Christopher Lum
lum@uw.edu

Lecture 06b
Understanding and Sketching the Root Locus

The YouTube video entitled ‘Understanding and Sketching the Root Locus’ that covers this lecture is located at https://youtu.be/gA-KOk3SAb0.

Outline

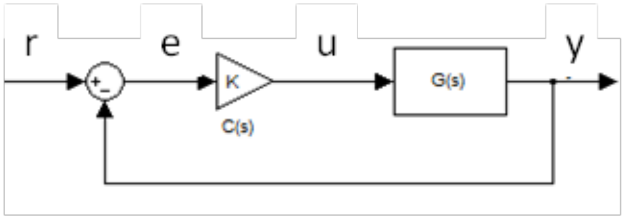
-Root Locus Refresher
-Sketching the Root Locus
  -Rule 1: Number of Poles Is Unchanged
  -Rule 2: Closed Loop Poles Start at Open Loop Poles and go to Open Loop Zeros or Zeros at Infinity
  -Rule 3: Valid Regions on Real Axis are to the Left of Odd Numbered Pole/Zeros (start numbering from right to left)
  -Rule 4: Angle of Asymptotes and Centroid of Asymptotes
  -Rule 5: Angle of Departure from Complex Poles
-Summary of Rules 1-5
-Additional Rules
  -Rule 6: Angle of Arrival to Complex Zeros
  -Rule 7: Locus Breaks Out/In on Real Axis at 90^o
  -Rule 8: Complex Roots Come in Pairs
  -Rule 9: The Root Locus Will Not Cross Over Itself
  -Rule 10: Number of Lines
-Example: Complex Transfer Function
-Summary of Rules
-Additional Rules

Root Locus Refresher

Recall that root locus is simply a plot of pole locations as a parameter of the system changes. Typically, this tunable parameter is referred to as K and we consider values of K between 0 and $+\infty$.

Sketching the Root Locus

For our situation, we consider the classical control architecture as shown below where the tunable parameter K represents the gain of the controller.



In this situation, we can easily find the closed loop transfer function

$$T(s) = \frac{Y(s)}{R(s)} = \frac{K\,G(s)}{1+K\,G(s)} \qquad \textbf{(Eq.1)}$$

From Eq.1, we see that the poles of the closed loop system are values of s which satisfy the characteristic equation of

$$\Delta(s) = 1 + K\,G(s) = 0 \qquad \text{(characteristic equation of } T(s)\text{)} \qquad \textbf{(Eq.2)}$$

One simple method of finding the root locus is to simply solve Eq.2 iteratively for different values of K

Example

$$G_1(s) = \frac{1}{s(s+2)}$$

```
G1[s_] = 1/(s (s + 2));
T[s_] = (G1[s] K)/(1 + G1[s] K) // Together
Δ[s_] = Denominator[T[s]]
```

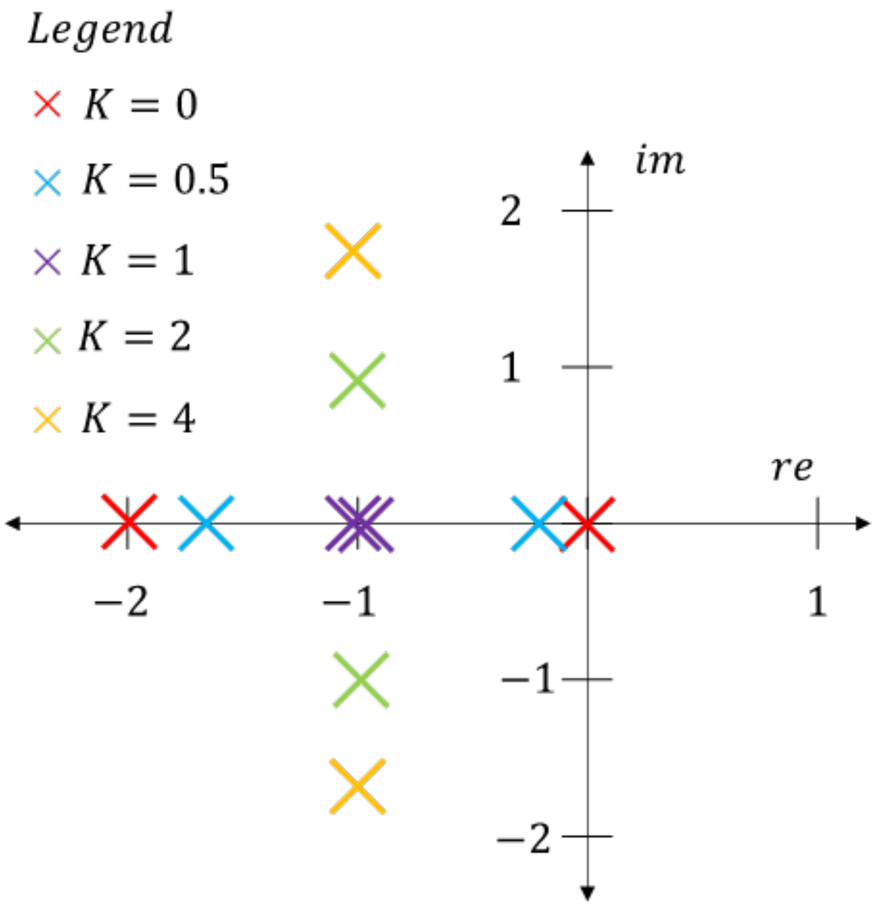
$$\frac{K}{K + 2\,s + s^2}$$

$$K + 2\,s + s^2$$

```
Solve[Δ[s] == 0 /. {K → 0}, s] // N
Solve[Δ[s] == 0 /. {K → 1 / 2}, s] // N
Solve[Δ[s] == 0 /. {K → 1}, s] // N
Solve[Δ[s] == 0 /. {K → 2}, s] // N
Solve[Δ[s] == 0 /. {K → 4}, s] // N
```

```
{{s → -2.}, {s → 0.}}

{{s → -1.70711}, {s → -0.292893}}

{{s → -1.}, {s → -1.}}

{{s → -1. - 1. i}, {s → -1. + 1. i}}

{{s → -1. - 1.73205 i}, {s → -1. + 1.73205 i}}
```

So we can sketch the locations of the roots as the parameter K changes.



However this technique of brute force solving the characteristic equation does not yield any physical understanding of the behavior of the root locus. For example, will the poles eventually become unstable? Could they come back to the real axis if we continue increasing the gain? There are many questions that are left unanswered which prompts us to explore further.

```
Clear[Δ, T, G1]
```

End Example

Recall that $G(s)$ is a complex polynomial of the form

$$G(s) = \frac{a(s)}{b(s)} \qquad \textbf{\textit{(Eq.3)}}$$

where $a(s)$ =numerator polynomial of $G(s)$
$b(s)$ =denominator polynomial of $G(s)$

We can therefore rewrite Eq.1 as

$$T(s) = \frac{K\frac{a(s)}{b(s)}}{1+K\frac{a(s)}{b(s)}}$$

$$= \frac{K\frac{a(s)}{b(s)}}{\frac{b(s)+K\,a(s)}{b(s)}}$$

$$= K \frac{a(s)}{b(s)} \frac{b(s)}{b(s)+K\,a(s)}$$

$$T(s) = \frac{K\,a(s)}{b(s)+K\,a(s)}$$

$$\frac{K\,\frac{a}{b}}{1+K\,\frac{a}{b}}$$ **// Simplify**

$$\frac{a\,K}{b + a\,K}$$

So an alternative expression for the characteristic equation of the closed loop system, $T(s)$, is given by

$\Delta(s) = b(s) + K\,a(s) = 0$ (characteristic equation of $T(s)$) ***(Eq.4)***

We can formulate several rules about the root locus by examining Eq.4.

Rule 1: Number of Poles Is Unchanged

Consider the open loop transfer function $G(s) = a(s)/b(s)$. If we assume that the open loop transfer function is proper, then the order of $a(s)$ should be less than or equal to the order of $b(s)$.

number of zeros of $G(s) = m =$ order of $a(s)$ ***(Eq.5)***

number of poles of $G(s) = n =$ order of $b(s)$ ***(Eq.6)***

Then we see that since we assume $G(s)$ is proper

$n \geq m$ (assume $G(s)$ is a proper transfer function)

If we now look at Eq.4, we see that the order of $\Delta(s)$ must remain at n (since $O(b(s)) \geq O(a(s))$. Therefore, we see that the number of closed loop poles is unchanged from the number of open loop poles. In other words,

Rule 1

number of poles of $T(s) =$ number of poles of $G(s) = n$

So we observe that the number of poles does not change, no matter what value K we have.

Example: Order of Closed Loop Characteristic Equation Does Not Change

Consider the transfer function

$$G_2(s) = \frac{a(s)}{b(s)} = \frac{2\,s^2+3\,s+4}{5\,s^3+2\,s^2+3}$$

We notice that the open loop system has a 3rd order characteristic equation and therefore, 3 poles.

The closed loop characteristic equation is given by Eq.4

$$\Delta(s) = b(s) + K\,a(s) = 0$$

$$= 5\,s^3 + 2\,s^2 + 3 + K\left(2\,s^2 + 3\,s + 4\right)$$

$$= 5\,s^3 + 2\,s^2 + 3 + 2\,K\,s^2 + 3\,K\,s + 4\,K$$

$$\Delta(s) = 5\,s^3 + (2 + 2\,K)\,s^2 + 3\,K\,s + (4\,K + 3)$$

So we see that this is still a 3rd order polynomial so the closed loop system so there are still 3 poles in the closed loop system.

Rule 2: Closed Loop Poles Start at Open Loop Poles and go to Open Loop Zeros or Zeros at Infinity

Again, consider the closed loop characteristic equation

$$\Delta(s) = b(s) + K\,a(s) = 0$$

We note that when $K = 0$, the closed loop characteristic equation (Eq.4), becomes

$$\Delta(s) = b(s) = 0$$

So we make the immediate observation that when $K = 0$, the poles of the closed loop system are the poles of the open loop system (recall that $b(s)$ was the characteristic equation of the open loop system, $G(s)$). This makes sense considering that $K = 0$ is effectively like saying that no control is being applied and the system should revert to the open loop, uncontrolled system.

