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Lecture09c
Eigenvalues and Modes of Linear Systems

Lecture is on YouTube

The YouTube video entitled ‘Eigenvalues and Modes of Linear Systems’ that covers this lecture is located at https://youtu.be/35BTWpaihkI.

Outline

-Solutions to Linear ODEs
-Exciting Specific Modes
 -Eigenvectors as Participation Factors
 -Reordering States

Solutions to Linear ODEs

For the system given by

$$\dot{\bar{x}}(t) = A\,\bar{x}(t) + B\,\bar{u}(t) \qquad \text{(linear, time invariant system)}$$

The solution of $\bar{x}(t)$ is given analytically by (see YouTube video entitled ‘Analytically Solving Systems of Linear Ordinary Differential Equations’ at https://youtu.be/i2QkjxtXKos)

$$\bar{x}(t) = e^{A(t-t_o)}\,\bar{x}(t_o) + \int_{t_o}^{t} e^{A(t-\tau)}\,B\,u(\tau)\,d\tau$$

Consider autonomous response ($\bar{u}(t) = \bar{0}$) and $t_o = 0$

$$\bar{x}(t) = e^{A\,t}\,\bar{x}(0)$$

where $\quad e^{A\,t} = \sum_{k=0}^{\infty} \frac{(A\,t)^k}{k!}$

Many times, this matrix exponential is difficult to calculate. Recall from our previous discussion of similarity transformations (see YouTube videos entitled ‘Similarity Transformation and Diagonalization’ at https://youtu.be/wvRlvDYDIgw and ‘Similarity Transformation of a Linear Dynamic System’ at https://youtu.be/XMkLNHUmTQM)

$$\overline{x} = T\,\overline{z}$$

$$A = T\,\tilde{A}\,T^{-1}$$

So the autonomous response can be written as

$$\overline{x}(t) = e^{T\,\tilde{A}\,T^{-1}\,t}\,\overline{x}(0)$$

Let's look closer at the matrix exponential

$$e^{T\,\tilde{A}\,T^{-1}\,t} = \sum_{k=0}^{\infty} \frac{\left(T\,\tilde{A}\,T^{-1}\,t\right)^k}{k!}$$

$$= I + \frac{\left(T\,\tilde{A}\,T^{-1}\,t\right)^1}{1!} + \frac{\left(T\,\tilde{A}\,T^{-1}\,t\right)^2}{2!} + \frac{\left(T\,\tilde{A}\,T^{-1}\,t\right)^3}{3!} + \ldots$$

$$= I + \frac{T\,\tilde{A}\,T^{-1}\,t}{1!} + \frac{\left(T\,\tilde{A}\,T^{-1}\,t\right)\left(T\,\tilde{A}\,T^{-1}\,t\right)}{2!} + \frac{\left(T\,\tilde{A}\,T^{-1}\,t\right)\left(T\,\tilde{A}\,T^{-1}\,t\right)\left(T\,\tilde{A}\,T^{-1}\,t\right)}{3!} + \ldots$$

$$= I + \frac{T\,\tilde{A}\,T^{-1}\,t}{1!} + \frac{\left(T\,\tilde{A}\,T^{-1}\,T\,\tilde{A}\,T^{-1}\,t^2\right)}{2!} + \frac{\left(T\,\tilde{A}\,T^{-1}\,T\,\tilde{A}\,T^{-1}\,T\,\tilde{A}\,T^{-1}\,t^3\right)}{3!} + \ldots \qquad \text{note: } T^{-1}\,T = I$$

$$= I + \frac{T\,\tilde{A}\,T^{-1}\,t}{1!} + \frac{\left(T\,\tilde{A}\,\tilde{A}\,T^{-1}\,t^2\right)}{2!} + \frac{\left(T\,\tilde{A}\,\tilde{A}\,\tilde{A}\,T^{-1}\,t^3\right)}{3!} + \ldots$$

$$= T\,T^{-1} + \frac{T\,\tilde{A}\,T^{-1}\,t}{1!} + \frac{\left(T\,\tilde{A}^{\,2}\,T^{-1}\,t^2\right)}{2!} + \frac{\left(T\,\tilde{A}^{\,3}\,T^{-1}\,t^3\right)}{3!} + \ldots$$

