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Lecture 05a
Line Integrals

The YouTube video entitled ‘Line Integrals’ that covers this lecture is located at https://y-outu.be/0sIsoJYmVVM

Introduction

A line integral is an integral where the function to be integrated is evaluated along a curve.

The function to be integrated may be a scalar field or a vector field. We investigate these two cases in the following sections.

Line Integral of a Scalar Function

In this scenario, the function to integrate is a scalar field (or function). In other words, at each point in the domain, it returns a scalar value.

Recall the concept of a standard, definite integral

$$\int_a^b f(x)\,dx \qquad \textbf{(Eq.1)}$$

Eq.1 describes integrating along the x-axis from $x = a$ to $x = b$ with the integrand $f(x)$ which is a function defined at each point $x \in [a, b]$.

Show demo using paper cutout of integral.

In the case of a line integral, instead of integrating along a straight line (the x-axis in the previous discussion) we shall integrate along a curve C in space and the integrand will be a function defined at each point of C. The curve C is called the path of integration.

A normal integral integrates along the x-axis but a line integral considers integrating along an arbitrary path $\bar{r}(t)$ instead of along the x-axis.

We first consider the case of a function defined in the xy plane and a curve C given by the parametric equations (see video entitled ‘Parameterizing Curves’ at https://youtu.be/MPcfaNIREN0).

$C = \{x, y \mid x = x(t),\ y = y(t),\ t \in [a, b]\}$

Note that C could alternatively be represented as

$\bar{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} \quad \text{for } t \in [a, b]$

If we assume that C is smooth (ie $\bar{r}'(t)$ is continuous and $\bar{r}'(t) \neq \bar{0}$), we can divide the parameter interval $[a, b]$ into n sub intervals $[t_{i-1}, t_i]$ each with length Δs_i.

We then choose a point x_i^* and y_i^* which is contained in the i^{th} sub arc which corresponds to the sub interval $[t_{i-1}, t_i]$.

We can then evaluate the function f at the point x_i^* and y_i^* (See Figure 1)

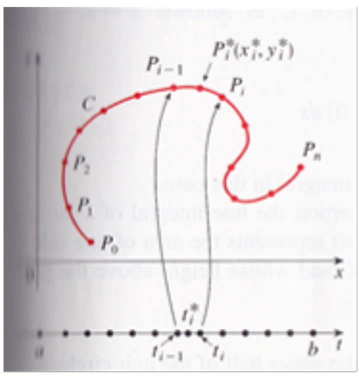


Figure 1

Finally we can form the sum

$\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$

which is similar to the Riemann sum. We now take the limit of these sums to define the line integral of the scalar function f along C as

$$\int_C f(x, y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i \qquad \textbf{\textit{(Eq.1)}}$$

Recall from our video entitled 'Arc Length (AKA Length of a Curve)' at https://youtu.be/FoiuvPkFppg we showed that the length of a curve can be given as

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \qquad \textbf{\textit{(Eq.2)}}$$

Considering small Δt_i segments allows us to use Eq.2 as the Δs_i terms in Eq.1.

Using the above arguments, Eq.1 can be rewritten as

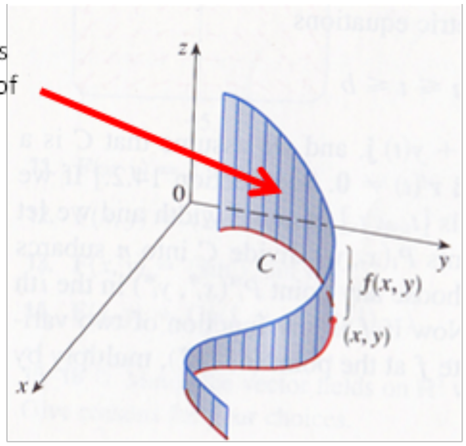
$$\int_C f(x,\, y)\, ds = \int_a^b f(x(t),\, y(t))\, \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \textbf{(Eq.3)}$$

Which gives a convenient expression to evaluate the line integral by integrating with respect to the parameter t

Just like a standard single integral, if the function f is positive over the curve C, we can interpret the line integral as the area of “fence” or “curtain” whose base is C and whose height above the point $(x,\, y)$ is $f(x,\, y)$.

