Christopher Lum
lum@uw.edu

# Lecture01f
# Row/Column Space, Linear Independence, and Rank of a Matrix

**Lecture is on YouTube**

The YouTube video entitled ‘Row/Column Space, Linear Independence, and Rank of a Matrix’ that covers this lecture is located at https://youtu.be/eIv8muz9Hsk.

---

## Introduction

Consider a matrix with *m* rows and *n* columns

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} - & \bar{r}_1 & - \\ - & \bar{r}_2 & - \\ - & \vdots & - \\ - & \bar{r}_m & - \end{pmatrix} = \begin{pmatrix} | & | & | & | \\ \bar{c}_1 & \bar{c}_2 & \dots & \bar{c}_n \\ | & | & | & | \end{pmatrix}$$

---

## Row Space of a Matrix

The row space of a matrix $A$ is the span of all its row vectors. In other words, this is the set of all possible linear combinations of the rows $\bar{r}_1$, $\bar{r}_2$, ..., $\bar{r}_m$.

The row space can be useful if we consider equations of the form

$$\bar{b} = \bar{x} A$$

$$\bar{b} = \begin{pmatrix} x_1 & x_2 & \dots & x_m \end{pmatrix} \begin{pmatrix} \bar{r}_1 \\ \bar{r}_2 \\ \vdots \\ \bar{r}_m \end{pmatrix}$$

$$\bar{b} = x_1 \bar{r}_1 + x_2 \bar{r}_2 + \dots + x_m \bar{r}_m$$

If we consider $x_1$, $x_2$, ..., $x_m$ to be scalars, we see that another way to interpret the previous equation is to see that $\bar{b}$ is comprised of a linear combination of the vectors $\bar{r}_1$, $\bar{r}_2$,..., $\bar{r}_m$. So we see that $\bar{b}$ must be able to be created by linearly combining $\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_m$ together in order for us to find a solution. We say that $\bar{b}$ must lie in the row space of $A$ in order for there to be a solution to this equation.

As we saw in our previous lecture entitled ‘Elementary Row Operations, Row Echelon Form, and Reduced Row Echelon Form’, we saw the elementary row operations do not change the fundamental nature of the rows of the matrix, and therefore one method of finding the row space of the matrix is to simply place the matrix in row echelon form or reduced row echelon form.

Consider the matrix

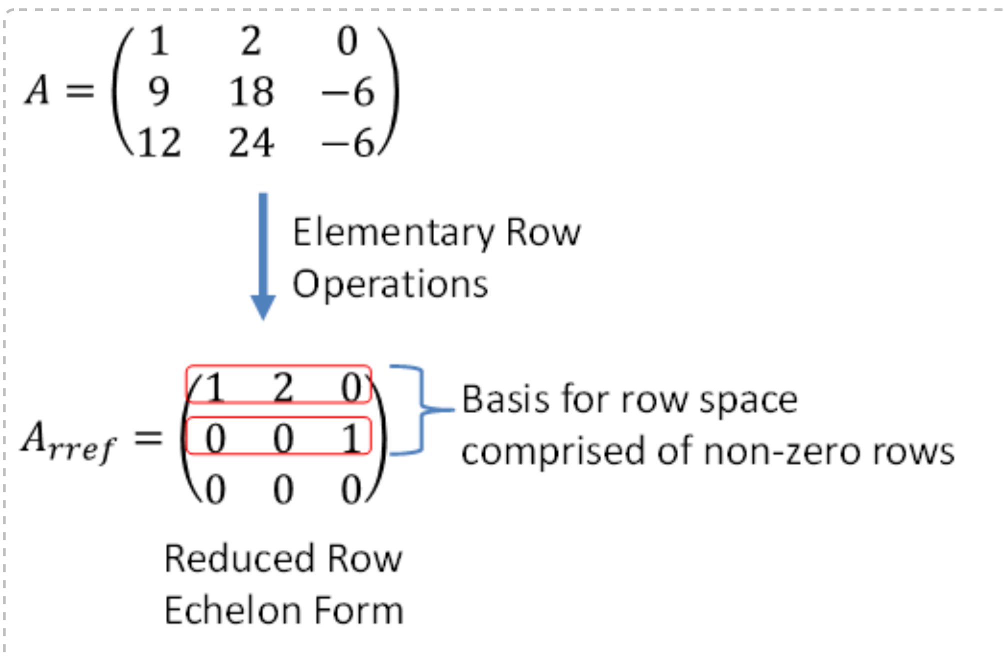
$$A = \begin{pmatrix} 1 & 2 & 0 \\ 9 & 18 & -6 \\ 12 & 24 & -6 \end{pmatrix}$$

