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Lecture 02d
The Laplace Transform

Lecture is on YouTube

The YouTube video entitled 'The Laplace Transform' that covers this lecture is located at https://youtu.be/q0nX8uIFZ_k.

Outline

-Laplace Transforms: Introduction
-Procedure for the Laplace Transform
-Laplace Transform of Common Functions

Roadmap

We will use the Laplace transform as a method to solve ordinary differential equations. The discussion that focuses on the Laplace Transform will be covered by several videos/lectures.

-Complex Numbers
-The Laplace Transform
-The Final Value Theorem
-Partial Fraction Expansion
-The Inverse Laplace Transform

From a software perspective, we'll cover

-2D Plotting in Mathematica
-Finding Roots of a Polynomial with Matlab and Mathematica

Introduction

The Laplace transform is an integral transformation invented by Pierre-Simon Laplace 1749 – 1827.

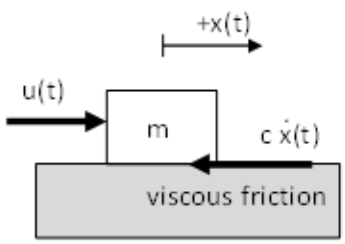
AKA “The French Newton”. This is typically used to transform a function in time to a function in complex variable s.

Laplace transforms are useful for solving linear, time-invariant (LTI) ordinary differential equations (ODEs). Recall LTI ODEs take the form of

$$a\,\ddot{x}(t) + b\,\dot{x}(t) + c\,x(t) = u(t)$$

This basically asks us to find a function, $x(t)$, such that linear combinations of its derivatives equals the input function, $u(t)$. In many scenarios, $u(t)$ is given but given the coupled nature of the problem, the solution for $x(t)$ is not immediately obvious.

How does this relate to our study of dynamic systems? Consider the simple scenario shown below.



If we are interested in the position of the block, $x(t)$, as a function of time, we can write from Newton's 2nd law

$$\sum F = m\,a(t) \qquad \text{recall: } a(t) = \ddot{x}(t)$$

$$u(t) - c\,\dot{x} = m\,\ddot{x}(t)$$

$$u(t) = m\,\ddot{x}(t) + c\,\dot{x}(t)$$

If we excite the block with a sinusoidal force of $u(t) = 3\sin(t)$, we might have

$$3\sin(t) = m\,\ddot{x}(t) + c\,\dot{x}(t)$$

This is a second order, ordinary differential equation. How do we typically solve these?

Typical method:

1. Start with differential equation $\dot{x}(t) = f(x, u, t)$
2. Input, $u(t)$, is given
3. Solve autonomous and particular solution using differential equation techniques
4. Obtain solution $x(t)$

It would be nice if we could use algebra to solve this. It turns out that the Laplace transform method will transform this differential equation into an algebraic equation.

Laplace Method:

1. Start with differential equation $\dot{x}(t) = f(x, u, t)$
2. Input, $u(t)$, is given

3. Apply Laplace transform to system and input to obtain $X(s)$ and $U(s)$
4. Solve solution using algebraic techniques to obtain $X(s)$
5. Apply inverse Laplace transform on $X(s)$ to obtain solution, $x(t)$

So the problem of

$$\dot{x}(t) = f(x, u, t)$$

Becomes a problem of

$$X = F(X, U)$$

which is a “static” or algebraic problem because there is no longer a dependence on time derivatives.

Procedure for the Laplace Transform

We first define some functions and terms for the Laplace transform procedure.

$f(t)$ =a function in the time domain such that $f(t) = 0$ for $t < 0$
s = a complex variable
L =Laplace transform operator
$F(s)$ =Laplace transform of $f(t)$

The Laplace transform of $f(t)$ is defined as

$$F(s) = L[f(t)] = \int_0^\infty f(t)\, e^{-s\,t}\, dt$$

One might wonder if there are conditions that need to be required in order for the Laplace transform to exist? The Laplace transform exists if $f(t)$ is piecewise continuous in every finite interval in the range $t > 0$ and if $f(t)$ is of exponential order as t approaches infinity. A function is of exponential order if a real, positive constant σ exists such that the function

$$\lim_{t\to\infty} e^{-\sigma\,t} \mid f(t) \mid \to 0$$

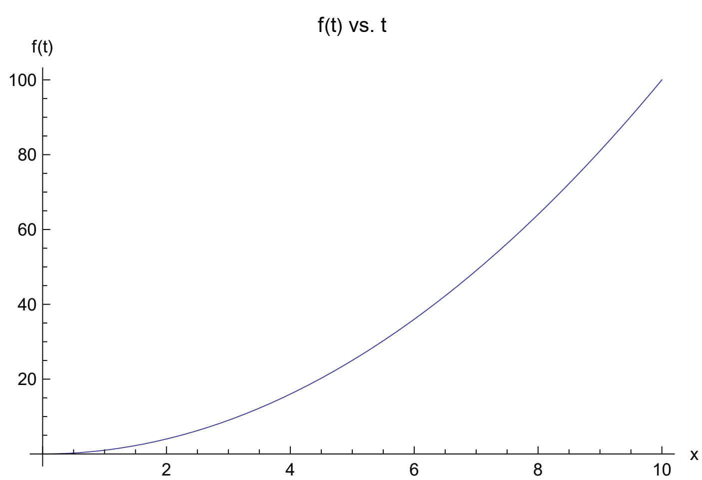
Basically this means that the function does not grow faster than an arbitrary exponential function (ie the $e^{-\sigma\,t}$ can "squash" it as $t \to \infty$). We can plot some example functions of $f(t)$ in this expression.

