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Lecture 08b
PDEs: 2D Wave Equation

The YouTube videos that cover this lecture are located at:

-'Derivation of the 2D Wave Equation' at https://youtu.be/KAS7JBztw8E.

-'Solving the 2D Wave Equation' at https://youtu.be/Whp6jolTu34.

Modeling: Membrane, Two-Dimensional Wave Equation

We now turn our attention to 2D membranes (such as a drum head). In this case, we consider finding a function $u(x, y, t)$ that describes the deflection of membrane in the out-of-plane direction (AKA z direction) at location x, y at time t.

<u>Physical Assumptions</u>

1. The mass of the membrane per unit area is constant (homogeneous membrane).

ρ = density per unit area $\left(\mathrm{kg}/m^2\right)$ = constant

2. The membrane is perfectly flexible and offers no resistance to bending.

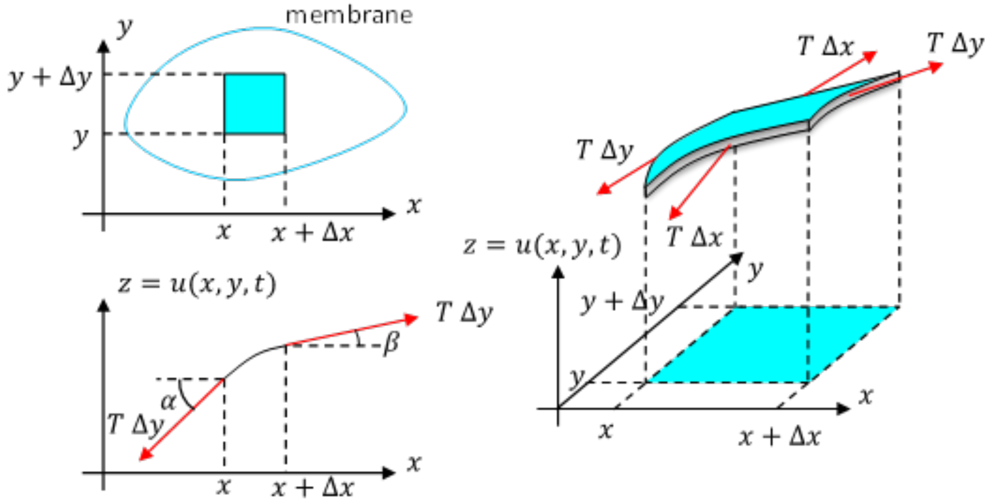
3. The membrane is stretched and then fixed along its entire boundary in the xy-plane. The

4. The tension per unit length T (ie N/m) caused by stretching the membrane is the same at all points and in all directions and does not change during the motion.

T = tension per unit length (N/m) = constant

5. The deflection $u(x, y, t)$ of the membrane during the motion is small compared to the size of the membrane, and all angles of inclination are small.

A diagram of the system is shown below



Note that since the geometry of the small portion of the membrane can vary with y, the values of α and β could be changing along the right and left side as y varies. As such, we use α and β as "average" values of the deflection angle at the right and left side. If Δy is small this variation should be small.

Derivation of the PDE of the Model ("Two-Dimensional Wave Equation") from Forces

Consider a small portion of the membrane with sides of length Δx and Δy. The tension T is the force per unit length. Hence the forces acting on the sides of the portion are approximately $T\,\Delta$x and $T\,\Delta$y. Since the membrane is perfectly flexible, these forces are tangent to the moving membrane at every instant.

Horizontal Component of Forces

The horizontal forces in the membrane are related to cosine of the inclination angle. Since we assume the inclination angle is small, these terms are all approximately 1. Therefore, the forces around the portion in the horizontal plane are all approximately equal and we conclude that the horizontal motion of the membrane is negligibly small and all motion will therefore occur in the vertical direction.

Vertical Component of Forces

The components along the right and left side are respectively

$T\,\Delta y \sin(\beta)$	right side
$-T\,\Delta y \sin(\alpha)$	left side

where T = tension force per unit length (N/m)

Since the angles are small, we have $\sin(\theta) \approx \theta \approx \tan(\theta)$. So the resultant vertical force (different in forces on either side) can be written as

$$\text{total vertical force on sides} = T\,\Delta y[\tan(\beta) - \tan(\alpha)]$$

Since $u(x, y)$ is the deflection of the membrane at a position x, y, then $u_x(x, y)$ is the slope of the membrane in the x direction at the point x, y. Since $\tan(\alpha)$ is the average slope of the membrane at x

and $\tan(\beta)$ is the average slope of the membrane at $x + x\,\Delta x$, the previous expression can be written as

$$\text{total vertical force on sides} = T\,\Delta y[u_x(x+\Delta x,\ y_1) - u_x(x,\ y_2)] \qquad \textbf{(Eq.1)}$$

where y_1 and y_2 are values between y and $y + \Delta\, y$

Note that $u_x(x + \Delta x,\ y)$ is actually a function of y as the slope in the x-direction can change as y changes. Therefore, in order to obtain a single slope, we chose a value of y_1 which is between y and $y + \Delta\, y$ and is such that $u_x(x + \Delta x,\ y_1) = \tan(\beta)$.

