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Lecture05c
Double Integrals

The YouTube video entitled ‘Double Integrals’ that covers this lecture is located at https://youtu.be/C-yfMjxxsz0 .

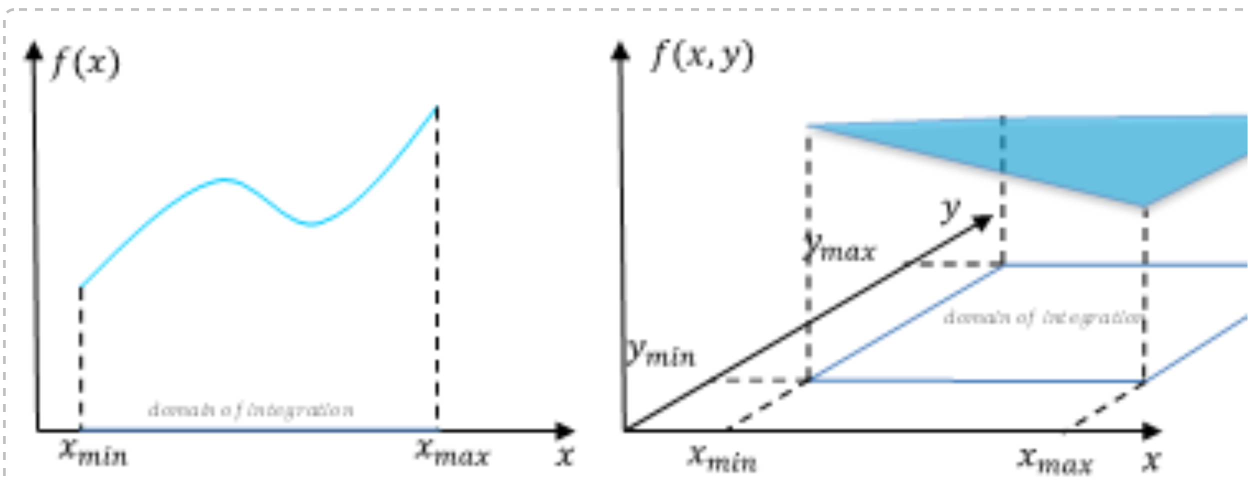
Double Integrals

Recall that a definite single integral computes the area under a curve.   In the picture below, we can approximate the integral via Riemann Sums and simply discretize the domain and write

$$J = \sum_{k=1}^{K} f(x_k)\,\Delta x_k$$

As the discretization become infinitesimally small, the limit approaches the integral value and thus defines the quantity

$$\int_{x_{\min}}^{x_{\max}} f(x)\, dx$$



A double integral computes the volume under a surface. A double integral is quite similar to a single integral. If we have a function *f* defined over the region *R*, we subdivide the region *R* and associate with each point (x_n, y_m) an area ΔA_k. We then form the sum

$$J = \sum_{k=1}^{K} f(x_n, y_m)\,\Delta A_k$$

Taking the limit as the infinitesimal areas approaches zero, we can define the double integral as

$$\iint_R f(x, y)\,dA = \iint_R f(x, y)\,dx\,dy = \iint_R f(x, y)\,dy\,dx$$

Just as the definite integral of a positive function of one variable represents the area of the region between the graph of the function and the x-axis, the double integral of a positive function of two variables represents the volume of the region between the surface defined by the function and the plane which contains its domain.

Note that the order of integration does not matter but in some situations it may be easier to integrate over x first and then y (or vice versa).

Note that since the concept of an anti-derivative is only defined for functions of a single real variable, the usual definition of indefinite integrals does not immediately extend to the multiple (double) integral.

