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Lecture 04c
Understanding and Sketching Individual Bode Plots Components

Lecture is on YouTube

The YouTube video entitled ‘Understanding and Sketching Individual Bode Plot Components ’ that covers this lecture is located at https://youtu.be/aoFakXGYOH0.

Outline

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-Summary

Introduction

Recall that the steady state response of a LTI system to an input $u(t) = A \sin(\omega t)$ is given as

$$y_{ss}(t) = A \, | G(j\omega) | \, \sin(\omega t + \theta) \qquad \textbf{\textit{(Eq.1)}}$$

where $| G(j\omega) | = \sqrt{(\text{Re}[G(j\omega)])^2 + (\text{Im}[G(j\omega)])^2}$

$\theta = \angle G(j\omega) = \text{atan2}(\text{Im}[G(j\omega)], \text{Re}[G(j\omega)]) = \text{atan2}(y, x)$

Furthermore, recall that this is more useful to look at this in Bode plot form where we investigate the magnitude and phase of the system's response as a function of ω.

In previous lectures, we saw how to use brute force techniques (such as numerical Matlab tools) to generate a bode plot of a system.

1. Computing $G(j\omega)$
2. Compute real and imaginary parts of $G(j\omega)$
3. Compute the magnitude $| G(j\omega) | = \left(\text{Re}[G(j\omega)]^2 + \text{Im}[G(j\omega)]^2\right)^{1/2}$
4. Computing angle using $\theta = \angle G(j\omega) = \text{atan2}(\text{Im}[G(j\omega)], \text{Re}[G(j\omega)]) = \text{atan2}(y, x)$

While this method is effective, it does not yield insight or understanding to how a transfer function influences the bode plot.

A more elegant approach is to examine individual components of a transfer function (poles and zeros) and understand how each component influences the bode plot.

Sketching Individual Bode Plots Components

We will investigate several components that make up a transfer function

1. Single Real Pole
2. Single Real Zero
3. Pole at Origin
4. Zero at Origin
5. Pair of Complex Conjugate Poles
6. Pair of Complex Conjugate Zeros
7. Constant gain

Single Real Pole

Consider a transfer function of the form (notice that the DC gain is unity)

$$G(s) = \frac{1}{s/p+1}$$

```
G[s_] = 1 / (s / p + 1);
```

We can notice several things about this transfer function

-single pole at $s = -p$
-DC gain = 1
-constant term in denominator is 1

Because we are interested in the frequency response of the system, we should compute $G(j\,\omega)$

$$G(j\,\omega) = \left(\frac{1}{j\,\omega/p+1}\right)$$

$$= \left(\frac{1}{j\,\omega/p+1}\right)\left(\frac{-j\,\omega/p+1}{-j\,\omega/p+1}\right)$$

$$= \frac{-j\,\omega/p+1}{1+(\omega/p)^2}$$

$$= \left(\frac{1}{1+(\omega/p)^2}\right) + \left(\frac{-\omega/p}{1+(\omega/p)^2}\right) j$$

$$G(j\,\omega) = \alpha + \beta\, j \qquad \textbf{\textit{(Eq.1.1)}}$$

where $\alpha = \left(\frac{1}{1+(\omega/p)^2}\right)$

$\beta = \left(\frac{-\omega/p}{1+(\omega/p)^2}\right)$

```
α = 1 / (1 + (ω / p)^2);
β = -ω / p / (1 + (ω / p)^2);
G[ω I] == α + β I // Simplify
```

```
True
```

We can now simply calculate the magnitude associated with $G(j\,\omega)$ using

$$|\, G(j\,\omega)\,| = \left(\alpha^2 + \beta^2\right)^{1/2}$$

$$= \left(\left(\frac{1}{1+(\omega/p)^2}\right)^2 + \left(\frac{-\omega/p}{1+(\omega/p)^2}\right)^2\right)^{1/2}$$

$$| \, G(j\,\omega) \, | = \left(\frac{1}{1+(\omega/p)^2}\right)^{\frac{1}{2}}$$ ***(Eq.1.2)***

```
magGjω = (1 / (1 + (ω / p)^2))^(1/2);
magGjω == (α^2 + β^2)^(1/2) // Simplify
True
```

Recall that the magnitude plot of the Bode plot (the y-axis of the bode plot) is given by $20\log_{10}(\,|\,G(j\,\omega)\,|\,)$. Using Eq.1.2, we have

$$20\log_{10}(\,|\,G(j\,\omega)\,|\,) = 20\log_{10}\left(\left(\frac{1}{1+(\omega/p)^2}\right)^{\frac{1}{2}}\right)$$

$$= 20\log_{10}\left(\left(1+(\omega/p)^2\right)^{-\frac{1}{2}}\right)$$

$$20\log_{10}(\,|\,G(j\,\omega)\,|\,) = -10\log_{10}\left(1+(\omega/p)^2\right)$$ (y-axis value of the magnitude bode plot) ***(Eq.1.3)***

And for the phase, the angle associated with $G(j\,\omega)$ is given by

$$\angle G(j\,\omega) = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

$$= \tan^{-1}\left(\left(\frac{-\omega/p}{1+(\omega/p)^2}\right)\Big/\left(\frac{1}{1+(\omega/p)^2}\right)\right)$$

$$= \tan^{-1}\left(\left(\frac{-\omega/p}{1+(\omega/p)^2}\right)\left(\frac{1+(\omega/p)^2}{1}\right)\right)$$

$$\angle G(j\,\omega) = \tan^{-1}(-\omega/p)$$ (y-axis value of the phase bode plot) ***(Eq.1.4)***

```
-ω / p == β / α
angleGjω = ArcTan[-ω / p];
True
```

Small Frequencies (ω << p)

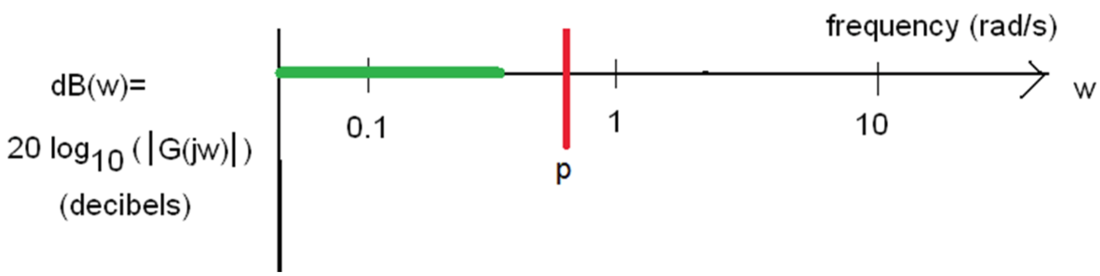
We now can now notice that for small frequencies (ω << p), the term $(\omega/p)^2 \approx 0$, so Eq.1.3 reduces to

$$20\log_{10}(\,|\,G(j\,\omega)\,|\,) = -10\log_{10}\left(1+(\omega/p)^2\right)$$ note: $(\omega/p)^2 \approx 0$

$\approx -10\log_{10}(1)$ recall: $\log_{10}(1) = 0$

$20\log_{10}(\,|\,G(j\,\omega)\,|\,) \approx 0$ (for $\omega << \mathrm{p}$)

We can now sketch the magnitude portion of the bode plot for frequencies smaller than p rad/s



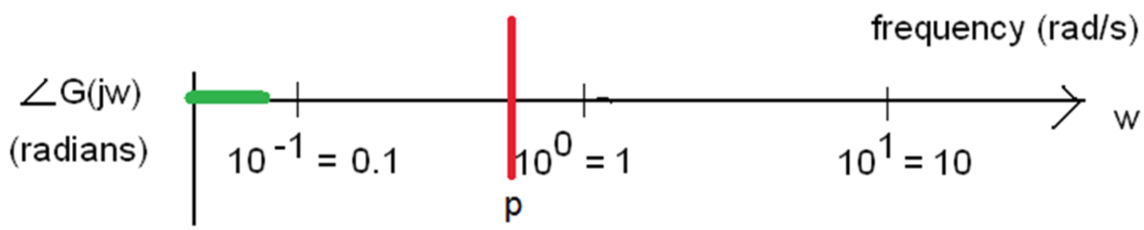
What about the phase? If $\omega << \mathrm{p}$, the term $\omega/p \approx 0$, so Eq.1.4 reduces to

$\angle G(j\,\omega) = \tan^{-1}(-\omega/p)$ note: $\omega/p \approx 0$

$\approx \tan^{-1}(0)$

$\angle G(j\,\omega) \approx 0$ ***(Eq.1.6)***

We can now sketch the phase portion of the bode plot for frequencies smaller than p. Keep in mind the assumption that $\omega/p \approx 0$ breaks down faster than the previous assumption of $(\omega/p)^2 \approx 0$, so we stop the approximation sooner in our phase plot



At Break/Corner Frequencies ($\omega = p$)

We can now investigate the bode plot at $\omega = p$. At this frequency, Eq.1.3 reduces to

$20\log_{10}(\,|\,G(p\,j)\,|\,) = -10\log_{10}\left(1 + (p/p)^2\right)$

$20\log_{10}(\,|\,G(p\,j)\,|\,) = -10\log_{10}(2) \approx -3.013$ (at $\omega = p$)

```
20 Log[10, magGjω] /. {ω → p} // N
-3.0103
```

So we see that when $\omega = p$, the magnitude plot should be at approximately -3 dB. This is why this frequency of $\omega = p =$ pole location is often called the -3 dB frequency. Other names include

break frequency
corner frequency
minus 3dB frequency

We can see how this relates to $|G(j\,\omega)|$

$$20\log_{10}(|G(j\,\omega)|) = -10\log_{10}(2)$$

```
Solve[20 Log[10, mag] == -10 Log[10, 2], mag]
```

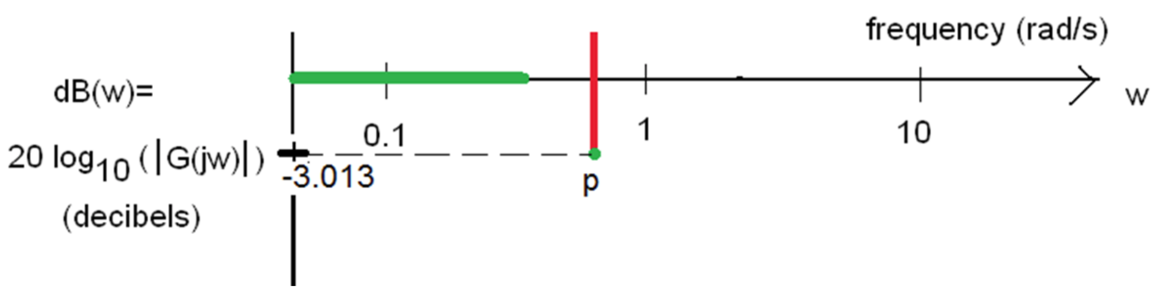
$$\left\{\left\{\text{mag} \to \frac{1}{\sqrt{2}}\right\}\right\}$$

We see that at the frequency of $\omega = p$, the magnitude of the system response is

$$|G(p\,j)| = \frac{1}{\sqrt{2}} \approx 0.707$$

So we see that the output has reduced to approximately 70% of its DC gain value.

