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# Lecture 02b
# Velocity & Acceleration in Non-Inertial Reference Frames (Coriolis & Centrifugal Acceleration)


The YouTube video entitled 'Velocity & Acceleration in Non-Inertial Reference Frames (Coriolis & Centrifugal Acceleration)' that covers this lecture is located at https://youtu.be/uTabQKD2WMs.

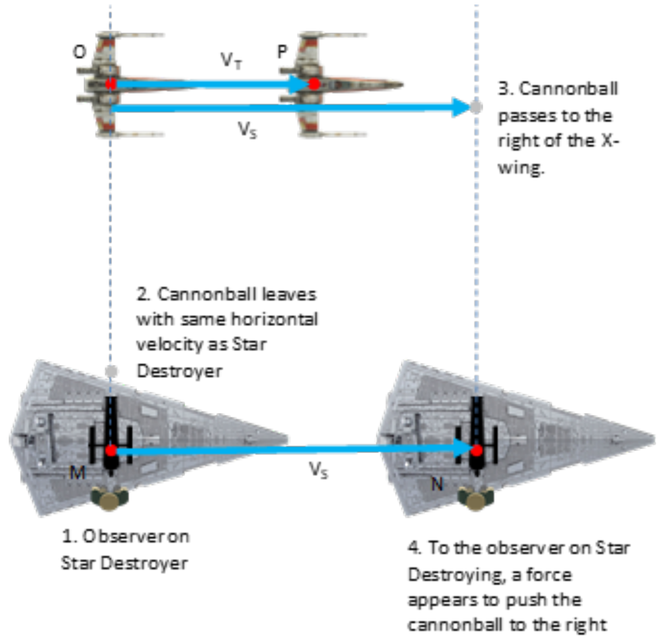
## Outline

-Coriolis "Force"
-Velocity and Acceleration in Moving Frames
 -Position
 -Velocity
 -Acceleration
 -Total Acceleration
-Newton's 2nd Law

## Coriolis "Force"

Consider the scenario where a Star Destroyer is traveling to the right at velocity $V_s$ and the X-wing is traveling to the right at velocity $V_T$ (which is less than $V_S$).

The Star Destroying is attempting to shoot an X-wing as shown below using a cannon. When the canon is fired (point M) the X-wing is lined up in it's sights. But as the cannonball travels, since the X-wing is traveling slower than the cannonball, the cannonball will miss by passing to the right of the X-wing. From the perspective of the observer on the Star Destroyer, it appears that some mysterious "force" deflects the cannonball deflects to the right.

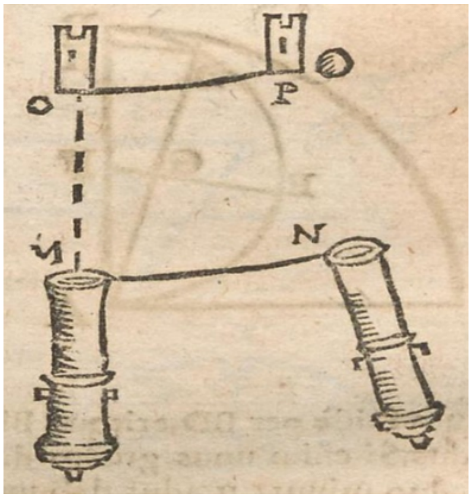



The above example is a simple case of relative motion and we can show that this can be applied to rotating reference frames as well. For example, Earth is a rotating reference frame and objects near the equator (the Star Destroyer in the previous example) are indeed traveling faster than objects at the poles (the X-wing in the previous example). As such, we expect to observe this phenomenon on Earth.

**Example: Throwing Balls in a Rotating Reference Frame**

We can perform a demo by moving in a circular motion and throwing balls. The observer running in a circle will see the ball move in an arc but the observers sitting in class will see the ball move in a straight line. View video at ”C:\KDriveCopy\AFSL\TechnicalDataPackage\Presentations\movies\Teaching\CoriolisEffect.mp4”

This effect was documented and described by Italian scientist Giovanni Battista Riccioli in connection with artillery in 1651. In 1674 Claude Francois Miliet Dechales described in *Cursus seu Mundus Mathematicus* how the rotation of the earth should cause a deflection in the trajectories of both falling bodies and projectiles aimed towards one of the plant's poles. Interestingly, Riccioli and Dechales used this description as an argument <u>against</u> Copernicus’ heliocentric model. In other words, they argued that if Earth was indeed rotating, then these effects should be present so failure to detect the effect was evidence for an immobile Earth. It turns out that these effect indeed take place but the magnitude of their effects are small for most applications such as the ones they studied.



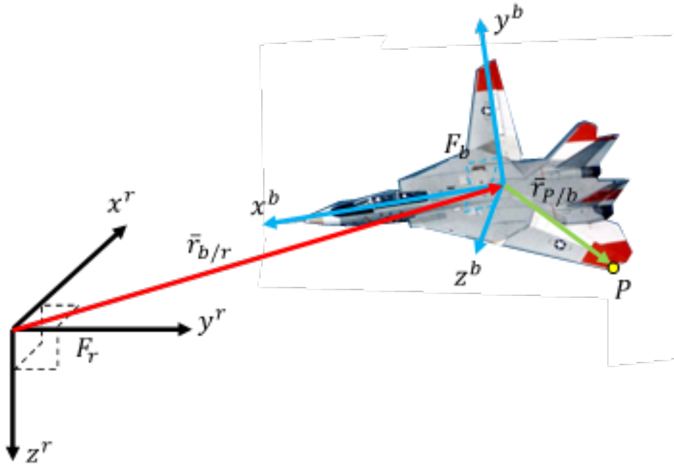

It was not until almost 100 years later (1749) when Euler applied this knowledge to develop the Coriolis acceleration equation.  Let us examine this effect in a formal fashion.

# Velocity and Acceleration in Moving Frames

We now consider deriving expressions for velocity and acceleration of an object from the perspective of both an inertial and moving frame.  We start with developing an expression for the position of a point.

## Position

Consider the previous picture of an F-14 with a point of interest on the wingtip of the vehicle (the picture is repeated here for convenience).




