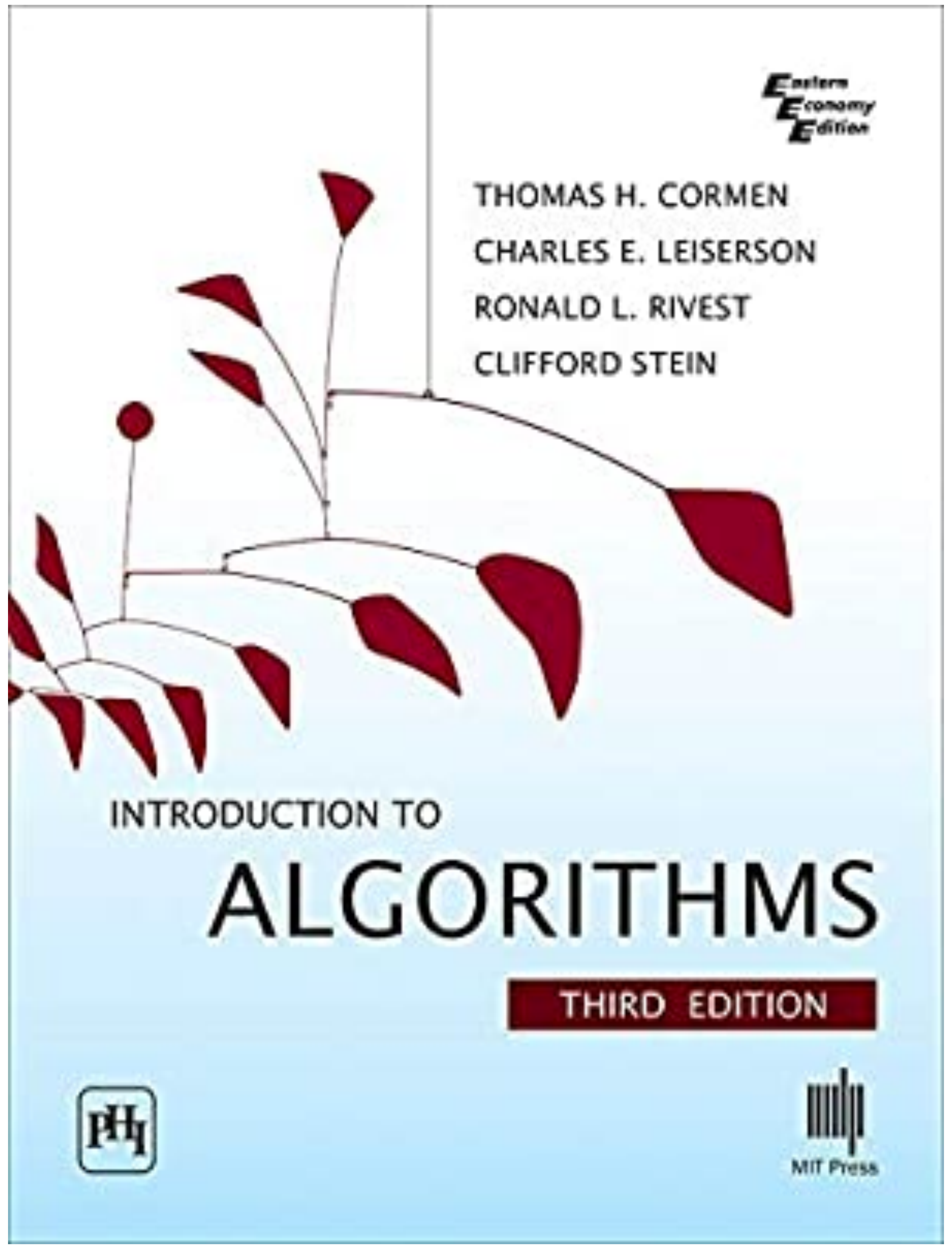
Lecture 5: Bellman-Ford

Overview

- Review: Notation
- Generic S.P. Algorithm
- Bellman-Ford Algorithm: Analysis & Correctness

