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Lecture05d
Green's Theorem

Lecture is on YouTube

The YouTube video entitled 'Green's Theorem: Relating Closed Line Integrals to Double Integrals' that covers this lecture is located at https://youtu.be/p7PSZW9NhLU

Green's Theorem

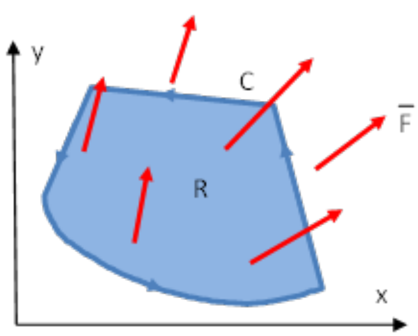
Green's Theorem relates double integrals over a plane region to line integrals over the boundary of the region.

Theorem: Green's Theorem in the Plane

Let R be a closed bounded region in the xy-plane whose boundary C consists of finitely many smooth curves and is positively oriented. Let $F_1(x, y)$ and $F_2(x, y)$ be functions that are continuous and have continuous partial derivatives $\partial F_1 / \partial y$ and $\partial F_2 / \partial x$ everywhere in some domain containing R. Then

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\, dy = \oint_C (F_1\, dx + F_2\, dy) \qquad \textbf{(Eq.1)}$$

Here we integrate along the entire boundary C of R in such a sense that R is on the left as we advance in the direction of integration (C is referred to as 'positively oriented').



Comments

Comment 1: If we interpret the functions $F_1(x, y)$ and $F_2(x, y)$ to be the the x and y components of a vector field, respectively, then we can write

$$\overline{F}(x, y) = \begin{pmatrix} F_1(x, y) \\ F_2(x, y) \end{pmatrix}$$

Recall from our discussion on line integrals (See YouTube video '***TBD***') that the work done by the vector field $\overline{F}$ by moving a particle along the curve C is given as

$$\text{Work} = \int_C \overline{F}(\overline{r}) \cdot d\overline{r}$$

So the right side of the theorem can be interpreted as the work done by moving along the closed curve C

Comment 2: If we write $\overline{F}$ in vector notation as $\overline{F} = \langle F_1, F_2 \rangle$ (this is vector notation, not an inner product) we can write Eq.1 in vector form as

$$\iint_R (\text{curl}\,\overline{F}) \cdot \hat{k}\, dx\, dy = \oint_C \overline{F} \cdot d\overline{r}$$

Note that the dot product between the curl $\overline{F}$ and the $\hat{k}$ vector will simply isolate the $\hat{k}$ element of the vector curl $\overline{F}$, thereby recovering the term $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$.

Comment 3: Note that the above assumes Cartesian coordinates, a more general formulation would be

$$\iint_R (\text{curl}\,\overline{F}) \cdot \hat{k}\, dA = \oint_C \overline{F} \cdot d\overline{r}$$

where $\quad dA$ = infinitesimal area of R

Applications of Green's Theorem

Work

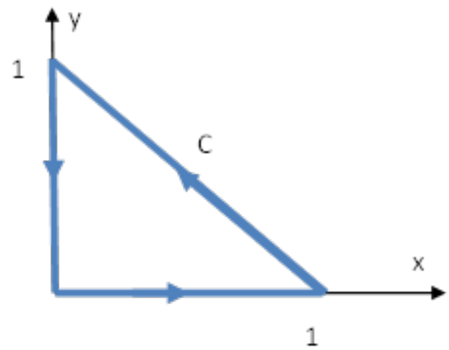
Green's Theorem can be used to calculate the work around a closed path C by switching to a double integral.

