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Lecture 03a
Computing Euler Angles: The Euler Kinematical Equations and Poisson's Kinematical Equations

Lecture is on YouTube

The YouTube video entitled 'Computing Euler Angles: The Euler Kinematical Equations and Poisson's Kinematical Equations' that covers this lecture is located at https://youtu.be/9GZjtfYOXao.

Outline

-Kinematic Matrix Relationships For Rotation
    -The Euler Kinematical Equations
    -Poisson's Kinematical Equations

Matrix Kinematic Relationships for Rotation

In a previous lecture, we discussed the definition of the Euler angles, ϕ, θ, and ψ. In many engineering applications we would like to keep track of the Euler angles. One challenge is that on a maneuvering, accelerating aircraft, we cannot directly measure Euler angles. The question now becomes, what can we measure? A common component in many aerospace systems is an inertial measurement unit, or IMU. One sensor on an IMU is a rate gyro ***<show Pixhwak>***. Since the rate gyro is attached to the body frame of the aircraft, it measures the angular velocity of how F_b is rotating with respect to F_v. If we express the vector in the body frame, we can call the quantities the roll, pitch, and yaw rate

$$\overline{\omega}_{b/v}{}^{b} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}^{b} = \begin{pmatrix} \text{"roll" rate} \\ \text{"pitch" rate} \\ \text{"yaw" rate} \end{pmatrix}$$

This is easy to physically interpret these quantities

p is "roll" rate (mostly generated when you activate the ailerons)
q is "pitch" rate (mostly generated when you activate the elevators)
r is "yaw" rate (mostly generated when you activate the rudder, which may have secondary effects

of introducing p and q)

However we need to be very careful about notation. Recall from our discussion of Euler angles, we have

ϕ = "bank" angle
θ = "pitch" angle
ψ = "yaw" angle

However, note that

$$p \neq \dot{\phi}$$
$$q \neq \dot{\theta}$$
$$r \neq \dot{\psi}$$

This terminology is commonly confused with pilots and engineers.

So the question becomes, what is relationship between rotation rates in the body to rates of change of Euler angles? In other words

What is relation between $\overline{\omega}_{b/v}$ and $\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$ (and therefore ϕ, θ, ψ)

If we can figure out this relationship, we can use the outputs of the IMU ($\overline{\omega}_{b/v}$) and generate $\dot{\phi}$, $\dot{\theta}$, $\dot{\psi}$ (and therefore ϕ, θ, and ψ). This is the beginnings of an Attitude and Heading Reference System (AHRS).

In general IMU < AHRS < INS.

Show how IMU (Pixhawk) directly measures $\overline{\omega}_{b/v}$ but not ϕ, θ, and ψ (these need to be calculated)

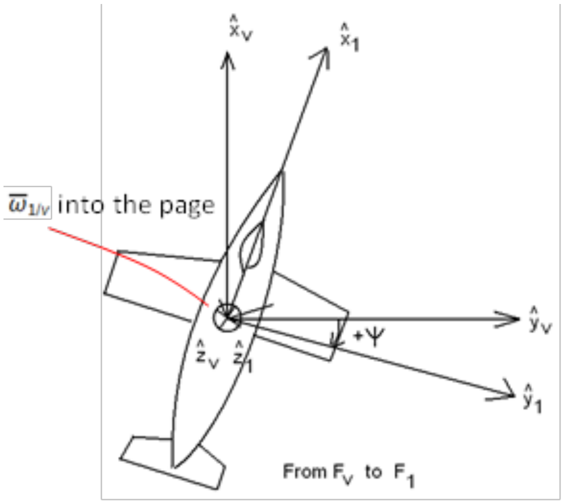
The Euler Kinematical Equations

Recall that we previously talked about how angular rotation vector is additive (we stated this without proof in the YouTube video entitled 'Vector Derivative (the Equation of Coriolis) and the Angular Velocity Vector' at https://youtu.be/-OyRCgv-hPs?t=4189). In other words

$$\overline{\omega}_{b/v} = \overline{\omega}_{b/2} + \overline{\omega}_{2/1} + \overline{\omega}_{1/v}$$

This is kinematics because we do not care about what generates these changes, this is a purely kinematic relationship.

