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Lecture09e
Aircraft Longitudinal & Lateral/Directional Models & Modes (Phugoid, Short Period, Dutch Roll, etc.)

Lecture is on YouTube

The YouTube video entitled ‘Aircraft Longitudinal & Lateral/Directional Models & Modes (Phugoid, Short Period, Dutch Roll, etc.)’ that covers this lecture is located at https://youtu.be/JZlqctmQ7is.

Outline

-Longitudinal and Lateral/Directional States
-Decoupled Systems
-Longitudinal Model
    -Phugoid and Short Period Modes
-Lateral/Directional Model
    -Dutch Roll, Roll Subsidence, and Spiral Divergence Modes

Longitudinal and Lateral/Directional States

As we will be working with aircraft models, be sure you have watched the videos below
‘A Nonlinear, 6 DOF Dynamic Model of an Aircraft: the Research Civil Aircraft Model (RCAM)’ at https://youtu.be/bFFAL9lI2IQ
‘Building a Matlab/Simulink Model of an Aircraft: the Research Civil Aircraft Model (RCAM)’ at https://youtu.be/m5sEln5bWuM

Recall that the linear model of the RCAM model about the straight and level flight condition was given by (dropping the Δ notation)

$$\dot{\bar{x}}(t) = A\,\bar{x}(t) + B\,\bar{u}(t)$$

where *A* and *B* are shown below

```
A matrix
   -0.0354         0    0.0612         0   -1.2298         0         0   -9.8089         0
         0   -0.1805         0    1.2713         0  -84.9905    9.8089         0         0
   -0.2199         0   -0.7063         0   82.2157         0         0   -0.1468         0
         0   -0.0286         0   -1.3460         0    0.5842         0         0         0
   -0.0010         0   -0.0336         0   -1.1073         0         0         0         0
         0    0.0077         0    0.0554         0   -0.5533         0         0         0
         0         0         0    1.0000         0    0.0150         0         0         0
         0         0         0         0    1.0000         0         0         0         0
         0         0         0         0         0    1.0001         0         0         0

B matrix
         0    0.1094         0    9.8100    9.8100
         0         0    2.3012         0         0
         0   -7.3157         0         0         0
   -0.9486         0    0.3640    0.0407   -0.0407
         0   -2.9193         0    0.3924    0.3924
   -0.0199         0   -0.4081    0.7804   -0.7804
         0         0         0         0         0
         0         0         0         0         0
         0         0         0         0         0
```

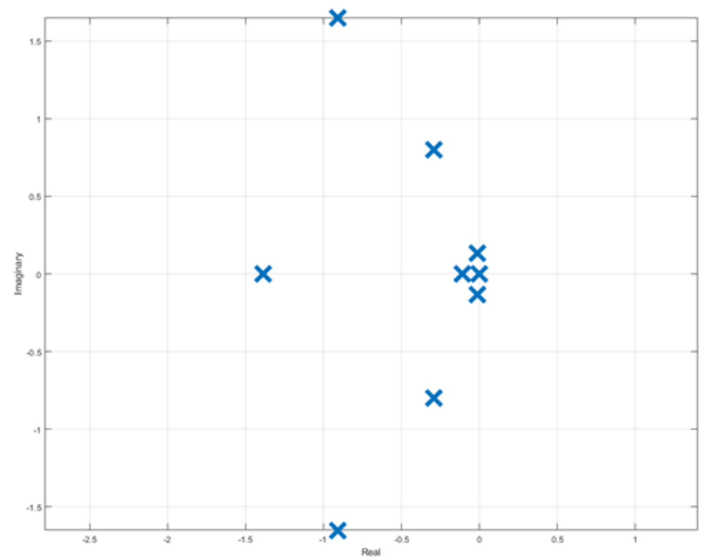
Recall the process was to first trim the model
'Trimming a Model of a Dynamic System Using Numerical Optimization' at https://youtu.be/YzZ-I1V2mJw8
'Trimming a Simulink Model Using the Linear Analysis Tool' at https://youtu.be/kypswO4RLkk

And then linearize the model
'Numerically Linearizing a Dynamic System' at https://youtu.be/1VmeijdM1qs?t=3759
'Linearizing a Simulink Model Using the Linear Analysis Tool and 'linmod' at https://youtu.be/M6FQfLmir0I

The eigenvalues are

```
lambda =

   0.0000 + 0.0000i
  -1.3873 + 0.0000i
  -0.2918 + 0.7999i
  -0.2918 - 0.7999i
  -0.1088 + 0.0000i
  -0.9097 + 1.6507i
  -0.9097 - 1.6507i
  -0.0148 + 0.1349i
  -0.0148 - 0.1349i
```

