

Ecuatii diferențiale

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Ecuatii liniare pe \mathbb{R}^n cu coeficienți

constante

Def: Fie matricea $A \in L(\mathbb{R}^n, \mathbb{R}^n)$, $\frac{dx}{dt} = Ax$. În coordonate avem:

$$\frac{dx_i}{dt} = \sum_{j=1}^n a_{ij}x_j, \quad i = 1, n \text{ sistem de ecuații liniare cu coeficienți constante.}$$

Caz particular $A(t) \equiv A \in L(\mathbb{R}^n, \mathbb{R}^n)$.

Teorema (E.U.G): $\forall (t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$, $\exists! \varphi(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^n$ soluție cu $\varphi(t_0) = x_0$.

$$S_A := \{ \varphi(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^n; \varphi(\cdot) \text{ soluție cu } x' = Ax \}$$

$S_A \subset C^1(\mathbb{R}, \mathbb{R}^n)$ subspațiu vectorial cu $\dim(S_A) = n$.

Obs: $\varphi(\cdot)$ soluție $\Rightarrow \varphi'(t) \equiv A\varphi(t) \Rightarrow \varphi''(t) \equiv A\varphi'(t) = A^2\varphi(t) = \dots \Rightarrow$
 $\Rightarrow \varphi^{(n)}(t) \equiv A^n\varphi(t), \forall n \in \mathbb{N}$.

$$\text{Dacă } \varphi(\cdot) \text{ analitică} \Rightarrow \varphi(t) = \sum_{k \geq 0} \frac{t^k}{k!} \varphi^{(k)}(0) = \sum_{k \geq 0} \frac{t^k}{k!} A^k \varphi(0) = \sum_{k \geq 0} \frac{(tA)^k}{k!} \varphi(0)$$

Exponențiala unei aplicații liniare

Prop: $\forall A \in L(\mathbb{R}^n, \mathbb{R}^n)$, $\exists \sum_{k \geq 0} \frac{A^k}{k!} = \lim_{m \rightarrow \infty} \sum_{k=0}^m \frac{A^k}{k!} =: \exp(A) (= e^A)$

Dem: (schita)

$$\text{Mai multe } L(\mathbb{R}^n, \mathbb{R}^n) \cong M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$$

$$\sum_{k \geq 0} \|U_k\| < \infty \Rightarrow \sum_{k \geq 0} U_k \text{ convergentă}$$

$$U_k = \frac{A^k}{k!}$$

$$\|A\| = \sup_{\|x\| \leq 1} \|Ax\|, \quad \|AB\| \leq \|A\| \cdot \|B\| \Rightarrow \|A^k\| \leq \|A\|^k, \quad \forall k$$

$$\sum_{k=0}^{\infty} \left\| \frac{A^k}{k!} \right\| \leq \sum_{k=0}^{\infty} \frac{\|A\|^k}{k!} = e^{\|A\|}$$

- Prop:
- $\exp(0) = \mathbb{I}_n$
 - $AB = BA \Rightarrow \exp(A+B) = \exp(A) \cdot \exp(B)$
 - $\forall A \in L(\mathbb{R}^n, \mathbb{R}^n), \exists (\exp(A))^{-1} = \exp(-A)$

Prop (legătura cu ecuația exponentială):

$A \in L(\mathbb{R}^n, \mathbb{R}^n), t \rightarrow \exp(tA)$ este derivabilă $(\exp(tA))' = A \exp(tA)$
 $(= (\exp(tA))A)$

Dem:

$$u(t) = \sum_{k \geq 0} u_k(t)$$

$\sum_{k \geq 0} u'_k(t)$ este uniform convergentă pe orice compactă $\Rightarrow \exists u'(t) = \sum$

$$\exp(tA) = \sum_{k \geq 0} \frac{(tA)^k}{k!}, \quad u_k(t) = \frac{t^k A^k}{k!}$$

$$\sum_{k \geq 1} \frac{t^k t^{k-1} A^k}{k!} = \sum_{k \geq 1} \frac{t^{k-1} A^k}{(k-1)!} = A \sum_{k \geq 1} \frac{t^{k-1} A^{k-1}}{(k-1)!} = A \exp(tA)$$