Example

$f(x, y) = x + 2y$ over the domain of $x \in [-1, 1]$ and $y \in [0, 2]$.

$$\iint_R f(x, y)\,dx\,dy = \int_{y_{\min}}^{y_{\max}}\left[\int_{x_{\min}}^{x_{\max}} f(x, y)\,dx\right]dy$$

$$= \int_0^2\left[\int_{-1}^{1} x + 2y\,dx\right]dy \qquad \text{note: consider } y \text{ constant}$$

$$= \int_0^2\left[\frac{x^2}{2} + 2yx\Big|_{x=-1}^{x=1}\right]dy$$

$$= \int_0^2\left[\left(\frac{x^2}{2} + 2yx\right)\Big|_{x=1} - \left(\frac{x^2}{2} + 2yx\right)\Big|_{x=-1}\right]dy$$

$$= \int_0^2\left[\left(\frac{1}{2} + 2y\right) - \left(\frac{1}{2} - 2y\right)\right]dy$$

$$= \int_0^2 4y\,dy$$

$$= 2y^2\Big|_{y=0}^{y=2}$$

$$\iint_R f(x, y)\,dx\,dy = 8$$

We can verify this with Mathematica

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In[•]:= Integrate[Integrate[x + 2 y, {x, -1, 1}], {y, 0, 2}]
Out[•]= 8
```

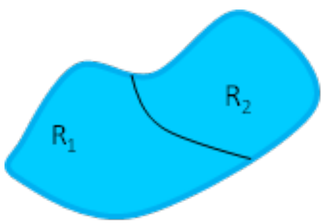
Properties of Double Integrals

Much like single integrals, some properties of double integrals include

$$\iint_R k\, f(x, y)\, dx\, dy = k \iint_R f(x, y)\, dx\, dy \quad (k \text{ constant})$$

$$\iint_R f(x, y) + g(x, y)\, dx\, dy = \iint_R f(x, y)\, dx\, dy + \iint_R g(x, y)\, dx\, dy \qquad \textbf{(Eq.1)}$$

$$\iint_R f(x, y)\, dx\, dy = \iint_{R_1} f(x, y)\, dx\, dy = \iint_{R_2} f(x, y)\, dx\, dy$$



Mean Value Theorem
Furthermore, if R is simply connected, then there exists at least one point $(x_0, y_0) \in R$ such that

$$\iint_R f(x, y)\, dx\, dy = f(x_0, y_0)\, A \quad \text{(Mean Value Theorem)} \qquad \textbf{(Eq.2)}$$

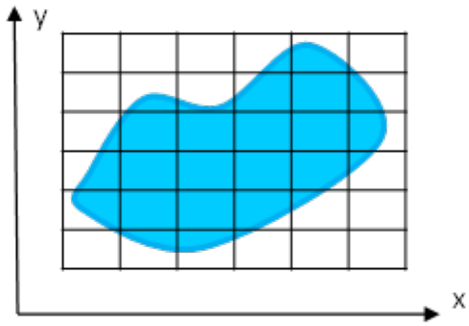
where $\quad A$ = area of R

Evaluation of Double Integrals by Two Successive Integrations

In the previous example, the region R was rectangular and could easily be described by

$$R = \{(x, y) \mid x_{min} \le x \le x_{max},\ y_{min} \le y \le y_{max}\}$$

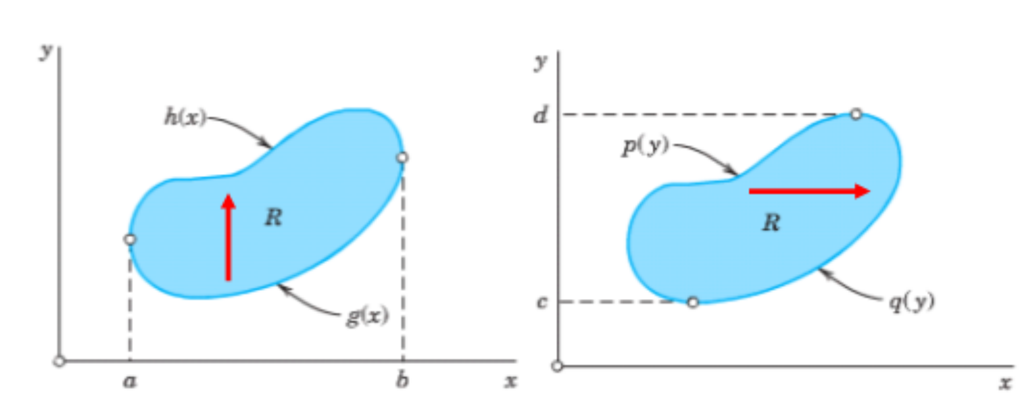
In general, the region R may be more complicated as shown below



We may first integrate over *y* and then *x* (or vice versa).

$$\iint_R f(x,\ y)\,dx\,dy = \int_a^b\left[\int_{g(x)}^{h(x)} f(x,\ y)\,dy\right]dx \qquad \text{(Type I)} \qquad \textbf{\textit{(Eq.3)}}$$

$$\iint_R f(x,\ y)\,dx\,dy = \int_c^d\left[\int_{p(y)}^{q(y)} f(x,\ y)\,dx\right]dy \qquad \text{(Type II)} \qquad \textbf{\textit{(Eq.4)}}$$



Integrate over y first
(Type I)

Integrate over x first
(Type II)

The red arrows are used to help visualize which integral is performed first.

Consider the case where we integrate with respect to x first (type II). The boundary curve of R is now represented by $x = p(y)$ and $x = q(y)$. Treating y as a constant, we first integrate $f(x,\ y)$ over x from $p(y)$ to $q(y)$ and then integrate the resulting function of y from $y = c$ to $y = d$

