# *Integration*

- Definite integral: no "+c", where you will find area under the graph for a interval
- Indefinite integral: '+c"
- $\int 1\, dx = x + c$

For continuous functions, the Fundamental Theorem of Calculus states that

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| If $\int f(x)dx = F(x) + c, then\ \int_a^b \boldsymbol{f(x)dx} = [\boldsymbol{F(x)}]_a^b = \boldsymbol{F(b) - F(a)}$, where $\frac{d}{dx}F(x) = f(x)$ |

Basic properties of definite and indefinite integrals

| Indefinite integrals | Definite integrals |
|---|---|
| $\int kf(x)dx = k\int f(x)dx, for\ any\ constant, k \neq 0$<br>$\int[f(x) \pm g(x)]dx = \int f(x)dx + \int g(x)dx$<br>$\int f'(x)dx = f(x) + c$ | $\int_a^a f(x)dx = 0$<br>$\int_a^b f(x)dx = -\int_b^a f(x)dx$<br>$\int_a^b kf(x)dx = k\int_a^b f(x)dx, k\ real\ constant$<br>$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx, where\ a < b < c$ |

N.B. To access definite integrals in GC, MATH 9/ ALPHA WINDOW 4.

SECTION 1.1 STANDARD AND GENERAL FORMS

| Differentiation | Integration | |
|---|---|---|
| | | *** |
| $\frac{d}{dx}x^{n+1} = (n+1)x^n$ | $\int x^n dx = \frac{x^{n+1}}{n+1} + c, \boldsymbol{n \neq -1}$ | $\int f'(x)(f(x))^n dx = \frac{(f(x))^{n+1}}{n+1} + c,\ \boldsymbol{n \neq -1}$<br>(Highly helpful and adaptable) |
| $\frac{d}{dx}(e^x) = e^x$ | $\int e^x\, dx = e^x + c$ | $\int f'(x)e^{f(x)}dx = e^{f(x)} + c$ |
| $\frac{d}{dx}(\ln x) = \frac{1}{x}$ | $\int \frac{1}{x}dx = ln\|x\| + c$ | $\int \frac{f'(x)}{f(x)}dx = ln\|f(x)\| + c$<br>(**Modulus sign** needed) |
| $\frac{d}{dx}(a^x) = a^x lna$ | $\int a^x dx = \frac{a^x}{lna} + c, a > 0$ | $\int f'(x)a^{f(x)}dx = \frac{a^{f(x)}}{lna} + c,\ a > 0$<br>(Easily forgotten) |
| $\frac{d}{dx}sinx = cosx$ | $\int cosx\, dx = sinx + c$ | $\int f'(x)\cos[f(x)]\, dx = \sin[f(x)] + c$ |
| $\frac{d}{dx}\cos x = -\sin x$ | $\int sinx\, dx = -cosx + c$ | $\int f'(x)\sin[f(x)]\, dx = -\cos[f(x)] + c$ |

