Christopher Lum
lum@uw.edu

Lecture 03e
Introduction to Frequency Domain Analysis

Lecture is on YouTube

The YouTube video entitled 'Introduction to Frequency Domain Analysis' that covers this lecture is located at https://youtu.be/yuT4Mg2NIQs

Outline

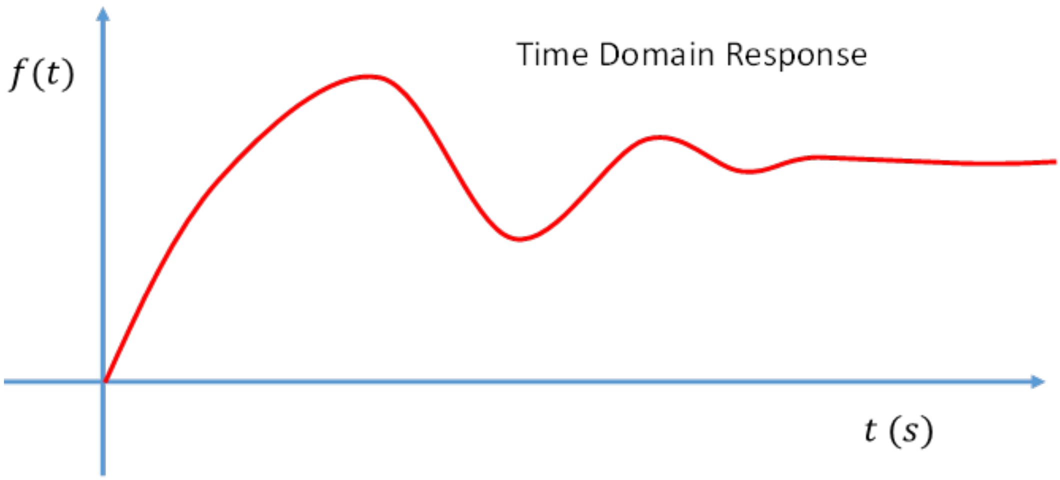
-Frequency Domain Analysis
    -Example of P(s)
-Understanding $G(j\,\omega)$
    -Example to Verify Results
-Summary

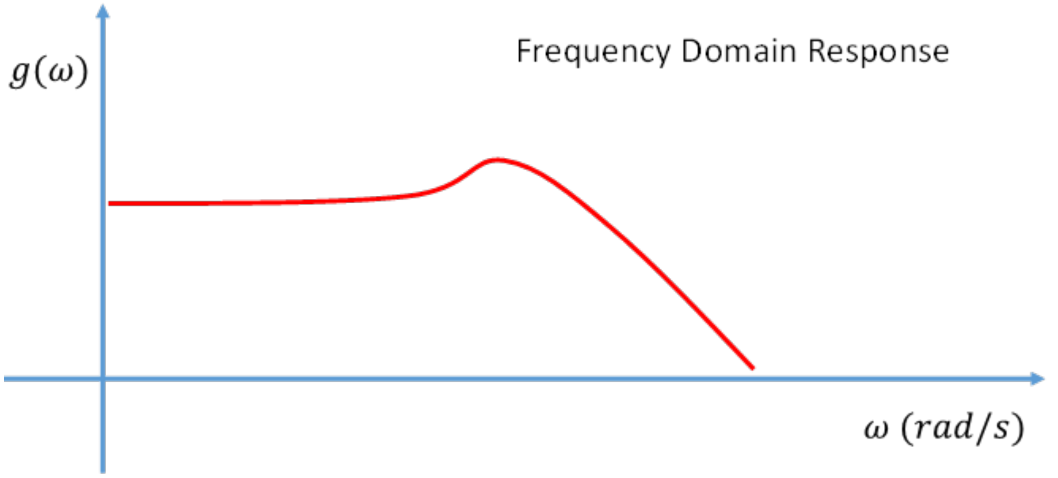
Frequency Domain Analysis

Previously, we looked at how systems reached to step and ramp inputs. Many systems are subjected to sinusoidal inputs (vibrations, rotations systems)

Frequency domain analysis is a subset of time domain analysis when the input is sinusoidal and we are interested in how the system responds to sinusoidal inputs at different frequencies.

To study the response, we vary the input sinusoid over many frequencies.





We now ask, if we are given a sinusoidal input to a system, can we predict the system response as the frequency of the sinusoid varies? To do this, we investigate the transfer function response.

Consider a general transfer function of the form

$$G(s) = \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)} \qquad \textbf{\textit{(Eq.1)}}$$

Note that the form of Eq.1 does not preclude repeated or complex conjugate poles. We will see that this does not affect the analysis.

Consider the input of

$$u(t) = A\sin(\omega t) \qquad \textbf{\textit{(Eq.2)}}$$

In the Laplace domain

$$U(s) = A\frac{\omega}{s^2+\omega^2} = \frac{A\omega}{(s+j\omega)(s-j\omega)} \qquad \textbf{\textit{(Eq.3)}}$$

```
LaplaceTransform[A Sin[ω t], t, s]
% == A ω / ((s + I ω) (s - I ω)) // Simplify
```

$$\frac{A\,\omega}{s^2+\omega^2}$$

True

So the output of the system in response to the input of $u(t) = A\sin(\omega t)$ is

$$Y(s) = G(s)\,U(s) \qquad \text{note: } U(s) = A\frac{\omega}{s^2+\omega^2}$$

$$= \frac{K(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)} A \frac{\omega}{s^2+\omega^2}$$

$$= \frac{A\,K\,\omega\,(s+z_1)(s+z_2)\ldots(s+z_m)}{(s+p_1)(s+p_2)\ldots(s+p_n)(s+j\,\omega)(s-j\,\omega)}$$

Once again we see that the poles of the function $Y(s)$ are a combination of the poles of $G(s)$ and the poles introduced by the sinusoidal input, $s + j\,\omega$ and $s - j\,\omega$. Therefore the partial fraction expansion of $Y(s)$ can be written as

$$Y(s) = P(s) + \left(\frac{c_1}{s+j\,\omega} + \frac{c_2}{s-j\,\omega}\right) \qquad \textbf{\textit{(Eq.4)}}$$

where $\quad P(s)$ =partial fraction expansion of poles from $G(s)$

Here, the term $P(s)$ is the partial fraction expansion of the poles of $G(s)$ and term in parenthesis is the partial fraction expansion of the poles from the sinusoidal input.

