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Lecture07a
Partial Differential Equations Introduction

Lecture is on YouTube

The YouTube video entitled 'Introduction to Partial Differential Equations' that covers this lecture is located at https://youtu.be/THjaxvPBGOU.

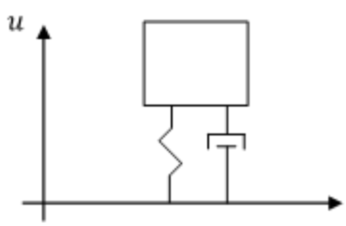
Basic Concepts of PDEs

Before discussing partial differential equations, we should recall our discussion of ordinary differential equations (ODEs). An example ODE might be something like

$$a \frac{d^2 u(t)}{dt^2} + b \frac{d u(t)}{dt} + c\, u(t) = f(t) \qquad \textbf{(Eq.1.1)}$$

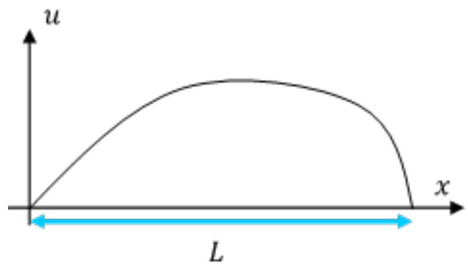
with $u(0) = u_o$
$\quad\dot{u}(0) = \dot{u}_o$

This might describe the vertical deflection of a block attached to a spring and damper as shown below



In this case, the problem is to find a function $u(t)$ that satisfies Eq.1.1. Note that the function $u(t)$ is a function of only a single independent variable, usually time, t. In this case, the block will move up and down as time progresses.

What if instead of a single point mass (the block), we instead had a continuous rope as shown below?



In this case, the deflection of the rope will be a function of both time, t, as well as the horizontal position, x. In this case, there are two independent variables (both x and t).

An equation involving one or more partial derivatives of an unknown function of two or more independent variables is called a **partial differential equation** (PDE). The order of the highest derivative is called the **order** of the equation.

A partial differential equation of the function $u(x_1, x_2, \ldots, x_n)$ (independent variables $x_1, x_2, \ldots, x_n$) has a general form of

$$F\left(x_1, x_2, \ldots, x_n, \frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_1}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \ldots \frac{\partial^2 u}{\partial x_1 \partial x_n}, \ldots\right) = 0 \qquad \textbf{(Eq.1.2)}$$

PDEs can be used to describe a wide variety of engineering and physical phenomena such as sound, heat, electrostatics, fluid flow, etc. ***<show animation of 1D wave, 2D plate, and heat equation solutions in PowerPoint>***

Notation

With PDEs, it is common to denote partial derivatives using subscripts. That is:

$$u_x = \frac{\partial u}{\partial x}$$

$$u_{xy} = \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left[\frac{\partial u}{\partial x}\right]$$

In physics and engineering, it is common to see "dot notation" to signify the derivative with respect to time

second derivative of u w.r.t. time $\Rightarrow \ddot{u} = \frac{\partial^2 u(x,t)}{\partial t^2}$

If there is only a single spatial dimension, you may see "prime notation" to signify the derivative with respect to the single spatial variable

second derivative of u w.r.t. space, x $\Rightarrow u'' = \frac{\partial^2 u(x,t)}{\partial x^2}$

Classification

Similar to ordinary differential equations, we say that a PDE is **linear** if it is a linear function of u and its derivatives. Another way to say this is if the function is of the first degree (note degree is NOT the order as discussed previous) in the dependent variable and its partial derivatives. In other words, we can have terms like u, u_x, u_{xy}, etc. as these are all of the first degree in the dependent variable and its partial derivatives. However, terms such as u^2, ${u_x}^{1/2}$, etc. are not of the first degree in the dependent variable and its partial derivatives.