Show demo using paper cutout of integral.

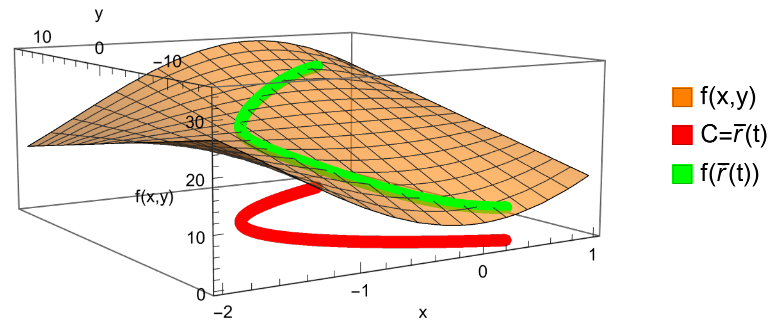
Line integral gives the surface area of one side of this shape



Here is a graphical example

In[ ]:=
```
f[x_, y_] = Cos[x] y + 20;

(*Plot options*)
opacity = 0.6;
lineThickness = 0.02;

(*Plot the surface*)
plotSurface = Plot3D[f[x, y], {x, -2, 1}, {y, -15, 15},
   (*Plot options*)
   AxesLabel → {"x", "y", "f(x,y)"},
   PlotStyle → Opacity[opacity],
   PlotRange → All];

(*Define path C*)
fx[t_] = 1/3 t^2 - 1;
fy[t_] = 6 t - 2;
fz[t_] = f[fx[t], fy[t]];

(*Plot the lines*)
plotLineDomain = ParametricPlot3D[{fx[t], fy[t], 0}, {t, -2, 2},
   (*Plot options*)
   PlotStyle → {Red, Thickness[lineThickness]}];
plotLineSurface = ParametricPlot3D[{fx[t], fy[t], fz[t]}, {t, -2, 2},
   (*Plot options*)
   PlotStyle → {Green, Thickness[lineThickness]}];

(*Composite figure*)
Legended[
 Show[plotSurface, plotLineDomain, plotLineSurface],

 (*Add legend information*)
 SwatchLegend[{Orange, Red, Green}, {"f(x,y)", "C=r̅(t)", "f(r̅(t))"}]
]
```

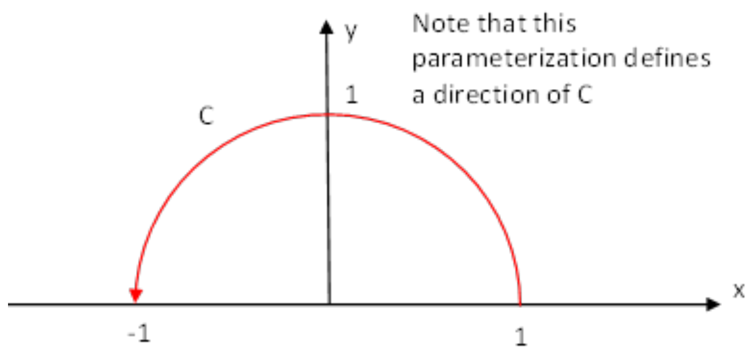
Out[ ]=



Example

Evaluate $\int_C (2 + x^2 y)\, ds$ where C is the upper half of the unit circle $x^2 + y^2 = 1$

The domain C is shown below



A convenient parametrization is

$$x(t) = \cos(t)$$
$$y(t) = \sin(t)$$

for $t \in [0, \pi]$

Note that this parametrization defines a direction of C as is shown in the figure.

