Christopher Lum
lum@uw.edu

Lecture 02g
Partial Fraction Expansion

Lecture is on YouTube

The YouTube video entitled ‘Partial Fraction Expansion/Decomposition’ that covers this lecture is located at https://youtu.be/vlCdCAEtRag.

Outline

-Partial Fraction Expansion
- -Distinct Real Poles
- -Repeated Real Poles
- -Complex Conjugate Poles

Partial Fraction Expansion

This is sometimes known as partial fraction decomposition (Wikipedia). The basic concept is to take a ratio of expressions and try to express them as a sum of a series of simpler fractions.

$$\frac{f(s)}{g(s)} = \sum_j \frac{f_j(s)}{g_j(s)}$$

The reason we want to do this is that often, the right side of the expression is simpler and easier to analyze than the left side. We will see this is directly the case when talking about the inverse Laplace transform.

For the problems we encounter in this class, the complex functions $F(s)$ are often of the form

$$F(s) = \frac{B(s)}{A(s)}$$

where $B(s)$ and $A(s)$ are polynomials in s and the degree of $B(s)$ is not higher than that of $A(s)$

For example,

$$F(s) = \frac{s^3+3\,s^2-2\,s+2}{3\,s^3-2}$$

To start the partial fraction expansion process, the easiest thing to do is to factor function $F(s)$ into the following form (often referred to as ZPK or Zero, Pole, Gain form).

$$F(s) = \frac{B(s)}{A(s)} = \frac{K(s+z_1)\,(s+z_2)\ldots\,(s+z_m)}{(s+p_1)\,(s+p_2)\ldots\,(s+p_n)} \qquad (m \text{ zeros and } n \text{ poles})$$

it is assumed that $n \geq m$

Note that if $n = m$, we typically have to do some manipulation by dividing the numerator by the denominator in order to produce a polynomial in s plus a remainder. The resulting polynomial in s will have a denominator degree strictly greater than the numerator. We will go over an example of this later.

With partial fraction expansion, we need to consider three cases

1. Distinct real poles
2. Repeated real poles
3. Complex conjugate poles

At this point, one might ask, how do we find the poles of a complex function? If you recall, poles are defined as values of s which make the function go to ∞. In other words, these are roots of the denominator polynomial. We can use the Matlab or Mathematica to help solve for the roots (see YouTube video at https://youtu.be/J8il5eB_VS8)

Distinct Real Poles

If the poles are distinct (ie no repeats), then we can expand it into the sum of simple partial fractions

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{s+p_1} + \frac{a_2}{s+p_2} + \ldots + \frac{a_n}{s+p_n}$$

Example: Distinct Real Roots (strictly proper polynomial)

$$F(s) = \frac{s^2+8\,s+15}{s^3+3\,s^2+2\,s}$$

We can find the poles of this system

```
B[s_] = s^2 + 8 s + 15;
A[s_] = s^3 + 3 s^2 + 2 s;
poles = Solve[A[s] == 0, s]
poles2 = Roots[A[s] == 0, s]
```

```
{{s → -2}, {s → -1}, {s → 0}}
```

```
s == 0 || s == -2 || s == -1
```

We can verify this with the Matlab ‘roots’ function (be sure to remember the constant coefficient in this example)

```
>> roots([1 3 2 0])

ans =

     0
    -2
    -1
```

So we have

pole 1 = 0 $\quad \Rightarrow p_1 = 0$

pole 2 = -1 $\quad \Rightarrow p_2 = 1$

pole 3 = -2 $\quad \Rightarrow p_3 = 2$

We can first factor the denominator

$$= \frac{s^2+8s+15}{s(s+1)(s+2)}$$

We would like to write it in the form

$$\frac{s^2+8s+15}{s(s+1)(s+2)} = \frac{a_1}{s} + \frac{a_2}{s+1} + \frac{a_3}{s+2}$$

Method 1 : Multiplication Method

Let us examine the left hand side. To get a_1, multiply each term by s (the first root)

$$s\left(\frac{s^2+8s+15}{s(s+1)(s+2)}\right) = s\left(\frac{a_1}{s} + \frac{a_2}{s+1} + \frac{a_3}{s+2}\right)$$

$$\frac{s^2+8s+15}{(s+1)(s+2)} = a_1 + \frac{a_2 s}{s+1} + \frac{a_3 s}{s+2}$$

This must be true at $s = 0$ (value of the first pole). Substituting $s = 0$ into the above equation immediately yields the constant a_1

$$\frac{15}{2} = a_1$$

We can repeat the same procedure for finding a_2. Multiply each term by $s + 1$ (the second root)

$$(s+1)\left(\frac{s^2+8s+15}{s(s+1)(s+2)}\right) = (s+1)\left(\frac{a_1}{s} + \frac{a_2}{s+1} + \frac{a_3}{s+2}\right)$$

$$\frac{s^2+8s+15}{s(s+2)} = \frac{a_1(s+1)}{s} + a_2 + \frac{a_3(s+1)}{s+2}$$

This must be true at $s = -1$ (value of the second pole)

$$\frac{(-1)^2+8\times(-1)+15}{(-1)\times((-1)+2)} = a_2$$

```
(s^2 + 8 s + 15)/(s (s + 2)) /. {s → -1}
```

