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Lecture09a
Similarity Transformation and Diagonalization

Lecture is on YouTube

The YouTube video entitled ‘Similarity Transformation and Diagonalization’ that covers this lecture is located at https://youtu.be/wvRlvDYDIgw.

Outline

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- -Property 3 : Similar eigenvectors
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Similarity Transformation

The mathematical definition of a similarity transformation within the context of linear algebra (https://en.wikipedia.org/wiki/Matrix_similarity) concerns two n – by – n matrices, A and $\tilde{A}$ are called similar if there exists an invertible n – by – n matrix T such that

$$\tilde{A} = T^{-1} A T \qquad \textbf{(Eq.1)}$$

As the name suggests, the two matrices A and $\tilde{A}$ are similar in the sense that they share some similar (or closely related) properties.

Property 1 : Same determinant

Recall the determinant of a matrix product (using notation that $|M| = \det(M)$) has the property that $|AB| = |A| \cdot |B|$. So applied to our similarity transformation we can write

$$|\tilde{A}| = |T^{-1} A T|$$

$$= |T^{-1}| \cdot |A| \cdot |T| \quad \text{recall:} \quad |M^{-1}| = |M|^{-1}$$

$$= |T|^{-1} \cdot |A| \cdot |T|$$

$$= \frac{1}{|T|} \cdot |A| \cdot |T|$$

$$|\tilde{A}| = |A| \qquad \textbf{\textit{(Eq.2)}}$$

So we see that the determinant of the matrix is not changed by the similarity transformation.

Property 2 : Same characteristic equations (and therefore same eigenvalues)

Recall the definition of eigenvalues/eigenvectors is

$$A\overline{v} = \lambda \overline{v}$$

$$(A - \lambda I)\overline{v} = \overline{0}$$

So eigenvalues are found by solving characteristic equation of

$$|A - \lambda I| = 0$$

Let's rewrite this using the fact that $T \tilde{A} T^{-1} = A$

$$|A - \lambda I| = |T \tilde{A} T^{-1} - \lambda I|$$

$$= |T \tilde{A} T^{-1} - \lambda T T^{-1}|$$

$$= |T \tilde{A} T^{-1} - T(\lambda I) T^{-1}|$$

$$= |T(\tilde{A} - \lambda I) T^{-1}| \qquad \text{recall:} \quad |AB| = |A| \cdot |B|$$

$$= |T| \cdot |\tilde{A} - \lambda I| \cdot |T^{-1}| \qquad \text{recall:} \quad |A^{-1}| = |A|^{-1}$$

$$= |T| \cdot |\tilde{A} - \lambda I| \cdot |T|^{-1}$$

$$= \left| T \right| \cdot \left| \tilde{A} - \lambda I \right| \cdot \frac{1}{|T|}$$

$$| A - \lambda I | = \left| \tilde{A} - \lambda I \right|$$

Since the characteristic equation of A and $\tilde{A}$ are the same, eigenvalues don't change under a similarity transformation. This holds true even with multiplicity.

$$\text{eigenvalues}(\tilde{A}) = \text{eigenvalues}(A) \qquad \textbf{\textit{(Eq.3)}}$$

Property 3 : Similar eigenvectors

Suppose that λ_i is an eigenvalue of A and $\overline{v}_i$ is the corresponding vector

λ_i, $\overline{v}_i$ = an eigenvalue and eigenvector pair of A

We can multiply $\tilde{A}$ by an arbitrary vector $\overline{u}_i$

$$\tilde{A}\,\overline{u}_i = T^{-1} A\, T\, \overline{u}_i \qquad \text{let } \overline{u}_i = T^{-1}\, \overline{v}_i$$

$$= T^{-1} A\, T\, T^{-1}\, \overline{v}_i$$

$$= T^{-1} A\, \overline{v}_i \qquad \text{recall by the definition of an eigenvalue/eigenvector } A\, \overline{v}_i = \lambda_i\, \overline{v}_i$$