So by Eq.3, we have

$$\int_C f(x, y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \int_0^{\pi} \left(2 + \cos^2(t)\sin(t)\right) \sqrt{(-\sin(t))^2 + (\cos(t))^2}\, dt$$

$$= \int_0^{\pi} \left(2 + \cos^2(t)\sin(t)\right) \sqrt{\sin^2(t) + \cos^2(t)}\, dt$$

$$= \int_0^{\pi} \left(2 + \cos^2(t)\sin(t)\right) dt$$

```
integrand = 2 + x^2 y /. {x → Cos[t], y → Sin[t]};
Integrate[integrand, {t, 0, π}]
Clear[integrand]
```

$\frac{2}{3} + 2\pi$

The physical interpretation of the line integral depends on the physical interpretation of the function f.

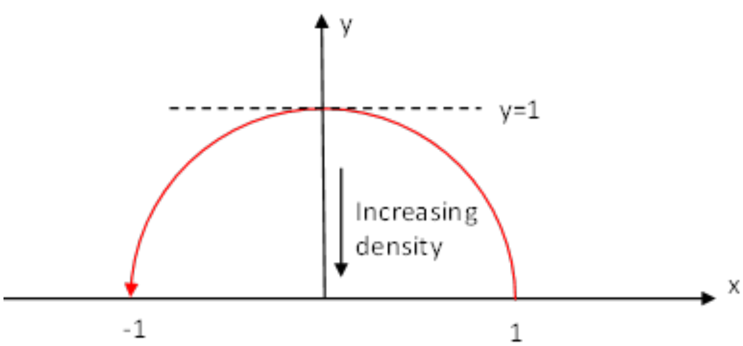
End Example

We now examine an example with a physical, engineering application.

Example: Weight of a variable thickness wire

A wire takes the shape of a semi-circle as shown below. The wire is denser near the x axis and the linear density (kg/m) is proportional to the distance from the line $y = 1$. Find

a. The mass of the wire
b. The center of mass of the wire



As with the previous example, we can use the parametrization

$$x(t) = \cos(t)$$
$$y(t) = \sin(t)$$

for $t \in [0, \pi]$

As stated, the linear density (kg/m) of the wire is given by

$$\rho(x, y) = k(1 - y) \qquad \text{where} \qquad k = \text{constant}$$

So the total mass of the wire is given by

$$m = \int_C \rho(x,\, y)\, ds$$

$$= \int_0^{\pi} k(1 - y(t))\, \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{recall: } y(t) = \sin(t)$$

$$= \int_0^{\pi} k(1 - \sin(t))\, \sqrt{\left(\frac{d}{dt}[\cos(t)]\right)^2 + \left(\frac{d}{dt}[\sin(t)]\right)^2}\, dt$$

$$= \int_0^{\pi} k(1 - \sin(t))\, \sqrt{\sin^2(t) + \cos^2(t)}\, dt$$

$$= \int_0^{\pi} k(1 - \sin(t))\, dt$$

```
m = Integrate[k (1 - y) /. {y → Sin[t]}, {t, 0, π}]
```

k (- 2 + π)

So we have

$$m = k(\pi - 2)$$

For the center of mass, we note that the general calculation of a center of mass is given by integrating an infinitesimal mass unit weighted by its position.

$$x_{cm} = \frac{1}{m} \int_C x\, \rho(x,\, y)\, ds$$

$$y_{cm} = \frac{1}{m} \int_C y\, \rho(x,\, y)\, ds$$

So for the x center of mass

$$x_{cm} = \frac{1}{k(\pi-2)} \int_0^{\pi} x(t)\, k(1 - y(t))\, ds \qquad \text{recall: } x(t) = \cos(t),\ y(t) = \sin(t)$$

$$= \frac{1}{k(\pi-2)} \int_0^{\pi} x(t)\, k(1 - y(t))\, \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \text{recall: } \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 1 \text{ for this example}$$

```
Integrate[x k (1 - y) /. {x → Cos[t], y → Sin[t]}, {t, 0, π}]
```

0

So we see that the integral is 0 so we obtain

$$x_{cm} = 0$$

which makes physical sense due to the symmetry of the problem.

