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Lecture 02d
Direction Cosine Matrix from North East Down to East North Up

Lecture is on YouTube

The YouTube video entitled 'Direction Cosine Matrix from North East Down to East North Up ' that covers this lecture is located at https://youtu.be/NI70-AWnO4w

Outline

-Euler Rotation Sequence from NED to Body
-Euler Rotation Sequence from NED to ENU
-Euler Rotation Sequence from body to ENU
-Euler Rotation Sequence from body to NED

Euler Rotation Sequence from NED to Body

Recall from the previous video entitled 'Euler Angles and the Euler Rotation Sequence' at https://youtu.be/GJBc6z6p0KQ that the rotation matrix/sequence to go from the NED frame (in the previous video this was referred to as the vehicle-carried-NED frame, F_v) to the body frame is given by

$$C_{b/\text{NED}}(\phi,\ \theta,\ \psi) = C_{b/2}(\phi)\, C_{2/1}(\theta)\, C_{1/\text{NED}}(\psi) \qquad \textbf{\textit{(Eq.1)}}$$

Note the previous video used the subscript 'v' that we now change to 'NED'

In[•]:=
```
C1NED[ψ_] = {{Cos[ψ], Sin[ψ], 0}, {-Sin[ψ], Cos[ψ], 0}, {0, 0, 1}};
C21[θ_] = {{Cos[θ], 0, -Sin[θ]}, {0, 1, 0}, {Sin[θ], 0, Cos[θ]}};
Cb2[ϕ_] = {{1, 0, 0}, {0, Cos[ϕ], Sin[ϕ]}, {0, -Sin[ϕ], Cos[ϕ]}};
CbNED[ϕ_, θ_, ψ_] = Cb2[ϕ].C21[θ].C1NED[ψ];
CbNED[ϕ, θ, ψ] // MatrixForm
```

$$
C1NED[\psi\_] = \begin{pmatrix} \cos[\psi] & \sin[\psi] & 0 \\ -\sin[\psi] & \cos[\psi] & 0 \\ 0 & 0 & 1 \end{pmatrix};
$$

$$
C21[\theta\_] = \begin{pmatrix} \cos[\theta] & 0 & -\sin[\theta] \\ 0 & 1 & 0 \\ \sin[\theta] & 0 & \cos[\theta] \end{pmatrix};
$$

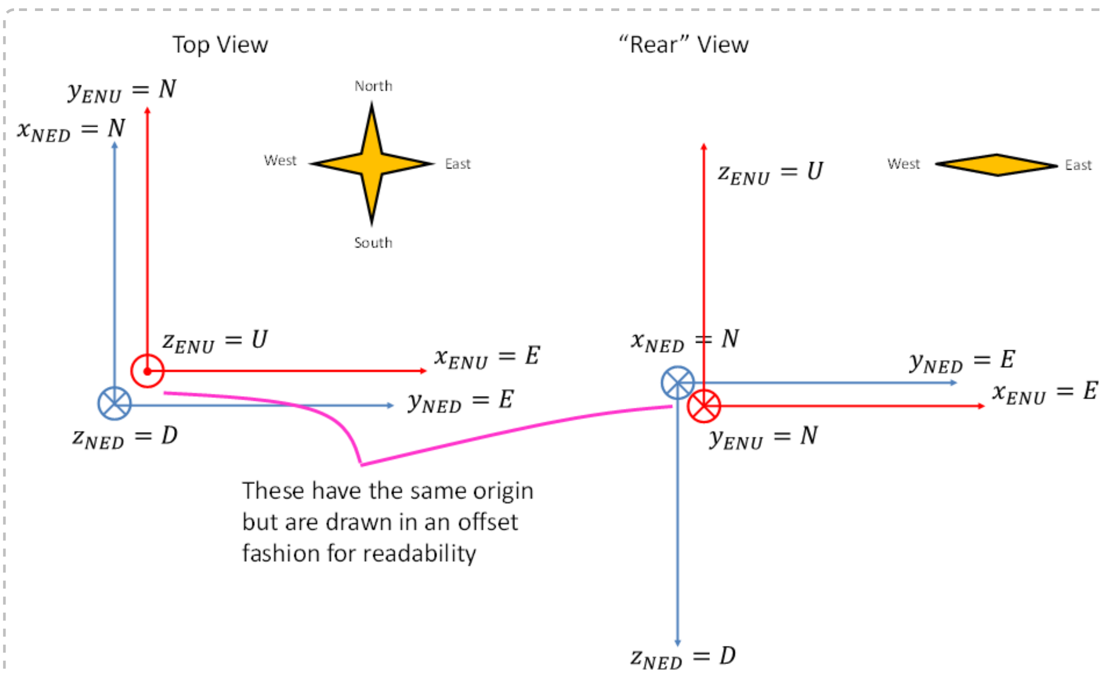
$$
Cb2[\phi\_] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos[\phi] & \sin[\phi] \\ 0 & -\sin[\phi] & \cos[\phi] \end{pmatrix};
$$

