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Lecture03f
Introduction to the Matrix Exponential

The YouTube video entitled 'Introduction to the Matrix Exponential' that covers this lecture is located at https://youtu.be/e_guF0dwwA4.

Outline

-Definition of the Matrix Exponential
-Properties of the Matrix Exponential

Definition of the Matrix Exponential

Recall the definition of the exponential function (the standard exponent with no matrices)

$$e^x = \exp(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

where $\quad x = \text{scalar}$

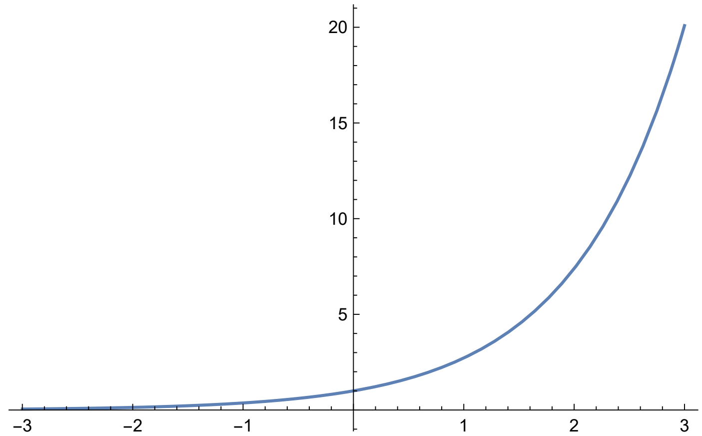
Writing out several terms of this expression yields

$$e^x = \frac{1}{0!} x^0 + \frac{1}{1!} x^1 + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \ldots$$

$$= 1 + \frac{1}{1!} x^1 + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \ldots$$

In[ ]:=
```
Plot[Exp[x], {x, -3, 3}]
```

Out[ ]=



Before we explore the solution to a state space system, it is appropriate to investigate the matrix exponential. This is a matrix function on square matrices which is analogous to the ordinary exponential function.

$$e^X = \exp(X) = \sum_{k=0}^{\infty} \frac{1}{k!} X^k \qquad \textbf{(Eq.1)}$$

where $\quad X = n - \text{by} - n$ square matrix

Writing out several terms of this expression yields

$$e^X = \frac{1}{0!} X^0 + \frac{1}{1!} X^1 + \frac{1}{2!} X^2 + \frac{1}{3!} X^3 + \ldots$$

$$= I + \frac{1}{1!} X^1 + \frac{1}{2!} X^2 + \frac{1}{3!} X^3 + \ldots$$

We note that this is the same definition as the scalar exponential but we note that terms such as X^2, X^3, are shorthand for matrix multiplication of $X\,X$ and $X\,X\,X$, respectively. They do not mean element-wise multiplication.

Example: Diagonal Matrix

Consider the special case of a diagonal matrix

$$X = \begin{pmatrix} 2\,t & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & \cos(t) \end{pmatrix}$$

We can compute the matrix exponential by applying Eq.1

$$e^X = \sum_{k=0}^{\infty} \frac{1}{k!} X^k$$

$$= I + \frac{1}{1!} X + \frac{1}{2!} X^2 + \frac{1}{3!} X^3 + \ldots$$

We now make the observation that because X is diagonal, then matrix multiplication becomes element-wise multiplication. Please be careful to note that this is only because X is diagonal, this is not true in general.

$$X^k = \begin{pmatrix} (2\,t)^k & 0 & 0 \\ 0 & 3^k & 0 \\ 0 & 0 & \cos^k(t) \end{pmatrix}$$

Substituting this into our expression yields

$$e^X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{1!}\begin{pmatrix} (2\,t)^1 & 0 & 0 \\ 0 & 3^1 & 0 \\ 0 & 0 & \cos^1(t) \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} (2\,t)^2 & 0 & 0 \\ 0 & 3^2 & 0 \\ 0 & 0 & \cos^2(t) \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} (2\,t)^3 & 0 & 0 \\ 0 & 3^3 & 0 \\ 0 & 0 & \cos^3(t) \end{pmatrix} + \ldots$$

