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Lecture 03c
Transfer Function to State Space

The YouTube video entitled 'Transfer Function to State Space' that covers this lecture is located at https://youtu.be/RG_tdz1VzwY.

Outline

-Introduction
-Transfer Function to State Space

Introduction

We would like to turn a transfer function model into a state space model. You should watch the following videos/lectures before proceeding on:

-State Space to Transfer Function (https://youtu.be/NNJ0sUmrKu8)
-State Space Representation of Differential Equations (https://youtu.be/pXvAh1IOO4U)
-The Laplace Transform (https://youtu.be/q0nX8uIFZ_k)

Transfer Functions from ODEs

Consider a transfer function of the form

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$$

where $\quad n \geq m$

For the purposes of the next several examples, let us consider the case of a strictly proper transfer function ($n > m$). It may be useful to consider a concrete example of $n = 3$, $m = 2$

$$G(s) = \frac{Y(s)}{U(s)}$$

$$= \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= \frac{b_2 s^2}{s^3 + a_2 s^2 + a_1 s + a_0} + \frac{b_1 s}{s^3 + a_2 s^2 + a_1 s + a_0} + \frac{b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

$$= b_2 s^2 \left(\frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} \right) + b_1 s \left(\frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} \right) + b_0 \left(\frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} \right)$$

$$= b_2 s^2 \left(\frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} \right) + b_1 s \left(\frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} \right) + b_0 \left(\frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} \right)$$

$$G(s) = b_2 s^2 G_c(s) + b_1 s G_c(s) + b_0 G_c(s)$$

where $\quad G_c(s) = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0}$

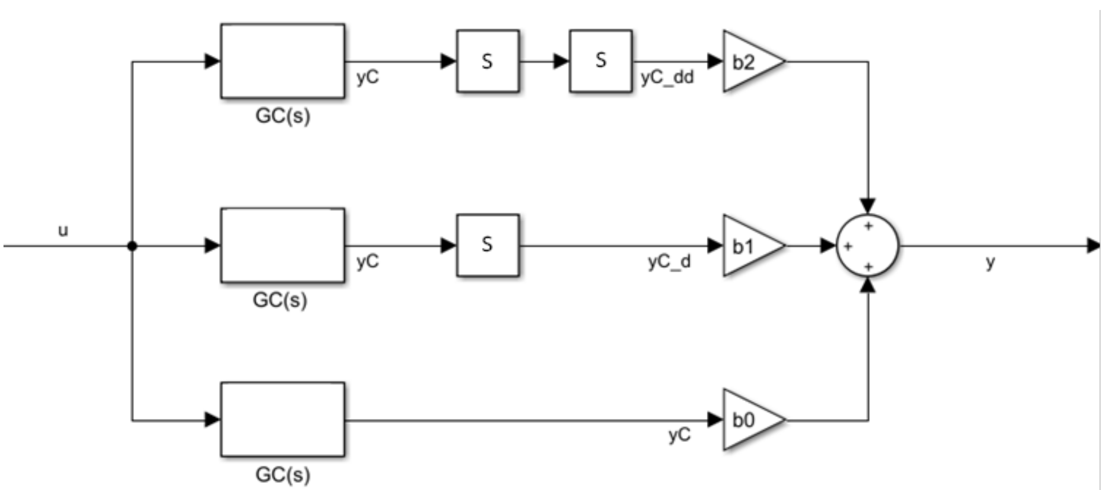
Note that we now write the output as

$$Y(s) = G(s) U(s)$$

$$= \left(b_2 s^2 G_c(s) + b_1 s G_c(s) + b_0 G_c(s) \right) U(s)$$

$$= b_2 s^2 G_c(s) U(s) + b_1 s G_c(s) U(s) + b_0 G_c(s) U(s)$$

We can now visualize the output of the system, $Y(s)$, as shown below



Let us focus on the signal $Y_C(s)$

$$Y_C(s) = G_C(s) U(s)$$

$$Y_C(s) = \frac{1}{s^3 + a_2 s^2 + a_1 s + a_0} U(s)$$

$$Y_C(s)\left(s^3 + a_2 s^2 + a_1 s + a_0\right) = U(s)$$

$$s^3 Y_C(s) + a_2 s^2 Y_C(s) + a_1 s Y_C(s) + a_0 Y_C(s) = U(s)$$

$$L^{-1}\left[s^3 Y_C(s) + a_2 s^2 Y_C(s) + a_1 s Y_C(s) + a_0 Y_C(s)\right] = L^{-1}[U(s)]$$

$$L^{-1}\left[s^3 Y_C(s)\right] + a_2 L^{-1}\left[s^2 Y_C(s)\right] + a_1 L^{-1}[s Y_C(s)] + a_0 L^{-1}[Y_C(s)] = L^{-1}[U(s)]$$