Consider the vector field

$$\overline{F}(x, y) = \langle x^2 - y, x^3 - y \rangle$$

```
In[ ]:= F[x_, y_] = {x^2 - y, x^3 - y};
```

And the curve shown below



Compute $\oint_C \overline{F} \cdot d\overline{r}$ (the work done by field F to move a particle along curve C)

Method 1: Brute Force via Line Integral

We can break up the curve C into three components

$$C_1 : \{(x, y) \mid x = t,\ y = 0,\ t \in [0, 1]\} \qquad \text{(horizontal curve)}$$
$$C_2 : \{(x, y) \mid x = 1 - t,\ y = t,\ t \in [0, 1]\} \qquad \text{(diagonal curve)}$$
$$C_3 : \{(x, y) \mid x = 0,\ y = 1 - t,\ t \in [0, 1]\} \qquad \text{(vertical curve)}$$

We can then break up the line integral into its three components

$$\oint_C \overline{F} \cdot d\overline{r} = \int_{C_1} \overline{F} \cdot d\overline{r} + \int_{C_2} \overline{F} \cdot d\overline{r} + \int_{C_3} \overline{F} \cdot d\overline{r}$$

For term 1, we have

$$\overline{r}(t) = \begin{pmatrix} t \\ 0 \end{pmatrix} \qquad t \in [0, 1]$$

So $\overline{r}'(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

```
In[]:= r1[t_] = {t, 0};
rprime1[t_] = D[r1[t], t]
Out[]= {1, 0}
```

Similarly, we have

$$\overline{F}(\overline{r}(t)) = < x^2 - y,\ x^3 - y > \qquad \text{recall: } x = t,\ y = 0$$

$$= < t^2,\ t^3 >$$

```
In[]:= F1[t_] = F[t, 0]
Out[]= {t^2, t^3}
```

So

$$\int_{C_1} \overline{F} \cdot d\overline{r} = \int_0^1 \overline{F}(\overline{r}(t)) \cdot \overline{r}'(t)\, dt$$

$$= \int_0^1 < t^2, t^3 > \cdot < 1, 0 > dt$$

$$= \int_0^1 t^2\, dt$$

$$\int_{C_1} \overline{F} \cdot d\overline{r} = \frac{1}{3}$$

In[ ]:=
```
term1 = Integrate[Dot[F1[t], rprime1[t]], {t, 0, 1}]
```

Out[ ]= $\frac{1}{3}$

Similarly for the second term, we have

$$\overline{r}(t) = \begin{pmatrix} 1 - t \\ t \end{pmatrix} \qquad t \in [0, 1]$$

So $\overline{r}'(t) = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

In[ ]:=
```
r2[t_] = {1 - t, t};
rprime2[t_] = D[r2[t], t]
```

Out[ ]= $\{-1, 1\}$

Similarly, we have $\overline{F}(\overline{r}(t))$

In[ ]:=
```
F2[t_] = F[1 - t, t]
```

Out[ ]= $\{(1 - t)^2 - t, (1 - t)^3 - t\}$

So the line integral is

In[ ]:=
```
term2 = Integrate[Dot[F2[t], rprime2[t]], {t, 0, 1}]
```

Out[ ]= $-\frac{1}{12}$

Similarly for the third term, we have

$x = 0,\ y = 1 - t$

$$\overline{r}(t) = \begin{pmatrix} 0 \\ 1 - t \end{pmatrix} \qquad t \in [0, 1]$$

So $\overline{r}'(t) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

In[∘]:=
```
r3[t_] = {0, 1 - t};
rprime3[t_] = D[r3[t], t]
```
Out[∘]= $\{0, -1\}$

Similarly, we have $\overline{F}(\overline{r}(t))$

In[∘]:=
```
F3[t_] = F[0, 1 - t]
```
Out[∘]= $\{-1 + t, -1 + t\}$

So the line integral is

In[∘]:=
```
term3 = Integrate[Dot[F3[t], rprime3[t]], {t, 0, 1}]
```
Out[∘]= $\frac{1}{2}$

