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Lecture 02h
Inverse Laplace Transform

Lecture is on YouTube

The YouTube video entitled ‘The Inverse Laplace Transform’ that covers this lecture is located at https://youtu.be/wZkrU1lPObM.

Outline

-Inverse Laplace Transform
- -Distinct Real Poles
- -Repeated Real Poles
- -Complex Conjugate Poles

-Example Solution of an ODE
-Example with an Improper Function

Inverse Laplace Transform

The inverse Laplace transform obtains the time domain function, $f(t)$, from the Laplace domain function, $F(s)$.

$$f(t) = L^{-1}[F(s)]$$

The inverse Laplace transform is given as the following line integral

$$f(t) = L^{-1}[F(s)] = \frac{1}{2\pi i} \lim_{T\to\infty} \oint_{\gamma - i T}^{\gamma + i T} e^{s t} F(s)\, ds$$

where the integration is done along the vertical line Re(s)=γ in the complex plane such that γ is greater than the real part of all singularities of $F(s)$ and $F(s)$ is bounded on the line.

While it is feasible to compute this integral, it is cumbersome. It is much easier to manipulate $F(s)$ into a form where its inverse Laplace transform is easily recognizable. ***<Talk about Transformers analogy here>***

If $F(s)$ is broken up into components

$$F(s) = F_1(s) + F_2(s) + \ldots + F_n(s)$$

And if the inverse Laplace transforms of $F_1(s), \ \ldots, \ F_n(s)$ are available, then

$$L^{-1}[F(s)] = L^{-1}[F_1(s)] + L^{-1}[F_2(s)] + \ldots + L^{-1}[F_n(s)]$$

$$f(t) = f_1(t) + f_2(t) + \ldots + f_n(t)$$

Example

Suppose we have a signal in the Laplace domain

$$F(s) = \frac{4}{s+3} + \frac{3}{s} + \frac{6}{s^2+9}$$

What is the signal in the time domain, $f(t)$?

Note that we can write $F(s)$ as

$$F(s) = 4\left(\frac{1}{s+3}\right) + 3\left(\frac{1}{s}\right) + 2\left(\frac{3}{s^2+3^2}\right)$$

So the inverse Laplace transform is given by

$$f(t) = L^{-1}[F(s)]$$

$$= L^{-1}\left[4\left(\frac{1}{s+3}\right) + 3\left(\frac{1}{s}\right) + 2\left(\frac{3}{s^2+3^2}\right)\right]$$

$$= L^{-1}\left[4\left(\frac{1}{s+3}\right)\right] + L^{-1}\left[3\left(\frac{1}{s}\right)\right] + L^{-1}\left[2\left(\frac{3}{s^2+3^2}\right)\right]$$

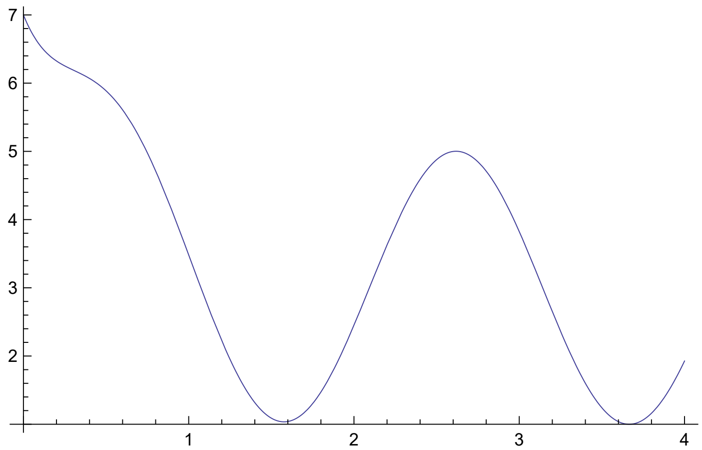
$$= 4\,L^{-1}\left[\frac{1}{s+3}\right] + 3\,L^{-1}\left[\frac{1}{s}\right] + 2\left[\frac{3}{s^2+3^2}\right]$$ note: these are all forms in a standard Laplace Transform table

