31.10.2017.

CURS 5 EC. DIFERENȚIALE

Existența și unicitatea globală a soluțiilor.

Teoremă (Unicitatea Globală)

Fie $f(\cdot,\cdot): D \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ cont $\frac{dx}{dt} = f(t,x)$

Atunci $f(\cdot,\cdot)$ admite UG a sol. pe $D \Leftrightarrow f(\cdot,\cdot)$ admite UL a sol pe D.

Dem: "$\Rightarrow$" Evident.

"$\Leftarrow$" $\varphi_1 : I_1 \to \mathbb{R}^n$
$\varphi_2 : I_2 \to \mathbb{R}^n$ sol pt care $\exists\, t_0 \in I_1 \cap I_2$ a.î. $\varphi_1(t_0) = \varphi_2(t_0)$

$\overset{?}{\Rightarrow} \varphi_1|_{I_1\cap I_2} = \varphi_2|_{I_1\cap I_2}$

Tehnică generală de globalizare

$I^* = \{t \in I_1\cap I_2;\ \varphi_1(t) = \varphi_2(t)\} \subset I_1 \cap I_2$ - interval $\to$ m. conexă

[M conexă $\Leftrightarrow$ deschisă, închisă, nu are submulțimi cu aceste propr.]

Arăt că: a) $I^* \neq \emptyset$
b) I^* închisă
c) I^* deschisă
$\Rightarrow I^* = I_1 \cap I_2$.

a) $I^* \neq \emptyset$ pt că $t_0 \in I_1 \cap I_2$

b) $I^* = \{t \in I_1\cap I_2;\ \varphi_1(t) = \varphi_2(t)\} = \{t \in I_1\cap I_2;\ (\varphi_1-\varphi_2)(t) = \dots$

$= \underbrace{(\varphi_1 - \varphi_2)}_{\text{cont.}}{}^{-1}(\underbrace{\{0\}}_{\text{închisă}})$ - închisă

c) I^* deschisă

$t_1 \in I^* \Rightarrow \varphi_1(t_1) = \varphi_2(t_1) \overset{U.L.}{\Longrightarrow} \exists\, I_0 \in \mathcal{V}(t_1)$ a.î.

$\varphi_1|_{I_0\cap(I_1\cap I_2)} = \varphi_2|_{I_0\cap(I_1\cap I_2)}$

$\Rightarrow I_0 \cap (I_1\cap I_2) \subset I^*$ deci I^* deschisă.

Prelungirea soluțiilor. Soluții maximale.

Def: a) $\varphi_1(\cdot): I_1 \subseteq \mathbb{R} \to \mathbb{R}^n$ s.n. prelungire a lui $\varphi_2: I_2 \subseteq \mathbb{R} \to \mathbb{R}^n$
dacă $I_1 \supset I_2$ și $\varphi_1(\cdot)$ ...

ii) $\varphi_1(\cdot): (a, b_1) \subseteq \mathbb{R} \to \mathbb{R}^n$ s.n. prelungire la dreapta a lui $\varphi_2(\cdot): (a, b_2) \subseteq \mathbb{R} \to \mathbb{R}^n$ dacă $b_2 < b_1$ şi $\varphi_1(\cdot)|_{(a,b_2)} = \varphi_2(\cdot)$.

$\varphi_1(\cdot) \succ_d \varphi_2$

c) $\varphi_1(\cdot): (a_1, b) \subseteq \mathbb{R} \to \mathbb{R}^n$ s.n. prelungire la stânga a lui $\varphi_2(\cdot): (a_2, b) \subseteq \mathbb{R} \to \mathbb{R}^n$ dacă $a_1 < a_2$ şi $\varphi_1(\cdot)|_{(a_2, b)} = \varphi_2(\cdot)$.

$\varphi_1(\cdot) \succ_s \varphi_2(\cdot)$

PROP: Fie $f(\cdot,\cdot): D \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ $\quad \frac{dx}{dt} = f(t,x)$

$$S_f := \{\varphi(\cdot);\ \varphi(\cdot): I \subseteq \mathbb{R} \to \mathbb{R}^n \ \varphi(\cdot) \text{ sol a ec } \frac{dx}{dt} = f(t,x)\}$$

Atunci $(S_f, \succ), (S_f, \succ_d), (S_f, \succ_s)$ sunt mulţimi ordonate

Dem: Ex!

Def: $\varphi(\cdot) \in S_f$ s.n. soluţie maximală dacă este un element maximal $(S_f, \ )$

Teorema asupra prelungirii soluţiilor:

Fie $f(\cdot,\cdot): D = \mathring{D} \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ cont $\quad \frac{dx}{dt} = f(t,x)$

Fie $\varphi(\cdot): (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ soluţie.

Atunci

$\to$ + se poate fi prelungită cu ea însăşi

1. $\varphi(\cdot)$ admite o prelungire strictă la dreapta $\Leftrightarrow b < +\infty$ $\exists\, t_0 \in (a, b)$, $\exists\, D_0 \subset D$ compactă a.î. $(t, \varphi(t)) \in D_0\ \forall t \in [t_0, b)$

graficul fc.

2. $\varphi(\cdot)$ admite o prelungire strictă la stânga $\Leftrightarrow a > -\infty$, $\exists\, t_0 \in (a, b)$, $\exists\, D_0 \subset D$ compactă a.î. $(t, \varphi(t)) \in D_0\ \forall t \in (a, t_0]$

Dem: 1) (2) analog)

"$\Rightarrow$" $\varphi(\cdot)$ admite o prelungire strictă la dreapta

$\Rightarrow \exists\, \varphi_1(\cdot): (a, b_1) \to \mathbb{R}^n$ sol $b < b_1$ $\varphi_1(\cdot) \succ_{\neq} \varphi(\cdot)$

$b < b_1 \Rightarrow b < +\infty$ Fie $t_0 \in (a, b)$

$D_0 := \{t, \varphi_1(t);\ t \in \underbrace{[t_0, b]}_{\text{compact}}\} \subset D$ ($\varphi_1(\cdot)$ cont) compactă

$t \in [t_0, b)$ $(t, \varphi(t)) = (t, \varphi_1(t)) \in D_0$ OK.

2. Aplicăm T Peano în $(b, \xi) \Rightarrow \exists\, \varphi_0(\cdot) : [b-\alpha, b+\alpha] \to \mathbb{R}^n$ sol a pb. $C(f, b, \xi)$.

3. $\varphi_1(t) := \begin{cases} \varphi(t), & t \in (a, b) \\ \varphi_0(t) & t \in [b, b+\alpha] \end{cases}$.