We can now update our magnitude portion of the bode plot with this single data point



Similarly, the phase at $\omega = p$ is simply

$$\angle G(p\,j) = \tan^{-1}(-p/p)$$

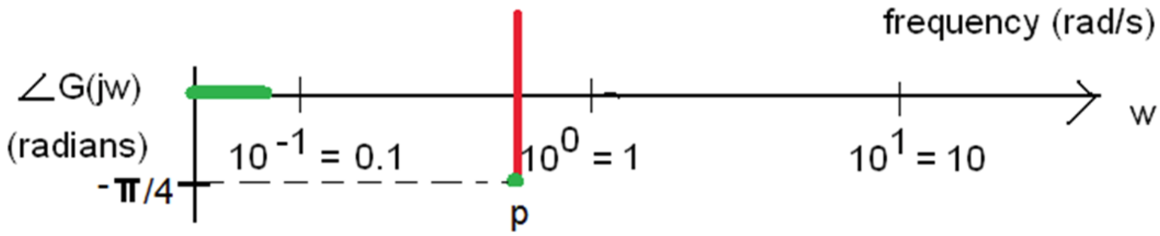
$$= \tan^{-1}(-1)$$

$$\angle G(p\,j) = \frac{-\pi}{4} = -45^o \qquad (\text{at } \omega = p)$$

```
angleGjω /. {ω → p}
```

$$-\frac{\pi}{4}$$

We can draw this point on the phase plot



Large Frequencies (ω >> p)

At large frequencies where ω >> p, the term $(\omega/p)^2$ dominates the argument of the $\log_{10}$ function. In other words, Eq.1.3 becomes

$$20\log_{10}(\,|\,G(j\,\omega)\,|\,) = -10\log_{10}\left(1+(\omega/p)^2\right) \qquad \text{note: } 1+(\omega/p)^2 \approx (\omega/p)^2$$

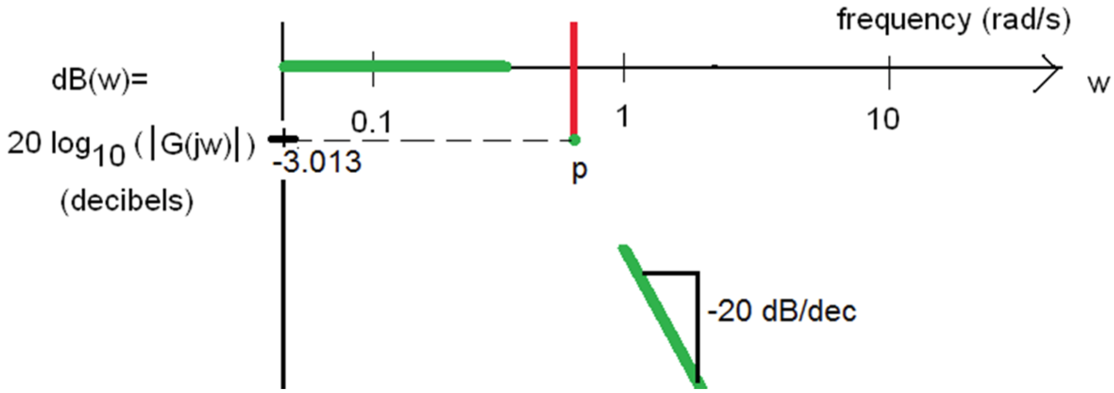
$$\approx -10\log_{10}\left((\omega/p)^2\right)$$

$$\approx -20\log_{10}(\omega/p) \qquad \text{recall: } \log_b(x/y) = \log_b(x) - \log_b(y)$$

$$\approx -20\,(\log_{10}(\omega) - \log_{10}(p))$$

$$20\log_{10}(\,|\,G(j\,\omega)\,|\,) \approx -20\log_{10}(\omega) + 20\log_{10}(p)$$

Note that since the bode plot x-axis is $\log_{10}(\omega)$ and the y-axis is $20\log_{10}(\,|\,G(j\,\omega)\,|\,)$, the above expression describes a line with slope of -20 dB (it looks like $y = m\,x + b$ where $m = -20$). The line decreases by 20 dB for every decade increase in frequency (a decade being an order of magnitude). So we can sketch the magnitude plot as



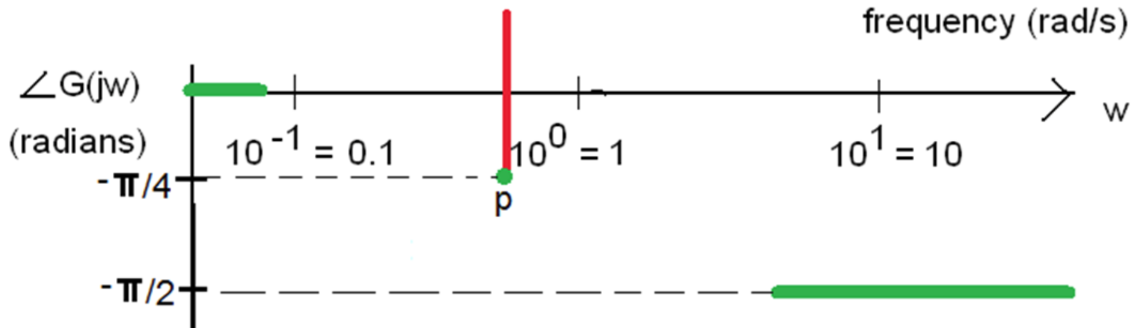
What about the phase? If ω >> p, the term ω/p becomes very large, so Eq.1.4 reduces to

$$\angle G(j\,\omega) = \tan^{-1}(-\omega/p) \qquad \text{note: } \omega \text{ is large}$$

$$\approx \tan^{-1}(-\infty)$$

$$\angle G(j\,\omega) \approx -\pi/2$$

We can add this information to our sketch for the phase

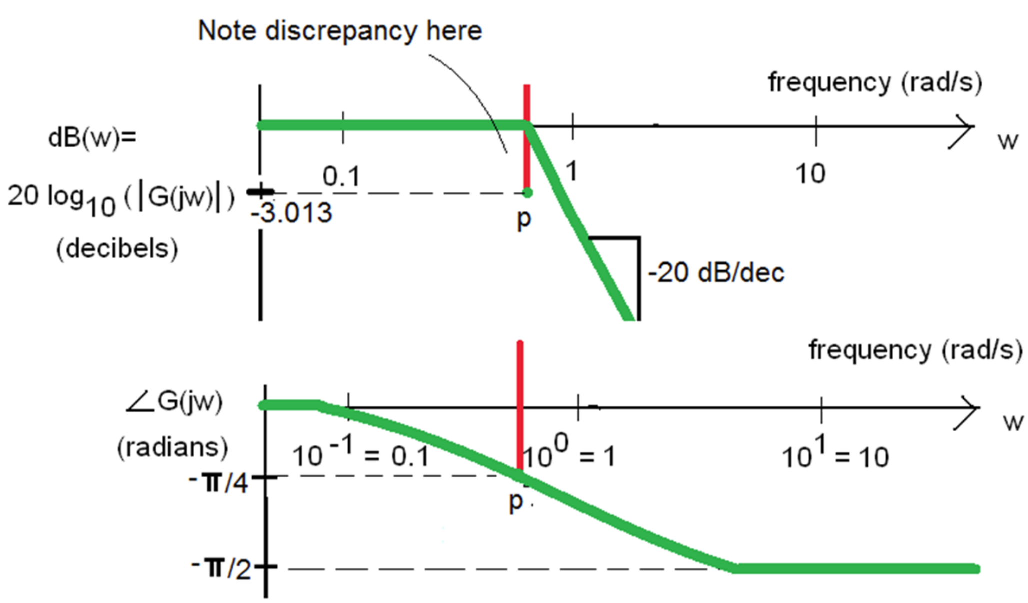


Approximate Plot for Single Real Pole

With all of this information, we can generate a few rules of how to sketch a bode plot of a single, real pole.

Magnitude Plot: remains at 0 dB until the break frequency $\omega = p$ and then decreases linearly at a rate of -20 dB/dec

Phase Plot: remains at 0 until approximately 1 decade before $\omega = p$ and then decreases to -π/2 approximately 1 decade after $\omega = p$ (thereby going through -π/4 at $\omega = p$)

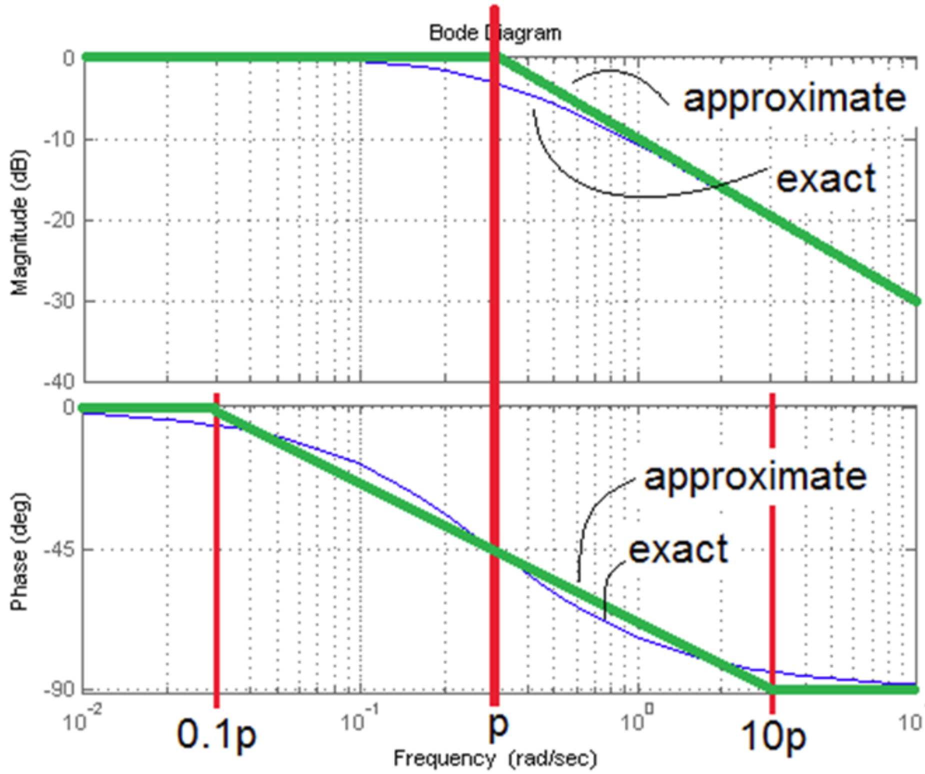


Notice that with this approximation rule, we realize that the magnitude plot at the break frequency is off by approximately 3 dB.

We can compare with the exact Bode plot. Let us return to our example problem where

$$G(s) = \frac{1}{s/p+1}$$

If we have $p = 0.3$, then the actual and approximate bode plot is shown below



As we can see, this approximate method provides a very good approximation of the bode plot.

```
Clear[angleGjω, magGjω, β, α, G]
```

Single Real Zero

We can repeat this analysis for a transfer function of the form (notice that the DC gain is unity)

$$G(s) = s/z + 1$$

```
G[s_] = s/z + 1;
```

We can notice several things about this transfer function

-single zero at $s = -z$
-DC gain = 1
-constant term in numerator is 1

Because we are interested in the frequency response of the system, we should compute $G(j\,\omega)$

$$G(j\,\omega) = (j\,\omega\,/z + 1)$$

$$G(j\,\omega) = \alpha + \beta\,j \qquad \textbf{\textit{(Eq.1.1)}}$$

where $\quad \alpha = 1$ and $\beta = \frac{\omega}{z}$

```
α = 1;
β = ω / z;
G[ω I] == α + β I // Simplify
True
```

We can now simply calculate the magnitude associated with $G(j\,\omega)$ using

$$|\ G(j\,\omega)\ | = \left(\alpha^2 + \beta^2\right)^{1/2}$$

$$= \left((1)^2 + \left(\tfrac{\omega}{z}\right)^2\right)^{1/2}$$

$$|\ G(j\,\omega)\ | = \left(1 + \left(\tfrac{\omega}{z}\right)^2\right)^{\frac{1}{2}} \qquad \textbf{\textit{(Eq.1.2)}}$$

```
magGjω = (1 + (ω/z)^2)^(1/2);
magGjω == (α^2 + β^2)^(1/2) // Simplify
True
```

Recall that the magnitude plot of the Bode plot (the y-axis of the bode plot) is given by $20\log_{10}(\ |\ G(j\,\omega)\ |\ )$. Using Eq.1.2, we have

$$20\log_{10}(\ |\ G(j\,\omega)\ |\ ) = 20\log_{10}\left(\left(1 + (\omega/z)^2\right)^{\frac{1}{2}}\right)$$

$$20\log_{10}(\ |\ G(j\,\omega)\ |\ ) = 10\log_{10}\left(1 + (\omega/z)^2\right) \qquad \text{(y-axis value of the magnitude bode plot)}$$

(Eq.1.3)

And for the phase, the angle associated with $G(j\,\omega)$ is given by

$$\angle G(j\,\omega) = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

$$= \tan^{-1}\left(\tfrac{\omega}{z}/1\right)$$

$$\angle G(j\,\omega) = \tan^{-1}(\omega/z) \qquad \text{(y-axis value of the phase bode plot)} \qquad \textbf{\textit{(Eq.1.4)}}$$

```
ω / z == β / α
angleGjω = ArcTan[ω / z];
```

```
True
```

We can now conduct the same analysis that we did with the single real pole. In fact, if we compare the expressions we obtained for $20\log_{10}(\,|\,G(j\,\omega)\,|\,)$ and $\angle G(j\,\omega)$ here with those that we obtained for a single real pole, we see some interesting parallels

<u>single real zero</u>

$20\log_{10}(\,|\,G(j\,\omega)\,|\,) = 10\log_{10}\left(1 + (\omega/z)^2\right)$

$\angle G(j\,\omega) = \tan^{-1}(\omega/z)$

<u>single real pole</u>

$20\log_{10}(\,|\,G(j\,\omega)\,|\,) = -10\log_{10}\left(1 + (\omega/p)^2\right)$