We now examine the other extreme and ask, what does $\Delta(s)$ look like as $K \to \infty$. For convenience, Eq.4 is repeated here

$$\Delta(s) = b(s) + K\,a(s) = 0$$

Note that from Eq.4, as K becomes larger and larger, the term $K\,a(s)$ begins to dominate the polynomial in relationship to $b(s)$.

For example, consider the following transfer function

$$G_3(s) = \frac{a(s)}{b(s)} = \frac{(s+1+2\,i)\,(s+1-2\,i)}{(s+2)\,(s-3)}$$

The associated characteristic polynomial for the closed loop system is therefore (Eq.4)

$$\Delta(s) = (s + 2)\,(s - 3) + K(s + 1 + 2\,i)\,(s + 1 - 2\,i)$$

Here, we see that

$b(s) = (s + 2)\,(s - 3)$	with roots of $s = -2, +3$ (open loop poles)
$a(s) = (s + 1 + 2\,i)\,(s + 1 - 2\,i)$	with roots of $s = -1 \pm 2\,i$ (open loop zeros)

```
Δexample[K_] = (s + 2) (s - 3) + K (s + 1 + 2 I) (s + 1 - 2 I);

Solve[Δexample[0] == 0, s] // N
Solve[Δexample[1] == 0, s] // N
Solve[Δexample[10] == 0, s] // N
Solve[Δexample[100] == 0, s] // N
Solve[Δexample[1000] == 0, s] // N
```

```
{{s → -2.}, {s → 3.}}

{{s → -1.}, {s → 0.5}}

{{s → -0.863636 - 1.80392 i}, {s → -0.863636 + 1.80392 i}}

{{s → -0.985149 - 1.98004 i}, {s → -0.985149 + 1.98004 i}}

{{s → -0.998501 - 1.998 i}, {s → -0.998501 + 1.998 i}}
```

So we see, that in the limit as $K \to \infty$, the characteristic equation of the closed loop transfer function becomes

$$\Delta(s) \to a(s) \qquad (\text{as } K \to \infty)$$

Since $a(s)$ is the numerator polynomial of the open loop system, we see that as $K \to \infty$, the closed loop poles go to the open loop zeros.

We can refine this observation further. Recall our first observation, that the number of closed loop poles is always equal to the number of open loop poles. Furthermore, we note that for most cases, the number of open loop zeros is less than the number of open loop poles. For example, if we add an open loop pole of −5 to the previous example and perform the same analysis

```
Δexample[K_] = (s + 2) (s - 3) (s + 5) + K (s + 1 + 2 I) (s + 1 - 2 I);

Solve[Δexample[0] == 0, s] // N
Solve[Δexample[1] == 0, s] // N
Solve[Δexample[10] == 0, s] // N
Solve[Δexample[100] == 0, s] // N
Solve[Δexample[1000] == 0, s] // N
```

```
{{s → -5.}, {s → -2.}, {s → 3.}}

{{s → -5.80846 + 0. i}, {s → -1.70941 + 0. i}, {s → 2.51787 + 1.4803 × 10^-16 i}}

{{s → -13.4411}, {s → -0.279442 + 1.18739 i}, {s → -0.279442 - 1.18739 i}}

{{s → -102.196}, {s → -0.902196 + 1.94552 i}, {s → -0.902196 - 1.94552 i}}

{{s → -1002.02}, {s → -0.99002 + 1.99495 i}, {s → -0.99002 - 1.99495 i}}
```

We see that two poles still go to the open loop zeros but the last pole goes to -∞.

Note: One key insight we gain here is the fact that zeros act as “magnets” in the sense that they have the ability to attract poles. Later, when we design compensator we can leverage this fact by using the zeros to pull the root locus in desirable directions.

Therefore, combining this analysis, we arrive at our second rule

Rule 2

Root Locus Start ($K = 0$): The root locus must start at the open loop poles

Root Locus End ($K \to \infty$): The root locus must end at either

a. The open loop zeros (at least m poles go to these zeros)
b. Zeros at infinity (remaining $n - m$ poles go to these zeros)

Rule 3: Valid Regions on Real Axis are to the Left of Odd Numbered Pole/Zeros (start numbering from right to left)

Let us consider the ratio of two complex variables of the form u/v where u, v are complex variables. Note that we can write this using Euler's formula $\left(z = \left| z \right| e^{i\theta}\right)$

$$\frac{u}{v} = \frac{|u| e^{i\theta_u}}{|v| e^{i\theta_v}} = \frac{|u|}{|v|} e^{(\theta_u - \theta_v) i}$$

In a similar fashion, we can easily compute the product of two complex numbers using Euler's formula

$$u\,v = \left| u \right| e^{i\theta_u} \left| v \right| e^{i\theta_v} = \left| u \right| \cdot \left| v \right| e^{(\theta_u + \theta_v) i}$$

So we see that the angle of the quotient and product of two complex number, u and v, can be written as

$$\angle \frac{u}{v} = \angle u - \angle v \qquad \textbf{\textit{(Eq.7)}}$$

$$\angle u\,v = \angle u + \angle v$$

In a similar fashion, the magnitude of the quotient and product of u and v can be written as

$$\left| \frac{u}{v} \right| = \left| u \right| / \left| v \right| \qquad \textbf{\textit{(Eq.8)}}$$

$$\left| u\,v \right| = \left| u \right| \; \left| v \right|$$

Example: Compute Complex Number Products/Quotients

$u = 2 + 4\,i$

$v = -3 - 1\,i$

```
u = 2 + 4 I;
v = -3 - 1 I;
```

We can easily compute u/v using traditional techniques (see lecture02)

```
u / v
```

$-1 - i$

We can now compute this using Euler's formula

```
magU = (Re[u]^2 + Im[u]^2)^(1/2);
magV = (Re[v]^2 + Im[v]^2)^(1/2);
θU = ArcTan[Re[u], Im[u]];
θV = ArcTan[Re[v], Im[v]];
magU/magV Exp[(θU - θV) I]
```

$\sqrt{2}\, e^{i\left(\pi - \text{ArcTan}\left[\frac{1}{3}\right] + \text{ArcTan}[2]\right)}$

If we simplify this, we see that this is the same as our previous result and we can verify that $\frac{u}{v} = \frac{|u|}{|v|} e^{(\theta_u - \theta_v) i}$

```
magU/magV Exp[(θU - θV) I] == u/v // FullSimplify
```

True

We can now verify that $\angle \frac{u}{v} = \angle u - \angle v$

```
ArcTan[Re[u/v], Im[u/v]] == θU - θV
```

False

Although this appears false, this is due to the fact that Mathematica uses a negative angle to represent θ_v

```
θV * 180 / π // N
```

-161.565

We see that the relationship is true if we simply subtract off 2π from the left side of the expression

```
ArcTan[Re[u/v], Im[u/v]] == θU - θV - 2 π
```

True

We can now verify that $\left| \frac{u}{v} \right| = |u| / |v|$

```
(Re[u/v]^2 + Im[u/v]^2)^(1/2) == magU / magV
```

True

The same is true for the product of these two numbers. We can first verify that $u\,v = |u| \cdot |v|\, e^{(\theta_u + \theta_v) i}$

```
u v
magU magV Exp[(θU + θV) I]
magU magV Exp[(θU + θV) I] == u v // FullSimplify
```

-2 - 14 i

$10\sqrt{2}\, e^{i\left(-\pi + \text{ArcTan}\left[\frac{1}{3}\right] + \text{ArcTan}[2]\right)}$

True

We can now verify that $\angle u\,v = \angle u + \angle v$

```
ArcTan[Re[u v], Im[u v]] == θU + θV
True
```

We can now verify that $|uv| = |u|\ |v|$

```
(Re[u v]^2 + Im[u v]^2)^(1/2) == magU magV
True
```

End Example

We can now apply our result of Eq.7, which stated that $\angle u/v = \angle u - \angle v$ and $\angle u\, v = \angle u + \angle v$, to our root locus sketch.

Consider a general open loop transfer function of the form (note that we use α as the open loop transfer function gain since we are already using K as the root locus parameter).

$$G(s) = \frac{a(s)}{b(s)} = \frac{\alpha(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

From Eq.4, we know that the closed loop characteristic equation is

$$\Delta(s) = b(s) + K\,a(s) = 0$$

$$(s+p_1)(s+p_2)\ldots(s+p_n) + K\,\alpha(s+z_1)(s+z_2)\ldots(s+z_m) = 0$$

We can rewrite this as

$$\frac{\alpha(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)} = -\frac{1}{K} \qquad \textbf{(Eq.9)}$$

So we see that the left side of Eq.9 is simply a complex number. Therefore, the angle associated with the left side of Eq.9 is simply given by applying Eq.7 ($\angle u/v = \angle u - \angle v$ and $\angle u\, v = \angle u + \angle v$)

$$\angle \frac{\alpha(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)} = \angle\alpha + \angle(s+z_1) + \angle(s+z_2) + \ldots + \angle(s+z_m) - \angle(s+p_1) - \angle(s+p_2) - \ldots - \angle(s+p_n)$$

For now, if we assume that α is positive and real, then $\angle\alpha = 0$. So the above equation becomes

$$\angle \frac{\alpha(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)} = \angle(s+z_1) + \angle(s+z_2) + \ldots + \angle(s+z_m) - \angle(s+p_1) - \angle(s+p_2) - \ldots - \angle(s+p_n)$$

Since we are assuming $K \in [0, +\infty)$, then the angle associated with the right side of Eq.9 is simply

$$\angle -\frac{1}{K} = -\pi \pm 2\pi k \qquad k \in Z$$

So we can write

$$\angle(s+z_1) + \angle(s+z_2) + \ldots + \angle(s+z_m) - \angle(s+p_1) - \angle(s+p_2) - \ldots - \angle(s+p_n) = -\pi \pm 2\pi k \qquad k \in Z$$

$$\sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i) = -\pi \pm 2\pi k \qquad k \in Z \qquad \textbf{(Eq.10)}$$

In Eq.10, z_i and p_i are fixed, so we are looking for values of s which satisfy Eq.10.

Let us first investigate one of the terms from Eq.10. For example, let us consider the term

$\angle(s + z_k)$

We first consider that z_k is on the real axis. This implies that there is an open loop zero at $-z_k$. For example, a term $(s + 3)$ in $a(s)$ implies there is a zero at the location $s = -3$.