For example, consider a first order PDE for an unknown function $u(x, y)$. This is linear if it can be expressed in the form

$$a(x, y)\frac{\partial u(x,y)}{\partial x} + b(x, y)\frac{\partial u(x,y)}{\partial y} + c(x, y)\,u(x, y) = d(x, y)$$

It is called **quasilinear** if it can be expressed in the form

$$a(x, y, u(x, y))\frac{\partial u(x,y)}{\partial x} + b(x, y, u(x, y))\frac{\partial u(x,y)}{\partial y} + c(x, y, u(x, y))\,u(x, y) = d(x, y)$$

A PDE which is neither linear nor quasilinear is said to be **nonlinear**.

If each term of such an equation contains either the dependent variable or one of its derivatives, the equation is said to be **homogeneous**; otherwise it is said to be **non-homogeneous**. This is similar to the idea of a forcing function in an ODE creating a non-homogeneous ODE.

Example 1: Important linear partial differential equations of second order

1) $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ One-dimensional wave equation

2) $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ One-dimensional heat equation

3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ Two-dimensional Laplace equation

4) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y)$ Two-dimensional Poisson equation

5) $\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ Two-dimensional wave equation

6) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$ Three-dimensional Laplace equation

Note: Eq.4 with $f(x, y) \neq 0$ is non-homogeneous while the other equations are homogeneous

We notice that solutions to these equations may not be unique. For example, if we examine the two-dimensional Laplace equation (Eq.3), we can show that the following 3 dissimilar functions are solutions to the 2D Laplace equation

Example: 3 Different Functions to Solve 2D Laplace Equation

Consider the 3 functions below

$$u_1 = x^2 - y^2$$
$$u_2 = e^x \cos(y)$$
$$u_3 = \ln(x^2 + y^2)$$

We can show that these all solve the 2D Laplace equation

```
(*Define functions*)
u1[x_, y_] = x^2 - y^2;
u2[x_, y_] = Exp[x] Cos[y];
u3[x_, y_] = Log[x^2 + y^2];

(*Define the 2D Laplacian operator*)
del2D[u_] := D[u, {x, 2}] + D[u, {y, 2}]

(*Verify solutions*)
del2D[u1[x, y]]
del2D[u2[x, y]]
del2D[u3[x, y]] // Simplify
```

```
0

0

0
```

Typically, we will need to apply additional constraints to narrow down the solution space. Often these constraints take the form of boundary or initial conditions.

Recall that for linear, homogeneous ordinary differential equations, we can apply superposition. A similar situation exists for linear homogeneous partial differential equations.

Theorem 1: Fundamental Theorem (Superposition or linearity principle)
If u_1 and u_2 are any solutions of a linear homogeneous partial differential equation in some region R, then

$$u = c_1 u_1 + c_2 u_2$$

where c_1 and c_2 are any constants

is also a solution of that equation in R

We note that a direct consequence of this is that we can generate a solution to a PDE from a family of solutions. In other words, the above theorem can be extended for 2, 3, or N solutions. In other words

$$u = \sum_{n=1}^{N} c_n u_n$$

is also a solution if each u_n is a solution. This will become very important in the future when we combine PDEs and Fourier analysis.

Example: Create Additional Solutions Using Superposition

Consider the same 3 functions that we previously demonstrated solved the 2D Laplace equation. We can use these to create additional solutions to the same PDE

```
(*Create additional solutions.  Notice that c1, c2,
and c3 are not defined.  This should work for any possible c1,
c2, and c3 as long as they are constant*)
u4[x_, y_] = c1 u1[x, y] + c2 u2[x, y];
u5[x_, y_] = c1 u1[x, y] + c2 u2[x, y] + c3 u3[x, y];

(*Verify these new solutions satisfy PDE*)
del2D[u4[x, y]]
del2D[u5[x, y]] // Simplify

(*clear variable*)
Clear[u1, u2, u3, u4, u5, del2D]
```

```
0
```

```
0
```

End Theorem 1 Discussion

Often, PDEs are difficult to solve and many times, there are no general solutions. However, if the system is special or simple enough, it may be possible to solve for general solution.

