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## Lecture 06d
## The Routh-Hurwitz Stability Criterion

**Lecture is on YouTube**

The YouTube video entitled ‘The Routh Hurwitz Stability Criterion’ that covers this lecture is located at ‘https://youtu.be/QWb9sq35cNk’

# Outline

-Introduction
-Routh-Hurwitz Stability Criterion
-Generalization
-Additional References

# Introduction

The root locus method gives us a tool which we can use to estimate how the poles of the closed loop system change as we vary the control gain, $K$. We saw that as we increase the gain to $K \to \infty$, the closed loop poles go to the open loop zeros or zeros at infinity. If any of the open loop zeros are in the right half plane or if any of the asymptotes to the zeros at infinity lead to the right half plane, then at some point, too large a gain will lead to an unstable closed loop system. The root locus will predict this behavior but it does not give us a method to determine the exact value of $K$ which will lead to an unstable system. To determine this critical gain $K$, we use the Routh-Hurwitz stability criterion.

# Routh - Hurwitz Stability Criterion

The Routh-Hurwitz criterion is a necessary and sufficient criterion for the stability of linear systems.

Consider a characteristic polynomial of the form

$$\Delta(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-1} + \ldots + a_1 s + a_0 = 0 \qquad \textbf{\textit{(Eq.1)}}$$

The Routh-Hurwitz criterion gives us a method to determine the number of roots of the polynomial that are in the right half plane without needing to explicitly find the roots of the characteristic equation. This is helpful as it allows us a technique to

**Math Joke: Left Half Plane**

A plane leaves Warsaw, Poland bound for Seattle, United states. As such, it is carrying half Polish citizens and half US citizens. The Polish people all sit on the port side of the plane and all the Americans sit on the starboard side. As the plane nears its destination in, the some of the American's look out their side of the plane and see a group of Orcas whales swimming around a ferry boat and start snapping photos of it. One of the Polish passengers overhears goes the over to have a look. As soon as he does this the plane experiences severe lateral instability and almost crashes but the pilot is able to recover. What is the moral of the story? Always keep <u>all</u> your poles in the left half plane.

**Step 1: Build an empty Routh array**

The Routh array will be a matrix with $n+1$ rows and $h$ columns where

$$h = \text{ceil}\left(\frac{n+1}{2}\right) \qquad \textbf{\textit{(Eq.2)}}$$

```
h[n_] := Ceiling[(n + 1)/2];
```

**Step 2: Fill in the first two rows of the Routh array**

The first two rows of the Routh array can be obtained directly from the coefficients $a_n$, $a_{n-1}$, ..., $a_0$

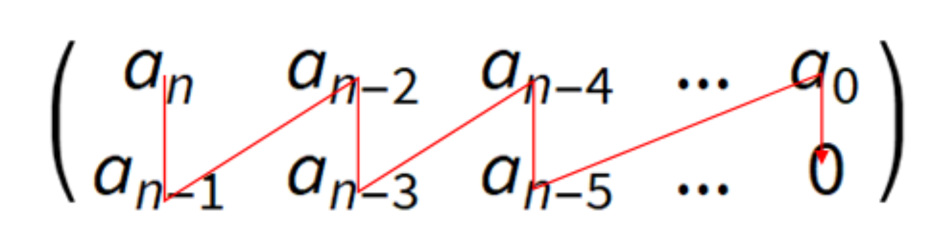
If $n$ is odd

$$\begin{pmatrix} a_n & a_{n-2} & a_{n-4} & \cdots & a_1 \\ a_{n-1} & a_{n-3} & a_{n-5} & \cdots & a_0 \end{pmatrix}$$

If $n$ is even

$$\begin{pmatrix} a_n & a_{n-2} & a_{n-4} & \cdots & a_0 \\ a_{n-1} & a_{n-3} & a_{n-5} & \cdots & 0 \end{pmatrix}$$

Note: Some people find it helpful to think of this as a zig-zag pattern.



