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Lecture 02c
Euler Angles and the Euler Rotation Sequence

Lecture is on YouTube

The YouTube video entitled ‘Euler Angles and the Euler Rotation Sequence’ that covers this lecture is located at https://youtu.be/GJBc6z6p0KQ.

Outline

-Euler Angles
-Euler Rotation Sequence
-Relation to Rodrigues’ Rotation Formula
-Intrinsic vs . Extrinsic Euler Angles

Euler Angles

The Euler angles are three angles introduced by Leonhard Euler to describe the orientation of a rigid body with respect to a fixed coordinate system.

Euler Rotation Sequence

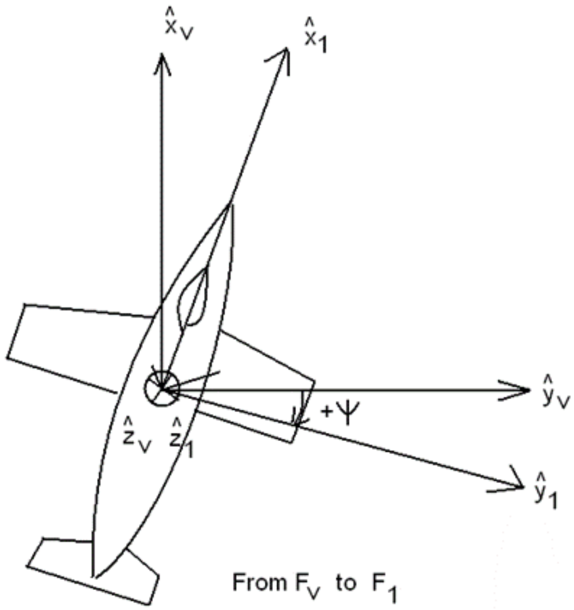
See Stevens and Lewis pg.25 for reference

<u>Euler Angle Rotation Sequence</u>
-Rotation to go from F_v (vehicle carried North-East-Down frame) to F_b (body fixed frame).
-Uses two intermediate frames

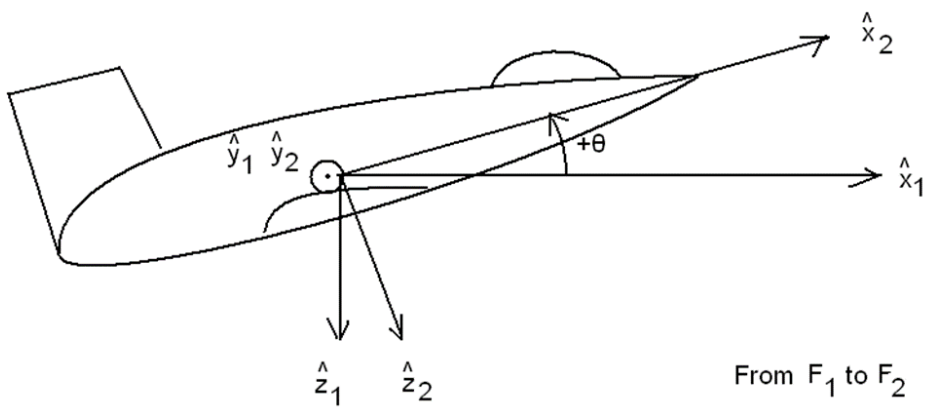
Start in F_v

1. Right handed rotation about z_v (same as z_1) axis through heading angle ψ (to go to F_1)
2. Right handed rotation about y_1 (same as y_2) axis through pitch angle θ (to go to F_2)
3. Right handed rotation about x_2 (same as x_b) axis through bank angle ϕ (to go to F_b)

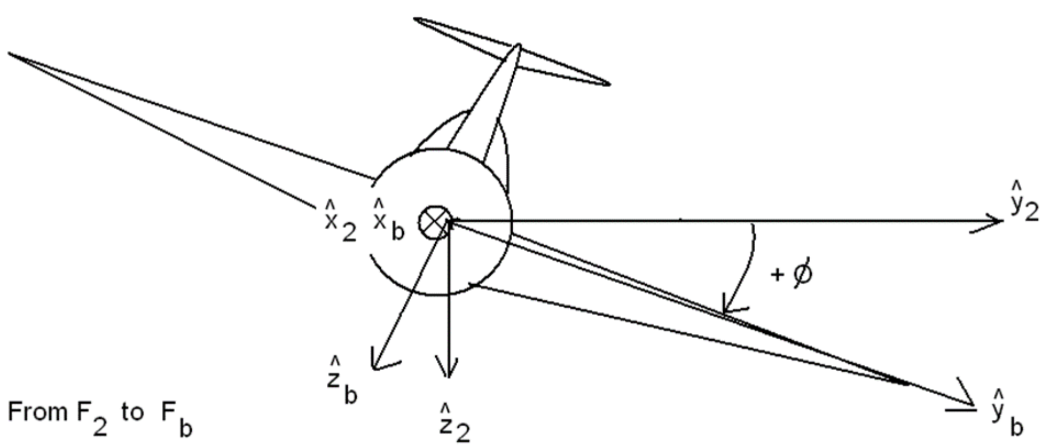
For example of rotation 1, $C_{1/v}(\psi)$, the picture is



