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Lecture03i
Computing the Matrix Exponential Using the Modal Method

Lecture is on YouTube

The YouTube video entitled ‘Computing the Matrix Exponential Using the Modal Method’ that covers this lecture is located at https://youtu.be/yf7ywxj7K20 .

Outline

-Computing the Matrix Exponential
    -Modal Method

Modal Method (Diagonalization)

We can now explore this problem using the modal transformation. This involves diagonalizing the A matrix. The first step is therefore to see if diagonalization is possible. In other words, we need to check that the eigenvectors span the space (which is guaranteed if the eigenvalues are distinct).

Again, we consider the matrix

$$A = \begin{pmatrix} -2 & 1 \\ -1 & 2 \end{pmatrix}$$

```
A = (-2  1
     -1  2);
temp = Eigensystem[A];
λ = temp[[1]]
TT = temp[[2]];
T = Transpose[TT];
T // MatrixForm
```

$\{-\sqrt{3}, \sqrt{3}\}$

$$\begin{pmatrix} 2+\sqrt{3} & 2-\sqrt{3} \\ 1 & 1 \end{pmatrix}$$

Since the eigenvalues are distinct, the eigenvectors should span the space and the matrix of eigenvectors, T can be uses in a similarity transformation to diagonalize A.

```
Atilde = Inverse[T].A.T // Simplify;
Atilde // MatrixForm
```

$$\begin{pmatrix} -\sqrt{3} & 0 \\ 0 & \sqrt{3} \end{pmatrix}$$

This shows that we can write

$$\tilde{A} = T^{-1} A\, T$$

or

$$T\, \tilde{A}\, T^{-1} = A \qquad \textbf{\textit{(Eq.7)}}$$

where $\tilde{A}$ = a diagonal matrix with eigenvalues on the diagonal.

Recall that the solution to the transition matrix for a time invariant system is

$$\phi(t) = e^{A\,t}$$

Substituting in Eq.7 yields

$$\phi(t) = e^{T\,\tilde{A}\,T^{-1}\,t}$$

$$= e^{T\left(\tilde{A}\,t\right)T^{-1}}$$

From the YouTube video ‘Introduction to the Matrix Exponential’ https://youtu.be/e_guF0d-wwA4?t=1113 we know that $e^{T\left(\tilde{A}\,t\right)T^{-1}} = T\, e^{\tilde{A}\,t}\, T^{-1}$. We now have a simplified expression for the transition matrix.

$$\phi(t) = T\, e^{\tilde{A}\,t}\, T^{-1}$$

We can easily calculate the matrix exponential of $e^{\tilde{A}\,t}$ since it is diagonal. Namely

$$e^{\tilde{A}\,t} = \begin{pmatrix} e^{-\sqrt{3}\,t} & 0 \\ 0 & e^{\sqrt{3}\,t} \end{pmatrix}$$

We can now use the transformation matrices to obtain the transition matrix.

$$\phi(t) = T\, e^{\tilde{A}\,t}\, T^{-1}$$

$$= \begin{pmatrix} 2+\sqrt{3} & 2-\sqrt{3} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^{-\sqrt{3}\,t} & 0 \\ 0 & e^{\sqrt{3}\,t)} \end{pmatrix} \begin{pmatrix} 2+\sqrt{3} & 2-\sqrt{3} \\ 1 & 1 \end{pmatrix}^{-1}$$

```
ϕmodal[t_] = T.( Exp[-√3 t]      0
                    0       Exp[√3 t] ).Inverse[T] // Simplify;

ϕmodal[t] // MatrixForm
```

$$\begin{pmatrix} \frac{e^{-\sqrt{3}\,t}\left(2+\sqrt{3}+\left(-2+\sqrt{3}\right)e^{2\sqrt{3}\,t}\right)}{2\sqrt{3}} & \frac{e^{-\sqrt{3}\,t}\left(-1+e^{2\sqrt{3}\,t}\right)}{2\sqrt{3}} \\ -\frac{e^{-\sqrt{3}\,t}\left(-1+e^{2\sqrt{3}\,t}\right)}{2\sqrt{3}} & \frac{e^{-\sqrt{3}\,t}\left(-2+\sqrt{3}+\left(2+\sqrt{3}\right)e^{2\sqrt{3}\,t}\right)}{2\sqrt{3}} \end{pmatrix}$$

This is the same result we obtained using the Laplace method.

We can verify this is the same as Mathematica's MatrixExp function

```
ϕmodal[t] == MatrixExp[A t] // Simplify
```

```
True
```

```
Clear[Atilde, T, TT, λ, temp, ϕmodal]
```