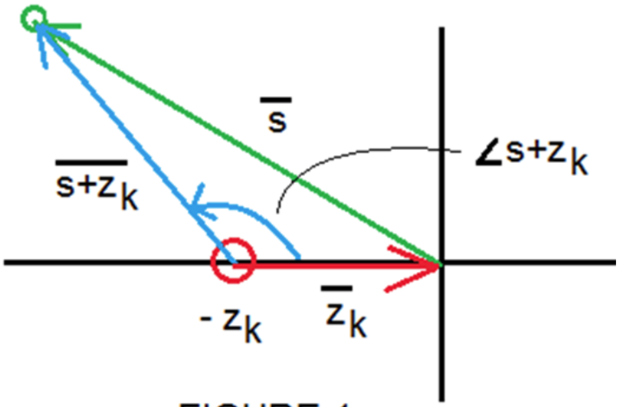


FIGURE 1
single zero on real axis

Graphically, we see that the vector $\overline{s + z_k}$ is obtained by simply drawing a vector from the zero location to the location of s. The angle can then be obtained by looking and the angle from horizontal to this vector

We can also consider the case where there is a complex conjugate pair (recall that we assume that all poles and zeros from the transfer function $G(s)$ are either real or come in complex conjugate pairs). In other words,

$z_{k,1}$ and $z_{k,2}$ are complex conjugate pairs

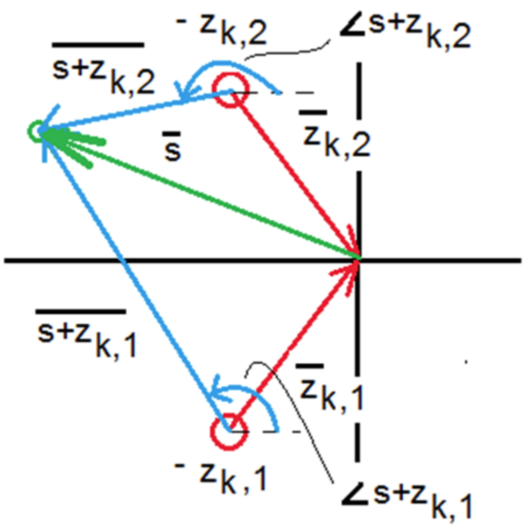


FIGURE 2
complex conjugate zeros

We see that the same graphical rules apply. Therefore, in order to find the angle associated with the quantity $\overline{s + z_k}$ for either a real or complex zero we proceed as follows.

1. Plot the location of the zero at $-z_k$
2. Plot the location of the candidate s location that you would like to investigate.
3. Draw a vector from $-z_k$ to s, this is the vector $\overline{s+z_k}$.
4. Compute $\angle\overline{s+z_k}$ by measuring angle from horizontal to the vector $\overline{s+z_k}$.

Let us now investigate the quantity $\angle(s+z_k)$ when s is on the real axis. We need to look at two situations,

Case 1. z_k is on the real axis

Case 2. z_k is one of a complex pair

Case 1: Zero is on the real axis

From Figure 1 we see that

$$\angle(s+z_k) = \begin{cases} 0 & \text{if } s \text{ is on the real axis to the right of the zero (at } -z_k) \\ \pi & \text{if } s \text{ is on the real axis to the left of the zero (at } -z_k) \end{cases}$$

Case 2: Zero is complex

From Figure 2 we see that if s is on the real axis, then the sum of the angle between s on the real axis and a complex conjugate pair is always zero.

$$\angle\left(s+z_{k,1}\right) + \angle\left(s+z_{k,2}\right) = 0$$

The same analysis holds true of the $(s+p_i)$ terms.

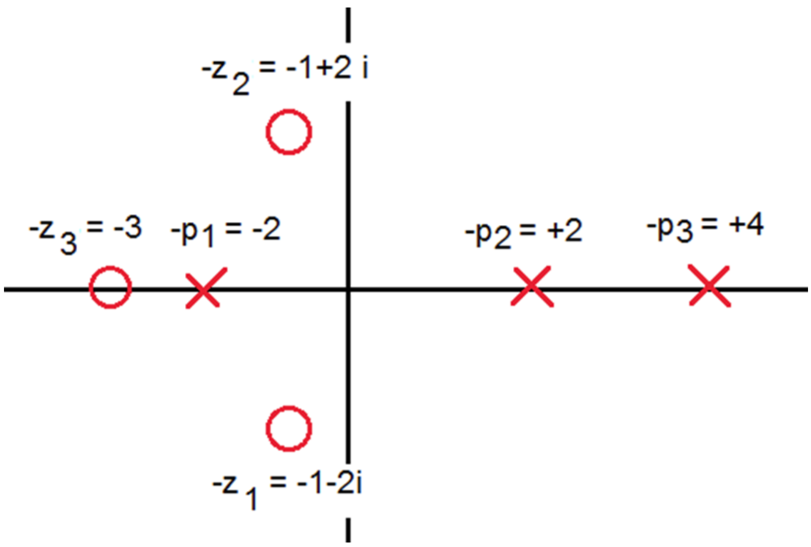
Let us now consider a more complicated example and see which locations on the real axis are valid regions for the root locus.

Example: Valid Regions on the Real Line

Consider the example open loop transfer function of

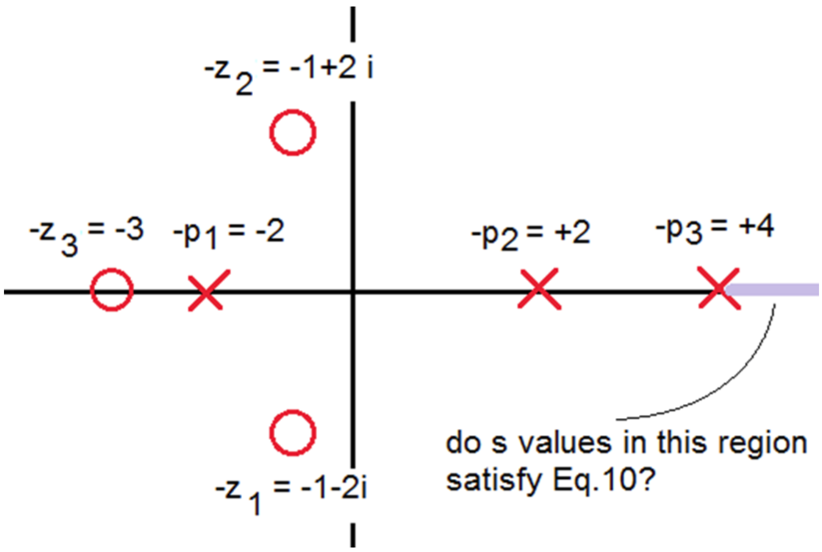
$$G_4(s) = \frac{(s+1+2\,i)(s+1-2\,i)(s+3)}{(s+2)(s-2)(s-4)} = \frac{(s+z_1)(s+z_2)(s+z_3)}{(s+p_1)(s+p_2)(s+p_3)}$$

We can sketch the pole/zero map of this transfer function as shown below



We now ask, "if s was on the real axis, which values of s will satisfy Eq.10?"

We first consider values of s which are to the right of the farthest right pole or zero. This location is shown below in purple



We have

$$\angle(s+z_1)+\angle(s+z_2)+\angle(s+z_3)-\angle(s+p_1)-\angle(s+p_2)-\angle(s+p_3)\overset{??}{=}-\pi\pm 2\pi k \qquad k\in Z$$

We can evaluate each term assuming that s is the purple region

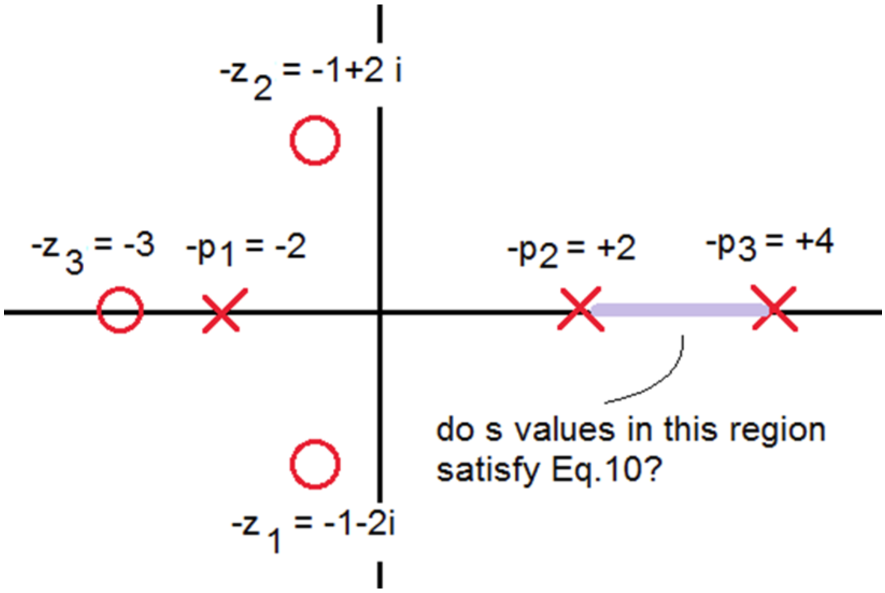
$$\angle(s+z_1)+\angle(s+z_2)=0 \qquad \text{(recall Figure 2)}$$
$$\angle(s+z_3)=0$$
$$\angle(s+p_1)=0$$
$$\angle(s+p_2)=0$$
$$\angle(s+p_3)=0$$

So we have

$$0\neq-\pi\pm 2\pi k$$

So we see that this is not a valid solution and therefore, the close loop poles cannot lie in the region shown in purple.

We can now move down by one section to the left (the section shown in purple) and ask the same question again.