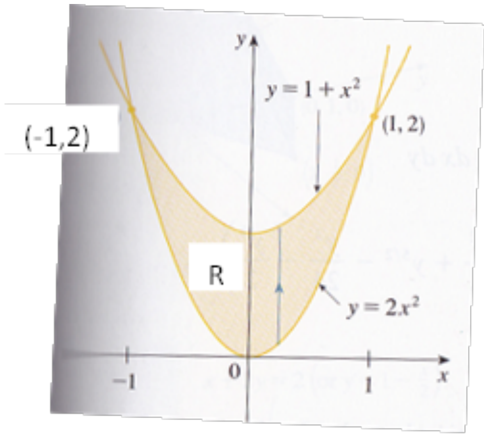
A region R is called **type I** or **type II** if the boundaries lie between two functions

$$R = \{(x,\ y)\ |\ a \le x \le b,\ g(x) \le y \le h(x)\} \qquad \text{(Type I)}$$
$$R = \{(x,\ y)\ |\ c \le y \le d,\ p(y) \le x \le q(y)\} \qquad \text{(Type II)}$$

In most cases it will be simplest to use Eq.3 for type I regions and Eq.4 for type II regions.

Example: Simple Region

Evaluate the double integral of the function $f(x,\ y) = x + 2\,y$ over the region shown below



We see that this is easiest described as a type I region

$$R = \{(x, y) \mid a \le x \le b,\ g(x) \le y \le h(x)\}$$

where $a = -1$
$b = 1$
$g(x) = 2x^2$
$h(x) = 1 + x^2$

So we use Eq.3 to perform the integration (integrate with respect to y first)

$$\iint_R f(x, y)\, dx\, dy = \int_a^b \left[\int_{g(x)}^{h(x)} f(x, y)\, dy\right] dx$$

$$= \int_{-1}^{1}\left[\int_{2x^2}^{1+x^2} x + 2y\, dy\right] dx \qquad \text{recall: consider } x \text{ constant}$$

$$= \int_{-1}^{1}\left[xy + y^2 \Big|_{y=2x^2}^{y=1+x^2}\right] dx$$

$$= \int_{-1}^{1}\left[(xy + y^2)\,|_{y=1+x^2} - (xy + y^2)\,|_{y=2x^2}\right] dx$$

$$= \int_{-1}^{1}\left[\left(x(1+x^2) + (1+x^2)^2\right) - \left(x(2x^2) + (2x^2)^2\right)\right] dx$$

$$= \int_{-1}^{1}\left[x + x^3 + 1 + 2x^2 + x^4 - 2x^3 - 4x^4\right] dx$$

$$= \int_{-1}^{1} 1 + x + 2x^2 - x^3 - 3x^4\, dx$$

$$= \int_{-1}^{1} 1 + x + 2x^2 - x^3 - 3x^4\, dx$$

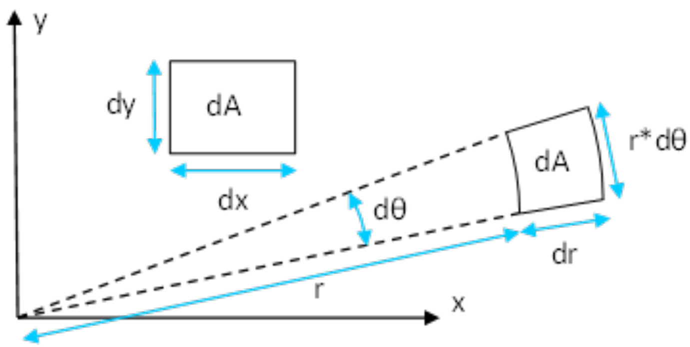
```
Integrate[1 + x + 2 x^2 - x^3 - 3 x^4, {x, -1, 1}]
```

$$\frac{32}{15}$$

Double Integrals in Polar Coordinates

In the previous example, the region was discretized using a Cartesian coordinate system and rectangular areas dA.

It may be useful to compute double integrals in polar coordinates (see YouTube video entitled ‘Cartesian, Polar, Cylindrical, and Spherical Coordinates’ at https://youtu.be/FLQXW6G9P8I). We note that $dA = dx\,dy = r\,dr\,d\theta$ (see below)



Therefore the integral in polar coordinates is given by

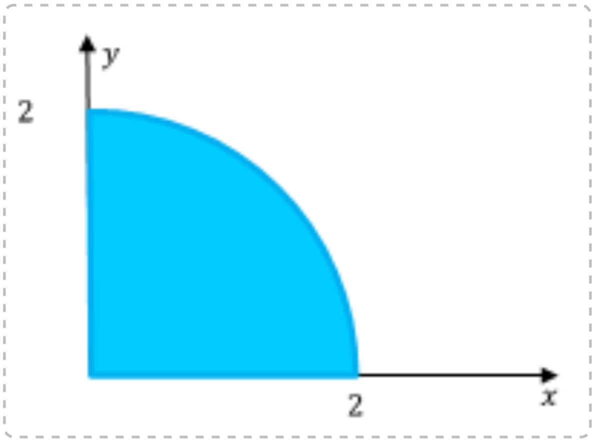
$$\iint_R f(x, y)\,dA = \iint_R f(x, y)\,dx\,dy = \iint_R f(r, \theta)\,r\,dr\,d\theta$$

Example

Evaluate $\iint_R f(x, y)\,dA$

where $\quad f(x, y) = \left(4 - x^2 - y^2\right)^{1/2}$

And R is shown below



We see that the region R is easily described in polar coordinates

$$R = \{(r, \theta) \mid r \in [0, 2], \theta \in [0, \pi/2]\}$$

Therefore we can convert the function $f(x, y)$ to polar coordinates. Recall to do this we make the substitution that

$$x = r\cos(\theta)$$
$$y = r\sin(\theta)$$

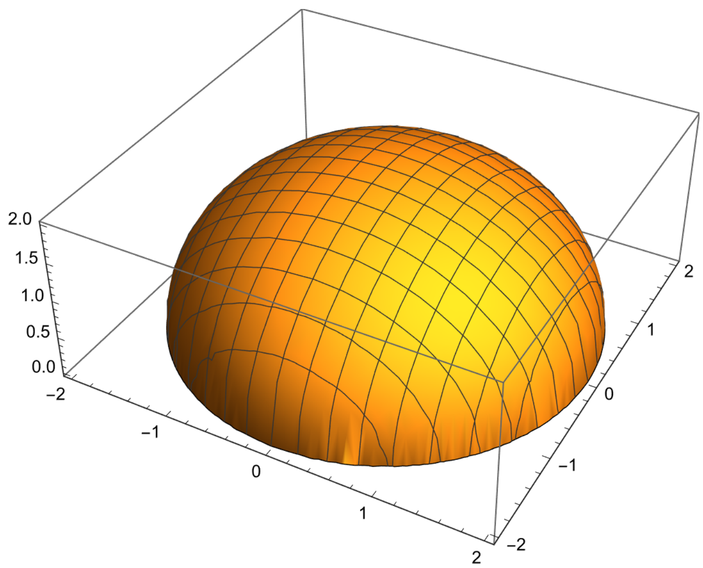
In[•]:=
```
fCartesian[x_, y_] = (4 - x^2 - y^2)^(1/2)
fPolar[r_, θ_] = fCartesian[r Cos[θ], r Sin[θ]] // Simplify