$\angle G(j\,\omega) = \tan^{-1}(-\omega/p)$

So we see that the arguments we used for the single real pole will apply to the single real zero with some minor variations

Small Frequencies ($\omega <<$ z)

We now can now notice that for small frequencies ($\omega <<$ z), the term $(\omega/z)^2 \approx 0$, so Eq.1.3 reduces to

$20\log_{10}(\,|\,G(j\,\omega)\,|\,) = 10\log_{10}\left(1 + (\omega/z)^2\right)$ note: $(\omega/z)^2 \approx 0$

$\approx 10\log_{10}(1)$ recall: $\log_{10}(1) = 0$

$20\log_{10}(\,|\,G(j\,\omega)\,|\,) \approx 0$ (for $\omega <<$ z)

What about the phase? If $\omega <<$ z, the term $\omega/z \approx 0$, so Eq.1.4 reduces to

$\angle G(j\,\omega) = \tan^{-1}(\omega/z)$ note: $\omega/z \approx 0$

$\approx \tan^{-1}(0)$

$\angle G(j\,\omega) \approx 0$ ***(Eq.1.6)***

At Break/Corner Frequencies ($\omega = z$)

We can now investigate the bode plot at $\omega = z$. At this frequency, Eq.1.3 reduces to

$$20\log_{10}(\,|\,G(z\,j)\,|\,) = 10\log_{10}\left(1 + (z/z)^2\right)$$

$$20\log_{10}(\,|\,G(z\,j)\,|\,) = 10\log_{10}(2) \approx 3.013 \qquad \text{(at } \omega = z\text{)}$$

```
20 Log[10, magGjω] /. {ω → z} // N
```

```
3.0103
```

Similarly, the phase at $\omega = z$ is simply

$$\angle G(z\,j) = \tan^{-1}(z/z)$$

$$= \tan^{-1}(1)$$

$$\angle G(z\,j) = \frac{\pi}{4} = 45^o \qquad \text{(at } \omega = z\text{)}$$

```
angleGjω /. {ω → z}
```

$\frac{\pi}{4}$

Large Frequencies (ω >> z)

At large frequencies where ω >> z, the term $(\omega/z)^2$ dominates the argument of the $\log_{10}$ function. In other words, Eq.1.3 becomes

$$20\log_{10}(\,|\,G(j\,\omega)\,|\,) = 10\log_{10}\left(1 + (\omega/z)^2\right) \qquad \text{note: } 1 + (\omega/z)^2 \approx (\omega/z)^2$$

$$\approx 10\log_{10}\left((\omega/z)^2\right)$$

$$\approx 20\log_{10}(\omega/z) \qquad \text{recall: } \log_b(x/y) = \log_b(x) - \log_b(y)$$

$$\approx 20\,(\log_{10}(\omega) - \log_{10}(z))$$

$$20\log_{10}(\,|\,G(j\,\omega)\,|\,) \approx 20\log_{10}(\omega) - 20\log_{10}(z)$$

Note that since the bode plot x-axis is $\log_{10}(\omega)$ and the y-axis is $20\log_{10}(\,|\,G(j\,\omega)\,|\,)$, the above expression describes a line with slope of 20 dB (it looks like $y = m\,x + b$ where $m = 20$). The line increases by 20 dB for every decade increase in frequency (a decade being an order of magnitude).

What about the phase? If ω >> z, the term ω/z becomes very large, so Eq.1.4 reduces to

$$\angle G(j\,\omega) = \tan^{-1}(\omega/z) \qquad \text{note: } \omega \text{ is large}$$

$$\approx \tan^{-1}(\infty)$$

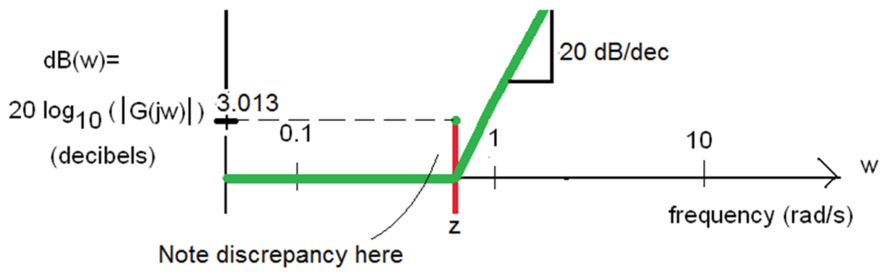
$\angle G(j\,\omega) \approx \pi/2$

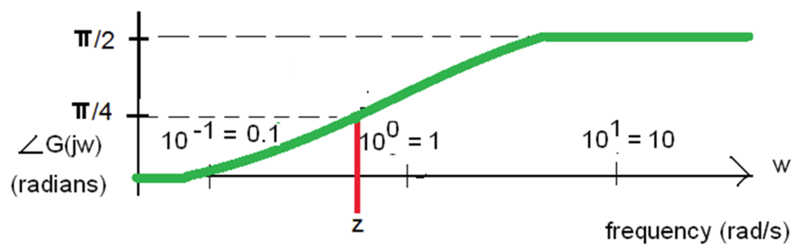
Approximate Plot for Single Real Zero

With all of this information, we can generate a few rules of how to sketch a bode plot of a single, real zero.

Magnitude Plot: remains at 0 dB until the break frequency $\omega = z$ and then increases linearly at a rate of +20 dB/dec

Phase Plot: remains at 0 until approximately 1 decade before $\omega = z$ and then increases to $\pi/2$ approximately 1 decade after $\omega = z$ (thereby going through $\pi/4$ at $\omega = z$)

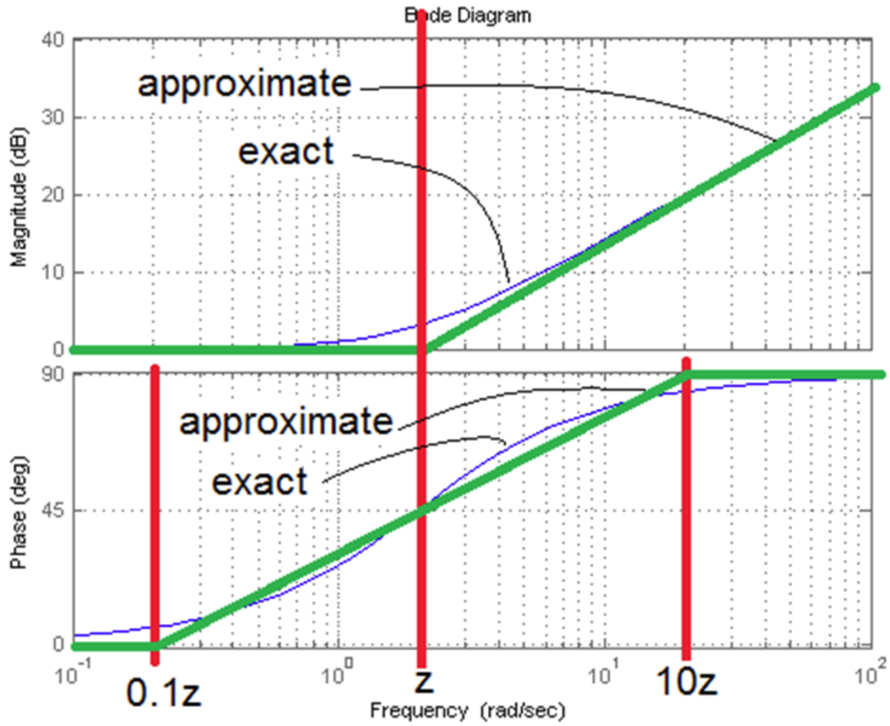




We can compare with the exact Bode plot. Consider a transfer function which consists of a single real zero of the form

$$G(s) = s/z + 1$$

If we have $z = 2$, then the actual and approximate bode plot is shown below




```
Clear[angleGjω, magGjω, β, α, G]
```

Poles at Origin

Consider a transfer function of the form

$$G(s) = 1/s$$

This is a single pole at the origin

```
G[s_] = 1 / s;
```

Because we are interested in the frequency response of the system, we should compute $G(j\,\omega)$

$$G(j\,\omega) = \frac{1}{j\,\omega}$$

$$= \frac{1}{j\,\omega}\,\frac{j}{j}$$

$$= -\frac{1}{\omega}\,j$$

$$G(j\,\omega) = \alpha + \beta\,j \qquad \textbf{(Eq.1.1)}$$

where $\alpha = 0$ and $\beta = -1/\omega$

```
α = 0;
β = -1 / ω;
G[ω I] == α + β I // Simplify
```

```
True
```

We can now simply calculate the magnitude associated with $G(j\,\omega)$ using

$$|\;G(j\,\omega)\;| = \left(\alpha^2 + \beta^2\right)^{1/2}$$

$$= \left((0)^2 + \left(\tfrac{-1}{\omega}\right)^2\right)^{1/2}$$

$$= \left(\left(\tfrac{-1}{\omega}\right)^2\right)^{1/2}$$

$$= \left(\tfrac{1}{\omega^2}\right)^{1/2}$$

$$|\;G(j\,\omega)\;| = \left(\tfrac{1}{\omega}\right) \qquad \textbf{\textit{(Eq.1.2)}}$$

```
magGjω = (1/ω);
Simplify[magGjω - (α^2 + β^2)^(1/2), ω > 0] == 0
```

```
True
```

Recall that the magnitude plot of the Bode plot (the y-axis of the bode plot) is given by $20\log_{10}(\;|\;G(j\,\omega)\;|\;)$. Using Eq.1.2, we have

$$20\log_{10}(\;|\;G(j\,\omega)\;|\;) = 20\log_{10}\left(\tfrac{1}{\omega}\right)$$

$$= 20\log_{10}\left(\omega^{-1}\right)$$

$$20\log_{10}(\;|\;G(j\,\omega)\;|\;) = -20\log_{10}(\omega) \qquad \text{(y-axis value of the magnitude bode plot)} \qquad \textbf{\textit{(Eq.1.3)}}$$

Note that with Eq.1.3, we can directly plot this <u>exactly</u> on the bode plot by noting that at $\omega = 1$ rad/s, $20\log_{10}(\;|\;G(1\,j)\;|\;) = 0$ dB. Therefore, we do not need to approximate the magnitude plot, it is simply a line with slope -20 dB/dec passing though 0 dB at $\omega = 1$ rad/s.

And for the phase, the angle associated with $G(j\,\omega)$ is given by

$$\angle G(j\,\omega) = \tan^{-1}\left(\tfrac{\beta}{\alpha}\right)$$

$$= \tan^{-1}\left(\tfrac{-1}{\omega}/0\right)$$

$$= \tan^{-1}(-\infty)$$

$\angle G(j\,\omega) = -\pi/2$ (y-axis value of the phase bode plot) ***(Eq.1.4)***

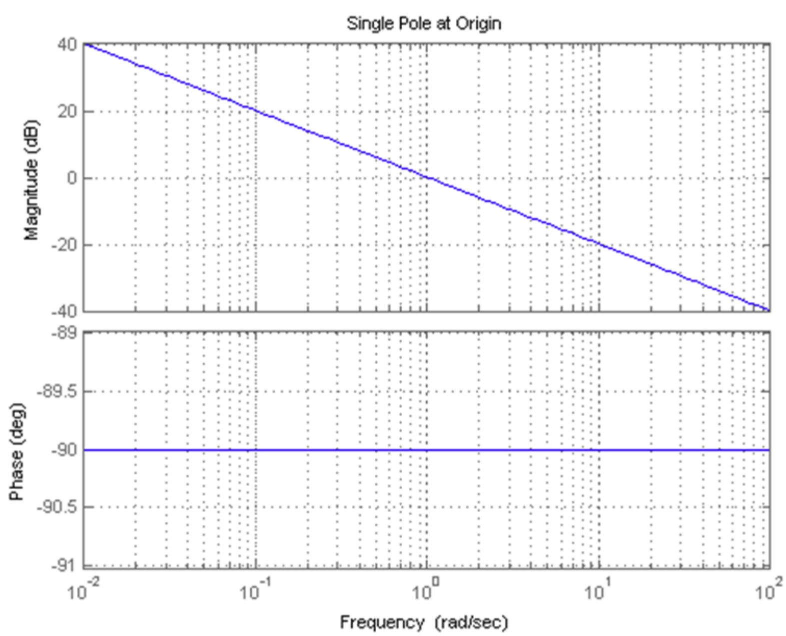
From Eq.1.4, we see that the phase is constant at -90 degrees.

Exact Plot for Single Pole at Origin

With all of this information, we can generate a few rules of how to sketch a bode plot of a single pole at the origin.

