Techniques

First Principles:

- Used to find gradient function $\frac{dy}{dx} = \lim_{\delta x \to 0} \frac{\delta y}{\delta x}$
- $\lim_{x \to a} f(x) = L \neq f(a)$, unless it's a continuous function
- Gradient = $\frac{f(x+\delta x)-f(x)}{x+\delta x - x}$ (means you're finding gradient at x)

Example 1:

Parametric equations of a curve are $x = at$, $y = at^2$, where a is a constant. The points P (ap, ap^2) and Q (aq, aq^2) lies on the curve. Find and simplify an expression in terms of p and q, for the gradient of chord Q. Deduce from your expression that the gradient of the tangent of the curve at P is 2p.

Gradient of PQ = $\frac{aq^2 - ap^2}{aq - ap}$

By first principles, Q → P, so q → p.

Gradient PQ → Gradient of tangent at P.

q + p → p + p = 2p (shown)

Graph sketching:

y = f(x)	y = f'(x)
y = f(x) is **strictly increasing**	y = f'(x) is **above** x -axis
y = f(x) is **strictly decreasing**	y = f'(x) is **below** y -axis
y = f(x) has a **stationary point** at x = a	y = f'(x) = 0 at x = a (**x-intercept**)
y = f(x) **changes in curvature** at x = a (concave upward to concave downward and vice versa)	**Stationary point** of y = f'(x) at x = a
y = f(x) has a **vertical asymptote** at x = a	y = f'(x) has a **vertical asymptote** at x = a
y = f(x) has a **horizontal asymptote** at **y =a**	y = f'(x) has a **horizontal asymptote** at **y = 0**

Differentiation rules:

Constant Rule	$\frac{d}{dx}[c] = 0$
Power Rule	$\frac{d}{dx}x^n = nx^{n-1}$
Product Rule	$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$
Quotient Rule	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$
Chain Rule	$\frac{d}{dx}[f(g(x))] = f'(g(x))g'(x)$

y	$\frac{dy}{dx}$	y	$\frac{dy}{dx}$
$sinx$	$cosx$	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$, $\lvert x \rvert < 1$ (MF 26)
$cosx$	$-sinx$	$cos^{-1}x$	$-\frac{1}{\sqrt{1-x^2}}$, $\lvert x \rvert < 1$ (MF 26)
$tanx$	$sec^2 x$	$tan^{-1}x$	$\frac{1}{1+x^2}$, $x \epsilon R$ (MF 26)
$cotx$	$-cosec^2x$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2-x^2}}, \lvert x \rvert < a$
$cosecx$	$-cosec^2xcotx$	$cos^{-1}\frac{x}{a}$	$-\frac{1}{\sqrt{a^2-x^2}}, \lvert x \rvert < a$
$secx$	$secxtanx\ (MF26)$	$tan^{-1}\frac{x}{a}$	$\frac{a}{a^2+x^2}$

y	$\frac{dy}{dx}$	y	$\frac{dy}{dx}$
$lnf(x)$	$\frac{f'(x)}{f(x)}$	$e^{f(x)}$	$e^{f(x)}$
$\log_a f(x)$	$\frac{f'(x)}{f(x)}\ log_a e$	$a^{f(x)}$	$f'(x)a^{f(x)}lna$

Implicit Differentiation (IMPT):

- $\frac{d}{dx}f(y) = \frac{d}{dy}f(y) \times \frac{dy}{dx}$
- $y=x^3+x^2+x+c$ → dy/dx= $3x^2+2x+1$. H/w, if y^2=x3+x^2+x+c → 2(dy/dx)= $3x^2+2x+1$ incorrect → <u>2**y**(dy/dx)=$3x^2+2x+1$ correct</u>

Parametric Differentiation:

- $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

- Don't confuse reciprocal, inverse and differential e.g. $\sin^{-1}(x+y) \neq \frac{1}{\sin(x+y)}$
- Other than LD, can sub values to see what values a graph tends towards, as long as results are not $\frac{0}{0}$ or $\frac{\infty}{\infty}$

Other trigonometric results:

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

Reciprocal identities :

$\csc x = \dfrac{1}{\sin x}$

$\sec x = \dfrac{1}{\cos x}$

$\cot x = \dfrac{1}{\tan x}$

Even - odd identities :

