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$\boxed{\text{EC. Dif} \\ \text{curs 12}}$

Ecuații ~~diferențiale~~ cu derivate parțiale de ordinul I

Def: $F(\cdot,\cdot,\cdot) : A \subseteq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ def. e.d.p. I

$$F\left(x, z, \frac{\partial z}{\partial x}\right) = 0$$

$\varphi(\cdot) : G = \mathring{G} \subseteq \mathbb{R}^n \to \mathbb{R}$ s.n. soluție a e.d.p. I dacă e derivabilă și dc. $F(x, \varphi(x), \partial\varphi(x)) \equiv 0$

În coordonate $x = (x_1, \dots, x_n)$ $F\left((x_1,\dots,x_n), z, \left(\frac{\partial z}{\partial x_1}, \dots, \frac{\partial z}{\partial x_n}\right)\right) = 0$

$\boxed{\text{Notații ale lui Monge}}$: $p_i = \frac{\partial z}{\partial x_i}$ $i = \overline{1,n}$ $p = (p_1, \dots, p_n)$

$$F(x, z, p) = 0$$

$\underline{n=2}$ $p = \frac{\partial z}{\partial x}$ $q = \frac{\partial z}{\partial y}$ $F((x,y), z, (p,q)) = 0$

Metoda caracteristicilor (a lui Cauchy) pentru e.d.p. I

Def. S.n. sistemul caracteristicilor asociat e.d.p. I $F\left(x, z, \frac{\partial z}{\partial x}\right) = 0$, unde $F(\cdot,\cdot,\cdot) : D = \mathring{D} \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ C^1 următorul sistem de ecuații diferențiale

$$\begin{cases} \frac{dx}{dt} = \frac{\partial F}{\partial p}(x,z,p) & x(0) = \alpha(\sigma) \\ \frac{dz}{dp} = \left\langle p, \frac{\partial F}{\partial p}(x,z,p) \right\rangle & z(0) = \beta(\sigma) \quad \Longleftrightarrow \\ \frac{dp}{dt} = -\frac{\partial F}{\partial x}(x,z,p) - p\frac{\partial F}{\partial z}(x,z,p) & p(0) = \gamma(\sigma) \end{cases}$$

$$\begin{cases} \frac{dx_i}{dt} = \frac{\partial F}{\partial p_i}(x,z,p) \\ \frac{dz}{dt} = \sum_{i=1}^{n} p_i \frac{\partial F}{\partial p_i}(x,z,p) \\ \frac{dp_i}{dt} = -\frac{\partial F}{\partial x_i}(x,z,p) - p_i \frac{\partial F}{\partial z}(x,z,p) \quad i = \overline{1,n} \end{cases}$$

$\underline{2n + 1 \text{ ec}}$

$$f(t,(x,z,p)) = \left(\frac{\partial F}{\partial p}(x,z,p), \langle p\, \frac{\partial F}{\partial p}(x,z,p)\rangle, \frac{-\partial F}{\partial x}(x,z,p) - p\frac{\partial F}{\partial z}(x,z,p)\right)'$$

Def S.n. caracteristica a e.d.p. I

$K(\cdot) = (x(\cdot), z(\cdot), P(\cdot)) : I \subseteq \mathbb{R} \to \Delta$ o soluție a caracteristicilor.

$c(\cdot) = pr_{1,2}\, K(\cdot) = (x(\cdot), z(\cdot))$ s.n. curbă caracteristică a e.d.p. I

Proprietăți ale sistemului caracteristicilor

PROP 1 (Proprietate fundamentală)

Fie $F(\cdot,\cdot,\cdot) : \Delta = \mathring{\Delta} \subseteq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ C^1 $F(x, z, \frac{\partial z}{\partial x}) = 0$

Fie $\varphi(\cdot) : G = \mathring{G} \subseteq \mathbb{R}^n \to \mathbb{R}$ C^2 Soluție a e.d.p. I

Atunci $\forall\, x_0 \in G$ $\exists\, K(\cdot) = (x(\cdot), z(\cdot), P(\cdot)) : I_0 \in \mathcal{V}(0) \to \Delta$ caracteristică a.î.

