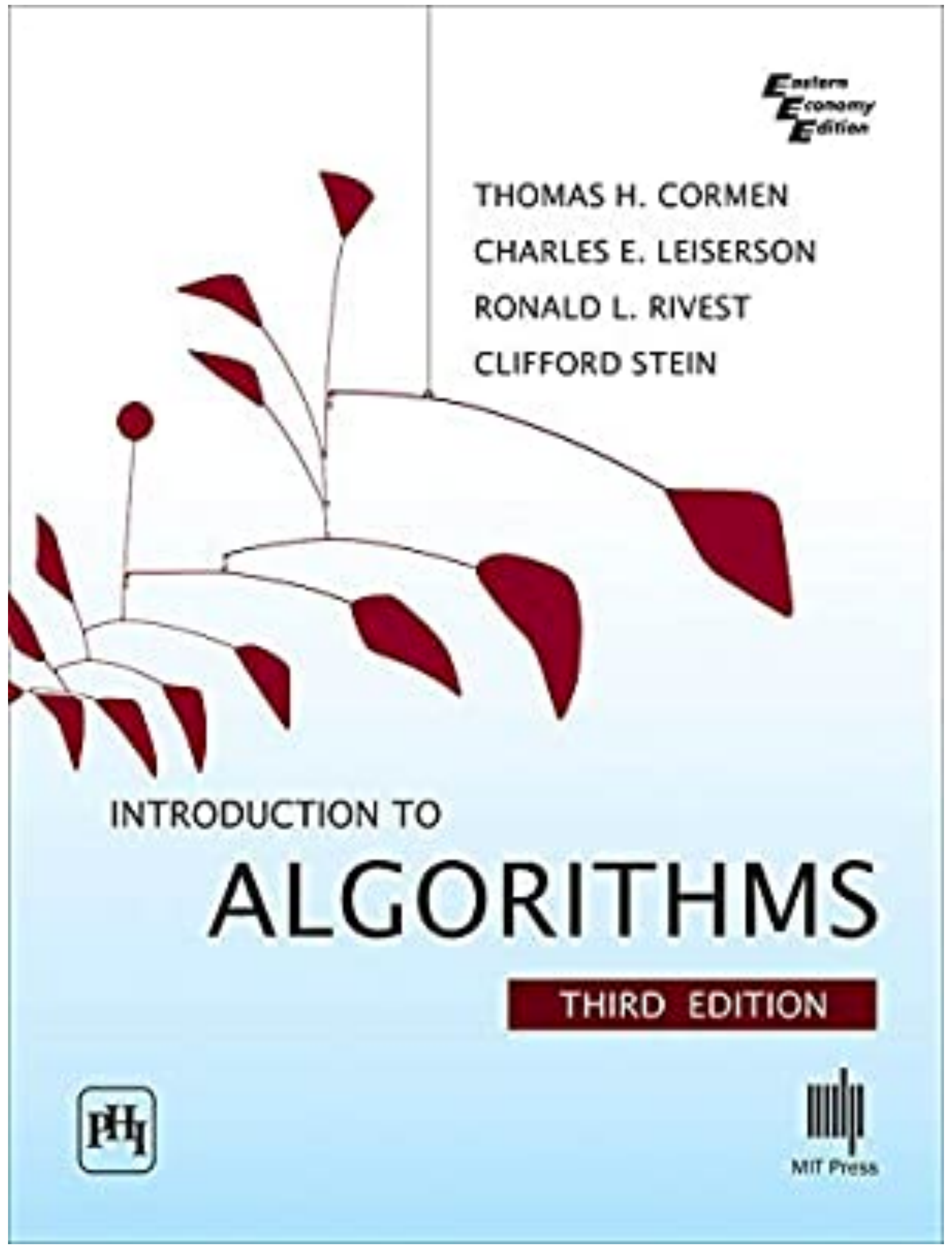
Lecture 4: Shortest Paths II - Dijkstra

Overview

- Review
- Shortest paths in DAGs
- Shortest paths in graphs without negative edges
- Dijkstra’s Algorithm

Readings



CLRS, Sections 24.2-24.3

Shortest path weight

Notation:

$v_0 \overset{p}{\longrightarrow} v_k$ means p is a path from v_0 to v_k. (v_0) is a path from v_0 to v_0 of weight 0.

