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Lecture 07c
Relationship Between Pole Locations and Performance of a Dynamic System

Lecture is on YouTube

YouTube video entitled 'Relationship Between Poles and Performance of a Dynamic System' covering this is located at https://youtu.be/0tbr4OIufK8.

Outline

-Relationship Between Pole Locations and System Response

    -Relationship between pole locations and ω_n and ζ

    -Relationship between pole locations and performance metrics

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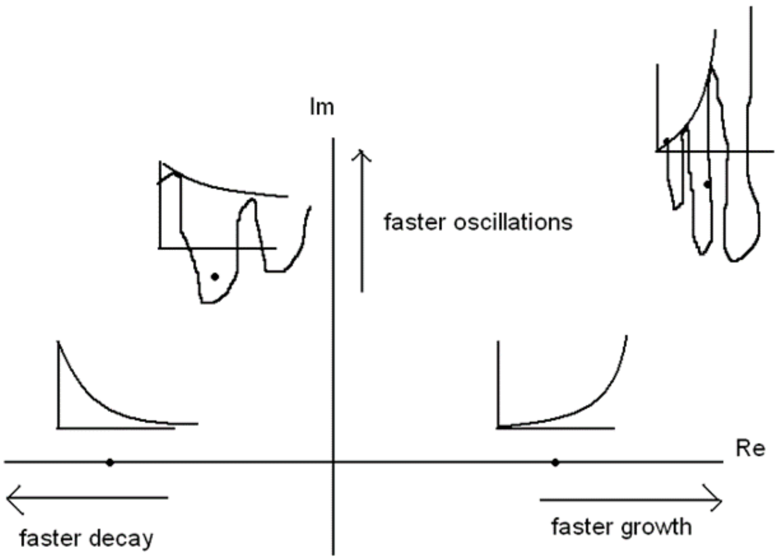
Relationship Between Pole Locations and System Response

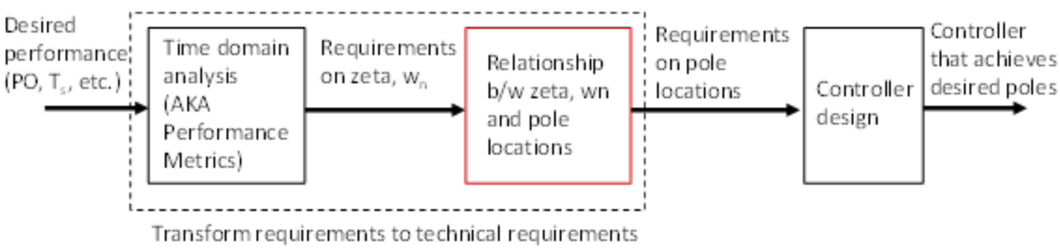
Based on prior experience with dynamic systems, we know that the response of the system was characterized by the poles of the system. If one knows the location of the poles of a system, one can predict the approximate response of the system without having to actually compute the analytical response

$p = x + y\,i$ (pole of system)

more negative $x \Longleftrightarrow$ faster decay
more positive $x \Longleftrightarrow$ faster growth
larger $|y| \Longleftrightarrow$ faster oscillations



In a previous lecture, we examined how ζ and ω_n relate to useful performance metrics such as percent overshoot, settling time, etc. We now seek to understand the relationship between ζ or ω_n and pole locations. If we understood this, we would have a firm link between desired performance and the requisite pole locations. This could then be used to find a controller that places the poles in the desired locations to achieve this performance. Therefore, the main focus of the following discussion is on the box outlined in red below.



Relationship Between Pole Locations and ω_n and ζ

For the remainder of the analysis, let us assume that the system is a second order system. Even if the system is a higher order system, often it can be approximately modeled as a pair of dominant poles.

We now investigate the relationship between pole locations and the damping ratio and natural frequency of the system.