In a similar fashion, the total force on the front and back are

$$\text{total vertical force on front/back} = T\,\Delta x\left[u_y(x_1,\ y + \Delta y) - u_y(x_2,\ y)\right] \qquad \textbf{(Eq.2)}$$

where x_1 and x_2 are values between x and $x + \Delta\, x$

Newton’s Second Law Gives the PDE of the Model

Once again, we can write Newton’s second law for the object

$$\rho \Delta x\, \Delta y\, \frac{\partial^2 u}{\partial t^2} = T\,\Delta y[u_x(x+\Delta x,\ y_1) - u_x(x,\ y_2)] + T\,\Delta x\left[u_y(x_1,\ y+\Delta y) - u_y(x_2,\ y)\right]$$

where $\quad \rho$ = density per unit area $\left(\text{kg}/m^2\right)$

Dividing by $\rho \Delta x\, \Delta y$ yields

$$\frac{\partial^2 u}{\partial t^2} = \frac{T}{\rho}\left(\frac{u_x(x+\Delta x, y_1) - u_x(x, y_2)}{\Delta x} + \frac{u_y(x_1, y+\Delta y) - u_y(x_2, y)}{\Delta y}\right)$$

As we let Δx and Δy approach 0, we obtain the PDE of the model

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \textbf{(Eq.3)}$$

where $\quad c^2 = T/\rho$

T = tension per unit length (N/m)

ρ = density of unit area $\left(\text{kg}/m^2\right)$

This is the 2-dimensional wave equation. The expression in parenthesis is the Laplacian of u, $\nabla^2 u = \Delta\, u$. Hence, an alternative form of Eq.3 is (warning: the textbook notation is incorrect)

$$\frac{\partial^2 u}{\partial t^2} = c^2\, \nabla^2 u = c^2\, \Delta\, u \qquad \text{(2D Wave Equation)} \qquad \textbf{(Eq.3')}$$

Note that if there is only a single dimension (for example x), then Eq.3’ becomes

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{(1D Wave Equation)}$$

Which is exactly the 1D wave equation we derived in a previous video.

Rectangular Membrane. Double Fourier Series

We now develop a solution for the 2D wave equation.

The problem can be stated as

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \qquad \text{(governing PDE)} \qquad \textbf{\textit{(Eq.1)}}$$

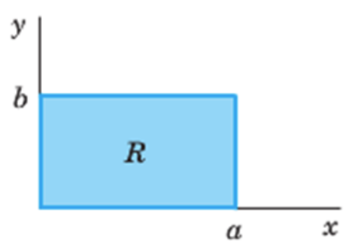
$$u = 0 \; \forall \; t \text{ on the boundary} \text{(boundary condition)} \qquad \textbf{\textit{(Eq.2)}}$$

$$u(x, y, 0) = f(x, y) \qquad \text{(initial displacement)} \qquad \textbf{\textit{(Eq.3a)}}$$

$$u_t(x, y, 0) = g(x, y) \qquad \text{(initial velocity)} \qquad \textbf{\textit{(Eq.3b)}}$$

Example: Rectangular Membrane

Let us consider the rectangular membrane as shown below



We will solve this problem in 3 steps

Step 1. Two separation of variable. First set $u(x, y, t) = F(x, y) G(t)$ and later $F(x, y) = H(x) Q(y)$. In other words, we obtain from the governing PDE (Eq.1) an ODE for G (temporal) and a PDE for F (spatial - which can then be broken into two additional ODEs).

Step 2. From the solutions of those ODEs we determine eigenfunctions, u_{mn}, that satisfy the boundary condition (Eq.2)

Step 3. We compose the u_{mn} in to a double Fourier series which solves the whole problem.