Consider the case of $\overline{\omega}_{1/v}$. This describes how F_1 is rotating w.r.t to F_v. Recall from our previous discussion on Euler angles that the F_1 frame is defined as shown below.



So we see that $\overline{\omega}_{1/v}$ is easily expressed in either F_1 or F_v.

$$\overline{\omega}_{1/v}{}^{1} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}^{1} \qquad \overline{\omega}_{1/v}{}^{v} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}^{v}$$

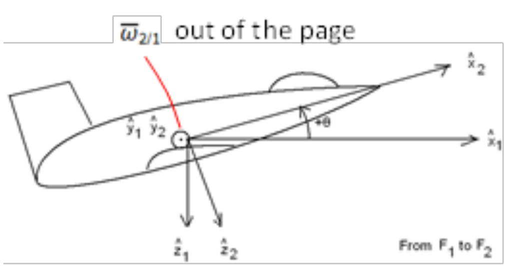
Note that we can check this by formally writing

$$\overline{\omega}_{1/v}{}^{1} = C_{1/v}(\psi)\,\overline{\omega}_{1/v}{}^{v} \qquad \text{recall:} \qquad C_{1/v}(\psi) = \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\psi) & \sin(\psi) & 0 \\ -\sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}$$

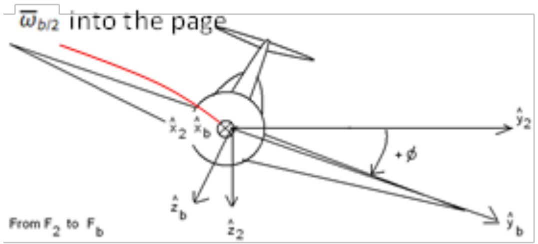
$$\overline{\omega}_{1/v}{}^{1} = \begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix} = \overline{\omega}_{1/v}{}^{v}$$

Similarly, for $\overline{\omega}_{2/1}$



$$\overline{\omega}_{2/1}{}^{2} = \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix}^{2} \qquad \overline{\omega}_{2/1}{}^{1} = \begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix}^{1}$$

Similarly, for $\overline{\omega}_{b/2}$



$$\overline{\omega}_{b/2}{}^{b} = \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix}^{b} \qquad \overline{\omega}_{b/2}{}^{2} = \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix}^{2}$$

So we can write

$$\overline{\omega}_{b/v}{}^{b} = \overline{\omega}_{b/2}{}^{b} + C_{b/2}(\phi)\,\overline{\omega}_{2/1}{}^{2} + C_{b/2}(\phi)\,C_{2/1}(\theta)\,\overline{\omega}_{1/v}{}^{1}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix}^{b} = \begin{pmatrix} \dot{\phi} \\ 0 \\ 0 \end{pmatrix}^{b} + C_{b/2}(\phi)\begin{pmatrix} 0 \\ \dot{\theta} \\ 0 \end{pmatrix}^{2} + C_{b/2}(\phi)\,C_{2/1}(\theta)\begin{pmatrix} 0 \\ 0 \\ \dot{\psi} \end{pmatrix}^{1}$$

We can use Mathematica to evaluate this

```
C21[θ_] = ( Cos[θ]  0  -Sin[θ]
              0     1    0
            Sin[θ]  0  Cos[θ] );
Cb2[ϕ_] = ( 1    0        0
            0  Cos[ϕ]   Sin[ϕ]
            0  -Sin[ϕ]  Cos[ϕ] );
wbvb = ( ϕdot
          0
          0 ) + Cb2[ϕ].( 0
                         θdot
                          0 ) + Cb2[ϕ].C21[θ].( 0
                                                 0
                                                ψdot );
wbvb // MatrixForm
```

```
( ϕdot - ψdot Sin[θ]
  θdot Cos[ϕ] + ψdot Cos[θ] Sin[ϕ]
  ψdot Cos[θ] Cos[ϕ] - θdot Sin[ϕ] )
```

