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Lecture 01e
Linear Transformations and Linear Systems

Lecture is on YouTube

The YouTube video entitled ‘Linear Transformations and Linear Systems’ that covers this lecture is located at https://youtu.be/DAzn8d_A-ic

Outline

-Linear Transformations

Linear Transformations

We've been talking about linear vs. nonlinear functions, so we should clarify what is meant by this.

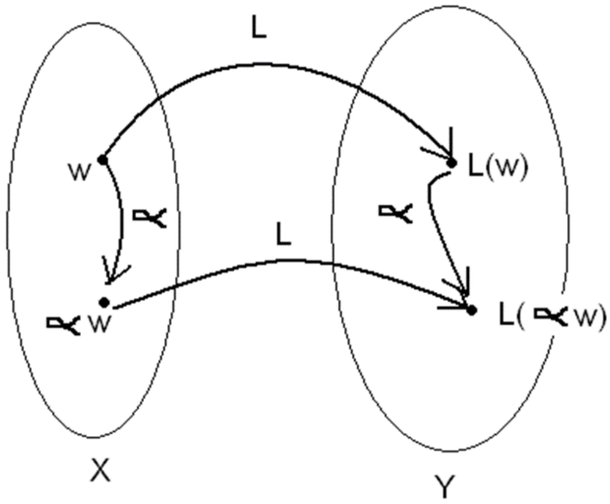
A transformation L of a linear space X into a linear space Y, where X and Y have the same scalar field F, is said to be a linear transformation if

(i)	$L(\alpha w) = \alpha L(w)$	$\forall \alpha \in F$ and $\forall w \in X$	(homogeneity)
(ii)	$L(w + p) = L(w) + L(p)$	$\forall w, p \in X$	(additivity)

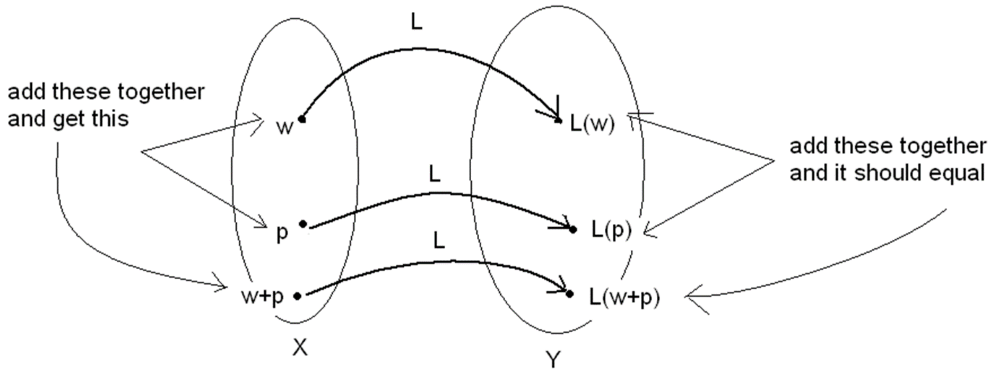
This is also referred to as a linear map or in some contexts as a linear function https://en.wikipedia.org/wiki/Linear_map

Written as $L : X \to Y$.

Picture for homogeneity is



Picture for additivity is

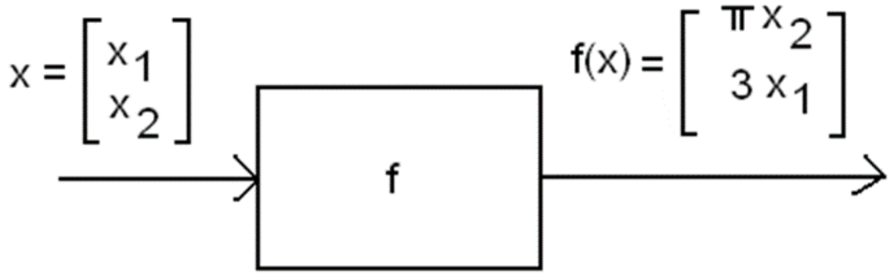


Example 1: Linear Transformation

$$f(x) = \begin{pmatrix} \pi\, x_2 \\ 3\, x_1 \end{pmatrix} \qquad \text{where} \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

So in this case $f : X \to Y$ where $X = \mathbb{R}^2$, $Y = \mathbb{R}^2$.

