Lab 6

$\{X(\omega) \mid \omega \in \Omega\} \rightarrow$ cel mult num
v.a. discretă
$\rightarrow$ inf nenum
v.a. continuă

$X : \begin{pmatrix} x_1 & x_2 & \dots & x_n & \dots \\ p_1 & p_2 & \dots & p_n & \dots \end{pmatrix}$ $\quad X : \begin{pmatrix} x \\ p_x \end{pmatrix}$ $\quad x \in \mathbb{R}$

Def $f : \mathbb{R} \rightarrow \mathbb{R}$ s.n. densitate de probabilitate dacă: 1) $f(x) \geq 0; \forall x \in \mathbb{R}$

2) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Def $F : \mathbb{R} \rightarrow \mathbb{R}$ s.n. fct. de repartiție dacă

$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(t)\,dt$

Obs Dacă F este derivabilă,

$$F'(x) = f(x)$$

Obs $P(a<X\le b) = F(b) - F(a)$

$$P(a\le X\le b) = P(a<X<b) = P(a\le X<b)$$

Ex 1) X v.a. cont.

$$f:\mathbb{R}\to\mathbb{R}\,; \quad f(x) = \begin{cases} ax, & x\in[0,1],\ a\in\mathbb{R} \\ 2-x, & x\in(1,2] \\ 0, & x\notin[0,2] \end{cases}$$

a) $\underset{+}{f}$ este densitate de prob

$\underset{\#}{a} = ?$

$$f \text{ dens. de prob} \Leftrightarrow \begin{cases} f(x)\ge 0, \ \forall x\in\mathbb{R} \\ \int_{-\infty}^{\infty} f(x)\,dx \end{cases}$$

$$f(x)\ge 0 \Leftrightarrow ax\ge 0,\ x\in[0,1] \Leftrightarrow a\ge 0$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \Leftrightarrow \int_0^1 ax\,dx + \int_1^2 2-x\,dx = 1 \Leftrightarrow$$

$$\Leftrightarrow a\frac{x^2}{2}\Big|_0^1 + 2x - \frac{x^2}{2}\Big|_1^2 = 1 \Leftrightarrow \frac{a}{2} - 0 + 2(2-1) - \left(2-\frac{1}{2}\right) = 1 \Leftrightarrow$$

$$\Leftrightarrow \frac{a}{2} + 2 - \frac{3}{2} = 1 \Leftrightarrow \frac{a}{2} + \frac{1}{2} = 1 \Leftrightarrow a = 1$$

$$f(x) = \begin{cases} 2x, & x \in [0,1] \\ 2-x, & x \in (1,2] \\ 0, & x \notin [0,2] \end{cases}$$

b)

$$f(t) = \begin{cases} 0, & (-\infty, 0) \\ t, & t \in [0,1] \\ 2-t, & t \in (1,2] \\ 0, & t \in (2,\infty) \end{cases}$$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

I $x < 0 \Rightarrow F(x) = \int_{-\infty}^{x} 0 \, dt$

II $x \in [0,1] \Rightarrow F(x) = \int_{-\infty}^{0} 0 \, dt + \int_{0}^{x} t\,dt = \frac{t^2}{2}\Big|_0^x = \frac{x^2}{2}$

III $x \in (1,2] \Rightarrow F(x) = \int_{-\infty}^{0} 0\,dt + \int_0^1 t\,dt + \int_1^x 2-t\,dt = 2x - \frac{x^2}{2} - 1$

IV $x \in (2,\infty) \Rightarrow F(x) = $ — // — $+ \int_2^{\infty} 0\,dt = 1$

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{2}, & x \in [0,1] \\ 2x - \frac{x^2}{2} - 1, & x \in (1,2] \\ 1, & x \in (2,\infty) \end{cases}$$

$$P(0.2 \le X \le 0.9 \mid 0.3 < X) = \frac{P(0.3 < X \le 0.9)}{P(0.3 < X)} =$$

$$= \frac{F(0.9) - F(0.3)}{1 - F(0.3)} = \frac{\frac{\left(\frac{9}{10}\right)^2}{2} - \frac{\left(\frac{3}{10}\right)^2}{2}}{1 - \frac{\left(\frac{3}{10}\right)^2}{2}} = \frac{72}{191}$$

$$f(x) = \begin{cases} x, & x \in (0,1] \\ 2-x, & x \in (1,2] \\ 0, & x \notin [0,2] \end{cases}$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_{-\infty}^{0} x \cdot 0\,dx +$$

$$+ \int_0^1 x \cdot x\,dx +$$

$$+ \int_1^2 x(2-x)\,dx +$$

$$+ \int_2^{\infty} x \cdot 0\,dx =$$

$$= \frac{x^3}{3}\Big|_0^1 + x^2 - \frac{x^3}{3}\Big|_1^2 = \frac{1}{3} + \left(2^2 - \frac{2^3}{3}\right) - \left(1^2 - \frac{1^3}{3}\right) =$$

$$= \frac{1}{3} + \frac{4}{3} - \frac{2}{3} = \frac{3}{3} = 1$$

$$E(X^2) = \frac{x^4}{4}\Big|_0^1 + \frac{2}{3}x^3 - \frac{x^4}{4}\Big|_1^2 = \frac{1}{4} + \left(\frac{2}{3} \cdot 8 - \frac{16}{4}\right) - \left(\frac{2}{3} - \frac{1}{4}\right) =$$

$$= \frac{7}{6}$$

$Var(X) = E(X^2) - E(X)^2 = \frac{7}{6} - 1^2 = \frac{1}{6}$

TEMĂ

X v.a. cont

$f: \mathbb{R} \to \mathbb{R}$; $f(x) = \begin{cases} k\, x^2 e^{-\frac{x}{2}} ; & x \geq 0 \\ 0 ; & x < 0 \end{cases}$

a) $k = ?$ a.î. f densitate

b) f dens de prob $\Leftrightarrow \begin{cases} f(x) \geq 0 ; \forall x \in \mathbb{R} \\ \int_{-\infty}^{\infty} f(x)\, dx = 1 \end{cases}$

Rez

a) $f(x) \geq 0 ; \forall x \in \mathbb{R} \Leftrightarrow k \underbrace{x^2 e^{-\frac{x}{2}}}_{\geq 0} \geq 0 ; \forall x \geq 0$

$\Leftrightarrow k > 0 ; \forall x \geq 0$

b) $\int_{-\infty}^{\infty} f(x)\, dx = 1 \Leftrightarrow \int_{-\infty}^{0} 0\, dx + \int_{0}^{\infty} k\, x^2 e^{-\frac{x}{2}} dx = 1$

Prop $\Gamma(a) = \int_0^{\infty} x^{a-1} e^{-x} dx ; \ a > 0$

$\Gamma(n) = (n-1)!\,, \quad \forall n \in \mathbb{N}^*$

$\Gamma(a) = (a-1)\,\Gamma(a-1)\,; \forall a > 1$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$