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Lecture 01c
Simple Vector Mechanics: Inner Product, Scalar/Vector Projection, and Cross Product

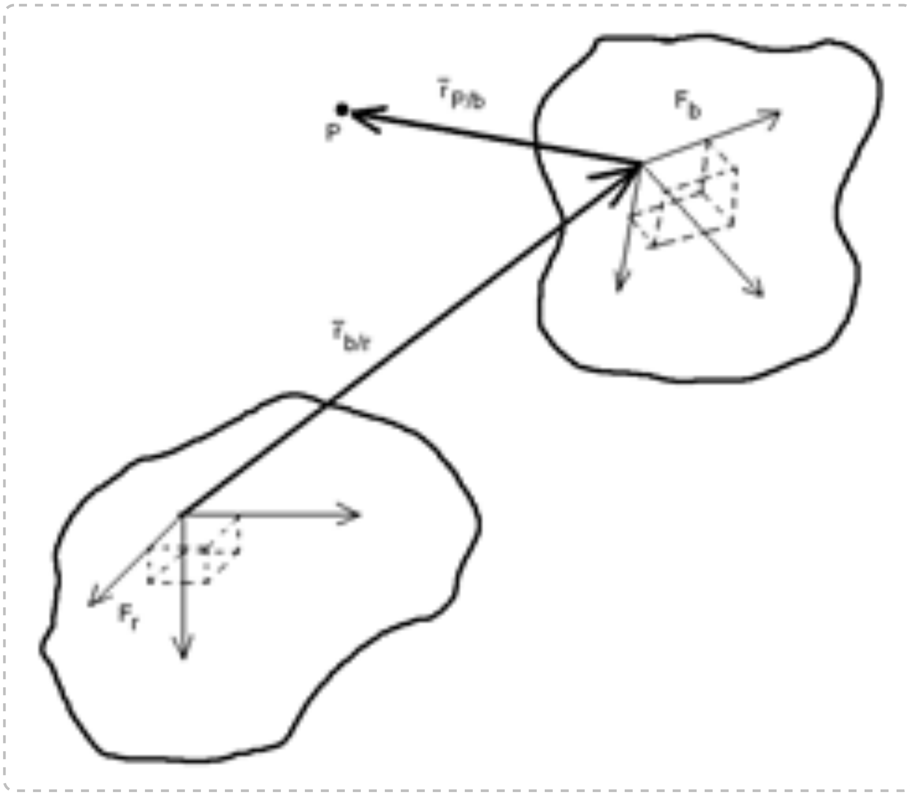
The YouTube video entitled ‘Simple Vector Mechanics: Inner Product, Scalar/Vector Projection, and Cross Product’ that covers this lecture is located at https://youtu.be/fAZZJgm096w.

Outline

-Inner Product (Dot Product)
-Scalar Projection
-Vector Projection
-Cross Product

Coordinate System

Frame of Reference: Establishes distances and directions
Coordinate System: A frame of reference with a point of reference (ie origin). Loosely referred to as a "frame". Can have multiple coordinate systems within a frame.



Magnitude of a Vector (2-Norm)

Recall that the magnitude of a vector is given as

$$|\ \overline{a}\ | = \left(\sum_{i=1}^{n} a_i^{\,2}\right)^{1/2}$$

Example

$$\mathbf{a} = \begin{pmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \end{pmatrix};$$

We can compute the magnitude as

$$|\ \overline{a}\ | = \sqrt{1^2 + 2^2 + 3^3} = \sqrt{14}$$

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Norm[a]
```

$\sqrt{14}$

Inner Product (Dot Product)

The inner product is a function which operates on two vectors and produces a scalar.

$$\overline{a} \cdot \overline{b} = <\overline{a}, \overline{b}> \triangleq \begin{cases} 0 & \text{if } \overline{a} = 0 \text{ or } \overline{b} = 0 \\ \sum_{i=1}^{n} a_i\, b_i & \text{otherwise} \end{cases}$$

Based on the definition of the inner product of two vectors, we can rewrite the magnitude of a vector in terms of its inner product as

$$|\ \overline{a}\ | = (\overline{a} \cdot \overline{a})^{1/2} = <\overline{a}, \overline{a}>^{1/2} \qquad \textbf{\textit{(Eq.1)}}$$

If $\overline{a}$ and $\overline{b}$ are column vectors, $\overline{a} \cdot \overline{b}$ can easily be calculated using

$$\overline{a} \cdot \overline{b} = \overline{a}^T\, \overline{b} = \overline{b}^T\, \overline{a}$$

Example

Consider the same $\overline{a}$ as used previously and now assume $\overline{b} = (\,4 \quad 5 \quad 6\,)^T$. So $\overline{a} \cdot \overline{b}$ is given by

$$\overline{a} \cdot \overline{b} = <\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}>$$

$$= 1 * 4 + 2 * 5 + 3 * 6$$

$$= 4 + 10 + 18$$

$$\overline{a} \cdot \overline{b} = 32$$