Definition:

Shortest path weight from u to v as

$$\delta(u,v) = \begin{cases} \min\left\{ w(p) : u \overset{p}{\longrightarrow} v \right\} & \text{if } \exists \text{ any such path} \\ \infty & \text{otherwise} \quad (v \text{ unreachable from } u) \end{cases}$$

General structure of S.P. Algorithms

Initialize: for $v \in V$: $\begin{array}{ccc} d[v] & \leftarrow & \infty \\ \Pi[v] & \leftarrow & \text{NIL} \end{array}$

$d[S] \leftarrow 0$

Main: repeat

select edge (u, v) [somehow]

"Relax" edge (u, v)

$\left[\begin{array}{l} \text{if } d[v] > d[u] + w(u, v) : \\ \quad d[v] \leftarrow d[u] + w(u, v) \\ \quad \pi[v] \leftarrow u \end{array}\right.$

until all edges have $d[v] \leq d[u] + w(u, v)$

Relaxation is Safe

Basic operation in shortest path computation is the relaxation operation:

$$\begin{array}{l} \mathrm{RELAX}(u, v, w) \\ \qquad \text{if } d[v] > d[u] + w(u, v) \\ \qquad\quad \text{then } d[v] \leftarrow d[u] + w(u, v) \\ \qquad\qquad\quad \Pi[v] \leftarrow u \end{array}$$

Lemma: The relaxation algorithm maintains the invariant that $d[v] \geq \delta(s, v)$ for all $v \in V$.

Proof: By induction on the number of steps. Consider RELAX(u, v, w). By induction $d[u] \geq \delta(s, u)$. By the triangle inequality, $\delta(s, v) \leq \delta(s, u) + \delta(u, v)$. This means that $\delta(s, v) \leq d[u] + w(u, v)$, since $d[u] \geq \delta(s, u)$ and $w(u, v) \geq \delta(u, v)$. So setting $d[v] = d[u] + w(u, v)$ is safe.

DAGs

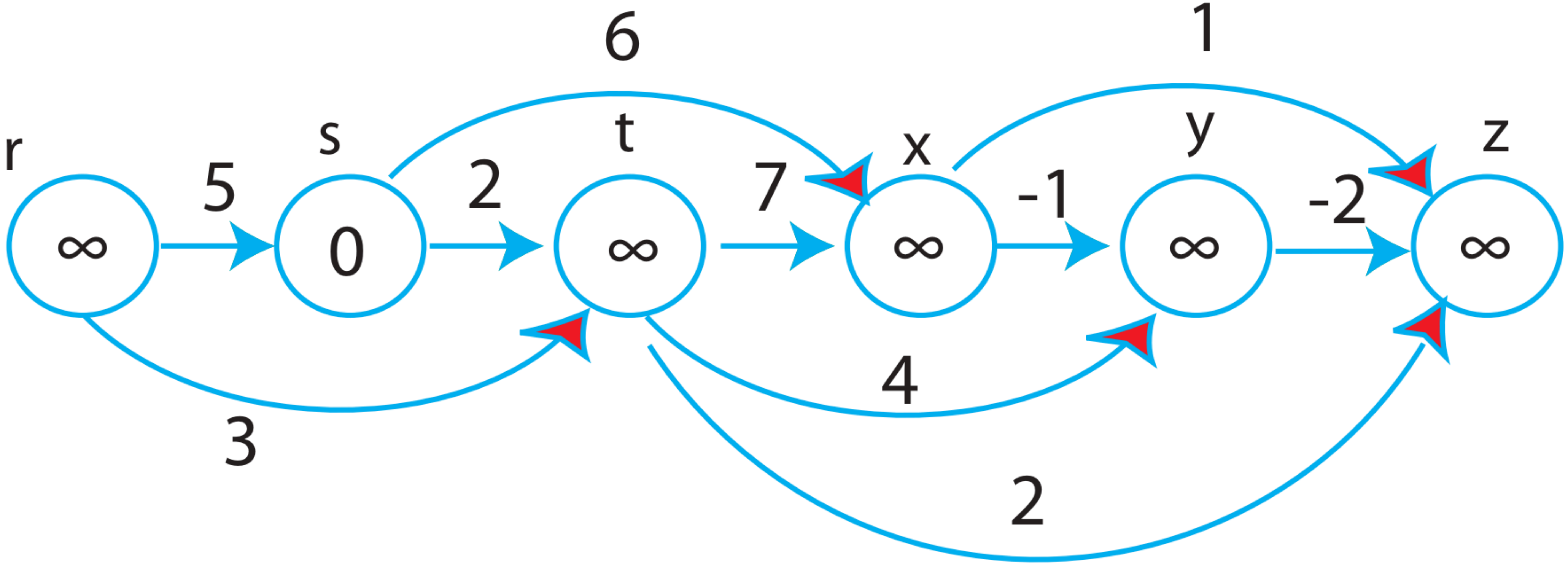
Can't have negative cycles because there are no cycles!