During these steps, there are many variables we will introduce. To help with notation, a glossary of relevant symbols is given here

Initial Problem Statement

T =tension in membrane per unit length (N/m)

ρ =density per unit area $\left(\text{kg}/m^2\right)$

a = width of membrane (m)

b = height of membrane (m)

$c = (T/\rho)^{1/2}$ = given parameter of PDE

$u(x, y, t)$ = displacement of membrane (m)

x = horizontal position (m)

y = vertical position (m)

t = time (s)

$f(x, y)$ = initial membrane displacement (m)

$g(x, y)$ = initial membrane velocity (m/s)

Step 1:

$u(x, y, t) = F(x, y)\, G(t)$

$-v^2$ = separation constant (first separation of variables, must be negative) (will show it must take specific values)

$\lambda = c\, v$ = parameter which is a function of separation constant

$F(x, y) = H(x)\, Q(y)$ = solution to Helmholtz equation

$-k^2$ = separation constant (second separation of variables, must be negative) (will show it must take specific values)

$p = \left(v^2 - k^2\right)^{1/2}$ = parameter which is a function of separation constants (will show it must take specific values)

Step 2:

$F_{\text{mn}}(x, y)$ = general solution at specific $m, n \in \mathbb{Z}$ values

$G_{\text{mn}}(t)$ = general solution at specific $m, n \in \mathbb{Z}$ values

$u_{\text{mn}}(x, y, t)$ = eigenfunction at specific $m, n \in \mathbb{Z}$ values

λ_{mn} = eigenvalue at specific $m, n \in \mathbb{Z}$ values

Step 3:

$K_m(y)$ = temporary variable used for Fourier analysis

Step 1. Three ODEs from the Wave Equation

To obtain ODEs from the original PDE (Eq.1), we apply two successive separations of variables. In the first separation, we set

$$u(x, y, t) = F(x, y)\, G(t)$$

Substituting this into Eq.1 yields

$$\frac{\partial^2}{\partial t^2}[F(x,\ y)\,G(t)] = c^2\left(\frac{\partial^2}{\partial x^2}[F(x,\ y)\,G(t)] + \frac{\partial^2}{\partial y^2}[F(x,\ y)\,G(t)]\right)$$

$$F(x,\ y)\,\frac{\partial^2}{\partial t^2}[G(t)] = c^2\left(\frac{\partial^2}{\partial x^2}[F(x,\ y)]\,G(t) + \frac{\partial^2}{\partial y^2}[F(x,\ y)]\,G(t)\right)$$

$$F\,\ddot{G} = c^2\left(F_{xx}\,G + F_{yy}\,G\right)$$

Dividing by $c^2\,F\,G$ yields

$$\frac{\ddot{G}}{c^2\,G} = \frac{1}{F\,G}\left(F_{xx}\,G + F_{yy}\,G\right)$$

$$\frac{\ddot{G}}{c^2\,G} = \frac{1}{F}\left(F_{xx} + F_{yy}\right)$$

The left side depends only on t and the right side is independent of t, therefore, both must equal a constant (separation constant) Simple investigation similar to the previous sections will show that the separation constant must be negative. Denoting the negative separation constant as $-v^2$, we have two equations

$$\frac{\ddot{G}}{c^2\,G} = -v^2 \qquad \Rightarrow \qquad \ddot{G} + \lambda^2\,G = 0 \qquad \text{(ODE, time function)} \qquad \textbf{\textit{(Eq.4)}}$$

where $\quad \lambda = c\,v$

$$\frac{1}{F}\left(F_{xx} + F_{yy}\right) = -v^2 \qquad \Rightarrow \qquad F_{xx} + F_{yy} + v^2\,F = 0 \qquad \text{(PDE, amplitude function)} \qquad \textbf{\textit{(Eq.5)}}$$

where $\quad v = \text{separation constant}$

The amplitude function is called the two-dimensional **Helmholtz equation**.

We can now perform a second separation of variables by focusing on the Helmholtz equation. We assume that F can be written as $F(x,\ y) = H(x)\,Q(y)$. Substituting this into Eq.5 yields

$$\frac{d^2}{d\,x^2}[H(x)\,Q(y)] + \frac{d^2}{d\,y^2}[H(x)\,Q(y)] + v^2\,H(x)\,Q(y) = 0$$

$$\frac{d^2\,H}{d\,x^2}\,Q + H\,\frac{d^2\,Q}{d\,y^2} + v^2\,H\,Q = 0$$

$$\frac{d^2\,H}{d\,x^2}\,Q = -\left(H\,\frac{d^2\,Q}{d\,y^2} + v^2\,H\,Q\right)$$

Dividing by $H\,Q$

$$\frac{1}{H\,Q}\left(\frac{d^2 H}{d x^2}\,Q\right) = -\frac{1}{H\,Q}\left(H\,\frac{d^2 Q}{d y^2} + v^2\,H\,Q\right)$$

$$\frac{1}{H}\,\frac{d^2 H}{d x^2} = -\frac{1}{Q}\left(\frac{d^2 Q}{d y^2} + v^2\,Q\right)$$