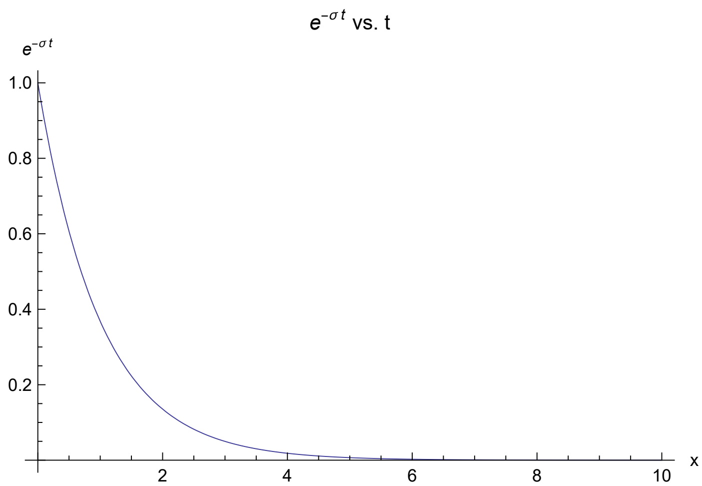
Note that this is a good opportunity to illustrate plotting in Mathematica (see https://youtu.be/j-utznrXmcY).

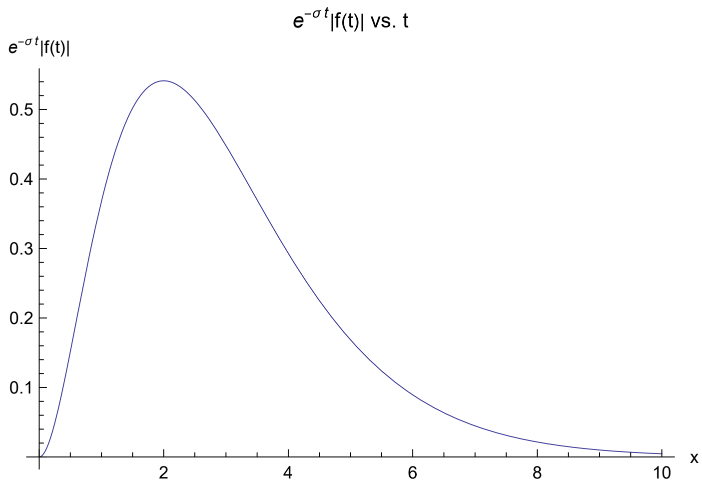
This is also a good time to discuss defining simple functions in Mathematica (see URL TBD)

We now return to our scenario where we would like to plot $\lim_{t\to\infty} e^{-\sigma\,t} \mid f(t) \mid \to 0$. We can use the previous discussion of defining a function and plotting to visualize using Mathematica.

```
(*Define a function, f(t)*)
f[t_] = t^2;

(*Define parameters of an exponential function*)
σ = 1;

(*Plot f(t)*)
Plot[f[t], {t, 0, 10},
 PlotRange → All,
 AxesLabel → {"x", "f(t)"},
 PlotLabel → "f(t) vs. t"
]

(*Plot the exponential function*)
Plot[Exp[-σ t], {t, 0, 10},
 PlotRange → All,
 AxesLabel → {"x", "e^(-σ t)"},
 PlotLabel → "e^(-σ t) vs. t"
]

(*Plot e^(-σ t)|f(t)|*)
Plot[Exp[-σ t] Abs[f[t]], {t, 0, 10},
 PlotRange → All,
 AxesLabel → {"x", "e^(-σ t)|f(t)|"},
 PlotLabel → "e^(-σ t)|f(t)| vs. t"
]
```







Signals like $e^{-c t}$, $\cos(\omega t)$, $e^{-c t} \sin(\omega t)$ all have Laplace transforms.

Signals like e^{t^2} or $t\, e^{t^2}$ do not have Laplace transforms.

Note : Although the function $f(t) = e^{t^2}$, $t \in [0, \infty)$ does not have a Laplace transform, the piecewise function

$$f(t) = \begin{cases} e^{t^2} & t \in [0, T] \\ 0 & t < 0,\ t > T \end{cases}$$

does have a Laplace transform since $f(t) = e^{t^2}$ only for a limited time interval $t \in [0, T]$. This signal can be physically generated and signals which can be physically generated have Laplace transforms.

Laplace Transform of Common Functions

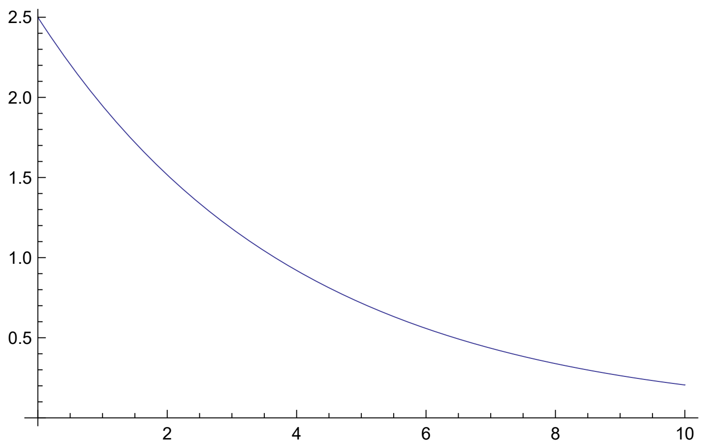
We can now investigate the Laplace transform for several common functions.