Out[•]//MatrixForm=

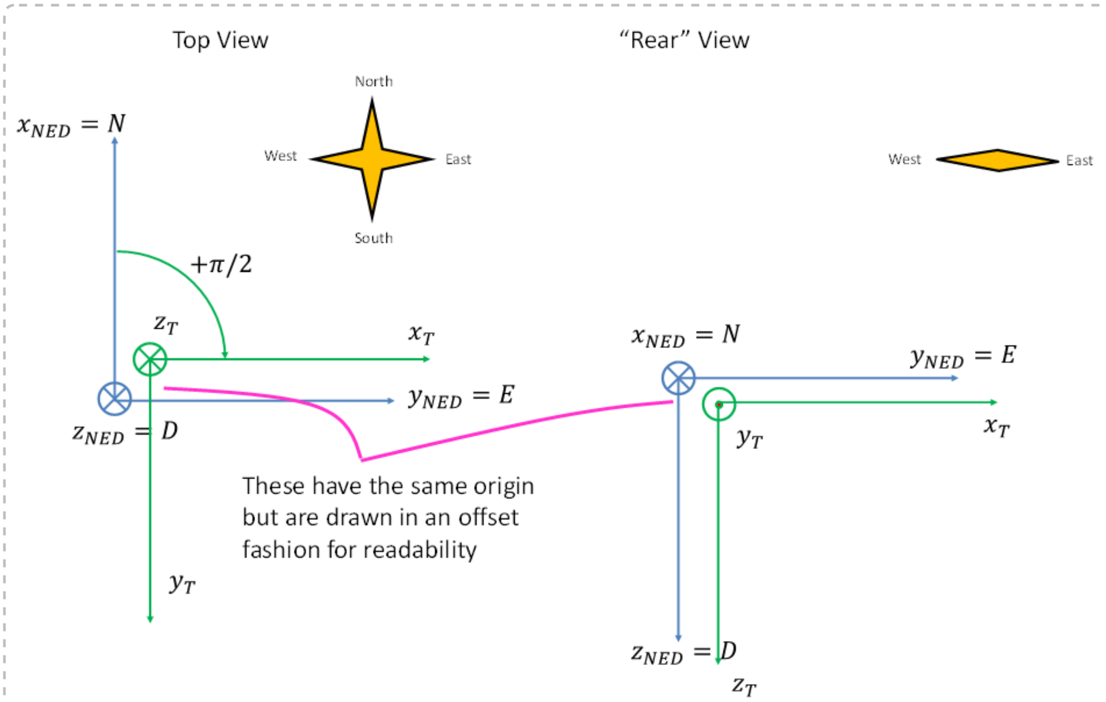
$$
\begin{pmatrix} \cos[\theta]\cos[\psi] & \cos[\theta]\sin[\psi] & -\sin[\theta] \\ \cos[\psi]\sin[\theta]\sin[\phi] - \cos[\phi]\sin[\psi] & \cos[\phi]\cos[\psi] + \sin[\theta]\sin[\phi]\sin[\psi] & \cos[\theta]\sin[\phi] \\ \cos[\phi]\cos[\psi]\sin[\theta] + \sin[\phi]\sin[\psi] & -\cos[\psi]\sin[\phi] + \cos[\phi]\sin[\theta]\sin[\psi] & \cos[\theta]\cos[\phi] \end{pmatrix}
$$

Euler Rotation Sequence from NED to ENU

In some situations, we may want to express a vector in the ENU frame as opposed to the NED frame. These NED and ENU frames are shown below



To develop a rotation matrix to go from NED to ENU, we first start with the NED frame and define a temporary frame, F_T, that is rotated about the z_{NED} axis through $\pi/2$ (90 deg)