For example of rotation 2, $C_{2/1}(\theta)$, the picture is



For example of rotation 3, $C_{b/2}(\phi)$, the picture is



Show movie (euler_rotations.m)

So we see that to transform from F_v to F_1, we have a rotation about $\hat{z}_v = \hat{z}_1$ axis. From HW1 we know this is given by

$$\overline{u}^1 = C_{1/v}(\psi)\,\overline{u}^v \quad \text{where} \quad C_{1/v}(\psi) = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly, to rotate about the y and x axes, we use the rotation matrices of

$$\overline{u}^2 = C_{2/1}(\theta)\,\overline{u}^1 \quad \text{where} \quad C_{2/1}(\theta) = \begin{pmatrix} \cos(\theta) & 0 & -\sin(\theta) \\ 0 & 1 & 0 \\ \sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

$$\overline{u}^b = C_{b/2}(\phi)\,\overline{u}^2 \quad \text{where} \quad C_{b/2}(\phi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\phi) & \sin(\phi) \\ 0 & -\sin(\phi) & \cos(\phi) \end{pmatrix}$$

So overall, we have

$$\overline{u}^b = C_{b/2}(\phi)\,\overline{u}^2 \qquad \text{recall: } \overline{u}^2 = C_{2/1}(\theta)\,\overline{u}^1$$

$$= C_{b/2}(\phi)\,C_{2/1}(\theta)\,\overline{u}^1 \qquad \text{recall: } \overline{u}^1 = C_{1/v}(\psi)\,\overline{u}^v$$

$$= C_{b/2}(\phi)\,C_{2/1}(\theta)\,C_{1/v}(\psi)\,\overline{u}^v$$

$$\overline{u}^b = C_{b/v}(\phi,\ \theta,\ \psi)\,\overline{u}^v$$

where $\quad C_{b/v}(\phi,\ \theta,\ \psi) = C_{b/2}(\phi)\,C_{2/1}(\theta)\,C_{1/v}(\psi)$

We refer to the product these three rotations as the **direction cosine matrix** (DCM)

$$C_{b/v}{}^{\text{ZYX}}(\phi,\ \theta,\ \psi) = C_{b/2}(\phi)\,C_{2/1}(\theta)\,C_{1/v}(\psi) =$$

$$\begin{pmatrix} \cos(\theta)\cos(\psi) & \cos(\theta)\sin(\psi) & -\sin(\theta) \\ -\cos(\phi)\sin(\psi) + \sin(\phi)\sin(\theta)\cos(\psi) & \cos(\phi)\cos(\psi) + \sin(\phi)\sin(\theta)\sin(\psi) & \sin(\phi)\cos(\theta) \\ \sin(\phi)\sin(\psi) + \cos(\phi)\sin(\theta)\cos(\psi) & -\sin(\phi)\cos(\psi) + \cos(\phi)\sin(\theta)\sin(\psi) & \cos(\phi)\cos(\theta) \end{pmatrix}$$

(Eq.1.1)

```
C1v[ψ_] = ( Cos[ψ]   Sin[ψ]  0
           -Sin[ψ]   Cos[ψ]  0
             0         0     1 );

C21[θ_] = ( Cos[θ]  0  -Sin[θ]
              0     1     0
            Sin[θ]  0   Cos[θ] );

Cb2[ϕ_] = ( 1     0        0
            0   Cos[ϕ]   Sin[ϕ]
            0  -Sin[ϕ]   Cos[ϕ] );

Cbv[ϕ_, θ_, ψ_] = Cb2[ϕ].C21[θ].C1v[ψ];
Cbv[ϕ, θ, ψ] // MatrixForm
```

$$\begin{pmatrix} \mathrm{Cos}[\theta]\,\mathrm{Cos}[\psi] & \mathrm{Cos}[\theta]\,\mathrm{Sin}[\psi] & -\mathrm{Sin}[\theta] \\ \mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi] - \mathrm{Cos}[\phi]\,\mathrm{Sin}[\psi] & \mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi] + \mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & \mathrm{Cos}[\theta]\,\mathrm{Sin}[\phi] \\ \mathrm{Cos}[\phi]\,\mathrm{Cos}[\psi]\,\mathrm{Sin}[\theta] + \mathrm{Sin}[\phi]\,\mathrm{Sin}[\psi] & -\mathrm{Cos}[\psi]\,\mathrm{Sin}[\phi] + \mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\psi] & \mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi] \end{pmatrix}$$