1. Topologically sort the DAG. Path from u to v implies that u is before v in the linear ordering.

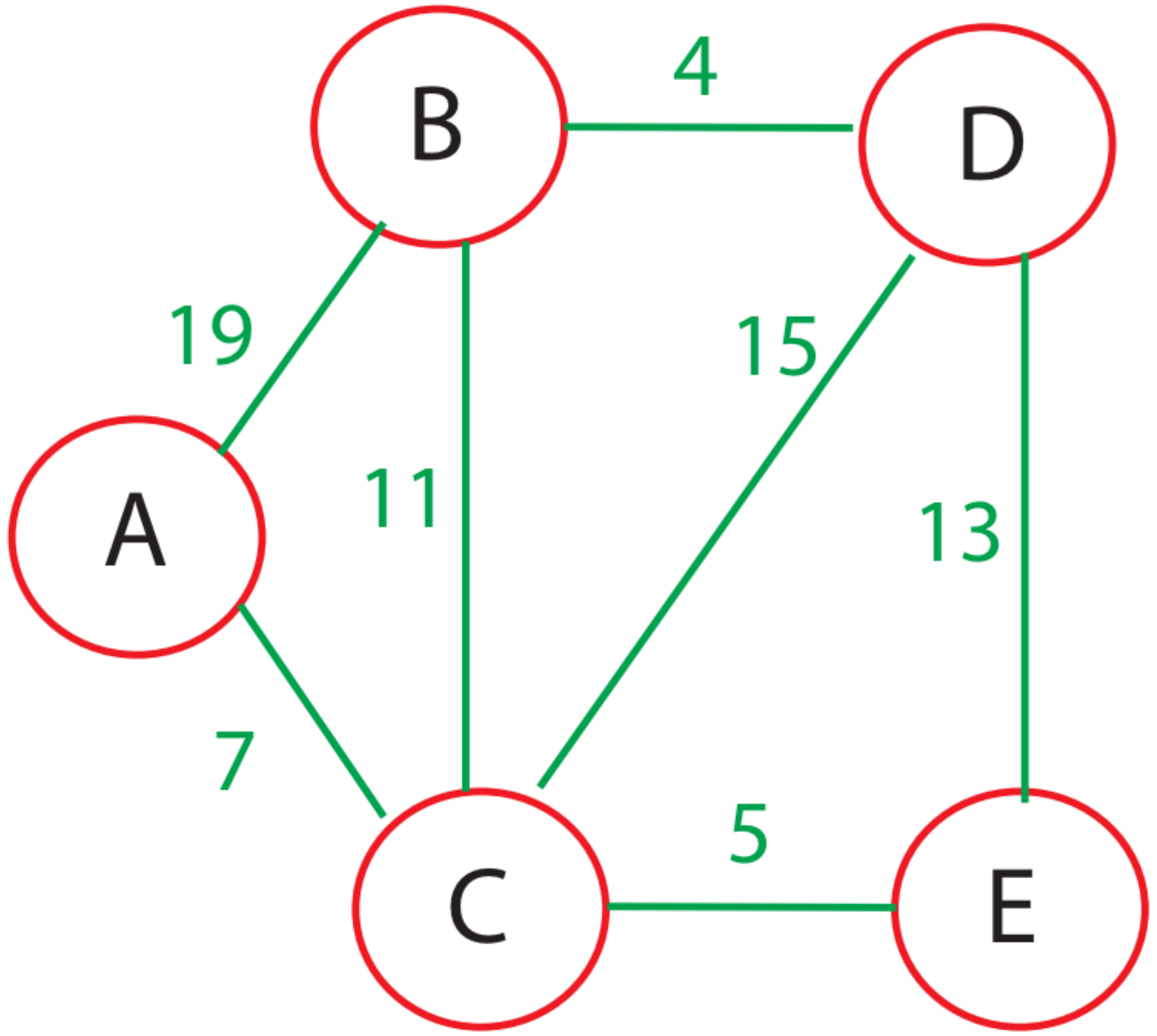
2. One pass over vertices in topologically sorted order relaxing each edge that leaves each vertex.