-8

So we obtain

$$-8 = a_2$$

Finally, repeat the same procedure for finding a_3. Multiply each term by $s + 2$ (the second root)

$$(s+2)\left(\frac{s^2+8s+15}{s(s+1)(s+2)}\right) = (s+2)\left(\frac{a_1}{s} + \frac{a_2}{s+1} + \frac{a_3}{s+2}\right)$$

$$\frac{s^2+8s+15}{s(s+1)} = \frac{a_1(s+2)}{s} + \frac{a_2(s+2)}{s+1} + a_3$$

This must be true at $s = -2$ (value of the third pole)

$$\frac{(-2)^2+8\times(-2)+15}{(-2)\times((-2)+1)} = a_3$$

```
(s^2 + 8 s + 15)/(s (s + 1)) /. {s → -2}
```

$\frac{3}{2}$

So we obtain

$$\frac{3}{2} = a_3$$

<u>Method 2 : Coefficients Method</u>

Let's get a common denominator on the right hand side

$$\frac{a_1}{s} + \frac{a_2}{s+1} + \frac{a_3}{s+2} = \frac{a_1(s+1)(s+2)}{s(s+1)(s+2)} + \frac{a_2(s+2)s}{s(s+1)(s+2)} + \frac{a_3(s+1)s}{s(s+1)(s+2)}$$

$$= \frac{a_1(s+1)(s+2)+a_2(s+2)s+a_3(s+1)s}{s(s+1)(s+2)}$$

Expand the numerator and collect terms of s

```
Collect[Expand[a1 (s + 1) (s + 2) + a2 (s + 2) s + a3 (s + 1) s], s]
```

2 a1 + (3 a1 + 2 a2 + a3) s + (a1 + a2 + a3) s^2

So we have

$$\frac{s^2+8\,s+15}{s(s+1)\,(s+2)} = \frac{(a_1+a_2+a_3)\,s^2+(3\,a_1+2\,a_2+a_3)\,s+2\,a_1}{s(s+1)\,(s+2)}$$

Equating the coefficients yields a systems of equations

$$1 = a_1 + a_2 + a_3$$
$$8 = 3\,a_1 + 2\,a_2 + a_3$$
$$15 = 2\,a_1$$

Writing in matrix form

$$\begin{pmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ 15 \end{pmatrix}$$

We can solve this using Gaussian elimination and back substitution. Alternatively, the matrix should not be singular so we can solve directly using the inverse.

```
Inverse[(1 1 1
         3 2 1
         2 0 0)].(1
                  8
                  15) // MatrixForm
```

$$\begin{pmatrix} \frac{15}{2} \\ -8 \\ \frac{3}{2} \end{pmatrix}$$

So we obtain the same results of

$$a_1 = 15/2$$
$$a_2 = -8$$
$$a_3 = 3/2$$

So we see that the partial fraction expansion in this case is given as

$$F(s) = \frac{s^2+8\,s+15}{s^3+3\,s^2+2\,s} = \frac{15/2}{s} + \frac{-8}{s+1} + \frac{3/2}{s+2}$$

I want to keep the focus of this lecture to be purely on the mathematical approach of applying partial fraction expansion so we will defer the discussion on applying this result to the inverse Laplace transform for a later lecture but keep this result in mind as we will revisit it shortly.

