Ecuații diferențiale

curs 7 : 14. 11. 2017

Ecuații liniare în $\mathbb{R}^n$

$\frac{dx}{dt} = A(t)x$, $A(\cdot): \mathcal{I} \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ continuă

Fie baza $B \subseteq \mathbb{R}^n$, $A_B = col(A(t)b_1, \dots, A(t)b_n) = (a_{ij}(t))_{\substack{i=\overline{1,n}\\ j=\overline{1,n}}}$

$B = \{(b_1, \dots, b_n)\}$

$$\underbrace{\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t) x_j, \quad i=\overline{1,n}}_{\text{sistem de ecuații liniare}}$$

• $n=1$; ecuații liniare scalare: $x' = a(t)x$

$a(\cdot): \mathcal{I} \subseteq \mathbb{R} \to \mathbb{R}$ continuă

$\varphi(\cdot)$ soluție $\Leftrightarrow \varphi(t) = c \cdot e^{\int_{t_0}^{t} a(s)\,ds}$, $t_0 \in \mathcal{I}$

Teoremă (E.U.G): Fie $A(\cdot): \mathcal{I} \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ continuă, $\frac{dx}{dt} = A(t)x$.

$\forall (t_0, x_0) \in \mathcal{I} \times \mathbb{R}^n$, $\exists!\ \varphi_{t_0,x_0}(\cdot): \mathcal{I} \to \mathbb{R}^n$ soluție cu $\varphi_{t_0,x_0}(t_0) = x_0$.

$S_{A(\cdot)} := \{\varphi(\cdot): \mathcal{I} \to \mathbb{R}^n;\ \varphi(\cdot)$ soluție, $x' = A(t)x\}$

$A(\cdot)$ continuă $\Rightarrow S_{A(\cdot)} \subset C^1(\mathcal{I}, \mathbb{R}^n)$ (clasă)

Prop (Soluția banală): Fie $A(\cdot): \mathcal{I} \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ continuă, $\frac{dx}{dt} = A(t)x$. dacă $\varphi(\cdot) \in S_{A(\cdot)}$ astfel încât $\exists\, t_0 \in \mathcal{I}$, $\varphi(t_0) = 0$ atunci $\varphi(t) \equiv 0$.

Dem: $\varphi(\cdot): \mathcal{I} \to \mathbb{R}^n$ soluție, $\varphi(t_0) = 0$

Fie $\psi(\cdot): \mathcal{I} \to \mathbb{R}^n$, $\psi(t) = 0$ soluție, $\psi(t_0) = 0$

$\overset{U.G.}{\Longrightarrow} \varphi(t) \equiv \psi(t)$ OK.

Teoremă (Spațiul soluțiilor): Fie $A(\cdot): \mathcal{I} \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ continuă, $\frac{dx}{dt} = A(t)x$.

Atunci $S_{A(\cdot)} \subset C^1(\mathcal{I}, \mathbb{R}^n)$ este subspațiu vectorial cu $\dim(S_{A(\cdot)}) = n$.

Dem: $\forall c_1, c_2 \in \mathbb{R}$, $\varphi_1(\cdot), \varphi_2(\cdot) \in S_{A(\cdot)} \Rightarrow c_1\varphi_1 + c_2\varphi_2 \in S_{A(\cdot)}$

$(c_1\varphi_1 + c_2\varphi_2)'(t) = c_1\varphi_1'(t) + c_2\varphi_2'(t) \equiv c_1 A(t)\varphi_1(t) + c_2 A(t)\varphi_2(t) =$

$= A(t)(c_1\varphi_1(t) + c_2\varphi_2(t)) = A(t)(c_1\varphi_1 + c_2\varphi_2)(t)$

Arătăm că $S_{A(\cdot)} \overset{\text{izomorf}}{\simeq} \mathbb{R}^n$.

Aplicația de evaluare în punctul $t_0 \in J$: $E_{t_0} : S_{A(\cdot)} \to \mathbb{R}^n$, $E_{t_0}(\varphi) := \varphi(t_0)$.