So we obtain

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix}^{b} = \begin{pmatrix} 1 & 0 & -\sin(\theta) \\ 0 & \cos(\phi) & \sin(\phi)\cos(\theta) \\ 0 & -\sin(\phi) & \cos(\phi)\cos(\theta) \end{pmatrix}\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}$$

```
Hinv[θ_, ϕ_] = ( 1     0          -Sin[θ]
                 0   Cos[ϕ]   Sin[ϕ] Cos[θ]
                 0  -Sin[ϕ]   Cos[ϕ] Cos[θ] );
```

```
wbvb == Hinv[θ, ϕ]. ( ϕdot
                      θdot
                      ψdot )
```

```
True
```

Note that we don't really express the term $\begin{pmatrix} \dot{\phi} & \dot{\theta} & \dot{\psi} \end{pmatrix}^T$ in a frame. This is because this is more a collection of quantities rather than a vector because it does not exist in any particular frame. Therefore, the 3x3 matrix is not a rotation matrix which implies that it may not be unitary which further implies that we must compute the full inverse in order to solve for $\begin{pmatrix} \dot{\phi} & \dot{\theta} & \dot{\psi} \end{pmatrix}^T$

The inverse transformation given by

$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & \tan(\theta)\sin(\phi) & \tan(\theta)\cos(\phi) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)/\cos(\theta) & \cos(\phi)/\cos(\theta) \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}^b \qquad \textit{\textbf{(Eq.1.3-22)}}$$

or also

$$\dot{\Phi} = H(\Phi)\,\overline{\omega}_{b/v}{}^b$$

where $$H(\Phi) = \begin{pmatrix} 1 & \tan(\theta)\sin(\phi) & \tan(\theta)\cos(\phi) \\ 0 & \cos(\phi) & -\sin(\phi) \\ 0 & \sin(\phi)/\cos(\theta) & \cos(\phi)/\cos(\theta) \end{pmatrix}$$

$\Phi = \begin{pmatrix} \phi & \theta & \psi \end{pmatrix}^T$ (CAUTION: Note the difference between Φ and ϕ)

These are referred to as the Euler Kinematical Equations.

```
H[θ_, ϕ_] = Inverse[Hinv[θ, ϕ]] // Simplify;
H[θ, ϕ] // MatrixForm
```

```
( 1  Sin[ϕ] Tan[θ]  Cos[ϕ] Tan[θ]
  0     Cos[ϕ]         -Sin[ϕ]
  0  Sec[θ] Sin[ϕ]  Cos[ϕ] Sec[θ] )
```

Example

For example, if aircraft is on an edge, $\phi = \pi/2$

```
H[0, π / 2] // MatrixForm
```

```
( 1  0   0
  0  0  -1
  0  1   0 )
```

What if there is only a “yaw rate” measured by the rate gyro, ie $r \neq 0$, $p = q = 0$

```
H[0, π / 2].(0, 0, r) // MatrixForm
```

$$\begin{pmatrix} 0 \\ -r \\ 0 \end{pmatrix}$$

So $\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ -r \\ 0 \end{pmatrix}$

In this case, a “yaw rate” (r) actually leads to change in the pitch Euler angle (θ) due to the orientation of the aircraft. For example, if the aircraft starts at

$$\Phi(0) = \begin{pmatrix} \phi(0) \\ \theta(0) \\ \psi(0) \end{pmatrix} = \begin{pmatrix} \pi/2 \\ 0 \\ 0 \end{pmatrix}$$

Then with a yaw rate of $r = -30 \frac{\pi}{180} \frac{\text{rad}}{s}$, Eq.1.3-22 yields