For the y center of mass, we have

$$y_{cm} = \frac{1}{k(\pi-2)} \int_0^\pi y(t)\, k(1 - y(t))\, ds$$

```
Grid[{{
   1/(k (π - 2)) Integrate[y k (1 - y) /. {x → Cos[t], y → Sin[t]}, {t, 0, π}]
  }}, Background → Pink, Frame → True]
```

$$-\frac{-4+\pi}{2\times(-2+\pi)}$$

So we have

$$y_{cm} = \frac{4-\pi}{2\,(\pi-2)}$$

End Example

We can extend this analysis into 3 dimensions. Similar analysis will yield the 3D equivalent of the line integral expression (Eq.3) as

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \qquad \textbf{(Eq.9)}$$

Note that in 3D, we lose the ability to physically describe the integrals as the area under the curve unless we consider the function f to be the distance from the curve at a given point.

Note that if we have the parameterized representation of the curve

$$\bar{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

Then its derivative with respect to t is given as

$$\bar{r}'(t) = \frac{d\,x(t)}{d\,t}\,\hat{i} + \frac{d\,y(t)}{d\,t}\,\hat{j} + \frac{d\,z(t)}{d\,t}\,\hat{k}$$

We notice that the norm of this vector is

$$|\,\bar{r}'(t)\,| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

Which is precisely the term within the integral in Eq.9. Therefore, Eq.9 can be expressed in the more compact form of

$$\int_C f(x, y, z)\, ds = \int_a^b f(\bar{r}(t)) \left|\, \bar{r}'(t) \,\right| dt \qquad \textbf{(Eq.10)}$$

Example

Evaluate the line integral $\int_C f(x, y, z)\, ds$ when

$$f(x, y, z) = y \sin(z)$$

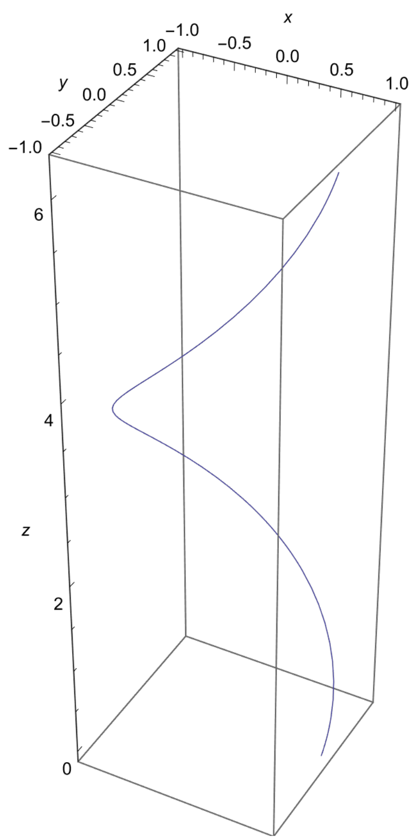
And C is the circular helix described by

$$x(t) = \cos(t)$$
$$y(t) = \sin(t)$$
$$z(t) = t$$

For $t \in [0, 2\pi]$

We can visualize C

```
ParametricPlot3D[{Cos[t], Sin[t], t}, {t, 0, 2 π},
 AxesLabel → {x, y, z}]
```



Eq.9 yields

$$\int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

$$= \int_a^b y(t) \sin(z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \qquad \text{recall: } x(t) = \cos(t) \Rightarrow \frac{dx}{dt} = -\sin(t)$$

$$y(t) = \sin(t) \Rightarrow \frac{dy}{dt} = \cos(t)$$
$$z(t) = t \Rightarrow \frac{dz}{dt} = 1$$

$$= \int_0^{2\pi} \sin(t) \sin(t) \sqrt{\sin^2(t) + \cos^2(t) + 1}\, dt$$

$$= \int_0^{2\pi} \sin^2(t) \sqrt{1 + 1}\, dt$$

$$= \sqrt{2} \int_0^{2\pi} \sin^2(t)\, dt \qquad \text{recall: } \sin^2(t) = \frac{1}{2} \times (1 - \cos(2t))$$

$$= \sqrt{2} \int_0^{2\pi} \frac{1}{2} \times (1 - \cos(2t))\, dt$$

```
Grid[{{√2 Integrate[1/2 × (1 - Cos[2 t]), {t, 0, 2 π}]}}, Background → Pink, Frame → True]
```