Since the original system $G(s)$ can have distinct, repeated, and complex conjugate poles, the term $P(s)$ can be written as

$$P(s) = \left[\frac{a_1}{s+p_{d,1}} + \frac{a_2}{s+p_{d,2}} + \ldots + \frac{a_{n_d}}{s+p_{d,n_d}}\right] + \left[\left(\frac{b_{1,1}}{(s+p_{r,1})} + \frac{b_{1,2}}{(s+p_{r,1})^2} + \ldots + \frac{b_{1,o_{r,1}}}{(s+p_{r,1})^{o_{r,1}}}\right) + \ldots + \left(\frac{b_{n_r,1}}{(s+p_{r,n_r})} + \frac{b_{n_r,2}}{(s+p_{r,n_r})^2} + \ldots + \frac{b_{n_r,o_{r,n_r}}}{(s+p_{r,n_r})^{o_{r,n_r}}}\right)\right] \qquad \textbf{\textit{(Eq.5)}}$$

where $\quad n_d$ = number of distinct poles

n_r = number of repeated poles

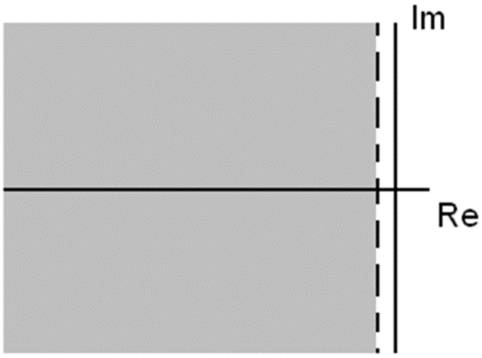
$o_{r,1}$ =order of first repeated pole

o_{r,n_r} = order of n_r th repeated pole

In Eq.5, the terms in the first set of square brackets is the partial fraction expansion of the distinct real and complex conjugate poles (since complex conjugate poles of the form $x + y\,j$ can be written as $s - x - y\,j$). The terms in the second set of square brackets is the partial fraction expansion of the repeated real and complex conjugate poles.

The main thing to notice with Eq.5 is that if we assume that the system is stable (more stringent than neutrally stable), then all poles are strictly in the left half plane. Therefore, no poles have the form of $s \pm j\,\omega$ (in other words, no poles lie on the imaginary axis or have positive real part).

{system is stable $\Rightarrow P(s)$ has no denominator terms of the form $s \pm j\,\omega$} ***(Eq.6)***



All poles are strictly in the left half plane
(no poles on imaginary axis)

Example of P(s)

If the system has the form of

$$G(s) = \frac{3\,(s+3)\,(s+2)\,(s-1)}{(s+1-2\,j)\,(s+1+2\,j)\,(s+2)\,(s+5)^2\,(s+1)^3}$$

The output of the system in response to a sinusoidal input is therefore

$$Y(s) = G(s)\,U(s) \qquad \text{note: } U(s) = \frac{A\,\omega}{(s+j\,\omega)\,(s-j\,\omega)}$$

$$= \left(\frac{3\,(s+3)\,(s+2)\,(s-1)}{(s+1-2\,j)\,(s+1+2\,j)\,(s+2)\,(s+5)^2\,(s+1)^3}\right)\left(\frac{A\,\omega}{(s+j\,\omega)\,(s-j\,\omega)}\right)$$

This then has a partial fraction expansion of

$$Y(s) = \frac{a_1}{s+1-2\,j} + \frac{a_2}{s+1+2\,j} + \frac{a_3}{s+2} + \frac{b_{1,1}}{(s+5)} + \frac{b_{1,2}}{(s+5)^2} + \frac{b_{2,1}}{(s+1)} + \frac{b_{2,2}}{(s+1)^2} + \frac{b_{2,3}}{(s+1)^3} + \frac{c_1}{s+j\,\omega} + \frac{c_2}{s-j\,\omega}$$

$$Y(s) = P(s) + \frac{c_1}{s+j\,\omega} + \frac{c_2}{s-j\,\omega}$$

where $P(s) = \frac{a_1}{s+1-2\,j} + \frac{a_2}{s+1+2\,j} + \frac{a_3}{s+2} + \frac{b_{1,1}}{(s+5)} + \frac{b_{1,2}}{(s+5)^2} + \frac{b_{2,1}}{(s+1)} + \frac{b_{2,2}}{(s+1)^2} + \frac{b_{2,3}}{(s+1)^3}$

Once again, we note that if $G(s)$ is stable, then $P(s)$ has no denominator terms of the form $s \pm j\,\omega$

End Example

We can use the multiplication method to find c_1. Multiplying both sides of Eq.4 by $s + j\,\omega$ (pole associated with c_1)

$$(s + j\,\omega)\,Y(s) = (s + j\,\omega)\left(P(s) + \frac{c_1}{s+j\,\omega} + \frac{c_2}{s-j\,\omega}\right) \qquad \text{recall: } Y(s) = G(s)\,\frac{A\,\omega}{(s+j\,\omega)\,(s-j\,\omega)}$$

$$(s + j\,\omega)\,G(s)\,\frac{A\,\omega}{(s+j\,\omega)\,(s-j\,\omega)} = (s + j\,\omega)\left(P(s) + \frac{c_1}{s+j\,\omega} + \frac{c_2}{s-j\,\omega}\right)$$

$$G(s)\,\frac{A\,\omega}{(s-j\,\omega)} = (s+j\,\omega)\,P(s) + c_1 + \frac{(s+j\,\omega)\,c_2}{(s-j\,\omega)}$$

This must be true at $s = -j\,\omega$ and since $G(s)$ is assumed stable, $P(s)$ has no denominator terms of the form $s \pm j\,\omega$ to cancel the $(s+j\,\omega)$ term in the numerator (Eq.6). So the equation can be written as

$$G(-j\,\omega)\,\frac{A\,\omega}{(-j\,\omega-j\,\omega)} = (-j\,\omega+j\,\omega)\,P(-j\,\omega) + c_1 + \frac{(-j\,\omega+j\,\omega)\,c_2}{(-j\,\omega-j\,\omega)}$$

$$G(-j\,\omega)\,\frac{A\,\omega}{-2\,j\,\omega} = (0)\,P(-j\,\omega) + c_1 + \frac{(0)\,c_2}{(-j\,\omega-j\,\omega)}$$

$$-\frac{A\,G(-j\,\omega)}{2\,j} = c_1 \qquad \textbf{\textit{(Eq.7)}}$$

We can use the multiplication method to find c_2. Multiplying both sides of Eq.4 by $s - j\,\omega$ (pole associated with c_2)

$$(s-j\,\omega)\,Y(s) = (s-j\,\omega)\left(P(s) + \frac{c_1}{s+j\,\omega} + \frac{c_2}{s-j\,\omega}\right) \qquad \text{recall: } Y(s) = G(s)\,\frac{A\,\omega}{(s+j\,\omega)\,(s-j\,\omega)}$$

$$(s-j\,\omega)\,G(s)\,\frac{A\,\omega}{(s+j\,\omega)\,(s-j\,\omega)} = (s-j\,\omega)\left(P(s) + \frac{c_1}{s+j\,\omega} + \frac{c_2}{s-j\,\omega}\right)$$

$$G(s)\,\frac{A\,\omega}{(s+j\,\omega)} = (s-j\,\omega)\,P(s) + \frac{(s-j\,\omega)\,c_1}{s+j\,\omega} + c_2$$