Readings



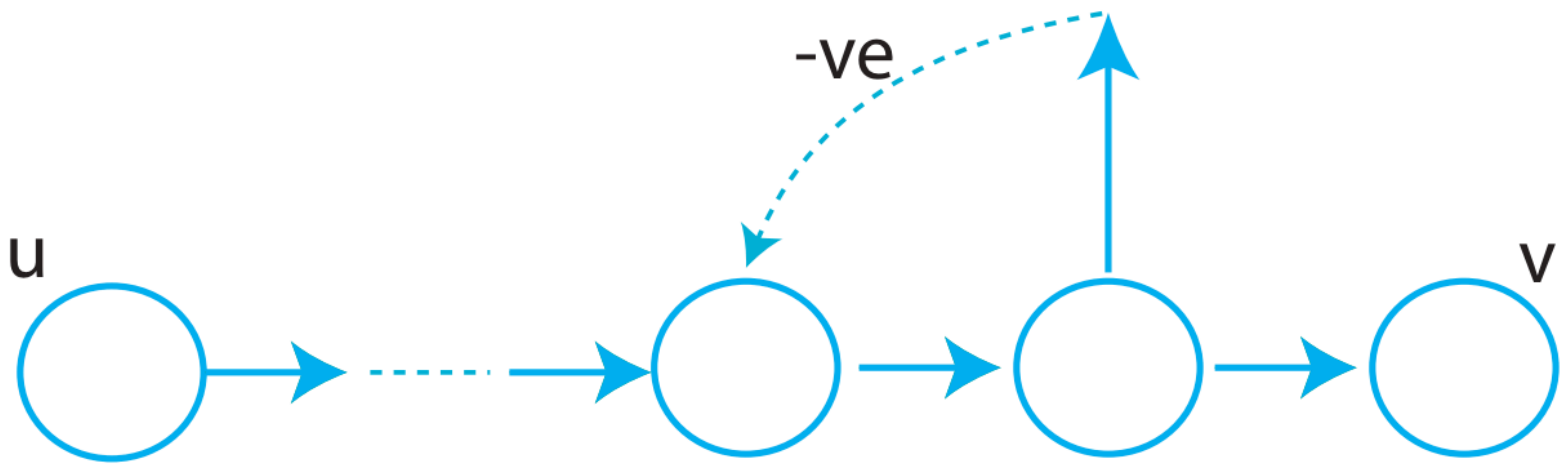
CLRS, Section 24

Review:Notation

$$\text{path } p = < v_0, v_1, \ldots v_k >$$

$$(v_i, v_{i+1}) \in E \quad \text{for} \quad 0 \leq i < k$$

$$w(p) = \sum_{i=0}^{k-1} w(v_i, v_{i+1})$$



General structure of S.P. Algorithms

Initialize: for $v \in V$: $\begin{array}{lcl} d[v] & \leftarrow & \infty \\ \Pi[v] & \leftarrow & \text{NIL} \end{array}$

$d[S] \leftarrow 0$

Main: repeat

select edge (u, v) [somehow]

"Relax" edge (u, v)
$$\left[\begin{array}{l} \text{if } d[v] > d[u] + w(u,v) : \\ \quad d[v] \leftarrow d[u] + w(u,v) \\ \quad \pi[v] \leftarrow u \end{array}\right.$$

until all edges have $d[v] \leq d[u] + w(u, v)$

Approach Problems

First: Time complexity could be exponential. How can we reduce it?

Second: What if we have negative cycles? How to terminate in this case? How can we approach graphs with negative cycles?