Using similar arguments, both sides must equal a constant (another separation constant) and this constant must be negative to avoid a trivial solution. Denoting this second separation constant as $-k^2$ we have

$$\frac{1}{H}\,\frac{d^2 H}{d x^2} = -k^2 \quad \Rightarrow \quad \frac{d^2 H}{d x^2} + k^2\,H = 0 \quad \text{(Spatial ODE \#1)} \qquad \textbf{(Eq.6)}$$

$$-\frac{1}{Q}\left(\frac{d^2 Q}{d y^2} + v^2\,Q\right) = -k^2 \quad \Rightarrow \quad \frac{d^2 Q}{d y^2} + p^2\,Q = 0 \quad \text{(Spatial ODE \#2)} \qquad \textbf{(Eq.7)}$$

where $\quad p^2 = v^2 - k^2 \quad$ (note p is another constant which is a function of the two separation constants)

Note that we again use the notation $p^2 = v^2 - k^2$ as we will see that p^2 must be positive in order to avoid a trivial solution.

Step 2. Satisfying the Boundary Condition

Using previously discussed arguments (recognizing that Eq.6 and Eq.7 are undamped oscillators), the general solution for Eq.6 can be shown to be

$$H(x) = A\cos(k\,x) + B\sin(k\,x)$$

```
In[1]:= H[x_] = A Cos[k x] + B Sin[k x];
```

Similarly, solutions for Eq.7 can be shown to be

$$Q(y) = C\cos(p\,y) + D\sin(p\,y)$$

```
In[2]:= Q[y_] = Cc Cos[p y] + Dd Sin[p y];
```

Recall that the boundary condition stated that

$$u(x,\,y,\,t) = F(x,\,y)\,G(t) = 0 \qquad \text{for } x,\,y \text{ on boundary for all } t$$

So we see that we require

$$F(x,\,y) = 0 \qquad \text{for } x,\,y \text{ on boundary}$$

Referring to our earlier diagram, we see that the boundary of this membrane is comprised of 4 edges: left ($x = 0$), right ($x = a$), bottom ($y = 0$), and top ($y = b$). Therefore, satisfying the boundary condition requires

$F(0, y) = 0 \Rightarrow \quad H(0)\,Q(y) = 0 \quad \Rightarrow \quad H(0) = 0 \qquad\qquad \text{(left)}$

$F(a, y) = 0 \quad \Rightarrow \quad H(a)\,Q(y) = 0 \quad \Rightarrow \quad H(a) = 0 \qquad\qquad \text{(right)}$

$F(x, 0) = 0 \Rightarrow \quad H(x)\,Q(0) = 0 \quad \Rightarrow \quad Q(0) = 0 \qquad\qquad \text{(bottom)}$

$F(x, b) = 0 \Rightarrow \quad H(x)\,Q(b) = 0 \quad \Rightarrow \quad Q(b) = 0 \qquad\qquad \text{(top)}$

From the boundary condition $H(0) = 0$, we see that

$A = 0$

In[3]:= `H[0]`

Out[3]= A

We now combine this result with the boundary condition $H(a) = 0$

In[4]:= `H[a] /. {A → 0}`

Out[4]= B Sin[a k]

Using the usual argument, we require $a\,k = m\,\pi$, $m = 1, 2, 3,$ ... in order to achieve a nontrivial solution. Therefore

$k = \frac{m\,\pi}{a} \qquad m \in Z \qquad\qquad$ ***(Eq.7a)***

In the same fashion, applying the boundary conditions to the bottom and top yields

$C = 0$

$p = \frac{n\,\pi}{b} \qquad n \in Z \qquad\qquad$ ***(Eq.7b)***

So general solutions take the form of

$H_m(x) = B \sin\left(\frac{m\,\pi}{a}\,x\right) \qquad m \in Z$

$Q_n(y) = D \sin\left(\frac{n\,\pi}{b}\,y\right) \qquad n \in Z$

Combining the two we obtain the general solution to the Helmholtz equation that satisfies the boundary conditions of zero deflection on the boundary of the membrane are given

$F_{\mathrm{mn}}(x, y) = B \sin\left(\frac{m\,\pi}{a}\,x\right) D \sin\left(\frac{n\,\pi}{b}\,y\right)$