Recall that the characteristic equation of any second order system is given by

$$s^2 + 2\,\zeta\,\omega_n\,s + {\omega_n}^2 = 0$$

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Solve[s^2 + 2 ξ ωn s + ωn^2 == 0, s]
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$$\left\{\left\{s \to -\xi\,\omega n - \sqrt{-\omega n^2 + \xi^2\,\omega n^2}\right\},\ \left\{s \to -\xi\,\omega n + \sqrt{-\omega n^2 + \xi^2\,\omega n^2}\right\}\right\}$$

So the pole locations are

$$\lambda_{1,2} = -\zeta\,\omega_n \pm \omega_n\,\sqrt{\zeta^2 - 1}$$

Let us assume that the system is stable, so $\omega_n > 0$ and $\zeta > 0$. Furthermore, let us assume that the poles are complex, so $\zeta < 1$

stable $\Longleftrightarrow$ $\omega_n > 0$, $\zeta > 0$
imaginary poles $\Longleftrightarrow$ $\zeta < 1$

In this case, we can write the pole location as

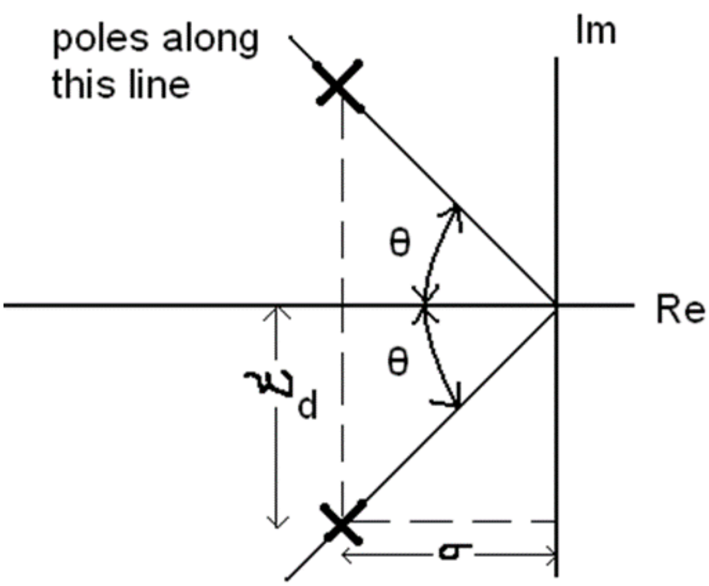
$$\lambda_{1,2} = -\zeta\,\omega_n \pm \omega_n\,\sqrt{1 - \zeta^2}\,i$$

$$\lambda_{1,2} = -\sigma \pm \omega_d\,i$$

where $\sigma = \zeta\,\omega_n$ (positive and real, represents the real part of the pole)
$\omega_d = \omega_n\,\sqrt{1 - \zeta^2}$ (positive and real, represents the imaginary part of the pole)

So we see that σ is the real part of the root and ω_d is the imaginary part of the root.

Consider pole locations along lines with angle θ. This is shown below



We can calculate the damping ratio for any pole along this line. For any pole along this line, we have the relationship

$$\tan(\theta) = \frac{\omega_d}{\sigma}$$

$$= \frac{\omega_n \sqrt{1-\zeta^2}}{\zeta\, \omega_n}$$

$$= \frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right) \qquad \textbf{\textit{(Eq.1)}}$$

This states that any pole along any line intersecting the origin has the same damping ratio. For example, any pole with damping of $\zeta = 1\big/\sqrt{2} \approx 0.707$ (considered optimal in many cases) must be along the line

$$\theta = \tan^{-1}\left(\frac{\sqrt{1-\left(1/\sqrt{2}\right)^2}}{1/\sqrt{2}}\right) = 45^o$$

In other words, if you want a system to have good damping of $\zeta = 1\big/\sqrt{2}$, the roots must be on the 45^o line.

Relationship Between Pole Locations and Performance Metrics

We can now extend this idea one step further and relate the damping ratio to a performance metric.

Percent Overshoot

For second order system, recall that the percent overshoot is given as

$$PO = 100\, e^{-\frac{\pi \zeta}{\sqrt{1-\zeta^2}}}$$

Since the percent overshoot is only a function of ζ, then all poles along the same line have the same percent overshoot. So a procedure for finding the acceptable pole locations for a specified percent overshoot should be

1. Based on required percent overshoot, calculate required ζ using $\zeta = -\frac{\ln\left(\frac{PO}{100}\right)}{\sqrt{\pi^2+\ln\left(\frac{PO}{100}\right)^2}}$

2. Based on ζ, calculate the required angle θ using $\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

3. Draw the line with angle θ on the real/imaginary plane. Choose a controller such that the poles end up on this line.

$$\xi\text{required} = -\frac{\text{Log}\left[\frac{\text{PO}}{100}\right]}{\sqrt{\pi^2 + \text{Log}\left[\frac{\text{PO}}{100}\right]^2}};$$

$$\theta\text{required} = \text{ArcTan}\left[\frac{\sqrt{1 - \xi^2}}{\xi}\right];$$