Exponential Function

$$f(t) = \begin{cases} 0 & t < 0 \\ A e^{-\alpha t} & t \geq 0 \end{cases}$$

where A, α are constants

```
f[t_] = A Exp[-α t];
Plot[f[t] /. {A → 2.5, α → 0.25}, {t, 0, 10}]
```



Applying the Laplace transform yields

$$L[f(t)] = L\left[A e^{-\alpha t}\right]$$

$$= \int_0^\infty A e^{-\alpha t} e^{-s t} \, dt$$

$$= A \int_0^\infty e^{-(s+\alpha) t} \, dt \qquad \text{note: } \int e^{-(s+\alpha) t} \, dt = \frac{-1}{s+\alpha} e^{-(s+\alpha) t}$$

$$= A\left(\frac{-1}{s+\alpha} e^{-(s+\alpha) t} \Big|_{t=\infty} + \frac{1}{s+\alpha} e^{-(s+\alpha) t} \Big|_{t=0}\right) \qquad \text{note: assume that } \mathrm{Re}[s+\alpha] > 0$$

$$F(s) = \frac{A}{s+\alpha}$$

Note that we needed to assume that $\mathrm{Re}[s] > -\alpha$ so that the integral converges.

Step Function

$$f(t) = \begin{cases} 0 & t < 0 \\ A & t \geq 0 \end{cases}$$

where A is a constant

Apply the Laplace transform

$$L[f(t)] = L[A]$$

$$= \int_0^\infty A\, e^{-s\,t}\, dt$$

$$= A \int_0^\infty e^{-s\,t}\, dt \qquad \text{note: } \int e^{-s\,t}\, dt = \frac{-1}{s}\, e^{-s\,t}$$

$$= A\left(\frac{-1}{s}\, e^{-s\,t}\,\Big|_{t=\infty} + \frac{1}{s}\, e^{-s\,t}\,\Big|_{t=0}\right) \qquad \text{note: assume that } s > 0$$

$$F(s) = \frac{A}{s}$$

Note that we can also argue that this is a special case of the exponential function with $\alpha = 0$ so the Laplace transform is

$$F(s) = \frac{A}{s}$$

If $A = 1$, the step function is sometime referred to as a unit step function, written $\mathbf{1}\,(t)$

Ramp Function

$$f(t) = \begin{cases} 0 & t < 0 \\ A\,t & t \geq 0 \end{cases}$$

where $\quad A$ is a constant

Apply Laplace transform

$$L[f(t)] = L[A\,t]$$

$$= \int_0^\infty A\,t\, e^{-s\,t}\, dt$$

$$= A \int_0^\infty t\, e^{-s\,t}\, dt$$

Now use integration by parts $\int_a^b u\, dv = u\, v\,\big|_a{}^b - \int_a^b v\, du$

$$u = t \qquad \Rightarrow \qquad du = dt$$
$$dv = e^{-s\,t}\, dt \quad \Rightarrow \quad v = \frac{-1}{s}\, e^{-s\,t}$$

So

$$A \int_0^\infty t\, e^{-s\,t}\, dt = A\left(\frac{-t}{s}\, e^{-s\,t}\,\Big|_0{}^\infty - \int_0^\infty \frac{-1}{s}\, e^{-s\,t}\, dt\right) \qquad \text{assume: } s > 0 \text{ so } \frac{-t}{s}\, e^{-s\,t}\,\Big|_0{}^\infty = 0$$

$$= A \int_0^\infty \frac{1}{s}\, e^{-s\,t}\, dt \qquad \text{note: } s \text{ is not a function of } t$$

$$= \frac{A}{s} \int_0^\infty e^{-s\,t}\, dt$$

$$= \frac{A}{s}\left(\frac{-1}{s}\, e^{-s\,t}\,\Big|_0{}^\infty\right)$$

$$= \frac{A}{s}\left(\frac{-1}{s}\, e^{-s\,t}\ |_{t=\infty} + \frac{1}{s}\, e^{-s\,t}\ |_{t=0}\right) \text{ recall: we assumed } s > 0$$

$$F(s) = \frac{A}{s^2}$$

Sin, Cos Function

Functions of

$$f_1(t) = \begin{cases} 0 & t < 0 \\ A \sin(\omega\, t) & t \geq 0 \end{cases}$$

$$f_2(t) = \begin{cases} 0 & t < 0 \\ A \cos(\omega\, t) & t \geq 0 \end{cases}$$

You will show in homework that

$$F_1(s) = \frac{A\ \omega}{s^2 + \omega^2}$$

$$F_2(s) = \frac{A\, s}{s^2 + \omega^2}$$

Translated Function

$$g(t) = \begin{cases} f(t - \alpha) & t \geq \alpha \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha \geq 0$

For example, consider the function

$$f(t) = 3 \sin(2\, t)$$

```
f[t_] = 3 Sin[2 t];
```

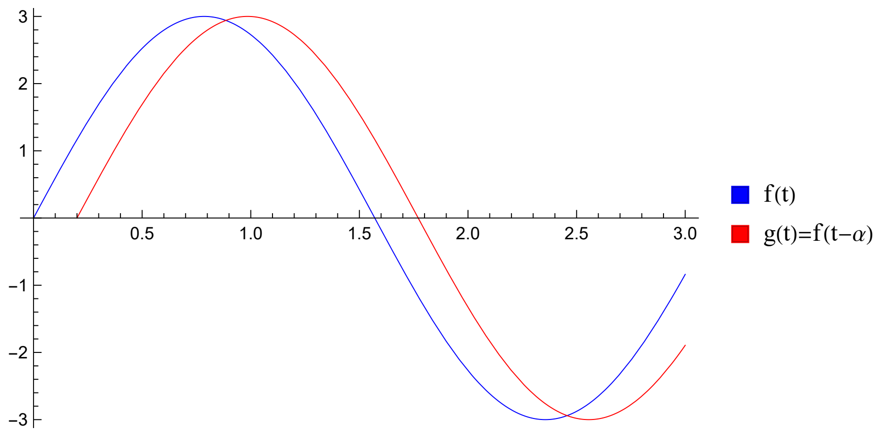
So the translated function for α is given by

$$g(t) = \begin{cases} f(t - \alpha) & t \geq \alpha \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} 3 \sin(2\,(t - \alpha)) & t \geq \alpha \\ 0 & \text{otherwise} \end{cases}$$