$= B\,D \sin\left(\frac{m\,\pi}{a}\,x\right) \sin\left(\frac{n\,\pi}{b}\,y\right) \qquad$ let: $\Gamma = B\,D$ (another arbitrary constant)

$$F_{mn}(x, y) = \Gamma \sin\left(\frac{m\pi}{a} x\right) D \sin\left(\frac{n\pi}{b} y\right) \qquad m, n \in Z \qquad \textbf{(Eq.8)}$$

We can quickly verify that this satisfies the boundary conditions

In[5]:=
```
(*Define the general solution*)
Fmn[x_, y_] = Γ Sin[(m π)/a x] Sin[(n π)/b y];

(*Verify that this satisfies the boundary condition*)
Print["Satisfies left boundary condition"]
Fmn[0, y] == 0
Print[" "]

Print["Satisfies right boundary condition"]
Simplify[Fmn[a, y] == 0, {Element[m, Integers], Element[n, Integers]}]
Print[" "]

Print["Satisfies bottom boundary condition"]
Fmn[x, 0] == 0
Print[" "]

Print["Satisfies top boundary condition"]
Simplify[Fmn[x, b] == 0, {Element[m, Integers], Element[n, Integers]}]
Print[" "]
```

```
Satisfies left boundary condition
```

Out[7]= True

```
Satisfies right boundary condition
```

Out[10]= True

```
Satisfies bottom boundary condition
```

Out[13]= True

```
Satisfies top boundary condition
```

Out[16]= True

Recall that the original Helmholtz equation that we just solved was given by

$$F_{xx} + F_{yy} + v^2 F = 0$$

where ν^2 = separation constant

We can determine what the separation constant ν^2 must be in order to satisfy this equation. Substituting Eq.8 into the Helmholtz equation yields

$$\frac{\partial^2}{\partial x^2}\left[\Gamma \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\right] + \frac{\partial^2}{\partial y^2}\left[\Gamma \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\right] + \nu^2\,\Gamma \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) = 0$$

$$\frac{\partial}{\partial x}\left[\cos\left(\frac{m\pi}{a}x\right)\right]\frac{m\pi}{a}\sin\left(\frac{n\pi}{b}y\right) + \frac{\partial}{\partial y}\left[\cos\left(\frac{n\pi}{b}y\right)\right]\frac{n\pi}{b}\sin\left(\frac{m\pi}{a}x\right) + \nu^2 \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) = 0$$

$$-\sin\left(\frac{m\pi}{a}x\right)\left(\frac{m\pi}{a}\right)^2\sin\left(\frac{n\pi}{b}y\right) - \sin\left(\frac{n\pi}{b}y\right)\left(\frac{n\pi}{b}\right)^2\sin\left(\frac{m\pi}{a}x\right) + \nu^2 \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) = 0$$

$$\nu^2 \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) = \left(\frac{m\pi}{a}\right)^2\sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) + \left(\frac{n\pi}{b}\right)^2\sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{m\pi}{a}x\right)$$

$$\nu^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

In[18]:=
```
Solve[D[Fmn[x, y], {x, 2}] + D[Fmn[x, y], {y, 2}] + νSquared Fmn[x, y] == 0, νSquared]
```

Out[18]= $\left\{\left\{\nu\text{Squared} \to \frac{\left(b^2 m^2 + a^2 n^2\right)\pi^2}{a^2 b^2}\right\}\right\}$

So we see that we require

$$\nu^2 = \frac{\left(b^2 m^2 + a^2 n^2\right)\pi^2}{a^2 b^2} = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \qquad \textbf{(Eq.8a)}$$

in order for Eq.8 to solve the Helmholtz equation

In[19]:= $\frac{\left(b^2 m^2 + a^2 n^2\right)\pi^2}{a^2 b^2} == \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$ `// Simplify`

Out[19]= `True`

Eigenfunctions and Eigenvalues

We now turn to Eq.5, the ODE for the time-function (repeated here for convenience)

$$\ddot{G} + \lambda^2 G = 0 \qquad \textbf{(Eq.5)}$$

where $\lambda = c\,\nu$

Recall that during the analysis for Eq.7, we assumed that

$$p^2 = \nu^2 - k^2$$

$$\nu = \left(p^2 + k^2\right)^{1/2}$$

Substituting this into the expression for λ yields

$$\lambda = c\,v$$

$$\frac{\lambda}{c} = \left(p^2 + k^2\right)^{1/2}$$

$$\lambda = c\left(p^2 + k^2\right)^{1/2}$$

Recall from the previous analysis of the Helmholtz equation we saw that k and p were constrained to specific values. Namely $k = \frac{m\pi}{a}$ and $p = \frac{n\pi}{b}$ with $m, n \in Z$ (Eq.7a and Eq.7b). So the expression for λ becomes

$$\lambda = \lambda_{\mathrm{mn}} = c\,\pi\left(\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2\right)^{1/2} \qquad m, n \in Z \qquad \textbf{(Eq.9)}$$