Corolar: $\varphi(\cdot) \in S_A \Leftrightarrow \exists x_0 \in \mathbb{R}^n$ astfel încât $\varphi(t) = (\exp(tA))x_0$

Dem: " \Leftarrow " Evident din propoziția anterioară

" \Rightarrow " Fixe $x_0 := \varphi(0)$

$$\text{Fixe } \Psi(t) = (\exp(tA))x_0$$

$$\begin{cases} \varphi(\cdot) \text{ soluție}, \varphi(0) = x_0 \\ \Psi(\cdot) \text{ soluție}, \Psi(0) = x_0 \end{cases} \xrightarrow{\text{u.f.}} \varphi(t) = \Psi(t)$$

Valeuri proprii. Vectori proprii. Soluții la
ecuațiilor liniare pe \mathbb{R}^n cu coeficienți constanți

Def: Fixe $A \in L(\mathbb{R}^n, \mathbb{R}^n)$, $\mathcal{T}(A) = \{ \lambda \in \mathbb{C}; \det(A - \lambda \mathbb{I}_n) = 0 \}$, $\lambda \in \mathcal{T}(A)$ se numește **valeură proprie** a lui A .

$$\mathcal{VPA}(\lambda) = \{ u \in \mathbb{C}^n \setminus \{0\}; (A - \lambda \mathbb{I}_n)u = 0 \}, \quad u \in \mathcal{VPA}(\lambda) \text{ se numește vector}$$

Prop 1: $A \in L(\mathbb{R}^n, \mathbb{R}^n)$

$$\lambda \in T(A) \cap \mathbb{R} \Rightarrow VP_A(\lambda) \cap \mathbb{R}^n \neq \emptyset$$

Dem:

$$u = v + iw \in VP_A(\lambda) \Rightarrow (A - \lambda I_n) \cdot u = 0, (A - \lambda I_n)(v + iw) = 0$$

$$(A - \lambda I_n)v + i(A - \lambda I_n)w = 0 \Rightarrow (A - \lambda I_n)v = 0 \quad \left. \begin{array}{l} \Rightarrow v \text{ si } w \text{ nu pot fi simultan} \\ (A - \lambda I_n)w = 0 \end{array} \right\} \text{egale cu } 0 \quad (\Rightarrow u = 0 \text{ x})$$

De exemplu: $v \neq 0 \Rightarrow v \in VP_A(\lambda) \cap \mathbb{R}^n$

Prop 2: $A \in L(\mathbb{R}^n, \mathbb{R}^n)$

$$\begin{aligned} \lambda = \alpha + i\beta \in T(A) \\ u = v + iw \in VP_A(\lambda) \end{aligned} \quad \left. \begin{array}{l} \Rightarrow \bar{\lambda} = \alpha - i\beta \in T(A) \\ \bar{u} = v - iw \in VP_A(\bar{\lambda}) \end{array} \right.$$

Dem:

$$\begin{aligned} \lambda \in T(A) \Rightarrow \det(A - \lambda I_n) = 0 \Rightarrow 0 = \overline{\det(A - \lambda I_n)} = \overline{\det(A - \bar{\lambda} I_n)} = \\ = \det(A - \bar{\lambda} I_n) \end{aligned}$$

$$u \in VP_A(\lambda) \quad (A - \lambda I_n) \cdot u = 0 \Rightarrow 0 = \overline{(A - \lambda I_n)u} = \overline{(A - \lambda I_n) \cdot u} = (A - \bar{\lambda} I_n) \cdot \bar{u}$$

Prop 3: (legătura cu ecuațiile diferențiale)

Fie $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ ce definește ecuația $\frac{dx}{dt} = Ax$.