**Step 3: Build the Routh array (rows 3 to $n+1$)**

The next rows (rows 3, 4, ..., $n+1$) can be constructed as follows. Consider the entry at row $k \in [3, n+1]$, column $i \in [1, h]$. Here, we denote the row directly above $k$ to be $y$ and the row above that one as $x$. Graphically we have

$$\begin{array}{ccccccccc}
 & & \text{col } 1 & \text{col } 2 & & \text{col } i & \text{col } i+1 & & \text{col } h \\
 & & \downarrow & \downarrow & & \downarrow & \downarrow & & \downarrow \\
x = \text{row } k-2 & \rightarrow & x_1 & x_2 & \ldots & \ldots & x_{i+1} & \ldots & x_h \\
y = \text{row } k-1 & \rightarrow & y_1 & y_2 & \ldots & \ldots & y_{i+1} & \ldots & y_h \\
\text{row } k & \rightarrow & \ldots & \ldots & \ldots & z_{k,i} & \ldots & \ldots & \ldots \\
 & & & & & \uparrow & & & \\
 & & & & & \text{computing} & & & \\
 & & & & & \text{this value} & & &
\end{array}$$

The entry at row $k$, column $i$ is given as

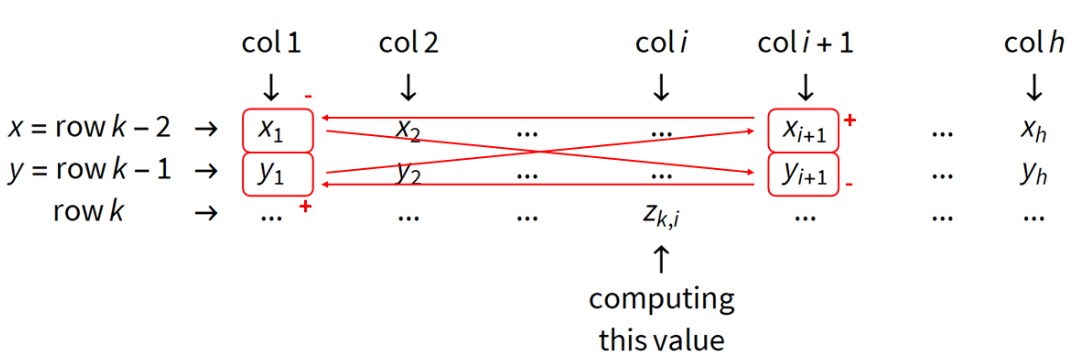
$$z_{k,i} = \begin{cases} \frac{y_1 x_{i+1} - x_1 y_{i+1}}{y_1} & i \in [1, h-1] \\ 0 & i = h \end{cases} \qquad \textbf{\textit{(Eq.3)}}$$

Note:

-This constantly uses the first column for the calculation ($x_1$ and $y_1$)

-The last column is always 0

-The relevant entries form a “figure 8” with the right side of the pattern expanding to the right as we increment $i$.




**Step 4: Count sign changes in the first column**

Isolate the first column of the array. As you proceed down the column, count the number of sign changes in the first column, this is the number of roots of the polynomial which has positive real parts.

**Example 1**

Consider a characteristic polynomial of the form

$$\Delta(s) = 2\, s^6 + 4\, s^5 + 2\, s^4 - s^3 + 2\, s - 2 = 0$$

```
Δ[s_] = 2 s^6 + 4 s^5 + 2 s^4 - s^3 + 2 s - 2;
```

So we see that the coefficients of the polynomials are

$$a_6 = 2$$
$$a_5 = 4$$

$a_4 = 2$
$a_3 = -1$
$a_2 = 0$
$a_1 = 2$
$a_0 = -2$

We first compute the number of columns

$$h = \text{ceil}\left(\tfrac{n+1}{2}\right)$$

$$= \text{ceil}(3.5)$$

$$h = 4$$

So we see that the total Routh array will have $n + 1 = 7$ rows and $h = 4$ columns

$$\begin{pmatrix} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{pmatrix} \qquad \text{(empty Routh array)}$$

We can now fill in the first two rows of the array.

Since $n$ is even

$$\begin{pmatrix} a_6 & a_4 & a_2 & a_0 \\ a_5 & a_3 & a_1 & 0 \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 & -2 \\ 4 & -1 & 2 & 0 \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{pmatrix} \qquad \text{(first two rows of Routh array)}$$

From Eq.3, we know that the last column for row 3 to 7 will be zero, so to make our calculations easier later, let’s just fill this in now

$$\begin{pmatrix} 2 & 2 & 0 & -2 \\ 4 & -1 & 2 & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \end{pmatrix}$$

(last column, rows 3 to 7 will be 0)