In[20]:=
```
λmn = c π ((n/b)^2 + (m/a)^2)^(1/2)
```

Out[20]= $c\sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}\,\pi$

So the time-function ODE can be written as

$$\ddot{G} + \lambda_{\mathrm{mn}}{}^2\,G = 0 \qquad \textbf{(Eq.5')}$$

where $\lambda_{\mathrm{mn}} = c\,\pi\left(\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2\right)^{1/2}$ $m, n \in Z$

We again recognize that Eq.5' defines an undamped oscillator with angular frequency λ_{mn} so the general solution is given by

$$G_{\mathrm{mn}}(t) = B_{\mathrm{mn}}\cos(\lambda_{\mathrm{mn}}\,t) + B_{\mathrm{mn}}{}^*\sin(\lambda_{\mathrm{mn}}\,t)$$

In[21]:=
```
Gmn[t_] = Bmn Cos[λmn t] + BmnStar Sin[λmn t];
```

We have now solved for general solutions for both F and G so the total general solution (eigenfunction) of this is given by

$$u_{\mathrm{mn}}(x, y, t) = F_{\mathrm{mn}}(x, y)\,G_{\mathrm{mn}}(t) \qquad m, n \in Z$$

$$= [B_{\mathrm{mn}}\cos(\lambda_{\mathrm{mn}}\,t) + B_{\mathrm{mn}}{}^*\sin(\lambda_{\mathrm{mn}}\,t)]\,\Gamma\sin\left(\frac{m\pi}{a}\,x\right)\sin\left(\frac{n\pi}{b}\,y\right)$$

$$= [\Gamma\,B_{\mathrm{mn}}\cos(\lambda_{\mathrm{mn}}\,t) + \Gamma\,B_{\mathrm{mn}}{}^*\sin(\lambda_{\mathrm{mn}}\,t)]\sin\left(\frac{m\pi}{a}\,x\right)\sin\left(\frac{n\pi}{b}\,y\right)$$ with no loss of generality we assume $\Gamma = 1$

$$u_{mn}(x, y, t) = [B_{mn} \cos(\lambda_{mn} t) + B_{mn}{}^* \sin(\lambda_{mn} t)] \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) \qquad m, n \in Z \quad \textbf{(Eq.10)}$$

where $\lambda_{mn} = c\,\pi\left(\left(\frac{n}{b}\right)^2 + \left(\frac{m}{a}\right)^2\right)^{1/2}$

$c = (T/\rho)^{1/2}$ = parameter of original PDE

In[22]:=
```
umn[x_, y_, t_] = Fmn[x, y] × Gmn[t] /. {Γ → 1}
```

Out[22]=
$$\left(\text{Bmn Cos}\left[c\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\,\pi\,t\right] + \text{BmnStar Sin}\left[c\sqrt{\frac{m^2}{a^2}+\frac{n^2}{b^2}}\,\pi\,t\right]\right) \text{Sin}\left[\frac{m\,\pi\,x}{a}\right] \text{Sin}\left[\frac{n\,\pi\,y}{b}\right]$$

The solutions of u_{mn} are the eigenfunctions and the values of λ_{mn} are the eigenvalues (these must take on specific, special values).

We can finally verify that Eq.10 satisfies the original, governing PDE and the boundary conditions (repeated here for convenience)

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \text{(governing PDE)} \qquad \textbf{(Eq.1)}$$

$u = 0\ \forall\ t$ on the boundary(boundary condition) ***(Eq.2)***

In[23]:=
```
Print["Satisfies original, governing PDE (2D wave equation)"]
LHS = D[umn[x, y, t], {t, 2}] // Simplify;
RHS = c^2 (D[umn[x, y, t], {x, 2}] + D[umn[x, y, t], {y, 2}]) // Simplify;
LHS == RHS // FullSimplify
Print[" "]

(*Verify that this satisfies the boundary condition*)
Print["Satisfies left boundary condition"]
umn[0, y, t] == 0
Print[" "]

Print["Satisfies right boundary condition"]
Simplify[umn[a, y, t] == 0, {Element[m, Integers], Element[n, Integers]}]
Print[" "]

Print["Satisfies bottom boundary condition"]
umn[x, 0, t] == 0
Print[" "]

Print["Satisfies top boundary condition"]
Simplify[umn[x, b, t] == 0, {Element[m, Integers], Element[n, Integers]}]
Print[" "]
```

Satisfies original, governing PDE (2D wave equation)

Out[26]= True

```
Satisfies left boundary condition
```

Out[29]= `True`

```
Satisfies right boundary condition
```

Out[32]= `True`

```
Satisfies bottom boundary condition
```

Out[35]= `True`

```
Satisfies top boundary condition
```

Out[38]= `True`

Discussion of Eigenfunctions

Depending on a and b, several functions F_{mn} may correspond to the same eigenvalue. Physically this means that there may exist vibrations having the same frequency but entirely different shapes/eigenfunctions. We investigate this in an example below.