$$f(t) = 4\,e^{-3\,t} + 3\mathbf{1}\,(t) + 2\sin(3\,t)$$

Back to our scenario, we wanted to plot $f(t) = L^{-1}[F(s)] = 4\,e^{-3\,t} + 3\mathbf{1}\,(t) + 2\sin(3\,t)$.

```
f[t_] = InverseLaplaceTransform[4/(s + 3) + 3/s + 6/(s^2 + 9), s, t]
Plot[f[t], {t, 0, 4}]
```

$3 + 4\, e^{-3\, t} + 2\, \mathrm{Sin}[3\, t]$



The easiest method is to use tables which outline the inverse Laplace transform. In order to do this, we need to get the function $F(s)$ into a form which is in the table.

Distinct Real Poles

Example: Distinct Real Roots (strictly proper polynomial)

Recall that in the previous lecture on partial fraction expansion (INSERT YOUTUBE URL), we examined a ratio of polynomials with distinct real roots and computed its partial fraction expansion as

$$F(s) = \frac{s^2+8\,s+15}{s^3+3\,s^2+2\,s} = \frac{7.5}{s} + \frac{-8}{s+1} + \frac{1.5}{s+2}$$

So the inverse Laplace transform of $F(s)$ can be written as

$$L^{-1}[F(s)] = L^{-1}\left[\frac{7.5}{s} + \frac{-8}{s+1} + \frac{1.5}{s+2}\right]$$

$$= 7.5\, L^{-1}\left[\frac{1}{s}\right] - 8\, L^{-1}\left[\frac{1}{s+1}\right] + 1.5\, L^{-1}\left[\frac{1}{s+2}\right]$$

We can easily find the inverse Laplace transform from the table. We see that

$$L^{-1}\left[\frac{1}{s+a}\right] = e^{-a\,t}$$

$$L^{-1}\left[\frac{1}{s}\right] = \mathbf{1}\,(t)$$

So we have

$$f(t) = L^{-1}[F(s)] = 7.5 \times \mathbf{1}\,(t) - 8\, e^{-t} + 1.5\, e^{-2\,t}$$

```
InverseLaplaceTransform[(s^2 + 8 s + 15)/(s^3 + 3 s^2 + 2 s), s, t]
```

$$\frac{15}{2} + \frac{3\, e^{-2t}}{2} - 8\, e^{-t}$$

Repeated Real Poles

Recall the previous example with repeated poles where we showed that

$$F(s) = \frac{s^2+2s+3}{(s+1)^3} = \frac{1}{(s+1)} + \frac{2}{(s+1)^3}$$

So we can write the function $F(s)$ as

$$F(s) = \frac{1}{(s+1)} + \frac{2}{(s+1)^3}$$

So the inverse Laplace transform is

$$f(t) = L^{-1}\left[\frac{1}{(s+1)}\right] + 2\,L^{-1}\left[\frac{1}{(s+1)^3}\right]$$

$$= e^{-t} + 2\left(\frac{1}{(3-1)!}\, t^{3-1}\, e^{-t}\right)$$

$$f(t) = e^{-t} + t^2\, e^{-t}$$

```
InverseLaplaceTransform[(s^2 + 2 s + 3)/(s + 1)^3, s, t] // Expand
```

$$e^{-t} + e^{-t}\, t^2$$

Complex Conjugate Poles

Again, recall the example from the previous lecture where we showed hat

$$F(s) = \frac{s-1}{s^2+2s+2} = \frac{s-1}{(s+1)^2+1}$$

If we look in the Laplace transform table, we see the closest form is

$$\frac{s+a}{(s+a)^2+\omega^2}$$

Note the sign difference, we can reconcile this by adding and subtracting a constant from the numerator

$$F(s) = \frac{s-1+2-2}{(s+1)^2+1}$$

$$= \frac{s+1}{(s+1)^2+1} - \frac{2}{(s+1)^2+1}$$

We can now inverse Laplace transform this

$$f(t) = L^{-1}\left[\frac{s+1}{(s+1)^2+1}\right] - 2\,L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$f(t) = e^{-t}\cos(t) - 2\,e^{-t}\sin(t)$$