$$= T\left(I + \frac{\tilde{A}\,t}{1!} + \frac{\left(\tilde{A}^{\,2}\,t^2\right)}{2!} + \frac{\left(\tilde{A}^{\,3}\,t^3\right)}{3!} + \ldots\right)T^{-1}$$

$$= T\left(I + \frac{\tilde{A}\,t}{1!} + \frac{\left(\tilde{A}\,t\right)^2}{2!} + \frac{\left(\tilde{A}\,t\right)^3}{3!} + \ldots\right)T^{-1}$$

$$= T\sum_{k=0}^{\infty} \frac{\left(\tilde{A}\,t\right)^k}{k!}\,T^{-1}$$

$$e^{T\,\tilde{A}\,T^{-1}\,t} = T\,e^{\tilde{A}\,t}\,T^{-1}$$

So we have

$$e^{A\,t} = T\,e^{\tilde{A}\,t}\,T^{-1}$$

what does the inner term of $e^{\tilde{A}\,t}$ look like?

$$e^{\tilde{A}\,t} = \sum_{k=0}^{\infty} \frac{\left(\tilde{A}\,t\right)^k}{k!}$$

If we chose T = eigenvectors(A) then $\tilde{A}$ is diagonal with eigenvalues along diagonal.

$$= \sum_{k=0}^{\infty} \frac{\left(\begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \ldots & .. & \vdots \\ 0 & \ldots & \ldots & \lambda_n \end{pmatrix} t\right)^k}{k!}$$

$$= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{(\lambda_1\,t)^k}{k!} & 0 & \ldots & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{(\lambda_1\,t)^k}{k!} & \ldots & 0 \\ \vdots & \ldots & .. & \vdots \\ 0 & \ldots & \ldots & \sum_{k=0}^{\infty} \frac{(\lambda_1\,t)^k}{k!} \end{pmatrix}$$

$$e^{\tilde{A}\,t} = \begin{pmatrix} e^{\lambda_1\,t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2\,t} & \ldots & 0 \\ \vdots & \ldots & .. & \vdots \\ 0 & \ldots & \ldots & e^{\lambda_n\,t} \end{pmatrix}$$

So solution is simply

$$\overline{x}(t) = e^{T\,\tilde{A}\,T^{-1}\,t}\,\overline{x}(0)$$

$$= T\left(e^{\tilde{A}\,t}\right)T^{-1}\,\overline{x}(0)$$

$$\overline{x}(t) = T\begin{pmatrix} e^{\lambda_1\,t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2\,t} & \ldots & 0 \\ \vdots & \ldots & .. & \vdots \\ 0 & \ldots & \ldots & e^{\lambda_n\,t} \end{pmatrix}T^{-1}\,\overline{x}(0)$$

So we see that the autonomous response of the system, $\overline{x}(t)$, is a function of the eigenvalues/modes. If λ_i has a positive real part, then $e^{\lambda_i\,t} \to \infty$ as $t \to \infty$, so system is considered unstable. Conversely, if λ_i has a negative real part, then $e^{\lambda_i\,t} \to 0$ as $t \to \infty$ as system is stable. See YouTube video entitled 'Deriving Percent Overshoot, Settling Time, and Other Performance Metrics' at https://youtu.be/QWCLthgJEbc for more discussion on how pole locations affect system performance.

The terms $e^{\lambda_i\,t}$ are often referred to as the **modes** of the system as they are the basis functions that make up the overall response $\overline{x}(t)$

Exciting Specific Modes

As we saw, the autonomous response of the system is composed of linear combinations of the modes.