Let us now suppose we have a point which is measured in $F_b$ and we wish to relate is position, velocity and acceleration in $F_r$.  From this figure, we see that the position is given by

$$\bar{r}_{P/r} = \bar{r}_{b/r} + \bar{r}_{P/b} \qquad \textbf{\textit{(Eq.3)}}$$

## Velocity

We can now look at the velocity of this vector. Since we are taking a derivative, we must specify with respect to what frame we are taking the derivative. Let us examine the derivative of the position vector from $F_r$

$$ {}^{r}\dot{\bar{r}}_{P/r} = \frac{d}{dt}\Big|_{r} \bar{r}_{b/r} + \frac{d}{dt}\Big|_{r} \bar{r}_{P/b} \qquad \textbf{(Eq.4)}$$

We recognize the first term as simply the translational velocity of $F_b$ with respect to $F_r$

$$\frac{d}{dt}\Big|_{r} \bar{r}_{b/r} = {}^{r}\dot{\bar{r}}_{b/r}$$

$$\triangleq \bar{V}_{b/r} \qquad \textbf{(Eq.5)}$$

However the second term is the change in position of the point $P$ with respect to $F_b$ when seen from $F_r$. This is the same effect that we saw previously in Eq.1.2-7 and is given by

$$\frac{d}{dt}\Big|_{r} \bar{r}_{P/b} = {}^{b}\dot{\bar{r}}_{P/b} + \bar{\omega}_{b/r} \times \bar{r}_{P/b} \quad \text{note: for sake of notation: } {}^{b}\dot{\bar{r}}_{P/b} \triangleq \bar{V}_{P/b} \qquad \textbf{(Eq.6)}$$

$$= \bar{V}_{P/b} + \bar{\omega}_{b/r} \times \bar{r}_{P/b}$$

Substituting Eq.5 and Eq.6 into Eq.4 yields

$$\bar{V}_{P/r} = \bar{V}_{b/r} + \bar{V}_{P/b} + \bar{\omega}_{b/r} \times \bar{r}_{P/b}$$

## Acceleration

We can differentiate this once more to obtain the acceleration. Recall that we are differentiating with respect to time but from the view of $F_r$.

$$ {}^{r}\dot{\bar{V}}_{P/r} = \frac{d}{dt}\Big|_{r} \bar{V}_{b/r} + \frac{d}{dt}\Big|_{r} \bar{V}_{P/b} + \frac{d}{dt}\Big|_{r} \left(\bar{\omega}_{b/r} \times \bar{r}_{P/b}\right) \qquad \textbf{(Eq.7)}$$

We need to evaluate each of these terms carefully. We start with the first term.

**Term 1:** $\frac{d}{dt}\Big|_{r} \bar{V}_{b/r}$

As can be seen, $\bar{V}_{b/r}$ is the translational velocity of $F_b$ relative to $F_r$ when seen from $F_r$. Therefore, if we take the derivative once more with respect to time from $F_r$ we simply obtain the acceleration of $F_b$

relative to $F_r$

$$\frac{d}{dt}\Big|_r \overline{V}_{b/r} = \overline{a}_{b/r} \qquad \textbf{(Eq.8)}$$

**Term 2:** $\frac{d}{dt}\Big|_r \overline{V}_{P/b}$

For the second term, we see that $\overline{V}_{P/b}$ is the velocity of $P$ with respect to $b$ when seen from $F_b$. If we want to see how this is changing with respect to time when seen from $F_r$, we need to use Eq.1.2-7 once more.

It is more convenient to write this term as

$$\frac{d}{dt}\Big|_r \overline{V}_{P/b} = {}^b\dot{\overline{V}}_{P/b} + \overline{\omega}_{b/r} \times \overline{V}_{P/b} \qquad \textbf{(Eq.9)}$$

**Term 3:** $\frac{d}{dt}\Big|_r \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right)$

Now for the last term we differentiate using the product rule

$$\frac{d}{dt}\Big|_r \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right) = \left(\frac{d}{dt}\Big|_r \overline{\omega}_{b/r} \times \overline{r}_{P/b}\right) + \left(\overline{\omega}_{b/r} \times \frac{d}{dt}\Big|_r \overline{r}_{P/b}\right) \qquad \textbf{(Eq.10)}$$

We notice that the time rate of change of the rotation between the two frames $\overline{\omega}_{b/r}$ with respect to time as seen from $F_r$ is the angular acceleration between the frames and is independent of frame from which it is being observed (see property (iv) on pg.10). So we have

$$\frac{d}{dt}\Big|_r \overline{\omega}_{b/r} = {}^r\dot{\overline{\omega}}_{b/r} = {}^b\dot{\overline{\omega}}_{b/r} \overset{\Delta}{=} \overline{\alpha}_{b/r} \qquad \textbf{(Eq.11)}$$

For the other derivative, we are trying to evaluate $\frac{d}{dt}\Big|_r \overline{r}_{P/b}$. Notice that we have already evaluated this previously which is repeated here for convenience

$$\frac{d}{dt}\Big|_r \overline{r}_{P/b} = \overline{V}_{P/b} + \overline{\omega}_{b/r} \times \overline{r}_{P/b}$$

So substituting Eq.11 and Eq.2 into Eq.10 yields

$$\frac{d}{dt}\Big|_r \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right) = \left(\overline{\alpha}_{b/r} \times \overline{r}_{P/b}\right) + \left[\overline{\omega}_{b/r} \times \left(\overline{V}_{P/b} + \overline{\omega}_{b/r} \times \overline{r}_{P/b}\right)\right] \qquad \textbf{(Eq.12)}$$

## Total Acceleration

Now, substituting Eq.8, Eq.9, and Eq.12 into Eq.7 (repeated here for convenience) yields

$$ {}^{r}\dot{\overline{V}}_{P/r} = \frac{d}{dt}\Big|_{r} \overline{V}_{b/r} + \frac{d}{dt}\Big|_{r} \overline{V}_{P/b} + \frac{d}{dt}\Big|_{r} \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right) \qquad \textbf{(Eq.7)} $$

$$ = \overline{a}_{b/r} + {}^{b}\dot{\overline{V}}_{P/b} + \overline{\omega}_{b/r} \times \overline{V}_{P/b} + \overline{\alpha}_{b/r} \times \overline{r}_{P/b} + \overline{\omega}_{b/r} \times \left(\overline{V}_{P/b} + \overline{\omega}_{b/r} \times \overline{r}_{P/b}\right) $$

Notice that we can simplify this to

$$ = \overline{a}_{b/r} + {}^{b}\dot{\overline{V}}_{P/b} + \overline{\omega}_{b/r} \times \overline{V}_{P/b} + \overline{\alpha}_{b/r} \times \overline{r}_{P/b} + \overline{\omega}_{b/r} \times \overline{V}_{P/b} + \overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right) $$

$$ = \overline{a}_{b/r} + {}^{b}\dot{\overline{V}}_{P/b} + 2\left(\overline{\omega}_{b/r} \times \overline{V}_{P/b}\right) + \overline{\alpha}_{b/r} \times \overline{r}_{P/b} + \overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right) $$

Creating one more definition of ${}^{b}\dot{\overline{V}}_{P/b} = \overline{a}_{P/b}$ yields the final form of

$$ \overline{a}_{P/r} = \overline{a}_{b/r} + \overline{a}_{P/b} + 2\left(\overline{\omega}_{b/r} \times \overline{V}_{P/b}\right) + \overline{\alpha}_{b/r} \times \overline{r}_{P/b} + \overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right) \qquad \textbf{(Eq.1.2-11)} $$

where $\overline{a}_{P/r} = {}^{r}\dot{\overline{V}}_{P/r}$ = true acceleration
$\overline{a}_{b/r}$ = acceleration of $F_b$ relative to $F_r$
$\overline{a}_{P/b} = {}^{b}\dot{\overline{V}}_{P/b}$ = relative acceleration as seen from $F_b$
$\overline{\omega}_{b/r}$ = angular velocity of $F_b$ relative to $F_r$
$\overline{V}_{P/b} = {}^{b}\dot{\overline{r}}_{P/b}$ = relative velocity as seen from $F_b$
$\overline{\alpha}_{b/r}$ = angular acceleration of $F_b$ relative to $F_r$
$\overline{r}_{P/b}$ = relative position

Two special terms are

$2\left(\overline{\omega}_{b/r} \times \overline{V}_{P/b}\right)$ = Coriolis acceleration
$\overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right)$ = Centrifugal acceleration

**Example: Merry-Go-Round**

Consider a children’s merry-go-round. If you haven’t already done, please watch the video entitled ‘Coriolis Effect Demonstration (with Drones)’ at https://youtu.be/okaxKzoyMK0.