This must be true at $s = j\,\omega$ and since $G(s)$ is assumed stable, $P(s)$ has no denominator terms of the form $s \pm j\,\omega$ to cancel the $(s-j\,\omega)$ term in the numerator (Eq.6). So the equation becomes

$$G(j\,\omega)\,\frac{A\,\omega}{(j\,\omega+j\,\omega)} = (j\,\omega-j\,\omega)\,P(j\,\omega) + \frac{(j\,\omega-j\,\omega)\,c_1}{j\,\omega+j\,\omega} + c_2$$

$$G(j\,\omega)\,\frac{A\,\omega}{2\,j\,\omega} = (0)\,P(j\,\omega) + \frac{(0)\,c_1}{j\,\omega+j\,\omega} + c_2$$

$$\frac{A\,G(j\,\omega)}{2\,j} = c_2 \qquad \textbf{\textit{(Eq.8)}}$$

We are now in a position to look at the steady state response of the system in response to the sinusoidal input. Taking the inverse Laplace transform of Eq.4, we see that

$$y(t) = L^{-1}\left[P(s) + \frac{c_1}{s+j\,\omega} + \frac{c_2}{s-j\,\omega}\right]$$

$$= L^{-1}[P(s)] + L^{-1}\left[\frac{c_1}{s+j\,\omega} + \frac{c_2}{s-j\,\omega}\right]$$

$$= L^{-1}[P(s)] + c_1\,e^{-j\,\omega\,t} + c_2\,e^{j\,\omega\,t}$$

We can investigate the $L^{-1}[P(s)]$ term

$$L^{-1}[P(s)] = L^{-1}\Big[\Big[\frac{a_1}{s+p_{d,1}} + \frac{a_2}{s+p_{d,2}} + \ldots + \frac{a_{n_d}}{s+p_{d,n}}\Big] + \Big[\left(\frac{b_{1,1}}{(s+p_{r,1})} + \frac{b_{1,2}}{(s+p_{r,1})^2} + \ldots + \frac{b_{1,o_{r,1}}}{(s+p_{r,1})^{o_{r,1}}}\right) + \ldots + \left(\frac{b_{n_r,1}}{(s+p_{r,n_r})} + \frac{b_{n_r,2}}{(s+p_{r,n_r})^2} + \ldots + \frac{b_{n_r,o_{r,n_r}}}{(s+p_{r,n_r})^{o_{r,n_r}}}\right)\Big]\Big]$$

$$= L^{-1}\Big[\frac{a_1}{s+p_{d,1}} + \frac{a_2}{s+p_{d,2}} + \ldots + \frac{a_{n_d}}{s+p_{d,n}}\Big] + L^{-1}\Big[\left(\frac{b_{1,1}}{(s+p_{r,1})} + \frac{b_{1,2}}{(s+p_{r,1})^2} + \ldots + \frac{b_{1,o_{r,1}}}{(s+p_{r,1})^{o_{r,1}}}\right)\Big] + \ldots + L^{-1}\Big[\left(\frac{b_{n_r,1}}{(s+p_{r,n_r})} + \frac{b_{n_r,2}}{(s+p_{r,n_r})^2} + \ldots + \frac{b_{n_r,o_{r,n_r}}}{(s+p_{r,n_r})^{o_{r,n_r}}}\right)\Big]$$

$$= \left[a_1\, e^{-p_{d,1}\, t} + a_2\, e^{-p_{d,2}\, t} + \ldots + a_{n_d}\, e^{-p_{d,n}\, t}\right] + \Big[b_{1,1}\, e^{-p_{r,1}\, t} + \frac{1}{(2-1)!}\, t\, e^{-p_{r,1}\, t} + \ldots + b_{1,o_{r,1}}\, \frac{1}{(o_{r,1}-1)!}\, t^{o_{r,1}-1}\, e^{-p_{r,1}\, t}\Big] + \ldots + \Big[b_{n_r,1}\, e^{-p_{r,n_r}\, t} + \frac{1}{(2-1)!}\, t\, e^{-p_{r,1}\, t} + \ldots + b_{n_r,o_{r,1}}\, \frac{1}{(o_{r,n_r}-1)!}\, t^{o_{r,n_r}-1}\, e^{-p_{r,n_r}\, t}\Big]$$

Although this is somewhat complex, the thing to notice that if $G(s)$ is stable, then all poles have a negative real part, so $p_{d,i} > 0$ and $\text{Re}(p_{r,i}) > 0$ for all poles, so all these terms eventually go to 0. Therefore

$$\lim_{t\to\infty} L^{-1}[P(s)] = 0 \qquad \textbf{\textit{(Eq.9)}}$$

So the response of $y(t)$ at steady state becomes

$$y_{ss}(t) = c_1\, e^{-j\,\omega\, t} + c_2\, e^{j\,\omega\, t}$$

$$= -\frac{A\, G(-j\,\omega)}{2\, j}\, e^{-j\,\omega\, t} + \frac{A\, G(j\,\omega)}{2\, j}\, e^{j\,\omega\, t}$$

$$y_{ss}(t) = \frac{A}{2\, j}\left(G(j\,\omega)\, e^{j\,\omega\, t} - G(-j\,\omega)\, e^{-j\,\omega\, t}\right) \qquad \textbf{\textit{(Eq.10)}}$$

As it is currently written, Eq.10 is not terribly useful. At this point, we need to investigate the terms $G(j\,\omega)$ and $G(-j\,\omega)$ to gain more insight into the problem.

Understanding $G(j\,\omega)$

To do this, let us consider a general transfer function. We know that a general transfer function can be written as

$$G(s) = \frac{b_m\, s^m + b_{m-1}\, s^{m-1} \ldots + b_1\, s + b_0}{a_n\, s^n + a_{n-1}\, s^{n-1} + \ldots + a_1\, s + a_0}$$

So the term $G(j\,\omega)$ is simply $G(s)$ with s replaced with $j\,\omega$

$$G(j\,\omega) = \frac{b_m (j\,\omega)^m + b_{m-1} (j\,\omega)^{m-1} \ldots + b_1 (j\,\omega) + b_0}{a_n (j\,\omega)^n + a_{n-1} (j\,\omega)^{n-1} + \ldots + a_1 (j\,\omega) + a_0}$$

Some insight may be obtained by studying a specific example. Consider a transfer function of the form

$$G(s) = \frac{b_8 s^8 + b_7 s^7 + b_6 s^6 + b_5 s^5 + b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0}{a_8 s^8 + a_7 s^7 + a_6 s^6 + a_5 s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0}$$

So evaluated at $s = j\,\omega$, we obtain

$$G(j\,\omega) = \frac{b_8 (j\,\omega)^8 + b_7 (j\,\omega)^7 + b_6 (j\,\omega)^6 + b_5 (j\,\omega)^5 + b_4 (j\,\omega)^4 + b_3 (j\,\omega)^3 + b_2 (j\,\omega)^2 + b_1 (j\,\omega) + b_0}{a_8 (j\,\omega)^8 + a_7 (j\,\omega)^7 + a_6 (j\,\omega)^6 + a_5 (j\,\omega)^5 + a_4 (j\,\omega)^4 + a_3 (j\,\omega)^3 + a_2 (j\,\omega)^2 + a_1 (j\,\omega) + a_0}$$

$$= \frac{b_8 j^8 \omega^8 + b_7 j^7 \omega^7 + b_6 j^6 \omega^6 + b_5 j^5 \omega^5 + b_4 j^4 \omega^4 + b_3 j^3 \omega^3 + b_2 j^2 \omega^2 + b_1 j\,\omega + b_0}{a_8 j^8 \omega^8 + a_7 j^7 \omega^7 + a_6 j^6 \omega^6 + a_5 j^5 \omega^5 + a_4 j^4 \omega^4 + a_3 j^3 \omega^3 + a_2 j^2 \omega^2 + a_1 j\,\omega + a_0}$$ note: $j^8 = 1$, $j^7 = -j$, $j^6 = -1$, $j^5 = j$, $j^4 = 1$, $j^3 = -j$, $j^2 = -1$