We now compute the remaining entries of the array using

$$z_{k,i} = \frac{y_1 x_{i+1} - x_1 y_{i+1}}{y_1}$$

**Row 3**

**Compute $z_{3,1}$**

Let us now build row 3. Starting with row 3, column 1, we compute

$$z_{3,1} = \frac{y_1 x_{1+1} - x_1 y_{1+1}}{y_1}$$

$$= \frac{y_1 x_2 - x_1 y_2}{y_1}$$

Graphically, the values used to compute this are shown below in bold

$$\begin{pmatrix} \mathbf{2} & \mathbf{2} & 0 & -2 \\ \mathbf{4} & \mathbf{-1} & 2 & 0 \\ z_{3,1} & \square & \square & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \end{pmatrix}$$

We obtain

```
z31 = ((4 * 2) - (2 * -1)) / 4
```

$$\frac{5}{2}$$

**Compute $z_{3,2}$**

For the next entry in this row, we compute $z_{3,2}$ as

$$z_{3,2} = \frac{y_1 x_{2+1} - x_1 y_{2+1}}{y_1}$$

$$= \frac{y_1 x_3 - x_1 y_3}{y_1}$$

Graphically, the values used to compute this are shown below in bold

$$\begin{pmatrix} \mathbf{2} & 2 & \mathbf{0} & -2 \\ \mathbf{4} & -1 & \mathbf{2} & 0 \\ \square & z_{3,2} & \square & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \end{pmatrix}$$

We obtain

```
z32 = ((4 * 0) - (2 * 2)) / 4
-1
```

**Compute $z_{3,3}$**

For the next entry in this row, we compute $z_{3,3}$ as

$$z_{3,3} = \frac{y_1 x_{3+1} - x_1 y_{3+1}}{y_1}$$

$$= \frac{y_1 x_4 - x_1 y_4}{y_1}$$

Graphically, the values used to compute this are shown below in bold

$$\begin{pmatrix} \mathbf{2} & 2 & 0 & \mathbf{-2} \\ \mathbf{4} & -1 & 2 & \mathbf{0} \\ \square & \square & z_{3,3} & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \end{pmatrix}$$

We obtain

```
z33 = ((4 * -2) - (2 * 0)) / 4
-2
```

We have now completed the 3rd row of the Routh array and we obtain

$$\begin{pmatrix} 2 & 2 & 0 & -2 \\ 4 & -1 & 2 & 0 \\ 5/2 & -1 & -2 & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \end{pmatrix}$$

**Row 4**

**Compute $z_{4,1}$**

Let us now build row 4.  Starting with row 4, column 1, we compute

$$z_{4,1} = \frac{y_1\, x_{1+1} - x_1\, y_{1+1}}{y_1}$$

$$= \frac{y_1\, x_2 - x_1\, y_2}{y_1}$$

Graphically, the values used to compute this are shown below in bold

$$\begin{pmatrix} 2 & 2 & 0 & -2 \\ \mathbf{4} & \mathbf{-1} & 2 & 0 \\ \mathbf{5/2} & \mathbf{-1} & -2 & 0 \\ z_{4,1} & \square & \square & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \end{pmatrix}$$

We obtain

$$\mathbf{z41} = \frac{\left(\frac{5}{2} * -1\right) - (4 * -1)}{\frac{5}{2}}$$

$$\frac{3}{5}$$

**Compute $z_{4,2}$**

For the next entry in this row, we compute $z_{4,2}$ as

$$z_{4,2} = \frac{y_1\, x_{2+1} - x_1\, y_{2+1}}{y_1}$$

$$= \frac{y_1\, x_3 - x_1\, y_3}{y_1}$$

Graphically, the values used to compute this are shown below in bold

$$\begin{pmatrix} 2 & 2 & 0 & -2 \\ \mathbf{4} & -1 & \mathbf{2} & 0 \\ \mathbf{5/2} & -1 & \mathbf{-2} & 0 \\ \square & z_{4,2} & \square & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \end{pmatrix}$$

We obtain

$$\mathbf{z42} = \frac{\left(\frac{5}{2} * 2\right) - (4 * -2)}{\frac{5}{2}}$$

$$\frac{26}{5}$$

**Compute $z_{4,3}$**

For the next entry in this row, we compute $z_{4,3}$ as

$$z_{4,3} = \frac{y_1 \, x_{3+1} - x_1 \, y_{3+1}}{y_1}$$

$$= \frac{y_1 \, x_4 - x_1 \, y_4}{y_1}$$

Graphically, the values used to compute this are shown below in bold

$$\begin{pmatrix} 2 & 2 & 0 & -2 \\ \mathbf{4} & -1 & 2 & \mathbf{0} \\ \mathbf{5/2} & -1 & -2 & \mathbf{0} \\ \square & \square & z_{4,3} & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \end{pmatrix}$$