$$\dddot{y}_C(t) + a_2 \ddot{y}_C(t) + a_1 \dot{y}_C(t) + a_0 y_C(t) = u(t)$$

$$\dddot{y}_C(t) = -a_2 \ddot{y}_C(t) - a_1 \dot{y}_C(t) - a_0 y_C(t) + u(t)$$

Recall the discussion in the video entitled 'State Space Representation of Differential Equations' (particularly at 47:13, direct link to this timestamp https://youtu.be/pXvAh1IOO4U?t=2833) where we showed how to turn an n^{th} order ODE into a state space this corresponded to an ODE of the form. We can define the state as

$$\overline{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} y_C(t) \\ \dot{y}_C(t) \\ \ddot{y}_C(t) \end{pmatrix}$$

So the state derivative is given as

$$\dot{\overline{x}}(t) = \begin{pmatrix} \dot{y}_C(t) \\ \ddot{y}_C(t) \\ \dddot{y}_C(t) \end{pmatrix}$$

$$= \begin{pmatrix} \dot{y}_C(t) \\ \ddot{y}_C(t) \\ -a_2 \ddot{y}_C(t) - a_1 \dot{y}_C(t) - a_0 y_C(t) + u(t) \end{pmatrix}$$

$$= \begin{pmatrix} x_2(t) \\ x_3(t) \\ -a_2 x_3(t) - a_1 x_2(t) - a_0 x_1(t) + u(t) \end{pmatrix}$$

$$\dot{\overline{x}}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(t)$$

$$\dot{\overline{x}}(t) = A\,\overline{x}(t) + B\,u(t)$$

where $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix}$ $\quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

We can now see from the signal flow diagram above that the total output of the system is

$$y(t) = b_0\, y_C(t) + b_1\, \dot{y}_C(t) + b_2\, \ddot{y}_C(t)$$

$$= \begin{pmatrix} b_0 & b_1 & b_2 \end{pmatrix} \begin{pmatrix} y_C(t) \\ \dot{y}_C(t) \\ \ddot{y}_C(t) \end{pmatrix}$$

$$y(t) = C\,\overline{x}(t) + D\,u(t)$$

where $C = \begin{pmatrix} b_0 & b_1 & b_2 \end{pmatrix}$ $\quad D = (0)$

This is known as **controllable canonical form**. See Modern Control Systems 10th Edition, pg. 144 for more information.

```
Amat = ( 0    1    0
         0    0    1
        -a0  -a1  -a2 );

Bmat = ( 0
         0
         1 );

Cmat = ( b0  b1  b2 );
Dmat = (0);

(*Check that we obtain the appropriate transfer function*)
temp = Simplify[Cmat.Inverse[s * IdentityMatrix[3] - Amat].Bmat + Dmat][[1, 1]];

(*Manipulate to get this in the right format*)
numG = Collect[Expand[Numerator[temp]], {s^2, s}];
denG = Collect[Expand[Denominator[temp]], {s^2, s}];

G[s_] = numG / denG
```

$$\frac{b0 + b1\, s + b2\, s^2}{a0 + a1\, s + a2\, s^2 + s^3}$$

Example: Recall the DC motor example

Recall the transfer function we considered earlier

$$G_V(s) = \frac{\dot{\theta}(s)}{V_a(s)} = \frac{K_T}{J_m L_a s^2 + (c L_a + J_m R + J_m R_m) s + (K_T K_V + c R + c R_m)}$$

$$= \frac{K_T}{J_m L_a \left(s^2 + \left(\frac{c L_a + J_m R + J_m R_m}{J_m L_a}\right) s + \left(\frac{K_T K_V + c R + c R_m}{J_m L_a}\right)\right)}$$

$$= \frac{\frac{K_T}{J_m L_a}}{s^2 + \left(\frac{c L_a + J_m R + J_m R_m}{J_m L_a}\right) s + \left(\frac{K_T K_V + c R + c R_m}{J_m L_a}\right)}$$

$$= \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}$$

where $b_0 = \frac{K_T}{J_m L_a}$

$b_1 = 0$

$a_0 = \frac{K_T K_V + c R + c R_m}{J_m L_a}$

$a_1 = \frac{c L_a + J_m R + J_m R_m}{J_m L_a}$

Using numerical values, we obtain

```
      4.616e04
--------------------
s^2 + 1021 s + 4845
```

Notice that this is only a 2nd order transfer function, and therefore we only expect a 2nd order state space representation.