Magnitude Plot: line with slope of -20 db/dec passing through 0 dB at $\omega = 1$ rad/s

Phase Plot: constant value of $-\pi/2 = -90^o$



Let us consider the physical meaning of the bode plot for a single pole at the origin. Recall that a transfer function of $1/s$ is a pure integrator. Therefore, the input/output relationship is

$$y(t) = \int u(t)\,dt$$

The bode plot describes the input output relationship for sinusoidal inputs, $u(t) = A\sin(\omega\,t)$. So if we integrate this input, we have

$$y(t) = \int A\sin(\omega\,t)\,dt$$

$$y(t) = -\frac{A}{\omega}\cos(\omega t)$$

So we see that the output is a negative cos wave, which is the same as a sin wave except phase shifted by -90 degrees and the amplitude is changed by $1/\omega$. This is precisely what the bode plot above states.

```
Clear[magGjω, β, α, G]
```

Zero at Origin

Consider a transfer function of the form

$$G(s) = s$$

This is a single zero at the origin

```
G[s_] = s;
```

Because we are interested in the frequency response of the system, we should compute $G(j\,\omega)$

$$G(j\,\omega) = j\,\omega$$

$$G(j\,\omega) = \alpha + \beta\,j \qquad \textbf{\textit{(Eq.1.1)}}$$

where $\alpha = 0$ and $\beta = \omega$

```
α = 0;
β = ω;
G[ω I] == α + β I // Simplify
True
```

We can now simply calculate the magnitude associated with $G(j\,\omega)$ using

$$|\,G(j\,\omega)\,| = \left(\alpha^2 + \beta^2\right)^{1/2}$$

$$= \left((0)^2 + (\omega)^2\right)^{1/2}$$

$$|\,G(j\,\omega)\,| = \omega \qquad \textbf{\textit{(Eq.1.2)}}$$

```
magGjω = ω;
Simplify[magGjω - (α^2 + β^2)^(1/2), ω > 0] == 0
True
```

Recall that the magnitude plot of the Bode plot (the y-axis of the bode plot) is given by $20\log_{10}(\,|\,G(j\,\omega)\,|\,)$. Using Eq.1.2, we have

$$20\log_{10}(\,|\,G(j\,\omega)\,|\,) = 20\log_{10}(\omega) \qquad \text{(y-axis value of the magnitude bode plot)} \qquad \textbf{\textit{(Eq.1.3)}}$$

Note that with Eq.1.3, we can directly plot this exactly on the bode plot by noting that at $\omega = 1\ \text{rad}/s$, $20\log_{10}(\,|\,G(1\,j)\,|\,) = 0$ dB. Therefore, we do not need to approximate the magnitude plot, it is simply a line with slope +20 dB/dec passing though 0 dB at $\omega = 1\ \text{rad}/s$.

And for the phase, the angle associated with $G(j\,\omega)$ is given by

$$\angle G(j\,\omega) = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

$$= \tan^{-1}(\omega/0)$$

$$= \tan^{-1}(\infty)$$

$$\angle G(j\,\omega) = \pi/2 \qquad \text{(y-axis value of the phase bode plot)} \qquad \textbf{\textit{(Eq.1.4)}}$$

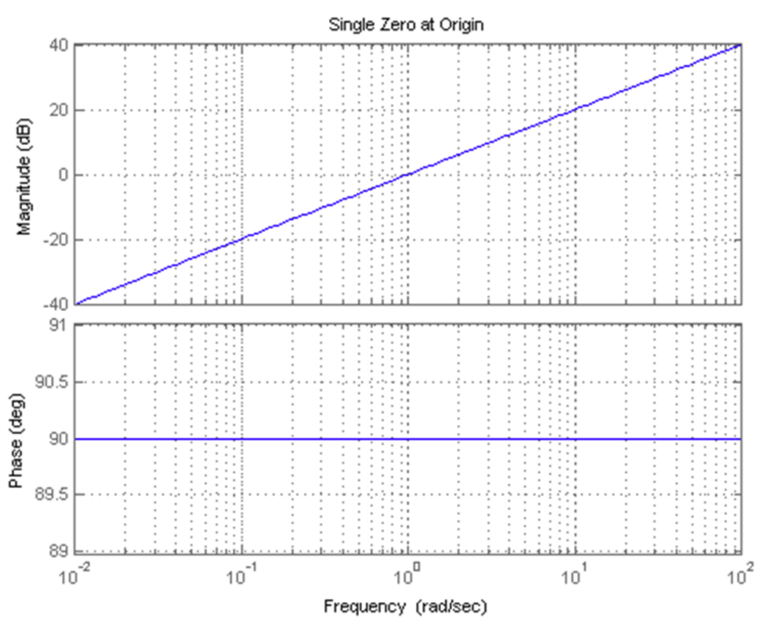
From Eq.1.4, we see that the phase is constant at +90 degrees.

Exact Plot for Single Zero at Origin

With all of this information, we can generate a few rules of how to sketch a bode plot of a single pole at the origin.

Magnitude Plot: line with slope of +20 db/dec passing through 0 dB at $\omega = 1\ \text{rad}/s$

Phase Plot: constant value of $+\pi/2 = +90^o$



Note that a transfer function of a zero at the origin, $G(s) = s$, is a pure differentiator.

$$y(t) = \frac{d}{dt}[u(t)]$$

So we can apply similar logic as we did for the pole at the origin to show that the bode plot is telling us this same information in the frequency domain for sinusoidal inputs.

```
D[A Sin[ω t], t]
```

A ω Cos[t ω]

```
Clear[magGjω, β, α, G]
```

Pair of Complex Conjugate Poles

Consider a transfer function of the form (notice that the DC gain is unity)

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\,\omega_n\,s + \omega_n^2} = \frac{1}{\frac{1}{\omega_n^2}s^2 + 2\frac{\zeta}{\omega_n}s + 1}$$

$$\texttt{G[s\_]} = \frac{1}{\frac{1}{\omega n^2}s^2 + 2\frac{\zeta}{\omega n}s + 1};$$

```
Solve[Denominator[G[s]] == 0, s]
```

$$\left\{\left\{s \to -\zeta\,\omega n - \sqrt{-1+\zeta^2}\;\omega n\right\},\ \left\{s \to -\zeta\,\omega n + \sqrt{-1+\zeta^2}\;\omega n\right\}\right\}$$

We can notice several things about this transfer function

-pair of complex poles at $s = -\zeta\,\omega_n \pm \omega_n\sqrt{1-\zeta^2}\;j$

-DC gain = 1

-constant term in denominator is 1

Because we are interested in the frequency response of the system, we should compute $G(j\,\omega)$. We can use Mathematica to skip the algebra

```
ComplexExpand[G[I ω]]
```

$$\frac{1}{\left(1-\frac{\omega^2}{\omega n^2}\right)^2 + \frac{4\zeta^2\omega^2}{\omega n^2}} - \frac{\omega^2}{\left(\left(1-\frac{\omega^2}{\omega n^2}\right)^2 + \frac{4\zeta^2\omega^2}{\omega n^2}\right)\omega n^2} - \frac{2\,i\,\zeta\,\omega}{\left(\left(1-\frac{\omega^2}{\omega n^2}\right)^2 + \frac{4\zeta^2\omega^2}{\omega n^2}\right)\omega n}$$

We can now extract α (the real part) and β (the imaginary part)

```
α = ComplexExpand[Re[G[I ω]]]
β = ComplexExpand[Im[G[I ω]]]
```

$$\frac{1}{\left(1-\frac{\omega^2}{\omega n^2}\right)^2+\frac{4\zeta^2\omega^2}{\omega n^2}}-\frac{\omega^2}{\left(\left(1-\frac{\omega^2}{\omega n^2}\right)^2+\frac{4\zeta^2\omega^2}{\omega n^2}\right)\omega n^2}$$

$$-\frac{2\zeta\omega}{\left(\left(1-\frac{\omega^2}{\omega n^2}\right)^2+\frac{4\zeta^2\omega^2}{\omega n^2}\right)\omega n}$$

So we have

$$G(j\,\omega)=\alpha+\beta\,j \qquad \textbf{\textit{(Eq.1.1)}}$$

where α = large expression above
β = large expression above

```
G[ω I] == α + β I // Simplify
```

```
True
```

We can now simply calculate the magnitude associated with $G(j\,\omega)$ using

$$|\ G(j\,\omega)\ |=\left(\alpha^2+\beta^2\right)^{1/2}$$

```
magGjω1 = FullSimplify[(α^2 + β^2)^(1/2), ωn > 0]
```

$$\omega n^2\sqrt{\frac{1}{\omega^4+2\times\left(-1+2\zeta^2\right)\omega^2\,\omega n^2+\omega n^4}}$$

$$|\ G(j\,\omega)\ |=\omega_n^{\,2}\left(\frac{1}{\omega^4+2\times\left(2\zeta^2-1\right)\omega^2\,\omega_n^{\,2}+\omega_n^{\,4}}\right)^{\frac{1}{2}}$$

$$=\omega_n^{\,2}\left(\frac{1}{\omega_n^{\,4}\left(\frac{\omega^4}{\omega_n^{\,4}}+2\times\left(2\zeta^2-1\right)\frac{\omega^2}{\omega_n^{\,2}}+1\right)}\right)^{\frac{1}{2}}$$

$$=\left(\frac{1}{\frac{\omega^4}{\omega_n^{\,4}}+2\times\left(2\zeta^2-1\right)\frac{\omega^2}{\omega_n^{\,2}}+1}\right)^{\frac{1}{2}}$$

$$|\ G(j\,\omega)\ |=\left(\frac{1}{\left(\frac{\omega}{\omega_n}\right)^4+2\times\left(2\zeta^2-1\right)\left(\frac{\omega}{\omega_n}\right)^2+1}\right)^{\frac{1}{2}}$$

$$|\ G(j\,\omega)\ |=\left(\left(\frac{\omega}{\omega_n}\right)^4+2\times\left(2\zeta^2-1\right)\left(\frac{\omega}{\omega_n}\right)^2+1\right)^{-\frac{1}{2}} \qquad \textbf{\textit{(Eq.1.2)}}$$

```
magGjω = ((ω/ωn)^4 + 2 × (2 ξ^2 - 1) (ω/ωn)^2 + 1)^(-1/2)
```

$$\frac{1}{\sqrt{1+\frac{\omega^4}{\omega n^4}+\frac{2\times(-1+2\,\xi^2)\,\omega^2}{\omega n^2}}}$$

```
Simplify[magGjω1 == magGjω, {ωn > 0, ω > 0, ξ > 0}]
```

True

Recall that the magnitude plot of the Bode plot (the y-axis of the bode plot) is given by $20\log_{10}(\,|\,G(j\,\omega)\,|\,)$. Using Eq.1.2, we have

$$20\log_{10}(\,|\,G(j\,\omega)\,|\,)=20\log_{10}\left(\left(\left(\frac{\omega}{\omega_n}\right)^4+2\times\left(2\,\zeta^2-1\right)\left(\frac{\omega}{\omega_n}\right)^2+1\right)^{-\frac{1}{2}}\right)$$

$$20\log_{10}(\,|\,G(j\,\omega)\,|\,)=-10\log_{10}\left(\left(\frac{\omega}{\omega_n}\right)^4+2\times\left(2\,\zeta^2-1\right)\left(\frac{\omega}{\omega_n}\right)^2+1\right)$$ (y-axis value of the magnitude bode plot) ***(Eq.1.3)***

```
FullSimplify[
 20 Log10[magGjω] == -10 Log10[(ω/ωn)^4 + 2 × (2 ξ^2 - 1) (ω/ωn)^2 + 1], {ωn > 0, ω > 0, ξ > 0}]
```

True

And for the phase, the angle associated with $G(j\,\omega)$ is given by

$$\angle G(j\,\omega)=\tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

However, we will see that we can encounter problems later if we do not use the 4 quadrant inverse tangent, so we store this as

$$\angle G(j\,\omega)=\text{ArcTan}(x,\,y)=\text{ArcTan}(\alpha,\,\beta)$$

```
angleGjω = ArcTan[α, β]
```

$$\text{ArcTan}\left[\frac{1}{\left(1-\frac{\omega^2}{\omega n^2}\right)^2+\frac{4\,\xi^2\,\omega^2}{\omega n^2}}-\frac{\omega^2}{\left(\left(1-\frac{\omega^2}{\omega n^2}\right)^2+\frac{4\,\xi^2\,\omega^2}{\omega n^2}\right)\omega n^2},\,-\frac{2\,\xi\,\omega}{\left(\left(1-\frac{\omega^2}{\omega n^2}\right)^2+\frac{4\,\xi^2\,\omega^2}{\omega n^2}\right)\omega n}\right]$$