The map of the eigenvalues of A are shown below

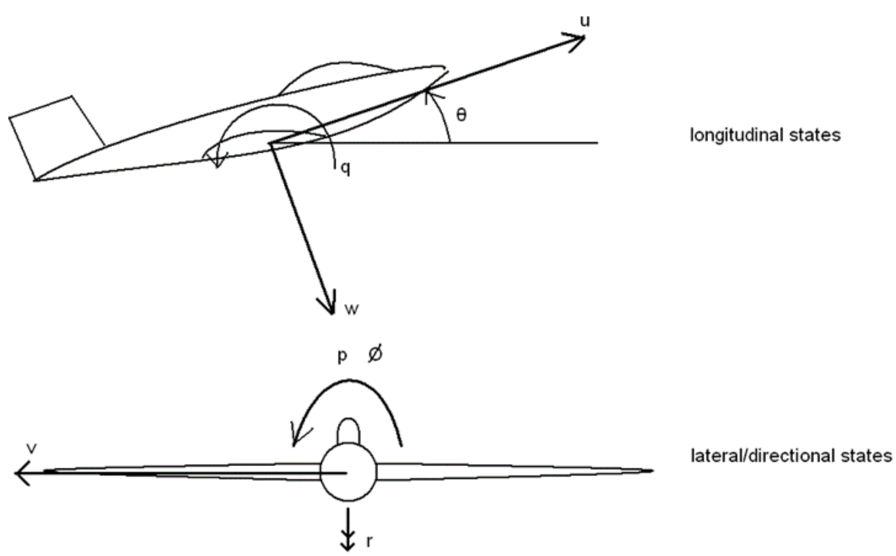


The state vector associated with this representation was the standard state vector of
$\overline{x}(t) = (u \quad v \quad w \quad p \quad q \quad r \quad \phi \quad \theta \quad \psi)^T$.

Certain states might be associated with the longitudinal or lateral/directional motion of the aircraft.

Longitudinal: Only motion in symmetric plane of aircraft (for conventional aircraft) (assume other states are zero)

Lateral/Directional: Other states



In our situation, we might be interested in looking at longitudinal/lateral directional states

$$\overline{x}_{\text{long}} = \begin{pmatrix} u \\ w \\ q \\ \theta \end{pmatrix} = \begin{pmatrix} x_1 \\ x_3 \\ x_5 \\ x_8 \end{pmatrix} \qquad \overline{x}_{\text{lat}} = \begin{pmatrix} v \\ p \\ r \\ \phi \end{pmatrix} = \begin{pmatrix} x_2 \\ x_4 \\ x_6 \\ x_7 \end{pmatrix}$$

When studying longitudinal, we assume that the lateral directional states are zero and vice versa. We will later investigate when this assumption is valid or not.

It may be useful to rearrange the states in the following order

$$\overline{z} = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_6 \\ z_7 \\ z_8 \\ z_9 \end{pmatrix} = \begin{pmatrix} u \\ w \\ q \\ \theta \\ v \\ p \\ r \\ \phi \\ \psi \end{pmatrix} = \begin{pmatrix} x_1 \\ x_3 \\ x_5 \\ x_8 \\ x_2 \\ x_4 \\ x_6 \\ x_7 \\ x_9 \end{pmatrix}$$

We can use a similarity transformation to rearrange the states (see ‘Similarity Transformation and Diagonalization’ at https://youtu.be/wvRlvDYDIgw and ‘Similarity Transformation of a Linear Dynamic System’ at https://youtu.be/XMkLNHUmTQM)

$$\overline{z} = \begin{pmatrix} 1 & \square & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & 1 & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & 1 & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & 1 & \square \\ \square & 1 & \square & \square & \square & \square & \square & \square & \square \\ \square & \square & \square & 1 & \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square & 1 & \square & \square & \square \\ \square & \square & \square & \square & \square & \square & 1 & \square & \square \\ \square & \square & \square & \square & \square & \square & \square & \square & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \\ x_9 \end{pmatrix}$$

$$\overline{z} = T^{-1}\,\overline{x}$$

Or

$$\overline{x} = T\,\overline{z}$$

```
In[•]:= Tinv = {{1, 0, 0, 0, 0, 0, 0, 0, 0},
               {0, 0, 1, 0, 0, 0, 0, 0, 0},
               {0, 0, 0, 0, 1, 0, 0, 0, 0},
               {0, 0, 0, 0, 0, 0, 0, 1, 0},
               {0, 1, 0, 0, 0, 0, 0, 0, 0},
               {0, 0, 0, 1, 0, 0, 0, 0, 0},
               {0, 0, 0, 0, 0, 1, 0, 0, 0},
               {0, 0, 0, 0, 0, 0, 1, 0, 0},
               {0, 0, 0, 0, 0, 0, 0, 0, 1}};

T = Inverse[Tinv];
T // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We see that the associated similarity transformation is therefore

We can now perform a similarity transformation to get to this new state representation.

$$\dot{\bar{z}}(t) = \tilde{A}\,\bar{z}(t) + \tilde{B}\,\bar{u}(t)$$

where $\tilde{A} = T^{-1} A\, T$

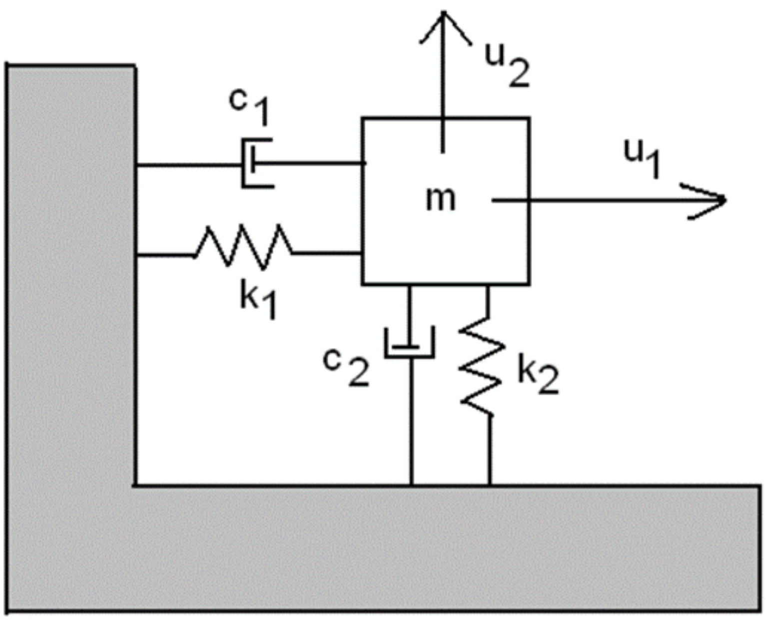
$\tilde{B} = T^{-1} B$

We should obtain results similar to those shown below