We have

$$\angle(s+z_1)+\angle(s+z_2)+\angle(s+z_3)-\angle(s+p_1)-\angle(s+p_2)-\angle(s+p_3)\overset{??}{=}-\pi\pm 2\pi k \qquad k\in Z$$

We can evaluate each term assuming that s is the purple region

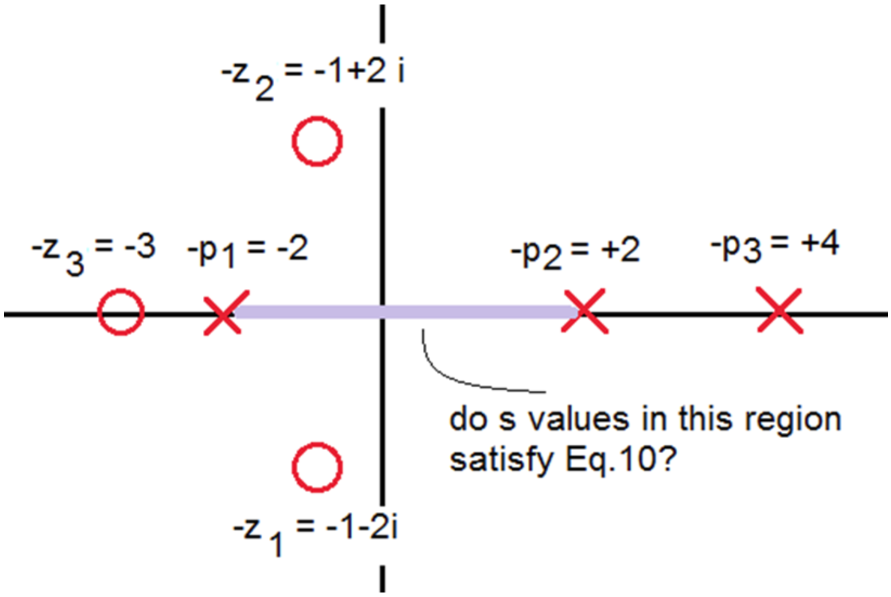
$$\angle(s+z_1)+\angle(s+z_2)=0 \qquad \text{(recall Figure 2)}$$
$$\angle(s+z_3)=0$$
$$\angle(s+p_1)=0$$
$$\angle(s+p_2)=0$$
$$\angle(s+p_3)=-\pi$$

So we have

$$\pi=-\pi\pm 2\pi k$$

So we see that this is a valid solution and therefore, the close loop poles can lie in the region shown in purple.

We can now move down by one section to the left (the section shown in purple) and ask the same question again.



We have

$$\angle(s+z_1)+\angle(s+z_2)+\angle(s+z_3)-\angle(s+p_1)-\angle(s+p_2)-\angle(s+p_3)\overset{??}{=}-\pi\pm 2\pi k \qquad k\in \mathbb{Z}$$

We can evaluate each term assuming that s is the purple region

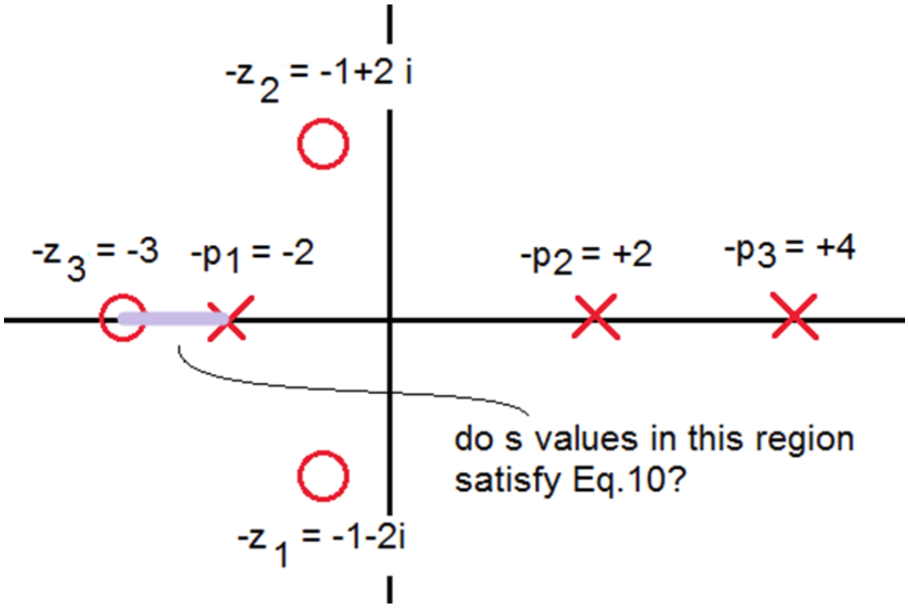
$$\angle(s+z_1)+\angle(s+z_2)=0 \qquad \text{(recall Figure 2)}$$
$$\angle(s+z_3)=0$$
$$\angle(s+p_1)=0$$
$$\angle(s+p_2)=-\pi$$
$$\angle(s+p_3)=-\pi$$

So we have

$$-2\pi\neq-\pi\pm 2\pi k$$

So we see that this is not a valid solution and therefore, the close loop poles cannot lie in the region shown in purple.

We can now move down by one section to the left (the section shown in purple) and ask the same question again.



We have

$$\angle(s+z_1)+\angle(s+z_2)+\angle(s+z_3)-\angle(s+p_1)-\angle(s+p_2)-\angle(s+p_3)\overset{??}{=}-\pi\pm 2\,\pi\,k \qquad k\in Z$$

We can evaluate each term assuming that s is the purple region

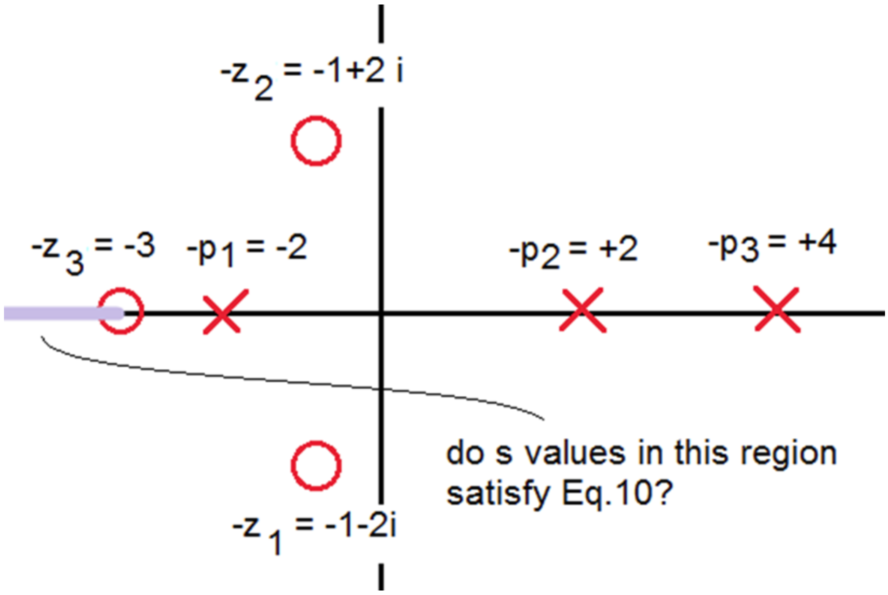
$$\angle(s+z_1)+\angle(s+z_2)=0 \qquad\qquad \text{(recall Figure 2)}$$
$$\angle(s+z_3)=0$$
$$\angle(s+p_1)=-\pi$$
$$\angle(s+p_2)=-\pi$$
$$\angle(s+p_3)=-\pi$$

So we have

$$-3\,\pi=-\pi\pm 2\,\pi\,k$$

So we see that this is a valid solution and therefore, the close loop poles can lie in the region shown in purple.

We can now move down by one section to the left (the section shown in purple) and ask the same question again.



We have

$$\angle(s+z_1)+\angle(s+z_2)+\angle(s+z_3)-\angle(s+p_1)-\angle(s+p_2)-\angle(s+p_3) \overset{??}{=} -\pi \pm 2\pi k \qquad k \in Z$$

We can evaluate each term assuming that s is the purple region

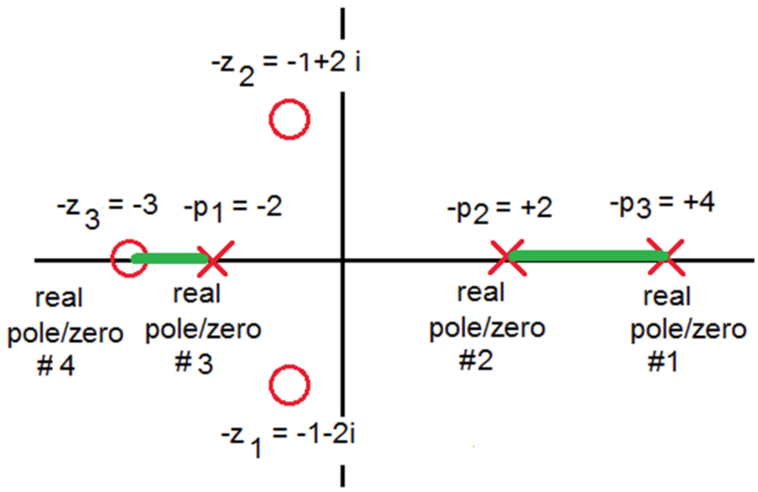
$$\angle(s+z_1)+\angle(s+z_2)=0 \qquad \text{(recall Figure 2)}$$
$$\angle(s+z_3)=-\pi$$
$$\angle(s+p_1)=-\pi$$
$$\angle(s+p_2)=-\pi$$
$$\angle(s+p_3)=-\pi$$

So we have

$$-4\pi \neq -\pi \pm 2\pi k$$

So we see that this is not a valid solution and therefore, the close loop poles cannot lie in the region shown in purple.

So the overall locations on the real axis where s can lie and still satisfy Eq.10 is shown below in green. We note that the angle associated with the pair of complex poles and/or zeros always sums to 0 so they are not important in the calculation.