$$= T^{-1}\, \lambda_i\, \overline{v}_i$$

$$\tilde{A}\,\overline{u}_i = \lambda_i\, T^{-1}\, \overline{v}_i \qquad \text{recall } \overline{u}_i = T^{-1}\, \overline{v}_i$$

$$\tilde{A}\,\overline{u}_i = \lambda_i\, \overline{u}_i$$

So we see that this is the definition of an eigenvalue/eigenvector for $\tilde{A}$ and therefore $\overline{u}_i$ is an eigenvector of $\tilde{A}$ and λ_i is the corresponding eigenvalue of $\tilde{A}$. We can then note that we previously used

$$\overline{u}_i = T^{-1}\, \overline{v}_i$$

So we see that the eigenvectors of $\tilde{A}$ are different from the eigenvectors of A but they are related through

$$\text{eigenvectors}(\tilde{A}) = T^{-1}\, \text{eigenvectors}(A) \qquad \textbf{\textit{(Eq.4)}}$$

Property 4 : Same trace

Recall that the trace of a matrix is simply the sum of the diagonal elements. Trace is only defined for square matrices.

$$\text{trace}(A) = \text{tr}(A) = \sum_{i=1}^{n} A_{\text{ii}}$$

Recall that if A is the product of two matrices ($A = B_{n\times m}\, C_{m\times n}$) then we can write