Perhaps it is easier to look at the transformed system of $\dot{\overline{z}}(t) = \tilde{A}\,\overline{z}(t) + \tilde{B}\,\overline{u}(t)$. Once again, considering the autonomous response of the system to initial conditions, we can write the solution of the transformed system as

$$\overline{z}(t) = e^{\tilde{A}\,t}\,\overline{z}(0) \qquad \text{recall: } e^{\tilde{A}\,t} \text{ is diagonal}$$

$$= \begin{pmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ \vdots & \ldots & .. & \vdots \\ 0 & \ldots & \ldots & e^{\lambda_n t} \end{pmatrix} \begin{pmatrix} z_1(0) \\ z_2(0) \\ \ldots \\ z_n(0) \end{pmatrix}$$

$$\overline{z}(t) = \begin{pmatrix} e^{\lambda_1 t} z_1(0) \\ e^{\lambda_2 t} z_2(0) \\ \vdots \\ e^{\lambda_n t} z_n(0) \end{pmatrix} \qquad \text{(series of } n \text{ scalar equations)}$$

Suppose we choose

$$\overline{z}(0) = \begin{pmatrix} 0 \\ \ldots \\ 0 \\ 1\left(\text{in } i^{\text{th}} \text{ position}\right) \\ 0 \\ \ldots \\ 0 \end{pmatrix}$$

In this case, the response becomes

$$\overline{z}(t) = \begin{pmatrix} 0 \\ \ldots \\ 0 \\ e^{\lambda_i t}\left(\text{in } i^{\text{th}} \text{ position}\right) \\ 0 \\ \ldots \\ 0 \end{pmatrix}$$

In this case, we have excited only the mode associated with λ_i.

If we want to excite only this mode with the original system, we simply need to transform the initial condition using the original transformation of $\overline{x} = T\,\overline{z}$

$\overline{x}(0) = T\,\overline{z}(0)$

Example

Consider the system with an A matrix of

$$A = \begin{pmatrix} -7 & 4 & 5 \\ -3 & 3 & 3 \\ -7 & 3 & 5 \end{pmatrix}$$

In[∘]:=
```
A = {{-7, 4, 5}, {-3, 3, 3}, {-7, 3, 5}};
```

We can easily obtain the eigenvalues and eigenvectors

In[∘]:=
```
Print["Eigenvalues A"]
λA = Eigenvalues[A]
Print[" "]

temp = Eigenvectors[A]; (*Mathematica returns eigenvectors in row format*)
T = Transpose[temp];

Print["Eigenvectors A (in column form)"]
T // MatrixForm
```

```
Eigenvalues A
```

Out[∘]= $\left\{-2, \frac{1}{2} \times \left(3 + i\sqrt{3}\right), \frac{1}{2} \times \left(3 - i\sqrt{3}\right)\right\}$

```
Eigenvectors A (in column form)
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & -\frac{9i-5\sqrt{3}}{3i+7\sqrt{3}} & -\frac{-9i-5\sqrt{3}}{-3i+7\sqrt{3}} \\ 0 & \frac{3\times\left(-7i+\sqrt{3}\right)}{3i+7\sqrt{3}} & \frac{3\times\left(7i+\sqrt{3}\right)}{-3i+7\sqrt{3}} \\ 1 & 1 & 1 \end{pmatrix}$$

So we see that the eigenvalues are

$$\lambda_1 = -2$$
$$\lambda_2 = \frac{3}{2} + \frac{\sqrt{3}}{2}\,i$$
$$\lambda_3 = \frac{3}{2} - \frac{\sqrt{3}}{2}\,i$$

If we attempt to use an arbitrary initial condition to excite the original system, we obtain

In[ ]:=
```
x0 = {{1}, {0}, {0}};
x[t_] = MatrixExp[A t].x0;
x[t] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 2\,e^{-2t} - e^{3t/2}\,\mathrm{Cos}\left[\frac{\sqrt{3}\,t}{2}\right] - \sqrt{3}\,e^{3t/2}\,\mathrm{Sin}\left[\frac{\sqrt{3}\,t}{2}\right] \\ -2\sqrt{3}\,e^{3t/2}\,\mathrm{Sin}\left[\frac{\sqrt{3}\,t}{2}\right] \\ 2\,e^{-2t} - 2\,e^{3t/2}\,\mathrm{Cos}\left[\frac{\sqrt{3}\,t}{2}\right] \end{pmatrix}$$

