\left(a\, e^{(-\omega\, t)} + b\, e^{(\omega\, t)}\right)$$

$$= e^{(-\sigma\, t)}\left(a\, e^{\left(-\sqrt{\left(-1+\zeta^2\right) \omega n^2}\; t\right)} + b\, e^{\left(\sqrt{\left(-1+\zeta^2\right) \omega n^2}\; t\right)}\right)$$

$$= e^{(-\sigma\, t)}\left(a\, e^{\left(-i \sqrt{\left(1-\zeta^2\right) \omega n^2}\; t\right)} + b\, e^{\left(i \sqrt{\left(1-\zeta^2\right) \omega n^2}\; t\right)}\right) \qquad \text{let } \omega d = \sqrt{\left(1-\zeta^2\right) \omega n^2}$$

$$= e^{(-\sigma\, t)}\left(a\, e^{(-i\, \omega d\, t)} + b\, e^{(i\, \omega d\, t)}\right)$$

$$= e^{(-\sigma t)}\, g(t) \qquad \textbf{(Eq.7c)}$$

where $g(t) = a\, e^{(-i\,\omega\mathrm{d}\,t)} + b\, e^{(i\,\omega\mathrm{d}\,t)}$

Examining the term in parenthesis ($g(t)$), we can convert using Euler's Formula $\left(e^{i\,x} = \cos(x) + i\sin(x)\right)$

In[19]:=
```
g[t_] = ExpToTrig[a Exp[-I ωd t] + b Exp[I ωd t]] // Simplify
```

Out[19]= $\mathrm{x0}\,\mathrm{Cos}[t\,\omega\mathrm{d}] + \dfrac{i\,(\mathrm{xDot0} + \mathrm{x0}\,\zeta\,\omega\mathrm{n})\,\mathrm{Sin}[t\,\omega\mathrm{d}]}{\sqrt{(-1+\zeta^2)\,\omega\mathrm{n}^2}}$

So we have

$$\mathrm{x0}\,\mathrm{Cos}[t\,\omega\mathrm{d}] + \frac{i\,(\mathrm{xDot0}+\mathrm{x0}\,\zeta\,\omega\mathrm{n})\,\mathrm{Sin}[t\,\omega\mathrm{d}]}{\sqrt{(-1+\zeta^2)\,\omega\mathrm{n}^2}} = \mathrm{x0}\,\mathrm{Cos}[t\,\omega\mathrm{d}] + \frac{i\,(\mathrm{xDot0}+\mathrm{x0}\,\zeta\,\omega\mathrm{n})\,\mathrm{Sin}[t\,\omega\mathrm{d}]}{i\,\sqrt{(1-\zeta^2)\,\omega\mathrm{n}^2}}$$

$$= \mathrm{x0}\,\mathrm{Cos}[t\,\omega\mathrm{d}] + \frac{i\,(\mathrm{xDot0}+\mathrm{x0}\,\zeta\,\omega\mathrm{n})\,\mathrm{Sin}[t\,\omega\mathrm{d}]}{i\,\omega\mathrm{d}}$$

In[20]:=
```
x0 Cos[t ωd] + I (xDot0 + x0 ζ ωn) Sin[t ωd] / (I ωd)
```

Out[20]= $\mathrm{x0}\,\mathrm{Cos}[t\,\omega\mathrm{d}] + \dfrac{(\mathrm{xDot0} + \mathrm{x0}\,\zeta\,\omega\mathrm{n})\,\mathrm{Sin}[t\,\omega\mathrm{d}]}{\omega\mathrm{d}}$

So we see that the imaginary terms are eliminated and we obtain a pure oscillatory term. Substituting this back into Eq.7c yields

$$x(t) = e^{(-\sigma t)}\left(\mathrm{x0}\cos(\omega\mathrm{d}\,t) + \frac{(\mathrm{xDot0}+\mathrm{x0}\,\zeta\,\omega\mathrm{n})}{\omega\mathrm{d}}\sin(\omega\mathrm{d}\,t)\right) \qquad \textbf{(Eq.7d)}$$