So the final result is the sum of these three terms

In[∘]:=
```
term1 + term2 + term3
```
Out[∘]= $\frac{3}{4}$

So we have

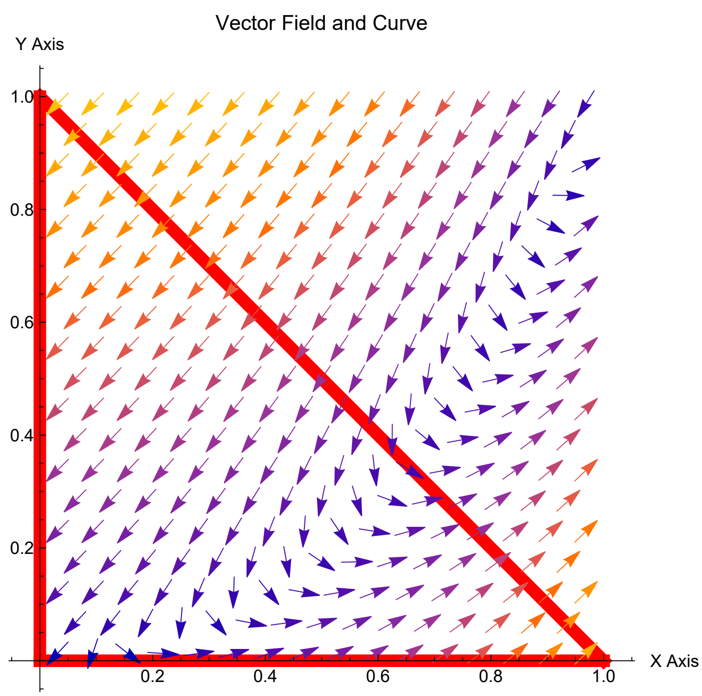
$$\oint_C \overline{F} \cdot d\overline{r} = 3/4$$

We can plot the scenario

In[∘]:=
```
F[x, y]
```
Out[∘]= $\{x^2 - y, x^3 - y\}$

In[ ]:=
```
Show[
 (*C1*)
 ParametricPlot[r1[t], {t, 0, 1}, PlotStyle → {Red, Thickness[0.02]}],

 (*C2*)
 ParametricPlot[r2[t], {t, 0, 1}, PlotStyle → {Red, Thickness[0.02]}],

 (*C3*)
 ParametricPlot[r3[t], {t, 0, 1}, PlotStyle → {Red, Thickness[0.02]}],

 (*Plot 2*)
 VectorPlot[F[x, y], {x, 0, 1}, {y, 0, 1}, VectorStyle → Blue],

 (*Plot Options*)
 PlotLabel → "Vector Field and Curve",
 AxesLabel → {"X Axis", "Y Axis"},
 PlotRange → {{0, 1}, {0, 1}}
]
```

Out[ ]=



Method 2: Green's Theorem to Convert to Double Integral

Recall that Green's Theorem relates the line integral to a double integral over this region

$$\oint_C \bar{F} \cdot d\bar{r} = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) d\,x\,d\,y$$

So we can evaluate the double integral

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) d\,x\,d\,y = \iint_R \left(\frac{\partial}{\partial x}\left[x^3 - y\right] - \frac{\partial}{\partial y}\left[x^2 - y\right]\right) d\,x\,d\,y$$

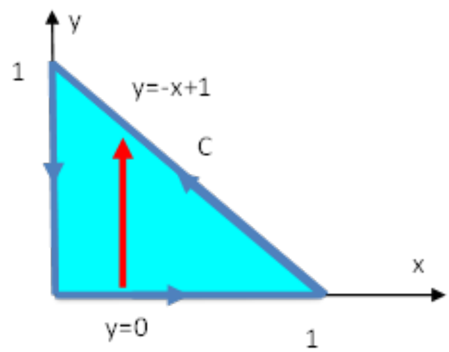
```
D[F[x, y][[2]], x] - D[F[x, y][[1]], y]
```

$1 + 3\,x^2$

$$= \iint_R 1 + 3\,x^2\,d\,x\,d\,y$$

This region can easily be treated as a type I or type II region. For practice, we will use both methods to verify they yield the same result.

Let us first try it as a type I region



So we see that we have

$$R = \{(x,\ y) \mid a \le x \le b,\ g(x) \le y \le h(x)\} \qquad \text{(Type I)}$$

where $a = 0$
$b = 1$
$g(x) = 0$
$h(x) = -x + 1$

So the double integral becomes

$$\iint_R f(x,\ y)\,d\,x\,d\,y = \int_a^b \left[\int_{g(x)}^{h(x)} f(x,\ y)\,d\,y\right] d\,x$$