① Dacă $\lambda \in T(A) \cap \mathbb{R}$ și $u_\lambda \in VP_A(\lambda) \cap \mathbb{R}^n \Rightarrow \varphi_\lambda(t) = e^{\lambda t} \cdot u_\lambda, \varphi_\lambda(t) \in S_A$

② Dacă $\lambda \in T(A), u_\lambda \in VP_A(\lambda) \Rightarrow \varphi_\lambda(t) = \operatorname{Re}(e^{\lambda t} \cdot u_\lambda)$
 $\varphi_{\bar{\lambda}}(t) = \operatorname{Im}(e^{\lambda t} \cdot u_\lambda), \varphi_\lambda(\cdot) \cdot \varphi_{\bar{\lambda}}(\cdot) \in S_A$

Dem:

$$① \quad \varphi_\lambda(t) = e^{\lambda t} \cdot u_\lambda$$

$$(\varphi_\lambda(t))' = \lambda e^{\lambda t} \cdot u_\lambda = e^{\lambda t} \underbrace{\lambda u_\lambda}_{= e^{\lambda t} A u_\lambda} = A e^{\lambda t} u_\lambda = A \varphi_\lambda(t)$$

$$u_\lambda \in VP_A(\lambda), (A - \lambda I_n) u_\lambda = 0 \Rightarrow \underbrace{A u_\lambda}_{= \lambda u_\lambda} = \lambda u_\lambda$$

$$② \quad \text{Fie } \tilde{\varphi}_\lambda(t) := e^{\lambda t} \cdot u_\lambda \quad (\text{la fel ca la 1.}) \quad \tilde{\varphi}'_\lambda(t) = A \tilde{\varphi}_\lambda(t)$$

$$= e^{\alpha t} (v \cos \beta t - w i \sin \beta t + i(v \sin \beta t + w \cos \beta t)) = \\ = e^{\alpha t} (v \cos \beta t - w \sin \beta t) + i e^{\alpha t} (w \sin \beta t + v \cos \beta t)$$

$$\text{Fie } \ell_\lambda(t) := \operatorname{Re}(\tilde{\ell}_\lambda(t)) = e^{\alpha t} (v \cos \beta t - w \sin \beta t)$$

$$\tilde{\ell}_\lambda(t) := \operatorname{Im}(\tilde{\ell}_\lambda(t)) = e^{\alpha t} (v \sin \beta t - w \cos \beta t)$$

$$\tilde{\ell}_\lambda(t) = \ell_\lambda(t) + i \varphi_\lambda(t) \quad \left\{ \Rightarrow \ell'_\lambda(t) + i \varphi'_\lambda(t) = A \ell_\lambda(t) + i A \varphi_\lambda(t) \Rightarrow \right.$$

$$\tilde{\ell}'_\lambda(t) = A \tilde{\ell}_\lambda(t)$$

$$\left. \begin{array}{l} \Rightarrow \ell'_\lambda(t) = A \ell_\lambda(t) \\ \ell'_\lambda(t) = A \varphi_\lambda(t) \end{array} \right\} \Rightarrow \ell_\lambda(\cdot), \varphi_\lambda(\cdot) \in S_A$$

Prop 4: (vectorii proprii linare dependenți): $A \in L(\mathbb{R}^n, \mathbb{R}^n)$

$$\textcircled{1} \quad \lambda_1, \dots, \lambda_k \in \text{T}(A) \text{ și } u_j \in V\text{P}_A(\lambda_j), j = \overline{1, k} \quad \left. \begin{array}{l} u_j \in V\text{P}_A(\lambda_j), j = \overline{1, k} \\ \lambda_j \neq \lambda_p, \forall j \neq p \end{array} \right\} \Rightarrow \{u_1, \dots, u_k\} \subset \mathbb{C}^k \text{ linier independent}$$

$$\textcircled{2} \quad \lambda_1, \dots, \lambda_m \in \text{T}(A) \cap \mathbb{R} \text{ și } u_j \in V\text{P}_A(\lambda_j) \cap \mathbb{R}^n, j = \overline{1, m}$$

$$\lambda_j = \alpha_j + i \beta_j \in \text{T}(A) \quad j = \overline{m+1, k}, u_j = v_j + i w_j \in V\text{P}_A(\lambda_j), j = \overline{m+1, k}$$

$$\Rightarrow \{u_1, \dots, u_m, u_{m+1}, \dots, u_k\} \subset \mathbb{R}^n \text{ linier independent}$$

Dem: \textcircled{1} Inductie după k .