Choose $F_r$ = earth frame (assumed inertial)
$F_b$ = body frame (attached to merry go round)

We can make several assumptions to simply the acceleration

$\overline{a}_{b/r} = 0$ (no relative acceleration between the frames)
$\overline{\alpha}_{b/r} = 0$ (constant merry-go-round speed)

With these simplifications Eq.1.2-11 simplifies to

$$\overline{a}_{P/r} = \overline{a}_{P/b} + 2\left(\overline{\omega}_{b/r} \times \overline{V}_{P/b}\right) + \overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right)$$

We now consider the apparent velocity of the object as observed from $F_b$. In other words, we solve for $\overline{a}_{P/b}$ to write

$$\overline{a}_{P/b} = \overline{a}_{P/r} - 2\left(\overline{\omega}_{b/r} \times \overline{V}_{P/b}\right) - \overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right) \qquad \textbf{\textit{(Eq.13)}}$$

In other words, an observer on $F_b$ will observe normal acceleration, $\overline{a}_{P/r}$, combined with Coriolis acceleration, $2\left(\overline{\omega}_{b/r} \times \overline{V}_{P/b}\right)$, and Centrifugal acceleration, $\overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right)$.

For numerical simplicity, let us consider that the merry-go-round makes 1 revolution every 6 seconds

$$\overline{\omega}_{b/r}{}^{b} = \begin{pmatrix} 0 \\ 0 \\ 2\pi/6 \end{pmatrix} \qquad \text{(rad/s)}$$

```
ωbrb = {0, 0, 2 π / 6};
```

Let us consider several scenarios

<u>Case 0: No rotation</u>

The simplest case to analyze is if there is no rotation, $\overline{\omega}_{b/r} = 0$. In this case, Eq.13 reduces to

$$\overline{a}_{P/b} = \overline{a}_{P/r}$$

And we see that there is difference between the acceleration observed by Randal (in $F_r$) or Brian (in $F_b$). In the case of throwing a ball, the only acceleration that both would observe is

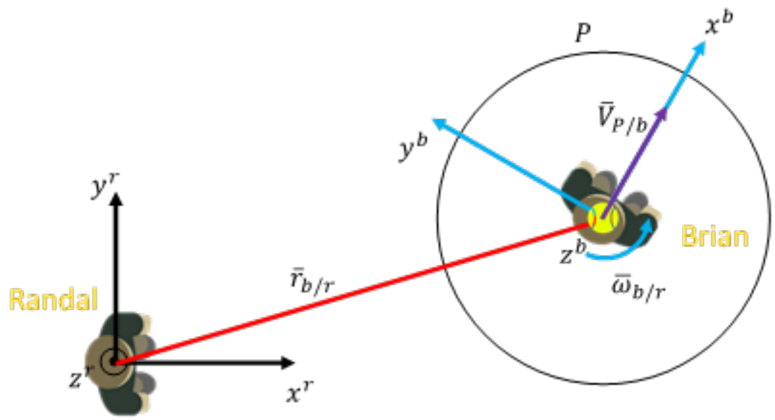
$$\overline{a}_{P/b}{}^{b} = \overline{a}_{P/r}{}^{b} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} \qquad \left(m/s^2\right)$$

Not that this is the same expressed in either frame

$$\overline{a}_{P/b}{}^{r} = \overline{a}_{P/r}{}^{r} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} \qquad \left(m/s^2\right)$$

<u>Case 1: Brian throws ball outwards (at ball leaves his hand at t = 0)</u>

Consider the scenario where Brian (riding on the merry-go-round) throws the ball outwards from the center at a constant velocity. Let us analyze the scenario as soon as the ball leaves his hand (at $t = 0$)




At the moment the ball leaves his hand, we see that

$$\bar{r}_{P/b}{}^b = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Assuming that he throws the ball  in his local x direction, we have

$$\bar{V}_{P/b}{}^b = \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} \quad \text{(m/s)}$$

Assuming no air resistance or other major affects, we see that the only “true” acceleration is gravity acting downwards.

$$\bar{a}_{P/r}{}^b = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}$$

So Eq.13 expressed in $F_b$ is given as

$$\bar{a}_{P/b}{}^b = \bar{a}_{P/r}{}^b - 2\left(\bar{\omega}_{b/r}{}^b \times \bar{V}_{P/b}{}^b\right) - \bar{\omega}_{b/r}{}^b \times \left(\bar{\omega}_{b/r}{}^b \times \bar{r}_{P/b}{}^b\right) \qquad \textbf{\textit{(Eq.14)}}$$

$$= \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} - 2\left(\begin{pmatrix} 0 \\ 0 \\ 2\pi/6 \end{pmatrix} \times \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}\right) - \begin{pmatrix} 0 \\ 0 \\ 2\pi/6 \end{pmatrix} \times \left(\begin{pmatrix} 0 \\ 0 \\ 2\pi/6 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\bar{a}_{P/b}{}^b = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} + \begin{pmatrix} 0 \\ -10.47 \\ 0 \end{pmatrix}$$

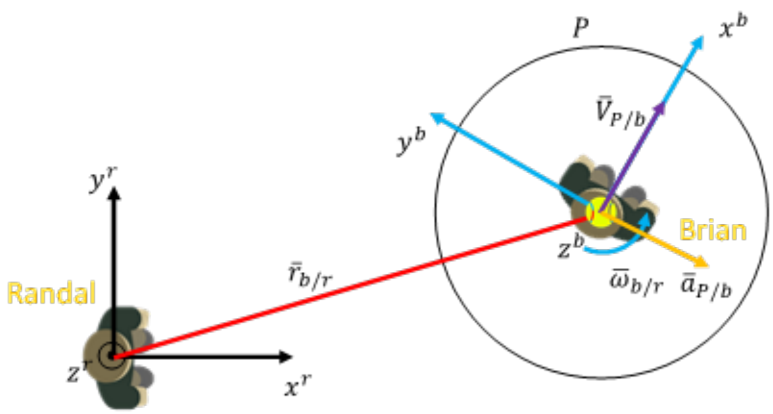
```
VPbb = {5, 0, 0};
rPbb = {0, 0, 0};

Print["-2 (ω̄b/r × V̄P/b)"]
coriolisAcceleration = -2 (Cross[ωbrb, VPbb]);
coriolisAcceleration // N

Print["- ω̄b/r × (ω̄b/r × r̄P/b)"]
centrifugalAcceleration = -Cross[ωbrb, Cross[ωbrb, rPbb]];
centrifugalAcceleration // N
```