$$= \begin{pmatrix} 1 + \frac{1}{1!}(2\,t)^1 + \frac{1}{2!}(2\,t)^2 + \frac{1}{3!}(2\,t)^3 + \ldots & 0 & 0 \\ 0 & 1 + \frac{1}{1!}3^1 + \frac{1}{2!}3^2 + \frac{1}{3!}3^3 + \ldots & 0 \\ 0 & 0 & 1 + \frac{1}{1!}\cos^1(t) + \frac{1}{2!}\cos^2(t) + \frac{1}{3!}\cos^3(t) + \ldots \end{pmatrix}$$

$$= \begin{pmatrix} \sum_{k=0}^{\infty} \frac{1}{k!}(2\,t)^k & 0 & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{1}{k!}(3)^k & 0 \\ 0 & 0 & \sum_{k=0}^{\infty} \frac{1}{k!}(\cos(t))^k \end{pmatrix}$$

$$e^X = \begin{pmatrix} e^{2\,t} & 0 & 0 \\ 0 & e^3 & 0 \\ 0 & 0 & e^{\cos(t)} \end{pmatrix}$$

So we see that for a case of a diagonal X matrix, we can computing the matrix exponential by simply exponentiating each element on the main diagonal.

Mathematica provides the ‘MatrixExp’ function to compute the matrix exponent

```
X = ( 2 t  0   0
      0    3   0
      0    0   Cos[t] );
MatrixExp[X] // MatrixForm
Clear[X]
```

$$\begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{3} & 0 \\ 0 & 0 & e^{\text{Cos}[t]} \end{pmatrix}$$

If we consider a general X matrix (not diagonal), we see that computing the matrix exponential is significantly more difficult (but Mathematica's MatrixExp still calculates it correctly).

In[ ]:=
```
A = ( 1  2
     -1  1 );
MatrixExp[A] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} e\,\text{Cos}[\sqrt{2}] & \sqrt{2}\, e\,\text{Sin}[\sqrt{2}] \\ -\frac{e\,\text{Sin}[\sqrt{2}]}{\sqrt{2}} & e\,\text{Cos}[\sqrt{2}] \end{pmatrix}$$

Properties of the Matrix Exponential

From the definition of the matrix exponential (Eq.1), we can show several properties of the matrix exponential

Standard Properties of the Matrix Exponential

$e^{0} = I$ (0 = zero matrix, I = identify matrix)

$e^{X^T} = (e^{X})^T$

$e^{aX} e^{bX} = e^{(a+b)X}$

$e^{X} e^{-X} = I$

if $X Y = Y X$ then $e^{X} e^{Y} = e^{Y} e^{X} = e^{(X+Y)}$

Derivatives of the Matrix Exponential

One of the most interesting properties is once again related to derivatives of the matrix exponential. Consider the matrix exponential e^{At}. We now take the derivative of this with respect to t

$$\frac{d}{dt}[e^{At}] = \frac{d}{dt}\left[\sum_{k=0}^{\infty} \frac{1}{k!}(A t)^k\right]$$

$$= \frac{d}{dt}\left[I + \frac{1}{1!} A t + \frac{1}{2!}(A t)^2 + \frac{1}{3!}(A t)^3 + \frac{1}{4!}(A t)^4 + \ldots\right]$$

$$= 0 + \frac{1}{1!} A + \frac{1}{2!} 2 (A t) A + \frac{1}{3!} 3 (A t)^2 A + \frac{1}{4!} 4 (A t)^3 A + \ldots$$

note: $\frac{2}{2!} = \frac{1}{1!}$, $\frac{3}{3!} = \frac{1}{2!}$, etc.

$$= A + \frac{1}{1!}(A\,t)\,A + \frac{1}{2!}(A\,t)^2\,A + \frac{1}{3!}(A\,t)^3\,A + \ldots$$

$$= A\left[I + \frac{1}{1!}(A\,t) + \frac{1}{2!}(A\,t)^2 + \frac{1}{3!}(A\,t)^3 + \ldots\right] \quad \text{or } = \left[I + \frac{1}{1!}(A\,t) + \frac{1}{2!}(A\,t)^2 + \frac{1}{3!}(A\,t)^3 + \ldots\right] A$$

$$= A\left[\sum_{k=0}^{\infty} \frac{1}{k!}(A\,t)^k\right] \quad \text{or } = \left[\sum_{k=0}^{\infty} \frac{1}{k!}(A\,t)^k\right] A$$

$$\frac{d}{d\,t}\left[e^{A\,t}\right] = A\,e^{A\,t} = e^{A\,t}\,A \qquad \textbf{\textit{(Eq.2)}}$$

<u>Example</u>

Consider the matrix

```
A = ( 1  2
     -1  1 );
```

We can compute $e^{A\,t}$

```
eAt = MatrixExp[A t];
```

We can now take the derivative of this. Note that we can take the derivative element-wise.