```
Atilde =

   -0.0354    0.0612   -1.2298   -9.8089         0         0         0         0         0
   -0.2199   -0.7063   82.2157   -0.1468         0         0         0         0         0
   -0.0010   -0.0336   -1.1073         0         0         0         0         0         0
         0         0    1.0000         0         0         0         0         0         0
         0         0         0         0   -0.1805    1.2713  -84.9905    9.8089         0
         0         0         0         0   -0.0286   -1.3460    0.5842         0         0
         0         0         0         0    0.0077    0.0554   -0.5533         0         0
         0         0         0         0         0    1.0000    0.0150         0         0
         0         0         0         0         0         0    1.0001         0         0


Btilde =

         0    0.1094         0    9.8100    9.8100
         0   -7.3157         0         0         0
         0   -2.9193         0    0.3924    0.3924
         0         0         0         0         0
         0         0    2.3012         0         0
   -0.9486         0    0.3640    0.0407   -0.0407
   -0.0199         0   -0.4081    0.7804   -0.7804
         0         0         0         0         0
         0         0         0         0         0
```

Decoupled Systems

Let's consider the two axis spring/mass/damper system shown below (recall we investigated this in a previous homework assignment)



We could write out the equations of motion as

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{-k_1}{m} x_1 - \frac{c_1}{m} x_2 + \frac{1}{m} u_1$$

$$\dot{x}_3 = x_4$$
$$\dot{x}_4 = \frac{-k_2}{m} x_3 - \frac{c_2}{m} x_4 + \frac{1}{m} u_2$$

where $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x \text{ position} \\ x \text{ velocity} \\ y \text{ position} \\ y \text{ velocity} \end{pmatrix}$

Writing this in linear system form of $\dot{\overline{x}} = A\,\overline{x} + B\,\overline{u}$ yields

$$\dot{\overline{x}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -k_1/m & -c_1/m & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -k_1/m & -c_1/m \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1/m & 0 \\ 0 & 0 \\ 0 & 1/m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Notice the block diagonal structure of the A matrix.

These two sets of equations are not related to each other in any way, so we can write them as two separate systems

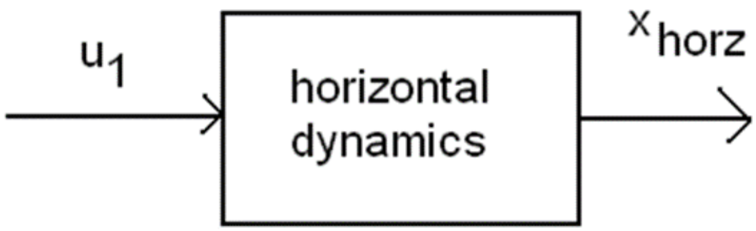
$$\dot{\overline{x}}_{\text{horz}} = A_{\text{horz}}\,\overline{x}_{\text{horz}} + B_{\text{horz}}\,u_{\text{horz}}$$
$$\dot{\overline{x}}_{\text{vert}} = A_{\text{vert}}\,\overline{x}_{\text{vert}} + B_{\text{vert}}\,u_{\text{vert}}$$

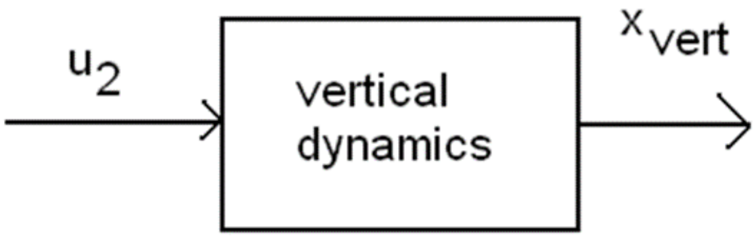
where $A_{\text{horz}} = \begin{pmatrix} 0 & 1 \\ -k_1/m & -c_1/m \end{pmatrix}$

$$A_{\text{vert}} = \begin{pmatrix} 0 & 1 \\ -k_2/m & -c_2/m \end{pmatrix}$$

$$B_{\text{horz}} = B_{\text{vert}} = \begin{pmatrix} 0 \\ 1/m \end{pmatrix}$$

In block diagram form





Longitudinal Model

Recall that the full linear model (with reordered states) was given by

$$\dot{\bar{z}} = A\,\bar{z} + B\,\bar{u}$$

We define the approximate longitudinal model as

$$\dot{\bar{x}}_{\text{long}} = A_{\text{long}}\,\bar{x}_{\text{long}} + B_{\text{long}}\,\bar{u}$$
$$\bar{y}_{\text{long}} = C_{\text{long}}\,\bar{x}_{\text{long}} + D_{\text{long}}\,\bar{u}$$

where $\quad \bar{x}_{\text{long}} = \begin{pmatrix} u \\ w \\ q \\ \theta \end{pmatrix} \qquad \bar{u} = \begin{pmatrix} \delta_A \\ \delta_s \\ \delta_R \\ \delta_{\text{th}_1} \\ \delta_{\text{th}_2} \end{pmatrix}$

How to obtain the A_{long}, B_{long}, C_{long}, and D_{long} matrices? One option is to take

$$A_{\text{long}} = A(1:4,\ 1:4)$$
$$B_{\text{long}} = B(1:4,\ :)$$
$$C_{\text{long}} = \text{eye}(4)$$
$$D_{\text{long}} = \text{zeros}(4,\ 5)$$

This is an approximation because this is saying that the lateral directional states are not coupled with the longitudinal states. In the case where the longitudinal states are decoupled from the lateral/directional states, this is not an approximation and is in fact exact. There are other situations where this generates significant errors.

Phugoid and Short Period Modes

By choosing T as matrix of eigenvectors of A_{long}, we can diagonalize the system. The resulting $\tilde{A}_{\text{long}}$ matrix is as shown below