So can visualize f as a black box where you stick in a vector in $\mathbb{R}^2$ and it spits out another vector in $\mathbb{R}^2$



Check homogeneity

$$f(\alpha\, x) = f\left(\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = f\left(\begin{pmatrix} \alpha\, x_1 \\ \alpha\, x_2 \end{pmatrix}\right) = \begin{pmatrix} \pi\, \alpha\, x_2 \\ 3\, \alpha\, x_1 \end{pmatrix}$$

$$\alpha f(x) = \alpha f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \alpha \begin{pmatrix} \pi x_2 \\ 3 x_1 \end{pmatrix} = \begin{pmatrix} \pi \alpha x_2 \\ 3 \alpha x_1 \end{pmatrix}$$

Check additivity

$$f(w + p) = f\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} + \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}\right) = f\left(\begin{pmatrix} w_1 + p_1 \\ w_2 + p_2 \end{pmatrix}\right) = \begin{pmatrix} \pi(w_2 + p_2) \\ 3 (w_1 + p_1) \end{pmatrix}$$

$$f(w) + f(p) = f\left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}\right) + f\left(\begin{pmatrix} p_1 \\ p_2 \end{pmatrix}\right) = \begin{pmatrix} \pi w_2 \\ 3 w_1 \end{pmatrix} + \begin{pmatrix} \pi p_2 \\ 3 p_1 \end{pmatrix} = \begin{pmatrix} \pi(w_2 + p_2) \\ 3 (w_1 + p_1) \end{pmatrix}$$

Both tests pass, so f is a linear transformation

Example 2: Nonlinear Transformation

$$f(x) = x_1 x_2 \qquad \text{where} \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

So in this case $f : X \to Y$ where $X = \mathbb{R}^2$, $Y = \mathbb{R}^1$.

Check homogeneity

$$f(\alpha x) = f\left(\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = f\left(\begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}\right) = \alpha^2 x_1 x_2$$

$$\alpha f(x) = \alpha f\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \alpha (x_1 x_2)$$

So this fails homogeneity, so is a nonlinear transformation.

Example 3: Line

Consider the equation of a line

$$f(x) = m x + b$$

Check homogeneity

$$f(\alpha x) = m \alpha x + b$$

$$\alpha f(x) = \alpha(m x + b) = m \alpha x + \alpha b$$

Surprisingly, this does not pass homogeneity if $b \neq 0$ and is therefore not a linear transformation. This is referred to as an affine transformation (https://en.wikipedia.org/wiki/Affine_transformation) which is a linear transformation plus a translation (the y-intercept)

Example 3: Matrix Multiplication

We can extend some of the ideas above to the concept of matrix multiplication

$$f(\overline{x}) = A \overline{x}$$

Check homogeneity

$$f(\alpha\, \overline{x}) = A(\alpha\, \overline{x}) = \alpha\, A\, \overline{x}$$

$$\alpha\, f(x) = \alpha\, A\, \overline{x}$$

Check additivity

$$f(\overline{w}\ +\ \overline{p}) = A(\overline{w} + \overline{p}) = A\, \overline{w} + A\, \overline{p}$$

$$f(\overline{w}) + f(\overline{p}) = A\, \overline{w} + A\, \overline{p}$$

Both tests pass, so f is a linear transformation

Example 4: Derivative

What if the operation/function is a derivative operator

$$f(x(t)) = \frac{d}{d\,t}[x(t)]$$

Check homogeneity

$$f(\alpha\, x(t)) = \frac{d}{d\,t}[\alpha\, x(t)] = \alpha\, \frac{d}{d\,t}[x(t)]$$