$\varphi_1(\cdot) \underset{\neq}{\succ} \varphi(\cdot)$

1. Criteriul lui Cauchy

$[\forall\, \varepsilon > 0\ \exists\, \delta_\varepsilon > 0$ a.î. $\forall\, t', t'' \in (b - \delta_\varepsilon, b)$ $\qquad \|\varphi(t') - \varphi(t'')\|$
$< \varepsilon \Rightarrow \exists \lim\limits_{\substack{t \to b \\ t < b}} \varphi(t)]$

$\varphi(\cdot)$ sol $\Rightarrow$ ec integrală asoc. $\varphi(t) = \varphi(t_0) + \int_{t_0}^{t} f(s, \varphi(s))\, ds$
$\forall\, t, t_0 \in (a, b)$

$\varphi(t') = \varphi(t_0) + \int_{t_0}^{t} f(s, \varphi(s))\, ds$

$\varphi(t'') = \varphi(t_0) + \int_{t_0}^{t''} f(s, \varphi(s))\, ds$

$\|\varphi(t') - \varphi(t'')\| = \left\| \int_{t''}^{t} f(s, \varphi(s))\, ds \right\| \le \left| \int_{t''}^{t} \underbrace{\|f(s, \varphi(s))\|}_{\le k} ds \right|$

$K := \max\limits_{(t,x) \in \Delta_0} \|f(t,x)\|$ $\qquad \le K|t' - t''|$
$\qquad (\delta_\varepsilon = \frac{\varepsilon}{K})$

2. $\varphi_1(\cdot)$ soluție

$\underline{\varphi_{1,s}'(b)} = \lim\limits_{\substack{t \to b \\ t < b}} \frac{\varphi_1(t) - \varphi_1(b)}{t - b} = \lim\limits_{\substack{t \to b \\ t < b}} \frac{\varphi(t) - \xi}{t - b} \overset{L'H}{=} \lim\limits_{\substack{t \to b \\ t < b}} \frac{\varphi'(t)}{1}$

$= \lim\limits_{\substack{t \to b \\ t < b}} f(t, \varphi(t)) = f(b, \xi) = \underline{\cancel{f(b, \varphi_1(b))}} = \varphi_{0,d}'(b) = \underline{\varphi_{1,d}'(b)}$

Teorema (Existența sol. maximale)

Fie $f(\cdot,\cdot) : D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont $\frac{dx}{dt} = f(t,x)$

Atunci $(\forall)\ \varphi(\cdot) \in S_f\ \exists\ \varphi_1(\cdot) \in S_f$ sol maximală $\varphi_1(\cdot) \succ \varphi(\cdot)$

Dem: Fie $\varphi(\cdot) \in S_f$

$S_{f,\varphi} = \{ \psi(\cdot) ; \psi(\cdot) \in S_f \ \psi(\cdot) \succ \varphi(\cdot) \} \neq \emptyset \ (\varphi(\cdot) \in S_{f,\varphi})$

Lema lui Zorn

$\forall \ \mathcal{P} \subset S_{f,\varphi}$ total ordonată admite un majorant $\Rightarrow \exists \varphi_1(\cdot) \in S_{f,\varphi}$ elem. maximal.

Fie $\mathcal{P} = \{ \varphi_j \}_{j \in \mathcal{J}} \subset S_{f,\varphi}$ total ordonată

$\forall \ i, j \in \mathcal{J}$ sau $\varphi_i(\cdot) \succ \varphi_j(\cdot)$

sau $\varphi_j(\cdot) \succ \varphi_i(\cdot)$

Fie $I_j := \operatorname{dom} \varphi_j(\cdot), \ j \in \mathcal{J}$

$I^* := \bigcup_{j \in \mathcal{J}} I_j \quad \varphi_*(\cdot) : I^* \to \mathbb{R}^n \quad \varphi_*(t) = \varphi_j(t)$ dacă $t \in I_j$

Arătăm a) I^* interval

b) $\varphi_*(\cdot)$ sol

a) I^* interval

$t_1, t_2 \in I^* \Rightarrow [t_1, t_2] \subset I^*$

$t_i \in I^*, \ i = \overline{1,2} \Rightarrow \exists \ j_1, j_2 \in \mathcal{J}$ a.î. $t_1 \in I_{j_1} = \operatorname{dom} \varphi_{j_1}(\cdot)$

$t_2 \in I_{j_2} = \operatorname{dom} \varphi_{j_2}(\cdot)$

sau $\varphi_{j_1}(\cdot) \succ \varphi_{j_2}(\cdot)$ $I_{j_1} \supset I_{j_2} \Rightarrow t_1, t_2 \in I_{j_1}$ interval

$\Rightarrow [t_1, t_2] \subset I_{j_1} \Rightarrow [t_1, t_2] \subset I^*$

sau $\varphi_{j_2}(\cdot) \succ \varphi_{j_1}(\cdot)$ $I_{j_2} \supset I_{j_1} \Rightarrow t_1, t_2 \in I_{j_2}$ interval

$\Rightarrow [t_1, t_2] \subset I_{j_2} \Rightarrow [t_1, t_2] \subset I^*$

b) $\varphi_*(\cdot)$ sol

$t \in I^* \Rightarrow \exists j \in \mathcal{J}$ a.î. $t \in I_j \Rightarrow \varphi_*(\cdot)|I_j = \varphi_j(\cdot)$

$\underline{\varphi_*'(t)} = \varphi_j'(t) = f(t, \varphi_j(t)) = \underline{f(t, \varphi_*(t))}$

PROP (intervalul de def al sol maximale)

Fie $f(\cdot,\cdot) : D = \overset{\circ}{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont $\frac{dx}{dt} = f(t,x)$

Fie $\varphi(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}^n$ ...

Dem: Fie $\varphi(\cdot): I \to \mathbb{R}^n$ sol maximală

Pp că I nu este deschis $\Rightarrow I = [a, b]$ sau $I = [a, b)$ sau $I = (a, b]$

Pp de ex $I = (a, b]$

Aplic T Peano în $(b, \varphi(b)) \Rightarrow \exists\ \varphi_1(\cdot): [b-\alpha, b+\alpha] \subseteq \mathbb{R}^n$ sol a pb Cauchy $(f, b, \varphi(b))$

Fie $\varphi_2(t) = \begin{cases} \varphi(t), & t \in (a, b] \\ \varphi_1(t), & t \in [b, b+\alpha] \end{cases}$

$\varphi_2(\cdot) \underset{\neq}{\succ} \varphi(\cdot)$ $\varphi_2(\cdot)$ sol $\not\!\circ$ ($\varphi(\cdot)$ maximală)

Teorema (Existența și unicitatea sol. maximale)

Fie $f(\cdot,\cdot): D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont $\frac{dx}{dt} = f(t, x)$ admite UL pe D.

Atunci:

1) $\forall\ \varphi(\cdot) \in S_f\ \exists!\ \varphi_1(\cdot) \in S_f$ maximală $\varphi_1(\cdot) \succ \varphi(\cdot)$

2) $\forall\ (t_0, x_0) \in D\ \exists!\ \varphi_{t_0,x_0}(\cdot): I(t_0, x_0) = (t^-(t_0,x_0), t^+(t_0,x_0)) \to \mathbb{R}^n$ sol maximală a pb Cauchy (f, t_0, x_0)

Dem: 1. Fie $\varphi(\cdot) \in S_f$ T. existența sol. maximale $\Rightarrow$

$\exists\ \varphi_1(\cdot): I_1 \subset \mathbb{R} \to \mathbb{R}^n$ maximală $\varphi_1(\cdot) \succ \varphi(\cdot)$

Pp. că $\exists\ \varphi_2(\cdot): I_2 \subset \mathbb{R} \to \mathbb{R}^n$ maximală $\varphi_2(\cdot) \succ \varphi(\cdot)$

$\varphi_1(\cdot) \neq \varphi_2(\cdot)$

$I_1 \neq I_2$ ($\varphi_1(\cdot), \varphi_2(\cdot) \succ \varphi(\cdot) \underset{U.G}{\overset{U.L}{\Rightarrow}} \varphi_1(\cdot)|_{I_1 \cap I_2} = \varphi_2|_{I_1 \cap I_2}$

$I_1 = I_2 \Rightarrow \varphi_1 = \varphi_2$ $\not\!\circ$)

Fie $\varphi_3(t) = \begin{cases} \varphi_1(t), & t \in I_1 \\ \varphi_2(t), & t \in (I_1 \cup I_2) \setminus I_1 \end{cases}$

$\varphi_3(\cdot) \underset{\neq}{\succ} \varphi_1(\cdot)$ $\not\!\circ$ ($\varphi_1(\cdot)$ maximală)

2. Fie $(t_0, x_0) \in D = \mathring{D}$, $f(\cdot,\cdot)$ continuă $\Rightarrow$ T. Peano

$\Rightarrow \exists\ \varphi_0(\cdot): [t_0 - a, t_0 + a] \to \mathbb{R}^n$ sol cu $\varphi_0(t_0) = x_0$

T. existența sol. maximale $\Rightarrow \exists\ \varphi_{t_0,x_0}(\cdot): I(t_0, x_0) \to \mathbb{R}^n$

sol maximală; $\varphi_{t_0,x_0}(\cdot) \succ \varphi_0(\cdot)$

PROP(intervalul de def al sol max) $I(t_0, x_0) := (t^-(t_0,x_0), t^+(t_0,x_0))$

Unicitatea: Analog cu 1.