Note that the above DCM contains the superscript “ZYX” to denote it was generated by the rotation sequence “ZYX” (first rotate about z-axis, then rotation about y-axis, then rotate about x-axis). We could derive the rotation matrix for any of the 12 combinations of rotations (for example XZY). This order of ZYX is standard in the aerospace industry and we therefore drop the superscript notation. Furthermore notice that we list the order of the input arguments as ϕ, θ, ψ. This ordering is more popular and standardized than the ordering of ψ, θ, ϕ but one should take care when using software or other tools as this ordering may not be always maintained.

$$C_{b/v}{}^{ZYX}(\phi, \theta, \psi) = C_{b/v}(\phi, \theta, \psi) \qquad \text{(understood to be for the ZYX rotation sequence)}$$

We have to be careful, Euler angles are not unique for a given orientation. This means that we can use two different sequences of angles to reach the same configuration.

Example

```
ϕ1 = 0 π/180;
θ1 = 180 π/180;
ψ1 = 0 π/180;
Cbv[ϕ1, θ1, ψ1] // MatrixForm
```

```
ϕ2 = 180 π/180;
θ2 = 0 π/180;
ψ2 = 180 π/180;
Cbv[ϕ2, θ2, ψ2] // MatrixForm
```

```
Cbv[ϕ1, θ1, ψ1] == Cbv[ϕ1, θ1, ψ1]
```

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

```
True
```

Note that this does not mean that the matrix is singular at some values. In fact, because the matrix is unitary, it is never singular.

One can show that this matrix is unitary

$$C_{b/v}(\phi,\ \theta,\ \psi)^{-1} = \left(C_{b/2}(\phi)\, C_{2/1}(\theta)\, C_{1/v}(\psi)\right)^{-1}$$

$$= C_{1/v}(\psi)^{-1}\, C_{2/1}(\theta)^{-1}\, C_{b/2}(\phi)^{-1}$$ recall: all these of these matrices are unitary, so $C^{-1} = C^{T}$

$$= C_{1/v}(\psi)^{T}\, C_{2/1}(\theta)^{T}\, C_{b/2}(\phi)^{T}$$

$$C_{b/v}(\phi,\ \theta,\ \psi)^{-1} = \left(C_{b/2}(\phi)\, C_{2/1}(\theta)\, C_{1/v}(\psi)\right)^{T}$$

Because $C_{b/v}(\phi,\ \theta,\ \psi)$ is unitary, we can compute the inverse using a simple transpose. This is the same as the rotation matrix to from F_b to F_v. Mathematically

$$C_{v/b}(\phi,\ \theta,\ \psi) = C_{b/v}(\phi,\ \theta,\ \psi)^{T}$$

If we know $C_{b/v}$, can reconstruct the Euler angles

$\phi = \text{atan2}(c_{23}, c_{33})$ ***(Eq.1.2)***
$\theta = -\sin^{-1}(c_{13})$ ***(Eq.1.3)***
$\psi = \text{atan2}(c_{12}, c_{11})$ ***(Eq.1.4)***

This is useful later in Poisson's Kinematical Equations

Relation to Rodrigues' Rotation Formula

If we compute the eigenvalues of $C_{b/v}$, we obtain

```
vals = Eigenvalues[Cbv[ϕ, θ, ψ]] // Simplify;
vals // MatrixForm
```

$$\begin{pmatrix} \frac{1}{8} \times \big(-4 + 2\,\text{Cos}[\theta - \phi] + 2\,\text{Cos}[\theta + \phi] + 2\,\text{Cos}[\theta - \psi] + 2\,\text{Cos}[\phi - \psi] + 2\,\text{Cos}[\theta + \psi] + 2\,\text{Cos}[\phi + \psi] - \text{Sin}[\theta \\ \frac{1}{8} \times \big(-4 + 2\,\text{Cos}[\theta - \phi] + 2\,\text{Cos}[\theta + \phi] + 2\,\text{Cos}[\theta - \psi] + 2\,\text{Cos}[\phi - \psi] + 2\,\text{Cos}[\theta + \psi] + 2\,\text{Cos}[\phi + \psi] - \text{Sin}[\theta \end{pmatrix}$$

We see that this always has one eigenvalue of magnitude 1. From the definition of the eigenvalue and eigenvector pair, we have

$C_{b/v}(\phi, \theta, \psi)\,\overline{v}_1 = \lambda_1\,\overline{v}_1$ recall: $\lambda_1 = 1$

$C_{b/v}(\phi, \theta, \psi)\,\overline{v}_1 = \overline{v}_1$ ***(Eq.1)***

This implies that the eigenvector $\overline{v}_1$ is unchanged by the rotation matrix $C_{b/v}$. This means that the rotation matrix does not actually rotate this vector. The question should be what kind of vector is unchanged by a rotation matrix?