So we see that the response contains all the modes of λ_1, λ_2, and λ_3

So we can diagonalize

In[ ]:=
```
Atilde = Inverse[T].A.T // Simplify;
Atilde // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & \frac{1}{2}\times\left(3+i\sqrt{3}\right) & 0 \\ 0 & 0 & \frac{1}{2}\times\left(3-i\sqrt{3}\right) \end{pmatrix}$$

So if we want to excite only the $\lambda_1 = -2$ mode, we chose

$$\bar{z}(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

In[ ]:=
```
z0 = {{1}, {0}, {0}};
z[t_] = MatrixExp[Atilde t].z0;
z[t] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} e^{-2t} \\ 0 \\ 0 \end{pmatrix}$$

If we were to transform this response back to the original states

In[ ]:=
```
x[t_] = T.z[t];
x[t] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} e^{-2t} \\ 0 \\ e^{-2t} \end{pmatrix}$$

We see that only $\lambda_1 = -2$ mode shows up in the response

The equivalent initial condition is therefore

$$\overline{x}(0) = T\,\overline{z}(0)$$

In[ ]:=
```
x0 = T.z0;
x0 // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

If we apply this to the original system we obtain

In[ ]:=
```
x[t_] = MatrixExp[A t].x0;
x[t] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} e^{-2t} \\ 0 \\ e^{-2t} \end{pmatrix}$$

The complex conjugates modes are slightly more interesting. If we want to excite only the $\lambda_2 = \frac{3}{2} + \frac{\sqrt{3}}{2}\,i$ mode, we chose

$$\overline{z}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

In[ ]:=
```
z0 = (0
      1
      0);
z[t_] = MatrixExp[Atilde t].z0;
z[t] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 \\ e^{\frac{1}{2}\,i\,\left(-3\,i+\sqrt{3}\right)\,t} \\ 0 \end{pmatrix}$$

While this achieves the desired effect that only the mode associated with λ_2 appears in the response, we should be concerned that the response is imaginary. We can more clearly see this if we transform $\overline{z}(0)$ to $\overline{x}(0)$

In[ ]:=
```
x0 = T.z0 // Simplify;
x0 // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{-9\,i+5\,\sqrt{3}}{3\,i+7\,\sqrt{3}} \\ \frac{3\times\left(-7\,i+\sqrt{3}\right)}{3\,i+7\,\sqrt{3}} \\ 1 \end{pmatrix}$$

As can be seen, this initial condition is imaginary and therefore not practical.

Let's go back and use an initial condition that excites both the λ_2 and λ_3 modes in equal proportion

In[∘]:=
```
z0 = (0
      1
      1);
x0 = T.z0 // Simplify;
x0 // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

This yields a response initial condition which in turn excites both the λ_2 and λ_3 modes

In[∘]:=
```
x[t_] = MatrixExp[A t].x0 // Simplify;
x[t] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} e^{3t/2}\left(\text{Cos}\left[\frac{\sqrt{3}\,t}{2}\right] + \sqrt{3}\,\text{Sin}\left[\frac{\sqrt{3}\,t}{2}\right]\right) \\ 2\sqrt{3}\,e^{3t/2}\,\text{Sin}\left[\frac{\sqrt{3}\,t}{2}\right] \\ 2\,e^{3t/2}\,\text{Cos}\left[\frac{\sqrt{3}\,t}{2}\right] \end{pmatrix}$$

Note: If $\overline{z}(0)$ did not have equal portions of λ_2 and λ_3, we may still obtain an imaginary initial condition of $\overline{x}(0)$. For example

In[∘]:=
```
z0 = (0
      1
      2);
x0 = T.z0 // Simplify;
x0 // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}\times\left(3 + i\sqrt{3}\right) \\ i\sqrt{3} \\ 3 \end{pmatrix}$$