```
-0.9096 + 1.6507i        0                  0                  0
      0           -0.9096 - 1.6507i         0                  0
      0                  0          -0.0148 + 0.1350i          0
      0                  0                  0          -0.0148 - 0.1350i
```

So the system is

$$\dot{\bar{z}} = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix} \bar{z} + \tilde{B}\,\bar{u}$$

This is a decoupled system

$$\begin{aligned} \dot{z}_1 &= \lambda_1 z_1 + \tilde{B}(1,\ :)\,\bar{u} \\ \dot{z}_2 &= \lambda_2 z_2 + \tilde{B}(2,\ :)\,\bar{u} \\ \dot{z}_3 &= \lambda_3 z_3 + \tilde{B}(3,\ :)\,\bar{u} \\ \dot{z}_4 &= \lambda_4 z_4 + \tilde{B}(4,\ :)\,\bar{u} \end{aligned}$$

As can be seen, the system has two pairs of imaginary roots. Furthermore, by looking at the real part, we see that two of the roots are fairly well damped, and two of them are poorly damped. The solution to the homogeneous system is

$$z_1(t) = z_{1_o}\, e^{(-0.9096+1.6507\, i)\, t} \qquad \textbf{\textit{(Eq.1.7)}}$$
$$z_2(t) = z_{2_o}\, e^{(-0.9096-1.6507\, i)\, t} \qquad \textbf{\textit{(Eq.1.8)}}$$
$$z_3(t) = z_{3_o}\, e^{(-0.0148+0.135\, i)\, t} \qquad \textbf{\textit{(Eq.1.9)}}$$
$$z_4(t) = z_{4_o}\, e^{(-0.0148-0.135\, i)\, t} \qquad \textbf{\textit{(Eq.1.10)}}$$

Therefore, the solution to $\bar{x}_{\text{long}}(t)$ is a linear combination of the response $\bar{z}(t)$. They are related through the definition of our similarity transformation

$$\begin{aligned} \bar{x}_{\text{long}}(t) &= T\,\bar{z}(t) \\ &= [\,\bar{v}_1 \quad \bar{v}_2 \quad \bar{v}_3 \quad \bar{v}_4\,] \begin{pmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{pmatrix} \end{aligned}$$

$$= \overline{v}_1\, z_1(t) + \overline{v}_2\, z_2(t) + \overline{v}_3\, z_3(t) + \overline{v}_4\, z_4(t)$$