$$= \int_0^1 \left[\int_0^{-x+1} 1 + 3\,x^2\,d\,y\right] d\,x$$

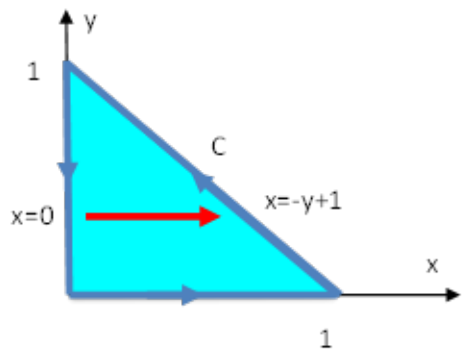
```
In[]:= Integrate[1 + 3 x^2, {y, 0, -x + 1}]
Out[]= (1 - x) × (1 + 3 x^2)
```

$$= \int_0^1 (1 - x) \times \left(1 + 3\,x^2\right) d\,x$$

In[∘]:= **Integrate[%, {x, 0, 1}]**

Out[∘]= $\frac{3}{4}$

If we treat this as a type II region we should obtain the same result.



$$R = \{(x, y) \mid c \leq y \leq d, p(y) \leq x \leq q(y)\} \qquad \text{(Type II)}$$

where $c = 0$

$d = 1$

$p(y) = 0$

$q(y) = -x + 1$

So the double integral becomes

$$\int\int_R f(x, y)\, d\,x\, d\,y = \int_c^d \left[\int_{p(y)}^{q(y)} f(x, y)\, dx\right] dy$$

$$= \int_0^1 \left[\int_0^{-y+1} 1 + 3\,x^2\, dx\right] dy$$

In[∘]:= **Integrate[1 + 3 x^2, {x, 0, -y + 1}]**

Out[∘]= $1 + (1 - y)^3 - y$

$$= \int_0^1 2 - 4\,y + 3\,y^2 - y^3\, dy$$

In[∘]:= **Integrate[%, {y, 0, 1}]**

Out[∘]= $\frac{3}{4}$

So we see that both method (either type I or type II) given the same result of

$$\int\int_R f(x, y)\, dx\, dy = 3/4$$

Comparing this with the line integral, we see that Green’s Theorem is true for this scenario.

Area

As an application of Green's Theorem, let us consider computing the area of R. Recall that area of R is given by the double integral of $f(x, y) = 1$ over the region R

$$\text{Area of } R = \iint_R dx\, dy$$

If this double integral is not convenient (ie it is not convenient to express R as a type I or type II region), it may be easier to use Green's Theorem and write this as a line integral. To apply Green's theorem, we notice that if we find functions F_1 and F_2 such that

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$$

Then Green's theorem becomes

$$\oint_C (F_1\, dx + F_2\, dy) = \iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy$$

$$= \iint_R dx\, dy$$

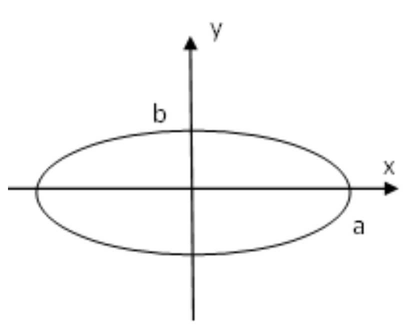
$$\oint_C (F_1\, dx + F_2\, dy) = \text{Area of } R$$

So we can alternatively calculate the area of R using the line integral rather than the double integral. Some popular choices include

F_1	F_2
0	x
$-y$	0
$-y/2$	$x/2$

Example: Area of an Ellipse

Compute the area of an ellipse shown below



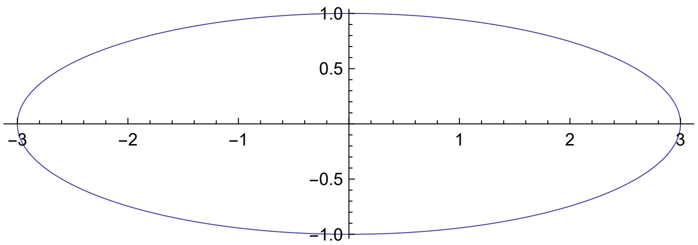
We recall that the ellipse is defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We can parameterize a curve around the ellipse as

$$x = a\cos(t)$$
$$y = b\sin(t)$$
$$t \in [0, 2\pi]$$

```
a = 3;
b = 1;
ParametricPlot[{a Cos[t], b Sin[t]}, {t, 0, 2 π}]
```