Arătăm că E_{t_0} izomorfism: - (a) liniară

- (b) injectivă

- (c) surjectivă

(a) $c_1, c_2 \in \mathbb{R}$, $\varphi_1, \varphi_2 \in S_{A(\cdot)}$:

$$E_{t_0}(c_1\varphi_1 + c_2\varphi_2) = (c_1\varphi_1 + c_2\varphi_2)(t_0) = c_1\varphi_1(t_0) + c_2\varphi_2(t_0) = c_1 E_{t_0}(\varphi_1) + c_2 E_{t_0}(\varphi_2)$$

(b) $E_{t_0}(\varphi_1) = E_{t_0}(\varphi_2)$, $\varphi_1(t_0) = \varphi_2(t_0) \overset{U.G.}{\Longrightarrow} \varphi_1 = \varphi_2$

(c) $\forall\, \xi \in \mathbb{R}^n$, $\exists\, \varphi \in S_{A(\cdot)}$ astfel încât $\underset{\varphi(t_0)}{\underset{\parallel}{E_{t_0}(\varphi)}} = \xi$

Deci T.E.G. aplicată în $(t_0, \xi) \Rightarrow \exists\, \varphi(\cdot) : \to \mathbb{R}^n$ soluție cu $\varphi(t_0) = \xi$

Def: Se numește sistem fundamental de soluții al ecuației $x' = A(t)x$,

$\{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\} \subset S_{A(\cdot)}$ bază.

Obs: $\varphi(\cdot) \in S_{A(\cdot)} \iff \exists\, c_i \in \mathbb{R}$, $i = \overline{1,n}$ astfel încât: $\varphi(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$

(soluția generală a ecuației)

Prop (Soluții liniar independente): Fie $A(\cdot) : J \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ continuă, $\frac{dx}{dt} = A$

Următoarele afirmații sunt echivalente:

(a) $\{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\} \subset S_{A(\cdot)}$ sunt liniar independente

(b) $\exists\, t_0 \in J$ astfel încât $\{\varphi_1(t_0), \dots, \varphi_n(t_0)\} \subset \mathbb{R}^n$ sunt liniari independente (ca vect. în $\mathbb{R}^n$)

(c) $\{\varphi_1(t), \dots, \varphi_n(t)\} \subset \mathbb{R}^n$ sunt liniar independenți, $\forall\, t \in J$

Dem: (a) $\Rightarrow$ (c): Fie $c_1\varphi_1(t) + \dots + c_n\varphi_n(t) = 0$

$c_1 E_t(\varphi_1) + \dots + c_n E_t(\varphi_n) = 0$

$E_t(c_1\varphi_1 + \dots + c_n\varphi_n) = 0 \Rightarrow c_1\varphi_1 + \dots + c_n\varphi_n = 0 \overset{a)}{\Rightarrow} c_1 = \dots = c_n = 0$

(c) $\Rightarrow$ (b) evident

(b) $\Rightarrow$ (a): $c_1\varphi_1 + \dots + c_n\varphi_n = 0$

$$E_{t_0}(c_1\varphi_1+\dots+c_m\varphi_m)=0$$

$$c_1\varphi_1(t_0)+\dots+c_m\varphi_m(t_0)=0 \overset{b)}{\Longrightarrow} c_1=\dots=c_m=0$$

Obs: a) $\{\varphi_1(\cdot),\dots,\varphi_m(\cdot)\}\subset C^1(J,\mathbb{R}^n)$ sunt liniar independente

$\Uparrow$ $\not\Downarrow$

b) $\exists\, t_0\in J$ astfel încât $\{\varphi_1(t_0),\dots,\varphi_m(t_0)\}\subset\mathbb{R}^n$ sunt liniar indep.

$\Uparrow$ $\not\Downarrow$

c) $\{\varphi_1(t),\dots,\varphi_m(t)\}\subset\mathbb{R}^n$ sunt liniar indep., $\forall t\in J$

Matrici de soluții. Soluții matriceale. WRONSKIAN

Def: a) $\varphi_1(\cdot),\dots,\varphi_m(\cdot)\in S_{A(\cdot)}$, $X(t)=\mathrm{col}(\varphi_1(t),\dots,\varphi_m(t))$ se numește matrice de soluții

b) $X(\cdot):J\to M_{n,m}(\mathbb{R})$ se numește soluție matriceală dacă $\exists\, B\subset\mathbb{R}^n$ bază a.î.