$$\text{tr}(B\,C) = \text{tr}(C\,B)$$

In[∘]:=
```
Bmat = ( b11  b12
         b21  b22
         b31  b32 ); Cmat = ( c11  c12  c13
                              c21  c22  c23 );
Amat = Bmat.Cmat;
Amat // MatrixForm

Tr[Bmat.Cmat] == Tr[Cmat.Bmat]
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \text{b11 c11} + \text{b12 c21} & \text{b11 c12} + \text{b12 c22} & \text{b11 c13} + \text{b12 c23} \\ \text{b21 c11} + \text{b22 c21} & \text{b21 c12} + \text{b22 c22} & \text{b21 c13} + \text{b22 c23} \\ \text{b31 c11} + \text{b32 c21} & \text{b31 c12} + \text{b32 c22} & \text{b31 c13} + \text{b32 c23} \end{pmatrix}$$

Out[∘]= True

For our similarity transformation we have

$$\text{tr}(\tilde{A}) = \text{tr}\left(T^{-1}\,A\,T\right) \qquad \text{let } B = T^{-1},\ C = A\,T$$

$$= \text{tr}(B\,C) \qquad \text{recall: tr}(B\,C) = \text{tr}(C\,B)$$

$$= \text{tr}(C\,B)$$

$$= \text{tr}\left(A\,T\,T^{-1}\right)$$

$$\text{tr}(\tilde{A}) = \text{tr}(A) \qquad \textbf{\textit{(Eq.5)}}$$

So we see that the trace of the matrix is not changed by the similarity transformation.

Property 5 : Same rank

We now consider the rank of $\tilde{A}$. To start, we can recall that rank (A) is the dimension of the vector space spanned by the columns of A. This corresponds to the maximal number of linearly independent columns of A. If we multiply the matrix A by an invertible matrix T, this is equivalent to transforming the vector space spanned by the columns but it does not reduce the number of linearly independent vectors. Therefore

$$\text{rank}(T\,A) = \text{rank}(A\,T) = \text{rank}(A) \qquad \text{for } T \text{ invertible} \qquad \textbf{\textit{(Eq.6.a)}}$$
$$\text{rank}\left(T^{-1}\,A\right) = \text{rank}\left(A\,T^{-1}\right) = \text{rank}(A) \quad \text{for } T^{-1} \text{ invertible}$$

An alterative way to prove this is to use the fact that rank($A\,B$) ≤ min(rank(A), rank(B)) for any two matrices A and B. If $B = T$ is invertible we can write

$$\text{rank}(A\,T) \leq \min(\text{rank}(A), \text{rank}(T))$$

We note that $\{T \text{ invertible}\} \Longleftrightarrow \{\text{rank}(T) = n\}$ and therefore $\text{rank}(T) = n$ must be greater than or equal to $\text{rank}(A)$ so we have

$$\text{rank}(A\,T) \leq \text{rank}(A) \qquad \textbf{\textit{(Eq.6.b)}}$$

We can make use of Eq.6.b by replacing the matrices A and T with $A \to A\,T$ and $T \to T^{-1}$

$$\text{rank}\left((A\,T)\,T^{-1}\right) \leq \text{rank}(A\,T)$$

$$\text{rank}(A) \leq \text{rank}(A\,T) \qquad \textbf{\textit{(Eq.6.c)}}$$

So we see that the only way that Eq.6.b and Eq.6.c are true simultaneously is if

$$\text{rank}(A\,T) = \text{rank}(A) \qquad \textbf{\textit{(Eq.6.d)}}$$

The same can be done to show $\text{rank}(T\,A) = \text{rank}(A)$ which yields Eq.6.a as previously described.

We can repeat this for a matrix T^{-1} to show that

$$\text{rank}\left(A\,T^{-1}\right) = \text{rank}\left(T^{-1}\,A\right) = \text{rank}(A)$$

So for our similar matrix, we can write

$$\text{rank}(\tilde{A}) = \text{rank}\left(T^{-1}\,A\,T\right)$$

$$= \text{rank}\left(\left(T^{-1}\,A\right)T\right) \qquad \text{recall: } \text{rank}\left(\left(T^{-1}\,A\right)T\right) = \text{rank}\left(T^{-1}\,A\right) \text{ pursuant to Eq.6.a}$$

$$= \text{rank}\left(T^{-1}\,A\right) \qquad \text{recall: } \text{rank}\left(T^{-1}\,A\right) = \text{rank}(A) \text{ pursuant to Eq.6.a}$$

$$\text{rank}(\tilde{A}) = \text{rank}(A) \qquad \textbf{\textit{(Eq.6)}}$$

Diagonalization

One particularly useful transformation matrix is a matrix of eigenvectors of the A matrix. Recall that an eigenvalue and an eigenvector satisfy the following equation

$$A\,\overline{v}_i = \lambda_i\,\overline{v}_i$$

$$A\,\overline{v}_i - \lambda_i\,\overline{v}_i = \overline{0}$$

Let's write this out as a series of equations ($i = 1, \ldots, n$)

$$A\,\overline{v}_1 - \lambda_1\,\overline{v}_1 = \overline{0}$$
$$A\,\overline{v}_2 - \lambda_2\,\overline{v}_2 = \overline{0}$$
$$\vdots$$
$$A\,\overline{v}_n - \lambda_n\,\overline{v}_n = \overline{0}$$

Stacking each equation into a separate column yields

$$[\,A\,\overline{v}_1 \quad A\,\overline{v}_2 \quad \ldots \quad A\,\overline{v}_n\,] - [\,\lambda_1\,\overline{v}_1 \quad \lambda_2\,\overline{v}_2 \quad \ldots \quad \lambda_n\,\overline{v}_n\,] = [\,\overline{0} \quad \overline{0} \quad \ldots \quad \overline{0}\,]$$