$$\overline{x}_{\text{long}}(t) = \overline{v}_1\, z_{1_o}\, e^{(-0.9096+1.6507\, i)\, t} + \overline{v}_2\, z_{2_o}\, e^{(-0.9096-1.6507\, i)\, t} + \overline{v}_3\, z_{3_o}\, e^{(-0.0148+0.135\, i)\, t} + \overline{v}_4\, z_{4_o}\, e^{(-0.0148-0.135\, i)\, t}$$

(Eq.1.11)

As discussed in the YouTube video entitled 'Eigenvalues and Modes of Linear Systems' https://y-outu.be/35BTWpaihkI?t=2527 , from Eq.1.11 the eigenvector can be seen as a "participation factor". That is, it shows how much of each mode shows up in the response of $\overline{x}_{\text{long}}(t)$. Since we know that $\overline{v}_1$ = conjugate($\overline{v_2}$) and $\overline{v}_3$ = conjugate($\overline{v}_4$), we can just look at $\overline{v}_1$ and $\overline{v}_3$. These are given by

$$\overline{v}_1 = \begin{pmatrix} 0.0154 - 0.0122\, i \\ 0.9995 \\ -0.0024 + 0.0201\, i \\ 0.0100 - 0.0040\, i \end{pmatrix} \qquad \text{(associated with fast mode of } e^{(-0.9096+1.6507\, i)\, t}\text{)} \qquad \textbf{(Eq.1.12)}$$

$$\overline{v}_3 = \begin{pmatrix} -0.9957 \\ 0.0919 - 0.0015\, i \\ -0.0019 - 0.0002\, i \\ 0.0029 - 0.0137\, i \end{pmatrix} \qquad \text{(associated with slow mode of } e^{(-0.0148+0.135\, i)\, t}\text{)} \qquad \textbf{(Eq.1.13)}$$

Recall

$$\overline{x}_{\text{long}} = \begin{pmatrix} u \\ w \\ q \\ \theta \end{pmatrix}$$

Since we know that the response $\overline{x}_{\text{long}}(t)$ is given by Eq.1.11, we see that the eigenvectors show how much of each mode appears in $x_{\text{long1}}(t)$, $x_{\text{long2}}(t)$, $x_{\text{long3}}(t)$, and $x_{\text{long4}}(t)$. For example, we see that $x_{\text{long1}}(t)$ is more associated with the slow mode by comparing the first element of $\overline{v}_1$ with the first element of $\overline{v}_3$ and seeing that $\overline{v}_3(1, 1)$ is relatively larger than $\overline{v}_1(1, 1)$ (note: we can compare these because the eigenvectors are already normalized) . Since $\overline{v}_3$ is associated with the slow mode, we see that $x_{\text{long1}}(t)$ will have more of a response from the slow mode in it.