This example illustrates the pattern that dictates where on the real axis the root locus may lie

Rule 3
Valid regions of the root locus on the real axis are those to the
left of odd number real poles/zeros (start numbering from right to left)

Rule 4: Angle of Asymptotes and Centroid of Asymptotes

We can combine rules 1, 2, and 3 to begin sketching a root locus for a simple system

Example:
Consider the example open loop transfer function of

$$G_5(s) = \frac{2\,(s+1)}{s^2+2\,s+2}$$

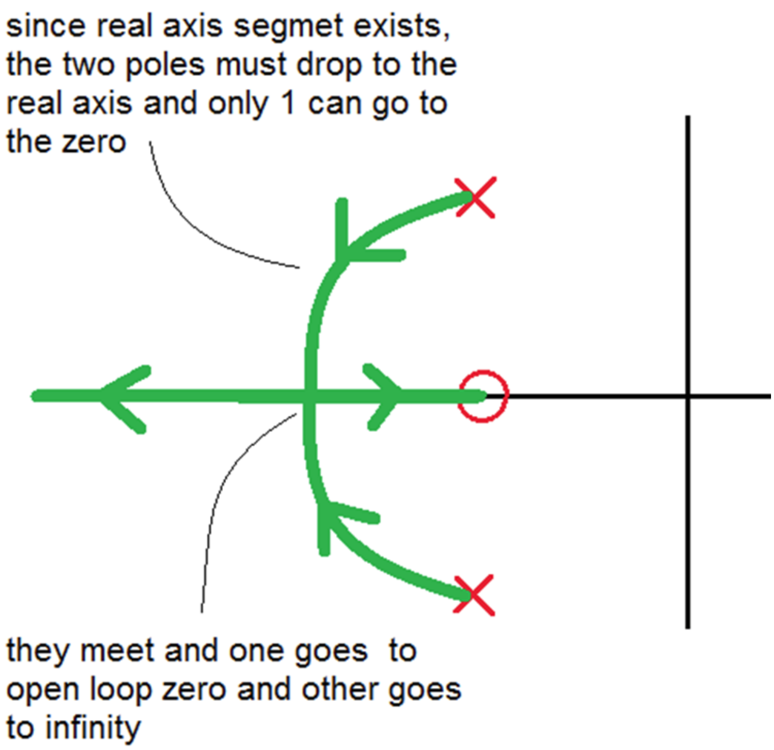
We can factor this as shown below

$$G_5(s) = \frac{2\,(s+1)}{(s+1+i)\,(s+1-i)}$$

We instantly see that

Rule 1: closed loop transfer function will have 2 poles
Rule 2: one of the closed loop poles must go to the single open loop zero,
other closed loop pole must go to a zero at infinity
Rule 3: valid region on the real axis is to the left of the only zero (#1) on real axis

We can quickly sketch the root locus based on these rules

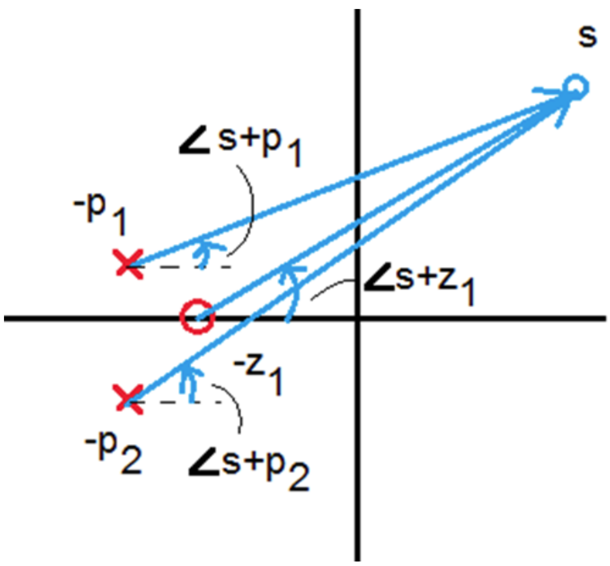


From our previous example, we saw that one of the closed loop poles goes to a zero at infinity. This leads to the question, “How many closed loop poles go to zeros at infinity?”. From our previous discussion, we can see that the number of closed loop poles that go to zeros at infinity must be the remaining poles that do not go to a open loop zero. In other words, the number of excess poles

$$\text{number zeros at infinity} = \text{num poles - num zeros} = n - m \qquad \textbf{(Eq.11)}$$

So we see that $n - m$ poles will tend towards infinity. We now ask, what is the angle of the asymptotes that these poles approach as they go to the zero at infinity.

Consider a pole/zero map of an open loop transfer function as shown below. In this case, we choose a value of s which is very far from away (tending towards infinity)



If s (our candidate point to be on the root locus) is far away, then the angles shown in the diagram are all nearly equal. Let us denote this angle as ϕ_A (angle of asymptote)

$\angle(s+p_1) \approx \phi_A$
$\angle(s+p_2) \approx \phi_A$ (when s is far away) ***(Eq.12)***
$\angle(s+z_1) \approx \phi_A$

So we see that if the point s is in the root locus, then this must satisfy the angle condition of Eq.10

$$\sum_{i=1}^{m} \angle(s+z_i) - \sum_{i=1}^{n} \angle(s+p_i) = -\pi \pm 2\pi k \qquad k \in Z$$

$$\angle(s+z_1) + \angle(s+z_2) + \ldots + \angle(s+z_m) - (\angle(s+p_1) + \angle(s+p_2) + \ldots + \angle(s+p_n)) = -\pi \pm 2\pi k$$

As s is moves far away, we can apply Eq.12 to obtain

$$\underbrace{\phi_A + \phi_A + \ldots + \phi_A}_{m \text{ times}} - (\underbrace{\phi_A + \phi_A + \ldots + \phi_A}_{n \text{ times}}) = -\pi \pm 2\pi k$$

$$m\,\phi_A - n\,\phi_A = -\pi \pm 2\pi k$$

$$(m-n)\,\phi_A = -\pi \pm 2\pi k$$

$$\phi_A = \frac{-\pi \pm 2\pi k}{m-n}$$

$$\phi_A = \frac{(1 \pm 2k)\pi}{n-m} \qquad k = 0, 1, \ldots, n-m-1$$

We saw that previously, when $n-m=1$ (1 excess pole), then there was 1 branch/asymptote where the closed loop poles tended towards a zero at infinity. Therefore, there should be $n-m$ asymptotes. So we can rewrite the previous expression for the angle of asymptotes as (taking care not to be off by 1 when counting the index k. ***Star Wars Joke***: This is sometimes called the O-B-1 or Obi-Wan error).

$$\phi_A = \frac{(1 \pm 2k)\pi}{n-m} \qquad k = 0, 1, \ldots, n-m-1 \qquad \textbf{(Eq.13)}$$

We can also compute the centroid of the asymptotes. The above analysis of the asymptote angle can be used to deduce that all the asymptotes must should converge at the location which is the centroid of the poles and zeros. In other words, the convergence point of the asymptotes is given by

$$\sigma_A = \frac{\sum \text{location of poles} - \sum \text{location of zeros}}{n-m} = \frac{\sum_{k=1}^{n}(-p_k) - \sum_{k=1}^{m}(-z_k)}{n-m} \qquad \textbf{(Eq.14)}$$

Rule 4

Angle of asymptotes and centroid of asymptotes can be computed as previously mentioned

Example

If we revisit our previous example where $n=2$, $m=1$

$$G_5(s) = \frac{2(s+1)}{(s+1+i)(s+1-i)}$$

Then we calculate the asymptotes as

$$\phi_A = \frac{(1 \pm 2\,k)\,\pi}{n-m} \qquad \text{recall: } n = 2,\ m = 1 \text{ so } k \text{ only goes to } 0$$

$$= \frac{(1 \pm 2 * 0)\,\pi}{2-1}$$

$$\phi_A = \frac{\pi}{1} = \pi$$

This states that there is 1 asymptote with an angle of π (which is exactly the situation we encountered in our previous sketch).

Example

We can consider another example of

$$G_6(s) = \frac{s+3}{s^4+8\,s^3+s^2-138\,s-232} = \frac{(s+3)}{(s+2)\,(s+5+2\,i)\,(s+5-2\,i)\,(s-4)}$$

We can determine the z_k and p_k terms

$$z_1 = 3$$

$$p_1 = 2$$
$$p_2 = 5 + 2\,i$$
$$p_3 = 5 - 2\,i$$
$$p_4 = -4$$

So we have

$$n = 4, m = 1 \qquad \Rightarrow n - m = 3 \text{ zeros at infinity}$$

We can compute the angle of asymptotes using Eq.13

$$\phi_A = \frac{(1 \pm 2\,k)\,\pi}{n-m} \qquad k = 0,\ 1,\ 2$$

```
Table[{"k=", k, " ϕA (rad)=", (1 + 2 k) π/(4 - 1), " ϕA (deg)=", (1 + 2 k) π/(4 - 1) * 180/π}, {k, 0, 2}] //
 TableForm
```

k=	0	ϕA (rad) =	$\frac{\pi}{3}$	ϕA (deg) =	60
k=	1	ϕA (rad) =	π	ϕA (deg) =	180
k=	2	ϕA (rad) =	$\frac{5\,\pi}{3}$	ϕA (deg) =	300

We can then compute the centroid of the asymptotes using Eq.14

$$\sigma_A = \frac{\sum_{k=1}^{n}(-p_k) - \sum_{k=1}^{m}(-z_k)}{n-m}$$

$$= \frac{-p_1-p_2-p_3-p_4-(-z_1)}{4-1}$$

$$= \frac{-2-(5+2\,i)-(5-2\,i)+4-(-3)}{4-1}$$

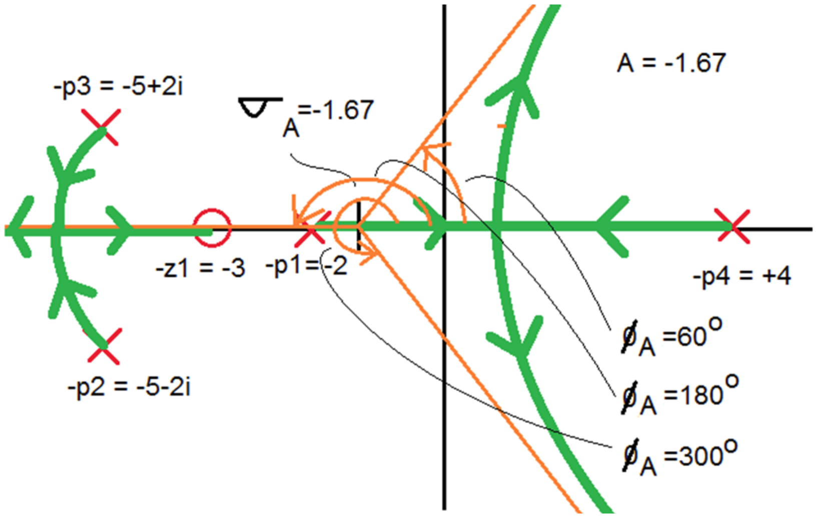
$$\frac{-2 - (5 + 2\,I) - (5 - 2\,I) + 4 - (-3)}{4 - 1}$$

$$-\frac{5}{3}$$

So we have

$$\sigma_A = -\frac{5}{3} \approx -1.67$$

We can now sketch the root locus as shown below. The asymptotes are shown in orange, the actual root locus is shown in green.