In[ ]:=
$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ 9 & 18 & -6 \\ 12 & 24 & -6 \end{pmatrix};$$

We can place this in reduced row echelon form

In[ ]:=
```
Arref = RowReduce[A];
Arref // MatrixForm
```

Out[ ]//MatrixForm=
$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

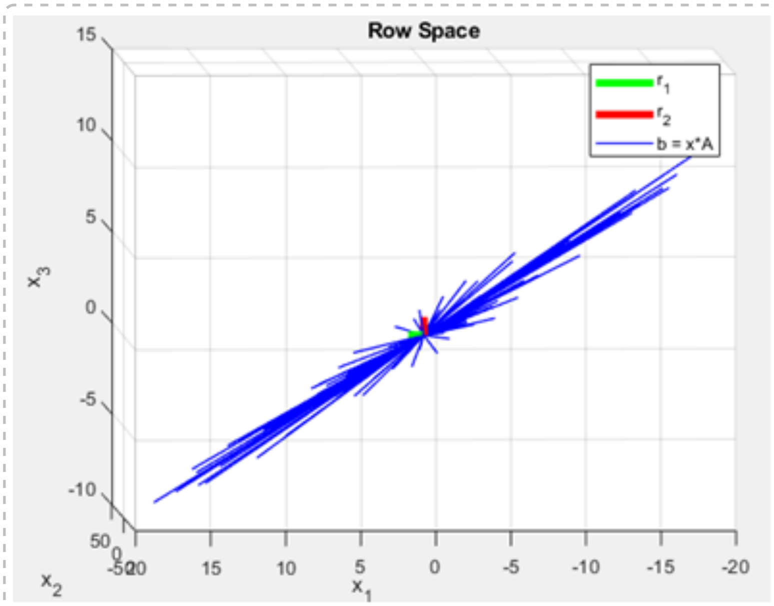



So we see that the row space of the matrix is spanned by the two non-zero row vectors of the reduced matrix.

row space of $A$ = rowsp($A$) = span($\bar{r}_1$, $\bar{r}_2$) **(Eq.1a)**

where $\bar{r}_1 = (1 \quad 2 \quad 0)$
$\bar{r}_2 = (0 \quad 0 \quad 1)$

We can visualize this row space by repeatedly computing $\bar{b} = \bar{x}\,A$ for random $\bar{x}$ values and then seeing how this compares with $\bar{r}_1$ and $\bar{r}_2$ (go to Matlab)




# Column Space of a Matrix

The column space of a matrix $A$ is the span of all its column vectors. In other words, this is the set of all possible linear combinations of the columns $\bar{c}_1$, $\bar{c}_2$, ..., $\bar{c}_n$.

The column space can be useful if we consider equations of the form

$$\bar{b} = A\,\bar{x}$$

$$\bar{b} = (\bar{c}_1 \quad \bar{c}_2 \quad \ldots \quad \bar{c}_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