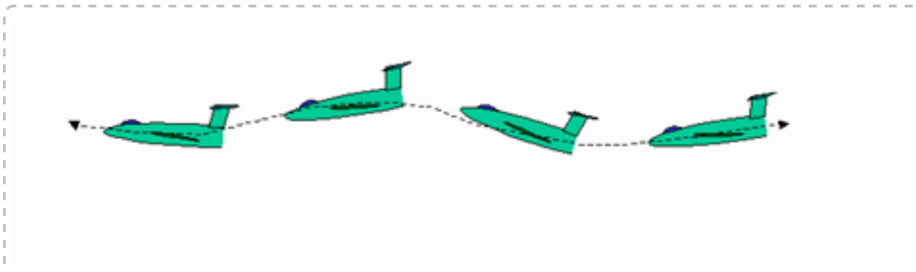
Performing a similar comparison for $x_{\text{long2}}(t)$, $x_{\text{long3}}(t)$, and $x_{\text{long4}}(t)$ yields that the slow mode appears more predominantly in $x_{\text{long1}}(t)$ and $x_{\text{long4}}(t)$ and the fast mode appears more in $x_{\text{long2}}(t)$ and $x_{\text{long3}}(t)$.

$x_{\text{long1}} = u$	slow mode
$x_{\text{long2}} = w$	fast mode
$x_{\text{long3}} = q$	both
$x_{\text{long4}} = \theta$	both

What are the physical description of these modes?

Short period mode ($\lambda = -0.9096 \pm 1.6507\, i$):
This is a very fast, usually heavily damped, oscillation with a period of a few seconds. The motion is a rapid pitching of the aircraft about the center of gravity. This is essentially an angle of attack oscillation.

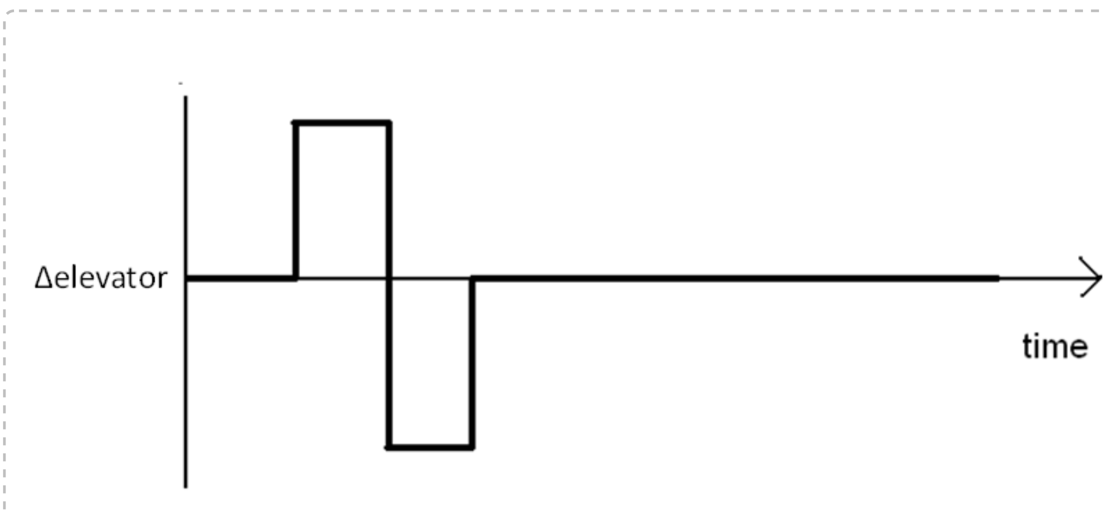
Phugoid mode ($\lambda = -0.0148 \pm 0.135\, i$):
Aircraft motion where the vehicle pitches up and climbs, and then pitches down and descends, accompanied by speeding up and slowdown as it goes "uphill" and "downhill" (ref: wikipedia). This is a constant angle of attack but a large amplitude change in airspeed, pitch angle, flight path angle, and altitude. This can be thought of as a slow trade off between kinetic and potential energy. Typical period for many aircraft is 20-60 seconds.



How do we excite these modes? There are several ways to do this. In flight testing, typical procedure is

1. Trim aircraft for straight and level flight
2. Input a singlet or doublet in elevator

What does a doublet in the elevator look like?

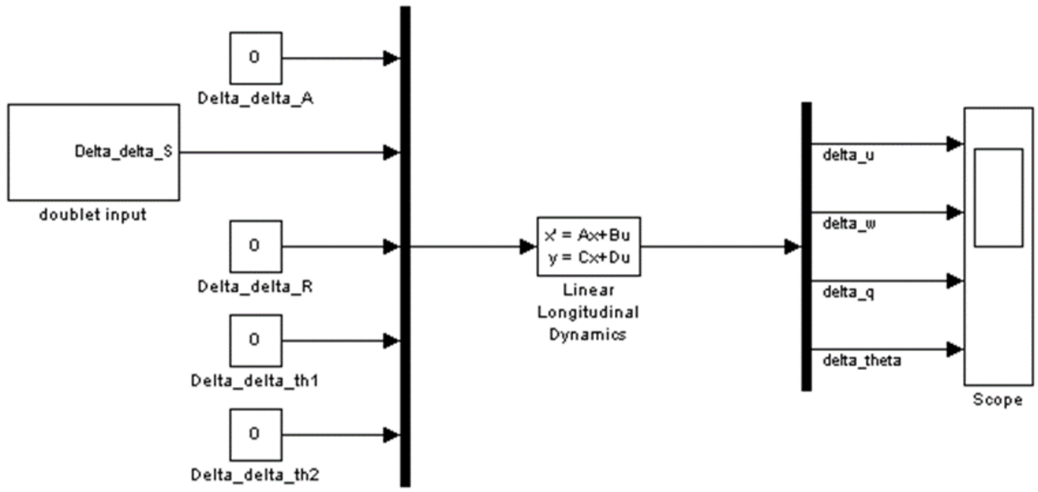


Keep in mind that these are Δ signals, so if you want to apply this to the nonlinear model, you need to take into account the trim conditions (see YouTube video entitled ‘TBD’)