We can therefore write a state space representation as

$$\dot{\overline{x}}(t) = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} \overline{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \overline{u}(t)$$

$$\overline{y}(t) = \begin{pmatrix} b_0 & b_1 \end{pmatrix} \overline{x}(t) + (0)\, \overline{u}(t)$$

Note that the state $\overline{x}(t)$ is not the same as the original state vector $\overline{x}(t)$. In other words, the first state is not the position, the second state is not the velocity, etc.

This yields the same input/output response as the original state space representation (go to Simulink).

There are other canonical forms such as

controllable canonical form
observable canonical form
etc.

We examine one of these alternative canonical forms next.

Modal Canonical Form

Another form is known as **modal canonical form**. Consider the case of a strictly proper TF with distinct poles.

$$G(s) = \frac{b_2 s^2 + b_1 s + b_0}{s^3 + a_2 s^2 + a_1 s + a_0}$$

Again, it maybe useful to consider a concrete example

$$G(s) = \frac{2 s^2 + 18 s + 40}{s^3 + 2 s^2 - 5 s - 6}$$

$$= \frac{2 (s+4) (s+5)}{(s+1) (s-2) (s+3)}$$

```
G[s_] = (2 s^2 + 18 s + 40) / (s^3 + 2 s^2 - 5 s - 6);
```

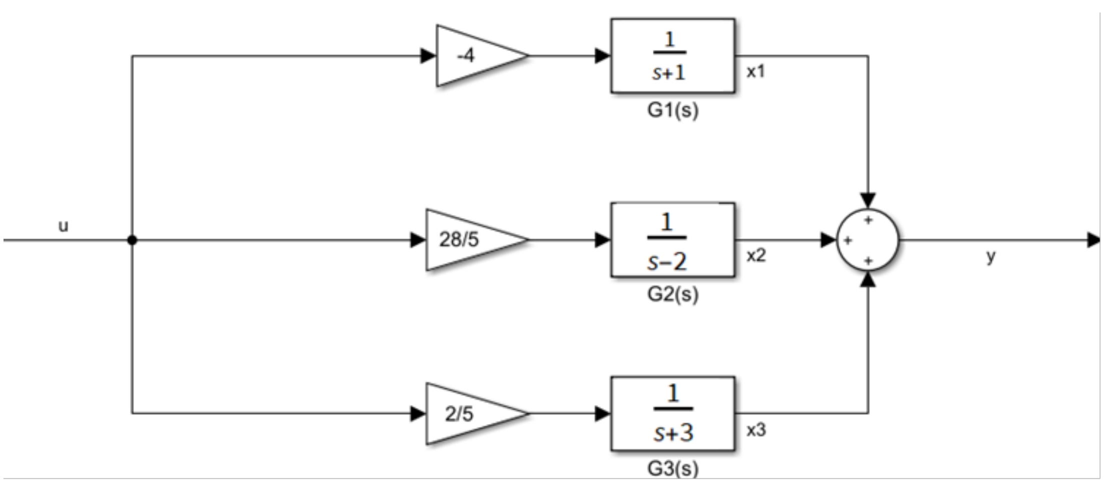
In this case, its partial fraction expansion can be written as (recall our previous videos/lectures entitled ‘Partial Fraction Expansion/Decomposition’ https://youtu.be/vlCdCAEtRag)

$$G(s) = \frac{-4}{s+1} + \frac{28/5}{s-2} + \frac{2/5}{s+3}$$

$$G(s) = (-4) \frac{1}{s+1} + \left(\frac{28}{5}\right) \frac{1}{s-2} + \left(\frac{2}{5}\right) \frac{1}{s+3}$$

```
-4/(s + 1) + (28/5)/(s - 2) + (2/5)/(s + 3) == G[s] // Simplify
True
```

As such, we can visualize this system as



So we can write

$$\dot{x}_1(t) = -1\, x_1(t) - 4\, u(t)$$

$$\dot{x}_2(t) = 2\, x_2(t) + \frac{28}{5}\, u(t)$$

$$\dot{x}_3(t) = -3\, x_3(t) + \frac{2}{5}\, u(t)$$

Side note: Let us check that this is correct

$$s\, X_1(s) = -X_1(s) - 4\, U(s)$$

$$s\, X_1(s) + X_1(s) = -4\, U(s)$$

$$(s+1)\, X_1(s) = -4\, U(s)$$

$$\frac{X_1(s)}{U(s)} = \frac{-4}{s+1}$$

So we can write the above 3 equations in matrix form

$$\dot{\overline{x}}(t) = \begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2\,(t) \\ \dot{x}_3\,(t) \end{pmatrix}$$

$$= \begin{pmatrix} -1\, x_1\,(t) - 4\, u\,(t) \\ 2\, x_2(t) + \frac{28}{5}\, u\,(t) \\ -3\, x_3\,(t) + \frac{2}{5}\, u\,(t) \end{pmatrix}$$

$$\dot{\overline{x}}(t) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} + \begin{pmatrix} -4 \\ 28/5 \\ 2/5 \end{pmatrix} u(t)$$

$$\dot{\overline{x}}(t) = A\, \overline{x}(t) + B\, u(t)$$

where $\quad A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \qquad B = \begin{pmatrix} -4 \\ 28/5 \\ 2/5 \end{pmatrix}$