```
b = (4
     5
     6);
```

```
Transpose[a].b
```

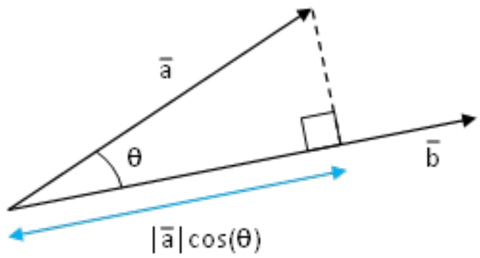
```
{{32}}
```

Alternatively, this can be expressed as

$$\overline{a} \cdot \overline{b} = |\ \overline{a}\ |\ \ |\ \overline{b}\ |\ \cos(\theta) \qquad \textbf{\textit{(Eq.2)}}$$

where $\theta \in [0, \pi]$ = angle between $\overline{a}$ and $\overline{b}$

From Eq.2, we can therefore interpret $\overline{a} \cdot \overline{b}$ as the length of the projection of $\overline{a}$ onto $\overline{b}$ multiplied by the length of $\overline{b}$.



Inner Product Rules

$\left(q_1 \overline{a} + q_2 \overline{b}\right) \cdot \overline{c} = q_1 \overline{a} \cdot \overline{c} + q_2 \overline{b} \cdot \overline{c}$ for all scalar q_1, q_2 (linearity)

$\overline{a} \cdot \overline{b} = \overline{b} \cdot \overline{a}$ (symmetry)

$\left(\overline{a} + \overline{b}\right) \cdot \overline{c} = \overline{a} \cdot \overline{c} + \overline{b} \cdot \overline{c}$ (distributivity)
$\overline{c} \cdot \left(\overline{a} + \overline{b}\right) = \overline{c} \cdot \overline{a} + \overline{c} \cdot \overline{b}$

$$\overline{a} \cdot \overline{a} = \begin{cases} 0 & \text{if and only if } \overline{a} = 0 \\ > 0 & \text{otherwise} \end{cases}$$ (positive definiteness)

A very famous inequality using inner products is known as the Cauchy-Schwartz inequality (https://en.wikipedia.org/wiki/Cauchy% E2 %80 %93 Schwarz_inequality). This states that the magnitude of a dot product is less than or equal to the product of the two individual norms.

$\left| \overline{a} \cdot \overline{b} \right| \leq | \overline{a} | \; | \overline{b} |$ (Cauchy-Schwartz inequality)

We can use this along with other inner product properties to generate other relationships. Recall Eq.1

$| \overline{c} | = (\overline{c} \cdot \overline{c})^{1/2}$

$| \overline{c} |^2 = \overline{c} \cdot \overline{c}$ let $\overline{c} = \overline{a} + \overline{b}$

$\left| \overline{a} + \overline{b} \right|^2 = \left(\overline{a} + \overline{b}\right) \cdot \overline{c}$ recall from distributivity: $\left(\overline{a} + \overline{b}\right) \cdot \overline{c} = \overline{a} \cdot \overline{c} + \overline{b} \cdot \overline{c}$

$= \overline{a} \cdot \overline{c} + \overline{b} \cdot \overline{c}$

$= \overline{a} \cdot \left(\overline{a} + \overline{b}\right) + \overline{b} \cdot \left(\overline{a} + \overline{b}\right)$

$= \overline{a} \cdot \overline{a} + \overline{a} \cdot \overline{b} + \overline{b} \cdot \overline{a} + \overline{b} \cdot \overline{b}$ recall from symmetry: $\overline{a} \cdot \overline{b} = \overline{b} \cdot \overline{a}$

$= \left| \overline{a} \right|^2 + 2 \overline{a} \cdot \overline{b} + \left| \overline{b} \right|^2$ note: $\overline{a} \cdot \overline{b}$ can be negative, so $\overline{a} \cdot \overline{b} \leq \left| \overline{a} \cdot \overline{b} \right|$

$\leq |\bar{a}|^2 + 2|\bar{a} \cdot \bar{b}| + |\bar{b}|^2$ recall $\bar{a} \cdot \bar{b} \leq |\bar{a}| \, |\bar{b}|$ (Cauchy-Schwartz)

$\leq |\bar{a}|^2 + 2|\bar{a}| \, |\bar{b}| + |\bar{b}|^2$