$\boxed{\sqrt{2}\,\pi}$

So we have

$$\int_C f(x, y, z)\, ds = \sqrt{2}\,\pi$$

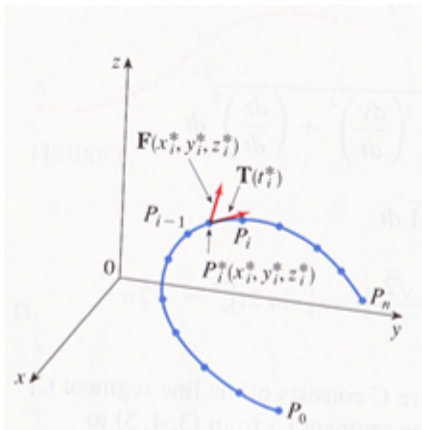
Line Integral of a Vector Function

Motivation of the Line Integral: Work by a Force and Line Integrals of Vector Fields

Previously, we looked at integrating a scalar function over an arbitrary curve in space. How about integrating a vector function $\overline{F}$ defined over a curve C? We can follow a similar procedure and sub-divide the curve into several sub arcs $P_{i-1} P_i$ with lengths Δs_i. Next, choose a point $P_i^*(x_i^*, y_i^*, z_i^*)$ on the i^{th} sub arc corresponding to the parameter value t_i^*. If Δs_i is small, then as we move from P_{i-1} to P_i along the curve, we proceed approximately in the direction of $\overline{T}(t_i^*)$, the unit tangent vector at P_i^*. Therefore, the “work” done by the force $\overline{F}$ in moving from P_{i-1} to P_i is given as

$$\overline{F}(x_i^*, y_i^*, z_i^*) \cdot \left(\Delta s_i\, \overline{T}(t_i^*)\right) = \left(\overline{F}(x_i^*, y_i^*, z_i^*) \cdot \overline{T}(t_i^*)\right) \Delta s_i \qquad \text{(infinitesimal work)}$$

where $\overline{F}(x_i^*, y_i^*, z_i^*)$ = vector of force at point x_i^*, y_i^*, z_i^*
$\overline{T}(t_i^*)$ = unit tangent vector at point x_i^*, y_i^*, z_i^* (corresponding to t_i)
Δs_i =infinitesimal distance



So the total work done is

$$\sum_{i=1}^{n}\left(\bar{F}(x_i^*,\ y_i^*,\ z_i^*)\cdot\bar{T}(t_i^*)\right)\Delta s_i$$

We can take the limit as the sub arcs become infinitesimally small to obtain

$$W = \int_C \bar{F}(x,\ y,\ z)\cdot\bar{T}(x,\ y,\ z)\,ds = \int_C \bar{F}\cdot\bar{T}\,ds \qquad \textbf{\textit{(Eq.11)}}$$

If the curve C is expressed as a parameterized vector equation of the form

$$\bar{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

Recall that from our previous video entitled ‘Tangent to a Curve’ at https://youtu.be/HH367um_Aho (also in Kreyszig section 9.5), the unit tangent vector to the curve is given by

$$\bar{T} = \frac{\bar{r}'(t)}{|\bar{r}'(t)|}$$

So the equation for total work (Eq.11) becomes

$$\int_C \bar{F}(x,\ y,\ z)\cdot\bar{T}(x,\ y,\ z)\,ds = \int_C \bar{F}(\bar{r}(t))\cdot\frac{\bar{r}'(t)}{|\bar{r}'(t)|}\,ds$$

recall: $ds = \left|\bar{r}'(t)\right| dt$ (see Eq.2.4 from ‘Arc Length’ video at https://youtu.be/FoiuvPkFppg?si=2x4f-f314OjuRCFIF&t=1003)

$$= \int_C \bar{F}(\bar{r}(t))\cdot\frac{\bar{r}'(t)}{|\bar{r}'(t)|}\left|\bar{r}'(t)\right| dt$$

$$= \int_C \bar{F}(\bar{r}(t))\cdot\bar{r}'(t)\,dt$$

This is often abbreviated as $\int_C \bar{F}(\bar{r}(t))\cdot d\bar{r}$.

