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Lecture 02e
Final Value Theorem

The YouTube video entitled ‘Final Value Theorem’ that covers this lecture is located at https://youtu.be/FgF-QfbP7zc.

Outline

-Final Value Theorem

Final Value Theorem

If $\lim_{t\to\infty} f(t)$ exists, then this theorem allows us to relate the steady-state behavior of $f(t)$ to the behavior of $s\,F(s)$. In other words, we can relate the steady state behavior of a system in the time domain to the behavior of the system in the s domain.

Theorem: if $f(t)$ and $d\,f(t)/d\,t$ are Laplace transformable, and if $\lim_{t\to\infty} f(t)$ exists then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\,F(s)$$

Proof: Let us examine the differentiation theorem

$$L[f'(t)] = s\,F(s) - f(0)$$

Using the definition of the Laplace transform, we can rewrite the left hand side

$$\int_0^\infty f'(t)\,e^{-s\,t}\,dt = s\,F(s) - f(0)$$

Taking the limit as $s \to 0$ of both sides

$$\lim_{s\to 0}\left[\int_0^\infty f'(t)\,e^{-s\,t}\,dt\right] = \lim_{s\to 0}[s\,F(s) - f(0)]$$

$$\lim_{s\to 0}\left[\int_0^\infty f'(t)\,e^{-s\,t}\,dt\right] = \lim_{s\to 0}[s\,F(s)] - f(0)$$

Since $\lim_{s\to 0} e^{-s\,t} = 1$, the left side can be simplified to $\int_0^\infty f'(t)\,dt$

$$\int_0^\infty f'(t)\,dt = \lim_{s\to 0}[s\,F(s)] - f(0) \qquad \textbf{(Eq.A)}$$

Using the rule for integration of a derivative, we know that

$$\int_0^\infty f'(t)\,dt = f(t)\,\big|_0^\infty = f(\infty) - f(0) \qquad \textbf{(Eq.B)}$$

If $\lim_{t\to\infty} f(t)$ exists, we can combine results from Eq.A and Eq.B to obtain

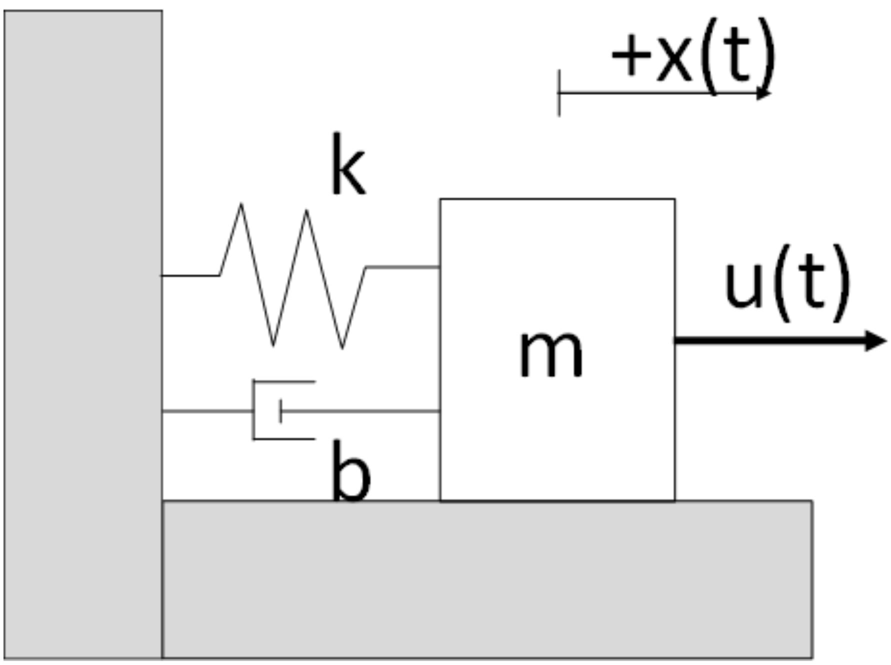
$$f(\infty) - f(0) = \lim_{s\to 0}[s\,F(s)] - f(0)$$

$$f(\infty) = \lim_{s\to 0} s\,F(s)$$

Note: The above proof assumed that $\lim_{t\to\infty} f(t)$ exists. We require the poles of the quantify $s\,F(s)$ must be strictly in the left half plane in order for $\lim_{t\to\infty} f(t)$ to exist. In many cases, such as when the input to a system is a ramp or sinusoid, the poles may not be strictly in the left half plane and therefore the steady state limit may not exist.