In[21]:=
```
xCheck3[t_] =
 Exp[-σ t] (x0 Cos[ωd t] + (xDot0 + x0 ζ ωn)/ωd Sin[ωd t]) /. {σ → ζ ωn, ωd → Sqrt[(1 - ζ^2) ωn^2]}
```

Out[21]= $e^{-t\,\zeta\,\omega\mathrm{n}}\left(\mathrm{x0}\,\mathrm{Cos}\left[t\sqrt{(1-\zeta^2)\,\omega\mathrm{n}^2}\right] + \dfrac{(\mathrm{xDot0}+\mathrm{x0}\,\zeta\,\omega\mathrm{n})\,\mathrm{Sin}\left[t\sqrt{(1-\zeta^2)\,\omega\mathrm{n}^2}\right]}{\sqrt{(1-\zeta^2)\,\omega\mathrm{n}^2}}\right)$

We can verify that this satisfies the original ODE (Eq.5)

In[22]:=
```
D[xCheck3[t], {t, 2}] + 2 ζ ωn D[xCheck3[t], t] + ωn^2 xCheck3[t] == 0 // Simplify
xCheck3[0] == x0
(D[xCheck3[t], t] /. {t → 0}) == xDot0
```

Out[22]= True

Out[23]= True

Out[24]= True

Finally, we can combine the two sinusoids in Eq.7d using the trig identity for a linear combination of cos and sin at the same frequency (https://socratic.org/trigonometry/trigonometric-identities-andequations/products-sums-linear-combinations-and-applications)

$$\alpha \cos(x) + \beta \sin(x) = \gamma \cos(x - \phi) \qquad \textbf{(Eq.8)}$$

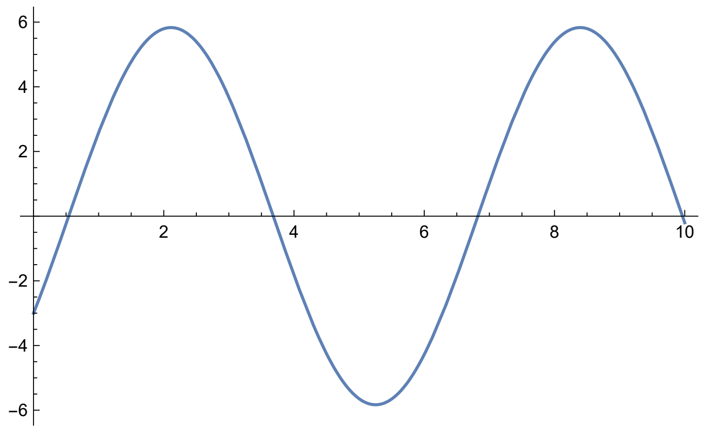
where $\gamma = \sqrt{\alpha^2 + \beta^2}$

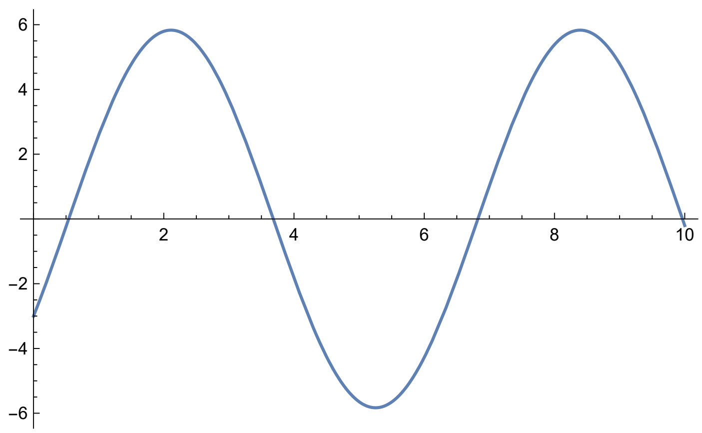
$\phi = \text{atan2}(\beta, \alpha) = \text{atan2}(y, x)$