We can examine this for a value of $\alpha = 0.2$

```
(*define g(t)*)
α = 0.2;
g[t_] = f[t - α];

(*Plot both f(t) and g(t)*)
Legended[
 Show[
  Plot[f[t], {t, 0, 3}, PlotStyle → Blue],
  Plot[g[t], {t, α, 3}, PlotStyle → Red]
 ],

 (*Legend info*)
 SwatchLegend[{Blue, Red}, {"f(t)", "g(t)=f(t-α)"}]
]
```



So we see that this is why it is called a translated function. Note that for $t < \alpha$, $g(t) = 0$, so we can add the term $\mathbf{1}(t - \alpha)$ and write this as

$$g(t) = f(t - \alpha)\,\mathbf{1}(t - \alpha)$$

Apply Laplace transform

$$L[g(t)] = L[f(t - \alpha)\,\mathbf{1}(t - \alpha)]$$

$$= \int_0^\infty f(t - \alpha)\,\mathbf{1}(t - \alpha)\,e^{-s\,t}\,dt$$

We can change the independent variable from t to $\tau = t - \alpha$ (thereby changing the limits of integration)

Lower limit of integration: $t = 0 \quad \Rightarrow \quad \tau = -\alpha$
Upper limit of integration: $t = \infty \quad \Rightarrow \quad \tau = \infty$

$$= \int_{-\alpha}^\infty f(\tau)\,\mathbf{1}(\tau)\,e^{-s\,(\tau+\alpha)}\,d\tau$$

$$= \int_{-\alpha}^{0} f(\tau)\, \mathbf{1}(\tau)\, e^{-s(\tau+\alpha)}\, d\tau + \int_{0}^{\infty} f(\tau)\, \mathbf{1}(\tau)\, e^{-s(\tau+\alpha)}\, d\tau \qquad \text{note: } \mathbf{1}(\tau) = \begin{cases} 0 & \tau \in [-\alpha, 0) \\ 1 & \tau \geq 0 \end{cases}$$

$$= \int_{0}^{\infty} f(\tau)\; e^{-s(\tau+\alpha)}\, d\tau$$

$$= \int_{0}^{\infty} f(\tau)\; e^{-s\tau} e^{-s\alpha}\, d\tau$$

$$= e^{-s\alpha} \int_{0}^{\infty} f(\tau)\; e^{-s\tau}\, d\tau \qquad \text{note: } \int_{0}^{\infty} f(\tau)\; e^{-s\tau}\, d\tau \triangleq L[f(t)] = F(s)$$

So we see that

$$L[f(t-\alpha)\, \mathbf{1}(t-\alpha)] = e^{-\alpha s} F(s)$$

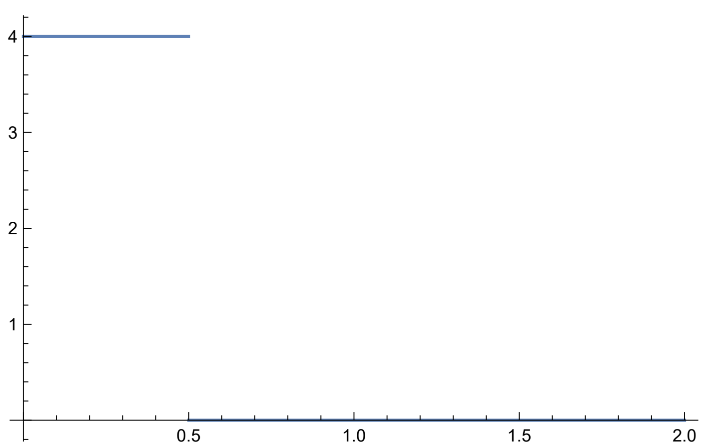
So we see that the Laplace transform of a function delayed by α seconds is equivalent to the Laplace transform of the original function multiplied by $e^{-\alpha s}$.

Pulse Function

$$f(t) = \begin{cases} A/t_0 & t \in (0, t_0) \\ 0 & t \in (-\infty, 0),\; t \in (t_0, \infty) \end{cases}$$

where A, t_0 are constants

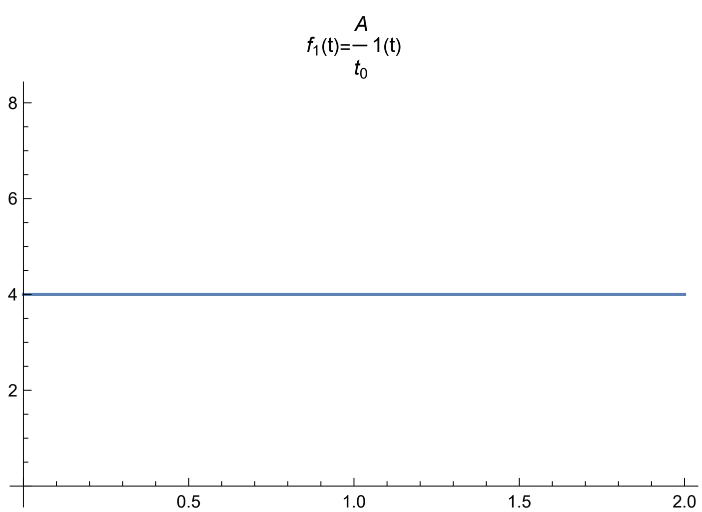
```
A = 2;
t0 = 0.5;
f[t_] = If[t < t0,
   (*value if true*)
   A / t0,

   (*value if false*)
   0
  ];

Plot[f[t], {t, 0, 2}]
```