$$A[\,\overline{v}_1 \quad \overline{v}_2 \quad \ldots \quad \overline{v}_n\,] - [\,\overline{v}_1 \quad \overline{v}_2 \quad \ldots \quad \overline{v}_n\,]\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \ldots & \ldots & \vdots \\ 0 & \ldots & \ldots & \lambda_n \end{pmatrix} = [\,\overline{0} \quad \overline{0} \quad \ldots \quad \overline{0}\,]$$

$$A[\,\overline{v}_1 \quad \overline{v}_2 \quad \ldots \quad \overline{v}_n\,] = [\,\overline{v}_1 \quad \overline{v}_2 \quad \ldots \quad \overline{v}_n\,]\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \ldots & \ldots & \vdots \\ 0 & \ldots & \ldots & \lambda_n \end{pmatrix}$$

let $T = [\,\overline{v}_1 \quad \overline{v}_2 \quad \ldots \quad \overline{v}_n\,]$

$$A\,T = T\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \ldots & \ldots & \vdots \\ 0 & \ldots & \ldots & \lambda_n \end{pmatrix}$$

If T is invertible (AKA if the eigenvectors span the space) then we can write

$$\tilde{A} = T^{-1}\,A\,T \qquad \textbf{\textit{(Eq.7)}}$$

where $\tilde{A} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \ldots & \ldots & \vdots \\ 0 & \ldots & \ldots & \lambda_n \end{pmatrix}$ (diagonal matrix with eigenvalues along the diagonal)

$T = [\,\overline{v}_1 \quad \overline{v}_2 \quad \ldots \quad \overline{v}_n\,]$ (vector of eigenvectors in column format)

So we see that using this transformation will yield a diagonal $\tilde{A}$ matrix with the eigenvalues on the diagonal. The similarity transform T = eigenvectors(A) is known as the modal matrix of A.

This helps us understand the result at https://en.wikipedia.org/wiki/Diagonalizable_matrix which states that:

"An $n \times n$ matrix A over a field F is diagonalizable if and only if the sum of the dimensions of its eigenspaces is equal to n, which is the case if and only if there exists a basis of F^n consistent of eigenvectors of A. If such a basis has been found, one can form the matrix P having these basis vectors as columns, and $P^{-1} A P$ will be a diagonal matrix whose diagonal entries are the eigenvalues of A. The matrix P is known as the modal matrix for A."

Examples

Example 1: Non-Defective Matrix

Consider the following non-defective matrix (meaning it can be diagonalized).

In[∘]:=
$$A = \begin{pmatrix} 3 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{pmatrix};$$

We can use a transformation matrix of

$$T = \begin{pmatrix} 1 & \frac{2}{7} & 4 & -3 \\ 2 & 1 & 2 & \frac{3}{4} \\ -2 & \frac{2}{3} & \frac{3}{4} & 2 \\ -\frac{5}{6} & -5 & 8 & 6 \end{pmatrix}$$

We can verify that T is invertible

In[∘]:=
```
Det[T]
```

Out[∘]=
$$\frac{17933}{56}$$

We can calculate $\tilde{A}$

In[∘]:=
```
Atilde = Inverse[T].A.T;
Atilde // MatrixForm
Atilde // MatrixForm // N
```

Out[∘]//MatrixForm=
$$\begin{pmatrix} \frac{235009}{71732} & \frac{768517}{753186} & -\frac{127303}{35866} & -\frac{332845}{71732} \\ -\frac{728105}{430392} & \frac{1653481}{645588} & -\frac{204022}{53799} & \frac{50267}{430392} \\ \frac{14635}{53799} & \frac{24730}{125531} & \frac{75763}{17933} & \frac{24300}{17933} \\ -\frac{245935}{322794} & \frac{1524821}{3389337} & -\frac{927161}{161397} & -\frac{342869}{322794} \end{pmatrix}$$

Out[∘]//MatrixForm=
$$\begin{pmatrix} 3.27621 & 1.02035 & -3.54941 & -4.64012 \\ -1.69173 & 2.5612 & -3.7923 & 0.116794 \\ 0.272031 & 0.197003 & 4.22478 & 1.35504 \\ -0.761895 & 0.449888 & -5.7446 & -1.06219 \end{pmatrix}$$

We can verify the aforementioned properties.

Property 1: Same determinant

In[]:=
```
detA = Det[A]
detAtilde = Det[Atilde]
detA == detAtilde
```

Out[]= 20

Out[]= 20

Out[]= True

Property 2: Same characteristic equations (and therefore same eigenvalues)

We can verify that $| A - \lambda I | = | \tilde{A} - \lambda I |$

In[]:=
```
charEqA = Det[A - λ * IdentityMatrix[4]]
charEqAtilde = Det[Atilde - λ * IdentityMatrix[4]]
charEqA == charEqAtilde
```

Out[]= $20 - 42\lambda + 30\lambda^2 - 9\lambda^3 + \lambda^4$

Out[]= $20 - 42\lambda + 30\lambda^2 - 9\lambda^3 + \lambda^4$

Out[]= True

Which means the eigenvalues are the same

In[]:=
```
eigA = Eigenvalues[A]
eigAtilde = Eigenvalues[Atilde]
eigA == eigAtilde
```

Out[]= {3 + i, 3 - i, 2, 1}

Out[]= {3 + i, 3 - i, 2, 1}

Out[]= True

Property 3: Similar eigenvectors

We can now verify that eigenvectors($\tilde{A}$) = T^{-1} eigenvectors(A)

In[∘]:=
```
(*Mathematica's Eigenvectors returns eigenvectors in row format,
transpose to get into col format*)
eigVecA = Transpose[Eigenvectors[A]];
eigVecAtilde = Transpose[Eigenvectors[Atilde]];