This gives the definition for the line integral of any continuous vector field over a curve C. Formally, we can write this as:

Definition: Let $\overline{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\overline{r}(t)$, $t \in [a, b]$. Then the line integral of $\overline{F}$ along C is

$$\int_C \overline{F}(\overline{r}) \cdot d\,\overline{r} = \int_a^b \overline{F}(\overline{r}(t)) \cdot \overline{r}\,'(t)\,dt = \int_C \overline{F} \cdot \overline{T}\,ds \qquad \textbf{(Eq.13)}$$

where $\quad \overline{T}$ = unit tangent vector

$t = a$ corresponds to start point of C

$t = b$ corresponds to end point of C

Recall that Eq.10 ($\int_C f(x, y, z)\,ds = \int_a^b f(\overline{r}(t)) \left| \overline{r}\,'(t) \right| dt$) gave the line integral of a general function $f(x, y, z) = f(\overline{r}(t))$ along a path. If we consider this function to be the the work done along the path

$$f(\overline{r}(t)) = \overline{F}(\overline{r}(t)) \cdot \overline{T} \qquad \text{recall: } \overline{T} = \frac{\overline{r}'(t)}{|\overline{r}'(t)|}$$

$$f(\overline{r}(t)) = \overline{F}(\overline{r}(t)) \cdot \frac{\overline{r}'(t)}{|\overline{r}'(t)|}$$

Substituting into Eq.10

$$\int_C f(x, y, z)\,ds = \int_a^b f(\overline{r}(t)) \left| \overline{r}\,'(t) \right| dt$$

$$= \int_a^b = \overline{F}(\overline{r}(t)) \cdot \frac{\overline{r}'(t)}{|\overline{r}'(t)|} \left| \overline{r}\,'(t) \right| dt$$

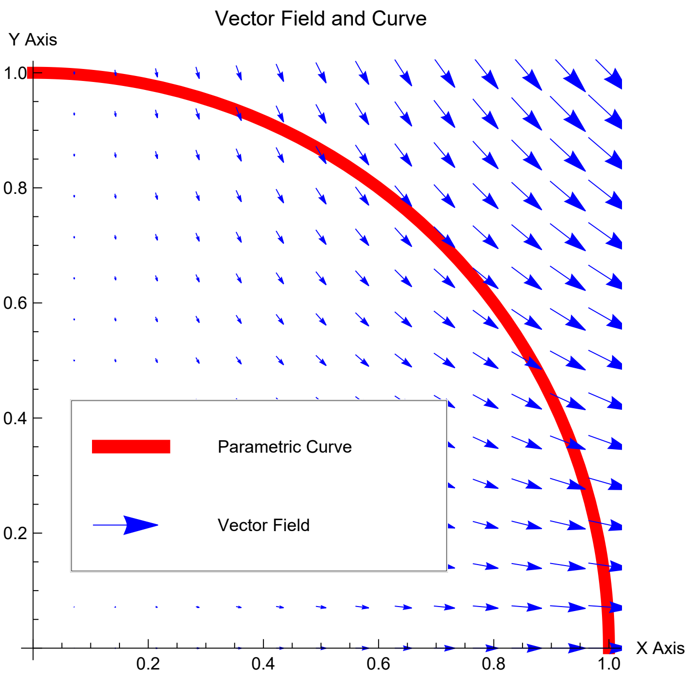
$$= \int_a^b = \overline{F}(\overline{r}(t)) \cdot \overline{r}\,'(t)\,dt$$

Which is what we previously wrote in Eq.13.