So we have

$$\angle G(j\,\omega)=\text{ArcTan}(x,\,y)=\text{ArcTan}\left(\frac{1}{\left(1-\frac{\omega^2}{\omega_n^{\,2}}\right)^2+\frac{4\,\zeta^2\,\omega^2}{\omega_n^{\,2}}}-\frac{\omega^2}{\left(\left(1-\frac{\omega^2}{\omega_n^{\,2}}\right)^2+\frac{4\,\zeta^2\,\omega^2}{\omega_n^{\,2}}\right)\omega n^2},\,-\frac{2\,\zeta\,\omega}{\left(\left(1-\frac{\omega^2}{\omega_n^{\,2}}\right)^2+\frac{4\,\zeta^2\,\omega^2}{\omega_n^{\,2}}\right)\omega_n}\right)$$ (y-axis value of the phase bode plot) ***(Eq.1.4)***

Small Frequencies ($\omega << \omega_n$)

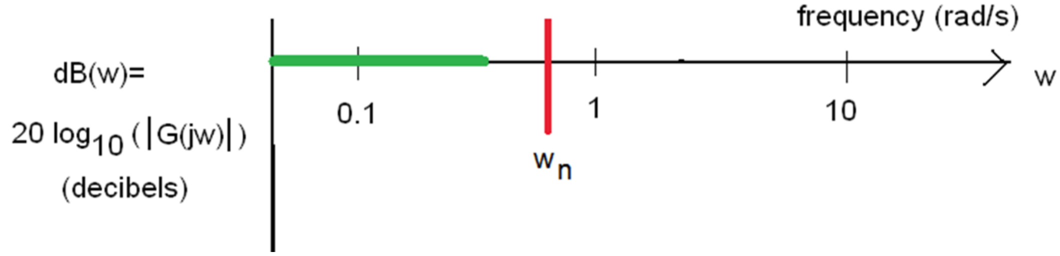
We now can now notice that for small frequencies ($\omega << \omega_n$), the term $(\omega/\omega_n)^4 \approx 0$ and $(\omega/\omega_n)^2 \approx 0$ so Eq.1.3 reduces to

$$20\log_{10}(\,|\,G(j\,\omega)\,|\,) = -10\log_{10}\left(\left(\frac{\omega}{\omega_n}\right)^4 + 2\times(2\zeta^2 - 1)\left(\frac{\omega}{\omega_n}\right)^2 + 1\right) \qquad \text{note: } (\omega/\omega_n)^4 \approx 0 \;\; (\omega/\omega_n)^2 \approx 0$$

$$\approx -10\log_{10}(1) \qquad \text{recall: } \log_{10}(1) = 0$$

$$20\log_{10}(\,|\,G(j\,\omega)\,|\,) \approx 0 \qquad (\text{for } \omega << \omega_n)$$

We can now sketch the magnitude portion of the bode plot for frequencies smaller than ω_n rad/s



What about the phase? Note that we can write the phase as (if we ignore the quadrant of the inverse tangent)

```
ArcTan[β / α] // Simplify
```

$$\text{ArcTan}\left[\frac{2\,\zeta\,\omega\,\omega n}{\omega^2 - \omega n^2}\right]$$

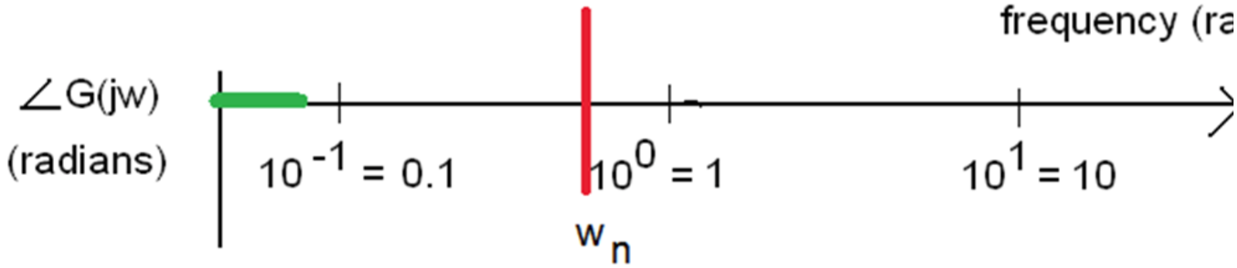
If $\omega << \omega_n$, the term $\omega\,\omega_n \approx 0$, so the above expression reduces to

$$\angle G(j\,\omega) = \tan^{-1}\left(\frac{2\,\zeta\,\omega\,\omega_n}{\omega^2 - \omega_n^2}\right) \qquad \text{note: } \omega\,\omega_n \approx 0$$

$$\approx \tan^{-1}(0)$$

$$\angle G(j\,\omega) \approx 0 \qquad \textbf{\textit{(Eq.1.6)}}$$

We can now sketch the phase portion of the bode plot for frequencies smaller than ω_n. Keep in mind the assumption that $\omega\,\omega_n \approx 0$ breaks down faster than the previous assumption of $(\omega/\omega_n)^2 \approx 0$, so we stop the approximation sooner in our phase plot



At Natural Frequency ($\omega = \omega_n$)

We can now investigate the bode plot at $\omega = \omega_n$. At this frequency, Eq.1.3 reduces to

$$20 \log_{10}(\,|\,G(\omega_n\,j)\,|\,) = -10 \log_{10}\left(\left(\frac{\omega_n}{\omega_n}\right)^4 + 2 \times \left(2\,\zeta^2 - 1\right)\left(\frac{\omega_n}{\omega_n}\right)^2 + 1\right)$$

$$= -10 \log_{10}\left(1 + 2 \times \left(2\,\zeta^2 - 1\right) + 1\right)$$

$$= -10 \log_{10}\left(1 + 4\,\zeta^2 - 2 + 1\right)$$

$$= -10 \log_{10}\left(4\,\zeta^2\right)$$

$$= -10 \log_{10}\left((2\,\zeta)^2\right)$$

$$20 \log_{10}(\,|\,G(\omega_n\,j)\,|\,) = -20 \log_{10}(2\,\zeta) \qquad \text{(at } \omega = \omega_n\text{)} \qquad \textbf{\textit{(Eq.1.7)}}$$

```
magGjωnDB = -20 Log10[2 ξ];
Simplify[(20 Log10[magGjω] /. {ω → ωn}) == magGjωnDB, {ξ > 0}]
```

```
True
```

This is an interesting result. It states that the magnitude (in dB) at the natural frequency is a function of the damping ratio. We can plot this for the range of $\zeta \in (0, 1)$. Recall that this analysis did not allow for the possibility of $\zeta = 0$ because then we would have a pole which is not strictly in the LHP and if we had $\zeta = 1$, we would actually have a real pole, which we have previously covered. We note that at these two extremes

$$-20 \log_{10}(2\,\zeta) = \begin{cases} \infty & \zeta = 0 \\ -20 \log_{10}(2) \approx -6.0206 & \zeta = 1 \end{cases}$$

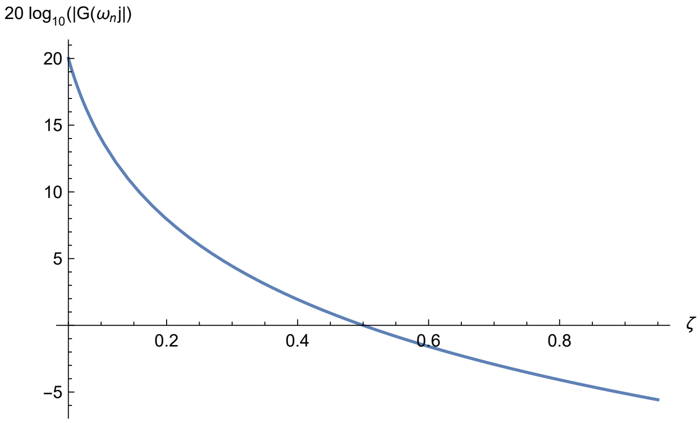
```
magGjωnDB /. {ξ → 0}
magGjωnDB /. {ξ → 1} // N
```

```
∞

-6.0206
```

We can plot this vs. ζ

```
Plot[magGjωnDB, {ζ, 0.05, 0.95}, PlotRange → All, AxesLabel → {ζ, "20 log10(|G(ωnj|)"}]
```

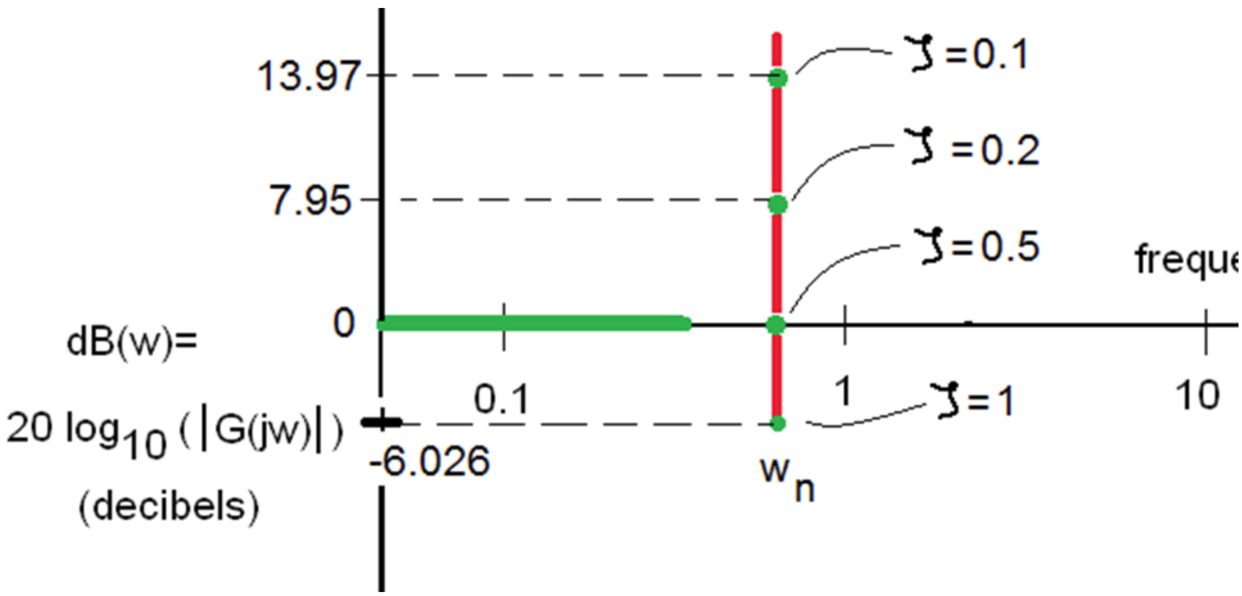


So we see, the magnitude at the natural frequency is larger if ζ is smaller, and vice versa.

```
Table[{"ζ=", ζ, " 20*log10(|G(j ωn)|)=", magGjωnDB, "dB"}, {ζ, 0.1, 0.9, 0.1}] //
 TableForm
```

ζ=	0.1	$20*\log_{10}(\lvert G(j\ \omega_n)\rvert)$ =	13.9794	dB
ζ=	0.2	$20*\log_{10}(\lvert G(j\ \omega_n)\rvert)$ =	7.9588	dB
ζ=	0.3	$20*\log_{10}(\lvert G(j\ \omega_n)\rvert)$ =	4.43697	dB
ζ=	0.4	$20*\log_{10}(\lvert G(j\ \omega_n)\rvert)$ =	1.9382	dB
ζ=	0.5	$20*\log_{10}(\lvert G(j\ \omega_n)\rvert)$ =	0.	dB
ζ=	0.6	$20*\log_{10}(\lvert G(j\ \omega_n)\rvert)$ =	−1.58362	dB
ζ=	0.7	$20*\log_{10}(\lvert G(j\ \omega_n)\rvert)$ =	−2.92256	dB
ζ=	0.8	$20*\log_{10}(\lvert G(j\ \omega_n)\rvert)$ =	−4.0824	dB
ζ=	0.9	$20*\log_{10}(\lvert G(j\ \omega_n)\rvert)$ =	−5.10545	dB

We can now update our magnitude portion of the bode plot with this single data point. Note that this is not the maximum value of magnitude plot (we will investigate this later).