Existența globală a soluțiilor

<u>Def</u>: $f(\cdot,\cdot) : I \times \mathbb{R}^n \to \mathbb{R}^n$, $I \subseteq \mathbb{R}$ interval are prop. de disipativitate (Δ) dacă $\exists\, r>0$ $\exists\, a(\cdot) : I \to \mathbb{R}_+$ continuă a.î.: $|\langle x, f(t,x)\rangle| \leq a(t)\cdot \|x\|^2$ $\forall\, t \in I$, $\forall\, x \in \mathbb{R}^n$ $\|x\| > r$

T(E. G)

Fie $f(\cdot,\cdot) : \underline{I \times \mathbb{R}^n} \to \mathbb{R}^n$, <u>continuă</u> cu <u>$(\Delta)$</u> $\frac{dx}{dt} = f(t,x)$

Atunci $f(\cdot,\cdot)$ admite E.G. a sol pe $I \times \mathbb{R}^n$ ($\forall\, (t_0, x_0) \in I \times \mathbb{R}^n$ $\exists\, \varphi(\cdot) : I \to \mathbb{R}^n$ sol cu $\varphi(t_0) = x_0$)

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Ec. Diferențiale
Seminar 5

$$x^{(n)} = f(t, x, x', \dots x^{(n-1)}) \quad f(\cdot,\cdot): D \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$$

T. Peano: $D = \mathring{D}$, $f(\cdot,\cdot)$ cont $\Rightarrow$ E.L. pe D ($\forall (t_0, (x_0, x_0^1, \dots, x_0^{n-1})) \in D$ $\exists$ $\varphi(\cdot): I_0 \in \mathcal{V}(t_0) \to \mathbb{R}$ sol cu $\varphi(t_0) = x_0$, $\varphi'(t_0) = x_0^1 \dots \varphi^{(n-1)}(t_0) = x_0^{n-1}$)

T. Cauchy-Lipschitz (EUL)

$D = \mathring{D}$, $f(\cdot,\cdot)$ cont, local-Lipschitz (II) $\Rightarrow$ EUL pe D
$\forall (t_0, (x_0, x_0^1, \dots x_0^{n-1})) \in D$ $\exists!$ $\varphi(\cdot): I_0 \in \mathcal{V}(t_0) \to \mathbb{R}$ sol cu $\varphi(t_0) = x_0$, $\varphi'(t_0) = x_0^1, \dots, \varphi^{(n-1)}(t_0) = x_0^{n-1}$)

1) $\forall n \in \mathbb{N}$ Să se det K_n = nr sol posibile ale pb:
$$x^{(n)} = t + x^3, \quad x(0) = 1, \quad x'(0) = 0.$$

$n = 0$: $x = t + x^3$, $x(0) = 1$, $x'(0) = 0$.

$x(t) = t + x^3(t)$, $\forall t$, $x(0) = 1$, $x'(0) = 0$.

$t = 0 \Rightarrow x(0) = x^3(0)$

$1 = 1$

$x'(t) = 1 + 3x^2(t) \cdot x'(t)$

$t = 0$ $\quad x'(0) = 1 + 3x^2(0)x'(0)$

$0 = 1$ Fals

$\Rightarrow K_0 = 0$.

$n = 1$: $x' = t + x^3$
$x(0) = 1$
$x'(0) = 0$

$t = 0 \Rightarrow x'(0) = x^3(0)$

$0 = 1$ Fals

$\Rightarrow K_1 = 0$

$n = 2$: $x'' = t + x^3$, $x(0) = 1$, $x'(0) = 0$ Pb. Cauchy $\Rightarrow$ Aplic T. C. Lipschitz

$f(t, (x_1, x_2)) = t + x_1^3$

T. C. Lipschitz $\exists!\ \varphi(\cdot): J_0 \in \mathcal{U}(0) \to \mathbb{R}$ sol. a. î. $\varphi(0)=1,\ \varphi'(0)=0$

$\Rightarrow k_2 = 1$.

Pt. $n=3$: $x''' = t + x^3,\ x(0)=1,\ x'(0)=0$

Definim $f(t,(x_1,x_2,x_3)) = t + x_1^3$

$f: \underbrace{\mathbb{R}\times\mathbb{R}^3}_{\text{deschisă}} \to \mathbb{R}$, $f(\cdot)$ continuă și local Lipschitz (II) și

de cls $C^1(\Pi) \xRightarrow{T.C.L} \forall a \in \mathbb{R},\ \exists!\ \varphi_a(\cdot): J_a \in \mathcal{U}(0) \to \mathbb{R}$.

sol. a. î. $\varphi_a(0) = 1,\ \varphi'_a(0) = 0,\ \varphi''_a(0) = 0 \Rightarrow a \in \mathbb{R}$ arbitrar,

$k_3 = \infty$.

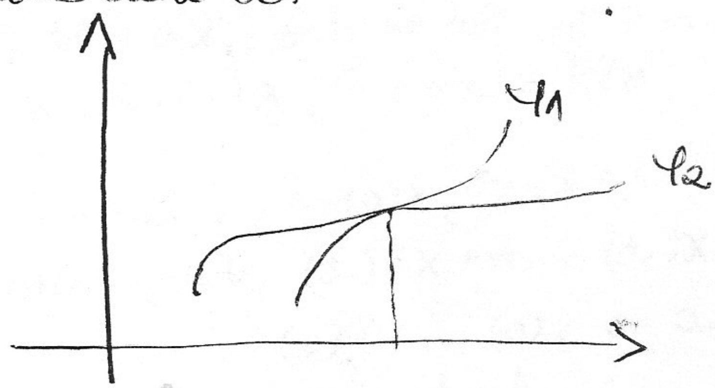
Pentru $n \geq 4 \Rightarrow k_n = \infty$.

2.) Să se studieze posibilitatea ca graficele a 2 sol. distincte să fie tangente pt fiecare din ec:

a) $x' = t^2 + x^4$

b) $x'' = t^2 + x^4$

c) $x''' = t^2 + x^4$



$\varphi_1: I_1 \to \mathbb{R},\ \varphi_2: I_2 \to \mathbb{R}$

$\exists\ t_0 \in I_1 \cap I_2$ a. î. $\varphi_1(t_0) = \varphi_2(t_0) = x_0$

$\varphi_1'(t_0) = \varphi_2'(t_0) = x_0^1$

a) $x' = t^2 + x^4$

$f(t,x) = t^2 + x^4$

$f: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$

cont, local-Lipschitz (II)

Cauchy-Lipschitz $\forall\ t_0, x_0 \in \mathbb{R}\ \exists!\ \varphi(\cdot): I_0 \to \mathbb{R}$ sol $\varphi(t_0) = x_0$ $\forall t_0$

b) $x'' = t^2 + x^4$

$f(t,(x_1,x_2)) = t^2 + x_1^4$

$f: \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$

cont, local Lipschitz (II)

$\varphi_1'(t) = 1$

$\varphi_2'(t) = 1 + 3t^2$

$1 = 1 + 3 \cdot 0$ O.K.