$$k=1, u_1 \in V\text{P}_A(\lambda_1) \Rightarrow u_1 \neq 0 \text{ deoarece}$$

$$k \mapsto k+1, A | c_1 u_1 + \dots + c_k u_k + c_{k+1} u_{k+1} = 0 \Rightarrow c_j = 0, \forall j = \overline{1, k+1}$$

$$c_1 \underbrace{A u_1}_{\lambda_1 u_1} + \dots + c_k \underbrace{A u_k}_{\lambda_k u_k} + c_{k+1} \underbrace{A u_{k+1}}_{\lambda_{k+1} u_{k+1}} = 0$$

$$c_1 \lambda_1 u_1 + \dots + c_k \lambda_k u_k + c_{k+1} \lambda_{k+1} u_{k+1} = 0$$

$$c_1 (\lambda_1 - \lambda_{k+1}) u_1 + \dots + c_k (\lambda_k - \lambda_{k+1}) u_k = 0$$

$$\Rightarrow c_j (\lambda_j - \lambda_{k+1}) = 0, j = \overline{1, k} \Rightarrow c_i = 0, i = \overline{1, k} \Rightarrow c_1 = 0$$

$$\textcircled{2} \quad \sum_{j=1}^m c_j u_j + \sum_{j=m+1}^k (c_j v_j + k_j w_j) = 0 \stackrel{?}{\Rightarrow} c_j = 0, k_j = 0$$

$$j = \overline{m+1, k} \quad \begin{cases} u_j = v_j + i w_j \\ \bar{u}_j = v_j - i w_j \end{cases} \Rightarrow v_j = \frac{1}{2}(u_j + \bar{u}_j) \Rightarrow w_j = \frac{1}{2i}(u_j - \bar{u}_j)$$

$$\Rightarrow \sum_{j=1}^m c_j u_j + \sum_{j=m+1}^k \left[\frac{c_j}{2}(u_j + \bar{u}_j) + \frac{k_j}{2i}(u_j - \bar{u}_j) \right] = 0$$

$$\sum_{j=1}^m c_j u_j + \sum_{j=m+1}^k \left[\left(\frac{c_j}{2} + \frac{k_j}{2i} \right) u_j + \left(\frac{c_j}{2} - \frac{k_j}{2i} \right) \bar{u}_j \right] = 0 \stackrel{?}{\Rightarrow}$$

$$\Rightarrow c_j = 0, j = \overline{1, m} \quad \begin{cases} \frac{c_j}{2} + \frac{k_j}{2i} = 0 \\ \frac{c_j}{2} - \frac{k_j}{2i} = 0 \end{cases}, j = \overline{m+1, k} \Rightarrow \begin{cases} c_j = 0 \\ k_j = 0 \end{cases}, j = \overline{m+1, k}$$

Teorema (Structura soluțiilor în cazul valorilor proprii simple):

$A \in L(\mathbb{R}^n, \mathbb{R}^n)$, $\frac{dx}{dt} = Ax$. Pre presupunem $\mathbb{T}(A) = \{\lambda_1, \dots, \lambda_m\}$ distincte. Definim

funcția, pentru $\lambda \in \mathbb{T}(A)$, $\varphi_\lambda(t) = \begin{cases} e^{\lambda t} \cdot u_\lambda, & \text{dacă } \lambda \in \mathbb{T}(A) \cap \mathbb{R}, u_\lambda \in VP_A(\lambda) \cap \mathbb{R}^n \\ Re(e^{\lambda t} \cdot u_\lambda) \\ Im(e^{\lambda t} \cdot u_\lambda) & \text{dacă } \lambda = \alpha + i\beta \in \mathbb{T}(A), u_\lambda \in VP_A(\lambda) \\ & \beta > 0 \end{cases}$

Atunci $\{\varphi_\lambda(\cdot)\}_{\lambda \in \mathbb{T}(A)} \subset S_A$ sistem fundamental de soluții

Dacă:

$\varphi_\lambda(\cdot) \in S_A$, $\forall \lambda \in \mathbb{T}(A)$ (Prop. 3)

$m =$ soluții \Rightarrow este suficient să verificăm $\{\varphi_\lambda(\cdot)\}_{\lambda \in \mathbb{T}(A)} \subset S_A$ liniar independent.