$$= \frac{b_8 \omega^8 - b_7 j\,\omega^7 - b_6 \omega^6 + b_5 j\,\omega^5 + b_4 \omega^4 - b_3 j\,\omega^3 - b_2 \omega^2 + b_1 j\,\omega + b_0}{a_8 \omega^8 - a_7 j\,\omega^7 - a_6 \omega^6 + a_5 j\,\omega^5 + a_4 \omega^4 - a_3 j\,\omega^3 - a_2 \omega^2 + a_1 j\,\omega + a_0}$$

$$= \frac{\left(b_0 - b_2 \omega^2 + b_4 \omega^4 - b_6 \omega^6 + b_8 \omega^8\right) + \left(b_1 \omega - b_3 \omega^3 + b_5 \omega^5 - b_7 \omega^7\right) j}{\left(a_0 - a_2 \omega^2 + a_4 \omega^4 - a_6 \omega^6 + a_8 \omega^8\right) + \left(a_1 \omega - a_3 \omega^3 + a_5 \omega^5 - a_7 \omega^7\right) j}$$

$$= \frac{b_{\text{even}}(\omega) + b_{\text{odd}}(\omega)\, j}{a_{\text{even}}(\omega) + a_{\text{odd}}(\omega)\, j} \qquad \textbf{\textit{(Eq.11)}}$$

where $b_{\text{even}}(\omega) = b_0 - b_2 \omega^2 + b_4 \omega^4 - b_6 \omega^6 + b_8 \omega^8$
$b_{\text{odd}}(\omega) = b_1 \omega - b_3 \omega^3 + b_5 \omega^5 - b_7 \omega^7$
$a_{\text{even}}(\omega) = a_0 - a_2 \omega^2 + a_4 \omega^4 - a_6 \omega^6 + a_8 \omega^8$
$a_{\text{odd}}(\omega) = a_1 \omega - a_3 \omega^3 + a_5 \omega^5 - a_7 \omega^7$

This pattern would continue if there were higher order terms. The result would be that the even terms would contain even powers of ω and the odd terms would contain odd terms of ω.

Despite the apparent complexity of Eq.11, we see that it is simply an imaginary number. We can eliminate the imaginary term in the denominator by multiplying top and bottom by the complex conjugate of the denominator.

$$G(j\,\omega) = \frac{b_{\text{even}}(\omega) + b_{\text{odd}}(\omega)\, j}{a_{\text{even}}(\omega) + a_{\text{odd}}(\omega)\, j} \left(\frac{a_{\text{even}}(\omega) - a_{\text{odd}}(\omega)\, j}{a_{\text{even}}(\omega) - a_{\text{odd}}(\omega)\, j}\right)$$

$$= \frac{\left(b_{\text{even}}(\omega) + b_{\text{odd}}(\omega)\, j\right)\left(a_{\text{even}}(\omega) - a_{\text{odd}}(\omega)\, j\right)}{a_{\text{even}}(\omega)^2 + a_{\text{odd}}(\omega)^2}$$

$$= \frac{a_{\text{even}}(\omega)\, b_{\text{even}}(\omega) - a_{\text{odd}}(\omega)\, b_{\text{even}}(\omega)\, j + a_{\text{even}}(\omega)\, b_{\text{odd}}(\omega)\, j - a_{\text{odd}}(\omega)\, b_{\text{odd}}(\omega)\, j^2}{a_{\text{even}}(\omega)^2 + a_{\text{odd}}(\omega)^2}$$

$$= \frac{\left(a_{\text{even}}(\omega)\, b_{\text{even}}(\omega) + a_{\text{odd}}(\omega)\, b_{\text{odd}}(\omega)\right) + \left(a_{\text{even}}(\omega)\, b_{\text{odd}}(\omega) - a_{\text{odd}}(\omega)\, b_{\text{even}}(\omega)\right) j}{a_{\text{even}}(\omega)^2 + a_{\text{odd}}(\omega)^2}$$

So the real and imaginary parts of the imaginary number $G(j\,\omega)$ would be

$$\text{Re}[G(j\,\omega)] = \frac{a_{\text{even}}(\omega)\, b_{\text{even}}(\omega) + a_{\text{odd}}(\omega)\, b_{\text{odd}}(\omega)}{a_{\text{even}}(\omega)^2 + a_{\text{odd}}(\omega)^2} \qquad \textbf{\textit{(Eq.12)}}$$

$$\text{Im}[G(j\,\omega)] = \frac{a_{\text{even}}(\omega)\, b_{\text{odd}}(\omega) - a_{\text{odd}}(\omega)\, b_{\text{even}}(\omega)}{a_{\text{even}}(\omega)^2 + a_{\text{odd}}(\omega)^2} \qquad \textbf{\textit{(Eq.13)}}$$

We can apply similar analysis to the complex number of $G(-j\,\omega)$. This is the same as $G(j\,\omega)$ except we use $-\omega$ instead of ω.

To make analysis simpler, it is interesting thing to notice that

$$b_{\text{even}}(-\omega) = b_0 - b_2\,(-\omega)^2 + b_4(-\omega)^4 - b_6(-\omega)^6 + b_8(-\omega)^8$$

$$= b_0 - b_2\,\omega^2 + b_4\,\omega^4 - b_6\,\omega^6 + b_8\,\omega^8$$

$$b_{\text{even}}(-\omega) = b_{\text{even}}(\omega)$$

Similarly

$$b_{\text{odd}}(-\omega) = b_1(-\omega) - b_3(-\omega)^3 + b_5(-\omega)^5 - b_7(-\omega)^7$$

$$= -b_1\,\omega + b_3\,\omega^3 - b_5\,\omega^5 + b_7\,\omega^7$$

$$= -\left(b_1\,\omega - b_3\,\omega^3 + b_5\,\omega^5 - b_7\,\omega^7\right)$$

$$b_{\text{odd}}(-\omega) = -b_{\text{odd}}(\omega)$$

So we see that

$$a_{\text{even}}(-\omega) = a_{\text{even}}(\omega) \qquad \textbf{\textit{(Eq.14a)}}$$
$$a_{\text{odd}}(-\omega) = -a_{\text{odd}}(\omega) \qquad \textbf{\textit{(Eq.14b)}}$$
$$b_{\text{even}}(-\omega) = b_{\text{even}}(\omega) \qquad \textbf{\textit{(Eq.14c)}}$$
$$b_{\text{odd}}(-\omega) = -b_{\text{odd}}(\omega) \qquad \textbf{\textit{(Eq.14d)}}$$

Therefore,

$$\text{Re}[G(-j\,\omega)] = \frac{a_{\text{even}}(-\omega)\,b_{\text{even}}(-\omega) + a_{\text{odd}}(-\omega)\,b_{\text{odd}}(-\omega)}{a_{\text{even}}(-\omega)^2 + a_{\text{odd}}(-\omega)^2}$$

$$= \frac{a_{\text{even}}(\omega)\,b_{\text{even}}(\omega) + a_{\text{odd}}(\omega)\,b_{\text{odd}}(\omega)}{a_{\text{even}}(\omega)^2 + a_{\text{odd}}(\omega)^2} \qquad \text{recall: } \text{Re}[G(j\,\omega)] = \frac{a_{\text{even}}(\omega)\,b_{\text{even}}(\omega) + a_{\text{odd}}(\omega)\,b_{\text{odd}}(\omega)}{a_{\text{even}}(\omega)^2 + a_{\text{odd}}(\omega)^2}$$

$$\text{Re}[G(-j\,\omega)] = \text{Re}[G(j\,\omega)] \qquad \textbf{\textit{(Eq.15)}}$$