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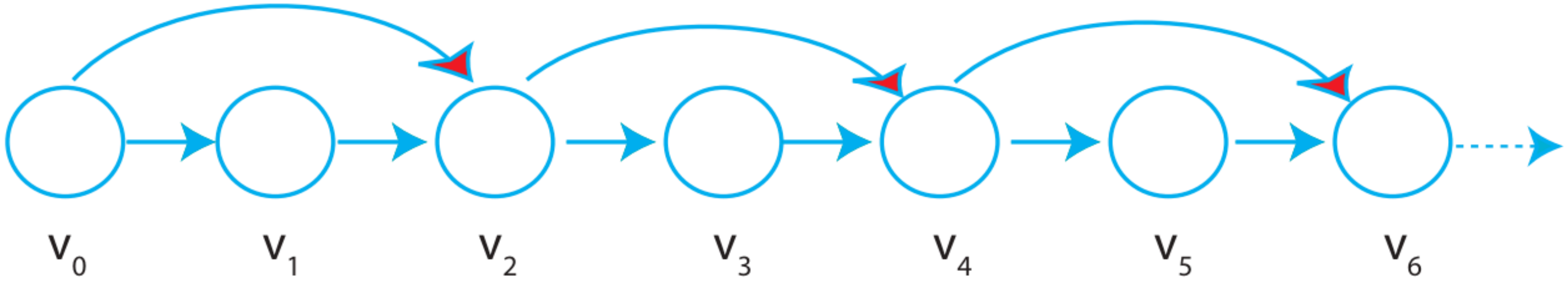
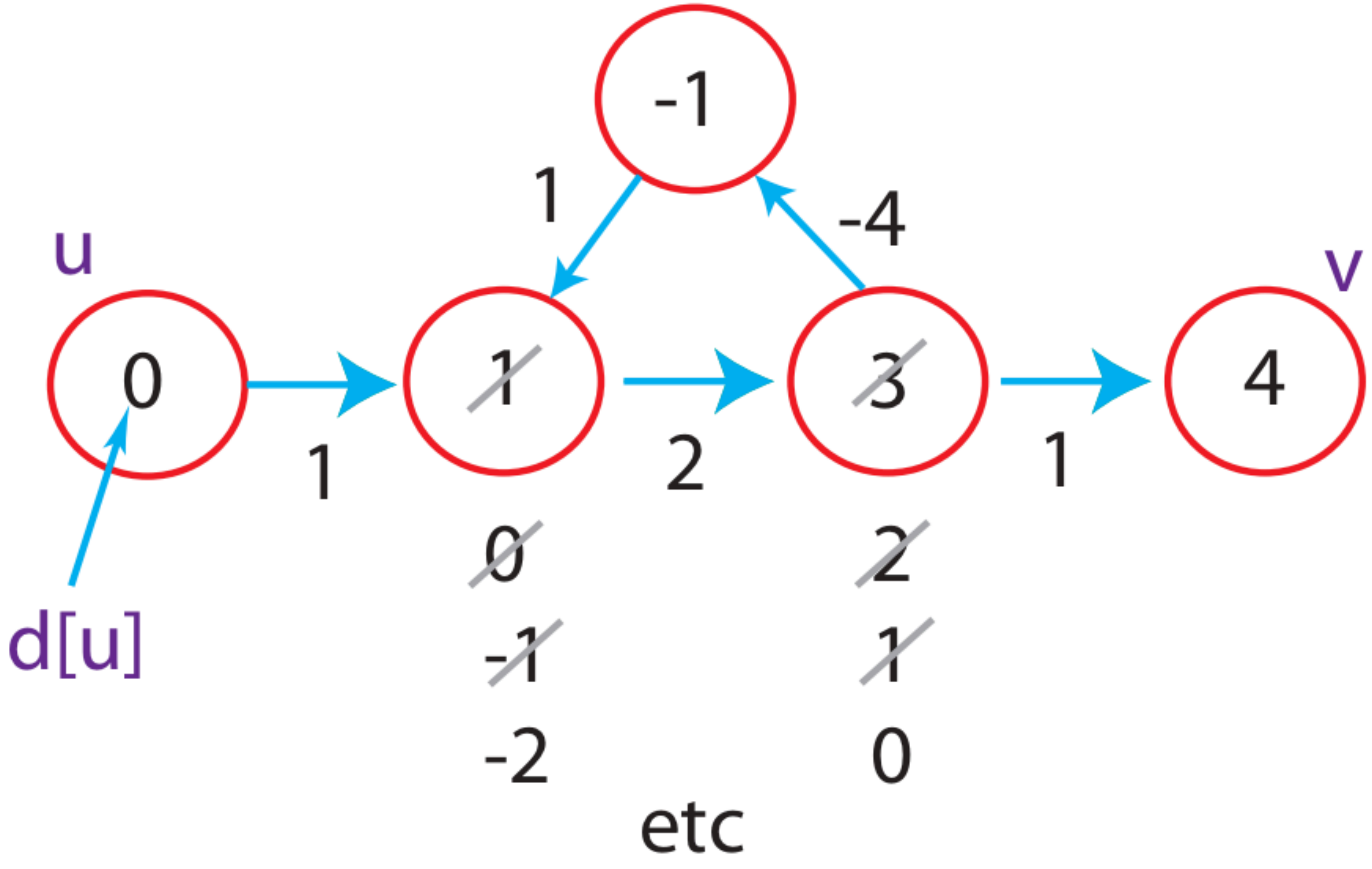