$\sin(-x) = -\sin x$

$\cos(-x) = \cos x$

$\tan(-x) = -\tan x$

Product to sum formulas :

$\sin x \cdot \sin y = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$

$\cos x \cdot \cos y = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right]$

$\sin x \cdot \cos y = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$

$\cos x \cdot \sin y = \frac{1}{2}\left[\sin(x+y) - \sin(x-y)\right]$

Sum to product :

$\sin x \pm \sin y = 2\sin\left(\dfrac{x \pm y}{2}\right)\cos\left(\dfrac{x \mp y}{2}\right)$

$\cos x + \cos y = 2\cos\left(\dfrac{x+y}{2}\right)\cos\left(\dfrac{x-y}{2}\right)$

$\cos x - \cos y = -2\sin\left(\dfrac{x+y}{2}\right)\sin\left(\dfrac{x-y}{2}\right)$

Double - angle formulas :

$\sin 2\theta = 2 \cdot \sin\theta\cos\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta = 2\cos^2\theta - 1$

$\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$

$\cos\left(\dfrac{\pi}{2} - x\right) = \sin x$

$\sin\left(\dfrac{\pi}{2} - x\right) = \cos x$

$\tan\left(\dfrac{\pi}{2} - x\right) = \cot x$

$\cot\left(\dfrac{\pi}{2} - x\right) = \tan x$

$\csc\left(\dfrac{\pi}{2} - x\right) = \sec x$

$\sec\left(\dfrac{\pi}{2} - x\right) = \csc x$

$\sin(x \pm 2\pi) = \sin x$

$\cos(x \pm 2\pi) = \cos x$

$\tan(x \pm \pi) = \tan x$

$\cot(x \pm \pi) = \cot x$

$\sec(x \pm 2\pi) = \sec x$

$\csc(x \pm 2\pi) = \csc x$

Sum and difference formulas :

$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$

$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$

$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$

Half - angle formulas :

$\sin\left(\dfrac{x}{2}\right) = \pm\sqrt{\dfrac{1-\cos x}{2}}$

$\cos\left(\dfrac{x}{2}\right) = \pm\sqrt{\dfrac{1+\cos x}{2}}$

$\tan\left(\dfrac{x}{2}\right) = \dfrac{(1-\cos x)}{\sin x}$

Law of sines :

$\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Law of cosines :

$a^2 = b^2 + c^2 - 2bc\cos A$

$A = \cos^{-1}\left(\dfrac{b^2 + c^2 - a^2}{2bc}\right)$

Area of triangle :

$\frac{1}{2}ab\sin C$

$\sqrt{s(s-a)(s-b)(s-c)},$

- Give reasons to support rejecting values
- If question asks you in words, answer with word statement
- Can use GC solve if question does not require exact
- Trigonometric function of power >1 can be directly differentiated with power rule. It is integration, where you can’t integrate directly.

Tangent and Normal

Perpendicular lines: $\boldsymbol{m_1 m_2 = -1}$
Equation of Tangent: $\boldsymbol{y - b = f'(x)(x - a)}$
Equation of Normal: $\boldsymbol{y - b = -\frac{1}{f'(x)}(x - a)}$, where gradient of curve at a point $f'(x) = \frac{dy}{dx}\Big|_{x=\_}$

Vertical lines: **gradient = ∞, x= k form**
Horizontal lines: **gradient = 0**, **y= k form**

Obtaining Cartesian from Parametric Equation:

- Means to remove parameter to express in x and y only
- Not always necessary to find cartesian. Can solve with parametric also.
- Be careful of Tstep (0.1/ 0.01) and domain (Tmin, Tmax)
- **Basic angle triangle** could be useful when manipulating parametric, especially for rate of change questions

- Techniques:
  1. Simultaneous equation (elimination/ substitution)
     - $x = \ln t, y = t + \frac{1}{t} - 2$: Then $t = e^x$ so, $y = e^x + e^{-x} - 2$.
     - Curve sketched with p varies for point $\left(p - \frac{1}{p}, p + \frac{1}{p}\right)$: Let $x = p - \frac{1}{p}$ and $y = p + \frac{1}{p}$. Now, $p = \frac{x+y}{2}$ so $y = \frac{x+y}{2} + \frac{2}{x+y}$.
  2. Trigonometric identities
     $1 + \tan^2\theta = \sec^2\theta$, so $1 + (x+4)^2 = \left(\frac{y}{3}\right)^2$.