And for the imaginary part

$$\text{Im}[G(-j\,\omega)] = \frac{a_{\text{even}}(-\omega)\,b_{\text{odd}}(-\omega) - a_{\text{odd}}(-\omega)\,b_{\text{even}}(-\omega)}{a_{\text{even}}(-\omega)^2 + a_{\text{odd}}(-\omega)^2}$$

$$= \frac{-a_{\text{even}}(\omega)\,b_{\text{odd}}(\omega) + a_{\text{odd}}(\omega)\,b_{\text{even}}(\omega)}{a_{\text{even}}(\omega)^2 + a_{\text{odd}}(\omega)^2}$$

$$= -\frac{a_{\text{even}}(\omega)\,b_{\text{odd}}(\omega) - a_{\text{odd}}(\omega)\,b_{\text{even}}(\omega)}{a_{\text{even}}(\omega)^2 + a_{\text{odd}}(\omega)^2} \qquad \text{recall: } \text{Im}[G(j\,\omega)] = \frac{a_{\text{even}}(\omega)\,b_{\text{odd}}(\omega) - a_{\text{odd}}(\omega)\,b_{\text{even}}(\omega)}{a_{\text{even}}(\omega)^2 + a_{\text{odd}}(\omega)^2}$$

$$\text{Im}[G(-j\,\omega)] = -\text{Im}[G(j\,\omega)] \qquad \textbf{\textit{(Eq.16)}}$$

Example to Verify Results

Consider the transfer function

$$G(s)=\frac{3\,(s+3)\,(s-2)}{(s+2)\,(s+2+2\,i)\,(s+2-2\,i)\,(s+4)\,(s+4)}$$

In[∘]:=
```
G[s_] = 3 (s + 3) (s - 2) / ((s + 2) (s + 2 + 2 I) (s + 2 - 2 I) (s + 4) (s + 4));
```

$G(j\,\omega)$

We can first compute $G(j\,\omega)$

In[∘]:=
```
num = Expand[Numerator[G[I ω]]]
den = Expand[Denominator[G[I ω]]]
```

Out[∘]= $-18+3\,i\,\omega-3\,\omega^2$

Out[∘]= $256+384\,i\,\omega-240\,\omega^2-80\,i\,\omega^3+14\,\omega^4+i\,\omega^5$

We can write the denominator in real and imaginary format

In[∘]:=
```
ReDen = Simplify[Re[den], ω ∈ Reals]
ImDen = Expand[Simplify[Im[den], ω ∈ Reals]]
```

Out[∘]= $256-240\,\omega^2+14\,\omega^4$

Out[∘]= $384\,\omega-80\,\omega^3+\omega^5$

So multiplying by the complex conjugate we have

In[∘]:=
```
denConjugate = ReDen - ImDen * I;
numAlt = ComplexExpand[num * denConjugate];
denAlt = Expand[den * denConjugate];

(*Break into real and imaginary parts*)
ReGjω = Simplify[Re[numAlt], ω ∈ Reals] / denAlt
ImGjω = Simplify[Im[numAlt], ω ∈ Reals] / denAlt

(*Check manipulations*)
ReGjω + ImGjω * I == G[I ω] // Simplify
```

Out[∘]= $\dfrac{-4608+4704\,\omega^2+228\,\omega^4-39\,\omega^6}{65\,536+24\,576\,\omega^2+3328\,\omega^4+448\,\omega^6+36\,\omega^8+\omega^{10}}$

Out[∘]= $\dfrac{3\,\omega\,\left(2560-336\,\omega^2-60\,\omega^4+\omega^6\right)}{65\,536+24\,576\,\omega^2+3328\,\omega^4+448\,\omega^6+36\,\omega^8+\omega^{10}}$

Out[∘]= True

So we see that $G(j\,\omega)$ is given by

$$G(j\,\omega)=\left(\frac{-4608+4704\,\omega^2+228\,\omega^4-39\,\omega^6}{65\,536+24\,576\,\omega^2+3328\,\omega^4+448\,\omega^6+36\,\omega^8+\omega^{10}}\right)+i\left(\frac{7680\,\omega-1008\,\omega^3-180\,\omega^5+3\,\omega^7}{65\,536+24\,576\,\omega^2+3328\,\omega^4+448\,\omega^6+36\,\omega^8+\omega^{10}}\right)$$

$G(-j\,\omega)$

We can now compute $G(-j\,\omega)$

In[*]:=
```
num2 = Expand[Numerator[G[-I ω]]]
den2 = Expand[Denominator[G[-I ω]]]
```

Out[*]= $-18 - 3\,i\,\omega - 3\,\omega^2$

Out[*]= $256 - 384\,i\,\omega - 240\,\omega^2 + 80\,i\,\omega^3 + 14\,\omega^4 - i\,\omega^5$

We can write the denominator in real and imaginary format

In[*]:=
```
ReDen2 = Simplify[Re[den2], ω ∈ Reals]
ImDen2 = Expand[Simplify[Im[den2], ω ∈ Reals]]
```

Out[*]= $256 - 240\,\omega^2 + 14\,\omega^4$

Out[*]= $-384\,\omega + 80\,\omega^3 - \omega^5$

So multiplying by the complex conjugate we have

In[*]:=
```
denConjugate2 = ReDen2 - ImDen2 * I;
numAlt2 = ComplexExpand[num2 * denConjugate2];
denAlt2 = Expand[den2 * denConjugate2];

(*Break into real and imaginary parts*)
ReGMinusjω = Simplify[Re[numAlt2], ω ∈ Reals] / denAlt2
ImGMinusjω = Simplify[Im[numAlt2], ω ∈ Reals] / denAlt2

(*Check manipulations*)
ReGMinusjω + ImGMinusjω * I == G[-I ω] // Simplify
```

Out[*]= $$\frac{-4608 + 4704\,\omega^2 + 228\,\omega^4 - 39\,\omega^6}{65\,536 + 24\,576\,\omega^2 + 3328\,\omega^4 + 448\,\omega^6 + 36\,\omega^8 + \omega^{10}}$$

Out[*]= $$-\frac{3\,\omega\,\left(2560 - 336\,\omega^2 - 60\,\omega^4 + \omega^6\right)}{65\,536 + 24\,576\,\omega^2 + 3328\,\omega^4 + 448\,\omega^6 + 36\,\omega^8 + \omega^{10}}$$

Out[*]= True

So we see that $G(-j\,\omega)$ is given by

$$G(-j\,\omega) = \left(\frac{-4608+4704\,\omega^2+228\,\omega^4-39\,\omega^6}{65\,536+24\,576\,\omega^2+3328\,\omega^4+448\,\omega^6+36\,\omega^8+\omega^{10}}\right) + i\left(\frac{-7680\,\omega+1008\,\omega^3+180\,\omega^5-3\,\omega^7}{65\,536+24\,576\,\omega^2+3328\,\omega^4+448\,\omega^6+36\,\omega^8+\omega^{10}}\right)$$