a) $x(0) = x_0$

b) $\varphi(x(t)) \equiv z(t)$

c) $\Delta(\varphi(x(t))) \equiv P(t)$

Dem: Fie $x_0 \in G$. Considerăm pb. Cauchy:

$$\begin{cases} \frac{dx}{dt} = \frac{\partial F}{\partial p}(x, \varphi(x), \Delta(\varphi(x))) = f_\varphi(t,x) \\ x(0) = x_0 \end{cases}$$

F C^1, $\varphi(\cdot)$ C^2 $\Rightarrow$ $f_\varphi(\cdot,\cdot)$ continuă

T. Peano $\Rightarrow$ $\exists\, x(\cdot) : I_0 \in \mathcal{V}(0) \to \mathbb{R}^n$ sol. a acestei pb.

Definim: $z(t) := \varphi(x(t))$

$P(t) := \Delta\varphi(x(t))$

Arătăm că $(x(\cdot), z(\cdot), P(\cdot))$ formează o sol. a sistemului caracteristicilor.

$$\underline{x'(t)} \equiv \frac{\partial F}{\partial P}(x(t), \underbrace{\varphi(x(t))}_{z(t)}, \underbrace{\Delta\varphi(x(t))}_{P(t)}) = \frac{\partial F}{\partial P}(x(t), z(t), P(t)) \quad \text{O.K.}$$

derivata

$$\underline{z'(t)} \equiv \Delta\varphi(x(t)) \cdot x'(t) \equiv P(t)\frac{\partial F}{\partial P}(x(t), z(t), P(t)) \quad \text{O.K.}$$

$$\underline{P'(t)} \equiv \Delta^2\varphi(x(t)) \cdot x'(t)$$

$\varphi(\cdot)$ sol. $F(x, \varphi(x), \Delta\varphi(x)) \equiv 0 \;\Big|\, \frac{\partial}{\partial x}$

$$\frac{\partial F}{\partial x}(x, \varphi(x), \Delta\varphi(x)) + \frac{\partial F}{\partial z}(x, \varphi(x), \Delta\varphi(x)) \cdot \Delta\varphi(x) + \frac{\partial F}{\partial p}(x, \varphi(x), \Delta\varphi(x))$$

$$\cdot\, \Delta^2\varphi(x) \equiv 0$$

$$\frac{\partial F}{\partial x}(x(t), z(t), p(t)) + \frac{\partial F}{\partial z}(x(t), z(t), p(t))\cdot p(t) + \underbrace{\frac{\partial F}{\partial p}(x(t), z(t), p(t))}_{x'(t)}\, \Delta^2 \varphi(x(t)) \equiv 0$$

$$\Rightarrow p'(t) \equiv \Delta^2\varphi(x(t))\cdot x'(t) \equiv -\frac{\partial F}{\partial x}(x(t), z(t), p(t)) - p(t)\frac{\partial F}{\partial z}(x(t), z(t), p(t)) \quad \text{o.k.}$$

PROP 2) $F(\cdot,\cdot,\cdot)$ este integrală primă pt. sistemul caracteristicilor

Dem: Criteriul .. $\frac{\partial F}{\partial t}(t,(x,z,p)) + \Delta F(t,(x,z,p))\, f(t,(x,z,p)) \equiv 0$?

$$\Delta F(x,z,p)\, f(t,(x,z,p)) \overset{?}{\equiv} 0$$

$$\Delta F(x,z,p)\, f(t,(x,z,p)) \equiv \left\langle \frac{\partial F}{\partial x}(x,z,p), \cancel{\frac{\partial F}{\partial p}(x,z,p)} \right\rangle + \frac{\partial F}{\partial z}(x,z,p)\left\langle \cancel{p}, \cancel{\frac{\partial F}{\partial p}(x,z,p)} \right\rangle + \left\langle \frac{\partial F}{\partial p}(x,z,p), -\cancel{\frac{\partial F}{\partial x}(x,z,p)} - p\,\cancel{\frac{\partial F}{\partial z}(x,z,p)} \right\rangle$$

$$\equiv 0 \Rightarrow \text{q.e.d.}$$

– Probleme la limită pt. e.d.p. I –

Def 1: S.n. problemă la limită generală asociată e.d.p. I (F, φ_0), unde

$F(\cdot,\cdot,\cdot): D \subseteq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ def. $F(x,z,\frac{\partial z}{\partial x}) = 0$

