Lecture 1
Median Finding

Overview

- Divide & Conquer Overview
- Master Theorem
- Median Finding Trivial Solution
  - Hoan Algorithm
- Median Finding (Blum, Floyd, Pratt, Rivest and Tarjan)

Divide & Conquer

Given a problem of size n divide it into subproblems of size $\frac{n}{b}$, $a \geq 1$, $b > 1$. Solve each subproblem recursively. Combine solutions of subproblems to get overall solution.

$$T(n) = aT(\frac{n}{b}) + [\text{work for merge}]$$

Master Theorem Definition

Master Theorem Examples

Median Finding Problem Definition

Median Finding Trivial Solution

Median Finding O(n) in average case by Toni Hoar

```
Select(S, i)
  1.  Pick x ∈ S // choose a pivot element. We can do it even randomly
  2.  B = {y ∈ S|y<x} // Set of values less than x
  3.  C = {y ∈ S|y>x} // Set of values greater than x
  4.  D = {y ∈ S|y==x} // Set of values equal to x
  5.  Compute k = rank(x) // We know how many elements less than x,so we know the rank!
  6.  if k = i:
  7.    return x
  8.  else if k>i:
  9.    return Select(B, i)
  10. else if k<i:
  11.   return Select(C, i - k)
```

Median Finding O(n) in average case by Toni Hoar

```
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  1.  Pick x ∈ S // choose a pivot element. We can do it even randomly
  2.  B = {y ∈ S|y<x} // Set of values less than x
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```

Example

```
Target array: l = [9,1,0,2,3,4,6,8,7,10,5]

len(l) == 11 -> we find the 6th smallest element.
Choose pivot randomly: l[3] = 2.
Divide the array according to the pivot:
[1,0,2], [9,3,4,6,8,7,10,5]

We find the 3rd (6 - rank(x)) smallest element in the right
array:
[9,3,4,6,8,7,10,5]
Choose pivot randomly: l[2]=6
Divide the array according to the pivot:
[3,4,5,6] [9,7,10]

We find the 3rd smallest element in the left array:
[3,4,5,6]
Choose pivot randomly: l[1]=4
Divide the array according to the pivot:
[3,4] [5,6]

We find the (3 - 2) smallest element in the right array:
[5,6]

Base variant where we just choose the min or max.
We need min return 5
```

$i = 6$, $k = 3$

$k = 4$, $i = 3$, $\boxed{k > i}$

$i = 3$, $k = 2$

6

1

```
Select(S, i)
1.   Pick x ∈ S randomly
2.   B = {y ∈ S|y<x}
3.   C = {y ∈ S|y>x}
4.   D = {y ∈ S|y==x}
5.   Compute k = rank(x)
6.   if k = i:
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```

$O(1)$

§7.1 → CLRS

$O(n)$

$$T(n) = O(n) + T(\quad)$$

$$T(n) = n + \frac{n}{2} + \frac{n}{4} + \frac{n}{8} + \ldots = 2n = O(n)$$

Median Finding O(n) in average case
Time Analysis

Worst case:

$T(n) = n + (n-1) + (n-2) + \dots \sim n^2 = O(n^2)$

$l = [1, 2, 3, \dots\dots, n]$

$1 \quad [2, 3, \dots, n]$

$2 \quad [3, \dots, n]$

Median Finding O(n) deterministic

```
Select(S, i)
  1.  Pick x ∈ S // Need to pick it cleverly!
  1.1 Arrange S into columns of size 5 (In ⌈n/5⌉ cols)
  1.2 Sort each column (bigger elements on top) (linear time)
  1.3 Find “median of medians” as x
```