$-2\ (\bar{\omega}_{b/r} \times \bar{V}_{P/b})$

{0., -10.472, 0.}

$-\ \bar{\omega}_{b/r} \times (\bar{\omega}_{b/r} \times \bar{r}_{P/b})$

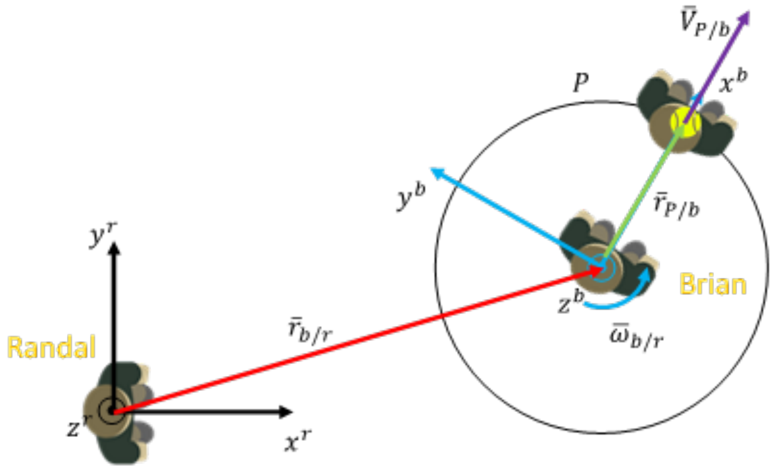
{0., 0., 0.}




So we see that as observed by Brian in $F_b$, he observes not only the normal vertical acceleration due to gravity, but also a significant acceleration to the right (the magnitude of which is even greater than gravity). This will appear to Brian as some additional “force” that accelerates the ball to the ball’s right.

Case 2: Edward throws ball outwards

Now consider Edward, another participant that throws the ball outwards at the same velocity as case 1 but Edward is standing at the edge of the merry-go-round.




In this case, the only difference is that $\bar{r}_{P/b} \neq 0$. We can assume that numerically we have

$$\bar{r}_{P/b}{}^b = \begin{pmatrix} 3/2 \\ 0 \\ 0 \end{pmatrix} \quad (m)$$

Again, everything expressed in $F_b$ as

$$\bar{a}_{P/b}{}^b = \bar{a}_{P/r}{}^b - 2\left(\bar{\omega}_{b/r}{}^b \times \bar{V}_{P/b}{}^b\right) - \bar{\omega}_{b/r}{}^b \times \left(\bar{\omega}_{b/r}{}^b \times \bar{r}_{P/b}{}^b\right)$$

In this case, both Coriolis and centrifugal force are non-zero and non-negligible

```
VPbb = {5, 0, 0};
rPbb = {3 / 2, 0, 0};

Print["-2 (ω̄b/r × V̄P/b)"]
coriolisAcceleration = -2 (Cross[ωbrb, VPbb]);
coriolisAcceleration // N

Print["- ω̄b/r × (ω̄b/r × r̄P/b)"]
centrifugalAcceleration = -Cross[ωbrb, Cross[ωbrb, rPbb]];
centrifugalAcceleration // N
```

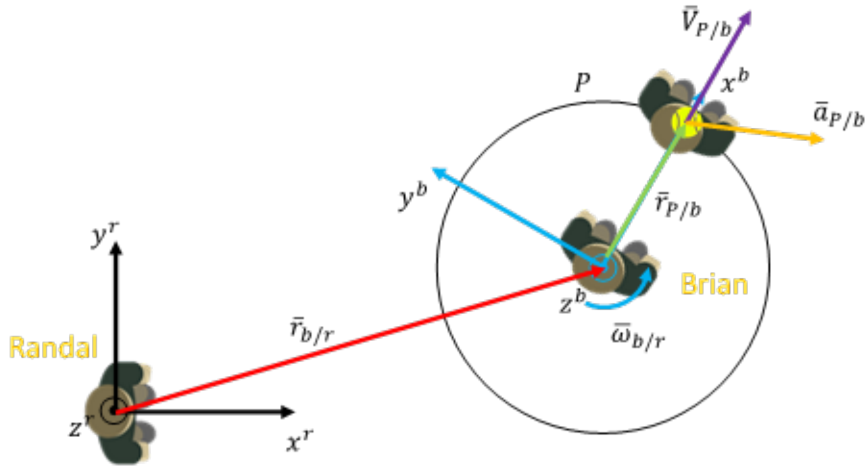
$-2\left(\bar{\omega}_{b/r} \times \bar{V}_{P/b}\right)$

{0., -10.472, 0.}

$-\ \bar{\omega}_{b/r} \times \left(\bar{\omega}_{b/r} \times \bar{r}_{P/b}\right)$

{1.64493, 0., 0.}

So we see that Coriolis acceleration still dominates but centrifugal acceleration is non-trivial at this point. If the merry-go-round was larger we would see that centrifugal acceleration would increase even further

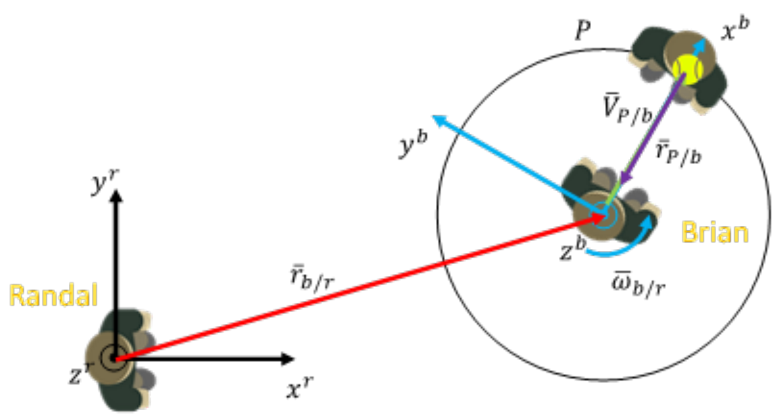



From Brian's perspective, not only does the ball appear to be accelerated to the right, but it appears to be accelerating away from him.