Sidenote: Check Trig Identity

$$-3\cos(t) + 5\sin(t) = \text{sign}(-3)\sqrt{(-3)^2 + (5)^2}\cos(t - \text{ArcTan}[-3, 5])$$

In[25]:=
```
Plot[-3 Cos[t] + 5 Sin[t], {t, 0, 10}]
Plot[Sqrt[(-3)^2 + (5)^2] Cos[t - ArcTan[-3, 5]], {t, 0, 10}]
```

Out[25]=


Out[26]=


End Sidenote

Applied to Eq.7d we have

$$x(t) = e^{(-\sigma t)}(\alpha \cos(\omega d\, t) + \beta \sin(\omega d\, t))$$

$$= e^{(-\sigma t)}(\gamma \cos(\omega d\, t - \phi))$$

$$x(t) = \gamma\, e^{-\sigma t} \cos(\omega\mathrm{d}\, t - \phi) \qquad \textbf{\textit{(Eq.7d)}}$$

where $\alpha = \mathrm{x0}$

$\beta = \frac{(\mathrm{xDot0} + \mathrm{x0}\, \zeta\, \omega\mathrm{n})}{\omega\mathrm{d}}$

$\phi = \mathrm{atan2}(\beta,\, \alpha) = \mathrm{atan2}(y,\, x)$

```
In[27]:= α = x0;
         β = (xDot0 + x0 ζ ωn) / ωd;
         γ = Sqrt[α^2 + β^2]
         ϕ = ArcTan[α, β]
          (*Note that Mathematica's ArcTan input parameter order is x,y not y,x*)
```

Out[29]= $\sqrt{\mathrm{x0}^2 + \frac{(\mathrm{xDot0} + \mathrm{x0}\, \zeta\, \omega\mathrm{n})^2}{\omega\mathrm{d}^2}}$

Out[30]= $\mathrm{ArcTan}\left[\mathrm{x0},\, \frac{\mathrm{xDot0} + \mathrm{x0}\, \zeta\, \omega\mathrm{n}}{\omega\mathrm{d}}\right]$

```
In[31]:= xCheck4[t_] = γ Exp[-σ t] Cos[ωd t - ϕ] /. {σ → ζ ωn, ωd → Sqrt[(1 - ζ^2) ωn^2]}
```

Out[31]= $e^{-t\, \zeta\, \omega\mathrm{n}} \sqrt{\mathrm{x0}^2 + \frac{(\mathrm{xDot0} + \mathrm{x0}\, \zeta\, \omega\mathrm{n})^2}{(1 - \zeta^2)\, \omega\mathrm{n}^2}}\ \mathrm{Cos}\left[t \sqrt{(1 - \zeta^2)\, \omega\mathrm{n}^2} - \mathrm{ArcTan}\left[\mathrm{x0},\, \frac{\mathrm{xDot0} + \mathrm{x0}\, \zeta\, \omega\mathrm{n}}{\sqrt{(1 - \zeta^2)\, \omega\mathrm{n}^2}}\right]\right]$

So our final form of the solution is. This shows that the solution can be expressed as a single sinusoid with a time varying amplitude.

$$x(t) = A\, e^{-\sigma t} \cos(\omega\mathrm{d}\, t\ - \phi) \qquad \textbf{\textit{(Eq.9)}}$$

where $\sigma = \zeta\, \omega\mathrm{n}$

$\omega\mathrm{d} = \sqrt{(1 - \zeta^2)\, \omega\mathrm{n}^2}$

$A = \sqrt{\mathrm{x0}^2 + \frac{(\mathrm{xDot0} + \mathrm{x0}\, \zeta\, \omega\mathrm{n})^2}{(1 - \zeta^2)\, \omega\mathrm{n}^2}}$