Rule 5: Angle of Departure from Complex Poles

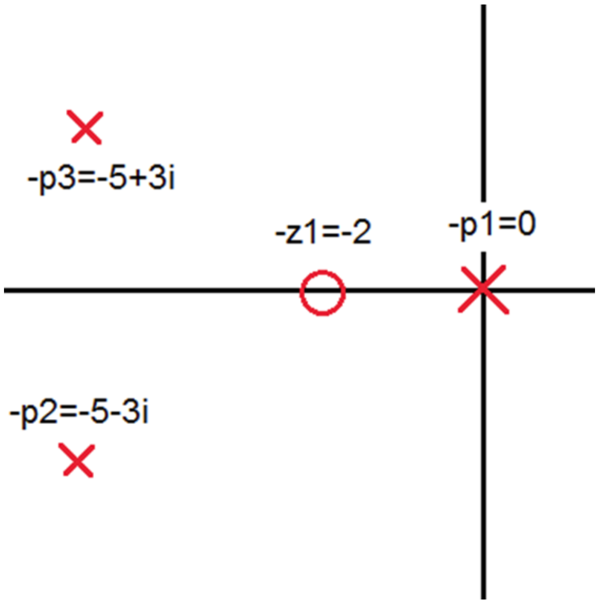
We can now consider the angle of departure from the complex poles.

Example: Angle of Departure and Full Root Locus Sketch

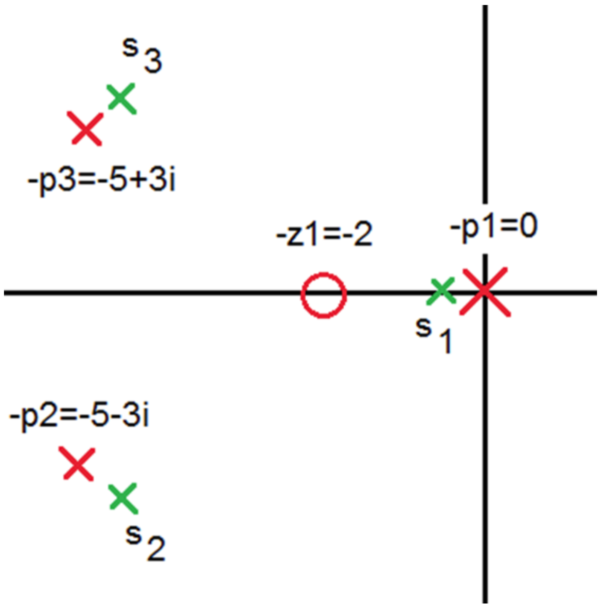
Consider the following example open loop transfer function

$$G_7(s) = \frac{s+2}{s^3+10\,s^2+34\,s} = \frac{(s+2)}{s(s+5+3\,i)\,(s+5-3\,i)}$$

We can sketch the open loop pole/zero map as shown below



The angle of departure from the open loop complex poles is defined as the angle where the roots leave the open loop complex poles. We see that this corresponds to the vector created from $-p_2$ to the root locus when K is slightly increased from 0. We can draw the root locus with $K = \varepsilon$ (small number). Therefore, the closed loop poles should be infinitely close to their open loop locations.



We already know that s_1 is on the root locus because it is to the left of an odd numbered pole/zero as we count from right to left (observation 3). However, we can ask, "what are the conditions of s_2 and s_3 for them to be included in the root locus?"

Recall that Eq.10 for this system becomes

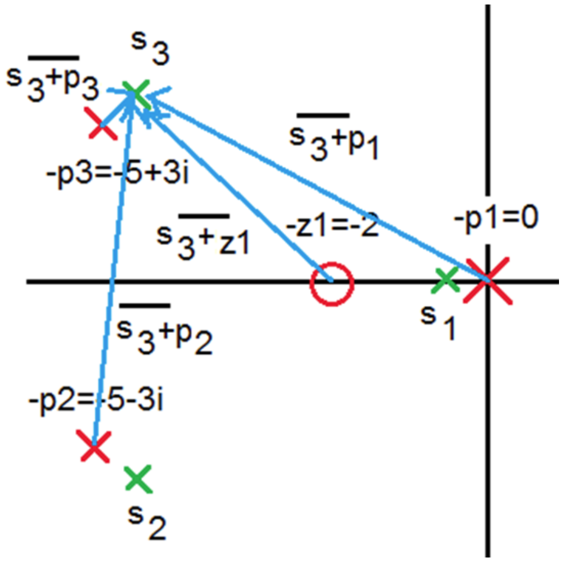
$$\sum_{i=1}^{m} \angle(s + z_i) - \sum_{i=1}^{n} \angle(s + p_i) = -\pi \pm 2\pi k \qquad k \in Z$$

$$\angle(s + z_1) - \angle(s + p_1) - \angle(s + p_2) - \angle(s + p_3) = -\pi \pm 2\pi k$$

We can now consider the candidate point of s_3 and see what are the conditions for it to be included in the root locus.

$$\angle(s_3 + z_1) - \angle(s_3 + p_1) - \angle(s_3 + p_2) - \angle(s_3 + p_3) = -\pi \pm 2\pi k \qquad \textbf{(Eq.15)}$$

We now assume that s_3 is very close to p_3 (meaning that the gain K is infinitesimally small). The picture associated with this becomes



We note that since s_3 is infinitely close to $-p_3$, then

$$\angle(s_3 + p_2) = 90 \frac{\pi}{180} \quad \text{(90 degrees).}$$

In a similar fashion, because s_3 is infinitely close to $-p_3$, we can compute $\angle s_3 + p_1$

$$\angle(s_3 + p_1) = \text{atan2}(x, y) = \text{atan2}(-5 - 0, 3 - 0) = 149.036 \frac{\pi}{180}$$

```
ArcTan[-5 - 0, 3 - 0] * 180 / π // N
149.036
```

In a similar fashion, because s_3 is infinitely close to $-p_3$, we can compute $\angle s_3 + z_1$

$$\angle(s_3 + z_1) = \text{atan2}(x, y) = \text{atan2}(-5 - (-2), 3 - 0) = 135 \frac{\pi}{180}$$

```
ArcTan[-5 - (-2), 3 - 0] * 180 / π // N
135.
```

Furthermore, if we see that $\angle s_3 + p_3$ is actually the angle of departure from the pole at $-p_3$ (because s_3 is infinitely close to $-p_3$). We can call this angle θ_3.

$$\angle s_3 + p_3 = \theta_3 \qquad \text{(angle of departure from pole at } -p_3)$$

So Eq.15 becomes (in degrees)

$$\angle(s_3 + z_1) - \angle(s_3 + p_1) - \angle(s_3 + p_2) - \angle(s_3 + p_3) = -180^o \pm 360^o k$$

$$135^o - 149.036^o - 90^o - \theta_3 = -180^o \pm 360^o k$$

$\theta_3 = 135^o - 149.036^o - 90^o + 180^o \pm 360^o k$

$\theta_3 = 75.964^o \pm 360\, k$

```
135 - 149.036 - 90 + 180
```

```
75.964
```

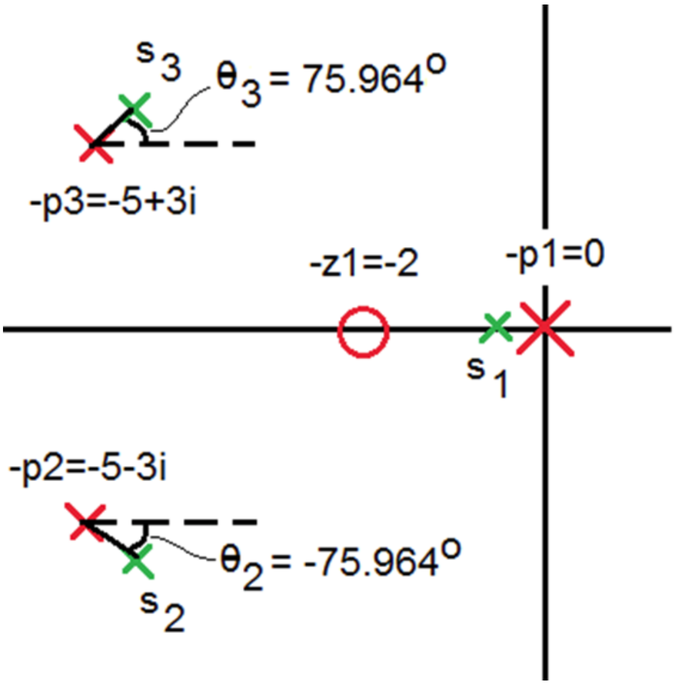
So we obtain

$\theta_3 = 75.964^o$ (angle of departure from pole at $-p_3$)

Since the root locus is symmetrical, we see that the angle of departure from the pole at $-p_2$ is

$\theta_2 = -75.964^o = 284.036^o$ (angle of departure from pole at $-p_2$)

So the root locus becomes



In general, we see that to compute the angle of departure from pole p_j we use

$$\theta_j = \angle(s_j + p_j) = \pi \pm 2\pi k + \sum_{i=1}^{m} \angle(p_j + z_i) - \sum_{i=1, i \neq j}^{n} \angle(p_j + p_i) \qquad \textbf{(Eq.16)}$$

Note that the angle of the poles are summed over all poles <u>except</u> the complex pole in question, p_j.

Rule 5

Angle of departure from a complex pole can be computed as previously mentioned

We can combine this with our other observations to flesh out the root locus.