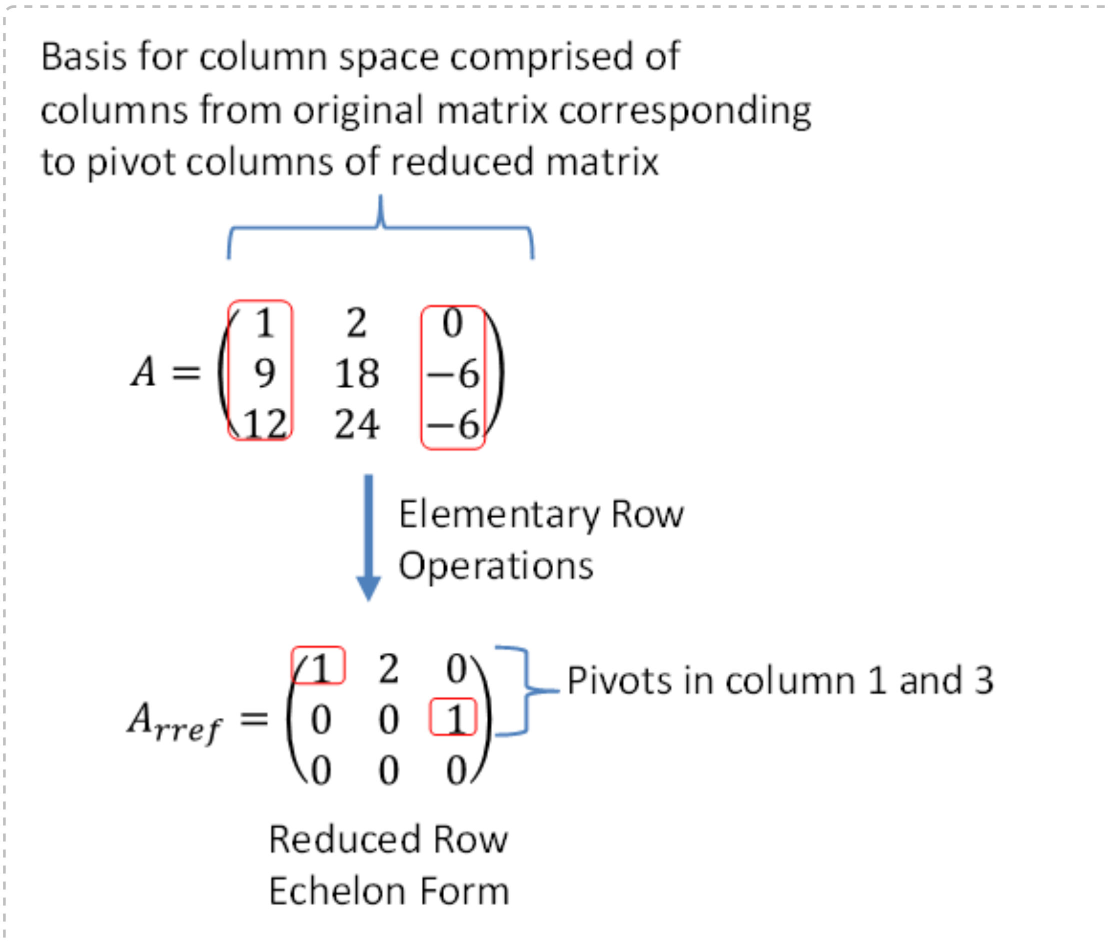
$$\bar{b} = x_1\,\bar{c}_1 + x_2\,\bar{c}_2 + \ldots + x_n\,\bar{c}_n$$

If we consider $x_1$, $x_2$, ..., $x_n$ to be scalars, we see that another way to interpret the previous equation is to see that $\bar{b}$ is comprised of a linear combination of the vectors $\bar{c}_1$, $\bar{c}_2$,..., $\bar{c}_n$. So we see that $\bar{b}$ must be able to be created by linearly combining $\bar{c}_1$, $\bar{c}_2$,..., $\bar{c}_n$ together in order for us to find a solution. We say that $\bar{b}$ must lie in the column space of $A$ in order for there to be a solution to this equation.

Row operations does change the column space of the matrix but we still use the reduced row echelon form of the matrix to find the column space.

1. Place the matrix in row echelon or reduced row echelon form.
2. Identify which columns have pivots in the reduced matrix.
3. The corresponding columns of the original matrix form a basis for the column space of the matrix.