Approach Problems

Second: What if we have negative cycles? How to terminate in this case? How can we approach graphs with negative cycles?



Bellman-Ford(G,W,s)

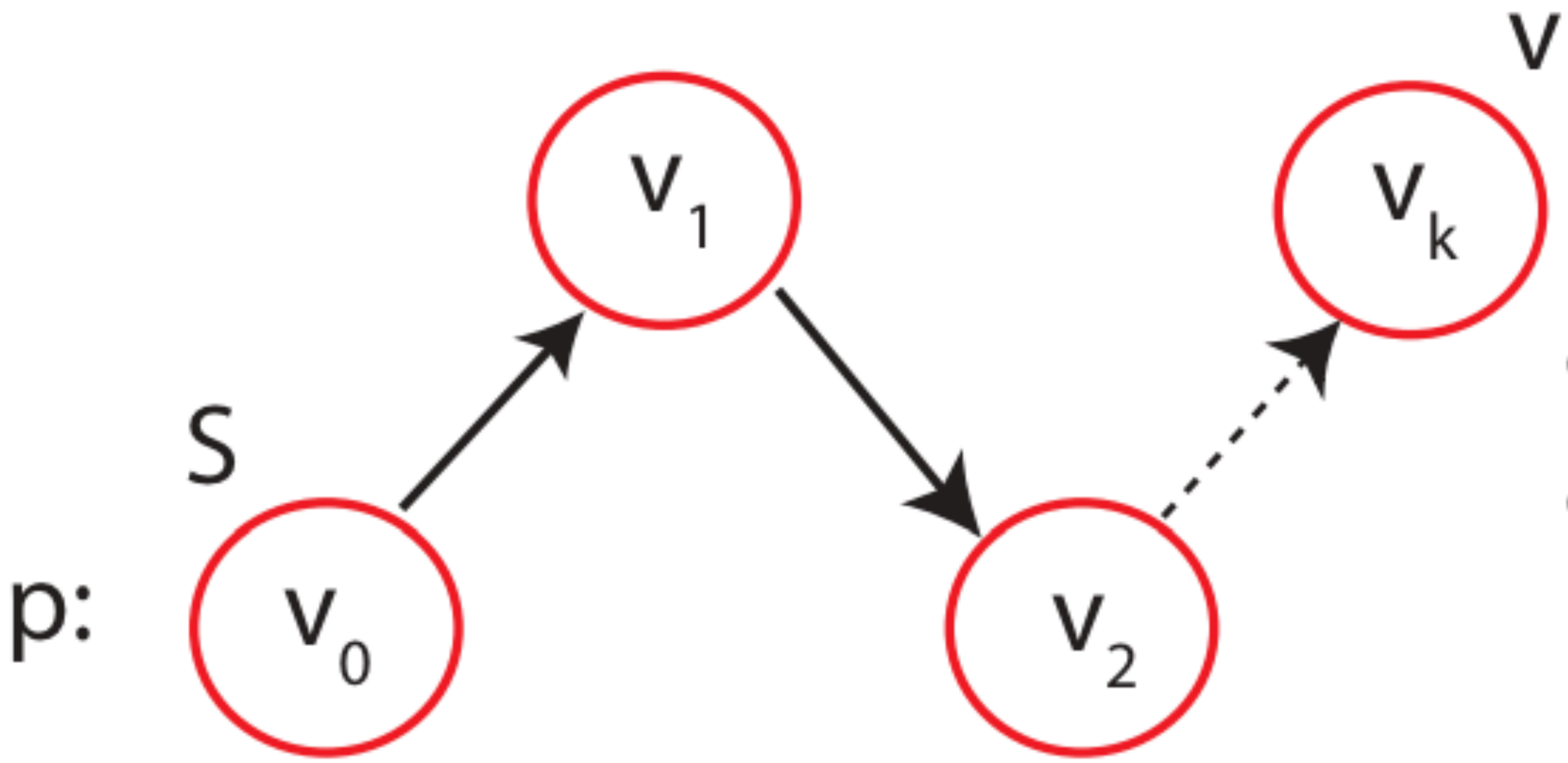
Initialize ()
for $i = 1$ to $|V| - 1$
    for each edge $(u, v) \in E$:
        Relax(u, v)
for each edge $(u, v) \in E$
    do if $d[v] > d[u] + w(u, v)$
        then report a negative-weight cycle exists