Repeated Real Poles

Consider the case

$$F(s) = \frac{s^2+2\,s+3}{(s+1)^3}$$

```
B[s_] = s^2 + 2 s + 3;
A[s_] = (s + 1)^3;
```

What happens if we try the same methods as the distinct poles case? In other words, what happens if we attempt a partial fraction expansion of the form

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{(s+1)} + \frac{a_2}{(s+1)} + \frac{a_3}{(s+1)}$$

<u>Method 1 : Multiplication Method</u>

Let' s look at the left hand side. To get a_1, multiply each term by $s + 1$ (the first root)

$$(s + 1)\left(\frac{s^2+2\,s+3}{(s+1)^3}\right) = (s + 1)\left(\frac{a_1}{(s+1)} + \frac{a_2}{(s+1)} + \frac{a_3}{(s+1)}\right)$$

$$\frac{s^2+2\,s+3}{(s+1)^2} = a_1 + a_2 + a_3$$

This must be true at $s = -1$ (value of the first pole), but now we get a divide by 0

$$\frac{s^2+2\,s+3}{0} = a_1 + a_2 + a_3 \;\; \textit{(Error!)}$$

<u>Method 2 : Coefficients Method</u>

Let's get a common denominator on the right hand side

$$\frac{s^2+2\,s+3}{(s+1)^3} = \frac{a_1}{(s+1)} + \frac{a_2}{(s+1)} + \frac{a_3}{(s+1)}$$

$$= \frac{a_1+a_2+a_3}{(s+1)}$$

$$= \frac{(a_1+a_2+a_3)\,(s+1)^2}{(s+1)\,(s+1)^2}$$

So we already see we have a problem since there are no higher powers of s so this will not work. This is because we need another type of partial fraction expansion for the repeated roots case.

<u>Method 1 : Derivative Method</u>

So instead, let us propose a partial fraction expansion of the form

$$F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{(s+1)} + \frac{a_2}{(s+1)^2} + \frac{a_3}{(s+1)^3}$$

Multiplying both sides by $(s + 1)^3$ yields

$$(s + 1)^3\,\frac{B(s)}{A(s)} = a_1(s + 1)^2 + a_2(s + 1) + a_3 \qquad \textbf{\textit{(Eq.2.8)}}$$

Evaluating Eq.2.8 at $s = -1$ (value of the repeated pole) yields

$$\left[(s+1)^3 \frac{B(s)}{A(s)}\right] \Big|_{s=-1} = \left[a_1 (s+1)^2 + a_2 (s+1) + a_3\right] \Big|_{s=-1} \qquad \textbf{\textit{(Eq.2.11)}}$$

$$\left[(s+1)^3 \frac{s^2+2s+3}{(s+1)^3}\right] \Big|_{s=-1} = \left[a_1 (s+1)^2 + a_2 (s+1) + a_3\right] \Big|_{s=-1}$$

$$\left[s^2 + 2s + 3\right] \Big|_{s=-1} = \left[a_1 (s+1)^2 + a_2 (s+1) + a_3\right] \Big|_{s=-1}$$

$$(-1)^2 + 2 \times (-1) + 3 = a_1 (-1+1)^2 + a_2 (-1+1) + a_3$$

$$2 = a_3$$

Note that the left hand side is not zero because $A(s)$ has enough poles to cancel the numerator $(s+1)^3$ term

```
a3 = (s + 1)^3 B[s]/A[s] /. {s → -1}
```

2

If we differentiate both sides once with respect to s

$$\frac{d}{ds}\left[(s+1)^3 \frac{B(s)}{A(s)}\right] = \frac{d}{ds}\left[a_1 (s+1)^2 + a_2 (s+1) + a_3\right]$$

$$\frac{d}{ds}\left[(s+1)^3 \frac{B(s)}{A(s)}\right] = 2 a_1 (s+1) + a_2 \qquad \textbf{\textit{(Eq.2.9)}}$$

$$\frac{d}{ds}\left[(s+1)^3 \frac{s^2+2s+3}{(s+1)^3}\right] = 2 a_1 (s+1) + a_2$$

$$\frac{d}{ds}\left[s^2 + 2s + 3\right] = 2 a_1 (s+1) + a_2$$

$$2s + 2 = 2 a_1 (s+1) + a_2 \qquad \text{note: this must be true at } s = -1 \text{ (value of the repeated pole)}$$

$$2 \times (-1) + 2 = a_2$$

$$a_2 = 0$$

```
a2 = D[(s + 1)^3 B[s]/A[s], s] /. {s → -1}
```

0

Differentiating both sides of Eq.2.9 once more with respect to s yields

$$\frac{d^2}{ds^2}\left[(s+1)^3 \frac{B(s)}{A(s)}\right] = \frac{d}{ds}\left[2\,a_1(s+1) + a_2\ \ \right]$$

$$\frac{d^2}{ds^2}\left[(s+1)^3 \frac{s^2+2s+3}{(s+1)^3}\right] = 2\,a_1 \qquad \textbf{\textit{(Eq.2.10)}}$$

$$\frac{d^2}{ds^2}\left[s^2 + 2s + 3\right] = 2\,a_1$$

$$\frac{d}{ds}[2s+2] = 2\,a_1$$

$$2 = 2\,a_1$$

$$a_1 = 1$$

```
a1 = 1/2 D[(s + 1)^3 B[s]/A[s], {s, 2}]
```

```
1
```

So we can write the function $F(s)$ as

$$F(s) = \frac{s^2+2s+3}{(s+1)^3} = \frac{1}{(s+1)} + \frac{2}{(s+1)^3}$$

In general, we have

$$a_l = \frac{1}{(k-l)!}\left(\frac{d^{k-l}}{ds^{k-l}}\left[(s-p_i)^k\,F(s)\right]\,|_{\,s=p_i}\right)$$