We can now check

In[*]:=
```
ReGMinusjω == ReGjω
ImGMinusjω == -ImGjω
```

Out[*]= True

Out[*]= True

Se we see that in this case

$\text{Re}[G(-j\,\omega)] = \text{Re}[G(j\,\omega)]$

$\text{Im}[G(-j\,\omega)] = -\text{Im}[G(j\,\omega)]$

End example

We return to $G(j\,\omega)$. We note that this is simply a complex number, and therefore it can be written using polar form as

$$G(j\,\omega) = \left| G(j\,\omega) \right| e^{j\,\theta} \qquad \textbf{\textit{(Eq.17)}}$$

where $\quad \theta = \text{atan2}(\text{Im}[G(j\,\omega)], \text{Re}[G(j\,\omega)]) = \text{atan2}(y, x)$

Similarly for the $G(-j\,\omega)$ term,

$$G(-j\,\omega) = \left| G(-j\,\omega) \right| e^{j\,\phi}$$

For the $G(-j\,\omega)$ term, the angle of the complex number, ϕ, is given by

$\phi = \text{atan2}(\text{Im}[G(-j\,\omega)], \text{Re}[G(-j\,\omega)])$ recall: $\text{Im}[G(-j\,\omega)] = -\text{Im}[G(j\,\omega)]$ and $\text{Re}[G(-j\,\omega)] = \text{Re}[G(j\,\omega)]$

$= \text{atan2}(-\text{Im}[G(j\,\omega)], \text{Re}[G(j\,\omega)])$ recall: $\text{atan2}(-y, x) = -\text{atan2}(y, x)$

$= -\text{atan2}(\text{Im}[G(j\,\omega)], \text{Re}[G(j\,\omega)])$ recall: $\theta = \text{atan2}(\text{Im}[G(j\,\omega)], \text{Re}[G(j\,\omega)])$

$$\phi = -\theta \qquad \textbf{\textit{(Eq.18)}}$$

Furthermore, if we look at the magnitude of the complex number $G(-j\,\omega)$

$\left| G(-j\,\omega) \right| = \sqrt{(\text{Re}[G(-j\,\omega)])^2 + (\text{Im}[G(-j\,\omega)])^2}$ $\text{Im}[G(-j\,\omega)] = -\text{Im}[G(j\,\omega)]$ and $\text{Re}[G(-j\,\omega)] = \text{Re}[G(j\,\omega)]$

$$= \sqrt{(\text{Re}[G(j\,\omega)])^2 + (-\text{Im}[G(j\,\omega)])^2}$$

$$= \sqrt{(\text{Re}[G(j\,\omega)])^2 + (\text{Im}[G(j\,\omega)])^2}$$

$$\left| G(-j\,\omega) \right| = \left| G(j\,\omega) \right| \qquad \textbf{\textit{(Eq.19)}}$$

With the results of Eq.18 and Eq.19, we can write $G(-j\,\omega)$ as

$$G(-j\,\omega) = \left| G(j\,\omega) \right| e^{-j\,\theta} \qquad \textbf{\textit{(Eq.20)}}$$

Finally, substituting, Eq.17 and Eq.20 back into Eq.10, we obtain

$$y_{ss}(t) = \frac{A}{2\,j}\left(G(j\,\omega)\,e^{j\,\omega\,t} - G(-j\,\omega)\,e^{-j\,\omega\,t}\right)$$

$$= \frac{A}{2j}\left(\left|G(j\omega)\right| e^{j\theta} e^{j\omega t} - \left|G(j\omega)\right| e^{-j\theta} e^{-j\omega t}\right)$$

$$= \frac{A}{2j}\left(\left|G(j\omega)\right| e^{j\theta+j\omega t} - \left|G(j\omega)\right| e^{-j\theta-j\omega t}\right)$$

$$= \frac{A|G(j\omega)|}{2j}\left(e^{(\omega t+\theta)j} - e^{-(\omega t+\theta)j}\right)$$ recall: Euler's Formula states that $e^{\varphi j} = \cos(\varphi) + \sin(\varphi)\, j$

$$= \frac{A|G(j\omega)|}{2j}[\cos(\omega t+\theta) + \sin(\omega t+\theta)\, j - (\cos(-(\omega t+\theta)) + \sin(-(\omega t+\theta))\, j)]$$ recall:

$\cos(-(\omega t+\theta)) = \cos(\omega t+\theta)$

$\sin(-(\omega t+\theta)) = -\sin(\omega t+\theta)$

$$= \frac{A|G(j\omega)|}{2j}[\cos(\omega t+\theta) + \sin(\omega t+\theta)\, j - (\cos(\omega t+\theta) - \sin(\omega t+\theta)\, j)]$$

$$= \frac{A|G(j\omega)|}{2j}[\cos(\omega t+\theta) + \sin(\omega t+\theta)\, j - \cos(\omega t+\theta) + \sin(\omega t+\theta)\, j]$$

$$= \frac{A|G(j\omega)|}{2j}[2\, j \sin(\omega t+\theta)]$$

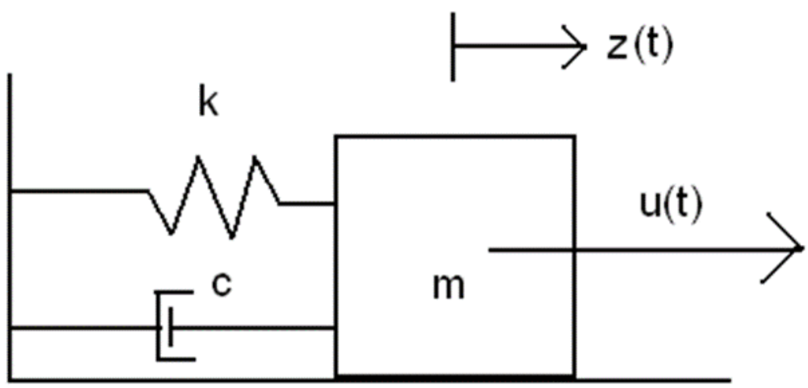
$$y_{ss}(t) = A\ \left|G(j\omega)\right|\ \sin(\omega t+\theta) \qquad \textbf{\textit{(Eq.21)}}$$

where $\left|G(j\omega)\right| = \sqrt{(\mathrm{Re}[G(j\omega)])^2 + (\mathrm{Im}[G(j\omega)])^2}$

$\theta = \angle G(j\omega) = \mathrm{atan2}(\mathrm{Im}[G(j\omega)], \mathrm{Re}[G(j\omega)]) = \mathrm{atan2}(y, x)$

So the final result is that the steady state output of the system (in response to an input of the form $u(t) = A\sin(\omega t)$) is a sin wave of the same frequency, ω, whose magnitude is multiplied by $\left|G(j\omega)\right|$ and whose phase is shifted by $\angle G(j\omega)$.