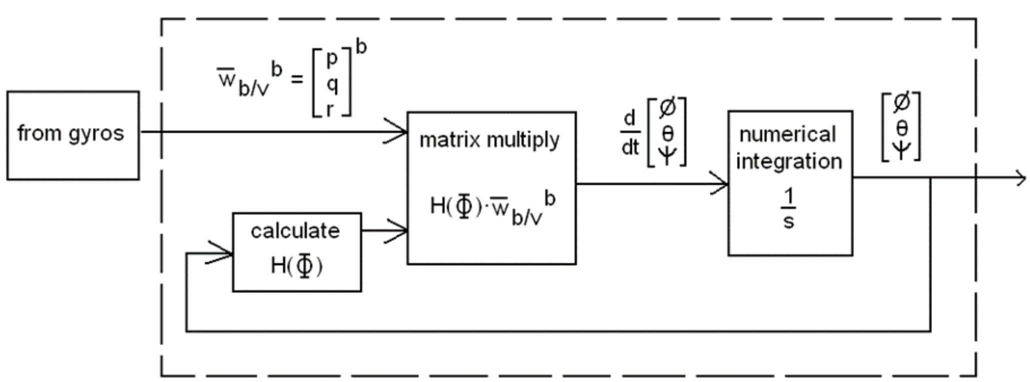
$$\begin{pmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 0 \\ -(-30 \frac{\pi}{180}) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 30 \frac{\pi}{180} \\ 0 \end{pmatrix}$$

After 2 seconds, we expect the Euler angles to be

$$\Phi(2) = \begin{pmatrix} \psi(2) \\ \theta(2) \\ \phi(2) \end{pmatrix} = \begin{pmatrix} 0 \\ 60 \frac{\pi}{180} \\ 0 \end{pmatrix}$$

This clearly shows that you cannot simply integrate each element of $\overline{\omega}_{b/v}{}^b$ to get the Euler angles.

In general, we cannot simply integrate the rates to get the Euler angles. Instead, we want to integrate $\dot{\Phi}$ to get Euler angles. A block diagram showing a pseudo-Simulink model to perform this integration is shown below.



Example: Singular H(Φ)

Because H(Φ) is not a rotation matrix, it is not unitary and therefore it is not guarantee to be singularity free. In fact, this does have a singularity when $\theta = \pm\pi/2$

```
Hinv[π / 2, ϕ] // MatrixForm
Eigenvalues[Hinv[π / 2, ϕ]]

Hinv[-π / 2, ϕ] // MatrixForm
Eigenvalues[-Hinv[π / 2, ϕ]]
```

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & \text{Cos}[\phi] & 0 \\ 0 & -\text{Sin}[\phi] & 0 \end{pmatrix}$$

{1, 0, Cos[ϕ]}

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & \text{Cos}[\phi] & 0 \\ 0 & -\text{Sin}[\phi] & 0 \end{pmatrix}$$

{-1, 0, -Cos[ϕ]}

```
H[π / 2, ϕ] // MatrixForm
H[-π / 2, ϕ] // MatrixForm
```

$$\begin{pmatrix} 1 & \text{ComplexInfinity} & \text{ComplexInfinity} \\ 0 & \text{Cos}[\phi] & -\text{Sin}[\phi] \\ 0 & \text{ComplexInfinity} & \text{ComplexInfinity} \end{pmatrix}$$

$$\begin{pmatrix} 1 & \text{ComplexInfinity} & \text{ComplexInfinity} \\ 0 & \text{Cos}[\phi] & -\text{Sin}[\phi] \\ 0 & \text{ComplexInfinity} & \text{ComplexInfinity} \end{pmatrix}$$

This is related to the concept of gimbal lock and shows why this integration scheme may not be appropriate if θ is near $\pm\pi/2$

Despise this disadvantage, the Euler Kinematical Equations are a popular way to compute rate of changes of Euler angles. That being said, this disadvantage leads to the discussion of Poisson's Kinematical Equations and quaternions.

Poisson' s Kinematical Equations

Previously we determined that the relation between p, q, and r with $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ was given by $\dot{\Phi} = H(\Phi)\,\overline{\omega}_{b/v}{}^{b}$ (Eq.1.3-22).

We can also ask the question, "Given rotation rate of F_b w.r.t. F_v, how does $C_{b/v}$ change in time and how is it related to $\overline{\omega}_{b/v} = (p, q, r)$?