$\phi = \mathrm{ArcTan}\left[\mathrm{x0},\, \frac{\mathrm{xDot0} + \mathrm{x0}\, \zeta\, \omega\mathrm{n}}{\sqrt{(1 - \zeta^2)\, \omega\mathrm{n}^2}}\right]$

```
In[32]:= xAlt[t_] =
          A Exp[-σ t] Cos[ωd t - ϕ] /. {σ → ζ ωn, ωd → Sqrt[(1 - ζ^2) ωn^2], A → Sqrt[x0^2 + (xDot0 + x0 ζ ωn)^2 / ((1 - ζ^2) ωn^2)]}
```

Out[32]= $e^{-t\, \zeta\, \omega\mathrm{n}} \sqrt{\mathrm{x0}^2 + \frac{(\mathrm{xDot0} + \mathrm{x0}\, \zeta\, \omega\mathrm{n})^2}{(1 - \zeta^2)\, \omega\mathrm{n}^2}}\ \mathrm{Cos}\left[t \sqrt{(1 - \zeta^2)\, \omega\mathrm{n}^2} - \mathrm{ArcTan}\left[\mathrm{x0},\, \frac{\mathrm{xDot0} + \mathrm{x0}\, \zeta\, \omega\mathrm{n}}{\sqrt{(1 - \zeta^2)\, \omega\mathrm{n}^2}}\right]\right]$

We can verify that this satisfies the original ODE (Eq.5)

In[33]:=
```
D[xAlt[t], {t, 2}] + 2 ζ ωn D[xAlt[t], t] + ωn^2 xAlt[t] == 0 // Simplify
xAlt[0] == x0
(D[xAlt[t], t] /. {t → 0}) == xDot0
```

Out[33]= True

Out[34]= True

Out[35]= True

Underdamped Scenario

We see that Eq.9 is most useful when $\zeta < 1$ (so the term $\sqrt{(1-\zeta^2)} > 0$.

Stable Case

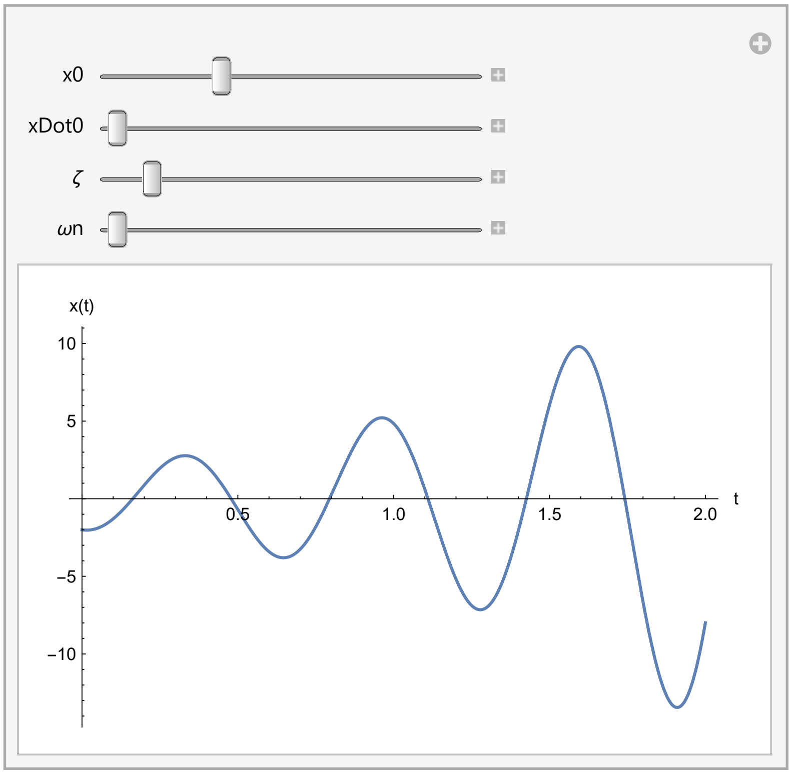
If we we have $\omega_n > 0$ and $\zeta \in [0, 1)$ then Eq.9 describes a stable system (the term $e^{-\sigma t}$ becomes something like $e^{-2.5 t}$ which is a decaying term).