For the matrix and results from above we have




So we see that the column space of the matrix is spanned by the two columns of the original matrix that correspond to pivot columns in the reduced matrix.

column space of $A$ = colsp($A$) = span($\overline{c}_1$, $\overline{c}_3$) **(Eq.2a)**

where $\overline{c}_1 = \begin{pmatrix} 1 \\ 9 \\ 12 \end{pmatrix}$ $\quad \overline{c}_3 = \begin{pmatrix} 0 \\ -6 \\ -6 \end{pmatrix}$

Recall that we obtained a physical intuition of the meaning of colsp($A$) by looking at the equation $\overline{b} = A\,\overline{x}$. Suppose we take the transpose of this equation

$$\overline{b}^T = (A\,\overline{x})^T$$

$$\overline{b}^T = \overline{x}^T A^T$$

This is the same equation which we examined when formulating the row space of a matrix.

As such, we note that we can also compute the column space of $A$ by computing the row space of $A^T$

$$\text{colsp}(A) = \text{rowsp}\left(A^T\right)$$

In[∘]:=
```
AT = Transpose[A];
Print["Aᵀ"]
AT // MatrixForm

Print["RowReduce[AT]"]
RowReduce[AT] // MatrixForm
```

$A^T$

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 9 & 12 \\ 2 & 18 & 24 \\ 0 & -6 & -6 \end{pmatrix}$$

RowReduce[AT]

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

So we have an alternate (but equivalent) representation of the column space as the nonzero rows of the row reduced matrix above

$$\text{column space of } A = \text{colsp}(A) = \text{span}\left(\overline{d}_1, \overline{d}_2\right) \qquad \textbf{\textit{(Eq.2a)}}$$

where $\overline{d}_1 = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \qquad \overline{d}_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

By symmetrical analysis, we can also compute the row space of $A$ by computing the column space of $A^T$

$$\text{rowsp}(A) = \text{colsp}\left(A^T\right)$$

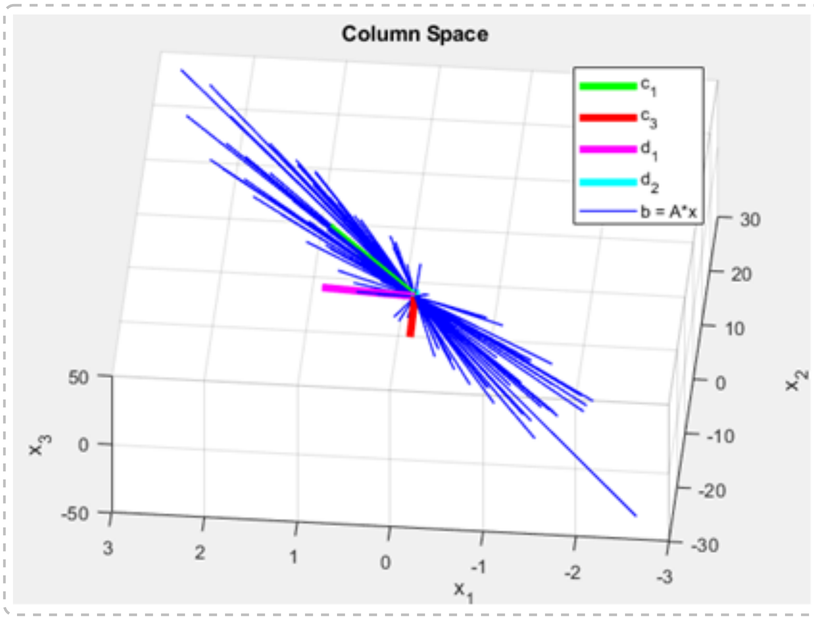
Applied to this system we see from the above result of $\text{RowReduce}\left(A^T\right)$ that there are pivots in columns 1 and 2 so the column space of $A^T$ is columns 1 and 2 of the original $A^T$ matrix. This is the row space of $A$.

$$\text{row space of } A - \text{rowsp}(A) = \text{span}(\overline{v}_1, \overline{v}_2)$$

where $\quad \overline{v}_1 = (\,1 \quad 2 \quad 0\,)$

$\overline{v}_2 = (\,9 \quad 18 \quad -6\,)$

We can visualize this column space by repeatedly computing $\overline{b} = A\,\overline{x}$ for random $\overline{x}$ values and then seeing how this compares with $\overline{c}_1$ and $\overline{c}_3$ (or alternatively $\overline{d}_1$ and $\overline{d}_2$) (go to Matlab).