For the derivation of the relationship, see Stevens and Lewis, 2nd Edition, pg. 28.

$$\dot{C}_{b/v} = -\Omega_{b/v}{}^{b}\, C_{b/v}$$

where $\quad \Omega_{b/v}{}^{b} = \begin{pmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{pmatrix}$

Note: this is 9 equations, not 3 like $\dot{\Phi} = H(\Phi)\,\overline{\omega}_{b/v}{}^{b}$.

$$\mathbf{Cbv} = \begin{pmatrix} \mathtt{C11} & \mathtt{C12} & \mathtt{C13} \\ \mathtt{C21} & \mathtt{C22} & \mathtt{C23} \\ \mathtt{C31} & \mathtt{C32} & \mathtt{C33} \end{pmatrix};$$

$$\mathbf{\Omega bv} = \begin{pmatrix} 0 & -\mathtt{r} & \mathtt{q} \\ \mathtt{r} & 0 & -\mathtt{p} \\ -\mathtt{q} & \mathtt{p} & 0 \end{pmatrix};$$

```
Cbvdot = -Ωbv.Cbv;
Cbvdot // MatrixForm
```

$$\begin{pmatrix} -\mathrm{C31}\ \mathrm{q} + \mathrm{C21}\ \mathrm{r} & -\mathrm{C32}\ \mathrm{q} + \mathrm{C22}\ \mathrm{r} & -\mathrm{C33}\ \mathrm{q} + \mathrm{C23}\ \mathrm{r} \\ \mathrm{C31}\ \mathrm{p} - \mathrm{C11}\ \mathrm{r} & \mathrm{C32}\ \mathrm{p} - \mathrm{C12}\ \mathrm{r} & \mathrm{C33}\ \mathrm{p} - \mathrm{C13}\ \mathrm{r} \\ -\mathrm{C21}\ \mathrm{p} + \mathrm{C11}\ \mathrm{q} & -\mathrm{C22}\ \mathrm{p} + \mathrm{C12}\ \mathrm{q} & -\mathrm{C23}\ \mathrm{p} + \mathrm{C13}\ \mathrm{q} \end{pmatrix}$$

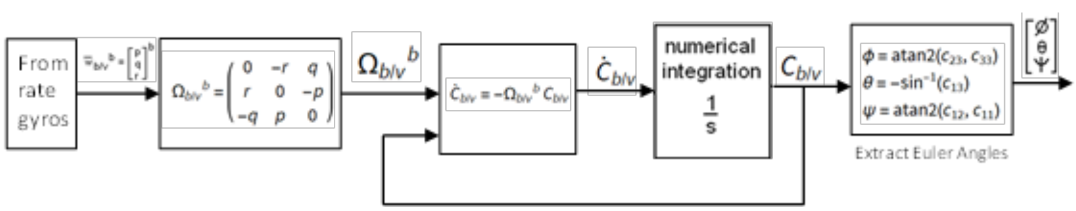
We can write this out as a scalar system of equations

$$\dot{C}_{11} = -q\,C_{31} + r\,C_{21}$$
$$\dot{C}_{12} = -q\,C_{32} + r\,C_{22}$$
$$\vdots$$
$$\dot{C}_{33} = -p\,C_{23} + q\,C_{13}$$

These are known as **Poisson's Kinematical Equations** or the inertial navigation **strapdown equations**.

We can obtain the Euler angles by solving for them in elements of the $C_{b/v}$ matrix. Recall from our discussion on Euler angles that we can extract the Euler angles from the DCM using

$$\phi = \operatorname{atan2}(c_{23}, c_{33}) \qquad \textbf{\textit{(Eq.1.2)}}$$
$$\theta = -\sin^{-1}(c_{13}) \qquad \textbf{\textit{(Eq.1.3)}}$$
$$\psi = \operatorname{atan2}(c_{12}, c_{11}) \qquad \textbf{\textit{(Eq.1.4)}}$$



<u>Advantages</u>
-Redundancy which can stabilize numerical errors
-We know that $C_{b/v}$ is a unitary matrix to check accuracy of integration process

<u>Disadvantages</u>
-9 equations instead of 3

-Euler angles are not directly calculated