Example 1: Eigenvalue and Eigenfunctions of the Square Membrane

Consider the square membrane with $a = b = 1$. For simplicity, we assume that $c = 1$.

In this situation , we see that $\lambda_{mn} = \lambda_{nm}$. We can look at various cases to get an idea of how the eigenfunctions behave

We can now look at the corresponding eigenfunction (and for now assuming that $B_{mn} = 1$ and $B_{mn}^* = 0$)

Show that some eigenfunctions have same frequency but different shapes (for example $m = 1$, $n = 2$ has the same frequency as $m = 2$, $n = 1$ but the mode shapes are differently oriented).

In[40]:=
```
(*Define the m and n that we would like to investigate*)
mPlot = 1;  (*Change these to change animation*)
nPlot = 3;  (*Change these to change animation*)

(*Define eigenvalue for this case*)
Print["Eigenvalue"]
λmnPlot = λmn /. {m → mPlot, n → nPlot, a → 1, b → 1, c → 1}
Print[" "]

Print["Frequency (Hz)"]
fHz = λmnPlot / (2 π) // N
Print[" "]

(*Define the eigenfunction for this case*)
Clear[umnPlot]
Print["Eigenfunction"]
umnPlot[x_, y_, t_] =
 umn[x, y, t] /. {m → mPlot, n → nPlot, a → 1, b → 1, c → 1, Bmn → 1, BmnStar → 0}
Print[" "]

(*Animate the eigenfunction*)
Manipulate[
 Plot3D[umnPlot[x, y, t], {x, 0, 1}, {y, 0, 1},

  (*Plot options*)
  PlotRange → {{0, 1}, {0, 1}, {-1, 1}},
  AxesLabel → {"x", "y", "u(x,y,t)"},
  PlotLabel → StringJoin["m = ", ToString[mPlot], ", n = ", ToString[nPlot]]
 ],

 {t, 0, 1 / fHz}
]
```

Eigenvalue

Out[43]= $\sqrt{10}\,\pi$

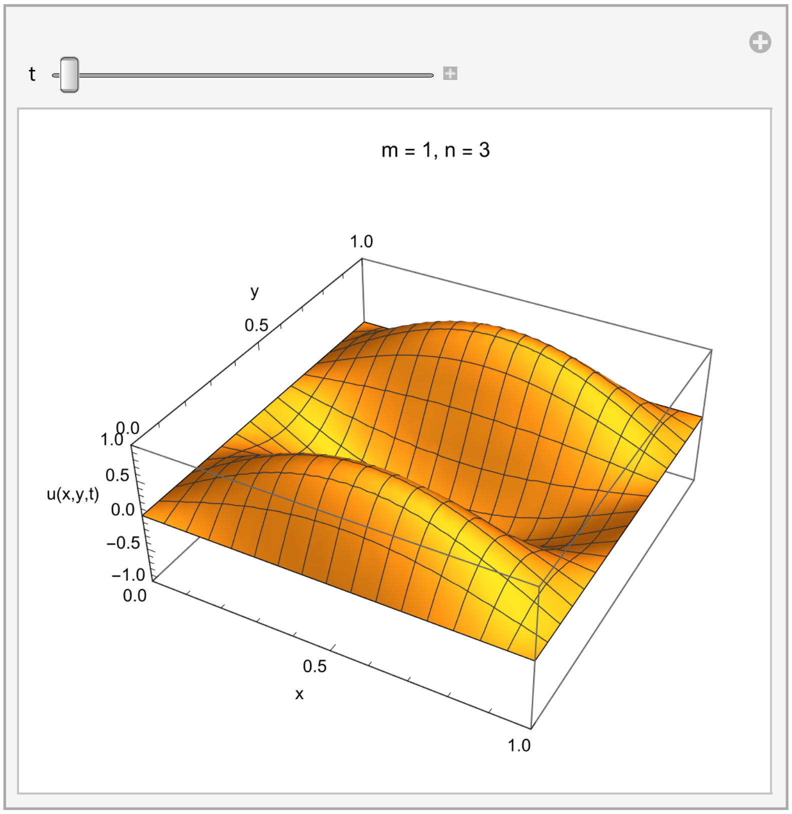
Frequency (Hz)