In an earlier discussion (and hw01) we investigated the use of the Rodrigues' Rotation Formula to rotate a vector, $\overline{u}$, about an arbitrary axis of rotation, $\overline{n}$ (unit vector), through an arbitrary left handed angle, μ, to obtain a new vector $\overline{v}$

$\overline{v} = (1 - \cos(\mu)) < \overline{u}, \overline{n} > \overline{n} + \cos(\mu)\,\overline{u} - \sin(\mu)\,(\overline{n} \times \overline{u})$ ***(Eq.1.2-6)***

If we investigate Rodrigues' Rotation Formula with $\overline{u} = \alpha\,\overline{n}$ (meaning that the vector we are rotating is aligned with the axis of rotation), we obtain

$$\overline{v} = (1 - \cos(\mu)) < \overline{u}, \overline{n} > \overline{n} + \cos(\mu)\,\overline{u} - \sin(\mu)\,(\overline{n} \times \overline{u})$$

$$= (1 - \cos(\mu)) < \alpha\,\overline{n}, \overline{n} > \overline{n} + \cos(\mu)\,\alpha\,\overline{n} - \sin(\mu)\,(\overline{n} \times \alpha\,\overline{n})$$

$$= (1 - \cos(\mu))\,\alpha < \overline{n}, \overline{n} > \overline{n} + \cos(\mu)\,\alpha\,\overline{n}$$

$= (1 - \cos(\mu))\,\alpha\ \left|\,\overline{n}\,\right|^2 \overline{n} + \cos(\mu)\,\alpha\,\overline{n}$ recall: $\left|\,\overline{n}\,\right| = 1$ ($\overline{n}$ is a unit vector)

$= (1 - \cos(\mu))\,\alpha\,\overline{n} + \cos(\mu)\,\alpha\,\overline{n}$

$= \alpha\,\overline{n} - \cos(\mu)\,\alpha\,\overline{n} + \cos(\mu)\,\alpha\,\overline{n}$

$= \alpha\,\overline{n}(1 - \cos(\mu) + \cos(\mu))$

$= \alpha\,\overline{n}$ recall: $\overline{u} = \alpha\,\overline{n}$

$\overline{v} = \overline{u}$

So we see that a vector aligned with the axis of rotation is unchanged by the rotation (which makes intuitive sense).

Combining this with the Eq.1, we see that the eigenvector $\overline{v}_1$ that is associated with the axis of rotation. If this is normalized to a unit vector, we have

$$\overline{n} = \frac{\overline{v}_1}{|\overline{v}_1|} \qquad \textbf{\textit{(Eq.2)}}$$

This shows that rotation to reach F_b from F_v can be achieved in a single rotation about the axis $\overline{n} = \overline{v}_1 / \left|\,\overline{v}_1\,\right|$.

Example

Recall the example we showed in Matlab with the following Euler angles.

$\psi = 70\,\frac{\pi}{180}$

$\theta = 130\,\frac{\pi}{180}$

$\phi = 25\,\frac{\pi}{180}$

```
ψGiven = 70 π/180;
θGiven = 130 π/180;
ϕGiven = 25 π/180;
```

We can calculate the rotation matrix

```
CbvGiven = Cbv[ϕGiven, θGiven, ψGiven];
CbvGiven // MatrixForm // N
```

$$\begin{pmatrix} -0.219846 & -0.604023 & -0.766044 \\ -0.740924 & 0.614196 & -0.271654 \\ 0.634586 & 0.507858 & -0.582563 \end{pmatrix}$$