$X'(t)\equiv A_B(t)X(t)$.

Prop: $X(\cdot):J\to M_{n,m}(\mathbb{R})$, $X(\cdot)$ este matrice de soluții $\Leftrightarrow$ $X(\cdot)$ este soluție matriceală.

Dem: "$\Rightarrow$" $X(t)=\mathrm{col}(\varphi_1(t),\dots,\varphi_m(t))$, $\varphi_i(\cdot)\in S_{A(\cdot)}$, $i=\overline{1,m}$

$X'(t)=\mathrm{col}(\varphi_1'(t),\dots,\varphi_m'(t))\equiv\mathrm{col}(A(t)\varphi_1(t),\dots,A(t)\varphi_m(t))\equiv A_B(t)\cdot\mathrm{col}(\varphi_1(t),\dots,\varphi_m(t))\equiv$

$\equiv A_B(t)X(t)$

"$\Leftarrow$" $X'(t)\equiv A_B(t)\cdot X(t)$

Fie $X(t)=\mathrm{col}(\varphi_1(t),\dots,\varphi_m(t))$ $\Bigg\}\Rightarrow \mathrm{col}(\varphi_1'(t),\dots,\varphi_m'(t))\equiv$

$\equiv A_B(t)\,\mathrm{col}(\varphi_1(t),\dots,\varphi_m(t))\equiv$

$\equiv\mathrm{col}(A(t)\varphi_1(t),\dots,A(t)\varphi_m(t))\Rightarrow$

$\Rightarrow\varphi_i'(t)\equiv A(t)\varphi_i(t)$, $i=\overline{1,m}$ OK.

Def: Se numește WRONSKIANUL soluțiilor $\varphi_1(\cdot),\dots,\varphi_n(\cdot)\in S_{A(\cdot)}$ funcția

$$W_{\varphi_1,\dots,\varphi_n}(t):=\det[\mathrm{col}(\varphi_1(t),\dots,\varphi_n(t))],\ t\in J$$

Teorema lui Liouville: Fie $A(\cdot):J\subseteq\mathbb{R}\to L(\mathbb{R}^n,\mathbb{R}^n)$ continuă, $\frac{dx}{dt}=A(t)x$. Fie $\varphi_1(\cdot),\dots,\varphi_n(\cdot)\in S_{A(\cdot)}$. Atunci $W_{\varphi_1,\dots,\varphi_n}(t)\equiv W_{\varphi_1,\dots,\varphi_n}(t_0)\cdot e^{\int_{t_0}^{t}\mathrm{Tr}(A(s))\,ds}$.

Tr – urma

Dem:

Obs: A demonstra teorema revine la a demonstra că $t\to W_{\varphi_1,\dots,\varphi_n}(t)$ este soluția ecuației scalare $\frac{dy}{dt}=\mathrm{Tr}(A(t))y$.

$$y(t) = c \cdot e^{\int_{t_0}^{t} Tr(A(s)) \cdot ds}$$

$$= y(t_0) \cdot e^{\int_{t_0}^{t} Tr(A(s)) \cdot ds}$$

$t = t_0 \Rightarrow y(t_0) = c$

$B \subset \mathbb{R}^n$ bază $\Rightarrow A_B(t) = \left(a_{ij}(t) \right)_{\substack{i=\overline{1,n} \\ j=\overline{1,n}}}$; $\varphi_i(\cdot) = \left(\varphi_i^j(\cdot) \right)_{j=\overline{1,n}}$, $i = \overline{1,n}$

$$W_{\varphi_1, \dots, \varphi_n}(t) = \det \left[col(\varphi_1(t), \dots, \varphi_n(t)) \right] = \det \left(\left(\varphi_i^j(t) \right)_{i,j=\overline{1,n}} \right) =$$

$$= \sum_{\tau \in S_n} sgn(\tau)\, \varphi^1_{\tau(1)}(t) \dots \varphi^j_{\tau(j)}(t) \dots \varphi^n_{\tau(n)}(t)$$

$$= \sum_{j=1}^{n} \sum_{\tau \in S_n} sgn\,\tau\, \varphi^1_{\tau(1)}(t) \dots \varphi^{j-1}_{\tau(j-1)}(t) \underbrace{\left(\varphi^j_{\tau(j)}(t) \right)'}_{= \sum_{k=1}^{n} a_{jk}(t)\, \varphi^k_{\tau(j)}(t)} \varphi^{j+1}_{\tau(j+1)}(t) \dots \varphi^n_{\tau(n)}(t) =$$