eigVecA // MatrixForm
eigVecAtilde // MatrixForm // N
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} i & -i & 2 & -9 \\ 0 & 0 & -1 & 5 \\ -1 & -1 & 1 & -2 \\ 1 & 1 & 0 & 2 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0.191206 + 1.77394\, i & 0.191206 - 1.77394\, i & 1.72126 & 1.22901 \\ 0.287534 - 2.35959\, i & 0.287534 + 2.35959\, i & -0.348722 & -0.0135712 \\ -0.709973 - 0.594141\, i & -0.709973 + 0.594141\, i & -0.788653 & -0.523043 \\ 1. & 1. & 1. & 1. \end{pmatrix}$$

In[∘]:=
```
(*Check that we extracted the eigenvectors correctly*)
v1 = Transpose[{eigVecA[[All, 1]]}];
v2 = Transpose[{eigVecA[[All, 2]]}];
v3 = Transpose[{eigVecA[[All, 3]]}];
v4 = Transpose[{eigVecA[[All, 4]]}];

vtilde1 = Transpose[{eigVecAtilde[[All, 1]]}];
vtilde2 = Transpose[{eigVecAtilde[[All, 2]]}];
vtilde3 = Transpose[{eigVecAtilde[[All, 3]]}];
vtilde4 = Transpose[{eigVecAtilde[[All, 4]]}];

v1 // MatrixForm
v2 // MatrixForm
v3 // MatrixForm
v4 // MatrixForm

vtilde1 // MatrixForm // N
vtilde2 // MatrixForm // N
vtilde3 // MatrixForm // N
vtilde4 // MatrixForm // N
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} i \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} -i \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Out[•]//MatrixForm=

$$\begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix}$$

Out[•]//MatrixForm=

$$\begin{pmatrix} -9 \\ 5 \\ -2 \\ 2 \end{pmatrix}$$

Out[•]//MatrixForm=

$$\begin{pmatrix} 0.191206 + 1.77394 i \\ 0.287534 - 2.35959 i \\ -0.709973 - 0.594141 i \\ 1. \end{pmatrix}$$

Out[•]//MatrixForm=

$$\begin{pmatrix} 0.191206 - 1.77394 i \\ 0.287534 + 2.35959 i \\ -0.709973 + 0.594141 i \\ 1. \end{pmatrix}$$

Out[•]//MatrixForm=

$$\begin{pmatrix} 1.72126 \\ -0.348722 \\ -0.788653 \\ 1. \end{pmatrix}$$

Out[•]//MatrixForm=

$$\begin{pmatrix} 1.22901 \\ -0.0135712 \\ -0.523043 \\ 1. \end{pmatrix}$$

In[•]:=
```
(*Extract the eigenvalues*)
λ1 = eigA[[1]]
λ2 = eigA[[2]]
λ3 = eigA[[3]]
λ4 = eigA[[4]]
```

Out[•]= 3 + i

Out[•]= 3 − i

Out[•]= 2

Out[•]= 1

We can verify that these are a valid set of eigenvectors by ensuring they satisfy

$$A\,\overline{v}_i = \lambda_i\,\overline{V}_i$$

In[ ]:=
```
(*Verify A v̄i = λi v̄i*)
A.v1 == λ1 * v1
A.v2 == λ2 * v2
A.v3 == λ3 * v3
A.v4 == λ4 * v4