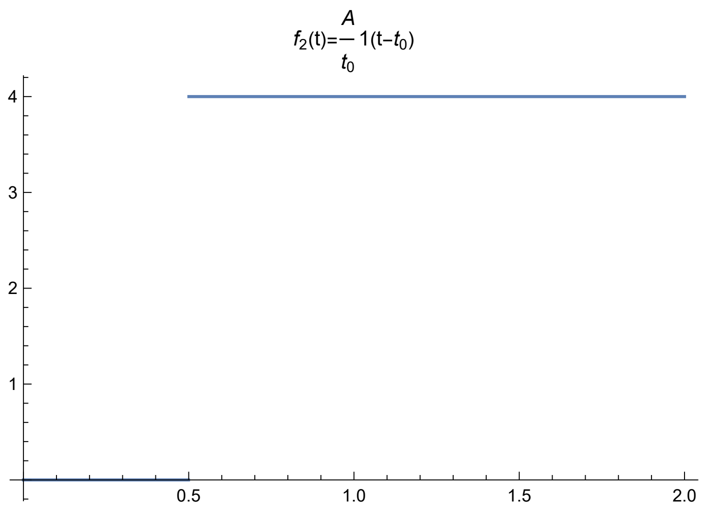


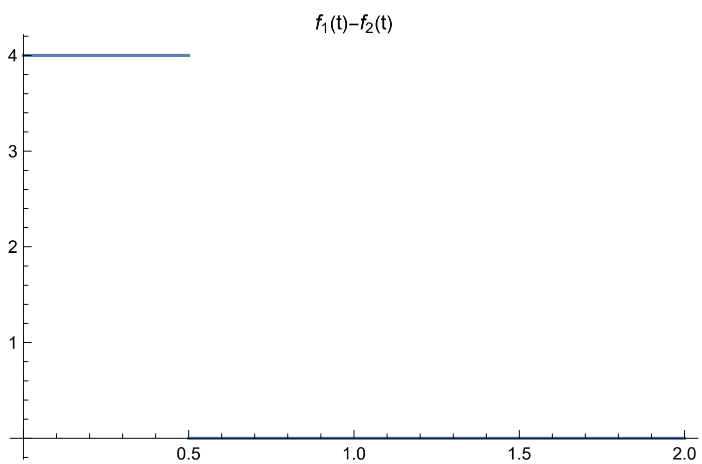
We can look at this as the superposition of two step functions.

$$f(t) = \frac{A}{t_0}\,\mathbf{1}(t) - \frac{A}{t_0}\,\mathbf{1}(t - t_0)$$

```
A = 2;
t0 = 0.5;
f1[t_] = A/t0;
f2[t_] = If[t > t0,
   (*value if true*)
   A / t0,

   (*value if false*)
   0
  ];

Plot[f1[t], {t, 0, 2}, PlotLabel → "f1(t) = A/t0 1(t)"]
Plot[f2[t], {t, 0, 2}, PlotLabel → "f2(t) = A/t0 1(t-t0)"]
Plot[f1[t] - f2[t], {t, 0, 2}, PlotLabel → "f1(t)-f2(t)"]
```







So applying the Laplace transform

$$L[f(t)] = L\left[\frac{A}{t_0}\,\mathbf{1}\,(t)\right] - L\left[\frac{A}{t_0}\,\mathbf{1}\,(t-t_0)\right] \qquad \text{recall: we already did a step function}$$

$$= \frac{A}{t_0\,s} - e^{-t_0\,s}\,\frac{A}{t_0\,s}$$

$$L[f\,(t)] \;=\; \frac{A}{t_0\,s}\left(1 - e^{-t_0\,s}\right)$$

Impulse Function

$$f(t) = \begin{cases} \lim_{t_0 \to 0} A/t_0 & t \in (0,\,t_0) \\ 0 & t \in (-\infty,\,0),\;\; t \in (t_0,\,\infty) \end{cases}$$

where $\quad A,\,t_0$ are constants

Note that this is a special case of a pulse function where $t_0 \to 0$. In this case, the area under the impulse is equal to A. From the Laplace transform of a pulse function, we have

$$L[f(t)] = \lim_{t_0 \to 0}\left[\frac{A}{t_0\, s}\left(1 - e^{-s\, t_0}\right)\right]$$

$$= \lim_{t_0 \to 0}\left[\frac{g(t_0)}{h(t_0)}\right]$$

where $\quad g(t_0) = A\left(1 - e^{-s\, t_0}\right)$

$h(t_0) = t_0\, s$

Note that $g(t_0) \to 0$ and $h(t_0) \to 0$ as $t_0 \to 0$, so we have an indeterminate form of type $\frac{0}{0}$.