| | | |
|---|---|---|
| $\frac{d}{dx}tanx = \sec^2 x$ | $\int \sec^2 x\, dx = \tan x + c$ | $\int f'(x) \sec^2[f(x)]\, dx = \tan[f(x)] + c$ |
| $\frac{d}{dx}secx = secxtanx$ | $\int secxtanx\, dx = secx + c$ | $\int f'(x)\, sec[f(x)]tan[\, f(x)]\, dx = \sec[f(x)] + c$ |
| $\frac{d}{dx}cosecx = -cosecxcotx$ | $\int cosecxcotx\, dx = -cosecx + c$ | $\int f'(x)cosec[f(x)] \cot[f(x)]\, dx = -\, cosec[f(x)] + c$ |
| $\frac{d}{dx}cotx = -cosec^2 x$ | $\int cosec^2\, x\, dx = -cotx + c$ | $\int f'(x)cosec^2\, [f(x)]dx = -\cot[f(x)] + c$ |
| $\frac{d}{dx}\sin^{-1}\frac{x}{a} = \frac{1}{\sqrt{a^2-x^2}}$ | $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\frac{x}{a} + c, \lvert x\rvert < \lvert a\rvert$ | |
| | $\int \frac{f'(x)}{\sqrt{a^2-(f(x))^2}}\, dx = \sin^{-1}\left(\frac{f(x)}{a}\right) + c, \lvert f(x)\rvert < \lvert a\rvert$ | |
| $\frac{d}{dx}\tan^{-1}\frac{x}{a} = \frac{a}{a^2+x^2}$ | $\int \frac{1}{x^2+a^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$ (MF26) | $\int\int \frac{f'(x)}{(f(x))^2+a^2}dx = \frac{1}{a}\tan^{-1}(\frac{f(x)}{a}) + c$ |
| $\frac{d}{dx}\ln(\frac{x-a}{x+a}) - \frac{2a}{x^2-a^2}$ | $\int \frac{1}{x^2-a^2}\, dx = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + c, x > a$ (MF26) | $\int \frac{f'(x)}{(f(x))^2-a^2}\, dx = \frac{1}{2a}\ln\left(\frac{f(x)-a}{f(x)+a}\right) + c, f(x) > a$ (Can be extended for $\lvert f(x)\rvert > 0$) |
| $\frac{d}{dx}\ln\left(\frac{a+x}{a-x}\right) = \frac{2a}{a^2 - x^2}$ | $\int \frac{1}{a^2-x^2} = \frac{1}{2a}ln\left(\frac{a+x}{a-x}\right) + c, \lvert x\rvert < \lvert a\rvert$ | $\int \frac{f'(x)}{a^2-(f(x))^2}\, dx = \frac{1}{2a}ln\left(\frac{a+f(x)}{a-f(x)}\right) + c, \lvert f(x)\rvert < \lvert a\rvert$ |

- Remember the "+c" and the "dx"!!
- Example to illustrate flexibility of first form: $\int sinxcos^2 x\, dx = -\int -sinx(cosx)^2 dx = \frac{\cos^3 x}{3} + c$
- May need to **re-express** the equation to obtain e.g. $f'(x)\big(f(x)\big)^n$ form for visualization.
  → Example: $\int \frac{2x^2}{\sqrt{1+x^3}}\, dx = \frac{2}{3}\int 3x^2(1+x^3)^{-0.5}$
- Don't be careless!!

[Other results from MF 26] Last page notes

$$\int tanx\, dx = \ln(secx), \lvert x\rvert < \frac{\pi}{2}$$

$$\int cotx\, dx = \ln(sinx),\ 0 < x < \pi$$

$$\int cosecx\, dx = -\ln(cosecx + cotx),\ 0 < x < \pi$$

$$\int secx = \ln(secx + tanx), \lvert x\rvert < \frac{\pi}{2}$$

SECTION 1.2 TRIGONOMETRIC FUNCTIONS

[Factor Formula, MF 26]

$$sinP + sinQ = 2\sin\left(\frac{1}{2}\right)(P+Q)\cos\left(\frac{1}{2}\right)(P-Q)$$

$$\sin P - \sin Q = 2cos(\frac{1}{2})(P+Q)\sin(\frac{1}{2})(P-Q)$$

$$\cos P + \cos Q = 2\cos\left(\frac{1}{2}\right)(P+Q)\cos\left(\frac{1}{2}\right)(P-Q)$$

$$\cos P - \cos Q = -2sin(\frac{1}{2})(P+Q)\sin(\frac{1}{2})(P-Q)$$

[Trigonometric functions raised to power n]

a. For $\int \sin^n x\, dx / \int \cos^n x\, dx$, double angle formula:

$$\int \sin^2 x\, dx = \int \frac{1}{2}(1 - cos2x)\, dx$$

$$\int \cos^2 x\, dx = \int \frac{1}{2}(1 + cos2x) dx$$

- If n is odd, <u>separate a single sinx or cosx from the original function</u>, and apply other results like $\sin^2 x + \cos^2 x = 1$. It is **likely possible to integrate x with coefficient 1**. Example: $\int \sin^5 x\, dx$, factorize to $\int sinx\,(\sin^2 x)^2\, dx = \int sinx(1 - cosx)^2 dx = \int sinx\,(1 - 2\cos^2 x + \cos^4 x) dx = -cosx + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$

- If n is even, apply results like double angle formula. It **is likely <u>not</u> possible to integrate x with coefficient 1**. Example: $\int \sin^2 x cos^2 x\, dx = \int \frac{1}{4}(1 - \cos^2 2x)\, dx = \int \frac{1}{4}\sin^2 2x = = \frac{1}{4}\int \frac{1}{2}(1 - cos4x)\, dx = \frac{1}{8}x - \frac{1}{32}sin4x + c$

b. For $\int \tan^2 x\, dx$ trigonometric identity, look out for ways to apply other results like $1 + \tan^2 x = \sec^2 x$:

$$\int \tan^2 x\, dx = \int \sec^2 x - 1\, dx$$

Takeaways:

1. General/ Standard Form
   Look out for $\int \boldsymbol{f'(x)\left(f(x)\right)^n dx}$ form and $\int \boldsymbol{\frac{f'(x)}{f(x)} dx}$ form

2. Partial Fractions
   Use when numerator and denominator are **polynomials**, and denominator is **factorisable**.

3. $\int \frac{1}{\sqrt{ax^2+bx+c}}$ or $\int \frac{1}{ax^2+bx+c}$
   **Complete the square** for $ax^2 + bx + c$, then apply standard forms **for $\sin^{-1}$, $\tan^{-1}$, ln** (Note: May have to factorize "-1" out to apply the standard forms)

4. $\int \frac{px+q}{\sqrt{ax^2+bx+c}}$ or $\int \frac{px+1}{ax^2+bx+c}$
   Difference from ②: denominator is **not factorisable**
   **Split px +q** to something like $k1 \int \frac{f\prime(x)}{f(x)}\, dx + k2 \int \frac{1}{f(x)}\, dx$ (it's alright if the quadratic in the denominator in a square root) and **complete the square** for $ax^2 + bx + c$, then apply standard forms **for $\sin^{-1}$, $\tan^{-1}$, ln**.
   To start solving, you may begin with: Let f(x) = Denominator (ignore square root if it's present). Then f'(x) = _______. Numerator = (manipulate f'(x) to be equal to numerator) e.g.
   $$(3x + 4) = \cdots = -\frac{1}{6}(-6 - 18x) + 3 = -\frac{1}{6}f'(x) + 3$$

- sin(-x) = -sinx
- Use **long division for improper fractions**.

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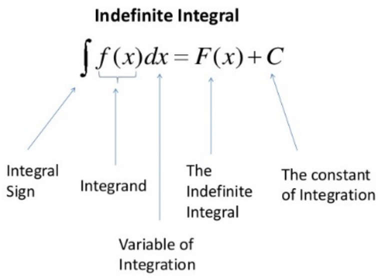


- Don't complicate things. Sometimes standard integration techniques is all you need to sovle questions.

> Tut Q13: $\int x^{n-1}e^{x^n}\,dx$ (Answer: $\frac{1}{n}e^{x^n} + c$)

- End goal is to find indefinite integral in **simplest form.**

> Example: Tut Q8e $\int \frac{x^2}{\sqrt{1-x^2}}dx$ with substitution x=sinθ, $\frac{dx}{d\theta} = cos\theta$
> $= \int \frac{sin^2\theta}{\sqrt{1-sin^2\theta}}(cos\theta)d\theta = \int \sin^2\theta\, d\theta = \int \frac{1-cos2\theta}{2}d\theta = \frac{\theta}{2} - \frac{sin2\theta}{4} + c = \frac{\sin^{-1}x}{2} - \frac{2sin\theta cos\theta}{4} + c$
> $= \frac{\sin^{-1}x}{2} - \frac{x\sqrt{1-x^2}}{2} + c$ (using basic angle triangle to substitute to original variable)
> But, we realise that we can also get an alternative answer $= \frac{\theta}{2} - \frac{\sin(2\sin^{-1}x)}{4} + c$, correct but not accepted as not simplest form.

- If there is <u>no need to put modulus sign after integration</u> **omit it**. Helps in DE topic later on.
- ln is inverse of e
- Don't be careless: don't forget your constants which you brought out of the integrand, don't forget to simplify, don't forget divide by f'(x) for inverse trigonometric function if needed

- Make use of **factorisation** to simplify integrands for integration.
- Note that although integration of an integrand may appear to yield 2 different results sometimes, they may actually be the same because their <u>+c values are different</u>.

> Example: $\int \frac{1}{2x+2}dx = \frac{1}{2}\int \frac{1}{x+1}dx = \frac{1}{2}ln\,|x+1| + c\ OR = \frac{1}{2}ln|2x+2| + c$ both are correct.

- Try to **expand** expression before integrating.