```
deAtdt = D[eAt, t];
deAtdt // MatrixForm
```

$$\begin{pmatrix} e^t \cos[\sqrt{2}\,t] - \sqrt{2}\,e^t \sin[\sqrt{2}\,t] & 2\,e^t \cos[\sqrt{2}\,t] + \sqrt{2}\,e^t \sin[\sqrt{2}\,t] \\ -e^t \cos[\sqrt{2}\,t] - \frac{e^t \sin[\sqrt{2}\,t]}{\sqrt{2}} & e^t \cos[\sqrt{2}\,t] - \sqrt{2}\,e^t \sin[\sqrt{2}\,t] \end{pmatrix}$$

We can now use the Eq.2 to check the calculation of the derivative.

```
check1 = A.eAt;
check2 = eAt.A;

check1 == deAtdt
check2 == deAtdt
```

```
True
```

```
True
```

Similarity Transformation and the Matrix Exponential

Another extremely useful property of the matrix exponential is related to similarity transformations (see YouTube video entitled ‘Similarity Transformation and Diagonalization’ https://youtu.be/wvRlvDY-DIgw). Consider a matrix which is a similar to another (ie $\tilde{A} = P^{-1}\,A\,P$ for some invertible P).

$$e^{\tilde{A}} = e^{P^{-1}\,A\,P}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!}\left(P^{-1}\,A\,P\right)^k$$

$$= I + \frac{1}{1!}P^{-1}\,A\,P + \frac{1}{2!}\left(P^{-1}\,A\,P\right)^2 + \frac{1}{3!}\left(P^{-1}\,A\,P\right)^3 + \ldots \qquad \text{note: } P^{-1}\,P = I$$

$$= P^{-1} P + \frac{1}{1!} P^{-1} A P + \frac{1}{2!} \left(P^{-1} A P\right)\left(P^{-1} A P\right) + \frac{1}{3!} \left(P^{-1} A P\right)\left(P^{-1} A P\right)\left(P^{-1} A P\right) + \ldots$$

$$= P^{-1} P + \frac{1}{1!} P^{-1} A P + \frac{1}{2!} \left(P^{-1} A P P^{-1} A P\right) + \frac{1}{3!} \left(P^{-1} A P P^{-1} A P P^{-1} A P\right) + \ldots$$

$$= P^{-1} P + \frac{1}{1!} P^{-1} A P + \frac{1}{2!} P^{-1} A^2 P + \frac{1}{3!} P^{-1} A^3 P + \ldots$$

$$= P^{-1}\left(I + \frac{1}{1!} A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \ldots\right) P$$

$$= P^{-1}\left(\sum_{k=0}^{\infty} \frac{1}{k!} (A)^k\right) P$$

$$= P^{-1} e^A P$$

So we obtain

$$e^{P^{-1} A P} = P^{-1} e^A P \qquad \textbf{\textit{(Eq.3)}}$$

for P non-singular

<u>Example</u>

Once again consider the matrix

```
A = ( 1   2
     -1   1 );
```

Now consider a similarity transformation of

```
P = ( 3   2
     -2  -6 );
```

We can quickly check that $e^{P^{-1} A P} = P^{-1} e^A P$

```
MatrixExp[Inverse[P].A.P] == Inverse[P].MatrixExp[A].P
```

```
True
```

With this framework in place, we will now investigate how the matrix exponential plays a role in solving state space systems (AKA systems of linear ordinary differential equations)

-Lecture03g: Solutions to State Space Systems

As a corollary, we will then investigate how to compute the matrix exponential using various techniques in

-Lecture03h: Computing the Matrix Exponential Using the Laplace Technique
-Lecture03i: Computing the Matrix Exponential Using the Modal Technique