$\varphi_0(\cdot): G_0 \subset \mathbb{R}^n \to \mathbb{R}$ dată

$\varphi(\cdot): G = \mathring{G} \subset \mathbb{R}^n \to \mathbb{R}$ s.n. soluție a problemei (F, φ_0) dacă

$$\begin{cases} F(x, \varphi(x), \Delta\varphi(x)) \equiv 0 \\ \varphi(\cdot)|_{G_0} = \varphi_0(\cdot) \end{cases}$$

Exemplu: Pb. Dirichlet $G_0 = G \setminus \text{int}\, G = \partial G$

Pb. Cauchy $G_0 = \{x_0^1\} \times H_0$, $H_0 \subset \mathbb{R}^{n-1}$

$\Gamma_{\varphi_0} = \text{graph}\,\varphi_0(\cdot) = \{(x, \varphi_0(x));\ x \in G_0\}$ s.n. Varietatea inițială

Def 2: S.n. parametrizare a varietății inițiale $(\alpha(\cdot), \beta(\cdot)): A \subset \mathbb{R}^{n-1} \to \mathbb{R}^n \times \mathbb{R}$ a.î. $\Gamma_{\varphi(\cdot)} = \{(\alpha(\tau), \beta(\tau));\ \tau \in A\}$

Obs. $\varphi(\cdot)|_{G_0} = \varphi_0 \iff \varphi(\alpha(\tau)) \equiv \beta(\tau)$

Def 3: S.n. problemă la limită parametrizată (F, α, β), unde

$F(\cdot,\cdot,\cdot): D \subseteq \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ def. $F(x,z,\frac{\partial z}{\partial x}) = 0$

$(\alpha(\cdot), \beta(\cdot)): A \subset \mathbb{R}^{n-1} \to \mathbb{R}^n \times \mathbb{R}$.

v.u.

$\varphi(\cdot)$ s.n. soluție a pb. (F,α,β) dacă:

$$F(x,\varphi(x),\Delta\varphi(x))\equiv 0$$
$$\varphi(\alpha(\tau))\equiv\beta(\tau)$$

Obs: $\varphi(\alpha(\tau))\equiv\beta(\tau)\ \Big|\ \frac{\partial}{\partial\tau}$ $\qquad \underbrace{\Delta\varphi(\alpha(\tau))}\,\Delta\alpha(\tau)\equiv\Delta\beta(\tau)$

Pb. $\alpha(\cdot),\beta(\cdot)$ derivabile $\qquad$!! $\gamma(\tau)$ $\quad \gamma(\cdot)$ s.n. funcție de compatibilitate

$\varphi(\cdot)$ sol. a e.d.p. I $\Rightarrow F(x,\varphi(x),\Delta\varphi(x))\equiv 0$, $x=\alpha(\tau)$ $\Rightarrow F(\alpha(\tau),\underbrace{\varphi(\alpha(\tau))}_{\beta(\tau)},\overbrace{\Delta\varphi(\tau)}^{=\gamma(\tau)})\equiv 0$

$$F(\alpha(\tau),\beta(\tau),\gamma(\tau))\equiv 0$$

$$(C_\tau)\begin{cases}\gamma(\tau)\,\Delta\alpha(\tau)\equiv\Delta\beta(\tau)\\ F(\alpha(\tau),\beta(\tau),\gamma(\tau))\equiv 0\end{cases}$$ condiții de compatibilitate

Def 4: S.n. pb. la limită compatibilă cu ecuația $F(\alpha,\cancel{\alpha},\beta,\gamma)$,

unde $F(\cdot,\cdot,\cdot):\Delta\subseteq\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$, $F(x,z,\frac{\partial z}{\partial x})=0$

$(\alpha(\cdot),\beta(\cdot),\gamma(\cdot)):A\subseteq\mathbb{R}^{n-1}\to\Delta$ care verifică (C_τ)

$\varphi(\cdot)$ s.n. soluție a pb. (F,α,β,γ) dacă:

- $F(x,\varphi(x),\Delta\varphi(x))\equiv 0$
- $\varphi(\alpha(\tau))\equiv\beta(\tau)$
- $\Delta\varphi(\alpha(\tau))\equiv\gamma(\tau)$