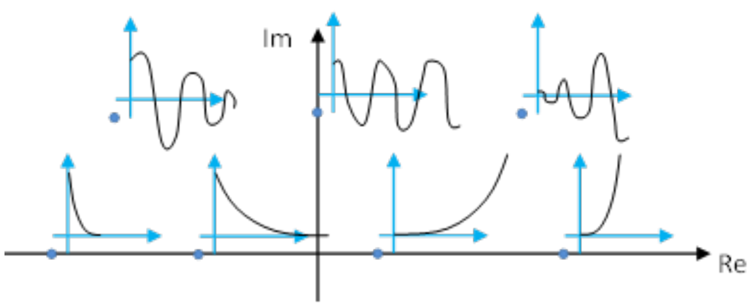
<u>Summary</u>

We can note that the types of poles actually influences the response of the system. For example, we see that distinct real roots tend to yield terms like $e^{-a\,t}$. Similarly, repeated real roots yield terms like $t\,e^{-a\,t}$ or $t^2\,e^{-a\,t}$. Finally, complex conjugate roots introduce oscillations via the sin/cos terms of $e^{-a\,t}\cos(\omega\,t)$ or $e^{-a\,t}\sin(\omega\,t)$

distinct, real roots $\Longleftrightarrow e^{-a\,t}$

repeated, real roots $\Longleftrightarrow \frac{1}{(n-1)!}\,t^{n-1}\,e^{-a\,t}$

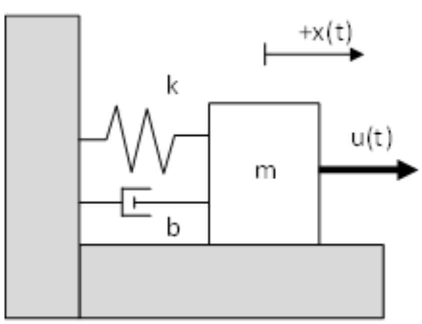
complex conjugate roots $\Longleftrightarrow e^{-a\,t}\cos(\omega\,t),\ e^{-a\,t}\sin(\omega\,t)$



Example Solution of an ODE

Example

Recall the example we considered during the video entitled ‘The Laplace Transform’.



We showed that the response of this system to initial conditions $x(0) = -2$, $\dot{x}(0) = 3\,m/s$, and input $u(t) = \mathbf{1}\,(t)$, was given as

$$X(s) = \frac{\frac{1}{m s} - 2 s + 3 - \frac{2 b}{m}}{s^2 + \frac{b}{m} s + \frac{k}{m}}$$

Let us chose $m = 1$, $k = 2$, and $b = 3$ as well as simplifying the numerator to obtain

```
temp = (1/(m s) - 2 s + 3 - (2 b)/m) / (s^2 + b/m s + k/m) /. {m → 1, k → 2, b → 3};
X[s_] = Simplify[Together[Numerator[temp]] / Denominator[temp]]
X[s] == temp // Simplify
```

$$\frac{1 - 3 s - 2 s^2}{2 s + 3 s^2 + s^3}$$

```
True
```

We first identify the roots

```
roots = Roots[Denominator[X[s]] == 0, s]
```

```
s == 0 || s == -2 || s == -1
```