Settling Time

We can perform a similar procedure for settling time. Recall that

$$T_s = \frac{-\ln(\delta)}{\zeta\,\omega_n} \qquad \text{recall: } \sigma = \zeta\,\omega_n$$

$$T_s = \frac{-\ln(\delta)}{\sigma}$$

$$\sigma = -\frac{\ln(\delta)}{T_s}$$

So if we are given a required settling time, we know where the real portion of the pole needs to be in order to satisfy the settling time requirement.

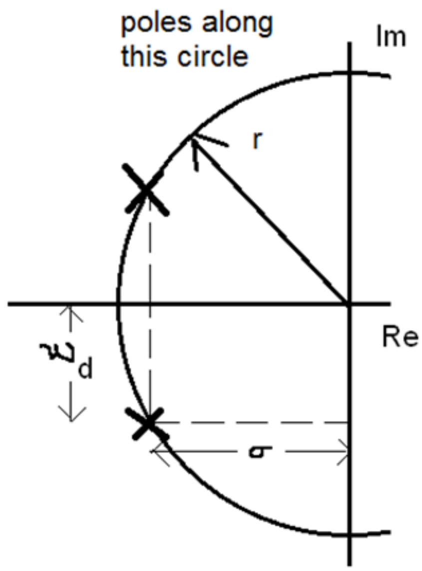
$$\sigma\text{required} = -\frac{\text{Log}[\delta]}{\text{Ts}}$$

$$-\frac{\text{Log}[\delta]}{\text{Ts}}$$

So a settling time requirement is a requirement on σ, the real part of the pole. Therefore the distance from the y-axis dictates the settling time of the system.

Natural Frequency

We can also investigate the natural frequency. Consider pole locations along a circle centered at the origin with radius r



From geometry, we see that

$$r = \left(\omega_d{}^2 + \sigma^2\right)^{1/2}$$

$$= \left(\left(\omega_n \sqrt{1-\zeta^2}\right)^2 + (\zeta\,\omega_n)^2\right)^{1/2}$$

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Simplify[((ωn Sqrt[1 - ξ^2])^2 + (ξ ωn)^2)^(1/2), ωn > 0]
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ωn

So we see that

$$r = \omega_n$$

In other words, the radial distance from the origin dictates the natural frequency of the system.

<u>Damped Natural Frequency</u>

Keep in mind that the damped natural frequency, ω_d, is the frequency that the system will oscillate at. If we recall that $\omega_d = \omega_n \sqrt{1-\zeta^2}$ we see that this is simply the imaginary component of the pole. Therefore the distance from the x-axis dictates the damped natural frequency of the system.

Example

Suppose that we require

PO < 20 × %
T_s < 10 seconds to within 10% of the final value
oscillation frequency of less than 0.1 rad/s

Using the percent overshoot constraint, we can compute the necessary damping ratio

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ξexample = ξrequired /. {PO → 20} // N
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0.45595

Using this, we can compute the required angle

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θexample = θrequired /. {ξ → ξexample};
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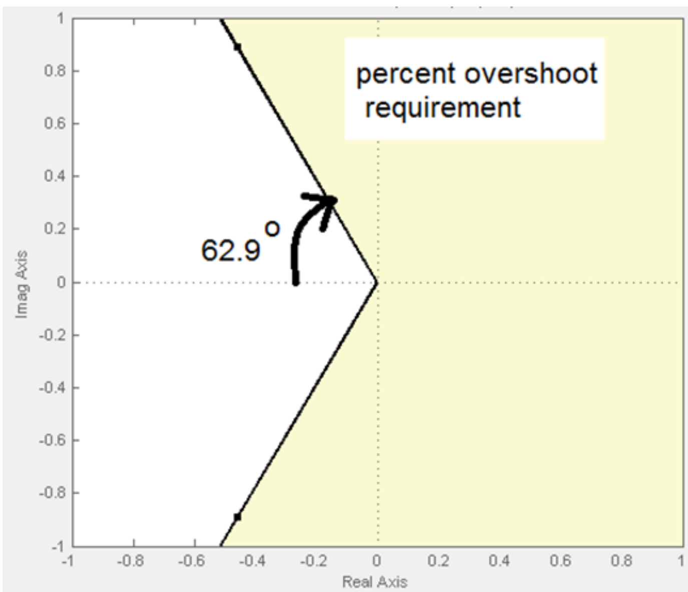
$$\theta\text{example} * \frac{180}{\pi}$$

62.8739

We can draw the inadmissible regions in yellow as shown below



Using the settling time constraint, we can compute the necessary real part of the pole.