- Example 2:

2018/MJC/JC1/Promo/6

A curve C has parametric equations

$$x = 2t + 1, \quad y = \frac{4}{t}.$$

(i) Show that the equation of the normal to C at the point M with coordinates $(3, 4)$ is $2y = x + 5$. [3]

(ii) The normal at M meets the curve again at the point N. Find the coordinates of the point N. [3]

(iii) The tangent at the point $P\left(2p+1, \frac{4}{p}\right)$ on C meets the x- and y- axes at the points Q and R respectively. Find the area of the triangle OQR in terms of p. [5]

(i) $x = 2t + 1$

$y = \frac{4}{t}$

$\frac{dy}{dt} = -\frac{4}{t^2}$

$\frac{dx}{dt} = 2$

$\frac{dy}{dx} = -\frac{4}{t^2} \times \frac{1}{2} = -\frac{2}{t^2}$

$\left.\frac{dy}{dx}\right|_{x=3} = -\frac{2}{1^2} = -2$

Gradient of normal at $x = 3$: $= \frac{-1}{-2} = \frac{1}{2}$

Eqn of normal to C at M:

$y - 4 = \frac{1}{2}(x - 3)$

$y = \frac{1}{2}x + \frac{5}{2}$

$2y = x + 5$ (shown)

(ii) When normal cuts C,

$2\left(\frac{4}{t}\right) = (2t + 1) + 5$

$8 = 2t^2 + 6t$

$t^2 + 3t - 4 = 0$

$(t - 1)(t + 4) = 0$

$t = 1$ or $t = -4$

When $t = -4$,

$x = 2(-4) + 1 = -7$

$y = \frac{4}{-4} = -1$

$N(-7, -1)$

(iii) When $t = p$,

$x = 2p + 1$

$y = \frac{4}{p}$

$\frac{dy}{dx} = -\frac{2}{p^2}$

Tangent: $y - \frac{4}{p} = -\frac{2}{p^2}(x - (2p + 1))$

When $y = 0$, $x = 2p + 2p + 1$

$x = 4p + 1$

So Q is at $(4p + 1, 0)$

When $x = 0$, $y = -\frac{2}{p^2}(-2p + 1) + \frac{4}{p}$

$= \frac{8}{p} + \frac{2}{p^2}$

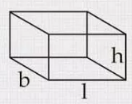
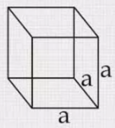
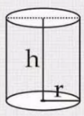
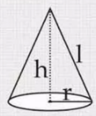
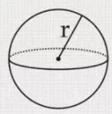
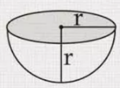
So P is at $(0, \frac{8}{p}, \frac{2}{p^2})$

Area OQR $= \frac{1}{2}(8 + \frac{2}{p^2})(4p + 1)$

$= \left(\frac{4p + 1}{p}\right)^2$ units^2

Rate of change

- Chain rule: $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$
- Rate of change is -ve for decreasing rate, but rate of decrease is +ve for decreasing rate
- Be familiar with area and volume formulas
- Be careful of **units** of final answer

Name of the solid	Figure	Volume	Laterial/Curved Surface Area	Total Surface Area
Cuboid		lbh	$2lh + 2bh$ or $2h(l+b)$	$2lh+2bh+2lb$ or $2(lh+bh+lb)$
Cube		a^3	$4a^2$	$4a^2+2a^2$ or $6a^2$
Right circular cylinder		$\pi r^2 h$	$2\pi rh$	$2\pi rh + 2\pi r^2$ or $2\pi r(h+r)$
Right circular cone		$\frac{1}{3}\pi r^2 h$	πrl	$\pi rl + \pi r^2$ or $\pi r(l+r)$
Sphere		$\frac{4}{3}\pi r^3$	$4\pi r^2$	$4\pi r^2$
Hemisphere		$\frac{2}{3}\pi r^3$	$2\pi r^2$	$2\pi r^2+\pi r^2$ or $3\pi r^2$