$\Theta(V + E)$ time

Example DAG



Dijkstra Example:



A	C	E	B	D
	7	12	18	22

D	B	E	C	A
	4	13	15	22

E	C	A	D	B
	5	12	13	16

Dijkstra's Algorithm

For each edge $(u, v) \epsilon E$, assume $w(u, v) \geq 0$, maintain a set S of vertices whose final shortest path weights have been determined. Repeatedly select $u \epsilon V - S$ with <u>minimum</u> shortest path estimate, add u to S, relax all edges out of u.

Pseudo-code

```
Dijkstra (G, W, s)        //uses priority queue Q
        Initialize (G, s)
        S ← ϕ
        Q ← V[G]        //Insert into Q
        while Q ≠ ϕ
              do u ← EXTRACT-MIN(Q)        //deletes u from Q
              S = S ∪ {u}
              for each vertex v ϵ Adj[u]
                      do RELAX (u, v, w)    ← this is an implicit DECREASE_KEY operation
```

Example

Dijkstra (G, W, s) //uses priority queue Q

  Initialize (G, s)

  $S \leftarrow \phi$

  $Q \leftarrow V[G]$ //Insert into Q

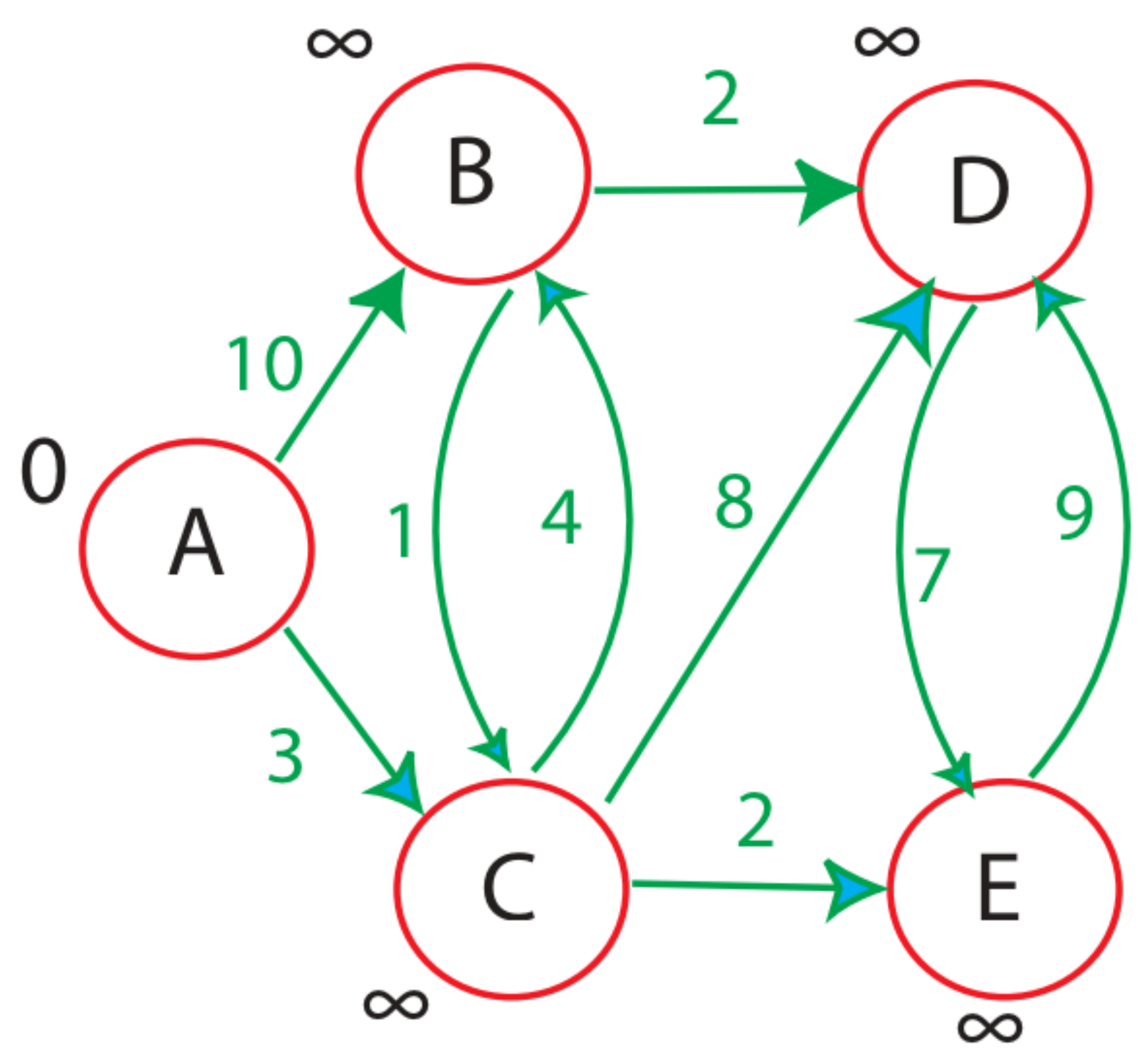
  while $Q \neq \phi$

    do $u \leftarrow$ EXTRACT-MIN(Q)