(*Plot both representations*)
Plot3D[fCartesian[x, y], {x, -2, 2}, {y, -2, 2}]
RevolutionPlot3D[fPolar[r, θ], {r, 0, 2}, {θ, 0, 2 π}]
```

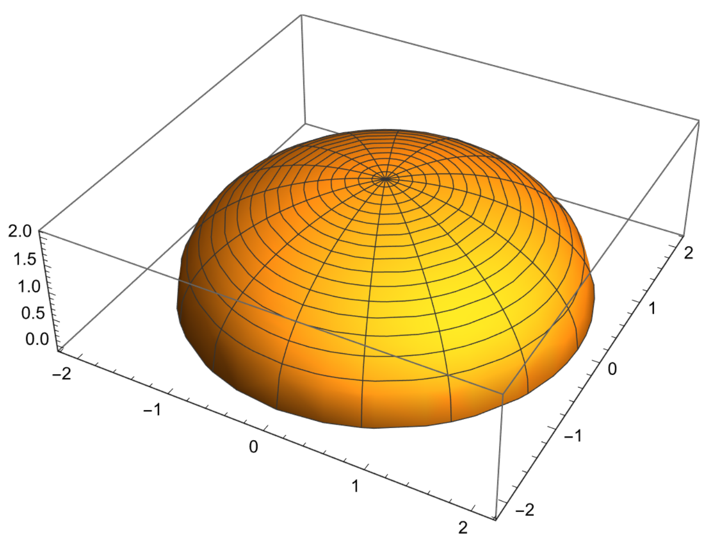
Out[•]= $\sqrt{4 - x^2 - y^2}$

Out[•]= $\sqrt{4 - r^2}$

Out[•]=



Out[•]=



We can now write the double integral as

$$\iint_R f(x, y)\, dA = \iint_R f(r, \theta)\, r\, dr\, d\theta$$

$$= \int_0^{\pi/2} \int_0^2 (4 - r^2)^{1/2} \, r \, dr \, d\theta$$

$$= \int_0^{\pi/2} \frac{8}{3} \, d\theta$$

$$= \frac{4\pi}{3}$$

In[ ]:=
```
t1 = Integrate[fPolar[r, θ] r, {r, 0 , 2}]
Integrate[t1, {θ, 0, π / 2}]
```

Out[ ]= $\frac{8}{3}$

Out[ ]= $\frac{4\pi}{3}$

We can check this answer by realizing that we are asked to calculate the volume of only 1/8 of a full sphere with radius 2

In[ ]:=
```
(*Volume of the entire sphere*)
V = 4/3 π r^3 /. {r → 2};