The rotation matrix is given as

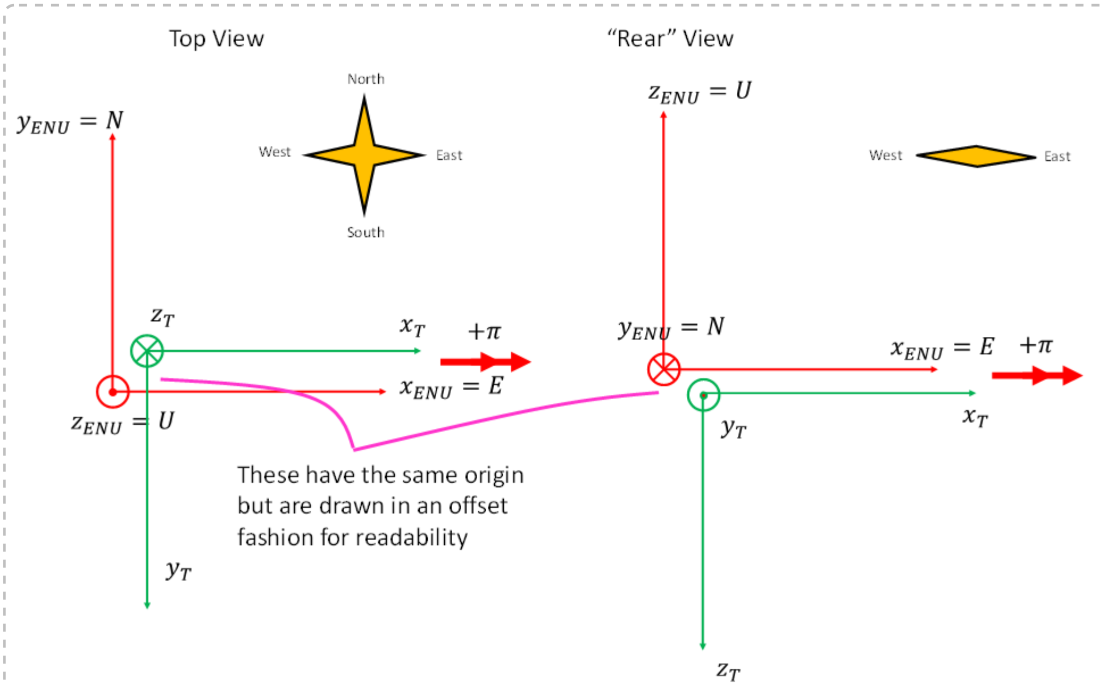
$$C_{T/\text{NED}} = \begin{pmatrix} \cos(\pi/2) & \sin(\pi/2) & 0 \\ -\sin(\pi/2) & \cos(\pi/2) & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \textbf{\textit{(Eq.2)}}$$

In[ ]:=
```
CTNED = ( Cos[π / 2]   Sin[π / 2]  0
         -Sin[π / 2]   Cos[π / 2]  0
          0            0           1 );
CTNED // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can define the ENU frame, F_{ENU}, that is rotated about the x_T axis through π (180 deg)



The rotation matrix is given as

$$C_{ENU/T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\pi) & \sin(\pi) \\ 0 & -\sin(\pi) & \cos(\pi) \end{pmatrix} \qquad \textbf{(Eq.3)}$$

In[∘]:=
```
CENUT = {{1, 0, 0}, {0, Cos[π], Sin[π]}, {0, -Sin[π], Cos[π]}};
CENUT // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

So the overall rotation matrix is given as

$$C_{ENU/NED} = C_{ENU/T}\, C_{T/NED} \qquad \textbf{(Eq.4)}$$

$$C_{NED/ENU} = C_{ENU/NED}{}^{T} \qquad \textbf{(Eq.5)}$$

In[ ]:=
```
CENUNED = CENUT.CTNED;
CENUNED // MatrixForm

CNEDENU = Transpose[CENUNED];
CNEDENU // MatrixForm
```

Out[ ]//MatrixForm=

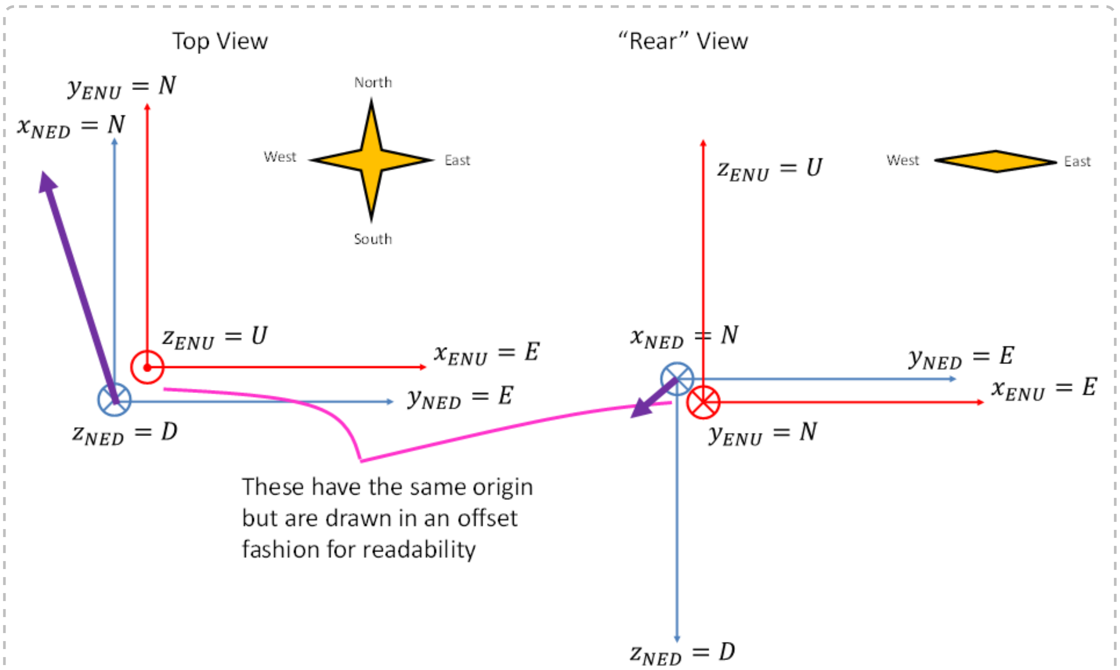
$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Out[ ]//MatrixForm=