We obtain

$$\mathbf{z42} = \frac{\left(\frac{5}{2} * 0\right) - (4 * 0)}{\frac{5}{2}}$$

0

We have now completed the 4rd row of the Routh array and we obtain

$$\begin{pmatrix} 2 & 2 & 0 & -2 \\ 4 & -1 & 2 & 0 \\ 5/2 & -1 & -2 & 0 \\ 3/5 & 26/5 & 0 & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \\ \square & \square & \square & 0 \end{pmatrix}$$

**Row 5, 6, 7**

So now, we see a pattern emerging and can fill in the rest of the Routh array to obtain the final solution of

$$\begin{pmatrix} 2 & 2 & 0 & -2 \\ 4 & -1 & 2 & 0 \\ 5/2 & -1 & -2 & 0 \\ 3/5 & 26/5 & 0 & 0 \\ -22.667 & -2 & 0 & 0 \\ 5.147 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 & -2 \\ 4 & -1 & 2 & 0 \\ 2.5 & -1 & -2 & 0 \\ 0.6 & 5.2 & 0 & 0 \\ -22.667 & -2 & 0 & 0 \\ 5.147 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \end{pmatrix}$$

We can verify this Routh array using the function located at https://github.com/clum/UWMatlab/blob/-master/UWMatlab/Controls_Functions/RouthArray.m.

We can now count the number of sign changes in the first column to determine the number of poles in the right half plane. We see that there are 3 sign changes

sign change #1: $+3/5 \to -22.667$
sign change #2: $-22.667 \to +5.147$
sign change #3: $+5.147 \to -2$

So we expect 3 roots in the right half plane. We can verify this by solving the characteristic equation.

```
Solve[Δ[s] == 0, s] // N
```

```
{{s → -1.45352}, {s → 0.650337}, {s → -0.961252 - 0.929103 i},
 {s → -0.961252 + 0.929103 i}, {s → 0.362844 - 0.678423 i}, {s → 0.362844 + 0.678423 i}}
```

So we see that there are indeed 3 roots in the right half plane.

**End Example**

**Example 2: Special Case**

Consider a characteristic polynomial of the form. Note that this example and the subsequent discussion is adapted from Brian Douglas' YouTube video entitled 'Routh-Hurwitz Criterion, Special Cases' (https://youtu.be/oMmUPvn6lP8).

$$\Delta(s) = s^4 + 2 s^3 + 2 s^2 + 4 s + 5 = 0$$

Let us first compute the roots so we know what the proper result from the Routh array should be

```
Δ[s_] = s^4 + 2 s^3 + 2 s^2 + 4 s + 5;
Solve[Δ[s] == 0, s] // N
```

```
{{s → 0.432673 - 1.38709 i}, {s → 0.432673 + 1.38709 i},
 {s → -1.43267 + 0.561907 i}, {s → -1.43267 - 0.561907 i}}
```

We see that there are 2 poles in the RHP, so we expect 2 sign changes.

We first compute the number of columns

$$h = \text{ceil}\left(\frac{n+1}{2}\right)$$

$$= \text{ceil}\left(\frac{6}{2}\right)$$

$$h = 3$$

So we see that the total Routh array will have $n + 1 = 5$ rows and $h = 3$ columns

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 0 \\ \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{pmatrix} \qquad \text{(empty Routh array)}$$

From Eq.3, we know that the last column for row 3 to 5 will be zero, so to make our calculations easier later, let's just fill this in now

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 0 \\ \square & \square & 0 \\ \square & \square & 0 \\ \square & \square & 0 \end{pmatrix} \qquad \text{(last column, rows 3 to 7 will be 0)}$$

**Row 3**

We can compute row 3

$$z_{3,1} = \frac{(2*2)-(1*4)}{2} = 0 \qquad \text{(this may cause issues later)}$$

$$z_{3,2} = \frac{(2*5)-(1*0)}{2} = 5$$

This zero in the first column will turn out to cause a slight issue later (because it will end up in the denominator of a future calculation). We now have