> Basics: recall $\int x^n dx = \frac{x^{n+1}}{n+1} + c$, $n \neq -1$. You must never divide by the variable x, because later on when you differentiate that expression you won't be getting back the integrand, as you will have to use quotient rule.

- When integrand is a trigonometric function of power n, need to apply things like double angle formula. Can only integrate an integrand of power n when there is f'(x) in the integrand as well.
- When differentiating inverse trigonometric functions, we have inverse trigo ($\frac{x}{a}$) forms and inverse trigo (x) forms. If it is 2x, for example, it doesn't matter whether you see it as (2x) or ($\frac{x}{\frac{1}{2}}$) .
- Upon integration, if you obtain a modulus sign for the ln of the expression, check to see if other products of integration have ln of the expression. If yes, do remember to include modulus sign for those expressions.

- Hence or otherwise questions: guiding you to integrate an integrand that has no known formula, where you must piece together integrals of different functions given by the question with appropriate +/- sign to derive answer (Q7iii manageable)

- If get stuck for 2 parts questions, where first part informs you on your second part, might want to consider reviewing your answer for the first part.

  Tut Q12: $\int cos2xcosx\ dx$. Hence find $\int 2xcos2xcosx\ dx$
  Here, we can see the first part helps us perform integration by parts in the later part. We note that we can apply double angle formula to solve the first part, or use factor formula. H/w, going by double angle formula, we will get stuck for the latter part, prompting us to use factor formula.
- Please also check your working if you get to a dead-end.

  Tut Q14b: Given that $x = 4\cos^2\theta + 7\sin^2\theta$ show that $7 - x = 3\cos^2\theta$ . Similarly, find an expression for x-4. Using substitution $x = 4\cos^2\theta + 7\sin^2\theta$, find $\int_4^7 \frac{1}{\sqrt{(x-4)(7-x)}}\ dx$. (Answer: $\pi$)

- Questions we found challenging:

Tut Q3jk: Require you to <u>guess which terms will be f'(x) which term will be f(x)</u> and <u>simplify appropriately</u> in preparation for integration
Q3j) $\int \tan^4 2x\ dx$ (Answer: $\frac{1}{6}\tan^3 2x + x - \frac{1}{2}tan2x + c$)
Q3k) $\int \sin^2 x\, cos^3 x\ \ dx$ (Answer: $\frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$)

Tut Q8f: Require to have an understanding on how to get rid of the power $\frac{3}{2}$ in the denominator
$\int \frac{1}{(4+x^2)^{\frac{3}{2}}}\ dx\ using\ x = 2tan\theta$ (Answer: $\frac{x}{4\sqrt{x^2+4}} + c$)

SECTION 1.3 SUBSTITUTION

- If you are required to execute this method, substitution will be provided by the question. Else, may want to consider looking for other simpler methods. (fyi: refer to page 18 of Integration 1 notes for table of substitution)

From Chain Rule,

$$\int f(x)\boldsymbol{dx} = \int f(x)\frac{dx}{du}\ \boldsymbol{du}$$

- Steps:
  1. Differentiate substitution provided w.r.t. x. (may need to take reciprocal to get differential w.r.t u)
  2. Replace terms in integrand. Change dx (in red) to du (in red).
  3. Integrate integrand w.r.t u. Here, you should be able to see apply Section 1.1.
  4. Replace all the terms in u to that in x.

- For integration of **definite integrals**, remember to **change the limits when performing substitution**. Upon which, you may realise that the upper limit may not be larger than the lower limit. It is okay, you're still on the right track!

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- Do not use substitution unless prompted by the question.
- Helpful way to think about substitution formula:

| What we use… | What was mentioned… |
|---|---|
| $\int f(x)\boldsymbol{dx}$ <br> $= \int f(x)\frac{dx}{du}\ \boldsymbol{du}$ <br><br> We observe that du and du will cancel. | Say for an integrand in terms of x and substitution in terms of u, we are able to find that <br> $\frac{dx}{du} = 2u$ <br> To help us re-express the integration, we think of $dx = 2udu$ (not accepted, as mathematically incorrect $\frac{dx}{du}$ is not a fraction, only a thought process) and we substitute accordingly. |

- Remember to convert all variables back to variable you were initially supposed to integrate w.r.t. to, do not leave in terms of variable used for substitution.
- If substitution involve trigonometric function, when converting variables in indefinite integral after substitution, we may need to **consider basic angle triangle** to be able to apply TOA CAH SOH. Take note of the angle in the triangle as well. For example, is it $\theta$ or $2\theta$?