We can apply l'Hopital' s rule to evaluate this limit. Recall the rule states if we have an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{t \to a}\left[\frac{g(t)}{h(t)}\right] = \lim_{t \to a}\left[\frac{g'(t)}{h'(t)}\right]$$

Therefore

$$L[f(t)] = \lim_{t_0 \to 0}\left[\frac{\frac{d}{d t_0}\left[A\left(1 - e^{-s\, t_0}\right)\right]}{\frac{d}{d t_0}[t_0\, s]}\right]$$

$$= \lim_{t_0 \to 0}\left[\frac{A\, s\, e^{-s\, t_0}}{s}\right]$$

$$= \frac{A\, s}{s}$$

$$L[f(t)] = A$$

Note that an impulse that has an infinite magnitude and zero duration is not physically realizable, it is simply a mathematical construct that will prove useful later in theoretical analysis. However, a good approximation of this is to apply a large input over a short time (ie hitting something with a hammer)

Comments on the Lower Limit of Integration

There are times when $f(t)$ contains an impulse function at time 0. In this case, we must be careful to denote the lower limit of integration

$$L_+[f(t)] = \int_{0+}^{\infty} f(t)\, e^{-s\, t}\, dt \qquad \text{(from the right)}$$

$$L_-[f(t)] = \int_{0-}^{\infty} f(t)\, e^{-s\, t}\, dt \qquad \text{(from the left)}$$

Note that we have

$$\int_{0-}^{\infty} f(t)\, e^{-s\, t}\, dt = \int_{0-}^{0+} f(t)\, e^{-s\, t}\, dt + \int_{0+}^{\infty} f(t)\, e^{-s\, t}\, dt$$

If the function has an impulse at 0, then

$$\int_{0-}^{0+} f(t)\, e^{-s\,t}\, dt \neq 0$$

So we see that

$$L_+[f(t)] \neq L_-[f(t)] \qquad \text{if function has impulse at } t = 0$$

Multiplication of $f(t)$ by $e^{-\alpha t}$

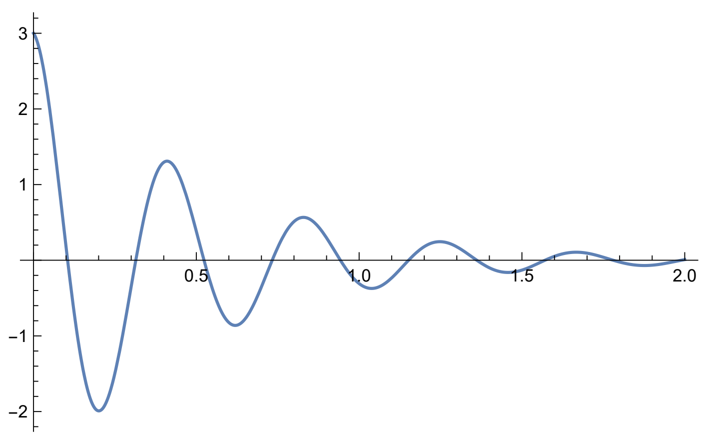
This is useful to model decaying systems (ie an underdamped mass/spring/damper looks like a sin wave with decaying amplitude).

Example: Decaying Cosine Wave

$$f(t) = e^{-\alpha t}(A\cos(\omega\, t))$$

where $A = 3$
$\omega = 15$
$\alpha = 2$

```
Plot[Exp[- α t] (A Cos[ω t]) /. {A → 3, ω → 15, α → 2}, {t, 0, 2}, PlotRange → All]
```



Recall the definition of the Laplace transform was

$$F(s) = L[f(t)] = \int_0^\infty f(t)\, e^{-s\,t}\, dt$$

It directly follows that

$$F(s+\alpha) = \int_0^\infty f(t)\, e^{-(s+\alpha)\,t}\, dt \qquad \textbf{\textit{(Eq.1)}}$$

Using an independent line of reasoning, if we write out the definition of the Laplace transform of the function $e^{-\alpha t} f(t)$ we have

$$L\left[e^{-\alpha t} f(t)\right] = \int_0^\infty e^{-\alpha t} f(t)\, e^{-s\,t}\, dt$$

$$L\left[e^{-\alpha t} f(t)\right] = \int_0^\infty f(t)\, e^{-(s+\alpha)\,t}\, dt \qquad \textbf{\textit{(Eq.2)}}$$

Comparing Eq.1 and Eq.2, we see that

$$L\left[e^{-\alpha t} f(t)\right] = F(s+\alpha) \qquad \textbf{\textit{(Eq.3)}}$$

We see that there are no restrictions on α, it can be positive, negative, real, or complex.

So returning to our example of $f(t) = e^{-\alpha t}(A\cos(\omega t))$, we recall from previous experience that the Laplace transform of $A\cos(\omega t)$ is simply

$$L[A\cos(\omega \mathrm{t})] = \frac{A s}{s^2+\omega^2}$$

Therefore via Eq.3, if we want to $e^{-\alpha t}(A\cos(\omega t))$, we simply write

$$L\left[e^{-\alpha t}(A\cos(\omega \mathrm{t}))\right] = \frac{A(s+\alpha)}{(s+\alpha)^2+\omega^2}$$

```
result = A (s + α) / ((s + α)^2 + ω^2);
check = LaplaceTransform[Exp[-α t] A Cos[ω t], t, s];
result == check
```

```
True
```

Differentiation Theorem

The Differentiation Theorem is perhaps the most important Laplace transform relationship when studying differential equation. To derive this relationship, we examine the definition of the Laplace integral

$$L[f(t)] = \int_0^\infty f(t)\, e^{-s t}\, dt$$

Recall differentiation by parts states

$$\int_a^b u\, dv = u\, v \,\Big|_a^b - \int_a^b v\, du$$

Let us use the following substitutions

$$u = f(t) \qquad \Rightarrow \qquad du = f'(t)\, dt$$
$$dv = e^{-s t}\, dt \qquad \Rightarrow \qquad v = \frac{-1}{s} e^{-s t}$$

$$\int_0^\infty f(t)\, e^{-s t}\, dt = \frac{-f(t)}{s} e^{-s t} \,\Big|_0^\infty + \int_0^\infty \frac{1}{s} e^{-s t} f'(t)\, dt$$

$$= \frac{-f(t)}{s} e^{-s t} \,\Big|_{t=\infty} + \frac{f(t)}{s} e^{-s t} \,\Big|_{t=0} + \int_0^\infty \frac{1}{s} e^{-s t} f'(t)\, dt$$