Out[46]= 1.58114

Eigenfunction

Out[50]= $\text{Cos}\left[\sqrt{10}\,\pi\,t\right]\,\text{Sin}[\pi\,x]\,\text{Sin}[3\,\pi\,y]$

Out[52]=



Step 3. Solution of the Model (1), (2), (3). Double Fourier Series

In a similar fashion to the 1D wave and 1D heat equation, the eigenfunctions of u_{mn} only solves the original PDE and the boundary conditions. Individually, they do not solve the general initial condition. In order to satisfy the initial condition, we consider the double series

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn}(x, y, t)$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} [B_{mn} \cos(\lambda_{mn} t) + B_{mn}^* \sin(\lambda_{mn} t)] \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) \quad \textbf{\textit{(Eq.14)}}$$

At $t = 0$ this should equal the initial displacement

$$u(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn} \sin\left(\frac{m\pi}{a} x\right) \sin\left(\frac{n\pi}{b} y\right) = f(x, y) \qquad \textbf{\textit{(Eq.15)}}$$

If $f(x, y)$ meets general requirements for Fourier analysis, then Eq.15 is called the **double Fourier series** of $f(x, y)$

We can find the coefficients of the double Fourier series as follows. First, set

$$K_m(y) = \sum_{n=1}^{\infty} B_{mn} \sin\left(\frac{n\pi}{b} y\right) \qquad \text{(function of } y \text{ only)} \qquad \textbf{\textit{(Eq.16)}}$$

we can then write Eq.15 in the form

$$f(x,\ y) = \sum_{m=1}^{\infty} K_m(y) \sin\left(\frac{m\pi}{a}\ x\right)$$

For a fixed y, we can consider $K_1(y)$, $K_2(y)$, ..., to be the coefficients that we are solving for to make this equation true. If this is the case, we then recognize this as the Fourier sine series of $f(x,\ y)$ (remember that we assume y fixed and therefore, $f(x,\ y)$ is considered to be a function of only x). From Eq.4 of Section 11.3, we see that the coefficients of this expansion ($K_m(y)$) are

$$K_m(y) = \frac{2}{a} \int_0^a f(x,\ y) \sin\left(\frac{m\pi}{a}\ x\right) dx \qquad \textbf{(Eq.17)}$$

Returning to Eq.16, we recognize Eq.16 as the Fourier sine series of $K_m(y)$ and from Eq.4 in section 11.3, it follows that the coefficients are

$$B_{\text{mn}} = \frac{2}{b} \int_0^b K_m(y) \sin\left(\frac{n\pi}{b}\ y\right) dy \qquad \text{recall: } K_m(y) \text{ was given by Eq.17}$$

$$= \frac{2}{b} \int_0^b \left(\frac{2}{a} \int_0^a f(x,\ y) \sin\left(\frac{m\pi}{a}\ x\right) dx\right) \sin\left(\frac{n\pi}{b}\ y\right) dy$$

$$B_{\text{mn}} = \frac{4}{a\,b} \int_0^b \int_0^a f(x,\ y) \sin\left(\frac{m\pi}{a}\ x\right) \sin\left(\frac{n\pi}{b}\ y\right) dx\,dy \qquad m,\ n \in Z \qquad \textbf{(Eq.18)}$$

The terms B_{mn} in Eq.14 are now determined by Eq.18. To determine the $B_{\text{mn}}{}^*$ terms, we differential Eq.14 term-wise with respect to t and set the result equal to the initial condition in velocity (Eq.3b) to obtain

$$\frac{\partial u}{\partial t}\ \Big|_{t=0} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{\text{mn}}{}^* \lambda_{\text{mn}} \sin\left(\frac{m\pi}{a}\ x\right) \sin\left(\frac{n\pi}{b}\ y\right) = g(x,\ y)$$

Assuming that $g(x,\ y)$ can be represented using a double Fourier series, we can proceed in a similar fashion as before to obtain the coefficients as (warning: textbook has a typo in the second sin term, these notes should be correct).

$$B_{\text{mn}}{}^* = \frac{4}{a\,b\,\lambda_{\text{mn}}} \int_0^b \int_0^a g(x,\ y) \sin\left(\frac{m\pi}{a}\ x\right) \sin\left(\frac{n\pi}{b}\ y\right) dx\,dy \qquad m,\ n \in Z \qquad \textbf{(Eq.19)}$$

We examine an example of this in the homework.