<u>Case 3:  Edward throws ball inward</u>

Now consider Edward throwing the ball inward instead of outward.  The only difference is that the velocity of $P$ is different

$$\overline{V}_{P/b}{}^{b} = \begin{pmatrix} -5\,m/s \\ 0 \\ 0 \end{pmatrix}$$



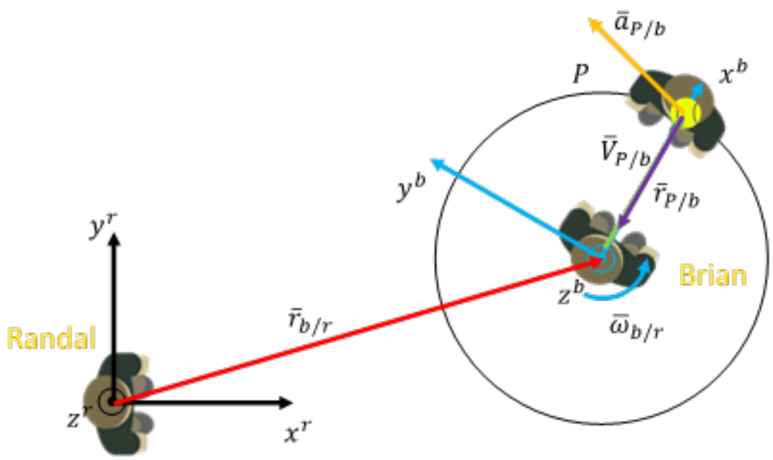
```
VPbb = {-5, 0, 0};
rPbb = {3 / 2, 0, 0};

Print["-2 (ω̄b/r × V̄P/b)"]
coriolisAcceleration = -2 (Cross[ωbrb, VPbb]);
coriolisAcceleration // N

Print["- ω̄b/r× (ω̄b/r×r̄P/b)"]
centrifugalAcceleration = -Cross[ωbrb, Cross[ωbrb, rPbb]];
centrifugalAcceleration // N
```

$-2\left(\bar{\omega}_{b/r}\times\bar{V}_{P/b}\right)$

{0., 10.472, 0.}

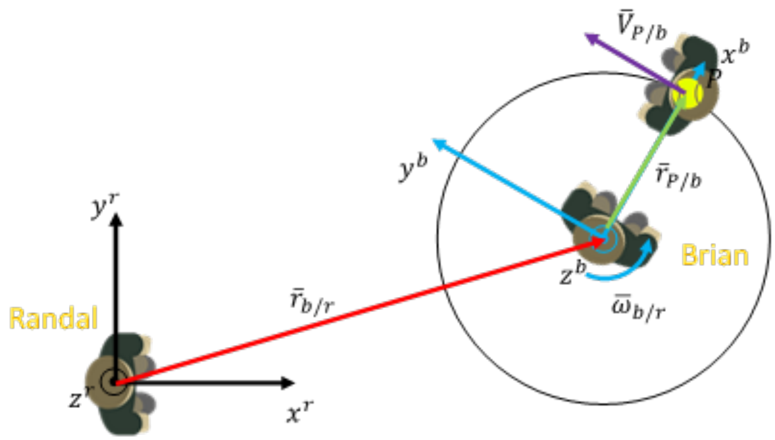
$-\bar{\omega}_{b/r}\times\left(\bar{\omega}_{b/r}\times\bar{r}_{P/b}\right)$

{1.64493, 0., 0.}




So again, the ball appears to deflect to its right (to Brian's left). Interesting, examining the component of acceleration towards him $\left(\bar{a}_{P/b}{}^{b}(1)\right)$, it appears that the ball is actually slowing down (even though we know it is traveling at a constant velocity.

Case 4: Tangentially in the direction of travel




Now consider Edward throwing tangentially to the merry-go-round in the direction of travel. The only difference is that the velocity of *P* is different

$$\overline{V}_{P/b}{}^{b} = \begin{pmatrix} 0 \\ 5\,m/s \\ 0 \end{pmatrix}$$

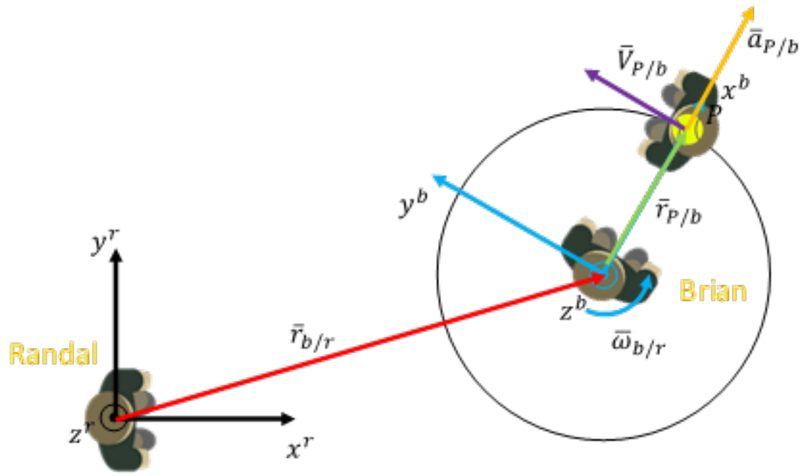
```
VPbb = {0, 5, 0};
rPbb = {3 / 2, 0, 0};

Print["-2 (ω̄b/r × V̄P/b)"]
coriolisAcceleration = -2 (Cross[ωbrb, VPbb]);
coriolisAcceleration // N

Print["- ω̄b/r × (ω̄b/r × r̄P/b)"]
centrifugalAcceleration = -Cross[ωbrb, Cross[ωbrb, rPbb]];
centrifugalAcceleration // N
```

$-2\left(\overline{\omega}_{b/r}\times\overline{V}_{P/b}\right)$

{10.472, 0., 0.}

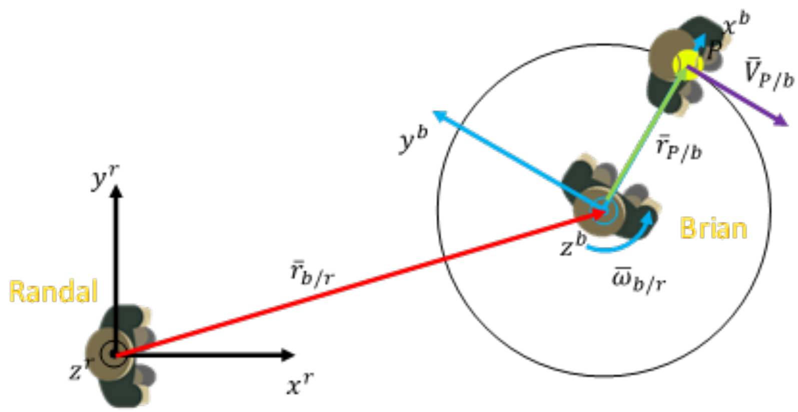
$-\ \overline{\omega}_{b/r}\times\left(\overline{\omega}_{b/r}\times\overline{r}_{P/b}\right)$

{1.64493, 0., 0.}




So again, the ball is deflected to its right.  Interestingly, both the Coriolis and centrifugal acceleration are in the same direction.