Assuming that $e^{-s t}$ shrinks faster than $-f(t)$ (ie $f(t)$ is of exponential order), first term goes to 0

$$= \frac{f(0)}{s} + \frac{1}{s} \int_0^\infty f'(t)\, e^{-s\,t}\, dt \qquad \text{note: } \int_0^\infty f'(t)\, e^{-s\,t}\, dt \triangleq L[f'(t)]$$

$$L[f(t)] = \frac{f(0)}{s} + \frac{1}{s} L[f'(t)]$$

$$L[f'(t)] = s\, L[f(t)] - f(0)$$

$$L[f'(t)] = s\, F(s) - f(0)$$

where $f'(t)$ = derivative of $f(t)$ w.r.t. t
$f(0)$ = initial condition of f in the time domain

We can repeat this process for a second derivative. We can define $g(t) = f'(t)$ so we can write

$$L[g'(t)] = s\, L[g(t)] - g(0)$$

$$L[f''(t)] = s\, L[f'(t)] - f'(0) \qquad \text{recall: } L[f'(t)] = s\, F(s) - f(0)$$

$$= s\,(s\, F(s) - f(0)) - f'(0)$$

$$L[f''(t)] = s^2\, F(s) - s\, f(0) - f'(0)$$

Repeating this for the n^{th} derivative we obtain

$$L\left[\frac{d^n}{d\,t^n} f(t)\right] = s^n\, F(s) - s^{n-1}\, f(0) - s^{n-2}\, \frac{d}{d\,t} f(0) - \ldots - \frac{d^{n-1}}{d\,t^{n-1}} f(0)$$

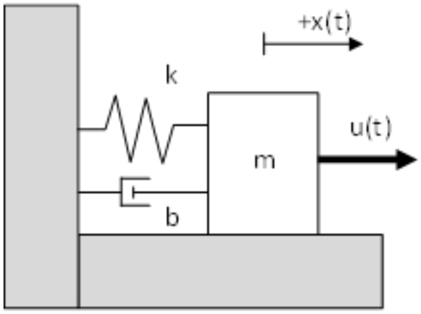
where $\frac{d^n}{d\,t^n} f(0) = \frac{d^n}{d\,t^n}[f(t)]\,|_{t=0}$

Go over how a time domain and Laplace domain representation of a function is like the light/dark world in Legend of Zelda: A Link to the Past (use SNES to show an example of how teleporting between the two worlds work or show pictures).

URL: https://www.youtube.com/watch?v=A16SJk7mJcU (fast forward to 7:00). Also search for “Legend of Zelda: A Link to the Past “Magic Mirror & Empty Bottles” [9 of 35]”

Example

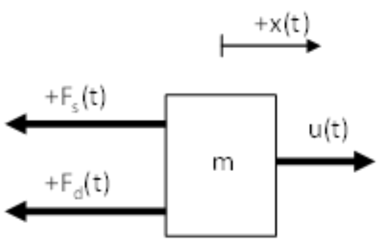
Consider a mass spring damper system shown below



Let us analyze this system. Recall the modeling procedure involves

1. Draw a schematic diagram (free body diagram) of the system and define variables.
2. Using physical laws, write equations for each component, combine them according to the system diagram to obtain a mathematical model.
3. Simulate and compare with experimental results.
4. Iterate until satisfactory.

The free body diagram is shown below



So we can write

$$\sum F(t) = m\, a(t) \qquad \text{note: } a(t) = \ddot{x}(t)$$

$$u\,(t) - F_s(t) - F_d(t) = m\,\ddot{x}(t) \qquad \text{note:} \qquad F_s(t) = k\,x(t)$$
$$F_d(t) = b\,\dot{x}(t)$$

$$u(t) - k\,x(t) - b\,\dot{x}(t) = m\,\ddot{x}(t)$$

Rewriting this yields the governing equation of motion of

$$\ddot{x}(t) + \frac{b}{m}\,\dot{x}(t) + \frac{k}{m}\,x(t) = \frac{1}{m}\,u(t)$$

Let us assume that we apply a step function to the block of unit magnitude. In other words

$$u(t) = \mathbf{1}\,(t)$$

As a side note, to visualize the dangers of applying a step function to a system, you can watch https://www.youtube.com/watch?v=JHbmIl9yQJQ (or search ‘Little girl is DEMOLISHED by dog’ on

YouTube). https://www.youtube.com/watch?v=I2-N2hkXVF0 (or search water ski dock start FAIL). Can also show video of Amara walking Gus.