Eigenvectors as Participation Factors

Another interesting observation can be made if we examine

$$\overline{x}(t) = T\,\overline{z}(t)$$

$$= [\,\overline{v}_1 \quad \overline{v}_2 \quad \overline{v}_3\,]\,e^{\tilde{A}t}\,\overline{z}(0)$$

$$= [\,\overline{v}_1 \quad \overline{v}_2 \quad \overline{v}_3\,]\begin{pmatrix} e^{\lambda_1 t} & 0 & 0 \\ 0 & e^{\lambda_2 t} & 0 \\ 0 & 0 & e^{\lambda_3 t} \end{pmatrix}\begin{pmatrix} z_1(0) \\ z_2(0) \\ z_3(0) \end{pmatrix}$$

$$= \overline{v}_1\, z_1(0)\, e^{\lambda_1 t} + \overline{v}_2\, z_2(0)\, e^{\lambda_2 t} + \overline{v}_3\, z_3(0)\, e^{\lambda_3 t}$$

As can be seen, the eigenvector can be seen as a "participation factor". That is, it shows how much of each mode shows up in the response of $\overline{x}(t)$.

For example

In[ ]:=
```
T // MatrixForm

v1 = Transpose[{T[[All, 1]]}];
v1 // MatrixForm // N

v2 = Transpose[{T[[All, 2]]}];
v2 // MatrixForm // N

v3 = Transpose[{T[[All, 3]]}];
v3 // MatrixForm // N
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & -\frac{9\,i-5\sqrt{3}}{3\,i+7\sqrt{3}} & -\frac{-9\,i-5\sqrt{3}}{-3\,i+7\sqrt{3}} \\ 0 & \frac{3\times\left(-7\,i+\sqrt{3}\right)}{3\,i+7\sqrt{3}} & \frac{3\times\left(7\,i+\sqrt{3}\right)}{-3\,i+7\sqrt{3}} \\ 1 & 1 & 1 \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1. \\ 0. \\ 1. \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0.5 - 0.866025\,i \\ 0. - 1.73205\,i \\ 1. \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0.5 + 0.866025\,i \\ 0. + 1.73205\,i \\ 1. \end{pmatrix}$$

Since we know that $\overline{v}_2$ = conjugate($\overline{v}_3$) we can just look at $\overline{v}_2$. These are given by

$$\overline{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \qquad \text{associated with mode of } e^{-2\,t}$$

$$\overline{v}_2 = \begin{pmatrix} 0.5 - 0.86\,i \\ 0 - 1.73\,i \\ 1 \end{pmatrix} \qquad \text{associated with mode of } e^{\left(\frac{3}{2}+\frac{\sqrt{3}}{2}\,i\right)t}$$

So we see that the mode of $e^{-2\,t}$ will only affect states 1 and 3 of the system whereas the modes of $e^{\left(\frac{3}{2}\pm\frac{\sqrt{3}}{2}\,i\right)t}$ will affect all 3 states.

For example, consider an arbitrary initial condition of

In[ ]:=
```
x0 = {{1}, {-2}, {3}};
x[t_] = MatrixExp[A t].x0 // Simplify;
x[t] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} e^{-2t} + \frac{4 e^{3t/2} \text{Sin}\left[\frac{\sqrt{3}\,t}{2}\right]}{\sqrt{3}} \\ 2 e^{3t/2} \left(-\text{Cos}\left[\frac{\sqrt{3}\,t}{2}\right] + \sqrt{3}\,\text{Sin}\left[\frac{\sqrt{3}\,t}{2}\right]\right) \\ e^{-2t} + 2 e^{3t/2} \text{Cos}\left[\frac{\sqrt{3}\,t}{2}\right] + \frac{2 e^{3t/2} \text{Sin}\left[\frac{\sqrt{3}\,t}{2}\right]}{\sqrt{3}} \end{pmatrix}$$

Note that there are no e^{-2t} terms in the 2nd state and there are oscillating sinusoids in all 3 states.