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σrequired /. {δ → 0.1, Ts → 10}
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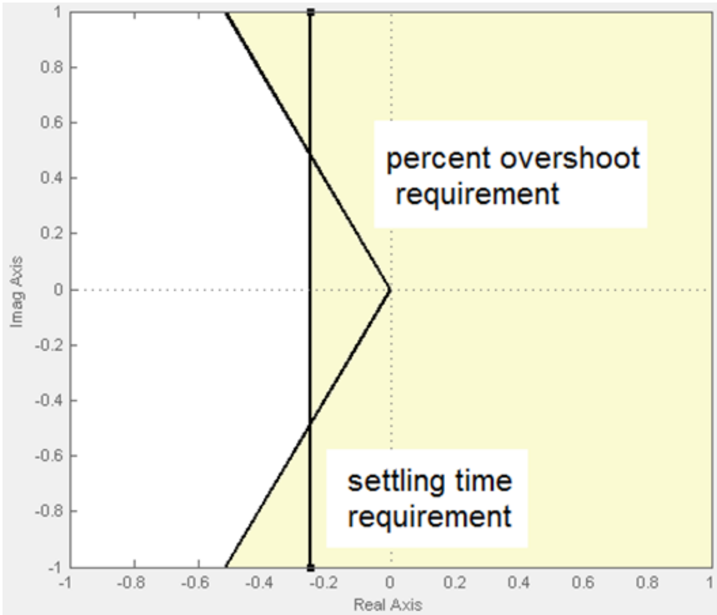
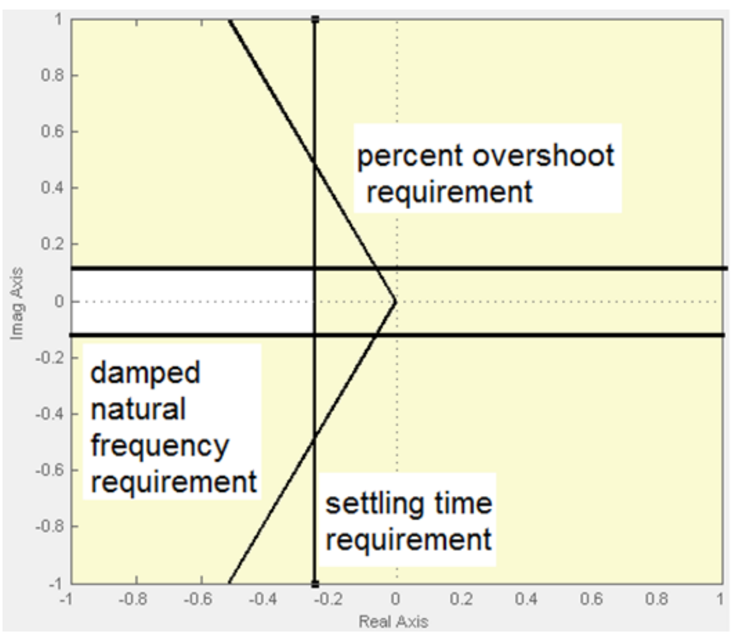
0.230259

So we see that the real part of the pole must be less than -0.23 in order to meet this requirement. Again, we can sketch the inadmissible regions in yellow



Finally, we can consider the oscillation frequency constraint. This translates to $\omega_d < 0.1$. Since ω_d is the y component of the poles, we see that the y component of the closed loop poles must be below 0.1. Once again, the inadmissible regions are shown in yellow



So we see that these performance requirements effectively dictate acceptable locations of the real/imaginary plane where the poles can lie. We investigate combining this analysis with the root locus technique to design a controller in the next lecture.

<u>WARNING:</u> All the aforementioned analysis assumes that the system is a 2nd order system. If there are additional poles/dynamics, then these regions may not map to precise performance metrics and should only be used as initial guidelines. Full simulation or analysis of higher order systems is required in order to ensure accuracy of results.