$$\alpha\, f(x(t)) = \alpha\, \frac{d}{d\,t}[x(t)]$$

Check additivity

$$f(w(t)\ + p(t)) = \frac{d}{d\,t}[w(t) + p(t)] = \frac{d}{d\,t}[w(t)] + \frac{d}{d\,t}[p(t)]$$

$$f(w(t)) + f(p(t)) = \frac{d}{d\,t}[w(t)] + \frac{d}{d\,t}[p(t)]$$

Both tests pass, so f is a linear transformation

Sidenote: Laplace Transforms are linear (see YouTube video ‘The Laplace Transform at https://y-outu.be/q0nX8uIFZ_k)

We investigate several other example in a homework problem.

Superposition

As we just saw, a function $f(x)$ that satisfies both additivity and homogeneity is referred to as a linear function. This is also referred to as the superposition property which states that the net output caused by two or more inputs is the sum of the outputs that would have been caused by each input individually.

In other word, for linear systems, superposition holds.

Consider a system with

$$y = f(u)$$

where u = input
y = output

So if superposition holds, then

$$y = f(\alpha u) = \alpha f(u) \qquad \text{(homogeneity)}$$

$$y_c = f(u_1 + u_2) = f(u_1) + f(u_2) \qquad \text{(additivity)}$$

We can consider both simultaneously with

$$f(\alpha u_1 + \beta u_2) = \alpha f(u_1) + \beta f(u_2)$$

The left side is the response of the system due to a complex input of $\alpha u_1 + \beta u_2$.

If superposition holds, then we can instead evaluate the right side which states that we can obtain the output the the single input u_1, repeat for another input u_2 and then scale and add them together arbitrarily to obtain another solution/output.

Linear Differential Equations

We would like to apply this our differential equations to determine if they are linear. Recall that we are interested in ordinary differential equations. These are functions which contains only one independent variable and one or more of its derivatives. This is distinctly different than partial differential equations which involve partial derivatives of several variables.

Example of an ordinary differential equation

$$3\ddot{x}(t) + 4\frac{\ln(\dot{x}(t))}{x(t)} + \cos(x(t)) = 2e^{2t} \qquad \text{(ODE)}$$

with $x(0) = x_o$
$\dot{x}(0) = \dot{x}_o$

Example of a partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 y}{\partial y^2} = 0 \qquad \text{(PDE)}$$

with $u(x, 0) = 0$
$\frac{\partial u(x,0)}{\partial y} = \frac{\sin(n x)}{n}$

Furthermore, we would like to look at a special class of ODEs, namely linear ODEs. These have the form

$$\frac{d^n x(t)}{d t^n} + a_n \frac{d^{n-1} x(t)}{d t^{n-1}} + \ldots + a_2 \frac{d x(t)}{d t} + a_1 x(t) = g(t) \qquad \textbf{\textit{(Eq.1)}}$$

Why are these called linear? Let us focus on the left side of the equation, the portion that concerns $x(t)$ and its derivatives. These are also referred to as the states of the system. Let us rewrite the left side as a function of n variables, namely

$$z_1 = x(t)$$
$$z_2 = \frac{d\,x(t)}{d\,t}$$
$$\vdots$$
$$z_n = \frac{d^{n-1}\,x(t)}{d\,t^{n-1}}$$
$$z_{n+1} = \frac{d^n\,x(t)}{d\,t^n}$$

So the left side of Eq.1 can be written as

$$\text{LHS} = z_{n+1} + a_n z_n + \ldots + a_2 z_2 + a_1 z_1$$

$$= \begin{pmatrix} a_1 & a_2 & \ldots & a_{n-1} & a_n & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_{n+1} \end{pmatrix}$$

$$\text{LHS} = f(\overline{z}) = \eta\,\overline{z}$$

where $\eta = \begin{pmatrix} a_1 & a_2 & \ldots & a_{n-1} & a_n & 1 \end{pmatrix}$
$\overline{z} = \begin{pmatrix} z_1 & z_2 & \ldots & z_{n+1} \end{pmatrix}^T$