$\Rightarrow x_0^1 = 1. \xrightarrow{CL}$ în $(0,(0,1)) \Rightarrow$ ∄

- Pt $n = 3$

$x''' = f(t,x)$

Pp ∃ f ca în enunț.

Fie $g(t,(x_1,x_2,x_3)) = f(t,x_1)$

$g : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}$ cont și local-Lipschitz (II)

$\xRightarrow{C.L.}$ EUL pe $\mathbb{R} \times \mathbb{R}^3 \Rightarrow \forall (t_0,(x_0,x_0^1,x_0^2)) \in \mathbb{R} \times \mathbb{R}^3$

∃! sol $\varphi : \mathcal{Y}_0 \in \mathcal{V}(t_0) \to \mathbb{R}$ a.î. $\varphi(t_0) = x_0$

$\varphi'(t_0) = x_0^1$

$\varphi''(t_0) = x_0^2$

? t_0 a.î. $\left.\begin{array}{l} \varphi_1(t_0) = \varphi_2(t_0) = x_0 \\ \varphi_1'(t_0) = \varphi_2'(t_0) = x_0^1 \\ \varphi_1''(t_0) = \varphi_2''(t_0) = x_0^2 \end{array}\right\}$ $t_0 = 0$, $x_0 = 0$, $x_0^1 = 1$

$\varphi_1''(t) = 0$

$\varphi_2''(t) = 6t$

$0 = 6 \cdot 0 \Rightarrow x_0^2 = 0$. ∄ CL. în $(0,(0,1,0))$

- Pt $n = 4$

$x^{(4)} = f(t,x)$

Pp. ∃ f ca în enunț.

Fie $g(t,(x_1,x_2,x_3,x_4)) = f(t,x_1)$

$g : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{R}$ cont și local Lipschitz (II)

$\xRightarrow{TCL}$ EUL pe $\mathbb{R} \times \mathbb{R}^4 \Rightarrow \forall (t_0,(x_0,x_0^1,x_0^2,x_0^3)) \in \mathbb{R} \times \mathbb{R}^4$

∃! sol $\varphi : \mathcal{Y}_0 \in \mathcal{V}(t_0) \to \mathbb{R}$ a.î. $\left.\begin{array}{l} \varphi(t_0) = x_0 \\ \varphi'(t_0) = x_0^1 \\ \varphi''(t_0) = x_0^2 \end{array}\right\}$ $\begin{array}{l} t_0 = 0 \\ x_0 = 0 \\ x_0^1 = 1 \\ x_0^2 = 0 \end{array}$

$\varphi'''(t_0) = x_0^3$

$\varphi_1'''(t) = 0$

$\varphi_2'''(t) = 6$

$\varphi_1'''(0) = \varphi_2'''(0)$

$0 = 6$ Fals

c) $x''' = t^2 + x^4$

$f(t,(x_1,x_2,x_3)) = t^2 + x_1^4$

$f: \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}$

cont, Local Lipschitz (II)

$\xrightarrow{TCL} \forall (t_0,(x_0,x_0^1,x_0^2)) \in \mathbb{R}\times\mathbb{R}^3 \ \exists!\ \varphi: I_0 \in \mathcal{V}(t_0) \to \mathbb{R}$

sol. a.î. $\varphi(t_0) = x_0$; $\varphi'(t_0) = x_0^1$, $\varphi''(t_0) = x_0^2$

Dacă $\varphi_1''(t_0) \neq \varphi_2''(t_0) \Rightarrow$ DA!

3) Să se det $n \in \mathbb{N}$ pt care $\exists\ f(\cdot,\cdot): \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ cont și local Lipschitz (II) a.î. $\begin{cases} \varphi_1(t) = t \\ \varphi_2(t) = t + t^3 \end{cases}$

sunt sol ale ec $x^{(n)} = f(t,x)$.

- Pt $n=0$ $\quad x = f(t,x)$

$\varphi_i(t) = f(t,\varphi_i(t)) \ \forall t, \forall i=\overline{1,2}$.

$t = f(t,t)$

$t+t^3 = f(t,t^3+t)$

$\Rightarrow f(t,x) = x$.

- Pt $n=1$ $\quad x' = f(t,x)$

Pp. $\exists f: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ cont și Local Lipschitz (II) $\xrightarrow{CL}$ EUL pe $\mathbb{R}\times\mathbb{R}$

$\Rightarrow \forall (t_0,x_0) \in \mathbb{R}\times\mathbb{R} \ \exists!\ \varphi: \mathcal{Y}_0 \in \mathcal{V}(t_0) \to \mathbb{R}$ sol cu $\varphi(t_0) = x_0$

? $\exists t_0$ a.î. $\varphi_1(t_0) = \varphi_2(t_0) = x_0$

$t_0 = t_0 + t_0^3$

$t_0 = 0 \Rightarrow x_0 = 0$.

$\Rightarrow$ ~~α~~ (C.L. în $(0,0)$).

- Pt $n=2$ $\quad x'' = f(t,x)$

Pp. că $\exists f$ ca în enunț

$g(t,(x_1,x_2)) = f(t,x_1)$

$g: \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}$ cont, local Lipschitz (II) $\Rightarrow$ EUL pe $\mathbb{R}\times\mathbb{R}^2$

$\forall (t_0,(x_0,x_0^1)) \in \mathbb{R}\times\mathbb{R}^2 \ \exists!\ \varphi: \mathcal{Y}_0 \in \mathcal{V}(t_0) \to \mathbb{R}$ sol cu

$\varphi(t_0) = x_0$

$\varphi_1^{(4)} = 0 = f(t,t)$

$\varphi_2^{(4)} = 0 = f(t, t+t^3)$

$f(t,x) \equiv 0$ verifică condițiile

- Pt $n \geq 4$: $f(t,x) \equiv 0$.

$n=3$ $\quad \varphi_i(\cdot)$ sunt sol. $\quad x''' = f(t,x)$

$0 = f(t,t) \ \forall t \qquad\qquad t=0 \quad 0 = f(0,0)$

$6 = f(t, t+t^3) \ \forall t \qquad\qquad 6 = f(0,0)$??

$n=2$ $\quad \varphi_i(\cdot)$ sol pt $x'' = f(t,x)$

$0 = f(t,t)$

$6t = f(t, t+t^3) \ \forall t.$

f Local-Lipschitz (II) $\Rightarrow \exists L>0$ a.î. $\|f(t,x_1) - f(t,x_2)\| \leq L\|x_1 - x_2\|$
în $(t_0, x_0) \in \mathbb{R}\times\mathbb{R}$ $\quad \forall (t,x_1), (t,x_2) \in D_0 \in \mathcal{U}(t_0,x_0)$

$x_1 \longrightarrow t$

$x_2 \longrightarrow t+t^3$

$\|0 - 6t| \leq L|t - t - t^3|$

$6|t| \leq L|t^3| \quad |: |t| \quad t \neq 0.$

$t_0 = 0, x_0 = 0 \quad D_0 \in \mathcal{U}(0,0)$

$\frac{6}{L} \leq |t^2| \ \forall t \in \mathcal{U}(0) \ t\neq 0$??