The output is simply

$$y(t) = x_1(t) + x_2(t) + x_3(t)$$

$$= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix}$$

$y(t) = C\,\overline{x}(t) + D\,u(t)$

where $C = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ $\qquad D = (0)$

```
Amat = ( -1  0  0
          0  2  0
          0  0 -3 );

Bmat = ( -4
         28/5
         2/5 );

Cmat = (1 1 1);
Dmat = (0);

(*Check that we obtain the appropriate transfer function*)
temp = Simplify[Cmat.Inverse[s * IdentityMatrix[3] - Amat].Bmat + Dmat][[1, 1]];

(*Manipulate to get this in the right format*)
numG = Collect[Expand[Numerator[temp]], {s^2, s}];
denG = Collect[Expand[Denominator[temp]], {s^2, s}];

G2[s_] = numG/denG

G2[s] == G[s]
```

$$\frac{40 + 18\,s + 2\,s^2}{-6 - 5\,s + 2\,s^2 + s^3}$$

```
True
```

We can generalize this to a system with distinct poles and therefore has the partial fraction expansion of

$$G(s) = \frac{k_1}{s+p_1} + \frac{k_2}{s+p_2} + \ldots + \frac{k_n}{s+p_n}$$

Has a modal canonical form of

$$A = \begin{pmatrix} -p_1 & 0 & \ldots & 0 \\ 0 & -p_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \ldots & 0 & -p_n \end{pmatrix} \quad B = \begin{pmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 & \ldots & 1 \end{pmatrix} \qquad D = (0)$$

Other Canonical Forms (Similarity Transformations)

We have now seen at least 2 different state space representations that result in the same transfer function. The question now becomes, “How many other state space realizations are there?”

Recall from our discussion on similarity transformations that we can define a transformation from states $\overline{x}(t)$ to states $\overline{z}(t)$ using

$$\overline{z}(t) = T\,\overline{x}(t)$$

where $\quad T$ = non-singular square matrix

The new state space representation is given as

$$\dot{\overline{z}}(t) = \tilde{A}\,\overline{z}(t) + \tilde{B}\,\overline{u}(t)$$
$$\overline{y}(t) = \tilde{C}\,\overline{z}(t) + \tilde{D}\,\overline{u}(t)$$

where $\quad \tilde{A} = T\,A\,T^{-1}$

$\tilde{B} = T\,B$

$\tilde{C} = C\,T^{-1}$

$\tilde{D} = D$

We can compute the transfer function of this new representation using

$$G(s) = \tilde{C}\left(s\,I - \tilde{A}\right)^{-1}\tilde{B} + \tilde{D}$$

$$= C\,T^{-1}\left(s\,I - T\,A\,T^{-1}\right)^{-1} T\,B + D$$

$$= C\,T^{-1}\left(s\,T\,T^{-1} - T\,A\,T^{-1}\right)^{-1} T\,B + D$$

$$= C\,T^{-1}\left[T\,(s\,I - A\,)\,T^{-1}\right]^{-1} T\,B + D$$

$$= C\,T^{-1}\left[T\,M\,T^{-1}\right]^{-1} T\,B + D \qquad \text{let } M = (s\,I - A\,)$$

$$= C\,T^{-1}[T\,N]^{-1}\,T\,B + D \qquad \text{let } N = M\,T^{-1},\ \text{recall: } (T\,N)^{-1} = N^{-1}\,T^{-1}$$

$$= C\,T^{-1}\,N^{-1}\,T^{-1}\,T\,B + D$$

$$= C\,T^{-1}\left[M\,T^{-1}\right]^{-1} T^{-1}\,T\,B + D \qquad \text{recall: } \left[M\,T^{-1}\right]^{-1} = T\,M^{-1}$$

$$= C\,T^{-1}\,T\,M^{-1}\,T^{-1}\,T\,B + D$$

$$= C\ M^{-1} B + D$$

$$G(s) = C\ (s\,I - A\,)^{-1} B + D$$

This is our original expression for the transfer function of the original system. Therefore we see that a similarity transformation does not affect the transfer function representation. Since there are infinite choices of transformation matrices T, there are infinite number of state space representations to a corresponding transfer function.