We have defined the function $f(\overline{z}) = f(z_1, z_2, \ldots, z_{n+1})$ and we can check if the function $f : \mathbb{R}^{n+1} \to \mathbb{R}^1$ is linear in $\overline{z}$

Check homogeneity

$$f(\alpha\,\overline{z}) = \eta\,(\alpha\,\overline{z}) = \alpha\,\eta\,\overline{z}$$

$$\alpha\,f(\overline{z}) = \alpha\,\eta\,\overline{z} \qquad \text{(passes homogeneity)}$$

Check additivity

$$f(\overline{y} + \overline{w}) = \eta(\overline{y} + \overline{w})$$

$$f(\overline{y}) + f(\overline{w}) = \eta\,\overline{y} + \eta\,\overline{w} = \eta(\overline{y} + \overline{w}) \qquad \text{(passes additivity)}$$

In this case, we say that the system is linear in the states $\left(x(t),\ \dot{x}(t),\ \ddot{x}(t), \text{etc.}\right)$.

Furthermore, we know that the derivative operator is linear, so we see that f is a combination of two linear operations, the derivative and matrix multiplication. As such, we expect superposition to hold.

Example: 2nd Order ODE

Consider the ODE of

$$\ddot{x}(t) + 5\,\dot{x}(t) + 6\,x(t) = 0$$

We know that a solution is

$$x_1(t) = e^{-2t}$$

In[∘]:=
```
x1[t]
```
Out[∘]= e^{-2t}

In[∘]:=
```
x1[t_] = Exp[-2 t];
D[x1[t], {t, 2}] + 5 D[x1[t], t] + 6 x1[t] == 0
```
Out[∘]= True

We know that another solution is

$$x_2(t) = e^{-3t}$$

In[∘]:=
```
x2[t_] = Exp[-3 t];
D[x2[t], {t, 2}] + 5 D[x2[t], t] + 6 x2[t] == 0
```
Out[∘]= True

Because this is a linear system, we expect superposition to hold and therefore a combination of these solutions should be another solution

$$u_h(t) = c_1\, x_1(t) + c_2\, x_2(t)$$

In[∘]:=
```
xh[t_] = c1 x1[t] + c2 x2[t];
D[xh[t], {t, 2}] + 5 D[xh[t], t] + 6 xh[t] == 0 // Simplify
```
Out[∘]= True

For more information see:
‘Homogeneous Linear Ordinary Differential Equations’ at https://youtu.be/3Kox-3APznI
‘Nonhomogeneous Linear Ordinary Differential Equations’ at https://youtu.be/t98ILS2YdrU

We can perform a similar action on the right hand side of Eq.1. These terms relate to the external inputs to the system. If the right hand side is linear as well we say that this is linear in the controls and therefore the overall differential equation is called linear.

The linear system of ODEs can be written in the form of

$$\dot{\overline{x}}(t) = A\, \overline{x}(t) + B\, \overline{u}(t)$$

For more information see:
‘State Space Representation of Differential Equations’ at https://youtu.be/pXvAh1IOO4U
‘Transfer Functions: Introduction and Implementation’ at https://youtu.be/Uh_-RZQIaEs

The reason why linear systems are so important is that we have tools and frameworks that allow us to solve these analytical (AKA in closed form)

For more information see:
‘Standard 2nd Order ODEs: Natural Frequency and Damping Ratio’ at https://youtu.be/eJMf9-CYHr6c
‘Analytically Solving Systems of Linear Ordinary Differential Equations’ at https://youtu.be/i2QkjxtXKos

For general systems, we need to resort to numerical techniques

For more information see:

'Getting Started with Simulink' at https://youtu.be/WLPvCefp6Qo

'Ordinary Differential Equations and Dynamic Systems in Simulink' at https://youtu.be/Cvu2zWk3gYw