\uparrow Prop (Soluții liniar independente)

$= \{\varphi_\lambda(0)\}_{\lambda \in \mathbb{T}(A)} \subset \mathbb{R}^n$ liniar independent

$= \{u_\lambda \mid \lambda \in \mathbb{T}(A) \cap \mathbb{R}, u_\lambda \in VP_A(\lambda) \cap \mathbb{R}^n\} \cup \{Re(u_\lambda), Im(u_\lambda) \mid \lambda = \alpha + i\beta \in \mathbb{T}(A)\}$

$u_\lambda \in VP_A(\lambda) \subset \mathbb{R}^n$ liniar independent

Algoritm: (cazul valorilor proprii simple)

$$\frac{dx}{dt} = Ax$$

- ① Rezolvă ecuația caracteristică: $\det(A - \lambda I_n) = 0 \Rightarrow \tau(A) = \{\lambda_1, \dots, \lambda_m\}$
- ② Dacă $\lambda \in \tau(A) \cap \mathbb{R}$ căută $u_\lambda \in \mathbb{R}^n \setminus \{0\}$ astfel încât distincțe
- $(A - \lambda I_n) u_\lambda = 0$. Iată $\varphi_\lambda(t) = e^{\lambda t} \cdot u_\lambda$
- ③ Dacă $\lambda = \alpha + i\beta \in \tau(A)$, $\beta > 0$ căută $u_\lambda \in \mathbb{C}^n \setminus \{0\}$ astfel încât
- $(A - \lambda I_n) u_\lambda = 0$. Iată soluții reale: $\varphi_\lambda(t) = \operatorname{Re}(e^{\lambda t} \cdot u_\lambda)$
- $\bar{\varphi}_\lambda(t) = \operatorname{Im}(e^{\lambda t} \cdot u_\lambda)$
- ④ Renumerotează $\{\varphi_\lambda(\cdot)\}_{\lambda \in \tau(A)} = \{\varphi_1(\cdot), \dots, \varphi_m(\cdot)\} \subset S_A$ sistem fundamental de soluții. Iată soluția generală $\Psi(t) = \sum_{i=1}^m c_i \varphi_i(t)$, $c_i \in \mathbb{R}$, $i = 1, m$

Ecuatii diferențiale

Seminar 3 : 21. 11. 2017.

Algoritru (casul valorilor proprii simple):

$$\frac{dx}{dt} = Ax, A \in L(\mathbb{R}^n, \mathbb{R}^n)$$

①. Rezolvă ecuația corect. $\det(A - \lambda I_n) = 0 \Rightarrow \tau(A) = \{\lambda_1, \dots, \lambda_n\}$ distințe

②. Dacă $\lambda \in \tau(A) \cap \mathbb{R}$ căci $u_\lambda \in \mathbb{C}^n \setminus \{0\}$ astfel încât $(A - \lambda I_n) \cdot u_\lambda = 0$

Se scrie soluția $f_\lambda(t) = e^{\lambda t} \cdot u_\lambda$

③. Dacă $\lambda = \alpha + i\beta \in \tau(A)$, $\beta > 0$ căci $u_\lambda \in \mathbb{C}^n \setminus \{0\}$ astfel încât $(A - \lambda I_n) u_\lambda = 0$.

Se scriu soluțiile: $f_\lambda(t) = \operatorname{Re}(e^{\lambda t} \cdot u_\lambda)$

$$f_\lambda(t) = \operatorname{Im}(e^{\lambda t} \cdot u_\lambda)$$

④. Renumerotarea că $\{f_\lambda(\cdot)\}_{\lambda \in \tau(A)} = \{f_1(\cdot), \dots, f_n(\cdot)\}$ sistem fundamental de soluții. Se scrie soluția generală $y(t) = \sum_{i=1}^n c_i f_i(t)$, $c_i \in \mathbb{R}$, $i = \overline{1, n}$

Exercitii: Se se determine soluția generală:

$$\begin{cases} x' = x + z - y \\ y' = x + y - z \\ z' = 2x - y \end{cases}$$

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix}$$

$$\det \left[\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{pmatrix} - \lambda \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] = 0$$

$$\begin{aligned} \det \begin{pmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 2 & -1 & -\lambda \end{pmatrix} &= \lambda(1-\lambda)^2 - 1 + 2 - 2(1-\lambda) - (1-\lambda) - \lambda = \\ &= -\lambda(1-2\lambda+\lambda^2) + 1 - 2 + 2\lambda - 1 + \lambda - \lambda = \\ &= -\lambda^3 + 2\lambda^2 - \lambda^3 - 2 + 2\lambda = -\lambda^3 + 2\lambda^2 + \lambda - 2 \end{aligned}$$