Atilde.vtilde1 == λ1 * vtilde1
Atilde.vtilde2 == λ2 * vtilde2
Atilde.vtilde3 == λ3 * vtilde3
Atilde.vtilde4 == λ4 * vtilde4
```

Out[ ]= True

Out[ ]= True

Out[ ]= True

Out[ ]= True

Out[ ]= True

Out[ ]= True

Out[ ]= True

Out[ ]= True

We can now verify that eigenvectors($\tilde{A}$) = T^{-1} eigenvectors(A)

In[ ]:=
```
Inverse[T].v1 // MatrixForm // N
vtilde1 // MatrixForm // N
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0.295266 - 0.102074\, i \\ -0.413958 + 0.0432954\, i \\ -0.0736073 + 0.144426\, i \\ -0.0391457 - 0.170666\, i \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0.191206 + 1.77394\, i \\ 0.287534 - 2.35959\, i \\ -0.709973 - 0.594141\, i \\ 1. \end{pmatrix}$$

At first glance, it appears that $T^{-1}\,\overline{v}_1 \neq \tilde{v}_1$ but we can show that it satisfies $\tilde{A}\ T^{-1}\,\overline{v}_1 = \lambda_1\, T^{-1}\,\overline{v}_1$ thereby showing it is an eigenvector associated with λ_1

In[ ]:=
```
Atilde.Inverse[T].v1 == λ1 * Inverse[T].v1
Atilde.Inverse[T].v2 == λ2 * Inverse[T].v2
Atilde.Inverse[T].v3 == λ3 * Inverse[T].v3
Atilde.Inverse[T].v4 == λ4 * Inverse[T].v4
```

Out[ ]= True

Out[ ]= True

Out[ ]= True

Out[ ]= True

Property 4: Same trace

In[ ]:=
```
TrA = Tr[A]
TrAtilde = Tr[Atilde]
TrA == TrAtilde
```

Out[ ]= 9

Out[ ]= 9

Out[ ]= True

Property 5: Same rank

In[ ]:=
```
rankA = MatrixRank[A]
rankAtilde = MatrixRank[Atilde]
rankA == rankAtilde
```

Out[ ]= 4

Out[ ]= 4

Out[ ]= True

Diagonalization

We can use the modal transformation of M = eigenvectors(A) to diagonalize the matrix using $M^{-1} A M$

In[ ]:=
```
M = eigVecA;
Inverse[M].A.M // MatrixForm
eigA
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 3+i & 0 & 0 & 0 \\ 0 & 3-i & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[ ]= $\{3+i, 3-i, 2, 1\}$

Example 2: Defective Matrix

Consider the following defective matrix (meaning it cannot be diagonalized). We use the same matrix as before except we change the (1, 1) element ($3 \to 5$)

In[∘]:=
```
A = {{5, 4, 2, 1}, {0, 1, -1, -1}, {-1, -1, 3, 0}, {1, 1, -1, 2}};
```

$$A = \begin{pmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{pmatrix};$$

We can use the same transformation matrix as before and calculate $\tilde{A}$

In[∘]:=
```
Atilde = Inverse[T].A.T;
Atilde // MatrixForm
Atilde // MatrixForm // N
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{220365}{71732} & \frac{724585}{753186} & -\frac{156591}{35866} & -\frac{288913}{71732} \\ -\frac{230279}{143464} & \frac{1669453}{645588} & -\frac{61796}{17933} & -\frac{61537}{430392} \\ \frac{30175}{53799} & \frac{35090}{125531} & \frac{96483}{17933} & \frac{8760}{17933} \\ -\frac{118705}{107598} & \frac{1194281}{3389337} & -\frac{382507}{53799} & -\frac{12329}{322794} \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 3.07206 & 0.962027 & -4.366 & -4.02767 \\ -1.60513 & 2.58594 & -3.44594 & -0.142979 \\ 0.560884 & 0.279533 & 5.38019 & 0.488485 \\ -1.10323 & 0.352364 & -7.10993 & -0.0381946 \end{pmatrix}$$