Unstable Case

If we we have $\omega_n < 0$ and $\zeta \in [0, 1)$ then Eq.9 describes an unstable system (the term $e^{-\sigma t}$ becomes something like $e^{2.5 t}$ which is a growing term).

In[36]:=
```
Manipulate[
 Plot[xAlt[t] /. {x0 → x0Given, xDot0 → xDot0Given, ζ → ζGiven, ωn → ωnGiven}, {t, 0, 2},
  AxesLabel → {"t", "x(t)"}],
 {{x0Given, -2, "x0"}, -5, 5},
 {{xDot0Given, -3, "xDot0"}, -3, 3},
 {{ζGiven, 1 / 10, "ζ"}, 0, 1},
 {{ωnGiven, -10, "ωn"}, -10, 10}
]
```

Out[36]=



In Eq.9 we notice that the amplitude is time varying and given by

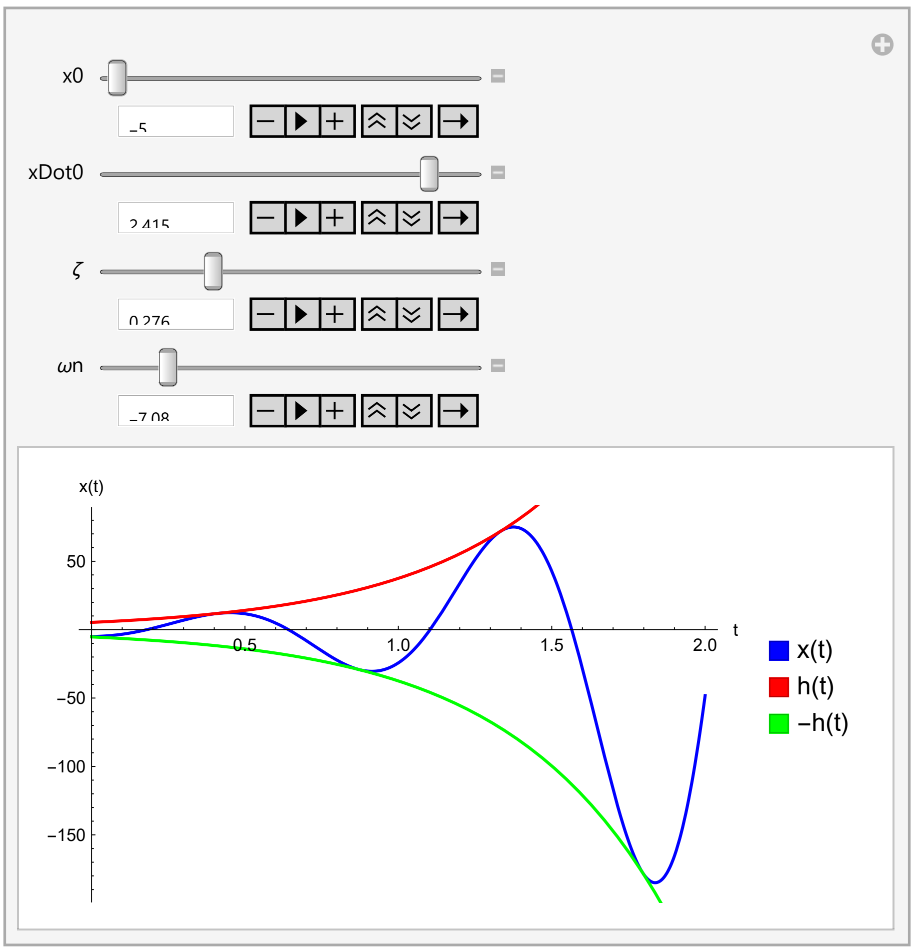
$h(t) = A\,e^{-\sigma t}$ (amplitude of sinusoidal oscillation)

In[37]:= $\texttt{h[t\_] = A Exp[-}\sigma\texttt{ t] /. }\left\{\sigma \to \zeta\,\omega n,\ A \to \sqrt{x0^2 + \frac{(xDot0 + x0\,\zeta\,\omega n)^2}{(1-\zeta^2)\,\omega n^2}}\right\};$