We can now calculate the angle of the asymptotes using Eq.13

$$\phi_A = \frac{(1 \pm 2k)\pi}{n-m} \qquad k = 0,\ 1,\ \ldots,\ n-m-1$$

$$= \frac{(1 \pm 2k)\pi}{3-1} \qquad k = 0,\ 1$$

$$\phi_A = \frac{\pi}{2},\ \frac{3\pi}{2} \qquad (90^o,\ 270^o)$$

We can compute the asymptote centroid using Eq.14

$$\sigma_A = \frac{\sum \text{location of poles} - \sum \text{location of zeros}}{n-m} = \frac{\sum_{k=1}^{n}(-p_k) - \sum_{k=1}^{m}(-z_k)}{n-m}$$

$$= \frac{\sum_{k=1}^{3}(-p_k) - \sum_{k=1}^{1}(-z_k)}{3-1}$$

$$= \frac{-p_1 - p_2 - p_3 - (-z_1)}{3-1}$$

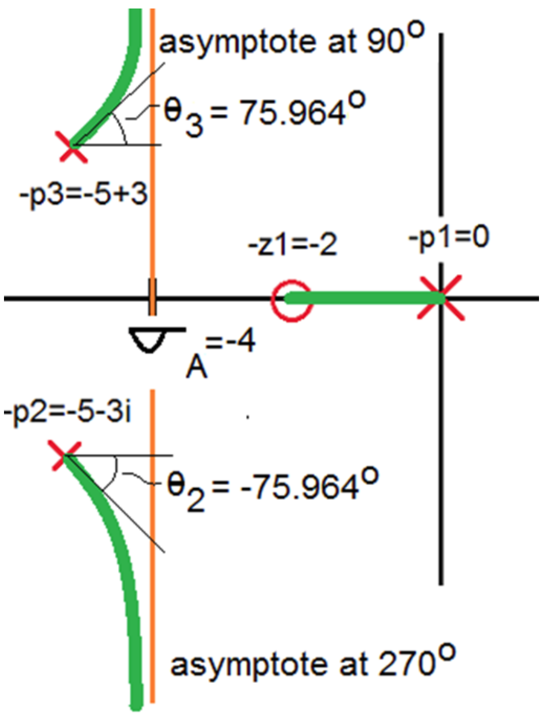
$$= \frac{-0-(5+3i)-(5-3i)-(-2)}{3-1}$$

$$\sigma_A = -4$$

$$\frac{\mathbf{0 - (5 + 3\,I) - (5 - 3\,I) - (-2)}}{\mathbf{3 - 1}}$$

-4

So the final root locus sketch appears as



The aforementioned 5 rules allow you to sketch a very reasonable root locus.

Summary of Rules 1-5

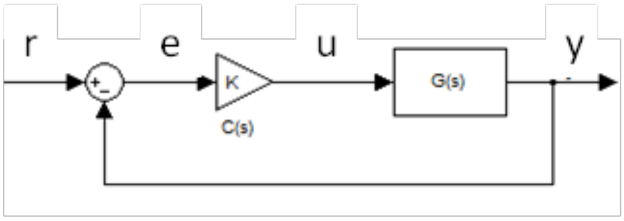
Consider an open loop transfer function of the form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{a(s)}{b(s)} = \frac{\alpha(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)}$$

m = number of zeros

n = number of poles

With the feedback architecture shown below



Rule 1 : Number of Poles Is Unchanged

number of poles of $T(s)$ = number of poles of $G(s) = n$

Rule 2: Closed Loop Poles Start at Open Loop Poles and Go to Open Loop Zeros

Root Locus Start ($K = 0$): The root locus must start at the open loop poles

Root Locus End ($K \to \infty$): The root locus must end at either

a. The open loop zeros (at least m poles go to these zeros)

b. Zeros at infinity (remaining $n - m$ poles go to these zeros)

Rule 3: Valid Regions On Real Axis Are to the Left of Odd Numbered Pole/Zeros (starting numbering from right to left)

Valid regions of the root locus on the real axis are those to the left of odd number real poles/zeros starting counting from right to left.

Rule 4: Angle of Asymptotes and Centroid of Asymptotes

The angle of asymptotes is given by

$$\phi_A = \frac{(1 \pm 2k)\pi}{n-m} \qquad k = 0, 1, \ldots, n-m-1$$

$$\sigma_A = \frac{\sum \text{location of poles} - \sum \text{location of zeros}}{n-m} = \frac{\sum_{k=1}^{n}(-p_k) - \sum_{k=1}^{m}(-z_k)}{n-m}$$

Rule 5: Angle of Departure from Complex Poles

$$\theta_j = \angle(s_j + p_j) = \pi \pm 2\pi k + \sum_{i=1}^{m} \angle(p_j + z_i) - \sum_{i=1, i \neq j}^{n} \angle(p_j + p_i)$$

Additional Rules

If additional fidelity is needed, additional rules can be developed.

Rule 6: Angle of Arrival to Complex Zeros

One can derive an expression for the angle of arrival to a pair of complex zeros.

Rule 7: Locus Breaks Out/In on Real Axis at 90^o

When the root locus leaves/enters the real axis, it does so perpendicularly at 90^o

Rule 8: Complex Roots Come in Pairs

Complex roots come in conjugate pairs. Therefore for the root locus is always symmetrical about the x-axis (the real axis).

Rule 9: The Root Locus Will Not Cross Over Itself

In other words, each value of K corresponds to a single solution set for s.

Rule 10: Number of Lines

There are max(n, m) lines

where n = order of denominator (number of open loop poles)

m = order of numerator (number of open loop zeros)

Example: Complex Transfer Function

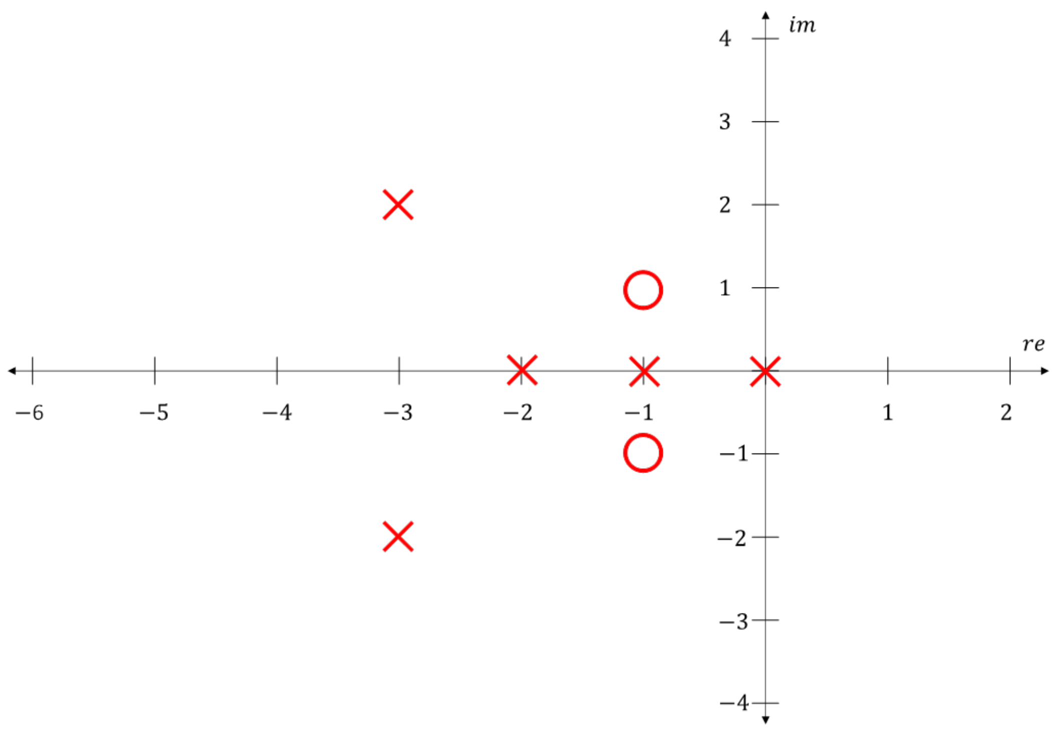
Let us consider an open loop transfer function of the form

$$G_8(s) = \frac{s^2+2\,s+2}{s^5+9\,s^4+33\,s^3+51\,s^2+26\,s} = \frac{(s+1+i)\,(s+1-i)}{(s+3+2\,i)\,(s+3-2\,i)\,(s+2)\,(s+1)\,s}$$

```
num8 = s^2 + 2 s + 2;
den8 = s^5 + 9 s^4 + 33 s^3 + 51 s^2 + 26 s;

(*find roots*)
Solve[num8 == 0, s]
Solve[den8 == 0, s]

(*check factorization*)
num8alt = (s + 1 + I) (s + 1 - I);
den8alt = (s + 3 + 2 I) (s + 3 - 2 I) (s + 2) (s + 1) s;

num8alt == num8 // Simplify
den8alt == den8 // Simplify
```

```
{{s → -1 - i}, {s → -1 + i}}

{{s → -3 - 2 i}, {s → -3 + 2 i}, {s → -2}, {s → -1}, {s → 0}}

True

True
```

The open loop pole/zero map is shown below



We can now apply our rules

Rule 1 : Number of Poles Is Unchanged

$n = 5$

Rule 2: Closed Loop Poles Start at Open Loop Poles and Go to Open Loop Zeros

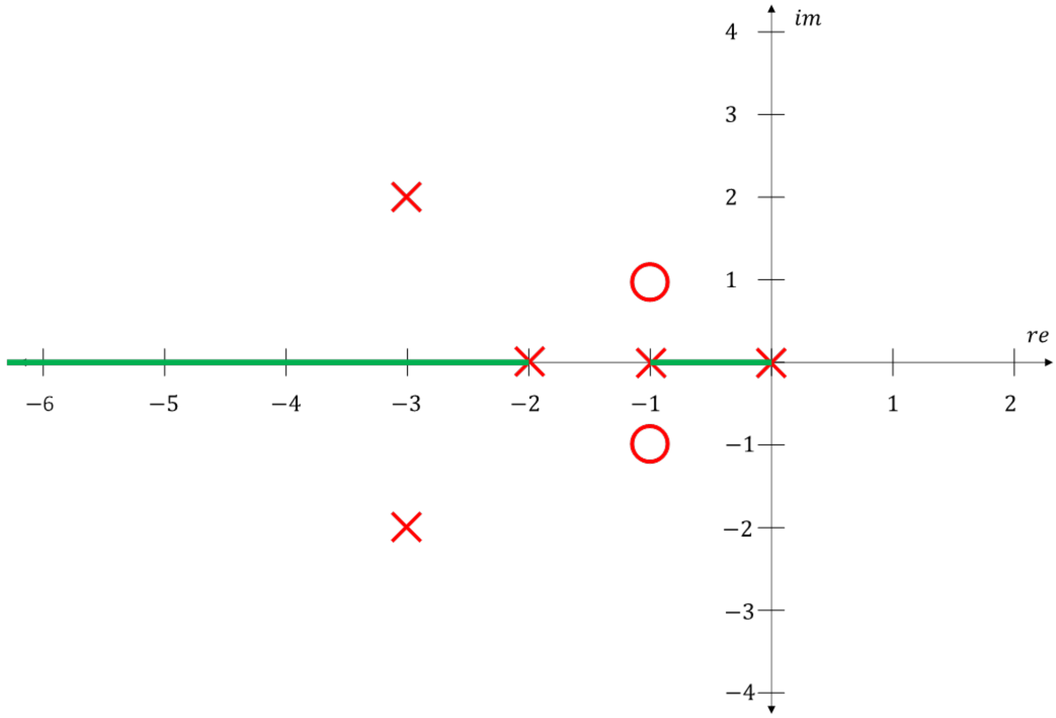
$m = 2$ poles go to open loop zeros

$n - m = 5 - 2 = 3$ poles go to zeros at infinity (3 asymptotes)