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Example

Assume we have a vector as shown below



Numerically, this is given as

In[ ]:= $\mathbf{rNED} = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix};$

This can be expressed in the ENU frame using

$$\bar{r}^{\text{ENU}} = C_{\text{ENU/NED}}\,\bar{r}^{\text{NED}}$$

In[ ]:=
```
rENU = CENUNED.rNED;
rENU // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -2 \\ 5 \\ -1 \end{pmatrix}$$

Euler Rotation Sequence from body to ENU

We can combine Eq.1 and Eq.5 to write

$$C_{b/\text{ENU}}(\phi, \theta, \psi) = C_{b/\text{NED}}(\phi, \theta, \psi)\, C_{\text{NED/ENU}}$$

In[ ]:=
```
CbENU[ϕ_, θ_, ψ_] = CbNED[ϕ, θ, ψ].CNEDENU;
CbENU[ϕ, θ, ψ] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\theta]\,\text{Sin}[\psi] & \text{Cos}[\theta]\,\text{Cos}[\psi] & \text{Sin}[\theta] \\ \text{Cos}[\phi]\,\text{Cos}[\psi] + \text{Sin}[\theta]\,\text{Sin}[\phi]\,\text{Sin}[\psi] & \text{Cos}[\psi]\,\text{Sin}[\theta]\,\text{Sin}[\phi] - \text{Cos}[\phi]\,\text{Sin}[\psi] & -\text{Cos}[\theta]\,\text{Sin}[\phi] \\ -\text{Cos}[\psi]\,\text{Sin}[\phi] + \text{Cos}[\phi]\,\text{Sin}[\theta]\,\text{Sin}[\psi] & \text{Cos}[\phi]\,\text{Cos}[\psi]\,\text{Sin}[\theta] + \text{Sin}[\phi]\,\text{Sin}[\psi] & -\text{Cos}[\theta]\,\text{Cos}[\phi] \end{pmatrix}$$

If we know $C_{b/\text{ENU}}$, can reconstruct the Euler angles

$\phi = \text{atan2}(c_{23}, c_{33})$ ***(Eq.2.2)***

$\theta = \sin^{-1}(c_{13})$ (different) ***(Eq.2.3)***

$\psi = \text{atan2}(c_{11}, c_{12})$ (different) ***(Eq.2.4)***

Euler Rotation Sequence from body to NED

For convenience, we can compare this with the DCM $C_{b/\text{NED}}(\phi, \theta, \psi)$

In[ ]:=
```
CbNED[ϕ, θ, ψ] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \text{Cos}[\theta]\,\text{Cos}[\psi] & \text{Cos}[\theta]\,\text{Sin}[\psi] & -\text{Sin}[\theta] \\ \text{Cos}[\psi]\,\text{Sin}[\theta]\,\text{Sin}[\phi] - \text{Cos}[\phi]\,\text{Sin}[\psi] & \text{Cos}[\phi]\,\text{Cos}[\psi] + \text{Sin}[\theta]\,\text{Sin}[\phi]\,\text{Sin}[\psi] & \text{Cos}[\theta]\,\text{Sin}[\phi] \\ \text{Cos}[\phi]\,\text{Cos}[\psi]\,\text{Sin}[\theta] + \text{Sin}[\phi]\,\text{Sin}[\psi] & -\text{Cos}[\psi]\,\text{Sin}[\phi] + \text{Cos}[\phi]\,\text{Sin}[\theta]\,\text{Sin}[\psi] & \text{Cos}[\theta]\,\text{Cos}[\phi] \end{pmatrix}$$

If we know $C_{b/\text{NED}}$, can reconstruct the Euler angles (see https://youtu.be/GJBc6z6p0KQ?si=zlNp12u72M-rFtttH&t=2890)

$\phi = \text{atan2}(c_{23}, c_{33})$ ***(Eq.1.2)***

$\theta = -\sin^{-1}(c_{13})$ ***(Eq.1.3)***

$\psi = \text{atan2}(c_{12}, c_{11})$ ***(Eq.1.4)***