unde B, R sunt mț. de indici coresp. unei baze care nu e dual admisibilă

$$\Rightarrow \sum_{j \in R} x_j + x_{m+1} = M, \quad x_{m+1} \geq 0$$

introducem în bază: x_j pt. care $r_j = \min\{r_k, r_k < 0\}$

iese din bază: x_{m+1}

Ex: $\inf \; -2x_1 - 3x_2$

$x_1 + x_2 + x_3 = 2$

$x_1 + \_ + x_4 = 1$

$x_i \geq 0$

$$B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

Ex: $x_2 + x_5 = M$

$\inf \; -2x_1 - 3x_2$

$x_1 + x_2 + x_3 = 2$

$x_1 + \dots + x_4 = 1$

$x_2 + x_5 = M$

$$\sum_{j \in R} x_j \leq M$$

$r_j < 0$

$$A' = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

iese x_5 din bază
intră x_2

$$B' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$B = \{1, 2, 4\}$

r

Curs 7 — 01 Aprilie 2019

DETERMINAREA UNEI BAZE DUAL ADMISIBILE

(1) $\inf \; c^T x$

$Ax = b$

$x \geq 0$

$\operatorname{rang} A = m \leq n$

Fie B baza care nu e dual admisibilă

Introducem ecuația: $\sum_{j\in R} x_j \le M$, M mare

adaug o var. $x_0 + \sum_{j\in R} x_j = M$

(2) $\inf c^T x$

$A_1 x = b$ $\qquad c_1 = \begin{pmatrix} 0 \\ c \end{pmatrix}$

$x_0, x_1, \dots, x_n \ge 0$

$x_0 + x_{j_1} + x_{j_2} + \dots + x_{j_{n-m}} = M$ $\qquad A_1 = \begin{pmatrix} 1 & 0 & e^T \\ 0 & \underbrace{B \quad R}_{A} \end{pmatrix}$

$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & B \end{pmatrix}$ $\qquad B_1^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & B^{-1} \end{pmatrix}$ $\qquad$ unde $e^T = (1, \dots, 1)$

B_1 - baza formată cu col. matricei A_1.

Este B_1 dual admisibilă?

$$r_j^{B_1} = c_j - c_1^T B_1^{-1} A_1^j$$
$$= c_j - (0 \; c^T)\begin{pmatrix} 1 & 0 \\ 0 & B^{-1} \end{pmatrix}\begin{pmatrix} 1 \\ A^j \end{pmatrix} \qquad j \in R$$
$$= c_j - (0 \; c^T)\begin{pmatrix} 1 \\ B^{-1}A^j \end{pmatrix} = c_j - c^T B^{-1} A^j = r_j^B$$

Pentru $j \in B$ voi avea:

$$r_j^{B_1} = c_j - c_1^T B_1^{-1} A^j = c_j - (0 \; c^T)\begin{pmatrix} 1 & 0 \\ 0 & B^{-1} \end{pmatrix}\begin{pmatrix} 0 \\ A_j \end{pmatrix}$$
$$= c_j - (0 \; c^T)\begin{pmatrix} 0 \\ B^{-1}A \end{pmatrix} = c_j - c^T B^{-1} A_j$$
$$= r_j^B = 0, \; j \in B$$

B_1 nu este dual admisibilă

Facem o sch. de bază: $\tilde{B}_1 = B_1 \cup \{k\} \setminus \{0\}$

x_0 părăsește baza, x_k intră în bază

k este ales a.î. $r_k = \min\{r_j, r_j < 0\}$

$$(*)\begin{cases} r_j^{\tilde{B}_1} = r_j^{B_1} - r_k^{B_1} \ge 0 \\ r_0^{B_1} = -r_k^{B_1} > 0 \end{cases} \longrightarrow \tilde{B}_1 \text{ dual admisibilă}$$

DEM: $\tilde{B}_1 = \begin{pmatrix} 1 & 0 \\ A^k & B \end{pmatrix}$; $\tilde{B}_1^{-1} = \begin{pmatrix} 1 & 0 \\ -B^{-1}A^k & B^{-1} \end{pmatrix}$