Example

Find the work done by the force field $\overline{F}(x, y) = x^2\,\hat{i} - x\,y\,\hat{j}$ in moving a particle along the quarter circle $\overline{r}(t) = \cos(t)\,\hat{i} + \sin(t)\,\hat{j}$ for $t \in [0, \pi/2]$.

```
(*Load the PlotLegends package*)
Needs["PlotLegends`"]

ShowLegend[
 Show[
  (*Plot 1*)
  ParametricPlot[{Cos[t], Sin[t]}, {t, 0, π / 2}, PlotStyle → {Red, Thickness[0.02]}],

  (*Plot 2*)
  VectorPlot[{x^2, -x y}, {x, 0, 1}, {y, 0, 1}, VectorStyle → Blue],

  (*Plot Options*)
  PlotLabel → "Vector Field and Curve",
  AxesLabel → {"X Axis", "Y Axis"}
 ],

 (*Add the legend information*)
 {
  {
   {Graphics[{Red, Thickness[0.2], Line[{{0, 0}, {2, 0}}]}], "Parametric Curve"},
   {Graphics[{Blue, Arrowheads[0.5], Arrow[{{0, 0}, {2, 0}}]}], "Vector Field"}
  },
  LegendPosition → {-0.8, -0.7},
  LegendSize → {1.1, 0.5},
  LegendShadow → False
 }
]
```



Note that $t = a = 0$ corresponds to the point (1,0) and $t = b = \pi/2$ corresponds to the point (0,1) so in effect, the curve $\overline{r}(t)$ goes in the CCW fashion.

We can parameterize $\overline{F}$ since $x = \cos(t)$ and $y = \sin(t)$

$$\overline{F}(\overline{r}(t)) = \cos^2(t)\,\hat{i} - \cos(t)\sin(t)\,\hat{j}$$

We can also calculate $\overline{r}\,'(t)$

$$\overline{r}\,'(t) = -\sin(t)\,\hat{i} + \cos(t)\,\hat{j}$$

So the work done is given by Eq.13

$$\int_C \overline{F}(\overline{r}(t)) \cdot d\overline{r} = \int_a^b \overline{F}(\overline{r}(t)) \cdot \overline{r}\,'(t)\,dt$$

$$= \int_0^{\pi/2} < \begin{pmatrix} \cos^2(t) \\ -\cos(t)\sin(t) \end{pmatrix}, \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} > dt \quad \text{note: } <\,,\,> \text{ notation means dot product}$$

$$= \int_0^{\pi/2} -\cos^2(t)\sin(t) - \cos^2(t)\sin(t)\,dt$$

$$= -2\int_0^{\pi/2} \cos^2(t)\sin(t)\,dt$$

We can calculate the indefinite integral

```
Integrate[Cos[t]^2 Sin[t], t]
```

$-\frac{1}{3}\text{Cos}[t]^3$

So we have

$$= \frac{2}{3}\cos^3(t)\,|_{t=\pi/2} - \frac{2}{3}\cos^3(t)\,|_{t=0}$$

$$\int_C \overline{F}(\overline{r}(t))\cdot d\overline{r} = \frac{-2}{3}$$

```
Grid[{{-2 Integrate[Cos[t]^2 Sin[t], {t, 0, π / 2}]}}, Background → Pink, Frame → True]
```

$\boxed{-\frac{2}{3}}$

Note that the negative shows that the work is negative because for the most part, the motion opposes the force field.

In general, the line integral is not independent of the path. In other words, even through the end points may be the same, the value of the line integral may depend on the path over which it is integrated. We investigate this in a future lecture.

General Properties of the Line Integral

Some general properties of line integrals are similar to standard integral properties

$$\int_C k\,\overline{F}\cdot d\overline{r} = k\int_C \overline{F}\cdot d\overline{r} \qquad (k\text{ constant}) \qquad \textit{\textbf{(Eq.8a)}}$$

$$\int_C (\overline{F}+\overline{G})\cdot d\overline{r} = \int_C \overline{F}\cdot d\overline{r} + \int_C \overline{G}\cdot d\overline{r} \qquad \textit{\textbf{(Eq.8b)}}$$

$$\int_C \overline{F}\cdot d\overline{r} = \int_{C_1} \overline{F}\cdot d\overline{r} + \int_{C_2} \overline{F}\cdot d\overline{r} \qquad \textit{\textbf{(Eq.8c)}}$$

where C is subdivided into two arcs C_1 and C_2 that have the same orientation as C