$\downarrow$

0

Curs 6 07. 11. 2017

Existența sol. algebrice

Def: $f(\cdot,\cdot): \dot{I} \times \mathbb{R}^n \to \mathbb{R}^n$. Spunem că are prop. de:

a) Disipativitate (D) dacă $\exists\, r>0$ $\exists\, a(\cdot): \dot{I} \to \mathbb{R}_+$ cont. a.î.

$|\langle x, f(t,x)\rangle| \le a(t)\cdot\|x\|^2$ $\forall t\in \dot{I}$, $\forall x\in\mathbb{R}^n$, $\|x\|>r$

b) Creștere liniară (CL) dacă $\exists\, r>0$ $\exists\, a(\cdot): \dot{I} \to \mathbb{R}_+$ continuă a.î. $\|f(t,x)\| \le a(t)\cdot\|x\|$, $\forall t\in \dot{I}$, $\forall x\in\mathbb{R}^n$, $\|x\|>r$

c) Creștere afină (CA) dacă $\exists\, r>0$ $\exists\, a(\cdot), b(\cdot): \dot{I} \to \mathbb{R}_+$ continuă a.î. $\|f(t,x)\| \le a(t)\|x\| + b(t)$, $\forall t\in \dot{I}$, $\forall x\in\mathbb{R}^n$, $\|x\|>r$

PROP 1. $CL \Leftrightarrow CA$

2. $CL \Rightarrow D$

3. $n=1$ $CL \Leftrightarrow CD$

4. $n>1$ $D \not\Rightarrow CL$

Dem: 1. $CL \Rightarrow CA$ $(b(t)\equiv 0)$

$CA \Rightarrow CL$ $\exists\, r>0$ $\exists\, a(\cdot): \dot{I}\to\mathbb{R}_+$ cont. a.î.

$$\|f(t,x)\| \le a(t)\|x\| + b(t) = \|x\|\left(a(t) + \frac{b(t)}{\|x\|}\right) \le$$

$$\le \|x\| \underbrace{\left(a(t) + \frac{b(t)}{r}\right)}_{= a_1(t)} \quad \forall t\in\dot{I},\ x\in\mathbb{R}^n,\ \|x\|>r$$

CL

2. inegalitatea C.B.S

$x, y\in\mathbb{R}^n$, $|\langle x, y\rangle| \le \|x\|\cdot\|y\|$

$$|\langle x, f(t,x)\rangle| \overset{CBS}{\le} \|x\|\cdot\|f(t,x)\| \le a(t)\cdot\|x\|^2,$$

$\forall t\in\dot{I}$, $x\in\mathbb{R}^n$, $\|x\|>r$ (D)

3. $n = 1$ CL $\Rightarrow$ CD

D: $\exists\, r > 0\ \exists\, a(\cdot): I \to \mathbb{R}_+$ a.î $|x \cdot f(t, x)| \leq a(t) \cdot |x^2|$, $\forall t \in I$, $\forall x \in \mathbb{R}$, $|x| > r$

$|f(t, x)| \leq a(t) \cdot |x|$, $\forall t \in I$, $\forall x \in \mathbb{R}$, $|x| > r$ adică CL

4. D $\not\Rightarrow$ CL

$n = 2$ $f = ?$ a.î $< x, f(t, x) > = 0$

$< (x_1, x_2), (f_1(t, x), f_2(t, x)) = 0$

$x_1 f_1(t, x) + x_2 f_2(t, x) = 0$

$f_1(\cancel{t}, x) = x_2$

$f_2(t, x) = -x_1$

$f(t(x_1, x_2) = (x_2 - x_1)$

$\| f(t, x) \| = \sqrt{x_2^2 + x_1^2} = \|x\|$ are CL

Alegem $f(t(x_1, x_2)) = (x_2, -x_1) \cdot \|x\|$

$< x, f(t, x) > = 0$, $\| f(a, x) \| = \|x\| \cdot \sqrt{x_1^2 + x_2^2} = \|x\|^2$

Pp. că $f(\cdot, \cdot)$ are C.L $\Rightarrow \exists\, r > 0$, $\exists\, a(\cdot): \mathring{I} \to \mathbb{R}_+$ cont

$\|x\|^2 \leq a(t) \cdot \|x\|$, $\forall t \in \mathring{I}$, $\forall x \in \mathbb{R}^2$, $\|x\| > r$

$\|x\| \leq a(t)$ $\forall t, \forall x$

Fie $t = t_0 \in \mathring{I} \Rightarrow \|x\| \leq a(t_0)$, $\forall x \in \mathbb{R}^n$, $\|x\| > 0$

∞ ∞ (Nu poate fi mărginită)

T (Existență Globală) E. G

$f(\cdot, \cdot): \mathring{I} \times \mathbb{R}^n \to \mathbb{R}^n$ cont cu (D)

$$\frac{dx}{dt} = f(t, x)$$

Atunci $\forall (t_0, x_0) \in \mathring{I} \times \mathbb{R}^n$ $\exists\, \varphi(\cdot): \mathring{I} \to \mathbb{R}^n$ sol cu $\varphi(t_0) = x_0$

Dem: Pp $\dot{I} = (\alpha, \beta)$ $\quad \alpha \quad (a \quad t_0 \quad b) \quad \beta$

Fie $(t_0, x_0) \in \dot{I} \times \mathbb{R}^n$ – mulț. deschisă, $f(\cdot,\cdot)$ continuă $\Longrightarrow$

T. Peano (E.L) $\exists\, \varphi_0(\cdot) : \dot{I}_0 \in V(t_0) \to \mathbb{R}^n$ sol cu $\varphi_0(t_0) = x_0$.

T. existența sol maximale $\Rightarrow \exists\, \varphi(\cdot) : \dot{I} \to \mathbb{R}^n$ sol maximală $\varphi(\cdot)$, $\varphi\ \varphi_0(\cdot) \Rightarrow \varphi(t_0) = \varphi_0(t_0) = x_0$

$\Downarrow$

φ_0 extindere a lui φ

PROP (intervalul de def. al sol maximale)

Arătăm că $J = \dot{I} \Longleftrightarrow a = \alpha, b = \beta$

De exemplu, arătăm că $a = \alpha$ (analog $b = \beta$)

Pp. $\alpha < a$

T. (asupra prelungirii sol)

$f(\cdot,\cdot) : D = \dot{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont. $\dfrac{dx}{dt} = f(t, x)$

$\varphi(\cdot) : (a, b) \subset \mathbb{R} \to \mathbb{R}^n$ sol. Atunci:

1. ...

2. $\varphi(\cdot)$ admite o prelungire strictă la stânga $\Longleftrightarrow a > -\infty$, $\exists\, t_0 \in (a, b)\ \exists\, D_0 \subset D$ compactă a.î. $(t, \varphi(t)) \in D_0$, $\forall t (a, b)$

$\alpha < a \Rightarrow a > -\infty$, $t_0 \in (a, b)$? $D_0 \subset D$ compactă a.î. $(t, \varphi(t)) \in D_0$, $\forall t \in [a, b]$.

Dacă da $\Longrightarrow$ T. asupra prelungirii sol $\Longrightarrow$

$\varphi(\cdot)$ admite o prelungire strictă la stânga (ab $\varphi(\cdot)$ maximală)

$\varphi(\cdot,\cdot)$ are $(\Delta) \Longrightarrow \exists\, r > 0\ \exists\, a(\cdot) : \dot{I} \to \mathbb{R}_+$ continuă a.î.

$|\langle x, f(t, x) \rangle| \le a(t) \|x\|^2$, $\forall t \in \dot{I}$, $\forall x \in \mathbb{R}^n\ \|x\| > r$.