Reordering States

Because we know that the mode associated with $\lambda_1 = -2$ only appears in the 1st and 3rd states, perhaps we can reorder the states

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_3 \\ x_2 \end{pmatrix}$$

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_3 \\ x_2 \end{pmatrix}$$

$$\overline{z} = T^{-1}\,\overline{x}$$

$$\overline{x} = T\,\overline{z}$$

```
Tinv = {{-1, 1, 0}, {0, 0, 1}, {0, 1, 0}};
T = Inverse[Tinv];
T // MatrixForm;
```

Try by brute force

In[ ]:=
```
T = ( T11  T12  T13
      T21  T22  T23
      T31  T32  T33 );
Atilde = Inverse[T].A.T // Simplify;
Atilde // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{3\,T12\,T21\,T23+5\,T12\,T23\,T31-4\,T21\,T23\,T32-5\,T23\,T31\,T32+T13\,(-3\,T21\,T22-5\,T22\,T31+3\,T21\,T32+3\,T31\,T32)-3\,T12\,T21\,T33+4\,T21\,T22\,T33-3\,T12\,T31}{-T13\,T22\,T31+T12\,T23\,T31+T13\,T21\,T32-T11\,T23\,T32-T12\,T21\,T33+T1} \\ \frac{T13\,\left(-3\,T21^2-2\,T21\,T31+3\,T31^2\right)+T11^2\,(-7\,T23+3\,T33)+(4\,T21+5\,T31)\,(-T23\,T31+T21\,T33)+T11\,(7\,T13\,T21+3\,T}{T13\,T22\,T31-T12\,T23\,T31-T13\,T21\,T32+T11\,T23\,T32+T12\,T21\,T33-T11} \\ \frac{T12\,\left(-3\,T21^2-2\,T21\,T31+3\,T31^2\right)+T11^2\,(-7\,T22+3\,T32)+(4\,T21+5\,T31)\,(-T22\,T31+T21\,T32)+T11\,(7\,T12\,T21+3\,T}{-T13\,T22\,T31+T12\,T23\,T31+T13\,T21\,T32-T11\,T23\,T32-T12\,T21\,T33+T1} \end{pmatrix}$$

We want a structure of

$$\tilde{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{pmatrix}$$

In[ ]:=
```
Atilde31[T11_, T12_, T13_, T21_, T22_, T23_, T31_, T32_, T33_] = Atilde[[3, 1]]
```

Out[ ]=

$$\Big(T12\,\left(-3\,T21^2-2\,T21\,T31+3\,T31^2\right)+T11^2\,(-7\,T22+3\,T32)+(4\,T21+5\,T31)\,(-T22\,T31+T21\,T32)+T11\,(7\,T12\,T21+3\,T21\,T22-3\,T12\,T31+12\,T22\,T31-10\,T21\,T32-3\,T31\,T32)\Big)\Big/(-T13\,T22\,T31+T12\,T23\,T31+T13\,T21\,T32-T11\,T23\,T32-T12\,T21\,T33+T11\,T22\,T33)$$

In[ ]:=
```
Solve[Atilde31[T11, 0, 0, 0, T22, 0, T31, T32, T33] == 0, T31]
```

Out[ ]=

$$\left\{\{T31 \to T11\},\ \left\{T31 \to \frac{7\,T11\,T22-3\,T11\,T32}{5\,T22}\right\}\right\}$$

Applying the similarity transformation yields

In[ ]:=
```
(*T= ( 1  0    0
       0  T22  0
       1  T32  T33 );*)
T = ( 1  0  0
      0  1  0
      1  1  1 );
A2 = Inverse[T].A.T;
A2 // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -2 & 9 & 5 \\ 0 & 6 & 3 \\ 0 & -7 & -3 \end{pmatrix}$$

In[ ]:=
```
Eigenvalues[A2] // MatrixForm
Transpose[Eigenvectors[A2]] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -2 \\ \frac{1}{2} \times \left(3 + i\sqrt{3}\right) \\ \frac{1}{2} \times \left(3 - i\sqrt{3}\right) \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & -\frac{-11 i + 9\sqrt{3}}{7 \times \left(-7 i + \sqrt{3}\right)} & -\frac{11 i + 9\sqrt{3}}{7 \times \left(7 i + \sqrt{3}\right)} \\ 0 & \frac{1}{14} \times \left(-9 - i\sqrt{3}\right) & \frac{1}{14} \times \left(-9 + i\sqrt{3}\right) \\ 0 & 1 & 1 \end{pmatrix}$$

$$\dot{z}_1 = f(z_1, z_2, z_3)$$