We can verify the aforementioned properties.

Property 1: Same determinant

In[∘]:=
```
detA = Det[A]
detAtilde = Det[Atilde]
detA == detAtilde
```

Out[∘]= 32

Out[∘]= 32

Out[∘]= True

Property 2: Same characteristic equations (and therefore same eigenvalues)

We can verify that $| A - \lambda I | = \left| \tilde{A} - \lambda I \right|$

In[∘]:=
```
charEqA = Det[A - λ * IdentityMatrix[4]]
charEqAtilde = Det[Atilde - λ * IdentityMatrix[4]]
charEqA == charEqAtilde
```

Out[∘]= $32 - 64\lambda + 42\lambda^2 - 11\lambda^3 + \lambda^4$

Out[∘]= $32 - 64\lambda + 42\lambda^2 - 11\lambda^3 + \lambda^4$

Out[∘]= True

Which means the eigenvalues are the same

In[ ]:=
```
eigA = Eigenvalues[A]
eigAtilde = Eigenvalues[Atilde]
eigA == eigAtilde
```

Out[ ]= {4, 4, 2, 1}

Out[ ]= {4, 4, 2, 1}

Out[ ]= True

Property 3: Similar eigenvectors

We can now verify that eigenvectors($\tilde{A}$) = T^{-1} eigenvectors(A)

In[ ]:=
```
(*Mathematica's Eigenvectors returns eigenvectors in row format,
transpose to get into col format*)
eigVecA = Transpose[Eigenvectors[A]];
eigVecAtilde = Transpose[Eigenvectors[Atilde]];

eigVecA // MatrixForm
eigVecAtilde // MatrixForm // N
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} -0.920784 & 0. & 1.16923 & 1.0804 \\ 1.76664 & 0. & 0.979668 & 0.562692 \\ -0.337537 & 0. & -0.419543 & -0.285776 \\ 1. & 0. & 1. & 1. \end{pmatrix}$$

In[ ]:=
```
(*Check that we extracted the eigenvectors correctly*)
v1 = Transpose[{eigVecA[[All, 1]]}];
v2 = Transpose[{eigVecA[[All, 2]]}];
v3 = Transpose[{eigVecA[[All, 3]]}];
v4 = Transpose[{eigVecA[[All, 4]]}];

vtilde1 = Transpose[{eigVecAtilde[[All, 1]]}];
vtilde2 = Transpose[{eigVecAtilde[[All, 2]]}];
vtilde3 = Transpose[{eigVecAtilde[[All, 3]]}];
vtilde4 = Transpose[{eigVecAtilde[[All, 4]]}];