Similarly, the phase at $\omega = \omega_n$ is simply

```
angleGjω /. {ω → ωn}
```

$\text{ArcTan}\left[0, -\frac{1}{2\zeta}\right]$

So we have

$$\angle G(j\,\omega_n) = \tan^{-1}(x,\ y)$$

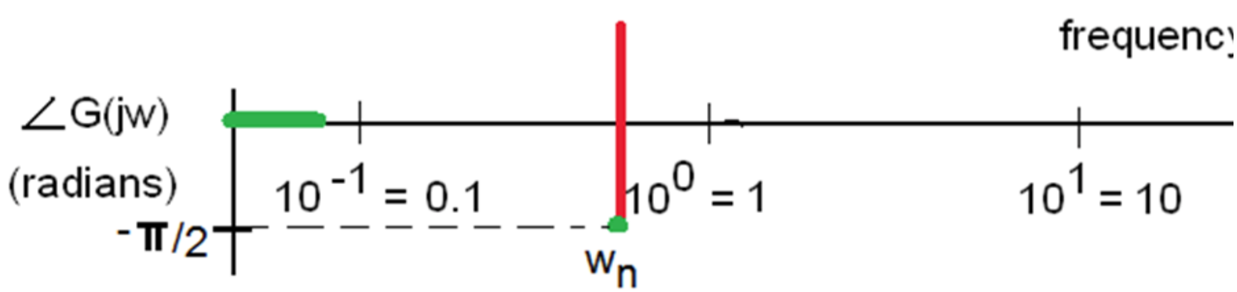
$$= \tan^{-1}\left(\frac{-1}{2\zeta}\Big/0\right)$$

$$= \tan^{-1}(-\infty)$$

$$\angle G(j\,\omega_n) = -\pi/2 = -90^o$$

Note that this is not a function of ζ and therefore, at $\omega = \omega_n$ the phase is exactly -90^o

We can draw this point on the phase plot



At Resonant Frequency ($\omega = \omega_r$)

As a side note, recall that we showed earlier that the maximum response from the system is actually at the resonant frequency, ω_r. Furthermore, recall that the resonant frequency was a real value only if $\zeta \in \left[0, 1\middle/\sqrt{2}\right]$.

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2} \qquad \textbf{(Eq.1.8)}$$

```
ωr = ωn Sqrt[1 - 2 ξ^2];
```

We note that for the most part, $\omega_r \neq \omega_n$ (unless $\zeta = 0$) which we precluded with an earlier discussion (as a side note, if $\zeta = 0$ and $\omega_n = \omega_r$ then the magnitude at $\omega_n = \omega_r$ would be infinite, so this is consistent with all of our previous analysis and showing that there is a single frequency of $\omega_n = \omega_r$ which will lead to an unbounded output given a bounded input).

We can now investigate the bode plot at $\omega = \omega_r$. At this frequency, Eq.1.3 reduces to (using Mathematica to skip the algebra)

```
magGjωrDB = Simplify[20 Log10[magGjω] /. {ω → ωr}, ξ > 0]
```

$$-\frac{10\,\mathrm{Log}\left[-4\,\xi^2\left(-1+\xi^2\right)\right]}{\mathrm{Log}[10]}$$

$$20\log_{10}(\,|\,G(\omega_r\,j)\,|\,) = -10\log_{10}(-4\,\zeta^2(\zeta^2-1)) \qquad (\text{at } \omega = \omega_r) \qquad \textbf{(Eq.1.9)}$$

```
Simplify[-10 Log10[-4 ξ^2 (ξ^2 - 1)] == magGjωrDB, {ξ > 0}]
```

```
True
```

This is an interesting result. It states that the magnitude (in dB) at the resonant frequency is a function of the damping ratio. We can plot this for the range of $\zeta \in \left[0, 1\middle/\sqrt{2}\right]$. Recall that this analysis did not allow for the possibility of $\zeta = 0$ because then we would have a pole which is not strictly in the LHP and if we had $\zeta \geq 1\big/\sqrt{2}$, the magnitude plot would not have a “maximum bump” and would instead be monotonically decreasing. We note that at these two extremes

$$-10\log_{10}(-4\,\zeta^2(\zeta^2-1)) = \begin{cases} \infty & \zeta = 0 \\ 0 & \zeta = 1\big/\sqrt{2} \end{cases}$$

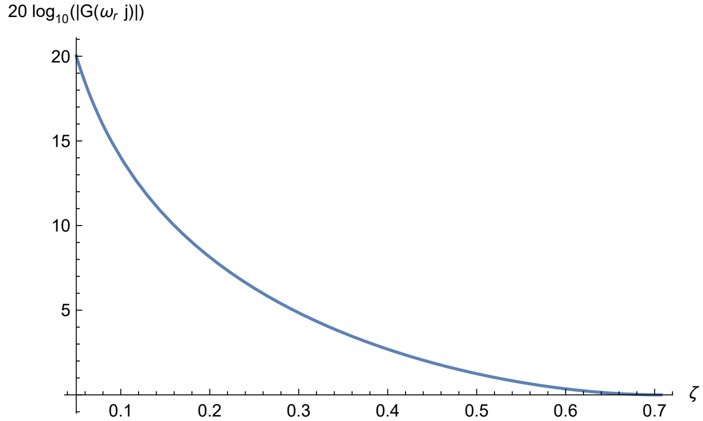
```
magGjωrDB /. {ξ → 0}
magGjωrDB /. {ξ → 1/Sqrt[2]}
```

```
∞
```

```
0
```

We can plot $20\log_{10}(\,|\,G(\omega_r\,j)\,|\,)$ for various values of ζ in the range of $[0,1/\sqrt{2}\,]$.

```
Plot[magGjωrDB, {ξ, 0.05, 1/ √2},
 PlotRange → All, AxesLabel → {"ξ", "20 log10(|G(ωr j)|)"}]
```

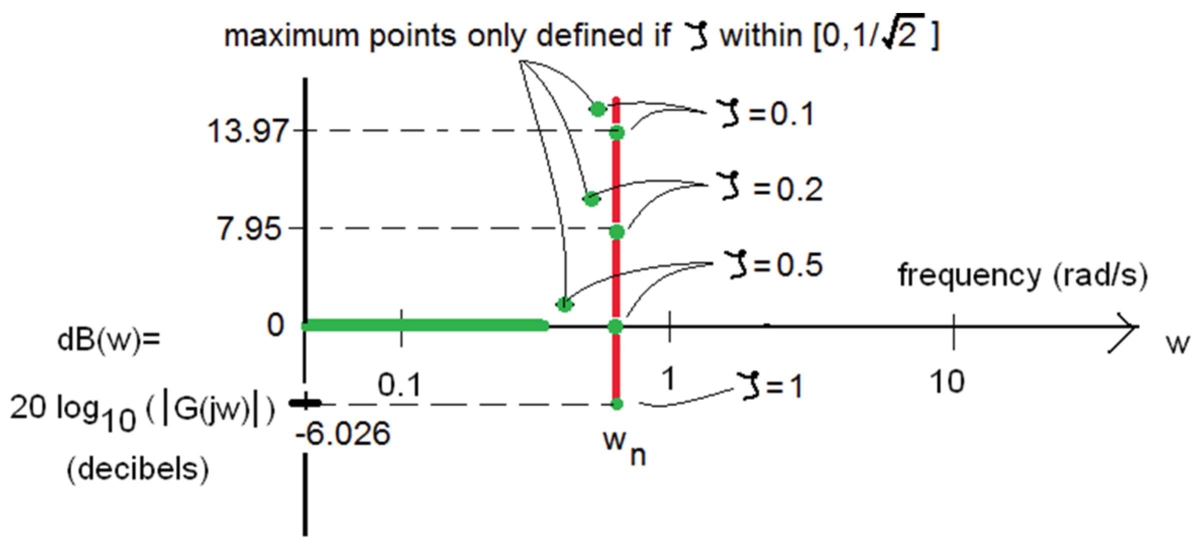


So we see, the magnitude at the resonant frequency is larger if ζ is smaller, and vice versa.

```
Table[{"ξ=", ξ, " 20*log10(|G(j ωr)|)=", magGjωrDB, "dB"},
  {ξ, 0.1, 1/√2, 0.1}] // TableForm
```

ξ=	0.1	20*log10(|G(j ωr)|)=	14.023	dB
ξ=	0.2	20*log10(|G(j ωr)|)=	8.13609	dB
ξ=	0.3	20*log10(|G(j ωr)|)=	4.84656	dB
ξ=	0.4	20*log10(|G(j ωr)|)=	2.69541	dB
ξ=	0.5	20*log10(|G(j ωr)|)=	1.24939	dB
ξ=	0.6	20*log10(|G(j ωr)|)=	0.354575	dB
ξ=	0.7	20*log10(|G(j ωr)|)=	0.00173753	dB

We can now update our magnitude portion of the bode plot with this single data point which shows the actual maximum response of the system if $\zeta \in \left[0, 1/\sqrt{2}\right]$



We are not concerned with the phase at $\omega = \omega_r$

Large Frequencies ($\omega >> \omega_n$)

At large frequencies where $\omega >> \omega_n$, the term $(\omega/\omega_n)^4$ dominates the argument of the $\log_{10}$ function. In other words, Eq.1.3 becomes

$$20\log_{10}(\,|\,G(j\,\omega)\,|\,) = -10\log_{10}\left(\left(\frac{\omega}{\omega_n}\right)^4 + 2\times(2\,\zeta^2 - 1)\left(\frac{\omega}{\omega_n}\right)^2 + 1\right)$$ note: $\left(\frac{\omega}{\omega_n}\right)^4 + 2\times(2\,\zeta^2 - 1)\left(\frac{\omega}{\omega_n}\right)^2 + 1 \approx \left(\frac{\omega}{\omega_n}\right)^4$

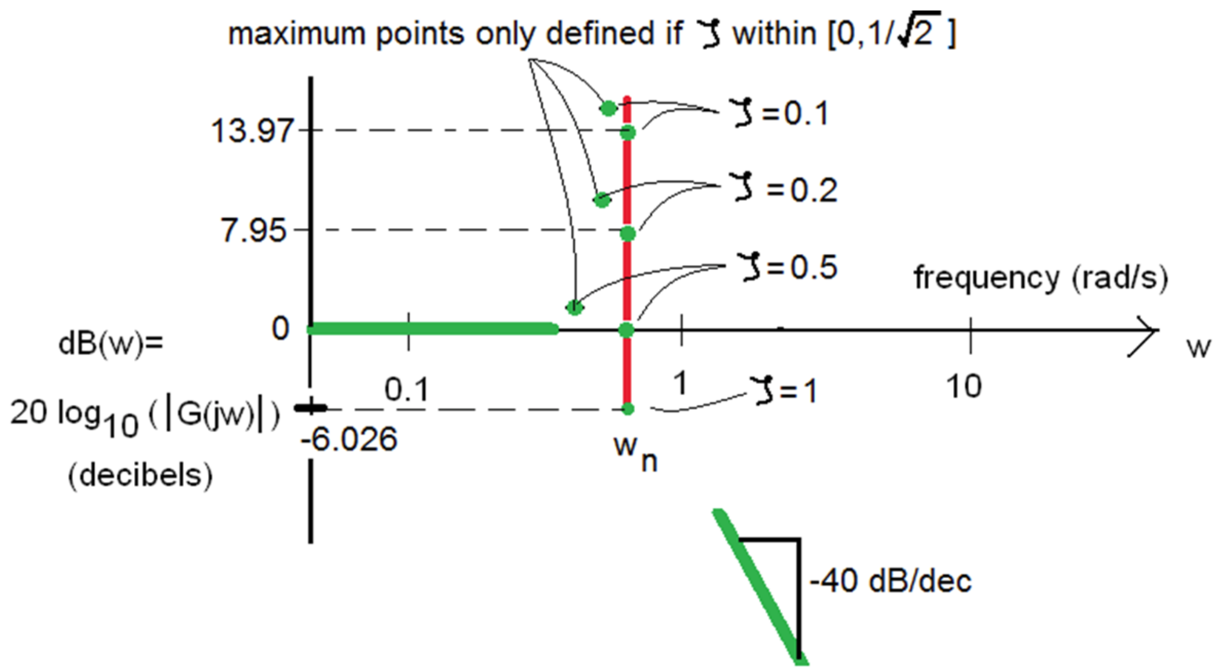
$$\approx -10\log_{10}\left(\left(\frac{\omega}{\omega_n}\right)^4\right)$$

$$\approx -40\log_{10}(\omega/\omega_n)$$ recall: $\log_b(x/y) = \log_b(x) - \log_b(y)$

$$\approx -40\,(\log_{10}(\omega) - \log_{10}(\omega_n))$$

$$20\log_{10}(\,|\,G(j\,\omega)\,|\,) \approx -40\log_{10}(\omega) + 40\log_{10}(\,\omega_n)$$

Note that since the bode plot x-axis is $\log_{10}(\omega)$ and the y-axis is $20\log_{10}(\,|\,G(j\,\omega)\,|\,)$, the above expression describes a line with slope of -40 dB (it looks like $y = m\,x + b$ where $m = -40$). The line decreases by 40 dB for every decade increase in frequency (a decade being an order of magnitude). So we can sketch the magnitude plot as



What about the phase? Recall that the phase angle was given by

$$\angle G(j\,\omega) = \text{atan2}\left(\frac{1}{\left(1-\frac{\omega^2}{\omega_n{}^2}\right)^2+\frac{4\,\zeta^2\,\omega^2}{\omega_n{}^2}} - \frac{\omega^2}{\left(\left(1-\frac{\omega^2}{\omega_n{}^2}\right)^2+\frac{4\,\zeta^2\,\omega^2}{\omega_n{}^2}\right)\omega n^2}, -\frac{2\,\zeta\,\omega}{\left(\left(1-\frac{\omega^2}{\omega_n{}^2}\right)^2+\frac{4\,\zeta^2\,\omega^2}{\omega_n{}^2}\right)\omega_n}\right)$$

Due to the nature of the 4 quadrant inverse tangent, it becomes difficult and tricky to compute limits exactly. One reason is that the inverse tangent has discontinuities around the values we are looking at (jumps from $-\pi$ to π). Therefore, a safer but somewhat less satisfying approach would be to simply use some value to determine the behavior at large ω.