SECTION 1.4 PARTS

From product rule in differentiation, we can derive a formula for integration by parts:

$$\int u\left(\frac{dv}{dx}\right)dx = uv - \int v\left(\frac{du}{dx}\right)dx$$

To solve:

- Must have **two** functions in x
  → **Need to simplify to 2 functions, if not already**.
- Note that not all integrands that are products of 2 terms need to be integrated by parts (e.g. $\int xe^{x^2}dx$ ), think of how you can break down the function further to help you solve (e.g. $\int \sec^4 x\, dx$ notes page 24 example 15ii)

- Decide on which function to integrate (i.e. be $\frac{dv}{dx}$) and which to differentiate (i.e. be u)
  In general, one of them must be directly integrable or must be differentiable but not directly integrable. Also, $\int v\left(\frac{du}{dx}\right)dx$ **must not be more complex than the original** $\int u\left(\frac{dv}{dx}\right)dx$**.**
  1. If **only one** of the two parts of the integrand is integrable, then integrate the one that is integrable. (i.e. the part is chosen to be $\frac{dv}{dx}$.
  2. If both parts of the integrand are integrable on their own, then choose the one that will **ultimately be simplified when differentiated** a sufficient number of times as the function u for differentiation.
  3. If both parts of the integrand are integrable on their own and neither simplify on repeated differentiation, then it **does not matter** which function we choose to integrate.
  4. Integrand that consists of only one part and cannot be integrated directly. Then **introduce "1" as a second part of the integral** and **integrate "1"**.

- Remember to add "+c" for indefinite integral, add near the end of your working when you remove the integrate the integrand
- ***Must be **consistent** in choosing what function you are going to differentiate and integrate, especially for integration by parts questions that require you to repeat integration by parts twice.

- Another helpful tool:

| Differentiate if closer to top | |
|---|---|
| L | Logarithm |
| I | Inverse trigonometric function |
| A | Algebra |
| T | Trigonometric function |
| E | Exponential function |
| Integrate if closer to bottom | |

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- Should we get back the term we sought to integrate at the beginning, we can **bring that term from RHS to LHS of equation** to continue solving.
- Don't be too obsessed with continuously integrating by parts. After integrating by parts once, it may be better for you to consider if **LD** will help the next round.

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| Tut Q10k: $\int xln(x+1)dx$ (Answer: $\frac{x^2}{2}\ln\lvert x+1\rvert - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}ln\lvert x+1\rvert + c$)<br>Tut Q10l: $\int x^2 \tan^{-1} x\, dx$ (Answer: $\frac{1}{3}x^3 \tan^{-1} x - \frac{1}{6}x^2 + \frac{1}{6}ln\lvert 1+x^2\rvert + c$) |

- Question that had difficulty in:

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| Actually, not very difficult, just applied concepts wrongly. Q10f: $\int cosec^3 dx$ (Answer: $-\frac{1}{2}cotxcosecx - \frac{1}{2}ln\lvert cosecx + cotx\rvert + c$). |

SECTION 1.5 INTEGRATION INVOLVING MODULUS

Applied to definite integrals.

1. Sketch graph of expression in the modulus, labelling the appropriate intercepts
2. Identify for which ranges of the expression you are to take, and whether they will be the positive of the expression or negative of the expression in different ranges.
3. Integrate accordingly after breaking up the modulus

SECTION 1.6 GENERAL POINTERS FOR AREA AND VOLUMES

- If question never provide you with sketch of graph and/or shaded area, **do draw in one yourself** for clarity of thought and visualization.
- If you can't find integrate exactly, even when question asks you to, **consider integration with respect to the other axes**.
- Concepts of finding area/ volume of revolution can easily be applied with parametric equations. Remember to change limits/ Tmin/ Tmax/ Tstep 0.01. To find area/ ~~volume~~ (not in syllabus) with parametric equations, typically **integration by substitution will come in handy**.