$$-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$$

$$-\lambda^3 + \lambda^2 + \lambda^2 - \lambda + 2\lambda - 2 = -\lambda^2(\lambda - 1) + \lambda(\lambda - 1) + 2(\lambda - 1) =$$

$$= (\lambda - 1)(-\lambda^2 + \lambda - 2) = 0$$

$$\lambda_1 = 1$$

$$\Delta = 1 + 4 \cdot 2 = 9$$

$$\lambda_2 = 2$$

$$\lambda_3 = -1$$

racăciuni reale, simple și distințe

- $\lambda = 1$, se caută $u = ?$ astfel încât $(A - \lambda I_3)u = 0$

$$u = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \left(\begin{array}{ccc} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & -1 \end{array} \right) \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left(\begin{array}{c} -b+c \\ a-c \\ 2a-b-c \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) \Leftrightarrow \left\{ \begin{array}{l} -b+c=0 \Rightarrow b=c \\ a-c=0 \Rightarrow c=a \\ 2a-b-c=0 \end{array} \right. \Rightarrow u = \begin{pmatrix} a \\ a \\ a \end{pmatrix} \quad a \in \mathbb{R} \setminus \{0\}$$

$$a=1 \Rightarrow u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \text{soluție } \varphi_1(t) = e^t \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

- $\lambda = 2$, se caută $u = ?$ astfel încât $(A - 2I_3) \cdot u = 0$

$$u = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \left(\begin{array}{ccc} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 2 & -1 & -2 \end{array} \right) \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left\{ \begin{array}{l} -a-b+c=0 \\ a-b-c=0 \\ 2a-b-2c=0 \end{array} \right. \Rightarrow b=0 \Rightarrow a=c \Rightarrow u = \begin{pmatrix} a \\ 0 \\ a \end{pmatrix}$$

$$\text{Pd. } a=1 \text{ avem soluția } \varphi_2(t) = e^{2t} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

- $\lambda = -1$, se caută $u = ?$ astfel încât $(A + I_3) \cdot u = 0$. $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\left(\begin{array}{ccc} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{array} \right) \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \left\{ \begin{array}{l} 2a-b+c=0 \\ a+2b-c=0 \\ 2a-b+c=0 \end{array} \right. \Rightarrow \left\{ \begin{array}{l} b=-3a \\ c=a+2b=-5a \end{array} \right.$$

$$\Rightarrow u = \begin{pmatrix} a \\ -3a \\ -5a \end{pmatrix}. \quad \text{Pd. } a=1 \text{ avem } \varphi_3(t) = e^{-t} \cdot \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$$

Soluția generală este: $\Psi(t) = c_1 \varphi_1(t) + c_2 \varphi_2(t) + c_3 \varphi_3(t)$

$$\varphi(t) = c_1 e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + c_3 e^{-t} \begin{pmatrix} -1 \\ 3 \\ 5 \end{pmatrix}$$

$$\varphi(t) = \begin{pmatrix} c_1 e^t + c_2 e^{2t} - c_3 e^{-t} \\ c_1 e^t + 3c_3 e^{-t} \\ c_1 e^t + c_2 e^{2t} + 5c_3 e^{-t} \end{pmatrix}$$

$$\Rightarrow x(t) = c_1 e^t + c_2 e^{2t} - c_3 e^{-t}$$

$$y(t) = c_1 e^t + 3c_3 e^{-t}$$

$$z(t) = c_1 e^t + c_2 e^{2t} + 5c_3 e^{-t}$$

Exercițiu: $x' = 2x + y$

$$2. \quad \begin{cases} y' = x + 3y - z \\ z' = 2y + 3z - x \end{cases}$$

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix}$$

$$A - \lambda \mathbb{I}_3 = \begin{pmatrix} 2-\lambda & 1 & 0 \\ 1 & 3-\lambda & -1 \\ -1 & 2 & 3-\lambda \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 1 & 0 \\ -1 & 3-\lambda & -1 \\ -1 & 2 & 3-\lambda \end{pmatrix}$$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ -1 & 3-\lambda & -1 \\ -1 & 2 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda)^2 - (3-\lambda) + 2(2-\lambda) + 3-\lambda$$