Example: Simple PDE

Solve the partial differential equation

$$u_{xy} = -u_x$$

Recall that this is shorthand notation for

$$\frac{\partial^2 u}{\partial x\, \partial y} = -\frac{\partial u}{\partial x} \qquad \textbf{(Eq.E.1)}$$

So we seek a function $u(x, y)$ which satisfies this relation.

Assume that if we take a derivative of u w.r.t. x we obtain a function called p

$$u_x = p \qquad \textbf{(Eq.E.2)}$$

If we take the partial derivative of both sides with respect to y, we obtain

$$u_{xy} = p_y \qquad \textbf{(Eq.E.3)}$$

Recall that we need to satisfy $u_{xy} = -u_x$ so using this to replace the left side of Eq.E.3 with $-u_x$ we obtain

$$-u_x = p_y$$

recall from Eq.E.2 that $u_x = p$ so multiplying by -1 yields $-u_x = -p$. Substituting this for the left side of the previous equation yields

$$-p = p_y$$

$$\frac{p_y}{p} = -1$$

Integrating both sides with respect to y yields

$$\int \frac{p_y}{p}\, d\,y = \int -1\, d\,y \qquad \text{recall: } \frac{d}{d\,t}[\ln(f(t))] = \frac{f'(t)}{f(t)}$$

```
D[Log[f[t]], t]
```

```
f′[t]
─────
f[t]
```

$$\ln(p) = -y + \tilde{c}(x) \qquad \text{where } \tilde{c}(x) = \text{arbitrary function of } x \text{ only}$$

$$e^{\ln(p)} = e^{-y+\tilde{c}(x)}$$

$$p = e^{-y}\, e^{\tilde{c}(x)} \qquad \text{let: } c(x) = e^{\tilde{c}(x)}$$

$$p = c(x)\, e^{-y} \qquad \text{recall: } p = u_x$$

$$u_x = c(x)\, e^{-y}$$

Now integrating both sides with respect to x yields

$$\int u_x\, d\,x = \int c(x)\, e^{-y}\, d\,x$$

$$\int u_x\, d\,x = e^{-y} \int c(x)\, d\,x$$

$$= e^{-y}(f(x) + r(y)) \qquad \text{where } f(x) = \int c(x)\, d\,x \text{ and } r(y) \text{ is a constant of integration}$$

$$= e^{-y}\, f(x) + e^{-y}\, r(y) \qquad \text{let: } g(y) = e^{-y}\, r(y)$$

$$u = e^{-y}\, f(x) + g(y)$$

Here, $f(x)$ and $g(y)$ are arbitrary provided that they are only functions of x and y, respectively.

We can check this candidate solution with a few test functions

```
(*Choose some arbitrary f(x) and g(y) terms*)
u1[x_, y_] = Exp[-y] fx + gy /. {fx → Cos[x], gy → y^2}
u2[x_, y_] = Exp[-y] fx + gy /. {fx → Log[x] x, gy → Tan[y^2]}
u3[x_, y_] = Exp[-y] fx + gy /. {fx → x^3 + 3 x + 1/Cot[Log[x]], gy → y^2 Exp[y^4]}

(*verify that we satisfy the PDE*)
D[u1[x, y], x, y] == -D[u1[x, y], x]
D[u2[x, y], x, y] == -D[u2[x, y], x]
D[u3[x, y], x, y] == -D[u3[x, y], x]

(*clear variables*)
Clear[u1, u2, u3]
```

$y^2 + e^{-y} \mathrm{Cos}[x]$

$e^{-y} x \mathrm{Log}[x] + \mathrm{Tan}[y^2]$

$e^{y^4} y^2 + e^{-y} (3 x + x^3 + \mathrm{Tan}[\mathrm{Log}[x]])$

```
True
True
True
```

Note that this example is somewhat simple. More complicated PDEs will require more formal methods to solve.   One popular technique is to treat the PDE like an ODE. We investigate these in the next section.