We see that we have distinct, real poles

We therefore propose the partial fraction expansion of

$$X(s) = \frac{-2 s^2 - 3 s + 1}{s^3 + 3 s^2 + 2 s} = \frac{-2 s^2 - 3 s + 1}{s(s+1)(s+2)} = \frac{a_1}{s} + \frac{a_2}{s+1} + \frac{a_3}{s+2}$$

```
(*Check that factorization of denominator is correct*)
s (s + 1) (s + 2) == Denominator[X[s]] // Simplify

(*Collect RHS coefficients*)
num[s_] = Collect[Expand[a1 (s + 1) (s + 2) + a2 (s) (s + 2) + a3 (s) (s + 1)], {s, s^2}]

(*Solve via coefficients method*)
results = Solve[
   {Coefficient[num[s], s^2] == -2, Coefficient[num[s], s] == -3, num[0] == 1}, {a1, a2, a3}];
a1 = a1 /. results[[1]]
a2 = a2 /. results[[1]]
a3 = a3 /. results[[1]]
```

True

$2\,a1 + (3\,a1 + 2\,a2 + a3)\,s + (a1 + a2 + a3)\,s^2$

$\frac{1}{2}$

-2

$-\frac{1}{2}$

So the inverse Laplace transform is simply

$$x(t) = L^{-1}[X(s)]$$

$$= L^{-1}\left[\frac{1/2}{s} - \frac{2}{s+1} - \frac{1/2}{s+2}\right]$$

$$= \frac{1}{2} L^{-1}\left[\frac{1}{s}\right] - 2\,L^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{2} L^{-1}\left[\frac{1}{s+2}\right]$$

$$x(t) = \frac{1}{2} - 2\,e^{-t} - \frac{1}{2}\,e^{-2\,t}$$

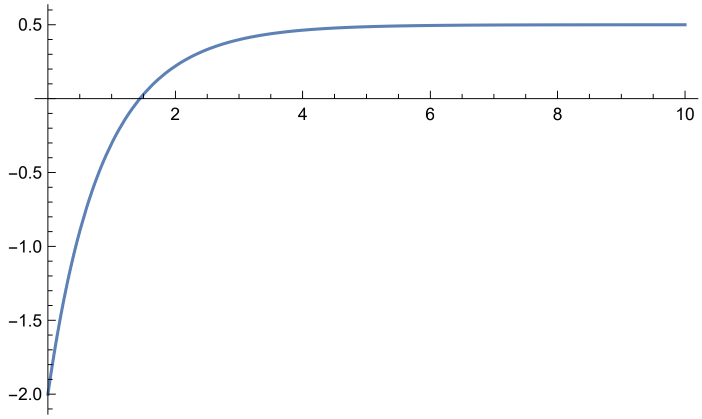
```
x[t_] = 1/2 - 2 Exp[-t] - 1/2 Exp[-2 t]

(*Check against InverseLaplaceTransform*)
x[t] == InverseLaplaceTransform[X[s], s, t] // Simplify
```

$\frac{1}{2} - \frac{e^{-2\,t}}{2} - 2\,e^{-t}$

True

We can plot and verify that this satisfies the initial conditions

```
Plot[x[t], {t, 0, 10}, PlotRange → All]
x[0] == -2
(D[x[t], t] /. {t → 0}) == 3
```




```
True
```

```
True
```

Example with Improper Complex Function

Again, recall the example we looked at previously with an improper transfer function.

$$F(s) = \frac{s^3+5s^2+9s+7}{(s+1)(s+2)} = s + 2 + \frac{2}{s+1} - \frac{1}{s+2}$$

So the inverse Laplace transform is

$$L^{-1}[F(s)] = L^{-1}[s] + 2L^{-1}[1] + 2L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right]$$

$$f(t) = \frac{d}{dt}[\delta(t)] + 2\,\delta(t) + 2\,e^{-t} - e^{-2t}$$

So we see that the fact that the transfer function was improper (numerator order is higher than denominator order) yields a signal which does not make must physical sense (what is the time rate of change of an impulse?). For this class, physically realizable signals that we are interested in analyzing should be proper or strictly proper (denominator order is or same order or higher, respectively, than the numerator)