We can calculate the eigenvalues

```
vals = Eigenvalues[CbvGiven] // Simplify;
vals // MatrixForm
```

$$\begin{pmatrix} \frac{1}{16}(-1)^{19/36}\left((1-i)+2(-1)^{1/18}-2(-1)^{5/36}+(-1)^{1/6}-3(-1)^{2/9}+2(-1)^{1/3}+2(-1)^{13/36}-(-1)^{4/9}\right. \\ \frac{1}{16}(-1)^{19/36}\left((1-i)+2(-1)^{1/18}-2(-1)^{5/36}+(-1)^{1/6}-3(-1)^{2/9}+2(-1)^{1/3}+2(-1)^{13/36}-(-1)^{4/9}\right. \\ \end{pmatrix}$$

```
temp = Eigenvectors[CbvGiven];
vecs = Transpose[temp];
vecs // N // MatrixForm
```

$$\begin{pmatrix} 0.0415333+0.876972\,i & 0.0415333-0.876972\,i & -5.69399 \\ -0.0746272+0.488073\,i & -0.0746272-0.488073\,i & 10.231 \\ 1. & 1. & 1. \end{pmatrix}$$

Extract the 3rd eigenvalue/vector

```
λ3 = vals[[3]]
v3 = Transpose[{vecs[[All, 3]]}] // Simplify ;
v3 // MatrixForm // N // Chop
```

1

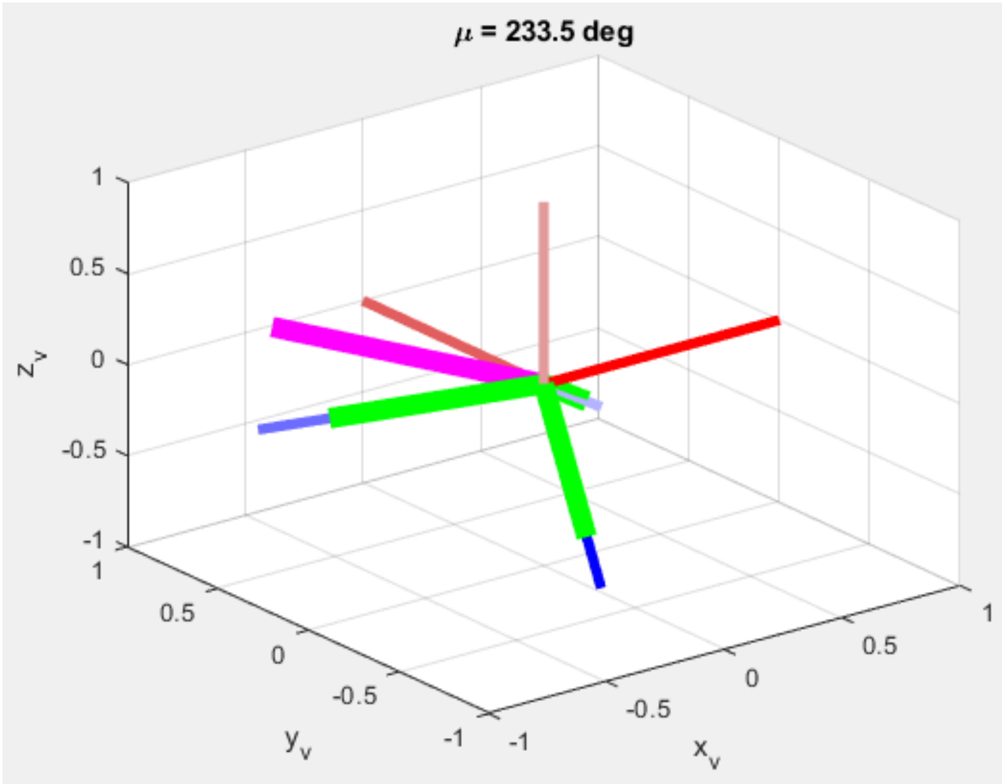
$$\begin{pmatrix} -5.69399 \\ 10.231 \\ 1. \end{pmatrix}$$

Ensure that this is an eigenvector

```
(CbvGiven.v3 - λ3 * v3) // N // Chop
```

{{0}, {0}, {0}}

Let us rotate the $\hat{x}^v = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ axis, $\hat{y}^v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $\hat{z}^v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ about the vector $\overline{v}_3$ using Rodrigues rotation formula.



Note that in the previous example, we used guess and check to determine that $\mu \approx 233.5 \frac{\pi}{180}$ is the correct rotation angle. We will revisit this example when we discuss Quaternions and see how to exactly calculate this angle.

Intrinsic vs. Extrinsic Euler Angles

The angles ϕ,θ,and ψ are sometimes referred to as intrinsic rotation angles because these are rotations about the axes of the rotating coordinate system which changes its orientation after each elemental rotation. This is opposed to extrinsic rotation angles which are rotations about axes of the original coordinate system which is assumed to remain motionless.

This sets the stage for our next discussion on ‘Kinematic Matrix Relationships for Rotation’