$\varphi_i(\cdot)$ soluție

$$= \sum_{j,k=1}^{n} a_{jk}(t) \underbrace{\sum_{\tau \in S_n} sgn\,\tau\, \varphi^1_{\tau(1)}(t) \dots \varphi^{j-1}_{\tau(j-1)}(t)\, \varphi^k_{\tau(j)}(t)\, \varphi^{j+1}_{\tau(j+1)}(t)}_{\Delta}$$

$$= \sum_{j,k=1}^{n} a_{jk}(t) \Delta_{jk}(t) = \sum_{j=1}^{n} a_{jj}(t)\, W_{\varphi_1, \dots, \varphi_n}(t) = Tr(A(t)) \cdot W_{\varphi_1, \dots, \varphi_n}(t)$$

$$\Delta_{jk}(t) = \begin{cases} W_{\varphi_1, \dots, \varphi_n}(t), & j = k \\ 0, & j \neq k \end{cases}$$

Prop (curent global al ecuațiilor liniare): Fie $A(\cdot): J \subset \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ continuă, $\frac{dx}{dt} = A(t)x$. Considerăm $\alpha_{A(\cdot)}(\cdot, \cdot, \cdot): J \times J \times \mathbb{R}^n \to \mathbb{R}^n$ curentul global al ecuației. Atunci $\alpha_{A(\cdot)}(t, \tau, \cdot) \in L(\mathbb{R}^n, \mathbb{R}^n)$, $\forall\, t, \tau \in J$.

Dem: Fie $c_1, c_2 \in \mathbb{R}$, $\xi_1, \xi_2 \in \mathbb{R}^n$, $\alpha_{A(\cdot)}(t, \tau, c_1\xi_1 + c_2\xi_2) = c_1 \alpha_{A(\cdot)}(t, \tau, \xi_1) + c_2 \alpha_A(t, \tau, \xi_2)$

$\tau \in J$

$t \to \alpha_{A(\cdot)}(t, \tau, c_1\xi_1 + c_2\xi_2)$ soluție globală a problemei Cauchy $(f_{A(\cdot)}, \tau,$

$t \to \alpha_{A(\cdot)}(t,\tau,\xi_1)$ soluție globală a problemei Cauchy $(f_{A(\cdot)},\tau,\xi_1)$ $\Big|$ $S_{A(\cdot)} \longrightarrow$

$t \to \alpha_{A(\cdot)}(t,\tau,\xi_2)$ soluție globală a problemei Cauchy $(f_{A(\cdot)},\tau,\xi_2)$ $\Big|$ sp. vectorial

$\Rightarrow t \to c_1\,\alpha_{A(\cdot)}(t,\tau,\xi_1)+c_2\cdot\alpha_{A(\cdot)}(t,\tau,\xi_2)$ soluție a problemei Cauchy $(f_{A(\cdot)},\tau,c_1\xi_1+c_2\xi_2)$ (2)

(1),(2) + U.G $\Rightarrow$ q.e.d

Def: Se numește rezolvanta ecuației liniare $x'=A(t)x$, funcția $\mathcal{R}_{A(\cdot)}(\cdot,\cdot): I\times I\to L(\mathbb{R}^n,\mathbb{R}^n)$

$$R_{A(\cdot)}(t,\tau)\,\xi := \alpha_{A(\cdot)}(t,\tau,\xi) = \varphi_{\tau,\xi}(t)$$

„Rezolvă ecuația" $\varphi_{\tau,\xi}(t) = R_{A(\cdot)}(t,\tau)\,\xi$

Def: Se numește matrice fundamentală de soluții a ecuației $x'=A(t)x$, matricea $X(t)=col(\varphi_1(t),\dots,\varphi_n(t))$ unde $\{\varphi_1(\cdot),\dots,\varphi_n(\cdot)\}\subset S_{A(\cdot)}$ sistem fundamental de soluții.