Def 5. S.n. curentul caracteristicilor asociat pb. la limită compatibilă cu ecuația (F,α,β,γ), funcția $K(\cdot,\cdot)=$
$=(X(\cdot,\cdot),Z(\cdot),P(\cdot,\cdot)):\Omega\subseteq\mathbb{R}\times A\to\Delta$ a.î.
$\forall\tau\in pr_2\,\Omega$ $K(\cdot,\tau)=(X(\cdot,\tau),Z(\cdot,\tau),P(\cdot,\tau))$ este caracteristică cu ~~ref~~:

$$K(0,\tau)=(X(0,\tau),Z(0,\tau),P(0,\tau))=(\alpha(\tau),\beta(\tau),\gamma(\tau))$$

Teorema asupra metodei caracteristicilor

Fie $F(\cdot,\cdot,\cdot):\Delta=\mathring{\Delta}\subseteq\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ C^2 def. $F(x,z,\frac{\partial z}{\partial x})=0$
$(\alpha(\cdot),\beta(\cdot),\gamma(\cdot))$ $A=\mathring{A}\subset\mathbb{R}^{n-1}\to\Delta$ C^1 a.î. (F,α,β,γ) pb. la limită compatibilă cu ecuația:

Fie $K(\cdot,\cdot)=(X(\cdot,\cdot),Z(\cdot,\cdot),P(\cdot,\cdot)):\Omega\to\Delta$ curentul caracteristicilor

Pp. $\exists\,\Omega_0\subset\Omega$ a.î. $X(\cdot,\cdot)|\Omega_0$ este difeomorfism cu
$(X(\cdot,\cdot)|\Omega_0)^{-1}=(T(\cdot),\Sigma(\cdot))$

Atunci $\varphi(x) := Z(T(x), \Sigma(x))$; $x \in G_0 = X(\Omega_0)$ este sol. a pb. $(F, \alpha, \beta, \gamma)$.

Dem: $(X(\cdot,\cdot)|_{\Omega_0})^{-1} = (T(\cdot), \Sigma(\cdot)) \iff$

$$X(T(x), \Sigma(x)) \equiv x$$

$$\begin{cases} T(X(t,\tau)) \equiv t \\ \Sigma(X(t,\tau)) \equiv \tau \end{cases}$$

1) $\varphi(\alpha(\tau)) \equiv \beta(\tau)$ #

$$\varphi(\alpha(\tau)) = Z(T(\alpha(\tau)), \Sigma(\alpha(\tau))) = Z(T(X(0,\tau)), \Sigma(X(0,\tau))) =$$
$$= Z(0,\tau) = \beta(\tau)$$

2) $\Delta\varphi(\alpha(\tau)) \equiv \beta(\tau)$ #

$$\Delta\varphi(x) = \Delta Z(T(x), \Sigma(x))(\Delta T(x), \Delta\Sigma(x)) \overset{\text{Lemă}}{=}$$

Lemă: $\Delta Z(t,\tau) \equiv P(t,\tau)\Delta X(t,\tau)$

Dem. lemei (cursul viitor)

Lemă $P(T(x),\Sigma(x)) \underbrace{\Delta X(T(x),\Sigma(x))(\Delta T(x), \Delta\Sigma(x))}_{= I_n}$

$$= P(T(x), \Sigma(x))$$

$$X(T(x), \Sigma(x)) = x \,\Big|\, \frac{d}{dx}$$

$$\Delta X(T(x),\Sigma(x))(\Delta T(x), \Delta\Sigma(x)) = I_n$$

$$\Delta\varphi(\alpha(\tau)) \equiv P(T(\alpha(\tau)), \Sigma(\alpha(\tau))) \equiv P(T(X(0,\tau)), \Sigma(X(0,\tau))) =$$
$$= P(0,\tau) = \gamma(\tau)$$

3) $F(x, \varphi(x), \Delta\varphi(x)) \equiv 0$ #

PROP2 $F(K(t,\tau)) \equiv c \equiv F(K(0,\tau)) \equiv F(\alpha(\tau), \beta(\tau), \gamma(\tau)) \overset{(C_\tau)}{=} 0$

$$F(X(t,\tau), Z(t,\tau), P(t,\tau)) \equiv 0$$

$t = T(x)$
$\tau = \Sigma(x)$
$$F(\underbrace{X(T(x),\Sigma(x))}_{x}, \underbrace{Z(T(x),\Sigma(x))}_{\varphi(x)}, \underbrace{P(T(x),\Sigma(x))}_{\Delta\varphi(x)}) \equiv 0$$

$$F(x, \varphi(x), \Delta\varphi(x)) \equiv 0$$

O.K.