    $S = S \cup \{u\}$

    for each vertex $v \in$ Adj$[u]$

      do RELAX (u, v, w)



S = { }	{ A	B	C	D	E }	= Q
S = { A }	0	∞	∞	∞	∞	
S = { A, C }	0	10	3	∞	∞	← after relaxing edges from A
S = { A, C }	0	7	3	11	5	← after relaxing edges from C
S = { A, C, E }	0	7	3	11	5	
S = { A, C , E, B}	0	7	3	9	5	← after relaxing edges from B

Dijkstra Algorithm Summary

***Strategy**: Dijkstra is a greedy algorithm: choose closest vertex in V – S to add to set S.*

***Correctness**: We know relaxation is safe. The key observation is that each time a vertex u is added to set S, we have d[u] = δ(s, u).*

Dijkstra Complexity
Θ(v) inserts into priority queue
Θ(v) EXTRACT MIN operations
Θ(E) DECREASE KEY operations

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Array impl:

$\Theta(v)$ time for extra min
$\Theta(1)$ for decrease key
Total: $\Theta(V.V + E.1) = \Theta(V^2 + E) = \Theta(V^2)$

Binary min-heap:

$\Theta(\lg V)$ for extract min
$\Theta(\lg V)$ for decrease key
Total: $\Theta(V \lg V + E \lg V)$