Example: Mass/Spring/Damper



Let us see how this system reacts to an input of the form

$$u(t) = A\sin(\omega t)$$

Equations of motion are

$$\ddot{z}(t) + \frac{c}{m}\dot{z}(t) + \frac{k}{m} z(t) = \frac{1}{m} u(t)$$

We can obtain the transfer function easily

$$\left(s^2 + \frac{c}{m} s + \frac{k}{m}\right) Z(s) = \frac{1}{m} U(s)$$

$$G(s) = \frac{Z(s)}{U(s)} = \frac{\frac{1}{m}}{s^2+\frac{c}{m} s+\frac{k}{m}}$$

We can choose constants of

$m = 1/3$
$c = 1/6$
$k = 4/3$

With these constants the transfer function can be written as

In[∘]:=
```
G[s_] = (1 / m) / (s^2 + c/m s + k/m) /. {m → 1 / 3, c → 1 / 6, k → 4 / 3};
G[s]
```

Out[∘]=
$$\frac{3}{4 + \frac{s}{2} + s^2}$$

So the steady state response should be

$$z_{ss}(t) = A \; | \; G(j\,\omega) \; | \; \sin(\omega\, t + \theta)$$

where $\theta = \angle \mathrm{G}(j\,\omega) = \mathrm{atan2}(\mathrm{Im}[G(j\,\omega)], \mathrm{Re}[G(j\,\omega)]) = \mathrm{atan2}(y, x)$

We first calculate the real and imaginary parts of $G(j\,\omega)$

$$G(j\,\omega) = \frac{3}{(j\,\omega)^2+\frac{1}{2}(j\,\omega)+4} \qquad \text{recall: } j^2 = -1$$

$$= \frac{3}{-\omega^2+\frac{1}{2}\omega\, j+4}$$

$$= \frac{3}{4-\omega^2+\frac{1}{2}\omega\, j}\left(\frac{4-\omega^2-\frac{1}{2}\omega\, j}{4-\omega^2-\frac{1}{2}\omega\, j}\right)$$

$$= \frac{12-3\,\omega^2-\frac{3\,\omega}{2}\, j}{16-\frac{31\,\omega^2}{4}+\omega^4}$$

$$= \frac{12-3\,\omega^2}{16-\frac{31\,\omega^2}{4}+\omega^4} + \frac{-\frac{3\,\omega}{2}}{16-\frac{31\,\omega^2}{4}+\omega^4}\, j$$

$$G(j\,\omega) = \alpha + \beta\, j$$

where $\quad \alpha = \text{Re}[G(j\,\omega)] = \frac{12-3\,\omega^2}{16-\frac{31\,\omega^2}{4}+\omega^4}$

$$\beta = \text{Im}[G(j\,\omega)] = \frac{-\frac{3\,\omega}{2}}{16-\frac{31\,\omega^2}{4}+\omega^4}$$

```
In[∘]:= realGjω = (12 - 3 ω^2) / (16 - (31 ω^2)/4 + ω^4);
        imagGjω = (-(3 ω)/2) / (16 - (31 ω^2)/4 + ω^4);
```

So the magnitude of $G(j\,\omega)$ is given by

$$|\,G(j\,\omega)\,| = \sqrt{\alpha^2 + \beta^2}$$

$$= \sqrt{\left(\frac{12-3\,\omega^2}{16-\frac{31\,\omega^2}{4}+\omega^4}\right)^2 + \left(\frac{-\frac{3\,\omega}{2}}{16-\frac{31\,\omega^2}{4}+\omega^4}\right)^2}$$

```
In[∘]:= magGjω = Sqrt[realGjω^2 + imagGjω^2] // Simplify
```

Out[∘]= $6\sqrt{\frac{1}{64-31\,\omega^2+4\,\omega^4}}$

So we have

$$|\,G(j\,\omega)\,| = 6\sqrt{\frac{1}{64-31\,\omega^2+4\,\omega^4}}$$

We can look at a specific magnitude and frequency. For example

$$A = 6$$
$$\omega = 3$$

For the specific case of $\omega = 3$, we have

```
In[∘]:= Agiven = 6;
        ωgiven = 3;
        magGjω /. {ω → ωgiven} // N
Out[∘]= 0.574696
```

So

$$|\,G(3\,j)\,| = 0.574$$

We can also calculate the phase shift,

$$\theta = \angle G(j\,\omega) = \text{atan2}(\text{Im}[G(j\,\omega)], \text{Re}[G(j\,\omega)]) = \text{atan2}(y,\, x)$$

For the case of $\omega = 3$, we see that

In[ ]:=
```
realGjω /. {ω → ωgiven} // N
imagGjω /. {ω → ωgiven} // N
```

Out[ ]= -0.550459

Out[ ]= -0.165138

So the angle is in the 3rd quadrant. We use the 4 quadrant inverse tangent to automatically take care of the quadrant

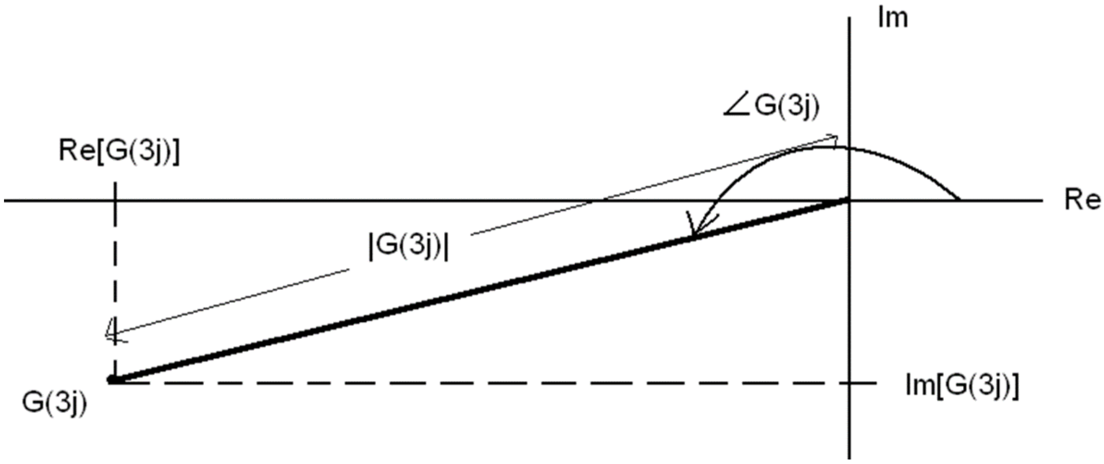
In[ ]:=
```
(*Note that Mathematica uses ArcTan[x,y]=ArcTan[Re,Im]*)
(ArcTan[realGjω, imagGjω] /. {ω → ωgiven}) 180/π // N
```

Out[ ]= -163.301

So we have

$$\theta = \angle G(3\,j) = -163.3^{o}$$

So in this case, we visualize the imaginary number $G(j\,\omega)$ with $\omega = 3$ as



So the response of the system at steady state is (at $\omega = 3$)

$$z_{ss}(t) = A \;|\; G(j\,\omega) \;|\; \sin(3\,t + \theta)$$

$$z_{ss}(t) = 0.574\,A \sin\!\left(3\,t - 163.3\,\frac{\pi}{180}\right)$$

We can confirm this using traditional techniques. The output in response to the sinusoidal input is

$$Z(s) = G(s)\,U(s)$$

$$Z(s) = \frac{3}{s^2+\frac{1}{2}\,s+4}\left(\frac{A\,\omega}{s^2+\omega^2}\right)$$

In[∘]:=
```
Z[s_] = G[s] × LaplaceTransform[A Sin[ω t], t, s]
```

Out[∘]=
$$\frac{3 A \omega}{\left(4 + \frac{s}{2} + s^2\right)\left(s^2 + \omega^2\right)}$$