So we have

$$\ddot{x}(t) + \frac{b}{m}\,\dot{x}(t) + \frac{k}{m}\,x(t) = \frac{1}{m}\,\mathbf{1}\,(t)$$

Let us apply the Laplace transform to both sides

$$L\left[\ddot{x}(t) + \frac{b}{m}\,\dot{x}(t) + \frac{k}{m}\,x(t)\right] = L\left[\frac{1}{m}\,\mathbf{1}\,(t)\right]$$

$$L\left[\ddot{x}(t)\right] + L\left[\frac{b}{m}\,\dot{x}(t)\right] + L\left[\frac{k}{m}\,x(t)\right] = L\left[\frac{1}{m}\,\mathbf{1}\,(t)\right]$$

$$L\left[\ddot{x}(t)\right] + \frac{b}{m}\,L\left[\dot{x}(t)\right] + \frac{k}{m}\,L[x(t)] = \frac{1}{m}\,L[\mathbf{1}\,(t)]$$

$$\left(s^2\,X(s) - s\,x(0) - \dot{x}(0)\right) + \frac{b}{m}\,(s\,X(s) - x(0)) + \frac{k}{m}\,X(s) = \frac{1}{m}\left(\frac{1}{s}\right)$$

$$s^2\,X(s) - s\,x(0) - \dot{x}(0) + \frac{b\,s}{m}\,X(s) - \frac{b}{m}\,x(0) + \frac{k}{m}\,X(s) = \frac{1}{m\,s}$$

$$\left(s^2 + \frac{b}{m}\,s + \frac{k}{m}\right)X(s) = \frac{1}{m\,s} + s\,x(0) + \dot{x}(0) + \frac{b}{m}\,x(0)$$

$$X(s) = \frac{\frac{1}{m\,s} + s\,x(0) + \dot{x}(0) + \frac{b}{m}\,x(0)}{s^2 + \frac{b}{m}\,s + \frac{k}{m}}$$

For example, if we started with block with initial conditions of

$$x(0) = -2\text{ m}$$
$$\dot{x}(0) = 3\text{ m/s}$$

We would have

$$X(s) = \frac{\frac{1}{m\,s} - 2\,s + 3 - \frac{2\,b}{m}}{s^2 + \frac{b}{m}\,s + \frac{k}{m}}$$

Recall the Laplace method for solving ODEs is given as:

1. Start with differential equation $\dot{x}(t) = f(x,\,u,\,t)$
2. Given input $u(t)$
3. Apply Laplace transform to system and input to obtain $X(s)$ and $U(s)$
4. Solve solution using algebraic techniques to obtain $X(s)$
5. Apply inverse Laplace transform on $X(s)$ to obtain output $x(t)$

So following the example above

1. For our system, $\ddot{x}(t) + \frac{b}{m}\,\dot{x}(t) + \frac{k}{m}\,x(t) = \frac{1}{m}\,u(t)$
2. Input $u(t) = \mathbf{1}\,(t)$
3. Apply Laplace transform to LHS and RHS
4. $X(s) = \frac{\frac{1}{m\,s} - 2\,s + 3 - \frac{2\,b}{m}}{s^2 + \frac{b}{m}\,s + \frac{k}{m}}$
5. ??

At this point, we could apply the final value theorem if we were interested in the final position of the block ($\lim_{t\to\infty} x(t)$), but how do we get the signal $X(s)$ back into the time domain where it is useful?

Many common Laplace transforms and properties are tabulated in various textbooks and online resources

Laplace Transform Table

$f(t)$		$F(s)$
Unit impulse $\delta(t)$		1
Unit step $\mathbf{1}(t)$		$\frac{1}{s}$
t		$\frac{1}{s^2}$
$\frac{t^{n-1}}{(n-1)!}$	$n = 1, 2, 3, \ldots$	$\frac{1}{s^n}$
t^n		$\frac{n!}{s^{n+1}}$
$e^{-a t}$		$\frac{1}{s+a}$
$t e^{-a t}$		$\frac{1}{(s+a)^2}$
$\frac{1}{(n-1)!} t^{n-1} e^{-a t}$	$n = 1, 2, 3, \ldots$	$\frac{1}{(s+a)^n}$
$t^n e^{-a t}$	$n = 1, 2, 3, \ldots$	$\frac{n!}{(s+a)^{n+1}}$
$\sin(\omega t)$		$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$		$\frac{s}{s^2+\omega^2}$
$\sinh(t)$		$\frac{\omega}{s^2-\omega^2}$
$\cosh(t)$		$\frac{s}{s^2-\omega^2}$
$e^{-a t}\sin(\omega t)$		$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-a t}\cos(\omega t)$		$\frac{s+a}{(s+a)^2+\omega^2}$
$1 - \cos(\omega t)$		$\frac{\omega^2}{s\left(s^2+\omega^2\right)}$
$\omega t - \sin(\omega t)$		$\frac{\omega^3}{s^2\left(s^2+\omega^2\right)}$

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Properties of Laplace Transforms

$L[A\,f(t)] = A\,F(s)$

$L[f_1(t) \pm f_2(t)] = F_1(s) \pm F_2(s)$

$L_\pm\left[\frac{d}{dt} f(t)\right] = s\,F(s) - f(0\,\pm)$

$L_\pm\left[\frac{d^2}{dt^2} f(t)\right] = s^2\,F(s) - s\,f(0\,\pm) - \dot{f}(0\,\pm)$

$L_\pm\left[\frac{d^n}{dt^n} f(t)\right] = s^n\,F(s) - \sum_{k=1}^{n} s^{n-k}\,f^{(k-1)}(0\,\pm)$ where $f^{(k-1)}(t) = \frac{d^{k-1}}{dt^{k-1}} f(t)$

$L_\pm\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \frac{\left[\int f(t)\,dt\right]_{t=0\pm}}{s}$

$L\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}$

$L\left[e^{-a\,t}\,f(t)\right] = F(s+a)$

$L[f(t-\alpha)\,1\,(t-\alpha)] = e^{-\alpha\,s}\,F(s) \quad \alpha \geq 0$

$L[t\,f(t)] = -\frac{d\,F(s)}{ds}$

$L\left[t^2\,f(t)\right] = \frac{d^2}{ds^2}\,F(s)$

$L\left[f\left(\frac{t}{a}\right)\right] = a\,F(a\,s)$