Rule 3: Valid Regions On Real Axis Are to the Left of Odd Numbered Pole/Zeros (starting

numbering from right to left)

The root locus becomes



Rule 4: Angle of Asymptotes and Centroid of Asymptotes

$$\phi_A = \frac{(1 \pm 2\,k)\,\pi}{n-m} \qquad k = 0,\ 1,\ \ldots,\ n-m-1$$

With $n = 5$, $m = 2$, we have $k = 0,\ 1,\ 2$

$$\phi_{A,0} = \frac{(1 \pm 2 * 0)\,\pi}{3} = \frac{\pi}{3} = 60^o \qquad (k = 0)$$

$$\phi_{A,1} = \frac{(1 \pm 2 * 1)\,\pi}{3} = \pi = 180^o \qquad (k = 1)$$

$$\phi_{A,2} = \frac{(1 \pm 2 * 2\,)\,\pi}{3} = \frac{5\,\pi}{3} = \pi = 300^o \qquad (k = 2)$$

The centroid of asymptotes

$$\sigma_A = \frac{\sum_{k=1}^{n}(-p_k) - \sum_{k=1}^{m}(-z_k)}{n-m}$$

$$= \frac{\sum_{k=1}^{5}(-p_k) - \sum_{k=1}^{2}(-z_k)}{5-2}$$

$$= \frac{(-p_1)+(-p_2)+(-p_3)+(-p_4)+(-p_5)-[(-z_1)+(-z_2)]}{5-2}$$

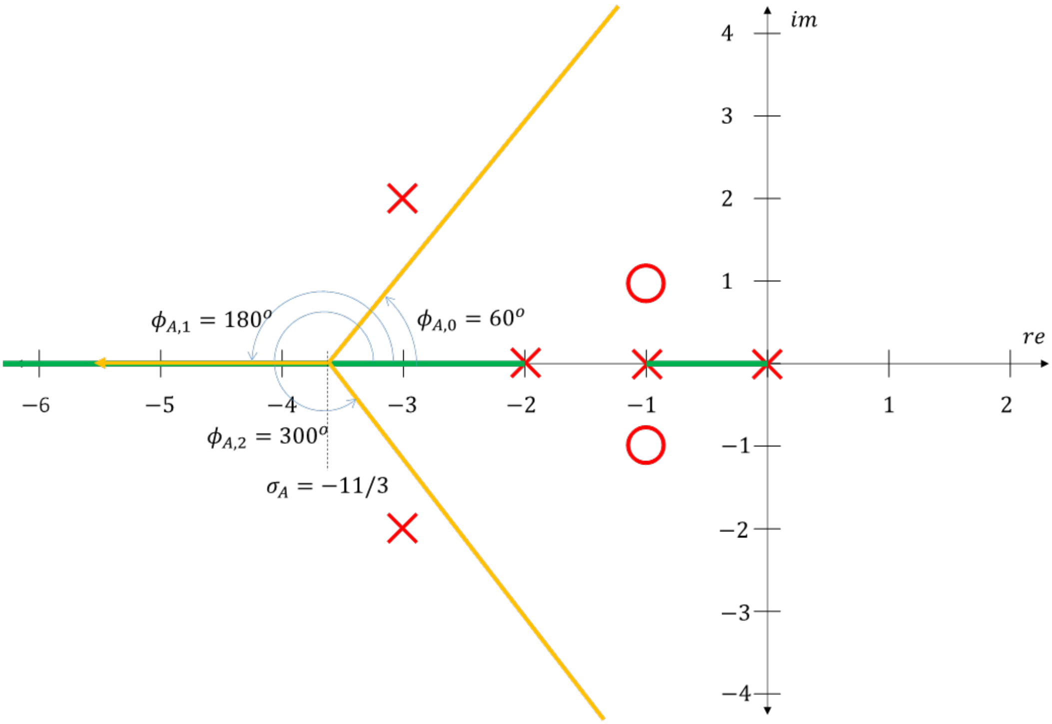
$$= \frac{(-(3+2\,i))+(-(3-2\,i))+(-2)+(-1)+(-0)-[(-(1+i))+(-(1-i))]}{3}$$

```
σA = ((- (3 + 2 I)) + (- (3 - 2 I)) + (-2) + (-1) + (0) + ((- (1 + I)) + (- (1 - I))))/3
σA // N
```

$-\frac{11}{3}$

-3.66667

So the root locus becomes



Rule 5: Angle of Departure from Complex Poles

Recall that the angle of departure is given by

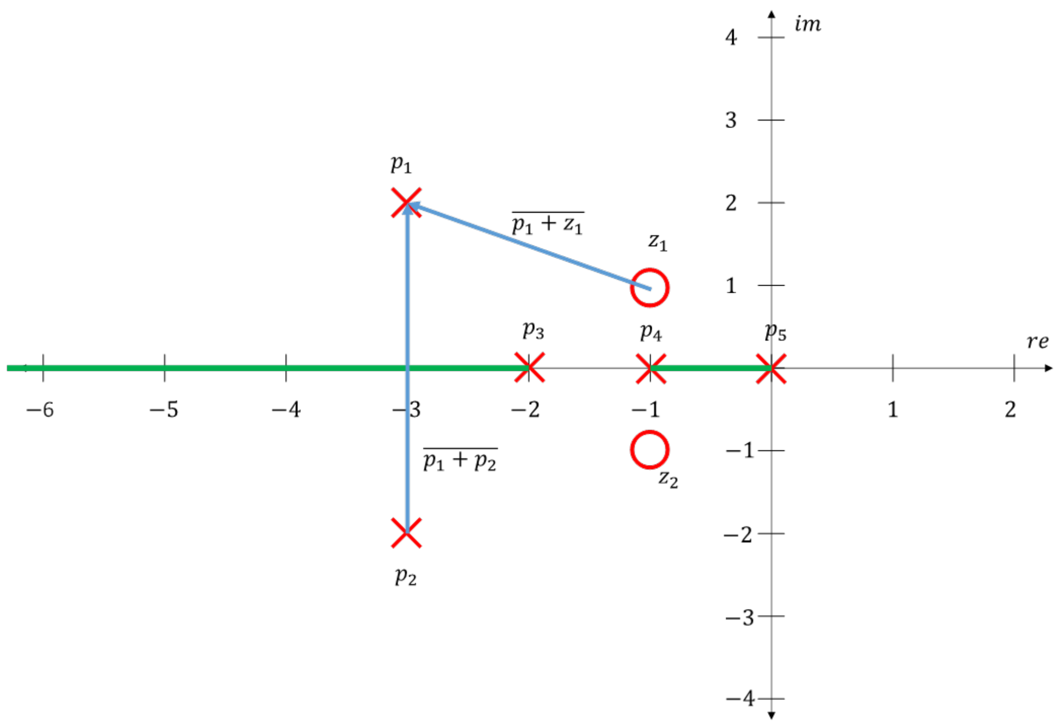
$$\theta_j = \angle(s_j + p_j) = \pi \pm 2\pi k + \sum_{i=1}^{m} \angle(p_j + z_i) - \sum_{i=1, i\neq j}^{n} \angle(p_j + p_i)$$

If we denote the top complex pole at p_1, we can write ($j = 1$)

$$\theta_1 = \angle(s_1 + p_1) = \pi \pm 2\pi k + \sum_{i=1}^{2} \angle(p_1 + z_i) - \sum_{i=1, i\neq j}^{5} \angle(p_1 + p_i)$$

$$= \pi \pm 2\pi k + [\angle(p_1 + z_1) + \angle(p_1 + z_2)] - [\angle(p_1 + p_2) + \angle(p_1 + p_3) + \angle(p_1 + p_4) + \angle(p_1 + p_5)]$$

If we label the poles/zeros as shown below, we can also measure the relevant angles (we only show the vectors for one pole and one zero)




```
anglep1z1 = ArcTan[-2, 1];
anglep1z2 = ArcTan[-2, 3];
anglep1p2 = ArcTan[0, 4];
anglep1p3 = ArcTan[-1, 2];
anglep1p4 = ArcTan[-2, 2];
anglep1p5 = ArcTan[-3, 2];

(*In degrees*)
anglep1z1 * 180 / π // N
anglep1z2 * 180 / π // N
anglep1p2 * 180 / π // N
anglep1p3 * 180 / π // N
anglep1p4 * 180 / π // N
anglep1p5 * 180 / π // N
```

```
153.435

123.69

90.

116.565

135.

146.31
```

So we have

$$\theta 1 = \pi + (\text{anglep1z1} + \text{anglep1z2}) - (\text{anglep1p2} + \text{anglep1p3} + \text{anglep1p4} + \text{anglep1p5})$$

$$\theta 1 * \frac{180}{\pi} \; // \; N$$

$$-\frac{\pi}{4} - \text{ArcTan}\left[\frac{1}{2}\right] + \text{ArcTan}\left[\frac{2}{3}\right] - \text{ArcTan}\left[\frac{3}{2}\right] + \text{ArcTan}[2]$$

$$-30.75$$

So we can update our root locus sketch as shown below