<u>Case 5:  Tangentially against the direction of travel</u>



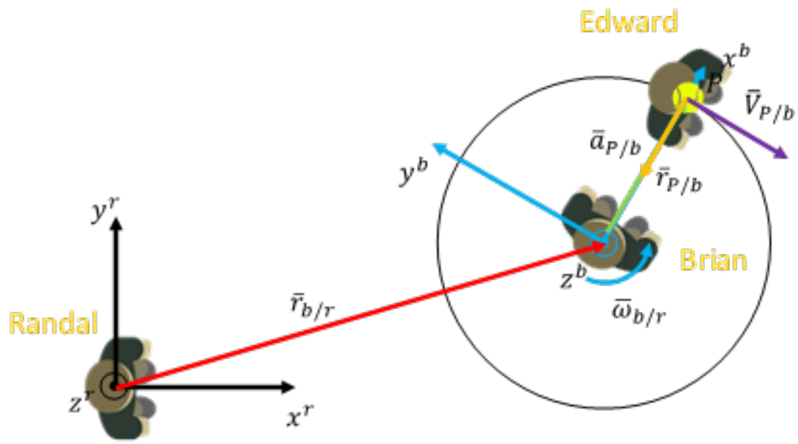

```
VPbb = {0, -5, 0};
rPbb = {3 / 2, 0, 0};

Print["-2 (ω̄b/r × V̄P/b)"]
coriolisAcceleration = -2 (Cross[ωbrb, VPbb]);
coriolisAcceleration // N

Print["- ω̄b/r × (ω̄b/r × r̄P/b)"]
centrifugalAcceleration = -Cross[ωbrb, Cross[ωbrb, rPbb]];
centrifugalAcceleration // N
```

$-2\ (\bar{\omega}_{b/r} \times \bar{V}_{P/b})$

{-10.472, 0., 0.}

$-\ \bar{\omega}_{b/r} \times (\bar{\omega}_{b/r} \times \bar{r}_{P/b})$

{1.64493, 0., 0.}




Note that the results of both case 2 - 4 are the same if Edward is standing just off the merry-go-round and not actually rotating with the merry-go-round (the only difference is $|\,\bar{r}_{P/b}\,|$ is slightly larger).

Case 6: Straight Up

Let us examine the total acceleration equation again (Eq.14 repeated here for convenience)

$$\bar{a}_{P/b}{}^b = \bar{a}_{P/r}{}^b - 2\left(\bar{\omega}_{b/r}{}^b \times \bar{V}_{P/b}{}^b\right) - \bar{\omega}_{b/r}{}^b \times \left(\bar{\omega}_{b/r}{}^b \times \bar{r}_{P/b}{}^b\right)$$

We note that if Brian throws the ball straight up from the center, we have

$$\bar{V}_{P/b}{}^b = \begin{pmatrix} 0 \\ 0 \\ V \end{pmatrix}$$

$$\bar{r}_{P/b}{}^b = \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix} \qquad h = \text{height of the ball at any given time}$$

So Eq.14 can be written as

$$\bar{a}_{P/b}{}^b = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} - 2\left(\begin{pmatrix} 0 \\ 0 \\ 2\pi/6 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ V \end{pmatrix}\right) - \begin{pmatrix} 0 \\ 0 \\ 2\pi/6 \end{pmatrix} \times \left(\begin{pmatrix} 0 \\ 0 \\ 2\pi/6 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ h \end{pmatrix}\right)$$

$$\bar{a}_{P/b}{}^b = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} = \bar{a}_{P/r}{}^b$$

```
VPbb = {0, 0, V};
rPbb = {0, 0, h};

Print["-2 (ω̄b/r × V̄P/b)"]
coriolisAcceleration = -2 (Cross[ωbrb, VPbb]);
coriolisAcceleration // N

Print["- ω̄b/r × (ω̄b/r × r̄P/b)"]
centrifugalAcceleration = -Cross[ωbrb, Cross[ωbrb, rPbb]];
centrifugalAcceleration // N
```

$-2\ (\overline{\omega}_{b/r} \times \overline{V}_{P/b})$

{0., 0., 0.}

$-\ \overline{\omega}_{b/r} \times (\overline{\omega}_{b/r} \times \overline{r}_{P/b})$

{0., 0., 0.}

In this case, both the Coriolis and centrifugal acceleration are zero, regardless of the velocity and the height. Therefore

$$\overline{a}_{P/b}{}^{b} = \overline{a}_{P/r}{}^{b} = \begin{pmatrix} 0 & 0 & -g \end{pmatrix}^{T}$$

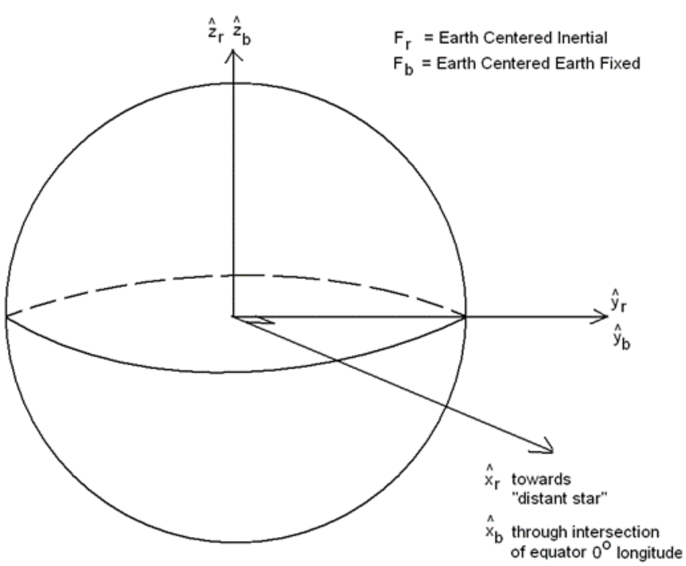
In other words, if Brian throws the tennis ball directly up, it will appear as a normal ball to both him and Randal (it would simply decelerate and fall back to the ground due to gravity).