$h(t)$ defines an envelope that contains the sinusoid. For example the plot of $x(t)$, $h(t)$, and $-h(t)$

In[38]:=
```
Manipulate[
 Legended[
  Show[
   Plot[xAlt[t] /. {x0 → x0Given, xDot0 → xDot0Given, ξ → ξGiven, ωn → ωnGiven}, {t, 0, 2},
    PlotStyle → Blue, AxesLabel → {"t", "x(t)"}],
   Plot[h[t] /. {x0 → x0Given, xDot0 → xDot0Given, ξ → ξGiven, ωn → ωnGiven}, {t, 0, 2},
    PlotStyle → Red],
   Plot[-h[t] /. {x0 → x0Given, xDot0 → xDot0Given, ξ → ξGiven, ωn → ωnGiven}, {t, 0, 2},
    PlotStyle → Green]
  ],
  SwatchLegend[{Blue, Red, Green}, {"x(t)", "h(t)", "-h(t)"}]
 ],
 {{x0Given, -2, "x0"}, -5, 5},
 {{xDot0Given, -3, "xDot0"}, -3, 3},
 {{ξGiven, 1/10, "ξ"}, 0, 1},
 {{ωnGiven, -10, "ωn"}, -10, 10}
]
```

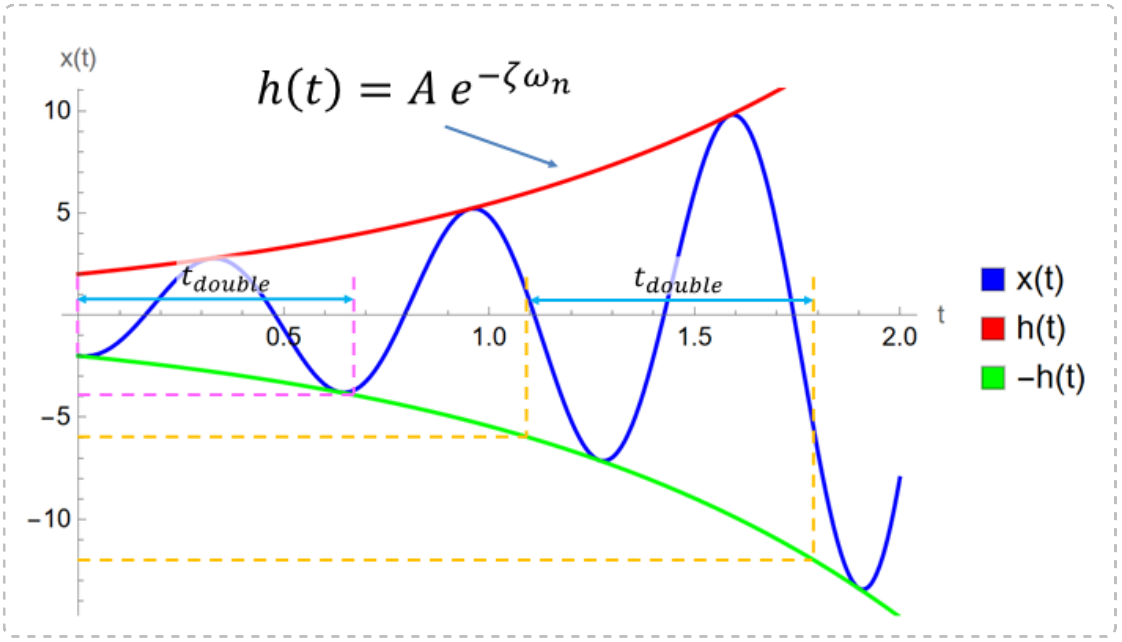
Out[38]=



Time to Double

We can compute the time needed for the signal/response to double in magnitude by looking at the amplitude envelope. This is similar to the first order system in Eq.3.