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 0 \\ 0 & 5 & 0 \\ \square & \square & 0 \\ \square & \square & 0 \end{pmatrix}$$

**Row 4**

We can compute row 4

$$z_{4,1} = \frac{(0*4)-(2*5)}{0} = -\infty$$

$$z_{4,2} = \frac{(0*0)-(2*0)}{0} = \text{indeterminate (NaN)}$$

If we proceed, the 4th row is given as

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 0 \\ 0 & 5 & 0 \\ -\infty & \text{NaN} & 0 \\ \square & \square & 0 \end{pmatrix}$$

**Row 5**

We can compute row 5

$$z_{5,1} = \frac{(-\infty*5)-(0*\text{NaN})}{-\infty} = \text{NaN}$$

$$z_{5,2} = \frac{(-\infty*0)-(0*0)}{-\infty} = \text{NaN}$$

If we proceed, the 5th row is given as

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 0 \\ 0 & 5 & 0 \\ -\infty & \text{NaN} & 0 \\ \text{NaN} & \text{NaN} & 0 \end{pmatrix}$$

We clearly see 1 sign change but what about the second sign change?

**Row 3 (revisited)**

The problem stemmed from the fact that we had a 0 in the first column. This eventually worked its way into the denominator which caused problem. Instead of using 0, chose $\epsilon$, a small value for this entry.

Let us now use $\epsilon$ instead of 0 for entry $z_{3,1}$ and the third row of the Routh array becomes

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 0 \\ \varepsilon & 5 & 0 \\ \square & \square & 0 \\ \square & \square & 0 \end{pmatrix}$$

**Row 4**

Row 4 now becomes

$$z_{4,1} = \frac{(\varepsilon * 4) - (2 * 5)}{\varepsilon} = -\infty$$

$$z_{4,2} = \frac{(\varepsilon * 0) - (2 * 0)}{\varepsilon} = 0$$

So the Routh array becomes

$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 0 \\ \varepsilon & 5 & 0 \\ -\infty & 0 & 0 \\ \square & \square & 0 \end{pmatrix}$$

**Row 5**

Row 5 now becomes

$$z_{5,1} = \frac{(-\infty * 5) - (\varepsilon * 0)}{-\infty} = 5$$

$$z_{4,2} = \frac{(\varepsilon * 0) - (2 * 0)}{\varepsilon} = 0$$

So the Routh array becomes

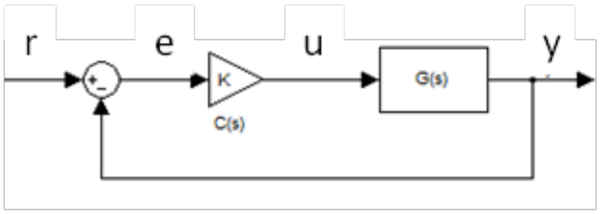
$$\begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 0 \\ \varepsilon & 5 & 0 \\ -\infty & 0 & 0 \\ 5 & 0 & 0 \end{pmatrix}$$

So now we see that there are indeed 2 sign changes, and therefore 2 poles in the RHP.