**Example: Rotating Earth**

Choose $F_r = F_{\text{ECI}}$ (earth centered inertial but non-rotating, good approximation of an inertial reference frame)

$F_b = F_{\text{ECEF}}$ (earth centered earth fixed)

Explain the difference between $F_{\text{ECI}}$ and $F_{\text{ECEF}}$




In this case,

$\overline{\alpha}_{b/r} = \overline{0}$ (rotation rate is not changing so all zero regardless of frame to express it in)

$\overline{a}_{b/r} = \overline{0}$ (two frames are not accelerating away from each other)

What about the angular rotation vector? Which direction is it pointing? What is the sign? What is the magnitude? How can we expressed it?

$$\overline{\omega}_{b/r}{}^{b} \approx \begin{pmatrix} 0 \\ 0 \\ \frac{2\pi}{24*60*60} \end{pmatrix} \quad \text{(approximately one rotation per day earth rotation)}$$

So Eq.1.2-11 simplifies to

$$\overline{a}_{P/r} = \overline{a}_{P/b} + 2\left(\overline{\omega}_{b/r} \times \overline{V}_{P/b}\right) + \overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right)$$

Recall Newton's 2nd law is valid in an inertial frame

$$\overline{F}_{\text{true}} = m \ {}^{r}\dot{\overline{V}}$$

$$= m\,\overline{a}_{P/r}$$

$$= m\left(\overline{a}_{P/b} + 2\left(\overline{\omega}_{b/r} \times \overline{V}_{P/b}\right) + \overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right)\right)$$

The interesting and confusing stems from fact that an observer on earth, we observe $\overline{a}_{P/b}$. This is the apparent acceleration that an observer on $F_b$ (ie earth) would see. So isolating the term $m\,\overline{a}_{P/b}$ yields

$$m\,\overline{a}_{P/b} = \overline{F}_{\text{true}} - 2\,m\left(\overline{\omega}_{b/r} \times \overline{V}_{P/b}\right) - m\left(\overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right)\right)$$

The term $m\,\overline{a}_{P/b}$ can be considered the apparent force that the observer in $F_b$ would see

$$\overline{F}_{\text{apparent}} = \overline{F}_{\text{true}} - 2\,m\left(\overline{\omega}_{b/r} \times \overline{V}_{P/b}\right) - m\left(\overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right)\right) \qquad \textbf{\textit{(Eq.8)}}$$

So it can be seen that the apparent force seen by an observer ($\overline{F}_{\text{apparent}}$) is affected by the Coriolis and Centripetal forces. These are sometimes referred to as fictitious or pseudo forces because they appear to act on an object when observing the object from a non-inertial frame. The force does not arise from any physical interaction, but rather from rotation of the non-inertial frame.

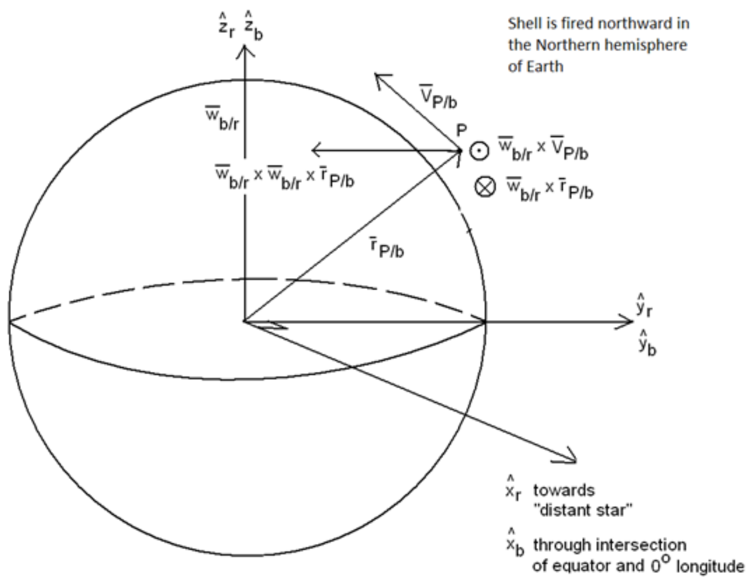
It may be easier to think about this from an acceleration perspective and divide by mass to obtain

$$\overline{a}_{P/b} = \frac{1}{m}\,\overline{F}_{\text{true}} - 2\left(\overline{\omega}_{b/r} \times \overline{V}_{P/b}\right) - \left(\overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right)\right) \qquad \textbf{\textit{(Eq.9)}}$$

**Example: Artillery Shell**

Choose $F_r = F_{\text{ECI}}$ (earth centered inertial)
$F_b = F_{\text{ECEF}}$ (earth centered earth fixed)

Let us examine an example of an artillery shell fired on Earth. The relevant geometry is shown below.




Realistic example values are

$$\overline{\omega}_{b/r}{}^{b} = \begin{pmatrix} 0 \\ 0 \\ \frac{2\pi}{24 \cdot 60 \cdot 60} \end{pmatrix}$$

$| \overline{\omega}_{b/r}{}^{b} | = 7.2722\, e^{-5}$ rad/s (earth rotation velocity)

$| \overline{V}_{P/b}{}^{b} | = 1600\, m/s$ (speed of artillery shell)

$| \overline{r}_{P/b} | = 6378.1$ km (radius of earth)

```
ωbrb = {0, 0, 2π/(24*60*60)};
normVPb = 1600;  (*m/s*)
normrPb = 6378.1*1000;  (*m*)
```

We can examine the possible magnitudes of the Coriolis and centrifugal acceleration

Coriolis Acceleration

Recall $\left| \overline{a} \times \overline{b} \right| = \left| \overline{a} \right| \cdot | \overline{b} | \sin(\theta)$ where $\theta$ is the angle between the two vectors. Let us consider look at worse case where $\sin(\theta) = 1$

$$\left| 2 \left( \overline{\omega}_{b/r} \times \overline{V}_{P/b} \right) \right| = 2 \cdot \left| \overline{\omega}_{b/r} \right| \cdot \left| \overline{V}_{P/b} \right| \cdot \sin(\theta)$$

Note that $\sin(\theta) = 1$ when $\theta = \pi/2$, so the Coriolis acceleration is maximal when $\overline{\omega}_{b/r}$ and $\overline{V}_{P/b}$ are perpendicular to each other. Since we know that $\overline{\omega}_{b/r}$ is pointing from the center of the earth and out the north pole, we see that the Coriolis effect is maximized in situations such as the artillery shell is located on the equator and fired directly vertical to the local horizon or at a pole and fired exactly horizontal to the local horizon. In this case, the magnitude is given by

$$\left| 2\left(\overline{\omega}_{b/r} \times \overline{V}_{P/b}\right) \right| \leq 2 \times \left(7.2722\, e^{-5}\, \frac{\text{rad}}{s}\right) \times \left(1600\, \frac{m}{s}\right)$$

$$2 \left| \overline{\omega}_{b/r} \times \overline{V}_{P/b} \right| \leq 0.23\, \frac{m}{s^2}$$

```
coriolisWorseCase = 2 Norm[ωbrb] normVPb // N
```

```
0.232711
```

In terms of a percentage of gravity we have

```
coriolisWorseCase / 9.81
```

```
0.0237218
```

So we see that this non-trivial (approximately 2.3% of local gravity).