- APGP/MOD is relevant whenever you are applying Riemann sum: **Write out the sum of rectangular strips first, deduce pattern/ progression, simplify** (Tut Q5/6/15)
- In the context of Riemann sum, "Using integration, obtain the values of limit of Area A as n tends to infinity…" ≠ pure limits method

| |
|---|
| Example of using integration: $\int_0^1 x^2dx$, Example of using pure limits: Area A= $\frac{1}{2} + \frac{1}{2n} + \frac{1}{6n^2} = \frac{1}{3} + 0 + 0$ as n tends to infinity. |

N.B. unlikely for question to require application of Riemann sum to volume

- When you have an expression in an integration sign, it is possible to change all the variables in the expression and change the variable you are integrating with respect to, as long as you are consistent.

Example (technique to Tut Q1): $\int_0^{\frac{\Pi}{2}} \frac{siny}{cosy+siny}\,dy = \int_0^{\frac{\Pi}{2}} \frac{sinx}{cosx+sinx}\,dx$. After which, question interesting requires you to apply this solving to solve for $2\int_0^{\frac{\Pi}{2}} \frac{sinx}{cosx+sinx}\,dx$.

- Hey, don't give up finding volumes of revolution and use GC! Think about how you can change the expression to something that is integratable.

Example (Tut Q9ii): $\pi\int_1^3 (\ln(x^2-x+1))^2 dx$ is not directly integratable, but if we consider $y=\ln(x^2-x+1)$, we can apply exponential throughout and use completing the square $x=\frac{1}{2}\pm\sqrt{e^y-\frac{3}{4}}$.

GC functions:

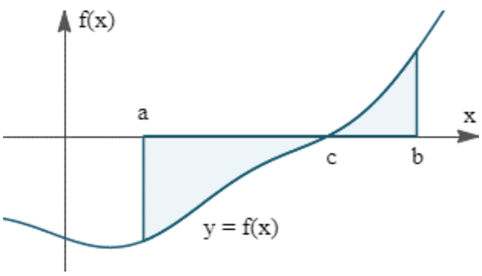
2ND INS to insert new things into expression without having to retype

0: Zoom Fit is useful when a graph is bounded within a range

## SECTION 1.7 AREA UNDER THE GRAPH

- For negative regions of graph, definite integral must have negative sign or modulus sign to keep area positive.
- Units of area should be given as $\boldsymbol{units^2}$

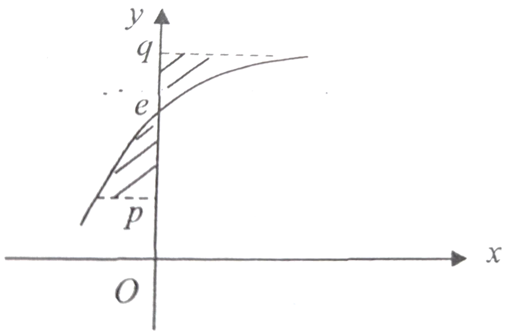
Area about x-axis




$$\text{Area}= -\int_a^c y\,dx + \int_c^b y\,dx \text{ OR } \left|\int_a^c y\,dx\right| + \int_c^b y\,dx$$

- For parametric curve, x= h(t) and y= g(t),
  Area= $\int_a^b y\,dx = \int_{t1}^{t2} g(t)\left(\frac{dx}{dt}\right)dt = \int_{t1}^{t2} g(t)h'(t)\,dt$

Area about y-axis



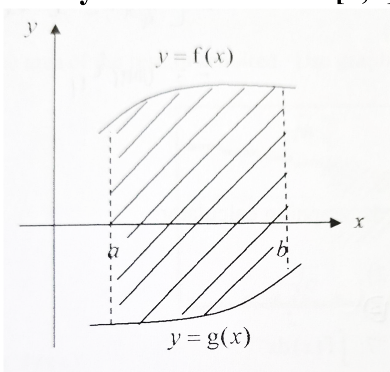

$$\text{Area} = -\int_p^e x\, dy + \int_e^q x\, dy \text{ OR } \left|\int_p^e x\, dy\right| + \int_e^q x\, dy$$

- For parametric curve, x= h(t) and y= g(t),
  Area= $\int_p^q x\, dy = \int_{t1}^{t2} h(t)\left(\frac{dy}{dt}\right) dt = \int_{t1}^{t2} h(t)g'(t)dt$