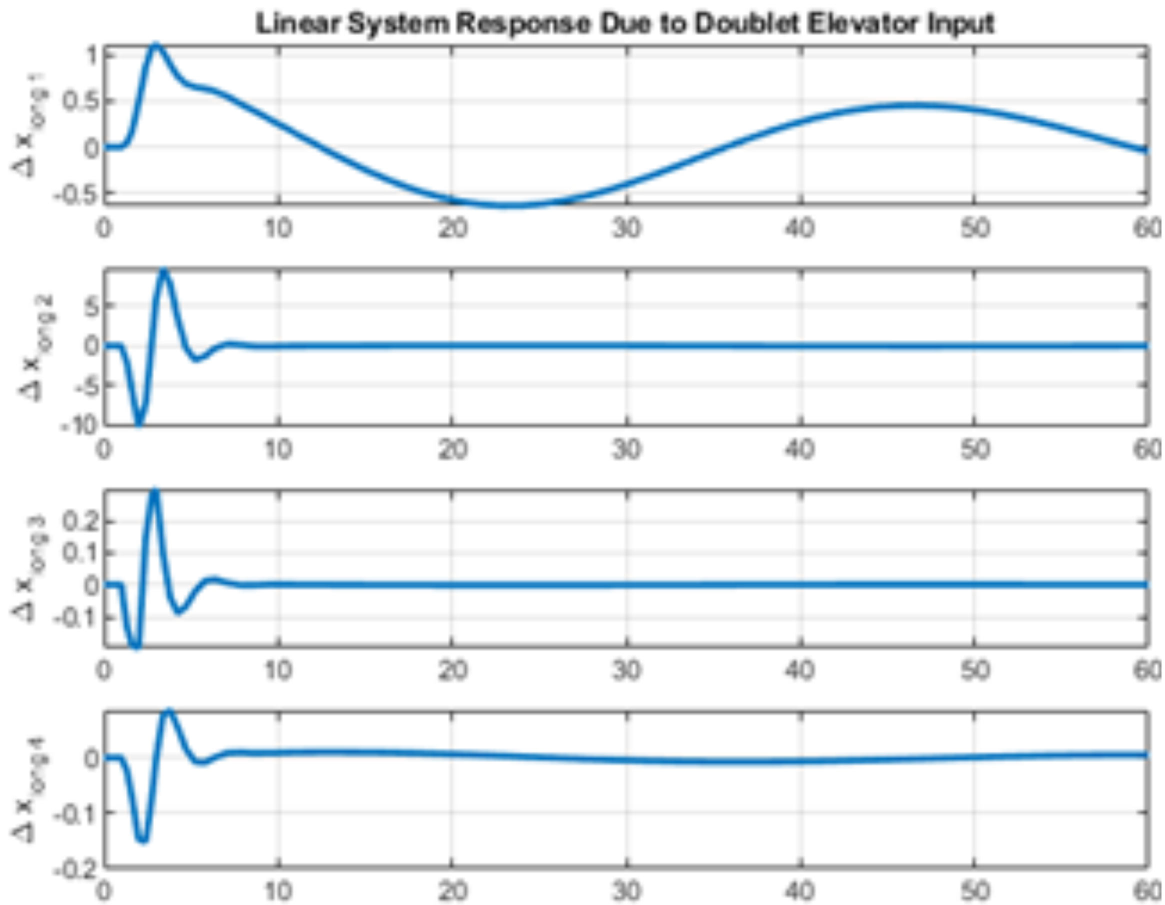
So in Simulink, we want to input a double to the elevator. We need to be careful about dimensions and which input and outputs we are monitoring. Recall our model was

$$\dot{\overline{x}}_{\text{long}} = A_{\text{long}}\,\overline{x}_{\text{long}} + B_{\text{long}}\,\overline{u}$$
$$\overline{y}_{\text{long}} = C_{\text{long}}\,\overline{x}_{\text{long}} + D_{\text{long}}\,\overline{u}$$

So the Simulink model to input a double in the elevator (or horizontal stabilizer in our case) is given as



The response to a ±10 degree doublet is shown below



As can be seen, this excited both the short period and phugoid modes .

Another option for exciting the two different modes is to use an appropriately chosen initial condition (again, see YouTube video entitled ‘Eigenvalues and Modes of Linear Systems’ https://youtu.be/35BTW-paihkI?t=1604). For example, to excite the mode associated with the eigenvalues $-0.9096 \pm 1.6507\,i$ (which are the first two eigenvalue), we can use the initial condition of

$$\bar{z}_o = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Then solution to system should be

$$\bar{x}(t) = \bar{v}_1\, z_{1_o}\, e^{(-0.9096+1.6507\,i)\,t} + \bar{v}_2\, z_{2_o}\, e^{(-0.9096-1.6507\,i)\,t}$$

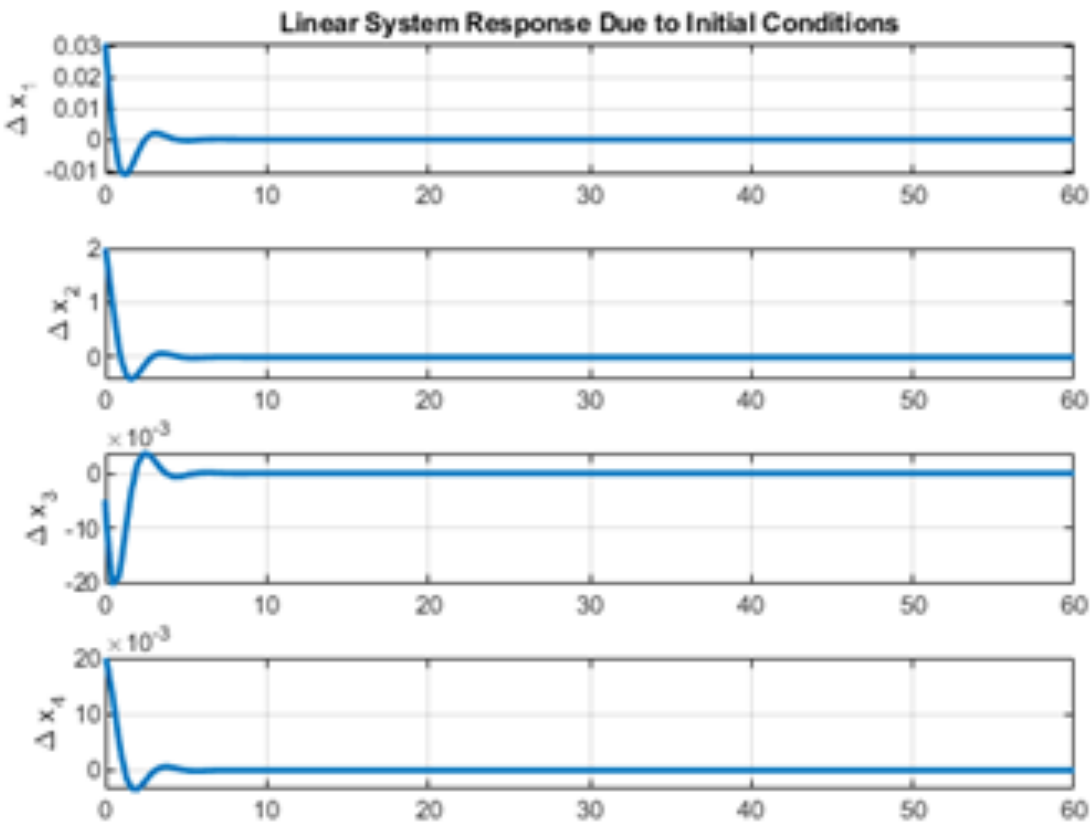
So only has effect of λ_1, $\lambda_2 = -0.9096 \pm 1.6507\,i$ mode in the response (excites the short period response)

Of course, to simulate the system, need to transform this initial condition

$$\overline{x}_o = T\,\overline{z}_o$$

$$\overline{x}_o = \begin{pmatrix} 0.0308 \\ 1.9991 \\ -0.0048 \\ 0.0199 \end{pmatrix} = \begin{pmatrix} u(0) \\ w(0) \\ q(0) \\ \theta(0) \end{pmatrix}$$

The response is shown below



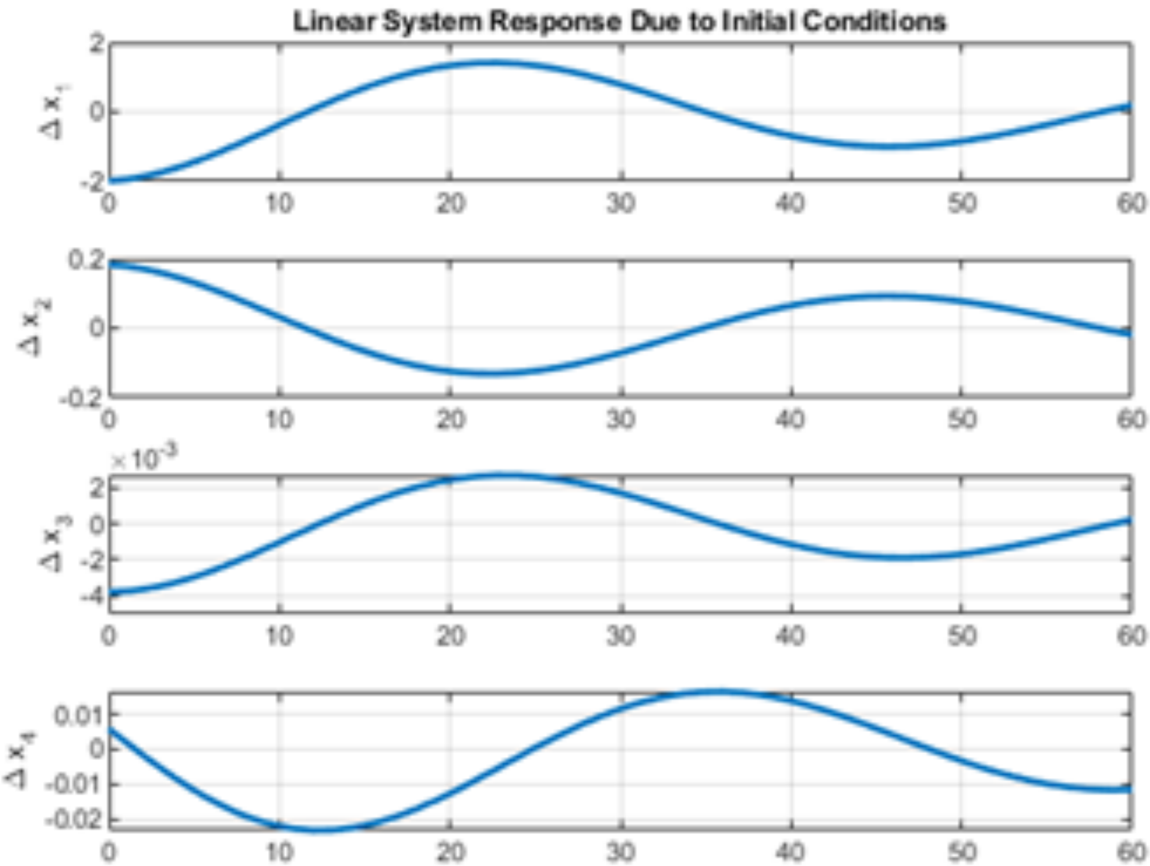
In a similar fashion, exciting only the phugoid mode is done by choosing