$\leq (|\bar{a}| + |b|)^2$

Finally taking the square root of both sides yields the final triangle inequality of

$|\bar{a} + \bar{b}| \leq |\bar{a}| + |\bar{b}|$ (Triangle Inequality) ***(Eq.3)***

This can be interpreted as the classical “the shortest distance between two points is a straight line” but extended to an arbitrary higher dimensional vector.

Scalar Projection

Since we interpret $<\bar{u}, \bar{v}>$ as the length of the projection of $\bar{u}$ onto $\bar{v}$ multiplied by the length of $\bar{v}$, if we normalized by the length of $\bar{v}$, we get the scalar projection of $\bar{u}$ onto $\bar{v}$.

$\text{comp}_{\bar{v}} \bar{u}$ = scalar projection of $\bar{u}$ onto $\bar{v}$

$\text{comp}_{\bar{v}} \bar{u} = \frac{<\bar{u},\bar{v}>}{|\bar{v}|}$ ***(Eq.4)***

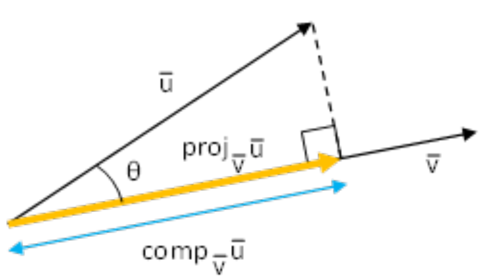
Alternatively, we can use Eq.2 to write

$\text{comp}_{\bar{v}} \bar{u} = \frac{<\bar{u},\bar{v}>}{|\bar{v}|}$

$= \frac{|\bar{u}| \, |\bar{v}| \cos(\theta)}{|\bar{v}|}$

$\text{comp}_{\bar{v}} \bar{u} = |\bar{u}| \cos(\theta)$ ***(Eq.5)***

Once again, we can physically interpret $\text{comp}_{\bar{v}} \bar{u}$ the length of the projection of $\bar{u}$ onto $\bar{v}$



Example

Consider the two vectors

$$\bar{u} = (\,1 \quad 3\,)^T$$
$$\bar{v} = (\,5 \quad 2\,)^T$$

Computing $\text{comp}_{\bar{v}}\,\bar{u}$ yields

$$\text{comp}_{\bar{v}}\,\bar{u} = \frac{<\bar{u},\bar{v}>}{|\bar{v}|} \qquad \text{recall:} \quad |\,\bar{v}\,| = (\bar{v}\cdot\bar{v})^{1/2} = <\bar{v},\,\bar{v}>^{1/2}$$

$$= \frac{<\bar{u},\bar{v}>}{<\bar{v},\bar{v}>^{1/2}}$$

$$= \frac{\bar{u}^T\,\bar{v}}{\left(\bar{v}^T\,\bar{v}\right)^{1/2}}$$

$$= \frac{\begin{pmatrix}1 & 3\end{pmatrix}\begin{pmatrix}5\\2\end{pmatrix}}{\left(\begin{pmatrix}5 & 2\end{pmatrix}\begin{pmatrix}5\\2\end{pmatrix}\right)^{1/2}}$$

$$= \frac{5*1+3*2}{(5*5+2*2)^{1/2}}$$

$$\text{comp}_{\bar{v}}\,\bar{u} = \frac{11}{(29)^{1/2}} \approx 2.04265$$

Vector Projection

Building on the $\text{comp}_{\bar{v}}\,\bar{u}$, we can create $\text{proj}_{\bar{v}}\,\bar{u}$ which is a vector

$$\text{proj}_{\bar{v}}\,\bar{u} = \text{vector projection of } \bar{u} \text{ onto } \bar{v}$$

$$= (\text{comp}_{\bar{v}}\,\bar{u}\,)\,\frac{\bar{v}}{|\,\bar{v}\,|}$$

$$\text{proj}_{\bar{v}}\,\bar{u} = \frac{<\bar{u},\bar{v}>}{|\,\bar{v}\,|^2}\,\bar{v} \qquad \textbf{\textit{(Eq.6)}}$$

Alternatively, we can use Eq.2 to write

$$\text{proj}_{\bar{v}}\,\bar{u} = \frac{<\bar{u},\bar{v}>}{|\,\bar{v}\,|^2}\,\bar{v}$$

$$= \frac{|\,\bar{u}\,|\;|\,\bar{v}\,|\,\cos(\theta)}{|\,\bar{v}\,|^2}\,\bar{v}$$

$$\text{proj}_{\overline{v}}\overline{u} = \frac{|\overline{u}|\cos(\theta)}{|\overline{v}|}\overline{v} \qquad \textbf{(Eq.7)}$$

Obviously, $|\text{proj}_{\overline{v}}\overline{u}| = \text{comp}_{\overline{v}}\overline{u}$

Example

Consider the same two vectors

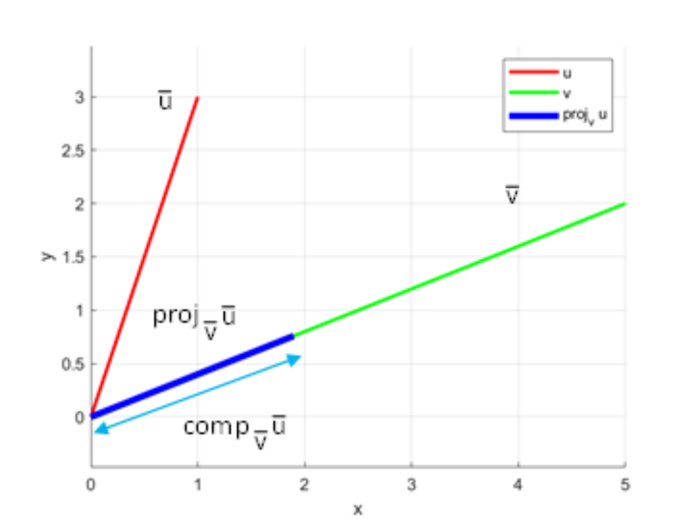
$$\text{proj}_{\overline{v}}\overline{u} = (\text{comp}_{\overline{v}}\overline{u})\frac{\overline{v}}{|\overline{v}|}$$

$$= \frac{11}{(29)^{1/2}}\frac{\begin{pmatrix}5\\2\end{pmatrix}}{(29)^{1/2}}$$