$\varphi(\cdot)\in S_{A(\cdot)} \Leftrightarrow \exists\, c\in\mathbb{R}^n$ a.î. $\varphi(t)\equiv X(t)\cdot c$

$$\left[\, c\in\mathbb{R}^n,\ c=\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix};\ X(\cdot) \text{ matrice fundamentală de soluții } X(t)c = col(\varphi_1(t),\dots,\varphi_n(t))\begin{pmatrix}c_1\\ \vdots\\ c_n\end{pmatrix} = \sum_{i=1}^{n} c_i\,\varphi_i(t) \,\right]$$

Teoremă (Proprietățile rezolvantei). Fie $A(\cdot): I\subseteq\mathbb{R}\to L(\mathbb{R}^n,\mathbb{R}^n)$ continuă, $\frac{dx}{dt}=A(t)x$, fie $R_{A(\cdot)}(\cdot,\cdot): I\times I\to L(\mathbb{R}^n,\mathbb{R}^n)$ rezolvanta ecuației. Atunci:

1. $R_{A(\cdot)}(t,t)=I_n$, $\forall t$
2. $R_{A(\cdot)}(t,\tau)\,R_{A(\cdot)}(\tau,s)=R_{A(\cdot)}(t,s)$, $\forall\, t,\tau,s\in I$
3. $\exists\,(R_{A(\cdot)}(t,\tau))^{-1}=R_{A(\cdot)}(\tau,t)$, $\forall\, t,\tau\in I$
4. $\forall\, B\subset\mathbb{R}^n$ bază și $\forall\, X(\cdot): I_1\to M_n(\mathbb{R})$ matrice fundamentală de soluții
$$R^B_{A(\cdot)}(t,\tau)=X(t)\,X^{-1}(\tau),\ \forall\, t,\tau\in I$$
5. $\forall\, B=\{b_1,\dots,b_n\}\subset\mathbb{R}^n$ bază, $\forall\,\tau\in I$, $t\to R^B_{A(\cdot)}(t,\tau)=col(\varphi_{\tau,b_1}(t),\dots,\varphi_{\tau,b_n}(t))$ matrice fundamentală de soluții
6. $$\det(R_{A(\cdot)}(t,\tau)) = e^{\int_\tau^t Tr(A(s))\,ds},\ \forall\,\tau,t\in I$$

Dem: (6) $\det(R_{A(\cdot)}(t,\tau)) = \det(R^{B}_{A(\cdot)}(t,\tau)) \overset{5)}{=} \det(\mathrm{col}(\varphi_{\tau,b_1}(t), \dots, \varphi_{\tau,b_n}(t))) =$

$= W_{\varphi_{\tau,b_1},\dots,\varphi_{\tau,b_n}}(t) \underset{\text{Liouville}}{\overset{T}{=}} W_{\varphi_{\tau,b_1},\dots,\varphi_{\tau,b_n}}(\tau) \cdot e^{\int_\tau^t Tr(A(s))\,ds} =$

$= \det(\mathrm{col}(\varphi_{\tau,b_1}(\tau), \dots, \varphi_{\tau,b_n}(\tau))) \cdot e^{\int_\tau^t Tr(A(s))\,ds} =$

$= \underbrace{\det(\mathrm{col}(b_1, \dots, b_n))}_{=1} \cdot e^{\int_\tau^t Tr(A(s))\,ds}$

baza canonică

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$\frac{dx}{dt} = A(t)x$, $A(\cdot) : J \subset \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ continuă

$S_{A(\cdot)} = \{ \varphi(\cdot) : J \to \mathbb{R}^n ;\ \varphi(\cdot)$ soluție $x' = A(t)x \}$

WRONSKIANUL asociat soluțiilor $\varphi_1(\cdot), \dots, \varphi_n(\cdot) \in S_{A(\cdot)}$

$W_{\varphi_1, \dots, \varphi_n}(t) := \det[\operatorname{col}(\varphi_1(t), \dots, \varphi_n(t))]$, $t \in J$