$(\|\varphi(t)\|^2)' = 2 \langle \varphi(t), \varphi'(t) \rangle = 2 \langle \varphi(t), f(t, \varphi(t)) \rangle$

Fie $A = \{ t \in (a, t_0] ; \|\varphi(t)\| > r \}$

Dacă $t \in A$ $(\|\varphi(t)\|^2)' = 2 < \varphi(t), f(t, \varphi(t)) > \geq$
$-2a(t) \cdot \|\varphi(t)\|^2$

C.B.S

Dacă $t \in B$ $(\|\varphi(t)\|^2)' = 2 < \varphi(t), f(t, \varphi(t)) > \geq$
$(|<x,y>| \leq \|x\| \cdot \|y\|)$

$\geq -2 \cdot \underbrace{\|\varphi(t)\|}_{\leq r} \cdot \underbrace{\|f(t, \varphi(t))\|}_{\leq K} \geq -2rK$

$K := \max_{(t,x) \in [a, t_0] \times \overline{B_r}(0)} \|f(t,x)\|$

Dacă $t \in (a, t_0]$ $(\|\varphi(t)\|^2)' \geq \min(-2a(t)\|\varphi(t)\|^2, -2rK) \geq -2rK - 2a(t) \cdot \|\varphi(t)\|^2 \Big| \int_t^{t_0}$

$\int_t^{t_0} (\|\varphi(s)\|^2)' ds \geq -2rK(t - t_0) - 2\int_t^{t_0} a(s) \cdot \|\varphi(s)\|^2 ds$

$\|\varphi(t_0)\|^2 - \|\varphi(t)\|^2 \geq -2rK(t - t_0) + 2\int_{t_0}^{t} a(s) \cdot \|\varphi(s)\|^2 ds$

$\|\varphi(t)\|^2 \leq \|\varphi(t_0)\|^2 + 2rK(t_0 - t) + 2\int_{t_0}^{t} a(s)\|\varphi(s)\|^2 ds \leq$

$\underbrace{\|\varphi(t_0)\|^2 + 2rK(t_0 - a)}_{= M} + 2 \cdot \int_{t_0}^{t} a(s) \cdot \|\varphi(s)\|^2 ds$

$t \in (a, t_0] \Rightarrow \|\varphi(t)\|^2 \leq M + \left| \int_{t_0}^{t} 2 \cdot a(s)\|\varphi(s)\|^2 ds \right| \xRightarrow[B6]{\text{Lema}}$

$\|\varphi(t)\| \leq M \cdot e^{2\left|\int_{t_0}^{t} a(s) ds\right|} \leq M \cdot e^{2\int_a^{t_0} a(s) ds} = \rho^2$

$\Rightarrow \|\varphi(t)\| \leq \rho, \forall t \in (a, t_0] \Leftrightarrow \varphi(t) \in \overline{B_\rho}(0)$

$D_0 := [a, t_0] \times \overline{B_\rho}(0)$ $(t, \varphi(t)) \in D_0, \forall t \in [a, t_0)$

Continuitatea sol maximale în raport cu datele inițiale și parametrii

$\frac{dx}{dt} = f(t,x)$ $\quad f(\cdot,\cdot): D = \mathring{D} \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ cont admite UL pe D.

$\forall(\tau,\xi)\in D \ \exists! \, \varphi_{\tau,\xi}(\cdot): I(\tau,\xi) = (t^-(\tau,\xi), t^+(\tau,\xi))$
$\to \mathbb{R}^n$ soluție maximală cu $\varphi_{\tau,\xi}(\tau)=\xi$

<u>Def</u>: s.n <u>curent maximal</u> al câmpului vectorial $f(\cdot,\cdot)$ funcția $\alpha_f(\cdot,\cdot,\cdot): D_f \subseteq \mathbb{R}\times D \to \mathbb{R}^n$ a.î. $\forall(\tau,\xi)\in D$

$\alpha_f(\cdot,\tau,\xi): \mathring{I}(\tau,\xi)\to\mathbb{R}$ sol. maximală unică a problemei Cauchy (f,τ,ξ).

$D_f = \{(t,\tau,\xi), (\tau,\xi)\in D, t\in \mathring{I}(\tau,\xi) = (t^-(\tau,\xi), t^+(\tau,\xi))\}$

$\alpha_f(\cdot,\tau,\xi) = \varphi_{\tau,\xi}(\cdot)$

$\to D_1\alpha_f(t,\tau,\xi) \equiv f(t,\alpha_f(t,\tau,\xi))$

$\to \alpha_f(\tau,\tau,\xi)=\xi$

$\to \alpha_f(\cdot,\tau,\xi)$ sol. maximală

Teorema asupra curentului maximal

Fie $f(\cdot,\cdot): D = \mathring{D} \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ continuă, $\frac{dx}{dt} = f(t,x)$

Pp. că a) $f(\cdot,\cdot)$ admite U.L pe D

b) $f(\cdot,\cdot)$ admite curent local continuu în fiecare punct din D

Fie $\alpha_f(\cdot,\cdot,\cdot): D_f \to \mathbb{R}^n$ curentul maximal.

Atunci:

1) $D_f \subseteq \mathbb{R}\times D$ deschisă

2) $\alpha_f(\cdot,\cdot,\cdot)$ continuă

<u>Obs</u>: Dacă f continuă, local-Lipschitz (II) $\Longrightarrow$ T. Cauchy-Lipschitz $\Longrightarrow$ EUL pe D adică a)

$\Longrightarrow$ Teorema (Existența, unicitatea și continuitatea curentului local parametrizat) $\longrightarrow f(\cdot,\cdot)$ admite curent local continuu în fiecare punct din D (adică b))

$$\frac{dx}{dt} = f(t,x,\lambda),\quad f(\cdot,\cdot,\cdot): D = \mathring{D} \subseteq \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^k \to \mathbb{R}^n \text{ continuă}$$

$\forall\, \lambda \in pr_3 D$, $f(\cdot,\cdot,\lambda)$ admite UL a sol.

$\forall (t_0, x_0, \lambda_0) \in D\ \exists!\ \varphi_{t_0,x_0,\lambda_0}(\cdot): I(t_0,x_0,\lambda_0) = (t^-(t_0,x_0,\lambda_0),$

$t^+(t_0,x_0,\lambda_0)) \to \mathbb{R}^n$ soluție maximală a pb. Cauchy

$(f(\cdot,\cdot,\lambda_0), t_0, \lambda_0)$

<u>Def</u>: Sn <u>curent maximal parametrizat</u> asociat c.v.p. $f(\cdot,\cdot,\cdot)$

funcția $\alpha_f(\cdot,\cdot,\cdot,\cdot): D_f \subseteq \mathbb{R}\times D \to \mathbb{R}^n$ a.î. $\forall (\tau,\xi,\lambda) \in D$,

$\alpha_f(\cdot,\tau,\xi,\lambda): I(\tau,\xi,\lambda) \to \mathbb{R}^n$ sol. maximală unică a pb. Cauchy $(f(\cdot,\cdot,\lambda), \tau, \xi)$

$\to D_1 \alpha_f(t,\tau,\xi,\lambda) = f(t, \alpha_f(t,\tau,\xi,\lambda), \lambda)$

$\to \alpha_f(\tau,\tau,\xi,\lambda) = \xi$

$\to$ sol. maximală

Metodă generală de studiu (transf. parametrilor în date inițiale)

$$(2)\begin{cases} \dfrac{dx}{dt} = f(t, x, \lambda) \\ \dfrac{d\lambda}{dt} = 0_{\mathbb{R}^k} \end{cases} \qquad \bar{x} = (x, \lambda)$$
$$\bar{f}(t, (x, \lambda)) = (f(t, x, \lambda), 0_{\mathbb{R}^k}).$$

$$(2)\ \frac{d\bar{x}}{dt} = \bar{f}(t, \bar{x})$$

$$\bar{f}(\cdot, \cdot) : D \subseteq \mathbb{R} \times (\mathbb{R}^n \times \mathbb{R}^k) \to \mathbb{R}^n \times \mathbb{R}^k$$