We can use this envelope to develop a conservative estimate of the time to double



We see that the relationship we see is

$$\frac{h(t+t_{\text{double}})}{h(t)} = 2 \qquad \textbf{(Eq.10)}$$

In[39]:= **LHS = $\frac{\texttt{h[t + tdouble]}}{\texttt{h[t]}}$ // Simplify**

Out[39]= $e^{-\text{tdouble}\,\zeta\,\omega n}$

So we have

$$e^{-\zeta\,\omega_n\,t_{\text{double}}} = 2$$

$$\ln\left(e^{-\zeta\,\omega_n\,t_{\text{double}}}\right) = \ln(2)$$

$$-\zeta\,\omega_n\,t_{\text{double}} = \ln(2)$$

$$t_{\text{double}} = \frac{\ln(2)}{-\zeta\,\omega_n} \quad \zeta\,\omega_n < 0 \qquad \textbf{(Eq.11)}$$

The interesting thing to note is that this doubling time is not a function of the initial condition, $x(0)$, and it is not a function of t. This means that the signal will take the same amount of time to double regardless of the IC or when the time starts.

Again, note that we are considering $\zeta\,\omega_n < 0$ which corresponds to a pole in the right half plane (AKA an unstable pole).

Example

Consider the figure shown above which was generated using

$$x_0 = -2$$
$$\dot{x}_0 = -3$$
$$\zeta = 1/10$$
$$\omega_n = -10$$

Using Eq.11 we have

```
In[40]:= tDouble = Log[2]/(-ζ ωn) /. {ζ → 1/10, ωn → -10} // N
Out[40]= 0.693147
```

Again, note that this is a conservative estimate based on the amplitude envelope. The actual time to double may be depending on how the time to double is defined (for example consider the case shown in orange).

Again, note that the quantity $-\zeta\,\omega_n$ is related to the pole location of the 2nd order system (Eq.5)

```
In[41]:= temp = Solve[s^2 + 2 ζ ωn s + ωn^2 == 0, s];
r1 = s /. temp[[1]]
r2 = s /. temp[[2]]
```

Out[42]= $-\zeta\,\omega n - \sqrt{-\omega n^2 + \zeta^2\,\omega n^2}$

Out[43]= $-\zeta\,\omega n + \sqrt{-\omega n^2 + \zeta^2\,\omega n^2}$

So we see the roots/poles are located at

$$r_{1,2} = -\zeta\,\omega_n \pm \sqrt{\left(\zeta^2 - 1\right)\omega_n^{\,2}}$$

Again, for the underdamped case ($\zeta \in [0,\,1)$) we can write this as

$$r_{1,2} = -\zeta\,\omega_n \pm i\,\sqrt{\left(1 - \zeta^2\right)\omega_n^{\,2}}$$

So the real part of the poles are given by the term $-\zeta\,\omega_n$. So we see that if the real part is positive (AKA ω_n is negative) then this is what governs the time to double. Using the numerical example from above

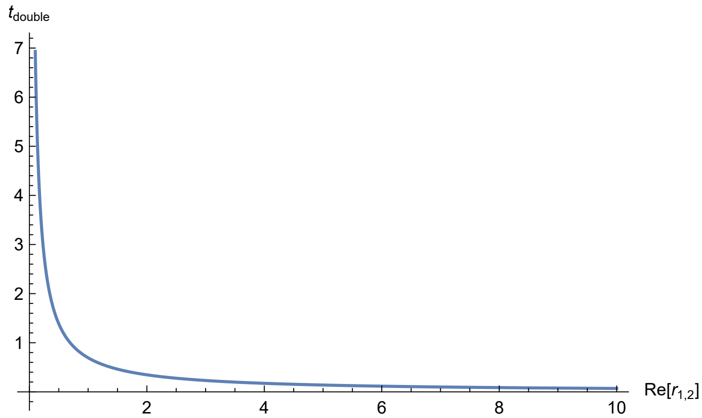
$$\mathrm{Re}\left[r_{1,2}\right] = -\zeta\,\omega_n$$
$$= -\left(\tfrac{1}{10}\right) \times (-10)$$