Teorema lui Liouville : $W_{\varphi_1, \dots, \varphi_n}(t) = W_{\varphi_1, \dots, \varphi_n}(\tau)\, e^{\int_\tau^t Tr(A(s))\cdot ds}$, $\forall\, t, \tau \in J$

Exercițiu : Fie ecuațiile : $\begin{cases} x' = e^t y - e^{2t} x \\ y' = (e^t - e^{3t})x + e^{2t} y \end{cases}$, $\varphi_1(t) = \begin{pmatrix} 1 \\ e^t \end{pmatrix}$ soluție

$\varphi_2(\cdot) = ?$ astfel încât $\{\varphi_1(\cdot), \varphi_2(\cdot)\}$ sistem fundamental de soluții.

$S_{A(\cdot)} \subset C^1(J, \mathbb{R}^n)$ subspațiu vectorial $\dim(S_{A(\cdot)}) = n$.

$\{\varphi_1(\cdot), \dots, \varphi_n(\cdot)\} \subset S_{A(\cdot)}$ bază se numește sistem fundamental de soluții

$\varphi(\cdot) \in S_{A(\cdot)} \iff \exists\, c_i \in \mathbb{R}$ a.î. $\varphi(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$ soluția generală

Fie $\varphi_2(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$; $A(t) = \begin{pmatrix} -e^{2t} & e^t \\ e^t - e^{3t} & e^{2t} \end{pmatrix}$

φ_2 soluție $\Rightarrow \begin{cases} a'(t) = e^t b(t) - e^{2t} a(t) \\ b'(t) = (e^t - e^{3t}) \cdot a(t) + e^{2t} \cdot b(t) \end{cases}$

$W(t) = \det[\operatorname{col}(\varphi_1(t), \varphi_2(t))] = \det\begin{pmatrix} 1 & a(t) \\ e^t & b(t) \end{pmatrix} = b(t) - e^t a(t)$

Din T. Liouville $\Rightarrow W(t) = W(\tau) \cdot e^{\int_\tau^t Tr(A(s))\cdot ds} = W(\tau) \cdot e^0 = W(\tau)$, $\forall\, t, \tau \in J$ $\Rightarrow$

$\Rightarrow \exists\, c \in \mathbb{R}$ a.î. $W(t) = c \iff b(t) - e^t a(t) = c \Rightarrow b(t) = c + e^t a(t)$

$a'(t) = e^t \cdot c + \cancel{e^{2t} a(t)} - \cancel{e^{2t} a(t)} = e^t \cdot c$

$a(t) = \int e^t \cdot c\, dt = c \cdot e^t + k$, $k \in \mathbb{R}$. , $k \in \mathbb{R}$

$b(t) = c + e^t (c \cdot e^t + k) = c + c \cdot e^{2t} + k \cdot e^t$, $k \in \mathbb{R}$

$\varphi_2(t) = \begin{pmatrix} c e^t + k \\ c + c \cdot e^{2t} + k \cdot e^t \end{pmatrix}$, $k \in \mathbb{R}$ $\to$ soluția generală

Exercițiu: Fie ecuațiile $\begin{cases} x' = y - tx \\ y' = (1-t^2)x + ty \end{cases}$. Să se determine soluția generală.

$y = x' + tx$

$y(t) = x'(t) + t \cdot x(t)$

$(x'(t) + t \cdot x(t))' = (1-t^2)\, x(t) + t \cdot (x'(t) + t \cdot x(t))$

$x''(t) + \cancel{x(t)} + \cancel{x'(t)\, t} = \cancel{x(t)} - \cancel{t^2 x(t)} + \cancel{t\, x'(t)} + \cancel{t^2 \cdot x(t)}$

$x''(t) = 0 \Rightarrow \exists\, c \in \mathbb{R}$ a.î. $x'(t) = c,\ c \in \mathbb{R}$

$x(t) = ct + c_2,\ c, c_2 \in \mathbb{R}$

$y(t) = c + ct^2 + c_2 t,\ c, c_2 \in \mathbb{R}$

deoarece urma matricei $= 0$

Exercițiu: Fie ecuațiile $\begin{cases} x' = y + tx \\ y' = (1-t^2)x + ty \end{cases}$

