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Lecture 04e
Tangent to a Curve

The YouTube video entitled ‘Tangent to a Curve’ that covers this lecture is located at https://youtu.be/H-H367um_Aho.

Outline

-Tangent to a Curve

Tangent to a Curve

Consider a curve in space that is parameterized by t and is described by $\bar{r}(t)$ (see YouTube video entitled ‘Parameterizing Curves’). As we discussed previously, provided that $\bar{r}(t)$ is differentiable, the derivative of this is given by (see YouTube video entitled ‘Scalar Functions, Vector Functions, and Vector Derivatives’)

$$\bar{r}'(t) = \lim_{\Delta t \to 0} \frac{\bar{r}(t+\Delta t) - \bar{r}(t)}{\Delta t}$$

We can now show that this is effectively the vector tangent to the curve at point $P = \bar{r}(t)$.

If the curve is parameterized (ie $C = \bar{r}(t)$) from geometry in Figure 180 (see below), we see that a tangent vector can be obtained by looking at the vector difference $\bar{r}(t + \Delta t) - \bar{r}(t)$ as Δt becomes small.

$$\text{tangent vector at point } P = \bar{r}(t) \text{ is given by } \lim_{\Delta t \to 0} \bar{r}(t + \Delta t) - \bar{r}(t)$$

If we scale (AKA multiply) this vector by the amount $1/\Delta t$, we have a tangent vector at point P given as $\lim_{\Delta t \to 0} \frac{\bar{r}(t+\Delta t) - \bar{r}(t)}{\Delta t}$. Comparing this with the previous expression for the derivative of the curve at point $P = \bar{r}(t)$, we see that they are the same. So we conclude that

$$\bar{r}'(t) = \text{a tangent vector to the curve at } P = \bar{r}(t)$$

If we would like the unit tangent vector, we simply need to normalize this vector

$$\bar{u}(t) = \frac{1}{|\bar{r}'(t)|}\,\bar{r}'(t) \qquad \text{(unit tangent vector at point } P = \bar{r}(t)\text{)} \qquad \textbf{\textit{(Eq.8)}}$$

Therefore, if we would like a point along this tangent line, we can find it using

$$T = \bar{q} = \bar{r}(t) + w\,\bar{u}(t) \qquad \textbf{\textit{(Eq.9)}}$$

where $\bar{u}(t)$ = unit tangent vector at point $P = \bar{r}(t)$ given by Eq.8

w = parameter determining the distance between $P = \bar{r}(t)$ and $\bar{q}(w)$

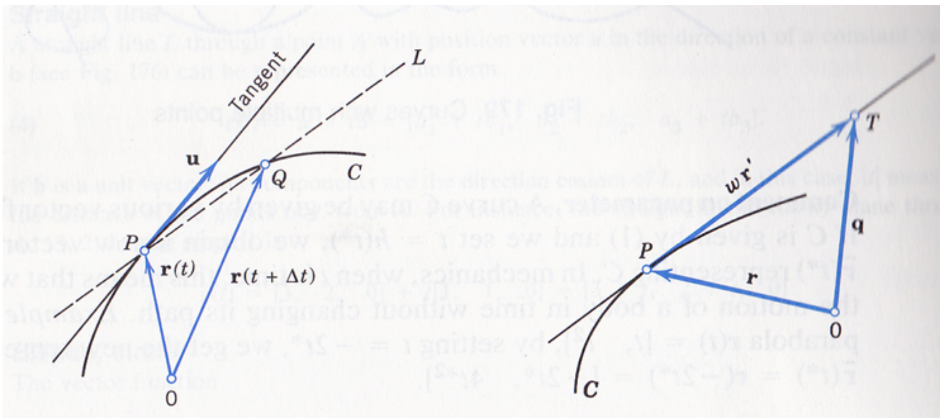

Fig. 180. Tangent to a curve

Fig. 181. Formula (9) for the tangent to a curve

Example 3: 3D Elliptical helix

Consider our previous example from the ‘Parameterizing Curves’ video.

$$\bar{r}(t) = \begin{pmatrix} a\cos(t) \\ b\sin(t) \\ c\,t \end{pmatrix}$$

In[1]:=
```
r[t_] = ( a Cos[t] )
        ( b Sin[t] ) ;
        (   c t    )
```

Let us compute the tangent to this curve using

$$\bar{r}'(t) = \frac{d}{dt}[\bar{r}(t)]$$

In[2]:=
```
rPrime[t_] = D[r[t], t];
rPrime[t] // MatrixForm
```

Out[3]//MatrixForm=

$$\begin{pmatrix} -a\,\mathrm{Sin}[t] \\ b\,\mathrm{Cos}[t] \\ c \end{pmatrix}$$

Using the same constants as the previous example ($a = 3$, $b = 1.2$, and $c = 1.2$) yields

In[4]:=
```
replaceString = {a → 3, b → 1.2, c → 1.2};

Manipulate[
 (*Plot the point of interest*)
 P = r[tC] /. replaceString;

 (*Compute the tip of the tangent vector*)
 u = rPrime[tC] /. replaceString;

 (*Draw on a single figure*)
 Show[
  (*Plot the curve*)
  ParametricPlot3D[{r[t][[1, 1]] /. replaceString,
    r[t][[2, 1]] /. replaceString, r[t][[3, 1]] /. replaceString}, {t, 0, 10},

   (*Plot Options*)
   AxesLabel → {"x", "y", "z"},
   PlotRange → {{-4, 4}, {-4, 4}, {0, 13}},
   PlotLabel → "||ū|| = " <> ToString[Norm[u]]],

  (*Plot the point of interest*)
  Graphics3D[
   {
    AbsolutePointSize[15], Green, Point[{P[[1, 1]], P[[2, 1]], P[[3, 1]]}]
   }
  ],

  (*Plot the curve tangent (draw an arrow from P to T=P+ū)*)
  Graphics3D[
   {
    Magenta, Arrow[{{P[[1, 1]], P[[2, 1]], P[[3, 1]]},
      {P[[1, 1]] + u[[1, 1]], P[[2, 1]] + u[[2, 1]], P[[3, 1]] + u[[3, 1]]}}]
   }
  ]
 ],
 {tC, 0, 10}]
```

Out[5]=