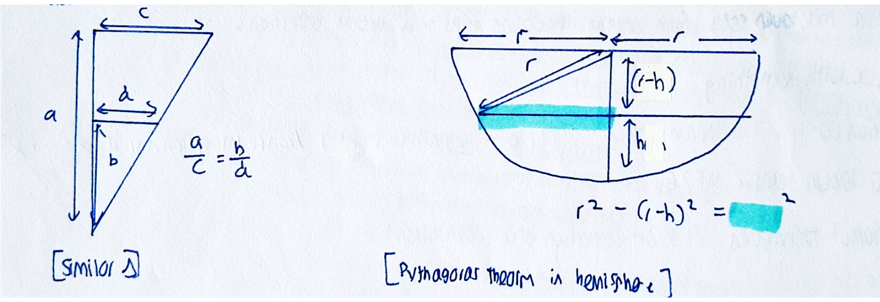
Maximum minimum problems

- Steps:
  1. Express quantity to be minimized or maximized as a function of **one variable**
  2. **Differentiate and obtain stationary value** of the variable
  3. **Check maximum and minimum** with first derivative test, $\frac{dy}{dx}$ or second derivative test, $\frac{d^2y}{dx^2}$

$\frac{d^2y}{dx^2}$	< 0	$= 0$	> 0
Result	x= ____ **gives a maximum/** is a maximum point.	Inconclusive, use first derivative test	x= ____ **gives a minimum/** is a minimum point.

N.B. May not have to sub values to prove sign of $\frac{d^2y}{dx^2}$ if you can do it otherwise e.g. $\frac{d^2y}{dx^2} = 4 + \frac{27000}{x^3}$, since x > 0, $\frac{d^2y}{dx^2} > 0$

- Look out for:



- If required to prove similar triangles: 2 pairs of corresponding angles equal/ 2 ratios of corresponding sides and included angle equal/ 3 ratios of corresponding sides equa

MacLaurin Series

Maclaurin expansion:

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^n}{n!}f^{(n)}(0) + \ldots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x|<1)$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots \qquad (\text{all } x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^r x^{2r+1}}{(2r+1)!} + \ldots \qquad (\text{all } x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^r x^{2r}}{(2r)!} + \ldots \qquad (\text{all } x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{r+1} x^r}{r} + \ldots \qquad (-1 < x \le 1)$$

Typical question structure:

(1) Show implicit differentiation

- **Differentiate** as many times as required by question
- Higher order derivatives can be expressed as lower order derivatives

(2) Apply MacLaurin Series

- Sub **x = 0** to find the desired derivative value
- Some series do not have Maclaurin series because they cannot be evaluated at x=0 e.g. y = $-\frac{1}{x}$

(3) Solve another problem**

(a) Type 1 (Link Maclaurin to a series expansion)

- Refer to <u>example 3;</u> Question could want you to find relate 2 series expansions and solve for constants a and b

(b) Type 2 (Finding another series using the series you have found)

i. **Substitute x with something**
   Refer to <u>example 4</u>; replacing "x" with "-x"

ii. **Differentiation**
   - Refer to <u>example 5</u>; power decrease
   - Can be integration if powers increase

iii. **Combine with known series in MF26**

iv. **Combination of Techniques**

- MacLaurin series is only valid for a narrow and small range of x, if you are doing for series of large x, need to have a series of many powers.

- Verifying maclaurin series:
  1. Use **standard series** (MF 26) to see if series you found tallies
  2. **Sub x = 0.01 into Maclaurin series** and use calculator to key in the same x value into the initial expression given to perform implicit differentiation to see if the calculated values tally
  3. **Use GC**: Maclaurin graph should not deviate too much from actual graph within zoom box

Example 3:

It is given that $y = -\sec 2x$

(i) Show that $\frac{d^2y}{dx^2} = -4$ when $x = 0$ (3)

$y = -\sec 2x$

$\frac{dy}{dx} = -2\sec 2x \tan 2x$

$\frac{d^2y}{dx^2} = -4\sec 2x \sec^2 2x \quad -4 \tan 2x \sec 2x \tan 2x$

$= -4(\sec^3(2x)) - \sec(2x)\tan^2(2x)$

When $x = 0$,

$\frac{d^2y}{dx^2} = -4(\sec^3 0 - \sec 0 \tan^2 0) = -4(1-0) = -4$ (shown) //

(ii) Find the first 2 non zero terms in the maclaurin series for y. (2)

When $x = 0$,

$y = -1$

$\frac{dy}{dx} = -2(0)$

$= 0$

$\therefore y = -1 - 2x^2 + \ldots$ //

(iii) It is given further that the first 2 non-zero terms in the series expansion of $(a + bx^2)^{\frac{1}{3}}$ where a and b are constants. Find the value of a and b [3].