$x' = y + tx \Rightarrow y(t) = x'(t) - t\,x(t)$

$(x'(t) - t\,x(t))' = (1-t^2)\,x + t(x'(t) - t\,x(t))$

$x''(t) - x(t) - t\,x'(t) = x - t^2 x + t\,x'(t) - t^2 x(t)$

$x''(t) - 2t\,x'(t) + (2t^2 - 2)\,x(t) = 0$

$x'' - 2t\,x' + (2t^2 - 2)\,x = 0$

Exercițiu: Fie ecuațiile $\begin{cases} x' = y - tx \\ y' = (1-t^2)x + ty \end{cases}$, $\varphi_1(t) = \begin{pmatrix} 1 \\ t^2+1 \end{pmatrix}$ soluție. Să se găsească $\varphi_2(\cdot) = ?$ astfel încât $\{\varphi_1(\cdot), \varphi_2(\cdot)\}$ sistem fundamental de soluții.

$\varphi_2(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$, $A(t) = \begin{pmatrix} -t & 1 \\ 1-t^2 & t \end{pmatrix}$

$\varphi_2(t)$ soluție $\Rightarrow \begin{cases} a'(t) = b(t) - t\,a(t) \\ b'(t) = (1-t^2)\,a(t) + t \cdot b(t) \end{cases}$

T. Liouville $\Rightarrow W(t) = W(\tau) \cdot e^{\int_\tau^t Tr(A(s))\, ds} = W(\tau) e^0 = W(\tau), \ \forall t, \tau \Rightarrow$

$\Rightarrow \exists\, c \in \mathbb{R}$ astfel încât $W(t) = c \iff t\, b(t) - (t^2+1) \cdot a(t) = c \Rightarrow$

$$\Rightarrow b(t) = \frac{c + (t^2+1) \cdot a(t)}{t}, \quad t > 0$$

$$a'(t) = \frac{c + (t^2+1) \cdot a(t)}{t} - t \cdot a(t)$$

$$a'(t) = \frac{c + (t^2+1) \cdot a(t) - t^2 \cdot a(t)}{t} \iff a'(t) = \frac{c}{t} + \frac{a(t)}{t}$$

$a' = \frac{a}{t} + \frac{c}{t}$ → ecuație afină scalară (rez. prin metoda variației constantelor)

$$\bar{a}' = \frac{\bar{a}}{t}$$

$$\bar{a}(t) = k \cdot e^{\int \frac{1}{t}\, dt} = k \cdot e^{\ln|t|} = k \cdot t, \quad k \in \mathbb{R}$$

$$a(t) = k(t) \cdot t \Rightarrow (k(t) \cdot t)' = \frac{k(t) \cdot t}{t} + \frac{c}{t}$$

$$k'(t) \cdot t + k(t) \cdot t' = k(t) + \frac{c}{t}$$

$$k'(t) \cdot t + \cancel{k(t)} = \cancel{k(t)} + \frac{c}{t}$$

$$k'(t) = \frac{c}{t^2} \Rightarrow k(t) = \int \frac{c}{t^2}\, dt = c \int \frac{1}{t^2}\, dt =$$

$$= -\frac{c}{t} + d, \ d \in \mathbb{R} \Rightarrow \boxed{a(t) = -c + dt}$$

$$b(t) = \frac{c}{d} + \frac{t^2+1}{t}(-c + dt) = \frac{c}{d} - \frac{c(t^2+1)}{t} + d(t^2+1) = -ct + dt^2 + d \Rightarrow$$

$$\Rightarrow \varphi_2(t) = \begin{pmatrix} -c + dt \\ -ct + dt^2 + d \end{pmatrix}, \quad c, d \in \mathbb{R}$$

$$\varphi_2(t) = c \underbrace{\begin{pmatrix} -1 \\ -t \end{pmatrix}}_{\varphi_2(t)} + d \underbrace{\begin{pmatrix} t \\ 1 + t^2 \end{pmatrix}}_{\varphi_1(t)}$$