Sidebar: compute $\bar{r}(t)$ and $\bar{r}'(t)$

$$\bar{r}(t) = \begin{pmatrix} a\cos(t) \\ b\sin(t) \end{pmatrix} \quad \Rightarrow \quad \bar{r}'(t) = \begin{pmatrix} -a\sin(t) \\ b\cos(t) \end{pmatrix}$$

$$\bar{F}(\bar{r}(t)) = \begin{pmatrix} 0 \\ x \end{pmatrix} \Big|_{x=a\cos(t), y=b\sin(t)} = \begin{pmatrix} 0 \\ a\cos(t) \end{pmatrix}$$

So integral can be written as

$$\oint_C (F_1\,dx + F_2\,dy) = \oint_C \bar{F}\cdot d\bar{r}$$

$$= \int_0^{2\pi} \bar{F}(\bar{r}(t))\cdot\bar{r}'(t)\,dt$$

$$= \int_0^{2\pi} \begin{pmatrix} 0 \\ a\cos(t) \end{pmatrix}\cdot\begin{pmatrix} -a\sin(t) \\ b\cos(t) \end{pmatrix} dt$$

$$= \int_0^{2\pi} a\,b\cos^2(t)\,dt$$

$$= \int_0^{2\pi} a\,b\cos^2(t)\,dt$$

$$= \pi\,a\,b$$

So, we can compute the area of the ellipse using

$$A = \oint_C (F_1\, dx + F_2\, dy)$$

We can choose $F_1 = 0$ and $F_2 = x$ to satisfy the requirement that $\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 1$. With this, we can write

$$A = \oint_C x\, dy$$

$$= \oint_C a\cos(t)\, dy \qquad \text{recall: } y = b\sin(t) \Rightarrow dy = b\cos(t)\, dt$$

$$= \int_0^{2\pi} a\cos(t)\,(b\cos(t))\, dt$$

$$= a\, b \int_0^{2\pi} \cos^2(t)\; dt$$

<u>Side Note</u>

A quick trick for remembering what the $\int_0^{2\pi} \cos^2(t)\; dt$ equals. Since we know that $\cos^2(t) + \sin^2(t) = 1$, we can write

$$\int_0^{2\pi} \cos^2(t) + \sin^2(t)\, dt = \int_0^{2\pi} 1\, dt = 2\pi$$

We can also write this as

$$\int_0^{2\pi} \cos^2(t)\, dt + \int_0^{2\pi} \sin^2(t)\, dt = 2\pi$$

Examining the integrands, we see that they are simply the same wave but shifted by 90^o. Since we are squaring them, they each have a non-zero value and must be equal to one another. In other words $\int_0^{2\pi} \cos^2(t)\, dt = \int_0^{2\pi} \sin^2(t)\, dt = \alpha$. So from inspection, we have

$$\alpha + \alpha = 2\pi$$

$$\alpha = \pi$$

$$\int_0^{2\pi} \cos^2(t)\, dt = \int_0^{2\pi} \sin^2(t)\, dt = \pi$$

```
Integrate[Cos[t]^2, {t, 0, 2 π}] == π
Integrate[Sin[t]^2, {t, 0, 2 π}] == π
```

```
True
```

```
True
```

<u>End Side Note</u>

So back to our problem, we have the total area given as

$$A = a\, b\, \pi$$

Which is the familiar area of a ellipse equation.

Discussion of Green’s Theorem

Simplifying Line Integrals

Recall that Green’s Theorem can be stated as

$$\int\int_R (\text{curl}\,\overline{F}) \cdot \hat{k}\, dA = \oint_C \overline{F} \cdot d\overline{r}$$

In many scenarios, curl $\overline{F}$ is typically easier to deal with than $\overline{F}$ because this has effectively “taken a derivative” of $\overline{F}$. The price you pay for this is the fact that you must now evaluate a double integral instead of a single line integral.

Application to Stoke’s Theorem

We will see that Green’s Theorem is actually a specialized version of Stoke’s Theorem. See YouTube video entitled “Stokes’ Theorem” at https://youtu.be/40UUPvrHN-c.