PROP (de echivalență)

$\varphi(\cdot)$ sol a ec (1) $\iff$ $\bar{\varphi}(\cdot) = (\varphi(\cdot), \lambda)$ este sol pt. (2)

Dem: "$\Rightarrow$" $\varphi'(t) \equiv f(t, \varphi(t), \lambda)$
$(\lambda)' \equiv 0$

"$\Leftarrow$" Fie $\varphi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot))$ sol. a ec (2) $\Rightarrow$

$$\Rightarrow \begin{cases} \varphi_1'(t) \equiv f(t, \varphi_1(t), \varphi_2(t)) \\ \varphi_2'(t) \equiv 0 \Rightarrow \exists \lambda \in pr_3 D \text{ a.î. } \varphi_2(t) \equiv \lambda \end{cases} \Rightarrow$$

$\Rightarrow \varphi_1'(t) = f(t, \varphi_1(t), \lambda)$ ok

Corolar: $\alpha_f(\cdot,\cdot,\cdot,\cdot) : \Delta_f \subseteq \mathbb{R} \times D \to \mathbb{R}^n$ curent maximal parametrizat al ec. (1) $\iff$ $\alpha_{\bar{f}}(\cdot,\cdot,\cdot,\cdot) : \Delta_{\bar{f}} = \Delta_f \to \mathbb{R}^n$,

$\alpha_{\bar{f}}(t, \tau, (\xi, \lambda)) := (\alpha_f(t, \tau, \xi, \lambda), \lambda)$ este curent maximal (neparametrizat) al ec (2)

T (continuitatea sol. maximale în raport cu parametrii)

Fie $f(\cdot,\cdot,\cdot) : D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ cont local Lipschit

$(1)\ \frac{dx}{dt} = f(t, x, \lambda).$

Atunci:

1. $D_f \subseteq \mathbb{R} \times D$ deschisă

2. $\alpha_f(\cdot,\cdot,\cdot,\cdot)$ continuă

Ecuații liniare pe $\mathbb{R}^n$

Def: Fie $A(\cdot): \overset{\circ}{I} \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$ def. ec. liniară $\frac{dx}{dt} = A(t) \cdot x$

Obs: $n=1$ $L(\mathbb{R}, \mathbb{R}) \simeq \mathbb{R}$

$x' = a(t) \cdot x$ ec. liniară scalară

$\varphi(\cdot)$ sol a ec $\Longleftrightarrow$ $\varphi(t) \equiv c \cdot e^{\int_{t_0}^{t} a(s) ds}$

$B \subset \mathbb{R}^n$ bază

$B = \{b_1, \dots b_n\}$

$L(\mathbb{R}^n, \mathbb{R}^n) \simeq M_n(\mathbb{R})$

$A_B(t) = (a_{ij}(t))_{\substack{i=\overline{1,n} \\ j=\overline{1,n}}}$ $A_B(t) = col(A(t) b_1 \dots A(t) b_n)$

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t) x_j, \quad i = \overline{1,n} \text{ (sistem de ecuații liniare)}$$

$\boxed{T(EUG)}$

Fie $A(\cdot): \overset{\circ}{I} \to L(\mathbb{R}^n, \mathbb{R}^n)$ cont, $\frac{dx}{dt} = A(t) x$

Atunci $\forall (t_0, x_0) \in \overset{\circ}{I} \times \mathbb{R}^n$, $\exists!$ $\varphi_0(\cdot): \overset{\circ}{I} \to \mathbb{R}^n$ sol cu $\varphi_0(t_0) = x_0$

Dem: $f_{A(\cdot)}(\cdot,\cdot)$ cont. local Lipschitz (II) (liniară (II)) $\Rightarrow$

T. Cauchy-Lipschitz EUL $\overset{T}{\Longleftrightarrow}$ UG

$\| f_{A(\cdot)}(t, x) \| = \| A(t, x) \| \le \| A(t) \| \cdot \| x \|$ C.L $\Rightarrow$ D $\overset{T}{\underset{EG}{\Longrightarrow}}$ EG

07.11.2017

Seminar 6

$f(\cdot,\cdot) : \mathring{I} \times \mathbb{R}^n \to \mathbb{R}^n$

DISIPATIVITATE (D)

$\exists r > 0$ $\exists a(\cdot) : \mathring{I} \to \mathbb{R}_+$ continuă a.î. $|\langle x, f(t,x)\rangle| \le a(t)\cdot\|x\|^2$ $\forall t \in \mathring{I}$, $\forall x \in \mathbb{R}^n$, $\|x\| > r$

T(EG)

$f(\cdot,\cdot) : \mathring{I} \times \mathbb{R}^n \to \mathbb{R}^n$ continuă cu (D) $\frac{dx}{dt} = f(t,x)$

Atunci $\forall (t_0, x_0) \in \mathring{I} \times \mathbb{R}^n$ $\exists$ $\varphi(\cdot) : \mathring{I} \to \mathbb{R}^n$ sol cu $\varphi(t_0) = x_0$

---

1) Fie ec. $\begin{cases} x_1' = -x_2^2 \\ x_2' = x_1 x_2 \end{cases}$

a) Să se arate că $\forall$ $\varphi(\cdot)$ sol. $\exists$ $c \in \mathbb{R}$ a.î. $\|\varphi(t)\| \equiv c$

b) Să se arate că admite EG

a) În general, dem că o fct. e ct dacă e derivabilă, și derivata sa este nulă.

Fie $\varphi(\cdot)$ soluție, $\varphi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot))$ (derivabilă și verifică ecuația). Deci:

$$\begin{cases} \varphi_1'(t) \equiv -\varphi_2^2(t) \\ \varphi_2'(t) \equiv \varphi_1(t)\cdot\varphi_2(t) \end{cases}$$

$\|\varphi(t)\|$ este ct.

$\|\varphi(t)\| = \sqrt{\varphi_1^2(t) + \varphi_2^2(t)}$

Fie $g(t) = \varphi_1^2(t) + \varphi_2^2(t)$

$g'(t) = 2\varphi_1(t)\,\varphi_1'(t) + 2\varphi_2(t)\,\varphi_2'(t)$

$= -2\varphi_1(t)\cdot\varphi_2^2(t) + 2\varphi_2^2(t)\cdot\varphi_1(t)$

$= 0$

$\Rightarrow \exists\ k \in \mathbb{R}$ ct. a.î $g(t) = k$

b)

$f(t,(x_1,x_2)) = (-x_2^2, x_1 x_2)$

$f: \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$
ok $(I=\mathbb{R}, n=2)$

f continuă

Verificăm prop. de disipativitate:

$|\langle x, f(t,x)\rangle| = |\langle (x_1,x_2), f(t,(x_1,x_2))\rangle| =$
$= |\langle (x_1,x_2), (-x_2^2, x_1x_2)\rangle$

$= -x_2^2\cdot x_1 + x_1x_2^2 = 0 \to$ satisface (D)