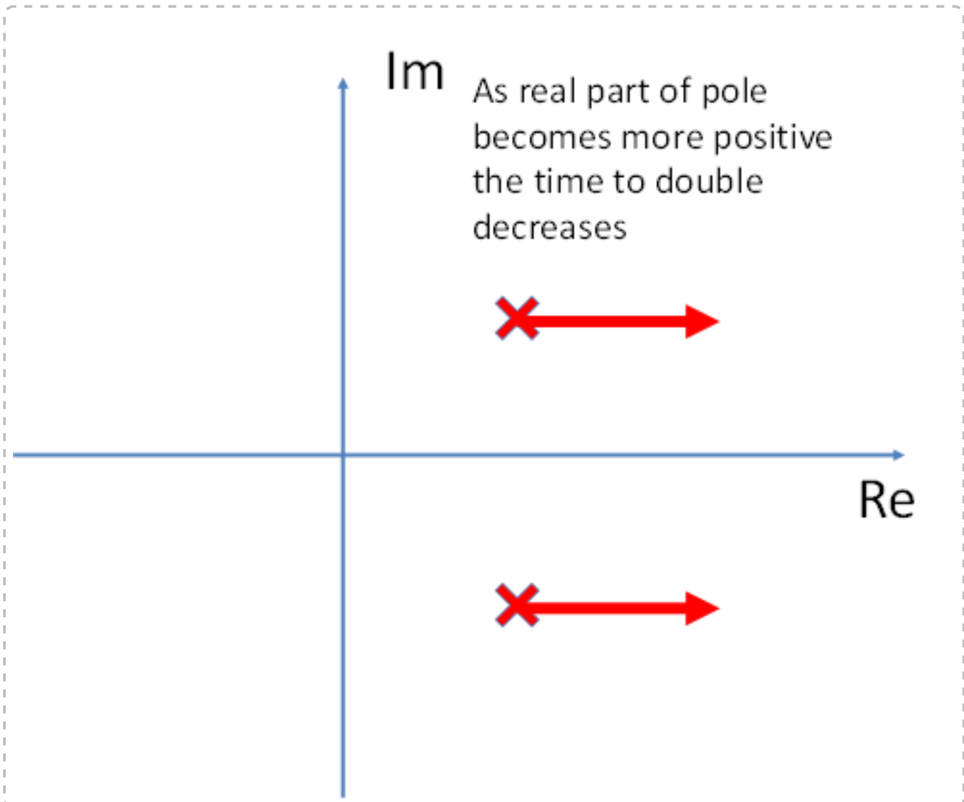
$$\mathrm{Re}\left[r_{1,2}\right] = 1$$

As the real part becomes more positive, the time to double decreases

```
In[44]:= Plot[Log[2]/RealPartPole, {RealPartPole, 1/10, 10},
  AxesLabel → {"Re[r1,2]", "tdouble"}, PlotRange → All]
```

Out[44]=





Notice that this is the same result as for a first order system. In other words, in both cases, the time to double depends on the real part of the pole.

Summary

First Order Unstable System

For a system of the form

$$\dot{x}(t) + p\, x(t) = 0 \qquad \textbf{\textit{(Eq.1)}}$$

with $x(0) = x_0$

When $p < 0$ the time to double is

$$t_{\text{double}} = \frac{\ln(2)}{-p} \qquad p < 0 \qquad \textbf{\textit{(Eq.4)}}$$

Second Order Unstable Underdamped System

For a system of the form

$$\ddot{x}(t) + 2\,\zeta\,\omega_n\,\dot{x}(t) + \omega_n^{\,2}\,x(t) = 0 \qquad \textbf{\textit{(Eq.5)}}$$

with $x(0) = x_0$

$\dot{x}(0) = \dot{x}_0$

When $\zeta \in [0, 1)$ and $\omega_n < 0$ the time to double is

$$t_{\text{double}} = \frac{\ln(2)}{-\zeta\,\omega_n} \qquad \zeta\,\omega_n < 0 \qquad \textbf{\textit{(Eq.11)}}$$