To be explicit, consider the case where the cannon is placed at the intersection of the equator and the prime meridian (where $\hat{x}^b$ pierces the Earth's surface) and fired vertically. In this case, we have

$$\overline{V}_{P/b}{}^{b} = \begin{pmatrix} 1600 \\ 0 \\ 0 \end{pmatrix}$$

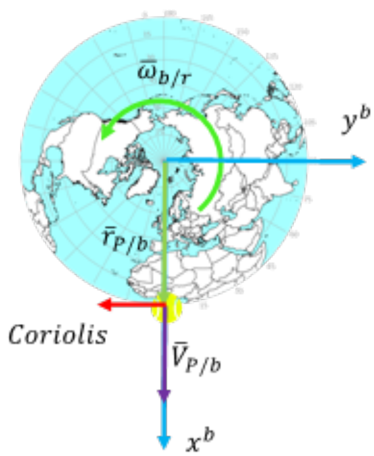
```
VPbb = {normVPb, 0, 0};
```

We can compute the Coriolis acceleration as seen from $F_b$ using $-2\left(\overline{\omega}_{b/r} \times \overline{V}_{P/b}\right)$

```
Print["-2 (ω̄b/r × V̄P/b)"]
coriolisAcceleration = -2 (Cross[ωbrb, VPbb]);
coriolisAcceleration // N
```

$-2\left(\overline{\omega}_{b/r} \times \overline{V}_{P/b}\right)$

```
{0., -0.232711, 0.}
```




So we see that the shell will not go straight up and come straight back down. In fact, performing some very rough calculations we can see the effect of this

```
(*Time of flight*)
tApogee = normVPb / 9.81
ΔT = 2 * tApogee
```

```
163.099
```

```
326.198
```

```
ΔV = coriolisWorseCase ΔT
ΔL = 1/2 coriolisWorseCase ΔT^2
```

```
75.9097
```

```
12380.8
```

So at impact back into the ground, it will be traveling at almost 170 MPH and land 12 km away from the point of departure. Of course, neglecting aerodynamic drag an other effects makes this calculation much more impressive than real life but for the most part, this shows the impact of the Coriolis effect.

Note that the previous calculation also neglected the centrifugal acceleration so we should now compute this and see how it affects the trajectory.

Centrifugal Acceleration

In a similar fashion, we can look at the contribution of centrifugal acceleration. Note that since there are two cross products, it is slightly more involved so we will simply look at the case where $\overline{r}_{P/b}$ is such that both cross products are maximized.

$$\left| \overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right) \right| \leq \left| \overline{\omega}_{b/r} \right| \cdot \left| \overline{\omega}_{b/r} \right| \cdot \left| \overline{r}_{P/b} \right|$$

$$\leq \left(7.2722\, e^{-5}\, \frac{\text{rad}}{s}\right) \times \left(7.2722\, e^{-5}\, \frac{\text{rad}}{s}\right) \times \left(6378.1\,\text{km} \left| \frac{1000\, m}{\text{km}} \right|\right)$$

$$\left| \overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right) \right| \leq 0.0337\, \frac{m}{s^2}$$

```
centripetalWorseCase = Norm[ωbrb] * Norm[ωbrb] * normrPb // N
```

```
0.0337306
```

Again, in terms of a fraction of gravity we see that

```
centripetalWorseCase / 9.81
```

```
0.00343839
```

You might be interested as to where on the earth the centrifugal acceleration is maximized. Let us examine the centrifugal acceleration expression of $-\left(\overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right)\right)$. We see that shell is exactly at the equator then $\left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right)$ will point from the origin to the plane of the equator. Therefore, $\overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right)$ is again in the plane of the equator.

```
rPbb = {normrPb, 0, 0};

Print["- ω̄b/r×(ω̄b/r×r̄P/b)"]
centrifugalAcceleration = -Cross[ωbrb, Cross[ωbrb, rPbb]];
centrifugalAcceleration // N
```

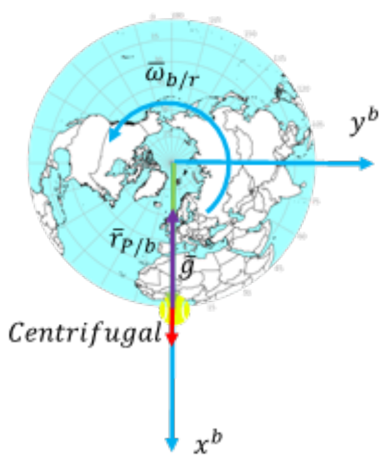
$- \overline{\omega}_{b/r} \times (\overline{\omega}_{b/r} \times \overline{r}_{P/b})$

{0.0337306, 0., 0.}

If we consider the case of the non-moving shell (AKA $\overline{V}_{P/b} = \overline{0}$). In this case, Eq.9 reduces to

$$\overline{a}_{P/b} = \frac{1}{m}\overline{F}_{\text{true}} - 2\left(\overline{\omega}_{b/r} \times \overline{V}_{P/b}\right) - \left(\overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right)\right) \qquad \text{note: } \overline{V}_{P/b} = \overline{0}$$

$$\overline{a}_{P/b} = \overline{g} - \left(\overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right)\right)$$




So we see that you actually weigh less on the equator due to centrifugal acceleration (it opposes gravity). This effect vanishes at the poles (because $\overline{\omega}_{b/r} \times \overline{r}_{P/b} = \overline{0} \Rightarrow \overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right) = \overline{0}$). Suppose you had a mass of 70 kg.

```
m = 70;

Print["Weight in Newtons"]
(*Weight at the poles (N)*)
WPoles = 9.81 * m

(*Weight at the equator (N)*)
WEquator = (9.81 - centripetalWorseCase) * m

Print["Weight in lbf"]
(*Convert to lbf*)
WPoles * .224809
WEquator * .224809
```

```
Weight in Newtons

686.7

684.339

Weight in lbf

154.376

153.846
```

This is over a half lbf difference.

# Newton's 2nd Law

Recall that Newton's second law states that the net force on an object is equal to the rate of change of its linear momentum in an inertial reference frame.

$$\overline{F} = \frac{d}{dt}\Big|_r \left(m\,\overline{V}\right)$$

This is valid only if $F_r$ is an inertial frame.

For a constant mass system in an inertial frame, this can be simplified to.

$$\overline{F} = m\ {}^r\dot{\overline{V}}$$

Consider an accelerometer ***<show small accelerometer>***. What does an accelerometer measure? Since these are basically force transducers with a specific mass, they measure the acceleration in the inertial frame, in other words

accelerometers measure ${}^r\dot{\overline{V}}$

Consider the acceleration equation (repeated here for convenience)

$$\overline{a}_{P/r} = \overline{a}_{b/r} + \overline{a}_{P/b} + 2\left(\overline{\omega}_{b/r} \times \overline{V}_{P/b}\right) + \overline{\alpha}_{b/r} \times \overline{r}_{P/b} + \overline{\omega}_{b/r} \times \left(\overline{\omega}_{b/r} \times \overline{r}_{P/b}\right)$$

So in this case we see that accelerometer measures $\overline{a}_{P/r}$. However, if you would like to navigate on the body frame $F_b$, you may be interested in inferring $\overline{a}_{P/b}$. Therefore, all the other acceleration terms need to be taken into account. What these terms mean depend on the frames in question.