Let's look at the angle at ω =10,000 rad/s with ω_n = 10 and see if the angle depends on ζ

```
Table[{"ξ=", ξ, " ∠G(jω)=", angleGjω * 180 / π /. {ωn → 10, ω → 10000}, "deg"},
  {ξ, 0.1, 0.9, 0.1}] // TableForm
```

ξ=	0.1	∠G(jω)=	−179.989	deg
ξ=	0.2	∠G(jω)=	−179.977	deg
ξ=	0.3	∠G(jω)=	−179.966	deg
ξ=	0.4	∠G(jω)=	−179.954	deg
ξ=	0.5	∠G(jω)=	−179.943	deg
ξ=	0.6	∠G(jω)=	−179.931	deg
ξ=	0.7	∠G(jω)=	−179.92	deg
ξ=	0.8	∠G(jω)=	−179.908	deg
ξ=	0.9	∠G(jω)=	−179.897	deg

It appears that the final value does not depend on ζ.

We can also look as we vary ω_n. Let's look at the angle at ω =10,000 rad/s with ζ = 0.1 and see if the angle depends on ω_n

```
Table[{"ωn=", ωn, " ∠G(jω)=", angleGjω * 180 / π /. {ξ → 0.1, ω → 10000}, "deg"},
  {ωn, 10, 1000, 100}] // TableForm
```

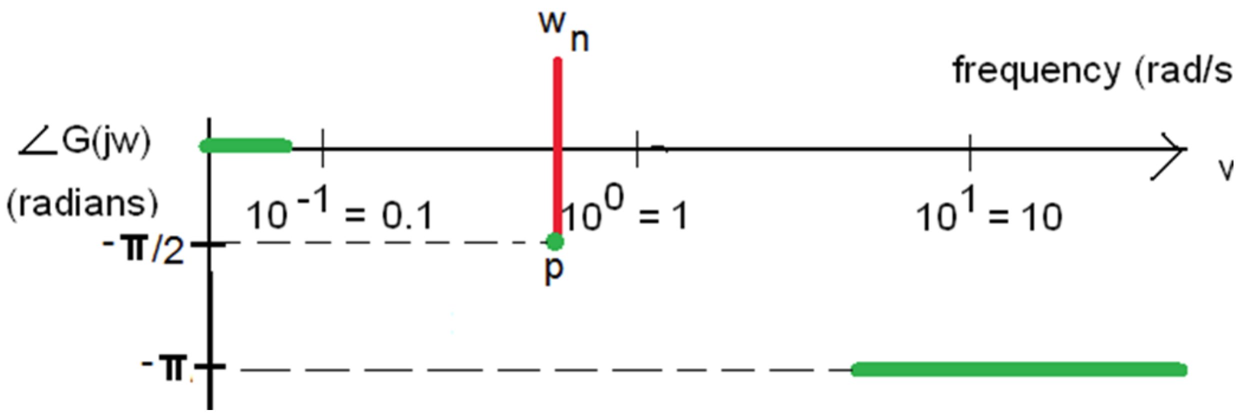
ωn=	10	∠G(jω)=	−179.989	deg
ωn=	110	∠G(jω)=	−179.874	deg
ωn=	210	∠G(jω)=	−179.759	deg
ωn=	310	∠G(jω)=	−179.644	deg
ωn=	410	∠G(jω)=	−179.529	deg
ωn=	510	∠G(jω)=	−179.414	deg
ωn=	610	∠G(jω)=	−179.298	deg
ωn=	710	∠G(jω)=	−179.182	deg
ωn=	810	∠G(jω)=	−179.066	deg
ωn=	910	∠G(jω)=	−178.949	deg

So we see that once again, if $\omega >> \omega_n$, the final angle does not depend on ω_n.

So Eq.1.4 reduces to

$$\angle G(j\,\omega) \approx -\pi \qquad (\text{for } \omega >> \omega_n)$$

We can add this information to our sketch for the phase



Approximate Plot for Pair of Complex Conjugate Poles

With all of this information, we can generate a few rules of how to sketch a bode plot of a pair of complex conjugate poles

Magnitude Plot:

1. Remains at 0 dB when $\omega << \omega_n$

2. Compute magnitude (in dB) at $\omega = \omega_n$ using

   $$20 \log_{10}(\,|\,G(\omega_n\, j)\,|\,) = -20 \log_{10}(2\,\zeta)$$

3. Compute magnitude (in dB) at $\omega = \omega_r$ using

   If $\zeta \in \left[0, 1 / \sqrt{2}\right]$

   Compute ω_r using

   $$\omega_r = \omega_n \sqrt{1 - 2\,\zeta^2}$$

   Compute magnitude (in dB) at $\omega = \omega_r$ using

   $$20 \log_{10}(\,|\,G(\omega_r\, j)\,|\,) = -10 \log_{10}\left(-4\,\zeta^2(\zeta^2 - 1)\right)$$

   else

   magnitude plot is monotonically decreasing

   end

4. Decreases at slope of -40 dB/dec when $\omega >> \omega_n$

Phase Plot:

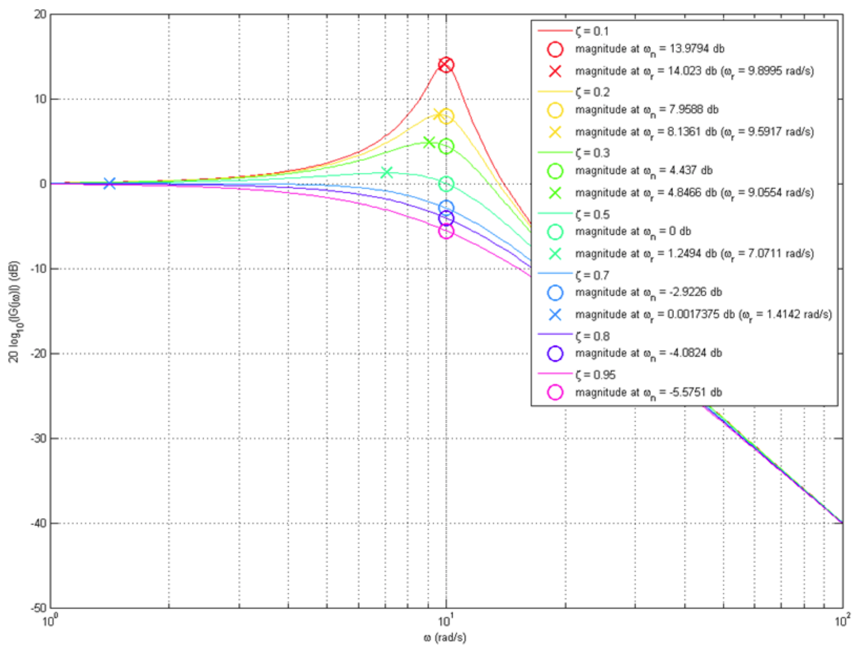
1. Remains at 0 when $\omega << \omega_n$

2. $\angle G(j\,\omega_n) = -\pi/2 = -90^o$

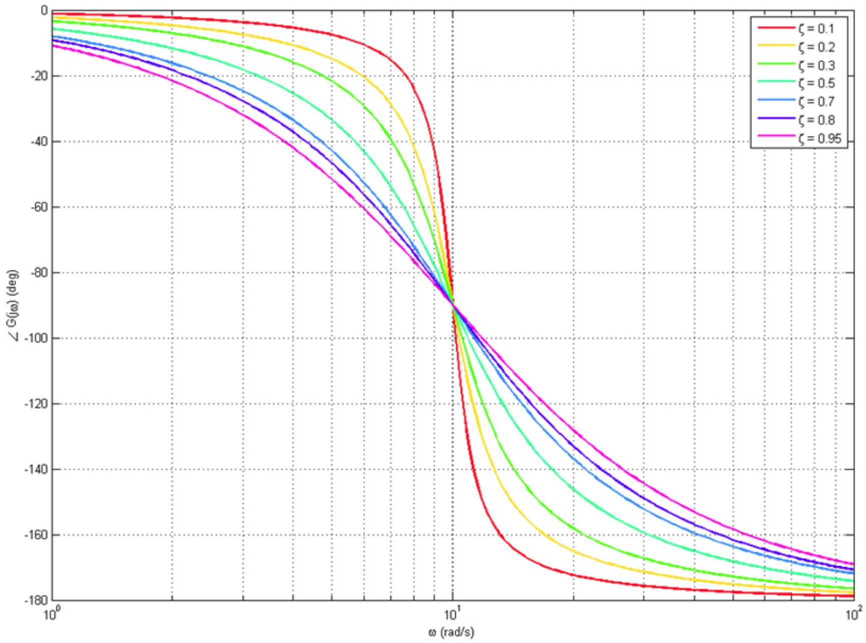
3. Remains at $-\pi = -180^o$ when $\omega >> \omega_n$

4. Damping ratio determines “aggressiveness" of the cutoff from 0 to -180^o.

The magnitude plot with $\omega_n = 10$ is shown below



The phase plot with $\omega_n = 10$ is shown below




```
Clear[magGjωrDB, ωr, magGjωnDB, angleGjω, magGjω, magGjω1, α, β, G]
```

Pair of Complex Conjugate Zeros

This is the reflection of a pair of complex poles (the analysis is the similar to that shown previously which illustrated the duality between a real pole and real zero).

To convince ourselves of this, let us compute $G(j\,\omega)$

Consider a transfer function of the form (notice that the DC gain is unity)

$$G(s) = \frac{s^2 + 2\,\zeta\,\omega_n\,s + \omega_n^2}{\omega_n^2} = \frac{1}{\omega_n^2}\,s^2 + 2\,\frac{\zeta}{\omega_n}\,s + 1$$

```
G[s_] = 1/ωn^2 s^2 + 2 ξ/ωn s + 1;
Solve[G[s] == 0, s]
```

$$\left\{\left\{s \to -\xi\,\omega n - \sqrt{-1 + \xi^2}\,\omega n\right\}, \left\{s \to -\xi\,\omega n + \sqrt{-1 + \xi^2}\,\omega n\right\}\right\}$$

We can notice several things about this transfer function

-pair of complex zeros at $s = -\zeta\,\omega_n \pm \omega_n\,\sqrt{1 - \zeta^2}\,j$

-DC gain = 1

-constant term in numerator is 1

Because we are interested in the frequency response of the system, we should compute $G(j\,\omega)$. We can use Mathematica to skip the algebra

```
ComplexExpand[G[I ω]]
```

$$1-\frac{\omega^2}{\omega n^2}+\frac{2\,i\,\zeta\,\omega}{\omega n}$$

We can now extract α (the real part) and β (the imaginary part)

```
α = ComplexExpand[Re[G[I ω]]]
β = ComplexExpand[Im[G[I ω]]]
```

$$1-\frac{\omega^2}{\omega n^2}$$

$$\frac{2\,\zeta\,\omega}{\omega n}$$

So we have

$$G(j\,\omega)=\alpha+\beta\,j \qquad \textit{\textbf{(Eq.1.1)}}$$

where $\alpha=1-\left(\frac{\omega}{\omega_n}\right)^2$

$\beta=2\,\zeta\,\frac{\omega}{\omega_n}$

```
G[ω I] == α + β I // Simplify
True
```