<u>Method 2 : Coefficient Method</u>

Once again, we can use the coefficient method by obtaining a common denominator of the partial fraction expansion

$$\frac{s^2+2s+3}{(s+1)^3} = \frac{a_1}{(s+1)} + \frac{a_2}{(s+1)^2} + \frac{a_3}{(s+1)^3}$$

$$= \frac{a_1(s+1)^2}{(s+1)^3} + \frac{a_2(s+1)}{(s+1)^3} + \frac{a_3}{(s+1)^3}$$

$$= \frac{a_1(s+1)^2+a_2(s+1)+a_3}{(s+1)^3}$$

```
Clear[a1, a2, a3]
Collect[Expand[a1 (s + 1)^2 + a2 (s + 1) + a3], {s, s^2}]
```

```
a1 + a2 + a3 + (2 a1 + a2) s + a1 s^2
```

So we have

$$\frac{s^2+2\,s+3}{(s+1)^3} = \frac{a_1\,s^2+(2\,a_1+a_2)\,s+(a_1+a_2+a_3)}{(s+1)^3}$$

Comparing coefficients of various powers of s, we see that

$$1 = a_1$$
$$2 = 2\;a_1 + a_2$$
$$3 = a_1 + a_2 + a_3$$

In matrix form

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

```
Inverse[(1 0 0
         2 1 0
         1 1 1)].(1
                  2
                  3) // MatrixForm
```

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

So we have

$$a_1 = 1$$
$$a_2 = 0$$
$$a_3 = 2$$

which is the same result

Complex Conjugate Poles

Consider a second order denominator of the form

$$a\,s^2 + b\,s + c$$

Recall from the quadratic equation the roots are given by

$$\frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a}$$

If $4\,a\,c > b^2$, the poles are imaginary (but come in complex conjugates). Therefore, if you attempt to factor the denominator, you obtain imaginary numbers.

We can complete the square to remedy this problem. For a quadratic of the form

$$s^2 + d\,s + e$$

We would like it to be of the form

$$(s+\alpha)^2+\omega^2$$

where $\quad \alpha = d/2$

$$\omega = \frac{\sqrt{4e-d^2}}{2}$$

Example

$$F(s) = \frac{s-1}{s^2+2s+2}$$

We can find the roots of this characteristic equation

```
Solve[s^2 + 2 s + 2 == 0, s]
```

```
{{s → -1 - i}, {s → -1 + i}}
```

So we see that the roots are complex and therefore we need to complete the square

We can complete the squares of the denominator

```
d = 2;
e = 2;
α = d / 2
ω = Sqrt[4 e - d^2] / 2
```

```
1
```

```
1
```

So we obtain

$$s^2+2s+2=(s+1)^2+1^2$$

$$F(s) = \frac{s-1}{(s+1)^2+1}$$

Example with Improper Complex Function

What if the numerator and denominator are same order (or numerator is higher)?

$$F(s) = \frac{s^3+5s^2+9s+7}{(s+1)(s+2)}$$

we can use synthetic division to obtain

$$F(s) = s+2+\frac{s+3}{(s+1)(s+2)}$$

We can now follow same procedure to break up last term using partial fraction expansion

$$\frac{s+3}{(s+1)(s+2)} = \frac{a_1}{s+1} + \frac{a_2}{s+2}$$

```
Collect[Expand[a1 (s + 2) + a2 (s + 1)], s]
```

```
2 a1 + a2 + (a1 + a2) s
```

```
Solve[{a1 + a2 == 1, 2 a1 + a2 == 3}, {a1, a2}]
```

```
{{a1 → 2, a2 → -1}}
```

So we can write

$$F(s) = s + 2 + \frac{2}{s+1} - \frac{1}{s+2}$$

$$s + 2 + \frac{2}{s+1} - \frac{1}{s+2} \text{ // Together}$$

$$\frac{7 + 9\,s + 5\,s^2 + s^3}{(1 + s) \times (2 + s)}$$

So we see that it is feasible to perform the partial fraction expansion but the physical implications of this will be examined when we look at the inverse Laplace Transform of this function.