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Lecture 04l
Summary of Vector Derivative Operations and Formulation in Cylindrical and Spherical Coordinates

The YouTube video entitled ‘Summary of Vector Derivative Operations and Formulation in Cylindrical and Spherical Coordinates’ that covers this lecture is located at https://youtu.be/JGDaIzC0o0c

Outline

-Summary of Vector Derivative Operations
-Grad, Div, Curl, Laplacian in Cylindrical and Spherical Coordinates

Summary of Vector Derivative Operations

It may be useful to summarize all of these functions and operators

Dot Product

$$<\overline{u}, \overline{v}> = \overline{u} \cdot \overline{v} = |\overline{u}||\overline{v}| \cos(\theta) = \sum_{i=1}^{n} u_i v_i$$

Input: two vectors
Output: scalar

Scalar Projection

$$\text{comp}_{\overline{v}} \overline{u} = |\overline{u}| \cos(\theta) = \frac{<\overline{u}, \overline{v}>}{|\overline{v}|}$$

Input: two vectors
Output: scalar

Vector Projection

$$\text{proj}_{\overline{v}} \overline{u} = \frac{|\overline{u}| \cos(\theta)}{|\overline{v}|} \overline{v} = \frac{<\overline{u}, \overline{v}>}{|\overline{v}|^2} \overline{v}$$

Input: two vectors
Output: vector

Cross Product

$$\overline{u} \times \overline{v} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$$

Input: two vectors
Output: vector

Gradient

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k}$$

Input: scalar $f(x, y, z)$
Output: vector

Directional Derivative

$$D_{\overline{b}} f = \frac{\overline{b}}{|\overline{b}|} \cdot \nabla f$$

Input: scalar $f(x, y, z)$ and a vector $\overline{b}$
Output: scalar

Laplacian

$$\nabla^2 f = \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Input: scalar $f(x, y, z)$
Output: scalar

Divergence

$$\text{div } \overline{v} = \nabla \cdot \overline{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}$$

Input: vector
Output: scalar

Curl

$$\text{curl } \overline{v} = \nabla \times \overline{v} = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) \hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) \hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) \hat{k}$$

Input: vector
Output: vector

Grad, Div, Curl, Laplacian in Cylindrical and Spherical Coordinates

Ensure you have watch the video entitled ‘Cartesian, Polar, Cylindrical, and Spherical Coordinates’ at https://youtu.be/FLQXW6G9P8I.

Note: A helpful Wikipedia link for some of these identities
http://en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates
WARNING: The Wikipedia article may use different definitions of θ and ϕ (they use φ = azimuth angle, θ = polar angle).

The following definitions use notation that is consistent with the textbook.

θ = azimuth angle
ϕ = inclination/polar angle

We can transform the gradient, divergence, and curl operators to work in cylindrical and spherical coordinates (see text for derivations)

Gradient

Recall that for the gradient, in Cartesian coordinates, assuming we have a scalar function in Cartesian coordinates $f(x, y, z)$ the gradient is given as $\text{grad } f = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = \begin{pmatrix} \partial f/\partial x \\ \partial f/\partial y \\ \partial f/\partial z \end{pmatrix}$

Assuming that we have a scalar function in cylindrical coordinates, $f(r, \theta, z)$, the gradient can be computed using

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{\partial f}{\partial z}\hat{e}_z \qquad \text{(cylindrical coordinates)}$$

Similarly, if we have a scalar function in spherical coordinates, $f(r, \theta, \phi)$, the gradient can be computed using

$$\text{grad } f = \nabla f = \frac{\partial f}{\partial r}\hat{e}_r + \frac{1}{r\sin(\phi)}\frac{\partial f}{\partial \theta}\hat{e}_\theta + \frac{1}{r}\frac{\partial f}{\partial \phi}\hat{e}_\phi \qquad \text{(spherical coordinates)}$$

Divergence

Recall for divergence, that in Cartesian coordinates, assuming we have a vector function in Cartesian coordinates $\overline{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$ the divergence of a vector field is given as $\text{div } \overline{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$.