$(a + bx^2)^{\frac{1}{3}} = a^{\frac{1}{3}}\left(1 + \frac{bx^2}{a}\right)^{\frac{1}{3}}$

$= a^{\frac{1}{3}}\left(1 + \frac{\left(\frac{1}{3}\right)}{1!}\left(\frac{bx^2}{a}\right) + \ldots\right)$

$= a^{\frac{1}{3}} + \frac{a^{\frac{1}{3}}bx^2}{3a} + \ldots$

$a^{\frac{1}{3}} = -1$

$\frac{a^{\frac{1}{3}}b}{3a} = -2$

$a = -1$

$b = 6$

Binomial series is quite commonly applied. See other possible applications below.

$y = \frac{\ln(1+3x)}{1-x}$

$= \ln(1+3x)(1-x)^{-1}$

$= \left[\left(\frac{3x}{1}\right) - \frac{(3x)^2}{2} + \frac{(3x)^3}{3} + \ldots\right]\left[1 + x + x^2 + x^3 + \ldots\right]$

$e^{-\sin^{-1}(2x)} = \left[e^{\sin^{-1}(2x)}\right]^{-1}$

Example 4:

If $y = e^{\cos^{-1}x}$, where $\cos^{-1}x$ denotes the principal values show that $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0$

$$\sqrt{1-x^2}\left(\frac{dy}{dx}\right) = -e^{\cos^{-1}x} = -y$$

Do reexpress/ simplify expressions as you go along differentiating, to make further differentiation more manageable.

$$\sqrt{1-x^2}\frac{dy}{dx} + y = 0 \quad (1)$$

Diff wrt x again,

$$\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)\frac{dy}{dx} + \sqrt{1-x^2}\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$$

Multiply $\sqrt{1-x^2}$ throughout,

$$-x\left(\frac{dy}{dx}\right) + (1-x^2)\left(\frac{d^2y}{dx^2}\right) + \sqrt{1-x^2}\frac{dy}{dx} = 0$$

$$(1-x)^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0 \text{ (shown)}$$

Diff wrt x again,

$$(1-x^2)\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2}(-2+x) - x\left(\frac{d^2y}{dx^2}\right) + \frac{dy}{dx}(1) - \frac{dy}{dx} = 0$$

When $x = 0$,

$$y = e^{\cos^{-1}0} = e^{\frac{\pi}{2}}$$

$$\frac{dy}{dx} = -e^{\frac{\pi}{2}}$$

$$\frac{d^2y}{dx^2} = e^{\frac{\pi}{2}}$$

$$\frac{d^3y}{dx^3} = -2e^{\frac{\pi}{2}}$$

By further differentiation, obtain the expansion of y in ascending powers of x up to and including the term x^3 in terms of $e^{\frac{\pi}{2}}$. Hence, show that for small values of x, where the value of k is to be found,

$$e^{\cos^{-1}x} \approx e^{\frac{\pi}{2}-x} + kx^3$$

Need to observe when you need to bring in standard series from MF26 (question may not always state explicitly)

$$e^{\frac{\pi}{2}-x} = e^{\frac{\pi}{2}}(e^{-x}) = e^{\frac{\pi}{2}}\left(1 + \frac{(-x)}{1!} + \frac{(-x)^2}{2!} + \frac{(-x)^3}{3!} + \dots\right)$$

$$= e^{\frac{\pi}{2}}\left(1 - x + \frac{1}{2}x^2 - \frac{x^3}{6} + \dots\right)$$

By Maclaurin series expansion, $y = e^{\frac{\pi}{2}}\left(1 - x - \frac{x^2}{2} - \frac{3x^3}{3!}\right)$ (up to x^3 term)

$= e^{\frac{\pi}{2}}\left(1 - x + \frac{1}{2}x^2 - \frac{1}{3}x^3\right)$ from previous part

So $y = e^{\cos^{-1}x}$

$$\approx e^{\frac{\pi}{2}}\left(1 - x + \frac{x^2}{2} - \frac{x^3}{3}\right)$$

$$= e^{\frac{\pi}{2}}\left(1 - x + \frac{x^2}{2} - \frac{x^3}{6}\right) - e^{\frac{\pi}{2}}\frac{x^3}{6}$$

$$= e^{\frac{\pi}{2}-x} - \frac{e^{\frac{\pi}{2}}}{6}x^3$$

$$\therefore k = -\frac{e^{\frac{\pi}{2}}}{6}$$

Include approximation sign for the first line when you do approximation

Example 5:

It is given $\ln y = \tan^{-1}(2x)$

(i) Prove $(1+4x^2)\frac{dy}{dx} = 2y$ (1)

$$\ln y = \tan^{-1}(2x)$$

$$\left(\frac{1}{y}\right)\left(\frac{dy}{dx}\right) = \frac{1}{1+(2x)^2}\cdot(2)$$

$$(1+4x^2)\frac{dy}{dx} = 2y \text{ (shown)}$$

(ii) By further differentiation of result in (i), obtain maclaurin series of y in ascending powers of x up to and including term in x^3. (5)

$$\frac{d^2y}{dx^2} + 4x^2\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} = 2\frac{dy}{dx}$$

$$(4x^2+1)\left(\frac{d^2y}{dx^2}\right) + (8x-2)\frac{dy}{dx} = 0$$

$$\frac{d^3y}{dx^3} + 4x^2\frac{d^3y}{dx^3} + 8x\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 8x\frac{d^2y}{dx^2} - 2\frac{d^2y}{dx^2} = 0$$

$$(1+4x^2)\frac{d^3y}{dx^3} + (16x-2)\frac{d^2y}{dx^2} + 8\frac{dy}{dx} = 0$$

When $x=0$,

$y=1$

$\frac{dy}{dx} = 2$, $\frac{d^2y}{dx^2} = 4$, $\frac{d^3y}{dx^3} = -8$ $\quad\therefore y = 1+2x+2x^2-\frac{4}{3}x^3+\ldots$

(iii) Hence, find the maclaurin series of $\frac{e^{\tan^{-1}x}}{1+4x^2}$ in ascending powers of x up to and including term in x^2 (2).

$$\frac{e^{\tan^{-1}(2x)}}{1+4x^2} = \frac{y}{1+4x^2}$$

From (i), $\frac{y}{1+4x^2} = \frac{1}{2}\frac{dy}{dx}$

$$= \frac{1}{2}(2+4x-4x^2)$$

$$= 1+2x-2x^2+\ldots$$

Alternative: Binomial series + $y(1+4x^2)^{-1}$

Small angle approximation:

- Clue: "…where x is small…"
- x must be in **radians**
- Angle contained within the bracket must be small, else expand first then apply SAA
  e.g. sinx, sin2x can be considered small angle. H/w, sin $(\frac{\pi}{4} + x)$ may not be considered a small angle

$$sinx \approx x$$
$$\cos x \approx 1 - \frac{x^2}{2}$$
$$tanx \approx x$$

- Question types:

1. **Show a trigonometric expression**
   i. **Cosine rule**
   ii. **Sine rule**
   Usually if 2 angles are provided, use sine rule
   iii. **TOA CAH SOH**
   Right angle triangle

2. Given that x is sufficiently small **show another expression by applying SAA**
   Binomial series is a common technique used here; look out for power -1

Example 6:

Shown in previous part $\frac{BC}{DC} = \frac{1}{\cos\theta - \sqrt{3}\sin\theta}$. Show that $\frac{BC}{DC} \approx 1 + a\theta + b\theta^2$ (3)

For small θ, $\cos\theta \approx 1 - \frac{\theta^2}{2}$, $\sin\theta \approx \theta$

$$\frac{BC}{DC} = \frac{1}{\cos\theta - \sqrt{3}\sin\theta}$$
$$\approx \frac{1}{1 - \frac{\theta^2}{2} - \sqrt{3}\theta}$$
$$= (1 - \sqrt{3}\theta - \frac{\theta^2}{2})^{-1}$$
$$\approx 1 + (-1)(-\sqrt{3}\theta - \frac{\theta^2}{2}) + \frac{(-1)(-2)}{2!}(-\sqrt{3}\theta - \frac{\theta^2}{2})^2$$
$$= 1 + \sqrt{3}\theta + \frac{\theta^2}{2} + 3\theta^2$$
$$= 1 + \sqrt{3}\theta + \frac{7}{2}\theta^2$$

$\therefore a = \sqrt{3}, b = \frac{7}{2}$