Correctness

Theorem: If $G = (V, E)$ contains no negative weight cycles, then after Bellman-Ford executes $d[v] = \delta(s, v)$ for all $v \in V$.

Proof: Let $v \in V$ be any vertex. Consider path $p = v_0, v_1, .., v_{ki}$ from $v_0 = s$ to $v_k = v$ that is a shortest path with minimum number of edges.

No negative weight cycles $\Rightarrow$ p is simple $\Rightarrow$ $k \leq |V|-1$

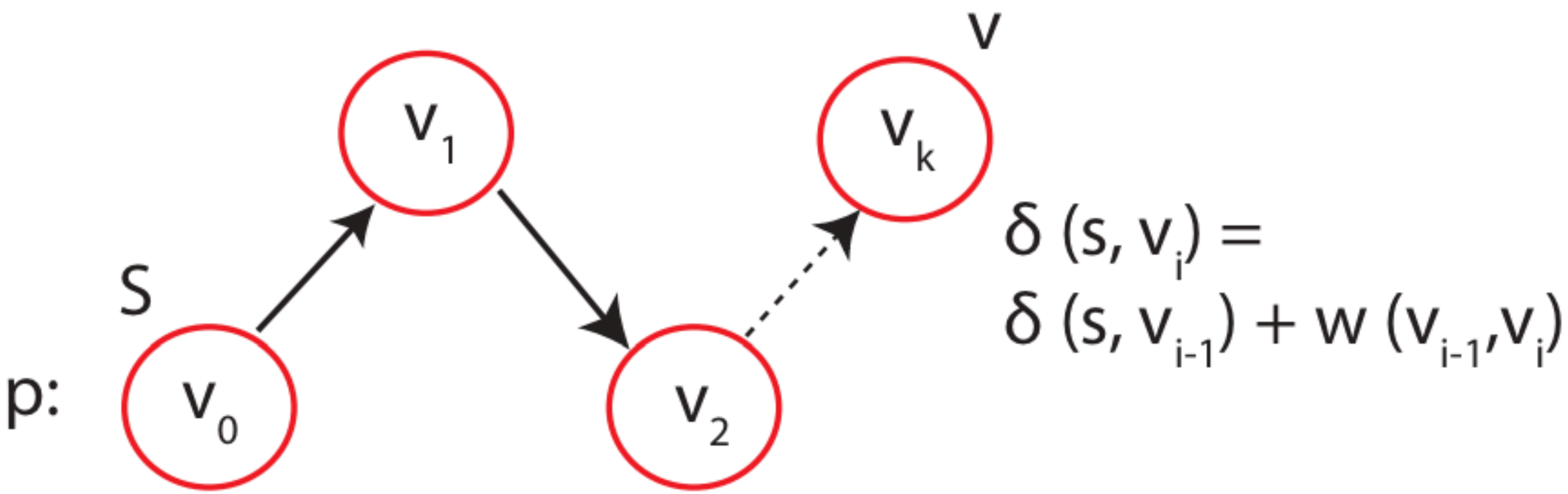


$\delta(s, v_i) =$
$\delta(s, v_{i-1}) + w(v_{i-1}, v_i)$

Correctness

Theorem: If G = (V, E) contains no negative weight cycles, then after Bellman-Ford executes d[v] = δ(s, v) for all v ∈ V.

Proof: After 1 pass through E, we have d[v1] = δ(s, v1), because we will relax the edge (v0, v1) in the pass, and we can't find a shorter path than this shortest path.
After i passes through E, we have d[vi] = δ(s, vi). etc.