Median Finding O(n) deterministic time analysis

```
Select(S, i)
  1.  Pick x ∈ S // Need to pick it cleverly!
  1.1 Arrange S into columns of size 5 (In [n/5] cols)
  1.2 Sort each column (bigger elements on top) (linear time)
  1.3 Find "median of medians" as x
  2.  B = {y ∈ S|y<x}
  3.  C = {y ∈ S|y>x}
  4.  D = {y ∈ S|y==x}
  5.  Compute k = rank(x)
  6.  if k = i:
  7.    return x
  8.  else if k>i:
  9.    return Select(B, i)
  10. else if k<i:
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```

$O(n)$

$O(n)$

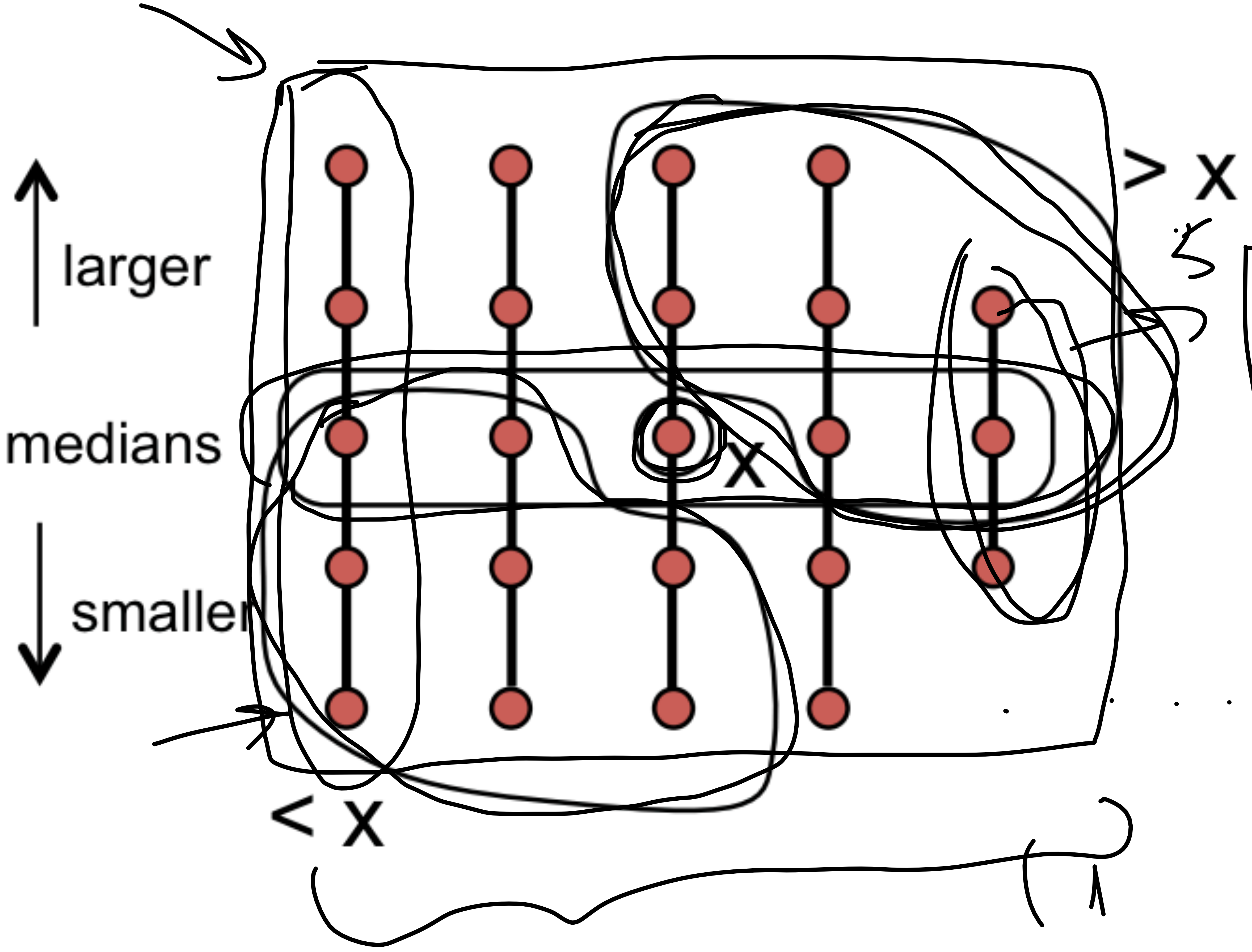
$2 \cdot n$

$n = 5$

$n \log n$

$O(n)$

Median Finding O(n) deterministic time analysis



How many el-s > x

5 $\lceil 3(\frac{n}{10}) \rceil > x$

$\lceil 3(\frac{n}{10}) \rceil$

$(\frac{7n}{10})$

Median Finding O(n) deterministic time analysis

$$\boxed{T(n) \le \underbrace{T(\lceil n/5 \rceil)}_{\text{find pivots}} + \underbrace{T\left(\tfrac{7}{10}n + 6\right)} + \underbrace{O(n)}_{\text{choose a rank of x}}.}$$