Area bounded by 2 curves

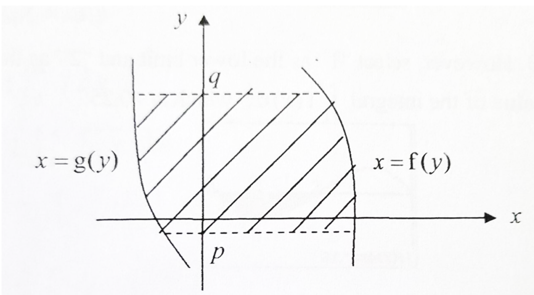
1. Sketch the curve
2. Find points of intersection of the curve
3. Determine area of enclosed area

- Integrating w.r.t. x-axis (**necessary to ensure that in [a,b]** $\boldsymbol{f(x) \geq g(x)}$)




$$Area = \int_a^b [f(x) - g(x)]dx$$

- Integrating w.r.t. y-axis




$$Area = \int_p^q [f(y) - g(y)]dx$$

Applied to solving problems

- If areas in positive and negative regions are symmetrical, you can integrate the positive area and multiply by 2 to obtain total area under curve of interest.

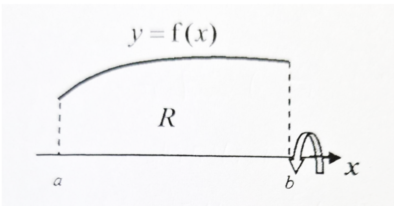
- To find the limit that an area tends to, MATH 0 to call out summation command (use APLHA X to call out variables other than X), fill in inputs in summation command, 2nd WINDOW to change table set up, 2nd GRAPH to view table, observe limit from table.

- If question does not require you to find exact area under curve, can use **GC**. Procedure: Plot graph, 2nd TRACE 7, input upper and lower limits, show appropriate workings.

- Tip: When plotting a curve of $f(x) = y^2$, you can input positive and negative square root to obtain plot easily. Alternatively, can write positive square root as y1, and negative APLHA TRACE y1 in y2.

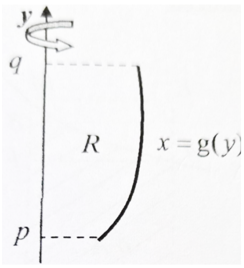
SECTION 1.8 VOLUMES OF REVOLUTION

- Units of area should be given as $\boldsymbol{unit^3}$ unless applied to a specific context like volume of wine cask
- Volume of revolution about x-axis



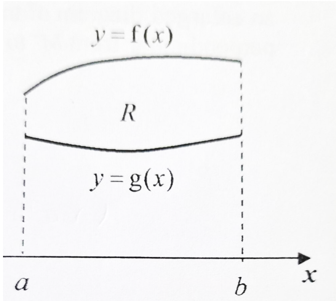

$$Volume = \pi \int_a^b y^2 dx$$

- Volume of revolution about y-axis




$$Volume = \pi \int_p^q x^2 dy$$

- Rotation of region bounded by two curves (**necessary to ensure that in [a,b] $f(x) \geq g(x)$**)




$$Volume\ of\ revolution\ about\ x - axis = \pi \int_a^b \{[f(x)]^2 - [g(x)]^2\}dx$$

<u>Applied to solving problems</u>

- For symmetrical shapes, can easily find 0.5 the area/ volume then multiply by 2, if need be.
- Also, take note that if a shaded region spans both +ve and -ve x or +ve and -ve y, finding the volume of revolution for the shape to rotate 180° is akin to the entire shape making a 360 revolution. **Calculations are highly affected by symmetry of shape**. If symmetrical about a particular axis, can directly apply volume of revolution. If not symmetrical, each part of the shape in either the +ve or -ve region will only rotate 180°, thus there is a need to multiply ½ to volume of revolution formula.

- Generally, to <u>find volumes</u>, we **add up the volumes of different section, then either subtract or sum the volumes**. If possible, **fall back on general formula of shapes** such as cones, cylinders, etc., to prevent making careless mistakes when using volumes of revolution formula.

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| Interesting example (desmos class activity): When we have a donut, we can actually cut it, stretch it , and rearrange it to a cylinder to find its volume. |

- Sometimes, we may be required/ able to **shift a graph**, such that we can apply volumes of revolution formula, which is with respect to the x/y axis. However, you <u>should not be shifting the graph blindly</u>! Use your sketch to make sure that the volume of revolution of the initial graph is equal to that of the shifted graph before proceeding.

- Extension: Question may guide you to calculate volume of revolution about an oblique asymptote (not in syllabus, but possible with guidance).