Example: Mass Spring Damper Step Response

Recall the mass spring damper system that we studied in our lecture on Laplace transforms



We showed that the ODE modeling the system was given as

$$\ddot{x}(t) + \frac{b}{m}\,\dot{x}(t) + \frac{k}{m}\,x(t) = \frac{1}{m}\,u(t)$$

We now desired to compute the final value of this system if it is subjected to a step input

$$u(t) = \mathbf{1}(t)$$

So the system we now want to solve is

$$\ddot{x}(t) + \frac{b}{m}\dot{x}(t) + \frac{k}{m}x(t) = \frac{1}{m}\mathbf{1}(t)$$

Let us apply the Laplace transform to both sides

$$L\left[\ddot{x}(t) + \frac{b}{m}\dot{x}(t) + \frac{k}{m}x(t)\right] = L\left[\frac{1}{m}\mathbf{1}(t)\right]$$

$$L\left[\ddot{x}(t)\right] + L\left[\frac{b}{m}\dot{x}(t)\right] + L\left[\frac{k}{m}x(t)\right] = L\left[\frac{1}{m}\mathbf{1}(t)\right]$$

$$L\left[\ddot{x}(t)\right] + \frac{b}{m}L\left[\dot{x}(t)\right] + \frac{k}{m}L[x(t)] = \frac{1}{m}L[\mathbf{1}(t)]$$

$$\left(s^2 X(s) - s\,x(0) - \dot{x}(0)\right) + \frac{b}{m}(s\,X(s) - x(0)) + \frac{k}{m}X(s) = \frac{1}{m}\left(\frac{1}{s}\right)$$

$$s^2 X(s) - s\,x(0) - \dot{x}(0) + \frac{b\,s}{m}X(s) - \frac{b}{m}x(0) + \frac{k}{m}X(s) = \frac{1}{m\,s}$$

$$\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right)X(s) = \frac{1}{m\,s} + s\,x(0) + \dot{x}(0) + \frac{b}{m}x(0)$$

$$X(s) = \frac{\frac{1}{m\,s} + s\,x(0) + \dot{x}(0) + \frac{b}{m}x(0)}{s^2 + \frac{b}{m}s + \frac{k}{m}}$$

```
X[s_] = (1/(m s) + s x0 + xDot0 + b/m x0)/(s^2 + b/m s + k/m);
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To get the state state position, we apply the final value theorem. We first compute $s\,X(s)$

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sX = Simplify[s X[s]]
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$$\frac{1 + b\,s\,x0 + m\,s\,(s\,x0 + xDot0)}{k + s\,(b + m\,s)}$$

Let us check that the poles of $s\,X(s)$ are strictly in the left half plane

```
temp = Solve[Denominator[sX] == 0, s] // Simplify;
pole1 = s /. temp[[1]]
pole2 = s /. temp[[2]]
```

$$-\frac{b + \sqrt{b^2 - 4\,k\,m}}{2\,m}$$

$$\frac{-b + \sqrt{b^2 - 4\,k\,m}}{2\,m}$$

Let us examine pole 1

$$-\frac{b+\sqrt{b^2-4\,k\,m}}{2\,m} < 0$$

if $m > 0$ we can write

$$-b-\sqrt{b^2-4\,k\,m} < 0$$

We see that if k and m are both strictly positive, the square root will be less than b or imaginary. Combine this with the assumption that $b > 0$ and we see that this is true and therefore this first pole is strictly in the left half plane.

Conducting similar analysis for the second pole

$$\frac{-b+\sqrt{b^2-4\,k\,m}}{2\,m} < 0 \qquad \text{assume: } m > 0$$

$$-b+\sqrt{b^2-4\,k\,m} < 0$$

Again, if k and m are both strictly positive, the square root be less than b or imaginary. Again, combining with the assumption that $b > 0$ and we see this is true so the second pole is also strictly in the left half plane.

```
pole1 /. {m → 0.1, b → 10, k → 0.1} // N
pole2 /. {m → 0.1, b → 10, k → 0.1} // N
```

```
-99.99
```

```
-0.010001
```

And now take the limit as $s \to 0$

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Simplify[sXs] /. {s → 0}
```

$$\frac{1}{k}$$

Note that this is not a function of initial condition nor the damping coefficient. This makes sense considering that we are only concerned with the steady state value where the spring force must counteract the input force of 1 newton.

Example: Mass Spring Damper Ramp Response

Note that if we repeat this analysis but use an input of a ramp, $u(t) = t$ and no initial conditions, we can write

$$\ddot{x}(t) + \frac{b}{m}\,\dot{x}(t) + \frac{k}{m}\,x(t) = \frac{1}{m}\,t$$

$$s^2\,X(s) + \frac{b}{m}\,s\,X(s) + \frac{k}{m}\,X(s) = \frac{1}{m\,s^2}$$

$$X(s) = \frac{1}{m\,s^2\left(s^2+\frac{b}{m}\,s+\frac{k}{m}\right)}$$

```
XRamp[s_] = 1/(m s^2 (s^2 + (b/m) s + k/m));
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$$\texttt{XRamp[s\_]} = \frac{1}{m\,s^2\left(s^2 + \frac{b}{m}\,s + \frac{k}{m}\right)};$$

So now computing $s\,X(s)$ yields

```
sXRamp = s XRamp[s]
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$$\frac{1}{m\,s\left(\frac{k}{m} + \frac{b\,s}{m} + s^2\right)}$$

Looking at the poles we have

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Solve[Denominator[sXRamp] == 0, s]
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$$\left\{\{s \to 0\},\ \left\{s \to \frac{-b - \sqrt{b^2 - 4\,k\,m}}{2\,m}\right\},\ \left\{s \to \frac{-b + \sqrt{b^2 - 4\,k\,m}}{2\,m}\right\}\right\}$$

We see that there is a pole at the origin which is not strictly in the left half plane so we cannot apply the final value theorem. Physically, this is because the output of the system will never reach a constant, steady state value due to the ever increasing force. The investigation of the behavior in response to a ramp is the topic of time domain analysis and will be covered in later lectures.