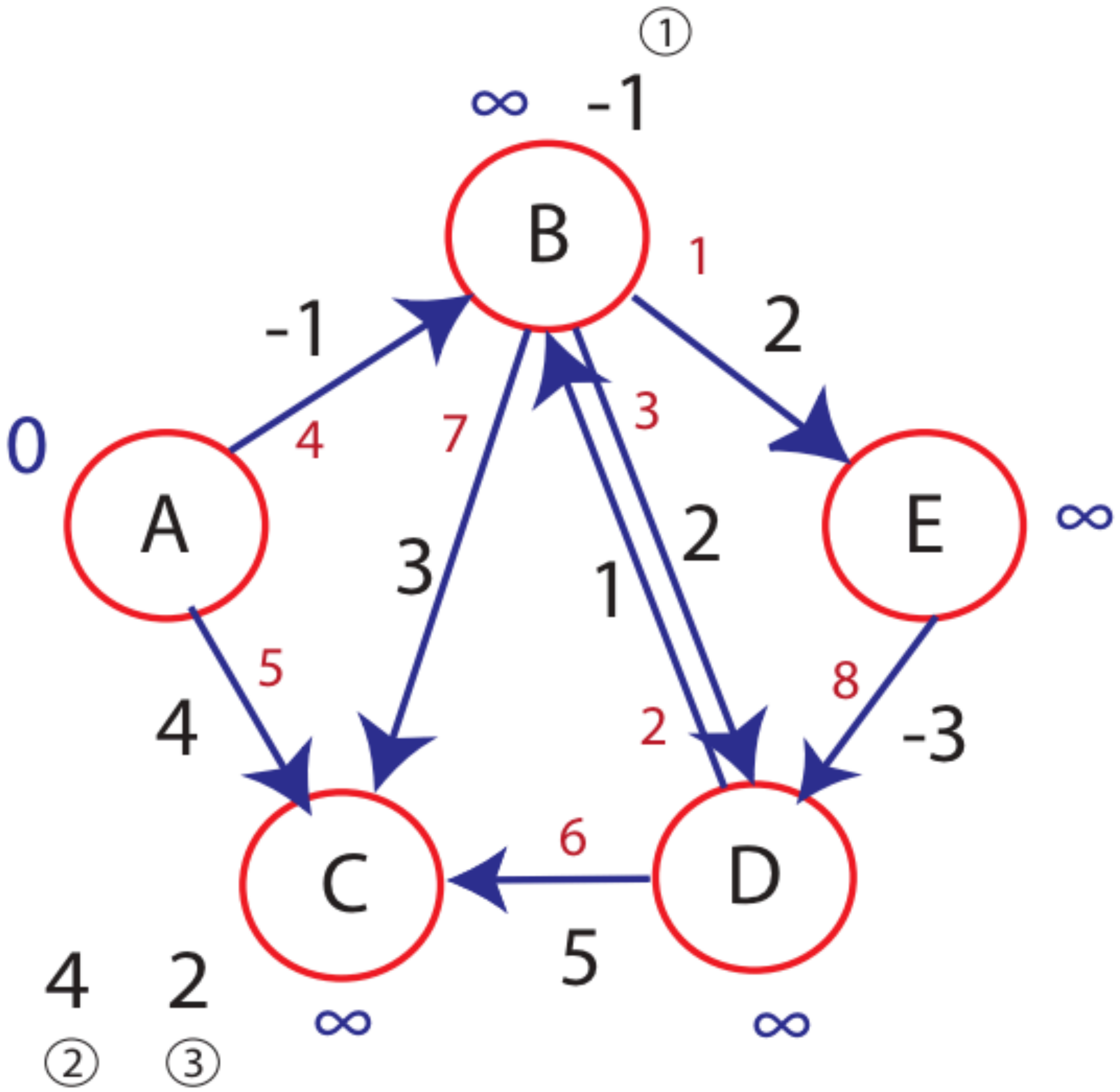


Corollary

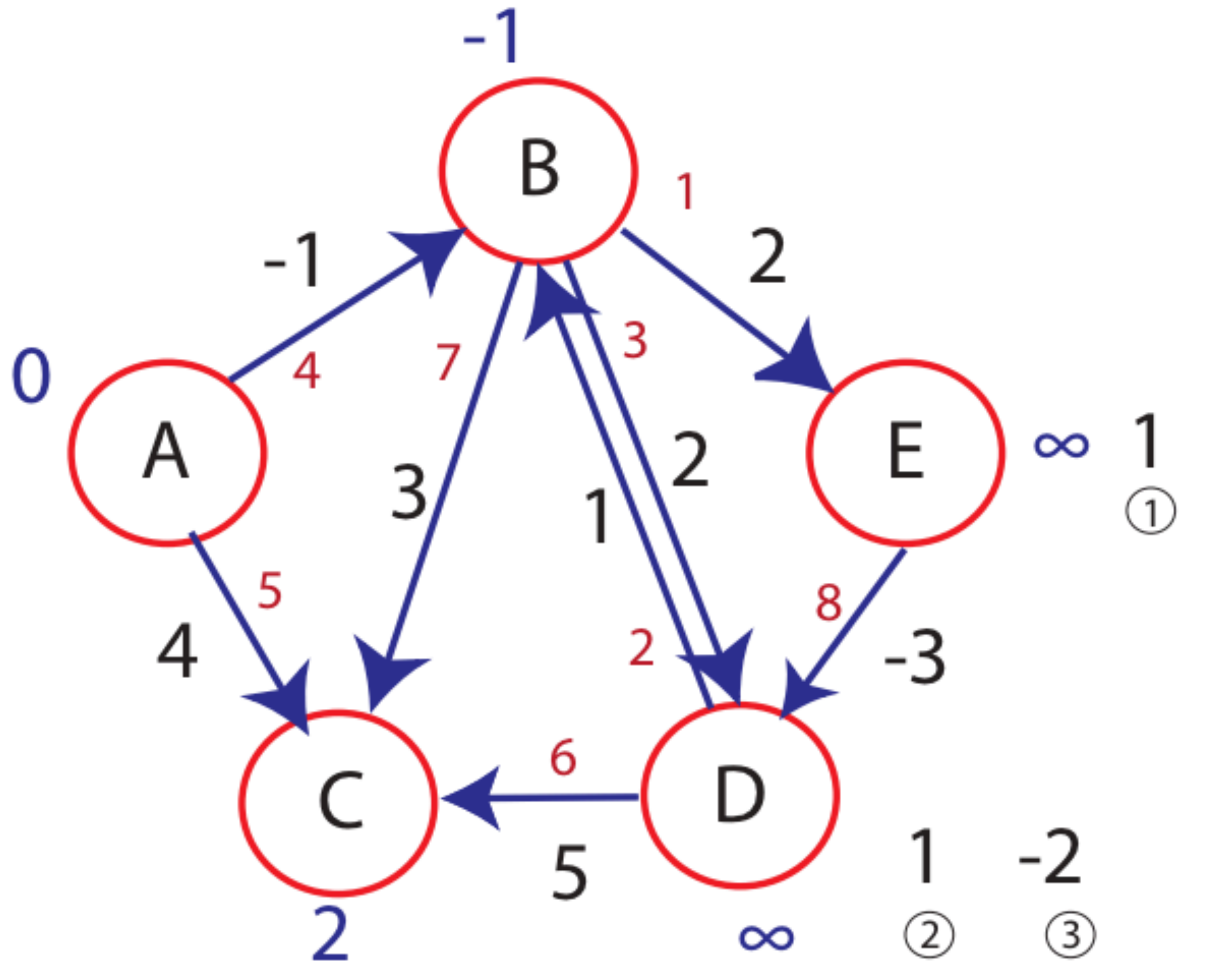
Theorem: If a value d[v] fails to converge after |V|−1 passes, there exists a negative-weight cycle reachable from s

Proof: After |V|-1 passes, if we find an edge that can be relaxed, it means that the current shortest path from s to some vertex is not simple and vertices are repeated. Since this cyclic path has less weight than any simple path the cycle has to be a negative-weight cycle.

Example



End of pass 1



End of pass 2 (and 3 and 4)