$$= \frac{11}{29}\begin{pmatrix}5\\2\end{pmatrix}$$

$$\text{proj}_{\overline{v}}\overline{u} = \begin{pmatrix}\frac{55}{29}\\\frac{22}{29}\end{pmatrix} \approx \begin{pmatrix}1.89655\\0.758621\end{pmatrix}$$

The picture that goes with this is shown below



Cross Product

The cross product of two vectors is given by the formula

$$\overline{u} \times \overline{v} = |\overline{u}||\overline{v}|\sin(\theta)\hat{n}$$

where $\hat{n}$ is the vector normal to both $\overline{u}$ and $\overline{v}$ according to the right hand rule
θ = angle between $\overline{u}$ and $\overline{v}$

Math Joke: The right hand rule is the engineering gang sign

Show movie (lecture_cross_product.m)

Can also use the following mnemonic to compute the cross product by hand

$$\overline{u} \times \overline{v} = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= i \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - j \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + k \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= i(u_2 v_3 - u_3 v_2) + j(u_3 v_1 - u_1 v_3) + k(u_1 v_2 - u_2 v_1)$$

$$\overline{u} \times \overline{v} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

Cross Product Rules

$k \overline{a} \times \overline{b} = k(\overline{a} \times \overline{b}) = \overline{a} \times k \overline{b}$ for all scalar k

$\overline{a} \times (\overline{b} + \overline{c}) = (\overline{a} \times \overline{b}) + (\overline{a} \times \overline{c})$ (distributive w.r.t. vector addition)

$(\overline{a} + \overline{b}) \times \overline{c} = (\overline{a} \times \overline{c}) + (\overline{b} \times \overline{c})$

$\overline{b} \times \overline{a} = -(\overline{a} \times \overline{b})$ (anti commutative)

Note that in most cases, it is not associative, that is

$\overline{a} \times (\overline{b} \times c) \neq (\overline{a} \times \overline{b}) \times \overline{c}$ (in most cases)

See Section 9.4 for some examples.

Math Joke: Cross Product

Question: What do you get when you cross a vector with a mountain climber?
Answer: Nothing, you can’t do this because a mountain climber is a scalar!

Question: What do you get when you cross a mountain goat with a mountain climber?
Answer: Nothing, you can’t do this because you can’t cross two scalars!

Mathematica provides the ‘Cross’ function to perform cross products. However note that the input to this function is a list, not a matrix.

```
(*Define u and v as Lists*)
Print["u̅ and v̅ as Lists"]
u = List[u1, u2, u3]  (*Use the List function to create a List*)
v = {v1, v2, v3}      (*Alternative syntax for defining a List*)

(*Perform the cross product operation*)
Print["u̅×v̅"]
uCrossv = Cross[u, v]

(*You can still MatrixForm to display a List*)
uCrossv // MatrixForm

(*Clear variables*)
Clear[u, v, uCrossv]
```

$\overline{u}$ and $\overline{v}$ as Lists

{u1, u2, u3}

{v1, v2, v3}

$\overline{u} \times \overline{v}$

{- u3 v2 + u2 v3, u3 v1 - u1 v3, - u2 v1 + u1 v2}

$$\begin{pmatrix} -u3\,v2 + u2\,v3 \\ u3\,v1 - u1\,v3 \\ -u2\,v1 + u1\,v2 \end{pmatrix}$$