$$= (2-\lambda)(9 - 6\lambda + \lambda^2) - 3 + \lambda + 4 - 2\lambda + 3 - \lambda$$

$$= (2-\lambda)(\lambda^2 - 6\lambda + 10)$$

$$\lambda_1 = 2$$

$$\lambda_{2,3} = 3 \pm i$$

• $\lambda = 2$, căută $u = ?$ astfel încât $(A - 2\mathbb{I}_3) \cdot u = 0$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & -1 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Leftrightarrow \begin{cases} b = 0 \\ a + b - c = 0 \Rightarrow a = c \end{cases} \Rightarrow u = \begin{pmatrix} a \\ 0 \\ a \end{pmatrix}$$

$$\alpha=1 \Rightarrow u = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \varphi_0(t) = e^{2t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\bullet \lambda = 3+i$, căutăm $u \in \mathbb{C}^3$ astfel încât $(A - (3+i)\mathbb{M}_3) \cdot u = 0$.

$$u = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \begin{pmatrix} -1-i & 1 & 0 \\ 1 & -i & -1 \\ -1 & 2 & -i \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} -a - ai + b = 0 \Rightarrow b = -a(1+i) \\ a - ib - c = 0 \Rightarrow c = a - ib = a(2-i) \\ -a + 2b - ic = 0 \end{cases} \Rightarrow u = \begin{pmatrix} a \\ a(1+i) \\ a(2-i) \end{pmatrix}$$

Pentru $a=1$ avem: $u = \begin{pmatrix} 1 \\ 1+i \\ 2-i \end{pmatrix}$

$$\tilde{\varphi}(t) = e^{(3+i)t} \cdot \begin{pmatrix} 1 \\ 1+i \\ 2-i \end{pmatrix} = e^{3t} \cdot e^{it} \cdot \begin{pmatrix} 1 \\ 1+i \\ 2-i \end{pmatrix} = e^{3t} \cdot (\cos t + i \sin t) \begin{pmatrix} 1 \\ 1+i \\ 2-i \end{pmatrix}$$

$$= e^{3t} \cdot \begin{pmatrix} \cos t + i \sin t \\ \cos t - \sin t + i(\sin t + \cos t) \\ 2\cos t + \sin t + i(2\sin t + \cos t) \end{pmatrix} =$$

$$= e^{3t} \begin{pmatrix} \cos t \\ \cos t - \sin t \\ 2\cos t + \sin t \end{pmatrix} + i \cdot e^{3t} \begin{pmatrix} \sin t \\ \sin t + \cos t \\ 2\sin t + \cos t \end{pmatrix}$$

$$\varphi_2(t) = e^{3t} \begin{pmatrix} \cos t \\ \cos t - \sin t \\ 2\cos t + \sin t \end{pmatrix} \quad \varphi_3(t) = e^{3t} \begin{pmatrix} \sin t \\ \sin t + \cos t \\ 2\sin t + \cos t \end{pmatrix}$$

Exercițiu:

$$\begin{cases} x' = x - 2y \\ y' = 3y + x \end{cases} \quad A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

$$\det(A - \lambda \mathbb{M}_2) = \begin{vmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = (1-\lambda)(3-\lambda) + 2 = 3 - \lambda - 3\lambda + \lambda^2 + 2 = \lambda^2 - 4\lambda + 5$$

$$\lambda^2 - 4\lambda + 5 = 0$$

$$\Delta = 16 - 20 = -4 \Rightarrow \lambda_{1,2} = 2 \pm i$$

$\bullet \lambda = 2+i$, se caută $u=?$ astfel încât $(A - (2+i)I) \cdot u = 0$.