Assuming we have a vector function in cylindrical coordinates, $\overline{F} = F_r\hat{e}_r + F_\theta\hat{e}_\theta + F_z\hat{e}_z$, the divergence can be computed using

$$\text{div } \overline{F} = \nabla \cdot \overline{F} = \frac{1}{r}\frac{\partial}{\partial r}[r F_r] + \frac{1}{r}\frac{\partial}{\partial \theta}[F_\theta] + \frac{\partial}{\partial z}[F_z] \qquad \text{(cylindrical coordinates)}$$

Similarly, if we have a vector function in spherical coordinates, $\overline{F} = F_r\,\hat{e}_r + F_\theta\,\hat{e}_\theta + F_\phi\,\hat{e}_\phi$, the divergence can be computed using

$$\text{div}\,\overline{F} = \nabla\cdot\overline{F} = \frac{1}{r^2}\,\frac{\partial}{\partial r}\left[r^2\,F_r\right] + \frac{1}{r\sin(\phi)}\,\frac{\partial}{\partial\theta}\,[F_\theta] + \frac{1}{r\sin(\phi)}\,\frac{\partial}{\partial\phi}\left[\sin(\phi)\,F_\phi\right] \quad \text{(spherical coordinates)}$$

Curl

Recall that in Cartesian coordinates, assuming we have a vector function in Cartesian coordinates $\overline{F} = F_x\,\hat{i} + F_y\,\hat{j} + F_z\,\hat{k}$ the curl of a vector field is given as
$\text{curl}\,\overline{v} = \nabla\times\overline{v} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{k}$

For curl $\overline{F} = \nabla\times\overline{F}$, assuming we have a vector function in cylindrical coordinates, $\overline{F} = F_r\,\hat{e}_r + F_\theta\,\hat{e}_\theta + F_z\,\hat{e}_z$, the curl can be computed using

$$\nabla\times\overline{F} = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\,\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ F_r & r\,F_\theta & F_z \end{vmatrix}$$

$$= \left(\frac{1}{r}\,\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}\right)\hat{e}_r + \left(\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right)\hat{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}\,[r\,F_\theta] - \frac{\partial F_r}{\partial\theta}\right)\hat{e}_z \quad \text{(cylindrical coordinates)}$$

Similarly, if we have a vector function in spherical coordinates, $\overline{F} = F_r\,\hat{e}_r + F_\theta\,\hat{e}_\theta + F_\phi\,\hat{e}_\phi$, the curl can be computed using

$$\nabla\times\overline{F} = \frac{1}{r^2\sin(\phi)}\begin{vmatrix} \hat{e}_r & r\,\hat{e}_\phi & r\sin(\phi)\,\hat{e}_\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial\theta} \\ F_r & r\,F_\phi & r\sin(\phi)\,F_\theta \end{vmatrix}$$

$$= \frac{1}{r\sin(\phi)}\left(\frac{\partial}{\partial\phi}\,[F_\theta\sin(\phi)] - \frac{\partial F_\phi}{\partial\theta}\right)\hat{e}_r + \frac{1}{r}\left(\frac{\partial}{\partial r}\left[r\,F_\phi\right] - \frac{\partial F_r}{\partial\phi}\right)\hat{e}_\theta + \frac{1}{r}\left(\frac{1}{\sin(\phi)}\,\frac{\partial F_r}{\partial\theta} - \frac{\partial}{\partial r}\,[r\,F_\theta]\right)\hat{e}_\phi \quad \text{(spherical coordinates)}$$

Laplacian

Recall that in Cartesian coordinates, the Laplacian is given as $\nabla^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

So for the Laplace operator assuming that we have a scalar function in cylindrical coordinates, $f(r,\ \theta,\ z)$, the Laplacian can be computed using

$$\nabla^2\,f = \Delta\,f = \frac{1}{r}\,\frac{\partial}{\partial r}\left[r\,\frac{\partial f}{\partial r}\right] + \frac{1}{r^2}\,\frac{\partial^2 f}{\partial\theta^2} + \frac{\partial^2 f}{\partial z^2}$$

Alternatively, if we notice that $\frac{\partial}{\partial r}\left[r\,\frac{\partial f}{\partial r}\right] = \frac{\partial}{\partial r}\,[r\,]\,\frac{\partial f}{\partial r} + r\,\frac{\partial}{\partial r}\left[\,\frac{\partial f}{\partial r}\right] = \frac{\partial f}{\partial r} + r\,\frac{\partial^2 f}{\partial r^2}$, we can write this as

$$\nabla^2 f = \Delta f = \frac{1}{r}\frac{\partial f}{\partial r} + \frac{\partial^2 f}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \qquad \text{(cylindrical coordinates)}$$

Similarly, if we have a scalar function in spherical coordinates, $f(r, \theta, \phi)$, the Laplacian can be computed using

$$\nabla^2 f = \Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 \frac{\partial f}{\partial r}\right] + \frac{1}{r^2 \sin^2(\phi)}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2 \sin(\phi)}\frac{\partial}{\partial \phi}\left[\sin(\phi)\frac{\partial f}{\partial \phi}\right]$$

Alternatively, by performing some of the differentiations using the product rule, we can alternatively write this as

$$\nabla^2 f = \Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2 \sin^2(\phi)}\frac{\partial^2 f}{\partial \theta^2} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\cot(\phi)}{r^2}\frac{\partial f}{\partial \phi} \qquad \text{(spherical coordinates)}$$