Now taking the inverse Laplace transform

In[∘]:=
```
z[t_] = InverseLaplaceTransform[Z[s], s, t] // Expand
```

Out[∘]=
$$\frac{6 A e^{-t/4} \omega \operatorname{Cos}\left[\frac{3\sqrt{7}\, t}{4}\right]}{64 - 31\omega^2 + 4\omega^4} - \frac{6 A \omega \operatorname{Cos}[t\,\omega]}{64 - 31\omega^2 + 4\omega^4} - \frac{62 A e^{-t/4} \omega \operatorname{Sin}\left[\frac{3\sqrt{7}\, t}{4}\right]}{\sqrt{7}\left(64 - 31\omega^2 + 4\omega^4\right)} + \frac{16 A e^{-t/4} \omega^3 \operatorname{Sin}\left[\frac{3\sqrt{7}\, t}{4}\right]}{\sqrt{7}\left(64 - 31\omega^2 + 4\omega^4\right)} + \frac{48 A \operatorname{Sin}[t\,\omega]}{64 - 31\omega^2 + 4\omega^4} - \frac{12 A \omega^2 \operatorname{Sin}[t\,\omega]}{64 - 31\omega^2 + 4\omega^4}$$

With $\omega = 3$

In[∘]:=
```
z[t] /. {ω → ωgiven}
```

Out[∘]=
$$-\frac{18}{109} A \operatorname{Cos}[3\,t] + \frac{18}{109} A e^{-t/4} \operatorname{Cos}\left[\frac{3\sqrt{7}\, t}{4}\right] - \frac{60}{109} A \operatorname{Sin}[3\,t] + \frac{246 A e^{-t/4} \operatorname{Sin}\left[\frac{3\sqrt{7}\, t}{4}\right]}{109\sqrt{7}}$$

So we have

$$z(t) = -\tfrac{18}{109} A \cos(3\,t) - \tfrac{60}{109} A \sin(3\,t) + \tfrac{18}{109} A\, e^{-\frac{1}{4}t} \cos\!\left(\tfrac{3\sqrt{7}}{4}\, t\right) + \tfrac{246\,A}{109\sqrt{7}}\, e^{-\frac{1}{4}t} \sin\!\left(\tfrac{3\sqrt{7}}{4}\, t\right)$$

We see that at steady state, the exponential terms will go to zero, so we have

$$z_{ss}(t) = -\tfrac{18}{109} A \cos(3\,t) - \tfrac{60}{109} A \sin(3\,t)$$

$$= -A\left(\tfrac{18}{109} \cos(3\,t) + \tfrac{60}{109} \sin(3\,t)\right) \qquad \text{let } a = \tfrac{18}{109},\ b = \tfrac{60}{109}$$

$$= -A(a \cos(3\,t) + b \sin(3\,t))$$

In[∘]:=
```
a = 18/109; b = 60/109; ω = 3;
```

We can now use the trig identity of $a\cos(\omega\,t) + b\sin(\omega\,t) = \sqrt{a^2 + b^2}\,\cos(\omega\,t - \phi)$ where $\phi = \tan^{-1}(b/a)$

$$= -A\left(\sqrt{a^2 + b^2}\,\cos(\omega\,t - \phi)\right)$$

In[∘]:=
```
Sqrt[a^2 + b^2]
ϕ = ArcTan[a, b] // N
```

Out[∘]=
$$\frac{6}{\sqrt{109}}$$

Out[∘]= 1.27934

So we have

$= -A\left(\frac{6}{\sqrt{109}}\cos(3t - \phi)\right)$ recall: $\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$

$= -A\left(\frac{6}{\sqrt{109}}\sin\left(\frac{\pi}{2} - (3t - \phi)\right)\right)$

$= -A\left(\frac{6}{\sqrt{109}}\sin\left(\frac{\pi}{2} - 3t + \phi\right)\right)$

$= -A\left(\frac{6}{\sqrt{109}}\sin\left(-\left(-\frac{\pi}{2} + 3t - \phi\right)\right)\right)$

$= A\left(\frac{6}{\sqrt{109}}\sin\left(3t - \frac{\pi}{2} - \phi\right)\right)$

$= A\,\frac{6}{\sqrt{109}}\sin\left(3t - \frac{\pi}{2} - \phi\right)$ recall: $\phi = 1.27934$ and $\frac{6}{\sqrt{109}} = 0.574$

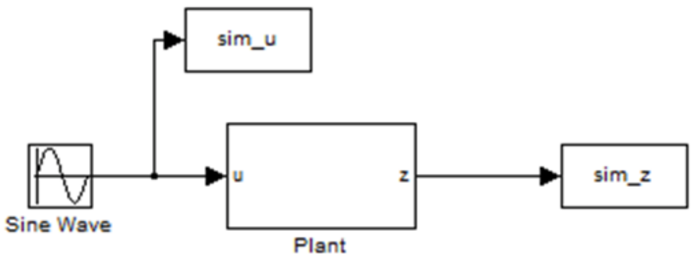
$z_{ss}(t) = 0.574\,A\sin\left(3t - 163.3\,\frac{\pi}{180}\right)$

This is precisely what we obtained when we used the frequency domain analysis approach.

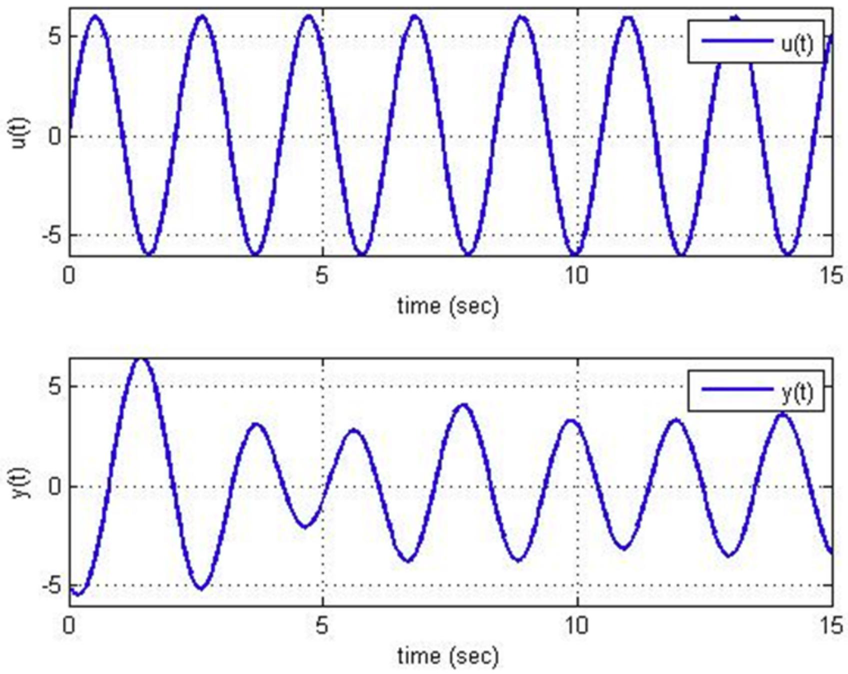
Finally, we can verify this numerically using Simulink. For extra practice, let's create an ODE model of this system with individual integrators and blocks. Recall the equation of motion is given by

$\ddot{z}(t) + \frac{c}{m}\dot{z}(t) + \frac{k}{m}z(t) = \frac{1}{m}u(t)$

$\ddot{z}(t) = -\frac{c}{m}\dot{z}(t) - \frac{k}{m}z(t) + \frac{1}{m}u(t)$



The output from the numerical simulation with initial conditions of $\overline{x}(0) = (\,-5 \quad -5\,)^T$ is shown below



As can be seen, the initial conditions die away and the steady state response is the attenuated and phase shifted sin wave.

Summary