We can now simply calculate the magnitude associated with $G(j\,\omega)$ using

$$|\;G(j\,\omega)\;|=\left(\alpha^2+\beta^2\right)^{1/2}$$

```
magGjω1 = FullSimplify[(α² + β²)^(1/2), {ωn > 0, ω > 0, ζ > 0}]
```

$$\sqrt{\left(-1+\frac{\omega^2}{\omega n^2}\right)^2+\frac{4\,\zeta^2\,\omega^2}{\omega n^2}}$$

$$\begin{aligned}
|\;G(j\,\omega)\;| &= \left(\left(\left(\frac{\omega}{\omega_n}\right)^2-1\right)^2+\left(\frac{2\,\zeta\,\omega}{\omega_n}\right)^2\right)^{1/2}\\
&= \left(\left(\left(\frac{\omega}{\omega_n}\right)^2-1\right)\left(\left(\frac{\omega}{\omega_n}\right)^2-1\right)+\left(\frac{2\,\zeta\,\omega}{\omega_n}\right)^2\right)^{1/2}\\
&= \left(\left(\frac{\omega}{\omega_n}\right)^2\left(\frac{\omega}{\omega_n}\right)^2-2\left(\frac{\omega}{\omega_n}\right)^2+1+\left(\frac{2\,\zeta\,\omega}{\omega_n}\right)^2\right)^{1/2}\\
&= \left(\left(\frac{\omega}{\omega_n}\right)^4-2\left(\frac{\omega}{\omega_n}\right)^2+1+4\,\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right)^{1/2}\\
&= \left(\left(\frac{\omega}{\omega_n}\right)^4+4\,\zeta^2\left(\frac{\omega}{\omega_n}\right)^2-2\left(\frac{\omega}{\omega_n}\right)^2+1\right)^{1/2}
\end{aligned}$$

$$= \left(\left(\frac{\omega}{\omega_n} \right)^4 + \left(4\,\zeta^2\; - 2\right) \left(\frac{\omega}{\omega_n} \right)^2 + 1 \right)^{1/2}$$

$$|\; G(j\,\omega)\;| = \left(\left(\frac{\omega}{\omega_n} \right)^4 + 2 \times \left(2\,\zeta^2\; - 1\right) \left(\frac{\omega}{\omega_n} \right)^2 + 1 \right)^{1/2} \qquad \textbf{\textit{(Eq.1.2)}}$$

```
magGjω = ((ω/ωn)^4 + 2 × (2 ξ^2 - 1) (ω/ωn)^2 + 1)^(1/2)
```

$$\sqrt{1 + \frac{\omega^4}{\omega \mathrm{n}^4} + \frac{2 \times \left(-1 + 2\,\xi^2\right)\,\omega^2}{\omega \mathrm{n}^2}}$$

```
Simplify[magGjω1 == magGjω, {ωn > 0, ω > 0, ξ > 0}]
```

True

Recall that the magnitude plot of the Bode plot (the y-axis of the bode plot) is given by $20\,\log_{10}(\;|\;G(j\,\omega)\;|\;)$. Using Eq.1.2, we have

$$20\,\log_{10}(\;|\;G(j\,\omega)\;|\;) = 20\,\log_{10}\left(\left(\left(\frac{\omega}{\omega_n} \right)^4 + 2 \times \left(2\,\zeta^2\; - 1\right) \left(\frac{\omega}{\omega_n} \right)^2 + 1 \right)^{1/2}\right)$$

$$20\,\log_{10}(\;|\;G(j\,\omega)\;|\;) = 10\,\log_{10}\left(\left(\left(\frac{\omega}{\omega_n} \right)^4 + 2 \times \left(2\,\zeta^2\; - 1\right) \left(\frac{\omega}{\omega_n} \right)^2 + 1 \right)\right)$$ (y-axis value of the magnitude bode plot) ***(Eq.1.3)***

```
FullSimplify[
 20 Log10[magGjω] == 10 Log10[(ω/ωn)^4 + 2 × (2 ξ^2 - 1) (ω/ωn)^2 + 1], {ωn > 0, ω > 0, ξ > 0}]
```

True

We can compare this with the magnitude expression for the pair of complex poles

$$20\,\log_{10}(\;|\;G(j\,\omega)\;|\;) = 10\,\log_{10}\left(\left(\left(\frac{\omega}{\omega_n} \right)^4 + 2 \times \left(2\,\zeta^2\; - 1\right) \left(\frac{\omega}{\omega_n} \right)^2 + 1 \right)\right) \qquad \text{(pair of complex zeros)}$$

$$20\,\log_{10}(\;|\;G(j\,\omega)\;|\;) = -10\,\log_{10}\left(\left(\frac{\omega}{\omega_n} \right)^4 + 2 \times \left(2\,\zeta^2 - 1\right) \left(\frac{\omega}{\omega_n} \right)^2 + 1 \right) \qquad \text{(pair of complex poles)}$$

We see that they are identical, except for the sign. Therefore, the pair of complex zeros has the same behavior as the pair of complex poles except it is flipped/reflected over the x-axis.

We perform a similar analysis on the phase. Ignoring the quadrant, we have

$$\angle \mathrm{G}(j\,\omega) = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$$

```
ArcTan[β / α] // Simplify
```

$$-\text{ArcTan}\left[\frac{2\,\zeta\,\omega\,\omega n}{\omega^2-\omega n^2}\right]$$

Again, comparing this with the result from the pair of complex poles, we have

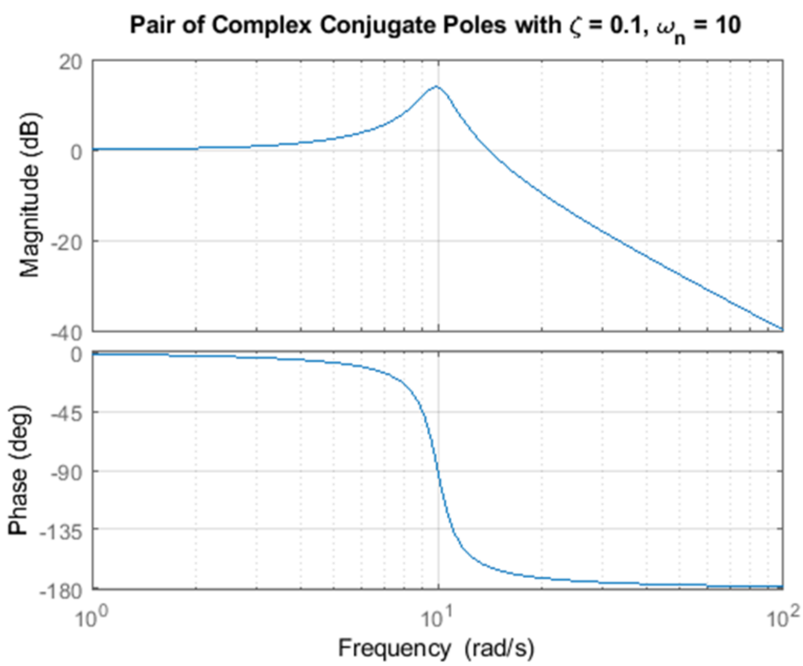
$$\angle G(j\,\omega) = -\tan^{-1}\left(\frac{2\,\zeta\,\omega\,\omega_n}{\omega^2-\omega_n^{\,2}}\right) \qquad \text{(pair of complex zeros)}$$

$$\angle G(j\,\omega) = \tan^{-1}\left(\frac{2\,\zeta\,\omega\,\omega_n}{\omega^2-\omega_n^{\,2}}\right) \qquad \text{(pair of complex poles)}$$

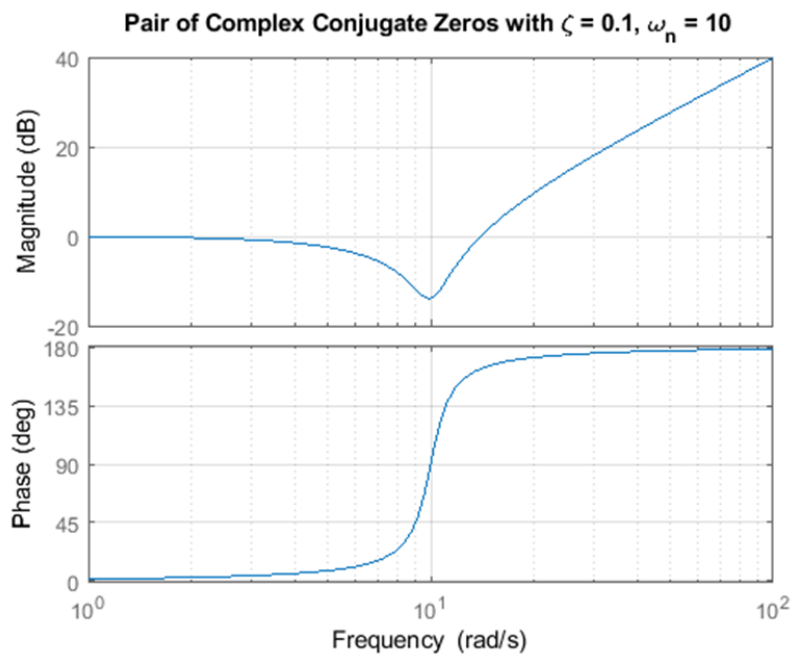
We see that they are identical, except for the sign. Therefore, the pair of complex zeros has the same behavior as the pair of complex poles except it is flipped/reflected over the x-axis.

Example

Complex poles with $\omega_n = 10$ and $\zeta = 0.1$



Complex zeros with $\omega_n = 10$ and $\zeta = 0.1$



Constant Gain

The last component we consider is a constant gain.

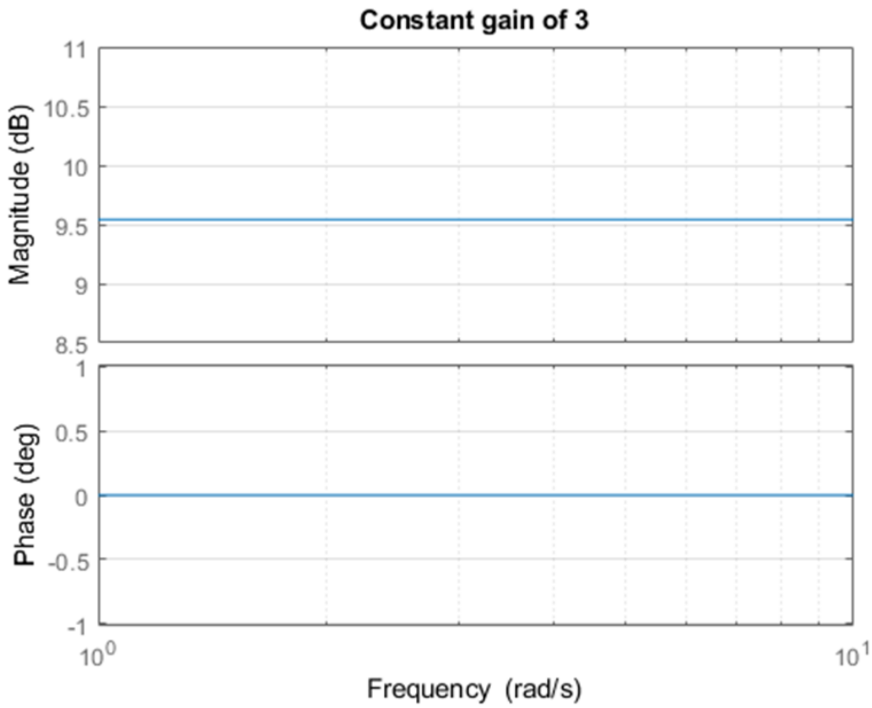
$$G(s) = K$$

This is actually the simplest component to analyze because we see that it is not a function of s. Therefore

$$20 \log_{10}(\,|\,G(j\,\omega)\,|\,) = 20 \log_{10}(K)$$
$$\angle G(j\,\omega) = 0$$

So the bode plot for a constant gain is simply straight line at $20 \log_{10}(K)$ for the magnitude and 0^o for the phase.

An example for a gain of $K = 3$ is shown below.



Summary

Recall that all previous analysis assumed that the transfer functions/components were written in standard bode plot format. Namely that the DC gain was 1 and that the constant in the num/den was 1

$G(s) = \frac{1}{s/p+1}$ (single real pole)

$G(s) = s/z + 1$ (single real zero)

$G(s) = \frac{1}{s}$ (single pole at origin)

$G(s) = s$ (Single zero at origin)

$G(s) = \frac{1}{\frac{1}{\omega_n^2} s^2 + 2\frac{\zeta}{\omega_n} s + 1}$ (Pair of complex conjugate poles)

$G(s) = \frac{1}{\omega_n^2} s^2 + 2\frac{\zeta}{\omega_n} s + 1$ (Pair of complex conjugate zeros)

$G(s) = K$ (Constant gain)

General trends we saw were

each pole breaks down at -20 dB/dec starting at the pole location or the natural frequency
each zero breaks up at +20 dB/dec starting at the pole location or the natural frequency

each pole will eventually introduce -90 degrees phase shift
each zero will eventually introduce +90 degrees phase shift

We will apply these techniques to look at how to sketch the bode plot for a more complex transfer function, such as

$$G(s) = \frac{30\,(s+3)\left(s^2+2\,s+25\right)}{s(s+5)}$$