$$u = \begin{pmatrix} a \\ b \end{pmatrix} \quad \begin{pmatrix} -1-i & -2 \\ 1 & 1-i \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -a - ai - 2b = 0 \\ a + b - bi = 0 \Rightarrow a = b(1+i) \end{cases}$$

$$\Rightarrow u = \begin{pmatrix} b(1+i) \\ b \end{pmatrix}$$

$$\text{Pentru } b=1 \Rightarrow u = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}. \text{ Fie } \tilde{\ell}(t) = e^{(2+i)t} \begin{pmatrix} 1+i \\ 1 \end{pmatrix} = e^{2t} \cdot e^{it} \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$$

$$= e^{2t} (\cos t + i \sin t) \begin{pmatrix} 1+i \\ 1 \end{pmatrix} = e^{2t} \cdot \begin{pmatrix} \cos t + \sin t + i(\sin t - \cos t) \\ \cos t + i \sin t \end{pmatrix}$$

$$= e^{2t} \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} + i \cdot e^{2t} \begin{pmatrix} \sin t - \cos t \\ \sin t \end{pmatrix}$$

$$\varphi_1(t) = e^{2t} \begin{pmatrix} \cos t + \sin t \\ \cos t \end{pmatrix} \quad \varphi_2(t) = e^{2t} \begin{pmatrix} \sin t - \cos t \\ \sin t \end{pmatrix}$$

Varianta a 2-a :

$$\left\{ \begin{array}{l} x' = x - 2y \\ y' = 3y + x \end{array} \right.$$

$$\left\{ \begin{array}{l} x' = x - 2y \\ y' = 3y + x \Rightarrow x = y' - 3y \end{array} \right.$$

$$(y' - 3y)' = y' - 3y - 2y$$

$$y'' - 3y' = y' - 5y$$

$$y'' - 4y' + 5y = 0$$

$y'' - 4y' + 5y = 0 \rightarrow$ ecuație liniară cu coeficienți constanti

Ecuția caracteristică este: $z^2 + 4z + 5 = 0$

$$z_{1,2} = 2 \pm i$$

$$y(t) = c_1 \cdot e^{2t} \cos t + c_2 \cdot e^{2t} \sin t, c_1, c_2 \in \mathbb{R}$$

$$x(t) = 2c_1 \cdot e^{2t} \cos t + c_1 \cdot e^{2t} (-\sin t) + 2c_2 \cdot e^{2t} \sin t + c_2 \cdot e^{2t} \cos t - 3c_1 \cdot e^{2t} \cos t - 3c_2 \cdot e^{2t} \sin t = (-c_1 + c_2) \cdot e^{2t} \cos t + (-c_1 - c_2) \cdot e^{2t} \sin t$$

Exercitiu:

$$\begin{cases} x^1 = 2x+y \\ y^1 = x+3y-z \\ z^1 = -x+2y+3z \end{cases}$$

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ -1 & 2 & 3 \end{pmatrix}$$

$$\det(A - \lambda I_3) = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 1 & 3-\lambda & -1 \\ -1 & 2 & 3-\lambda \end{vmatrix}$$

Exercitiu:

$$\begin{cases} x^1 = x-y-z \\ y^1 = x+y \\ z^1 = 3x+z \end{cases}$$

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix} \Rightarrow \det(A - \lambda I_3) = \begin{vmatrix} 1-\lambda & -1 & -1 \\ 1 & 1-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{vmatrix} =$$

$$= (1-\lambda)^3 + 3(1-\lambda) + (1-\lambda) = (1-\lambda)[(1-\lambda)^2 + 3 + 1] = (1-\lambda)(1-2\lambda+\lambda^2+4) =$$

$$= (1-\lambda)(\lambda^2-2\lambda+5)$$

$$1-\lambda=0 \Rightarrow \lambda_1=1$$

$$\lambda^2-2\lambda+5=0$$

$$\Delta = 4-20 = -16 \Rightarrow \lambda_2 = 1+2i$$

$$\sqrt{\Delta} = 4i \quad \lambda_3 = 1-2i$$

- $\lambda_1=1$, sănătăm $u=?$ astfel încât $(A - \lambda_1 I_3) \cdot u = 0$. $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 2 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} b=0 \\ a-c=0 \\ -a+2b=0 \end{cases} \Rightarrow \begin{cases} c=b \\ a=b \\ a=0 \end{cases}$$

$$\Rightarrow u = \begin{pmatrix} 0 \\ b \\ -b \end{pmatrix}. \text{ Pentru } b=1 \text{ avem } u = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix},$$

$$u = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \Rightarrow e^{tA} = e^{t \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ -1 & 2 & 0 \end{pmatrix}}$$