sign change #1: $+\varepsilon \to -\infty$
sign change #2: $-\infty \to +5$

# Application to Root Locus

Consider the system shown below

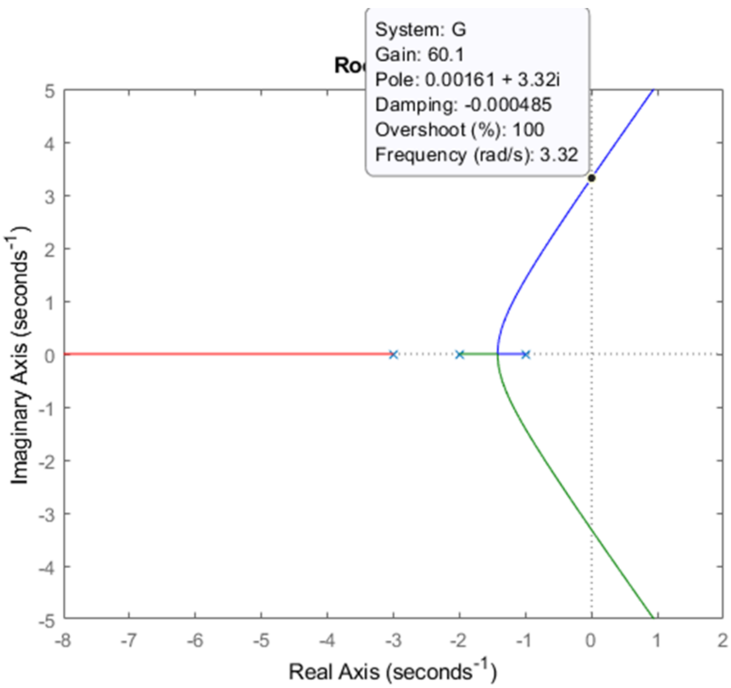



Suppose we have the following plant model

$$G(s) = \frac{1}{(s+1)(s+2)(s+3)}$$

```
G[s_] = 1/((s + 1) (s + 2) (s + 3));
```

We can easily compute the root locus (see ‘Using ‘rlocus’ in Matlab to Plot the Root Locus’ at https://youtu.be/im19KuzjWwo).




So we see that a gain of $K \approx 60$ yields two poles in the RHP. Let us see if Routh Hurwitz would predict this same result. We first compute the closed loop characteristic equation

$$\Delta(s) = 1 + K\,G(s) = b(s) + K\,a(s) = 0$$

```
a[s_] = Numerator[G[s]];
b[s_] = Denominator[G[s]];
Δ[s_] = b[s] + K a[s] // Expand
```

$6 + K + 11\,s + 6\,s^2 + s^3$

So we have

$$\Delta(s) = s^3 + 6\,s^2 + 11\,s + (6 + K)$$

We can now build the Routh array for this polynomial

$$h = \text{ceil}\left(\frac{n+1}{2}\right) = \text{ceil}\left(\frac{3+1}{2}\right) = 2$$

Initializing the Routh array

$$\begin{pmatrix} 1 & 11 \\ 6 & 6+K \\ \square & \square \\ \square & \square \end{pmatrix}$$

Filling in the right column with 0's

$$\begin{pmatrix} 1 & 11 \\ 6 & 6+K \\ \square & 0 \\ \square & 0 \end{pmatrix}$$

We compute the entries for row 3

$$\mathbf{z31} = \frac{(6 * 11) - (1 * (6 + K))}{6}$$

$$\frac{60 - K}{6}$$

$$\begin{pmatrix} 1 & 11 \\ 6 & 6+K \\ \frac{60-K}{6} & 0 \\ \square & 0 \end{pmatrix}$$

We compute the entries for row 4

$$\mathbf{z41} = \frac{\left(\left(\frac{60-K}{6}\right) * (6 + K)\right) - (6 * 0)}{\frac{60-K}{6}}$$

$$6 + K$$

So we obtain

$$\begin{pmatrix} 1 & 11 \\ 6 & 6+K \\ \frac{60-K}{6} & 0 \\ 6+K & 0 \end{pmatrix}$$

From inspection of the first column, we see that we have a sign change when

$$\frac{60-K}{6} < 0$$

$$K > 60$$

So if we use $K = 60 + \varepsilon$, the first column of the Routh array becomes

$$\begin{pmatrix} 1 & \square \\ 6 & \square \\ \frac{-\varepsilon}{6} & \square \\ 66+\varepsilon & \square \end{pmatrix}$$

And we see there are two sign changes which predict the two positive poles.

# Generalization

The procedure above is useful because it applies to any order polynomial.  However, if we have a reduced order polynomial, we can simply the process.  In other words, instead of constructing a Routh array for a given polynomial, let us consider a general polynomial and see if we can identify relationships on the coefficients of the polynomial to yield a stable system.

In the following, we assume the coefficient of the highest order, $a_n$, is positive.  If necessary, this can always be achieved by multiplication of the polynomial with -1.

Assumption 1: $a_n > 0$ **(A.1)**

**2nd Order Polynomial**

Consider the following 2nd order polynomial

$$P_2(s) = a_2 s^2 + a_1 s + a_0 = 0$$

The corresponding Routh array is

$$\begin{pmatrix} a_2 & a_0 \\ a_1 & 0 \\ b_1 & 0 \end{pmatrix}$$

where $b_1 = \frac{a_1 a_0 - a_2 * 0}{a_1} = a_0$

So we see that we can write the Routh array as

$$\begin{pmatrix} a_2 & a_0 \\ a_1 & 0 \\ a_0 & 0 \end{pmatrix}$$

Therefore, we see that in order for this polynomial to be stable, we require that all coefficient are positive.