Exercițiu: $J = ?$, $A(\cdot) = ?$, $A(\cdot): J \to L(\mathbb{R}^2, \mathbb{R}^2)$ astfel încât $\varphi_1(\cdot), \varphi_2(\cdot) \in S_{A(\cdot)}$, unde $\varphi_1(t) = \begin{pmatrix} 1 \\ t^2 \end{pmatrix}$ și $\varphi_2(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Metoda 1:

Fie $A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}$ → sistemul asociat este $\begin{cases} x' = a(t)x + b(t)y \\ y' = c(t)x + d(t) \cdot y \end{cases}$

φ_1 soluție $\Rightarrow \begin{cases} 1' = a(t) + t^2 \cdot b(t) \end{cases} \iff \begin{cases} 0 = a(t) + t^2 b(t) \end{cases}$

$\overset{T.}{=}$ Liouville $\Rightarrow W(t) = W(\tau)\cdot e^{\int_\tau^t Tr(A(s))\,ds} = W(\tau)\, e^0 = W(\tau)\ ,\ \forall t,\tau \Rightarrow$

$\Rightarrow \exists\, c\in\mathbb{R}$ astfel încât $W(t) = c \Leftrightarrow t\,b(t) - (t^2+1)\cdot a(t) = c \Rightarrow$

$$\Rightarrow b(t) = \frac{c+(t^2+1)\cdot a(t)}{t}\ ,\ t>0$$

$$a'(t) = \frac{c+(t^2+1)\cdot a(t)}{t} - t\cdot a(t)$$

$$a'(t) = \frac{c+(t^2+1)\cdot a(t) - t^2\cdot a(t)}{t} \Leftrightarrow a'(t) = \frac{c}{t} + \frac{a(t)}{t}$$

$a' = \frac{a}{t} + \frac{c}{t} \rightarrow$ ecuație afină scalară (rez. prin metoda variației constantelor)

$$\bar{a}' = \frac{\bar{a}}{t}$$

$$\bar{a}(t) = k\cdot e^{\int \frac{1}{t}\,dt} = k\cdot e^{\ln|t|} = k\cdot t\ ,\ k\in\mathbb{R}$$

$$a(t) = k(t)\cdot t \Rightarrow \big(k(t)\cdot t\big)' = \frac{k(t)\cdot t}{t} + \frac{c}{t}$$

$$k'(t)\cdot t + k(t)\cdot t' = k(t) + \frac{c}{t}$$

$$k'(t)\cdot t + \cancel{k(t)} = \cancel{k(t)} + \frac{c}{t}$$

$$k'(t) = \frac{c}{t^2} \Rightarrow k(t) = \int \frac{c}{t^2}\,dt = c\int\frac{1}{t^2}\,dt =$$

$$= -\frac{c}{t} + d\ ,\ d\in\mathbb{R} \Rightarrow \boxed{a(t) = -c + dt}$$

$$b(t) = \frac{c}{d} + \frac{t^2+1}{t}\left(-c+dt\right) = \frac{c}{d} - \frac{c(t^2+1)}{t} + d(t^2+1) = -ct + dt^2 + d \Rightarrow$$

$$\Rightarrow \varphi_2(t) = \begin{pmatrix} -c+dt \\ -ct + dt^2 + d \end{pmatrix}\ ,\ c,d\in\mathbb{R}$$

$$\varphi_2(t) = c\underbrace{\begin{pmatrix} -1 \\ -t \end{pmatrix}}_{\varphi_2(t)} + d\underbrace{\begin{pmatrix} t \\ 1+t^2 \end{pmatrix}}_{\varphi_1(t)}$$

Exercițiu: $J=?$, $A(\cdot)=?$, $A(\cdot): J\to L(\mathbb{R}^2,\mathbb{R}^2)$ astfel încât $\varphi_1(\cdot), \varphi_2(\cdot) \in S_{A(\cdot)}$, unde $\varphi_1(t) = \begin{pmatrix} 1 \\ t^2\end{pmatrix}$ și $\varphi_2(t) = \begin{pmatrix} 1 \\ 1\end{pmatrix}$.

Metoda 1:

Fie $A(t) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t)\end{pmatrix} \rightarrow$ sistemul asociat este $\begin{cases} x' = a(t)x + b(t)y \\ y' = c(t)x + d(t)\cdot y\end{cases}$

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