v1 // MatrixForm
v2 // MatrixForm
v3 // MatrixForm
v4 // MatrixForm

vtilde1 // MatrixForm // N
vtilde2 // MatrixForm // N
vtilde3 // MatrixForm // N
vtilde4 // MatrixForm // N
```

Out[ ]//MatrixForm=
$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Out[ ]//MatrixForm=
$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Out[ ]//MatrixForm=
$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Out[ ]//MatrixForm=
$$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Out[ ]//MatrixForm=
$$\begin{pmatrix} -0.920784 \\ 1.76664 \\ -0.337537 \\ 1. \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 0. \\ 0. \\ 0. \\ 0. \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1.16923 \\ 0.979668 \\ -0.419543 \\ 1. \end{pmatrix}$$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1.0804 \\ 0.562692 \\ -0.285776 \\ 1. \end{pmatrix}$$

In[∘]:=
```
(*Extract the eigenvalues*)
λ1 = eigA[[1]]
λ2 = eigA[[2]]
λ3 = eigA[[3]]
λ4 = eigA[[4]]
```

Out[∘]= 4

Out[∘]= 4

Out[∘]= 2

Out[∘]= 1

We can verify that these are a valid set of eigenvectors by ensuring they satisfy

$$A\,\overline{v}_i = \lambda_i\,\overline{V}_i$$

In[]:=
```
(*Verify A v̄i = λi v̄i*)
A.v1 == λ1 * v1
A.v2 == λ2 * v2
A.v3 == λ3 * v3
A.v4 == λ4 * v4

Atilde.vtilde1 == λ1 * vtilde1
Atilde.vtilde2 == λ2 * vtilde2
Atilde.vtilde3 == λ3 * vtilde3
Atilde.vtilde4 == λ4 * vtilde4
```

Out[]= True

Out[]= True

Out[]= True

Out[]= True

Out[]= True

Out[]= True

Out[]= True

Out[]= True

We can now verify that eigenvectors($\tilde{A}$) = T^{-1} eigenvectors(A)

In[]:=
```
Inverse[T].v1 // MatrixForm // N
vtilde1 // MatrixForm // N
```

Out[]//MatrixForm=

$$\begin{pmatrix} 0.193191 \\ -0.370662 \\ 0.0708192 \\ -0.209812 \end{pmatrix}$$

Out[]//MatrixForm=

$$\begin{pmatrix} -0.920784 \\ 1.76664 \\ -0.337537 \\ 1. \end{pmatrix}$$

At first glance, it appears that $T^{-1}\,\overline{v}_1 \neq \tilde{v}_1$ but we can show that it satisfies $\tilde{A}\;T^{-1}\,\overline{v}_1 = \lambda_1\,T^{-1}\,\overline{v}_1$ thereby showing it is an eigenvector associated with λ_1

In[ ]:=
```
Atilde.Inverse[T].v1 == λ1 * Inverse[T].v1
Atilde.Inverse[T].v2 == λ2 * Inverse[T].v2
Atilde.Inverse[T].v3 == λ3 * Inverse[T].v3
Atilde.Inverse[T].v4 == λ4 * Inverse[T].v4
```

Out[ ]= True

Out[ ]= True

Out[ ]= True

Out[ ]= True

Property 4: Same trace

In[ ]:=
```
TrA = Tr[A]
TrAtilde = Tr[Atilde]
TrA == TrAtilde
```

Out[ ]= 11

Out[ ]= 11

Out[ ]= True

Property 5: Same rank

In[ ]:=
```
rankA = MatrixRank[A]
rankAtilde = MatrixRank[Atilde]
rankA == rankAtilde
```

Out[ ]= 4

Out[ ]= 4

Out[ ]= True

Diagonalization

In this situation we cannot use the modal transformation of M = eigenvectors(A) to diagonalize the matrix because the eigenvectors do not span the space, meaning that M is singular so M^{-1} does not exist

In[ ]:=
```
M = eigVecA;
MatrixRank[M]
Inverse[M]
```

Out[ ]= 3

Inverse: Matrix {{1, 0, 1, −1}, {0, 0, −1, 1}, {−1, 0, 0, 0}, {1, 0, 1, 0}} is singular.

Out[ ]= Inverse[{{1, 0, 1, -1}, {0, 0, -1, 1}, {-1, 0, 0, 0}, {1, 0, 1, 0}}]

This can be ameliorated using Jordan Normal form (https://en.wikipedia.org/wiki/Jordan_normal-_form) but this is a topic for another video.