$a_2 > 0$ (automatically true as per assumption A.1) ***(Eq.4a)***

$a_1 > 0$ ***(Eq.4b)***

$a_0 > 0$ ***(Eq.4c)***

Note that in Eq.4, $a_2 > 0$ is automatically satisfied as per assumption A.1.

**3rd Order Polynomial**

This can be extended to a 3rd order polynomial

$$P_3(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

The corresponding Routh array is

$$\begin{pmatrix} a_3 & a_1 \\ a_2 & a_0 \\ b_1 & 0 \\ c_1 & 0 \end{pmatrix}$$

where $b_1 = \frac{a_2 a_1 - a_3 a_0}{a_2}$

$$c_1 = \frac{b_1 a_0 - a_2 * 0}{b_1} = a_0$$

So we see that we can write the Routh array as

$$\begin{pmatrix} a_3 & a_1 \\ a_2 & a_0 \\ \frac{a_2 a_1 - a_3 a_0}{a_2} & 0 \\ a_0 & 0 \end{pmatrix}$$

Therefore, we immediately see that in order for this polynomial to be stable, that we require

$a_3 > 0$ (automatically true as per assumption A.1) ***(Eq.5a)***

$a_2 > 0$ ***(Eq.5b)***

$a_0 > 0$ ***(Eq.5c)***

Also, we see that we require that the quantity

$\frac{a_2 a_1 - a_3 a_0}{a_2} > 0$ note: we previously assumed that $a_2 > 0$

$$a_2 a_1 - a_3 a_0 > 0$$

$a_2 a_1 > a_3 a_0$ ***(Eq.6)***

Interestingly, notice there is no constrain on $a_1$. However if we examine Eq.6

$a_2\,a_1 > a_3\,a_0$ assume $a_2 > 0$ is true per Eq.5b

$a_1 > \frac{a_3\,a_0}{a_2}$

Note that if $a_3 > 0$ (Eq.5a) and $a_0 > 0$ (Eq.5c), then shows that $a_1 > 0$. In other words

$$a_3 > 0,\ a_2 > 0,\ a_0 > 0,\ a_2\,a_1 > a_3\,a_1 \quad \Rightarrow \quad a_1 > 0 \qquad \textbf{\textit{(Eq.7)}}$$

In other words, Eq.5a, b, c, and Eq.6 being true will guarantee that $a_1 > 0$.

**Example**

We can apply this to the previous root locus problem

$$\Delta(s) = s^3 + 6\,s^2 + 11\,s + (6 + K)$$

| | | | |
|---|---|---|---|
| Eq.5a: | $a_3 > 0$ | $1 > 0$ | true |
| Eq.5b: | $a_2 > 0$ | $6 > 0$ | true |
| Eq.5c: | $a_0 > 0$ | $6 + K > 0$ | true if $K > -6$ |
| Eq.6: | $a_2\,a_1 > a_3\,a_0$ | | |

$$6 * 11 > 1 * (6 + K)$$

$$66 > 6 + K$$

$$60 > K$$

So we arrive at the same conclusion we had earlier, namely $K \geq 0$ will lead to positive poles.

**End Example**

**4th Order Polynomial**

This can be extended 4th order polynomials of the form

$$P_4(s) = a_4\,s^4 + a_3\,s^3 + a_2\,s^2 + a_1\,s + a_0 = 0$$

The empty array is

$$\begin{pmatrix} a_4 & a_2 & a_0 \\ a_3 & a_1 & 0 \\ \square & \square & 0 \\ \square & \square & 0 \\ \square & \square & 0 \end{pmatrix}$$

<u>Row 3</u>

$$z_{3,1} = \frac{a_3\,a_2 - a_4\,a_1}{a_3}$$

$$z_{3,2} = \frac{a_3\,a_0 - a_4 * 0}{a_3} = \frac{a_3\,a_0}{a_3} = a_0$$

So the array becomes

$$\begin{pmatrix} a_4 & a_2 & a_0 \\ a_3 & a_1 & 0 \\ b_1 & b_2 & 0 \\ \square & \square & 0 \\ \square & \square & 0 \end{pmatrix}$$