The **column space** of the matrix $A$ is defined as the set of all possible linear combinations of the column vectors of the matrix. This is also sometimes referred to as the **range of the matrix *A***.

# Linear Independence, Matrix Rank

## Linear Independence

Given any set of $m$ vectors $\overline{a}_1,\ \ldots,\ \overline{a}_m$ (each vector having the same number of components), a linear combination of these vectors is an expression of the form

$$c_1\,\overline{a}_1 + c_2\,\overline{a}_2 + \ldots + c_m\,\overline{a}_m$$

where $\quad c_k$ = any scalars

We now consider the equation

$$c_1\,\overline{a}_1 + c_2\,\overline{a}_2 + \ldots + c_m\,\overline{a}_m = 0 \qquad \textbf{(Eq.1)}$$

One obvious solution for this equation is if $c_k = 0$ for $k = 1,\ \ldots,\ m$. If this is the only solution for which Eq.1 is true, then the vectors $\overline{a}_1,\ \overline{a}_2,\ \ldots,\ \overline{a}_m$ are **linearly independent**. The left side of this equation is referred to as a linear combination of the vectors $\overline{a}_1,\ \ldots,\ \overline{a}_m$.

An alternate definition is as follows:
A set of vectors is said to be **linearly independent** if there exists no nontrivial linear combination of the vectors that equals the zero vector.

If there is some combination of coefficients where one or more are not zero and yields a solution to Eq.1, the set of vectors are said to be **linearly dependent**, and at least one of the vectors can be written as a combination of the others.

Suppose that the vectors $\overline{a}_1, \ldots, \overline{a}_m$ are linearly dependent. This implies that there exists at least one of the scalars $c_1, .., c_m$ that is non-zero. Denote this non-zero coefficient as $c_p$ and Eq.1 can be written as

$$\overline{a}_p = ((-c_1)/c_p)\,\overline{a}_1 + \ldots + -(c_m/c_p)\,\overline{a}_m$$

This showing that $\overline{a}_p$ can be written as a linear combination of the other vectors.

A linearly dependent set of vectors can be reduced to a set of linearly independent vectors by removing redundant vectors. Another name for a linearly independent set of vectors is a basis set since they form the smallest essential set of vectors (a basis from which other vectors can be constructed).

While the matrix $A$ may not be linearly independent, the row and column space are linearly independent as they derive from the reduced forms of the matrix which are by definition/construction linearly independent.

## Matrix Rank

If we consider a matrix to be comprised of a set of vectors (either row or column vectors), we denote the

**column rank** of the matrix is the maximum number of linearly independent columns of $A$
**row rank** of the matrix as the maximum number of linearly independent rows of $A$

The column rank is always equal to the row rank, so this is often referred to as simply the **rank** of the matrix.

We note that the number of linearly independent rows of the matrix does not change if we change the order of rows or multiply a row by a nonzero $c$ or take a linear combination by adding a multiple of a row to another row. This means that rank is invariant under elementary row operations. Therefore, we can find the number of linearly independent rows by transforming the matrix to row-reduced echelon form.

Combining with our previous discussion on row and columns space we have the following properties

rank($A$) = dim(rowsp($A$)) = dim(colsp($A$))
rank($A$) = number of pivots in any echelon form of $A$
rank($A$) = the maximum number of linearly independent rows or columns of $A$.

In addition, other properties that are helpful include

Row-equivalent matrices have the same rank

$\text{rank}(A) \le \min(m, n)$ ( $A$ is an $m$ – by – $n$) matrix