$$\overline{z}_o = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$\overline{x}_o = T\,\overline{z}_o$$

$$\overline{x}_o = \begin{pmatrix} -1.9913 \\ 0.1838 \\ -0.0038 \\ 0.0058 \end{pmatrix} = \begin{pmatrix} u(0) \\ w(0) \\ q(0) \\ \theta(0) \end{pmatrix}$$

The response is shown below



Helios crash (2003) was likely caused by exciting the phugoid mode. https://en.wikipedia.org/wiki/AeroVironment_Helios_Prototype

Lateral/Directional Model

We can perform a similar analysis with the lateral/directional model

We define the approximate lateral/directional model as

$$\dot{\bar{x}}_{lat} = A_{lat}\,\bar{x}_{lat} + B_{lat}\,\bar{u}$$
$$\bar{y}_{lat} = C_{lat}\,\bar{x}_{lat} + D_{lat}\,\bar{u}$$

where $\bar{x}_{lat} = \begin{pmatrix} v \\ p \\ r \\ \phi \end{pmatrix} = \begin{pmatrix} x_2 \\ x_4 \\ x_6 \\ x_7 \end{pmatrix}$ $\quad \bar{u} = \begin{pmatrix} \delta_A \\ \delta_s \\ \delta_R \\ \delta_{th_1} \\ \delta_{th_2} \end{pmatrix}$

How to obtain the A_{lat}, B_{lat}, C_{lat}, and D_{lat} matrices? One option is to take

$$A_{lat} = A(5:8,\ 5:8)$$
$$B_{lat} = B(5:8,\ :)$$
$$C_{lat} = \text{eye}(4)$$
$$D_{lat} = \text{zeros}(4,\ 5)$$

Dutch Roll, Roll Subsidence, and Spiral Divergence Modes

We can perform a similar analysis for the lateral/directional modes.

What are the physical description of these modes?

Roll Subsidence mode ($\lambda = -1.387$):
Simply damping of rolling motion.

Dutch Roll mode ($\lambda = -0.291 \pm 0.799\,i$):
Oscillatory motion and combines roll and yaw motion. Similarity to an ice-skating motion of dutch skaters.

Spiral Divergence mode ($\lambda = -0.108$):
Aircraft diverges and starts to spiral. Easily observed and corrected for in VFR conditions but can lead to crashes at night (JFK Jr. accident in 1999, departed at 8:39pm, half hour past sunset, hazy, flight over featureless, open water. Horizon visually blends with the water which causes spatial disorientation)