where $b_1 = \frac{a_3\,a_2 - a_4\,a_1}{a_3}$

$b_2 = a_0$

```
b1 = (a3 * a2 - a4 * a1) / a3;
b2 = a0;
```

<u>Row 4</u>

$$z_{4,1} = \frac{b_1\,a_1 - a_3\,b_2}{b_1}$$

$$z_{4,2} = \frac{b_1 * 0 - a_3 * 0}{b_1} = 0$$

So the array becomes

$$\begin{pmatrix} a_4 & a_2 & a_0 \\ a_3 & a_1 & 0 \\ b_1 & b_2 & 0 \\ c_1 & 0 & 0 \\ \square & \square & 0 \end{pmatrix}$$

where $c_1 = \frac{b_1\,a_1 - a_3\,b_2}{b_1}$

```
c1 = (b1 * a1 - a3 * b2) / b1 // Simplify
```

$$\frac{-a1\,a2\,a3 + a0\,a3^2 + a1^2\,a4}{-a2\,a3 + a1\,a4}$$

<u>Row 5</u>

$$z_{5,1} = \frac{c_1\,b_2 - b_1 * 0}{c_1} = \frac{c_1\,b_2}{c_1} = b_2$$

$$z_{5,2} = \frac{c_1 * 0 - b_1 * 0}{c_1} = 0$$

So the array becomes

$$\begin{pmatrix} a_4 & a_2 & a_0 \\ a_3 & a_1 & 0 \\ b_1 & b_2 & 0 \\ c_1 & 0 & 0 \\ d_1 & 0 & 0 \end{pmatrix}$$

where $\quad d_1 = b_2$

```
d1 = b2
```

```
a0
```

After making the relevant substitutions the Routh array is given as

$$\begin{pmatrix} a_4 & a_2 & a_0 \\ a_3 & a_1 & 0 \\ \frac{a_2 a_3 - a_1 a_4}{a_3} & a_0 & 0 \\ \frac{a_4 a_1^2 + a_0 a_3^2 - a_1 a_2 a_3}{a_1 a_4 - a_2 a_3} & 0 & 0 \\ a_0 & 0 & 0 \end{pmatrix}$$

Therefore, we immediately see that in order for this polynomial to be stable, that we require

$a_4 > 0$ (automatically true as per assumption A.1) ***(Eq.8a)***

$a_3 > 0$ ***(Eq.8b)***

$a_0 > 0$ ***(Eq.8c)***

Also, we see that we require that the quantity

$\frac{a_2 a_3 - a_1 a_4}{a_3} > 0$ note: we previously assumed that $a_3 > 0$

$a_2 a_3 - a_1 a_4 > 0$

$a_2 a_3 > a_1 a_4$ ***(Eq.9)***

In addition, we require

$\frac{a_4 a_1^2 + a_0 a_3^2 - a_1 a_2 a_3}{a_1 a_4 - a_2 a_3} > 0$ note: if Eq.9 is true, then denominator is less than 0

$a_4 a_1^2 + a_0 a_3^2 - a_1 a_2 a_3 < 0$

$a_1 a_2 a_3 > a_4 a_1^2 + a_0 a_3^2$ ***(Eq.10)***

Again, we note that there are no constraints on $a_1$ or $a_2$ individually.

**Summary**

All the roots are in the left half plane of each respective polynomial if

2nd order

$P_2(s) = a_2 s^2 + a_1 s + a_0 = 0$

Coefficients of $P_2(s)$ satisfy

$a_2 > 0$ (automatically true as per assumption A.1)

$a_1 > 0$

$a_0 > 0$

3rd order

$P_3(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$

Coefficient of $P_3(s)$ satisfy

$a_3 > 0$ (automatically true as per assumption A.1)

$a_2 > 0$

$a_1$ no restrictions although satisfying other restrictions will guarantee that $a_1 > 0$

$a_0 > 0$

$a_2 a_1 > a_3 a_0$

4th order

$P_4(s) = a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$

Coefficient of $P_4(s)$ satisfy

$a_4 > 0$ (automatically true as per assumption A.1)

$a_3 > 0$

$a_2$ no restrictions

$a_1$ no restrictions

$a_0 > 0$

$a_2 a_3 > a_1 a_4$

$a_1 a_2 a_3 > a_4 a_1^2 + a_0 a_3^2$

We investigate applying this knowledge to a real system in the homework.

# Additional References

-Brian Douglas' video on special cases https://youtu.be/oMmUPvn6lP8
-https://www.math24.net/routh-hurwitz-criterion/