(aici îmi lipsesc foi)

(2) $\min\ (-x_3 + x_4 + x_5)$

$$\begin{cases} x_0 + x_3 + x_4 + x_5 = M \\ x_1 + x_3 + x_4 - x_5 = -2 \\ x_2 - x_3 - x_4 + x_5 = 1 \end{cases}$$

$x_0, \dots, x_5 \geq 0$

$x = \begin{pmatrix} M+2 \\ 0 \\ -1 \\ -2 \\ 0 \\ 0 \end{pmatrix}$ ec. $\underbrace{x_0}_{M+2} + x_3 + x_4 + x_5 = M$

$2 + \underbrace{x_3}_{\geq 0} + \underbrace{x_4}_{\geq 0} + \underbrace{x_5}_{\geq 0} = 0$

eliminăm ac. ec.

Cazul $\nexists$ x_0 var de bază

în pr. 1) $B^* = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$ $x = (0, -1, -2, 0, 0)^T$

$(B^*)^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

Aplicăm alg. simplex dual:

- iese $x_3 = -2$ iese din bază

y^j $j = 1, 4, 5$

$y^1 = (B^*)^{-1} A^1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$y^4 = (B^*)^{-1} A^4 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$y^5 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$y_{3j} > 0 \ \forall j \in \{1, 4, 5\}$

$\Rightarrow$ pr. 1) nu are sol. admis.

TEOREMA ECARTURILOR COMPLEMENTARE

(1) $\inf\ c^T x$ $\quad$ (2) $\sup\ b^T w$

$Ax \geq b$ $\quad$ $A^T w \leq c$

$x \geq 0$ $\quad$ $w \geq 0$

Ex 1: (P) $\max\ 5x_1 + 10x_2$

$x_1 + 3x_2 \le 50 \to w_1$

$4x_1 + 2x_2 \le 60 \to w_2$

$x_1 \le 5 \to w_3$

$x_1, x_2 \ge 0$

(D) $\min\ 50w_1 + 60w_2 + 5w_3$

$w_1 + 4w_2 + w_3 \ge 5$

$3w_1 + 2w_2 \ge 10$

$w_1, \dots, w_3 \ge 0$

Presupunem $x^* = (5, 15)^T$ sol. optimă pt. (P)

Care este sol. optimă pt. D?

$u_1^* = 50 - x_1^* - 3x_2^* = 50 - 5 - 3 \cdot 15 = 0$

$u_2^* = 60 - 4x_1^* - 2x_2^* = 60 - 4 \cdot 5 - 2 \cdot 15 = 10 > 0$

$u_3^* = 5 - x_1^* = 0$

Dacă $w^* = (w_1^*, w_2^*, w_3^*)$ sol. opt. pt. (D).

$v_1^* = w_1^* + 4w_2^* + w_3^* - 5$

$v_2^* = 3w_1^* + 2w_2^* - 10$

u_1^*, u_2^*, u_3^* - ecarturi pt. P

v_1^*, v_2^* - ecarturi pt. D

T.E.C

$x_1^* > 0 \Rightarrow v_1^* = 0$

$x_2^* > 0 \Rightarrow v_2^* = 0$

$u_2^* > 0 \Rightarrow w_2^* = 0$

$$\begin{cases} w_1^* + \overbrace{4w_2^*}^{0} + w_3^* = 5 \\ 3w_1^* + \underbrace{2w_2^*}_{0} = 10 \end{cases} \Rightarrow w_1^* = \frac{10}{3}$$

$w_3^* = 5 - \frac{10}{3} = \frac{5}{3} \Rightarrow w^* = \left(\frac{10}{3}, 0, \frac{5}{3}\right)^T$

sol. opt. pt. (D)

Ne întoarcem cla pr. inițiale:

(P) (1) $\inf c^T x$

$Ax \geq b$

$x \geq 0$

$\operatorname{rang} A = m < n$

(D) (2) $\sup b^T w$

$A^T w \leq c$

$w \geq 0$

ECARTURILE CORESP:

(P): $\begin{cases} u = Ax - b \\ u \geq 0 \\ u \in \mathbb{R}^m \end{cases}$

(D): $\begin{cases} v = c - A^T w \\ v \geq 0 \\ v \in \mathbb{R}^n \end{cases}$

$$x^T v + w^T u = x^T (c - A^T w) + w^T (Ax - b)$$
$$= x^T c - \cancel{x^T A^T w} + \cancel{w^T A x} - w^T b =$$
$$= x^T c - w^T b$$

Dacă x e sol. opt. ptr. (P)

w e — " — (D)

$\Rightarrow x^T c - w^T b = 0$

$\Rightarrow \underbrace{x^T}_{\geq 0} v + \underbrace{w^T}_{\geq 0} u = 0 \Rightarrow \begin{cases} x^T v = 0 \\ w^T u = 0 \end{cases}$

$x^T v = 0 \Rightarrow \sum_{j=1}^{n} \underbrace{x_j}_{\geq 0} \underbrace{v_j}_{\geq 0} = 0$

$w^T u = 0 \Rightarrow \sum_{i=1}^{m} \underbrace{w_i}_{\geq 0} \underbrace{u_i}_{\geq 0} = 0$

$\Rightarrow \begin{cases} x_j \cdot v_j = 0, \ \forall j = \overline{1,n} \\ w_i u_i = 0, \ \forall i = \overline{1,m} \end{cases}$

Dacă $x_j > 0 \Rightarrow v_j = 0$

$v_j > 0 \Rightarrow x_j = 0$

$w_i > 0 \Rightarrow u_i = 0$

$u_i > 0 \Rightarrow w_i = 0$

Teorema ecarturilor complementare

x^* este sol. opt. pt. (1) și w^* e sol. pt. (2) $\Leftrightarrow$

$\begin{cases} x_j^* v_j^* = 0, \ j = \overline{1,n} \\ u_i^* w_i^* = 0, \ i = \overline{1,m} \end{cases}$

unde $v^* = c - A^T w^*$, $u^* = Ax^* - b$

08 Aprilie 2019

Curs 8 - Exerciții pt. examen

① Teorema ecarturilor complementare

(1) $\inf c^T x$ duala (2) $\sup b^T w$

$Ax \geq b$ $A^T w \leq c$

$x \geq 0$ $w \geq 0$

TEC : x^* e sol. optimă pt. (1) și w^* e sol opt. pt. (2)

$$\Leftrightarrow \begin{cases} (x^*)^T \overbrace{(c - A^T w^*)}^{v^*} = 0 \\ (w^*)^T \underbrace{(Ax^* - b)}_{u^*} = 0 \end{cases} \Leftrightarrow \begin{cases} (x^*)^T v^* = 0 \\ (w^*)^T u^* = 0 \end{cases} \Leftrightarrow \begin{cases} x_j^* v_j^* = 0, \forall j = \overline{1,n} \\ w_i^* u_i^* = 0, \forall i = \overline{1,m} \end{cases}$$

Obs : Dacă $x_j^* > 0 \Rightarrow v_j^* \leq 0$

$v_j^* > 0 \Rightarrow x_j^* = 0$

$w_i^* > 0 \Rightarrow u_i^* = 0$

$u_i > 0 \Rightarrow w_i^* = 0$

Ex 1 $\max x_1 + 2x_2 + 3x_3 + 4x_4$

$x_1 + 2x_2 + 2x_3 + 3x_4 \leq 20 \rightarrow w_1$

$2x_1 + x_2 + 3x_3 + 2x_4 \leq 20 \rightarrow w_2$

$x_1, \dots, x_4 \geq 0$

duala: $\min 20w_1 + 20w_2$

$w_1 + 2w_2 \geq 1$

$2w_1 + w_2 \geq 2$ $\rightarrow$ rez. grafic

$2w_1 + 3w_2 \geq 3$ sol. opt. $w^* = (1,2; 0,2)^T$

$3w_1 + 2w_2 \geq 4$

$w_1, w_2 \geq 0$

Obs : $w_1^* = 1,2 > 0 \Rightarrow$ ec. 1) din pr. primală e satisf. cu =

$w_2^* = 0,2 > 0 \Rightarrow$ ec. 2) —— " ——

(*)