(*we only want the volume of 1/8 of the sphere*)
V / 8
```

Out[ ]= $\frac{4\pi}{3}$

Applications of Double Integrals

The volume, V, beneath the surface $z = f(x, y) > 0$ and above the region R in the xy plane is

$$V = \iint_R f(x, y) \, dx \, dy \qquad \text{(volume)}$$

Area, A, of a region R in the xy plane can be evaluated as

$$A = \iint_R dx \, dy \qquad (f(x, y) = 1) \qquad \text{(area)}$$

Let $f(x, y)$ be the density (mass/unit area) of mass in the xy-plane. Then we can compute the total mass, M, in R. We can also compute the center of gravity to be at location $\overline{x}$, $\overline{y}$. Finally, we can compute the moments of inertia about the x and y axes, I_x and I_y.

$$M = \iint_R f(x, y) \, dx \, dy \qquad \text{(total mass)}$$

$$\overline{x} = \frac{1}{M} \iint_R x \, f(x, y) \, dx \, dy \qquad \text{(x center of gravity)}$$

$\overline{y} = \frac{1}{M} \iint_R y\, f(x, y)\, dx\, dy$ (y center of gravity)

$I_x = \iint_R y^2 f(x, y)\, dx\, dy$ (moment of inertia about the x axis)

$I_y = \iint_R x^2 f(x, y)\, dx\, dy$ (moment of inertia about the y axis)