$\xRightarrow{T.E.G}$ E.G pe $\mathbb{R}\times\mathbb{R}^2$

2) Fie ec. $\frac{dx_i}{dt} = \sum_{j,k=1}^{n} c_{ijk}x_jx_k,\ i=\overline{1,n}$

$c_{ijk} = -c_{kji}\ \forall i,j,k \in \{1,\dots n\}$

a) $\forall\ \varphi(\cdot)$ sol $\exists\ c\in\mathbb{R}$ a.î $\|\varphi(t)\| \equiv c$

b) Admite E.G

Fie $\varphi(\cdot)$ soluție, $\varphi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot), \dots \varphi_n(\cdot))$, $\varphi(\cdot)$ derivabilă și verifică ecuația. Deci:

$\varphi_i'(t) = \sum_{j,k=1}^{n} c_{ijk}\ \varphi_j(t)\cdot\varphi_k(t),\ \forall\ i=\overline{1,n}$

$$\|\varphi(t)\| = \sqrt{\varphi_1^2(t) + \varphi_2^2(t) + \dots + \varphi_n^2(t)}$$

Fie $g(t) = \varphi_1^2(t) + \dots + \varphi_n^2(t)$

g deriv.

$$g'(t) = 2 \cdot \sum_{i=1}^{n} \varphi_i(t) \cdot \varphi_i'(t)$$

$$= 2 \cdot \sum_{i=1}^{n} \varphi_i(t) \cdot \sum_{j,k=1}^{n} c_{ijk} \cdot \varphi_j(t) \cdot \varphi_k(t)$$

$$= 2 \cdot \sum_{i,j,k=1}^{n} c_{ijk} \cdot \varphi_i(t) \cdot \varphi_j(t) \cdot \varphi_k(t) \underset{c_{ijk}=c_{kij}}{=} 0$$

$\Rightarrow \exists\, k \in \mathbb{R}$ a.î. $g(t) \equiv k$.

b) $f(t, (x_1, \dots x_n)) = \left(\sum_{j,k=1}^{n} c_{1jk} x_j x_k, \dots, \sum_{j,k=1}^{n} c_{njk} x_j x_k \right)$

$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

f continuă

Verificăm prop. de disipativitate.

$$|\langle x, f(t,x) \rangle| = |\langle (x_1, \dots x_n), \left(\sum_{j,k=1}^{n} c_{1jk} x_j x_k \dots \sum_{j,k=1}^{n} c_{njk} x_j x_k \right)|$$

$$= x_1 \sum_{j,k=1}^{n} c_{1jk} x_{jk} + \dots + x_n \sum_{j,k=1}^{n} c_{njk} x_j x_k$$

$$= \left| \sum_{i=1}^{n} x_i \sum_{j,k}^{n} c_{ijk} x_j x_k \right| = \left| \sum_{i,j,k=1}^{n} c_{ijk} x_i x_j x_k \right| = 0 \Rightarrow (D)$$

$\xrightarrow{TEG}$ EG pe $\mathbb{R} \times \mathbb{R}^n$

3) Fie ec. $\begin{cases} x_1' = x_1 x_2 \\ x_2' = -2x_1^4 \end{cases}$

b) Admite EG.

a) $\varphi(\cdot) = (\varphi_1(\cdot), \varphi_2(\cdot))$ soluție $\Rightarrow \varphi(\cdot)$ deriv. și verifică ecuația.

Deci: $\begin{cases} \varphi_1'(t) = \varphi_1(t) \cdot \varphi_2(t) \\ \varphi_2'(t) = -2\varphi_1^4(t) \end{cases}$

$\varphi_1^4(t) + \varphi_2^2(t) \equiv C$.

$g(t) = \varphi_1^4(t) + \varphi_2^2(t)$.

$g(t)$ deriv.

$g'(t) = 4\varphi_1^3(t) \cdot \varphi_1'(t) + 2\varphi_2(t) \cdot \varphi_2'(t)$

$= 4\varphi_1^3(t) \cdot \varphi_1(t) \cdot \varphi_2(t) + 2 \cdot \varphi_2(t) \cdot (-2\varphi_1^4(t))$

$= 0$.

$\Rightarrow \exists k \in \mathbb{R}$ a.î. $g(t) = k$.

b) $f(t, (x_1, x_2)) = (x_1 x_2, -2x_1^4)$

$f : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$ OK

f cont

Verificăm prop. de disipativitate:

$|\langle x, f(t,x) \rangle| = |\langle (x_1, x_2), (x_1 x_2, -2x_1^4) \rangle|$

$= |x_1^2 \cdot x_2 - 2x_2 \cdot x_1^4| \leq a(t) \cdot \|x\|^2 = a(t) \cdot (x_1^2 + x_2^2)$

$\downarrow$

Pol. de grad 5 $\rightarrow$ Pol. de grad 2

Pp. că f are (D) $\Rightarrow \exists r > 0$ $\exists a(\cdot) : \mathbb{R} \to \mathbb{R}_+$ cont a.î.

$|x_1^2 \cdot x_2 - 2x_1^4 \cdot x_2| \leq a(t) \cdot (x_1^2 + x_2^2)$, $\forall t \in \mathbb{R}$, $\forall x \in \mathbb{R}^2$

$\|x\| > r$.

Fie $t = 0$.

$|x_1^2 x_2 - 2x_1^4 x_2| \leq a(0) \cdot (x_1^2 + x_2^2)$ $\forall x \in \mathbb{R}^2$, $\|x\| > r$

$x_n = (n, n)$

$\Rightarrow |n - x_n^3| \leq 2 \cdot a(0) \quad \forall n > n_0$

$\downarrow$

$\infty \qquad$ ∞

$\forall (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^2 \ \exists \varphi(\cdot) : \mathbb{R} \to \mathbb{R}^2$ sol, $\varphi(t_0) = x_0$.

Fie $(t_0, x_0) \in \underbrace{\mathbb{R} \times \mathbb{R}^2}_{\text{deschisă}}$, f cont $\Rightarrow$ T. Peano: $\exists \varphi_0(\cdot) : I_0 \to \mathbb{R}^2$ sol

$\varphi_0(t_0) = x_0$

T. (Existența sol maximale) $\Rightarrow \exists \varphi : I \to \mathbb{R}^2$ sol maximală, φ prelungire a lui $\varphi_0 \Rightarrow \varphi(t_0) \equiv \varphi_0(t_0) = x_0$.

PROP (int. def al sol. max) $\Rightarrow I$ deschis, $I = (a, b)$

Arătăm că $I = \mathbb{R}$, $a = -\infty$, $b = +\infty$

Dem $b = +\infty$.

Pp. că $b < +\infty$ $t_0 \in (a, b)$? $\exists D_0 \subset \mathbb{R} \times \mathbb{R}^2$ compactă a.î. $(t, \varphi(t)) \in D_0 \ \forall t \in [t_0, b)$

Dacă am găsit D_0 cu această prop aplicăm T asupra prelungirii sol $\Rightarrow$ $\left.\begin{array}{l} \varphi(\cdot) \text{ admite o prelungire strictă la dreapta} \\ \varphi(\cdot) \text{ sol maximală} \end{array}\right\} \text{∞}$

$\overset{a)}{\Rightarrow} \exists c \in \mathbb{R}$ a.î. $\varphi_1^4(t) + \varphi_2^2(t) \leq c \Rightarrow \varphi_1^4(t) \leq c \ \forall t \Rightarrow$

$\varphi_2^2(t) \leq c \ \forall (t)$

$\Rightarrow \varphi_1^2(t) \leq \sqrt{c}, \forall (t)$

$\Rightarrow \varphi_1^2(t) + \varphi_2^2(t) \leq c + \sqrt{c}$

$\|\varphi(t)\| \leq k, \ \forall (t) \Leftrightarrow \varphi(t) \in \overline{B_k}(0)$

$D_0 = [t_0, b] \times \overline{B}_k(0)$