TO:

FROM: Ecuații diferențiale - 12.12.2017 - curs

Ecuații afine de ordin superior

(1) $x^{(n)} = \sum_{j=1}^{n} a_j(t)\, x^{(n-j)} + b(t)$

$a_1(\cdot), \dots, a_n(\cdot), b(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}$

Sistemul canonic asociat

$$(2)\begin{cases} \frac{dx_1}{dt} = x_2 \\ \frac{dx_2}{dt} = x_3 \\ \dots \\ \frac{dx_{n-1}}{dt} = dx_n \\ \frac{dx_n}{dt} = \sum_{j=1}^{n} a_j(t)\, x_{n-j+1} + b(t) \end{cases}$$

$\tilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $A(t) = \text{comp}(a_1(t), \dots, a_n(t))$ $\tilde{b}(t) = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ b(t) \end{pmatrix}$

$(2)\ \frac{d\tilde{x}}{dt} = A(t)\tilde{x} + \tilde{b}(t)$

PROP (de echivalență):
$\varphi(\cdot)$ sol. a ec. (1) $\Leftrightarrow \tilde{\varphi}(\cdot) = (\varphi(\cdot), \varphi'(\cdot), \dots, \varphi^{(n-1)}(\cdot))$ sol. a ec. (2)

Th. (E.U.G.)
$a_1(\cdot), \dots, a_n(\cdot), b(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}$ cont. def. (1)
Atunci $\forall (t_0, (x_0, x_0', \dots, x_0^{(n-1)})) \in I \times \mathbb{R}^n\ \exists!\ \varphi(\cdot) : I \to \mathbb{R}$ sol. cu $\varphi(t_0) = x_0$, $\varphi'(t_0) = x_0'$, ... $\varphi^{(n-1)}(t_0) = x_0^{n-1}$.

$S_{a_1(\cdot), \dots, a_n(\cdot), b(\cdot)} = \{\varphi(\cdot) : I \to \mathbb{R} ;\ \varphi(\cdot) \text{ sol. ec. } (1)\}$

Th. (varietatea sol.)

$S_{a_1(\cdot), \dots, a_n(\cdot), b(\cdot)} = S_{a_1(\cdot), \dots, a_n(\cdot)} + \{\varphi_0(\cdot)\}\ \forall \varphi_0(\cdot) \in S_{a_1(\cdot), \dots, a_n(\cdot), b(\cdot)}$

Th. (principiul variației constantelor)

Fie $\{\bar{\varphi}_1(\cdot), \dots, \bar{\varphi}_n(\cdot)\}$ sistem fundamental de sol. pt. ec. liniară asociată $\bar{x}^{(n)} = \sum_{j=1}^{n} a_j(t)\, \bar{x}^{(n-j)}$

At. $\varphi(\cdot) \in S_{a_1(\cdot), \dots, a_n(\cdot), b(\cdot)} \Leftrightarrow \exists\ c(\cdot) = \begin{pmatrix} c_1(\cdot) \\ \vdots \\ c_n(\cdot) \end{pmatrix}$ primitivă a funcției

$$t \longmapsto \begin{pmatrix} \bar{\varphi}_1(t) & \dots & \bar{\varphi}_n(t) \\ \bar{\varphi}_1'(t) & \dots & \bar{\varphi}_n'(t) \\ \vdots & & \\ \bar{\varphi}_1^{(n-1)}(t) & \dots & \bar{\varphi}_n^{(n-1)}(t) \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b(t) \end{pmatrix}$$ a. î. $\varphi(t) = \sum_{i=1}^{n} c_i(t)\, \bar{\varphi}_i(t)$

Dem: Th. (Principiul var. const. pt. ec. afine pe $\mathbb{R}^n$) apl. lui (2) + Prop. de echivalență

$\tilde{\bar{\varphi}}_j(\cdot) = \begin{pmatrix} \bar{\varphi}_j(\cdot) \\ \bar{\varphi}_j'(\cdot) \\ \vdots \\ \bar{\varphi}_j^{(n-1)}(\cdot) \end{pmatrix} \Rightarrow \{\tilde{\bar{\varphi}}_1(\cdot), \dots, \tilde{\bar{\varphi}}_n(\cdot)\} \subset S_{A(\cdot)}$ sist. fundamental de sol.

pt. $\frac{d\tilde{x}}{dt} = A(t)\tilde{x}$  $A(t) = \text{comp}(a_1, \dots, a_n) \Rightarrow X(t) = \text{col}[\tilde{\varphi}_1(t), \dots \tilde{\varphi}_n(t)]$

matrice fundamentală de sol. pt. $\frac{dx}{dt} = A(t)\tilde{x}$

Aplic Th (Pr. var. const. pt. ec. afine pe $\mathbb{R}^n$) lui (2) $\left[\frac{dx}{dt} = A(t)\tilde{x} + \tilde{b}(t)\right]$ $\tilde{\varphi}(\cdot)$

sol. ec. (2) $\Leftrightarrow \exists\, c(\cdot)$ primitivă a lui $t \mapsto X^{-1}(t)\tilde{b}(t)$ a.î.

$\tilde{\varphi}(t) = X(t)c(t)$

$c(t) = \begin{pmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{pmatrix}$

$\Rightarrow$ Se reține doar egalitatea dintre prima comp. din stânga cu prima comp. din dreapta $\Rightarrow$ q.e.d

Algoritm

$$x^{(n)} = \sum_{j=1}^{n} a_j(t)x^{(n-j)} + b(t)$$

1. Considerăm ec. liniară asociată $\bar{x}^{(n)} = \sum_{j=1}^{n} a_j(t)\bar{x}^{(n-j)}$

   Determină $\{\bar{\varphi}_1(\cdot), \dots, \bar{\varphi}_n(\cdot)\}$ sistem fundamental de sol.

Obs: Dacă $a_j(t) \equiv a_j \in \mathbb{R}$ $\forall j = \overline{1,n} \rightarrow$ vezi Algoritm

Scrie sol. generală $\bar{x}(t) = \sum_{i=1}^{n} c_i \bar{\varphi}_i(t)$  $c_i \in \mathbb{R}, i = \overline{1,n}$

2. Variația const. propriu-zisă

   Caută în sol. de forma $x(t) = \sum_{i=1}^{n} c_i(t)\bar{\varphi}_i(t)$

Rezolvă sistemul algebric

$$\begin{cases} \sum_{i=1}^{n} c_i'(t)\bar{\varphi}_i(t) = 0 \\ \sum_{i=1}^{n} c_i'(t)\bar{\varphi}_i'(t) = 0 \\ \dots \\ \sum_{i=1}^{n} c_i'(t)\bar{\varphi}_i^{(n-2)}(t) = 0 \\ \sum_{i=1}^{n} c_i'(t)\bar{\varphi}_i^{(n-1)}(t) = \varphi \end{cases} \xrightarrow{\text{Kramer}} c_i'(t) = \dots \quad i = \overline{1,n}$$

$\Rightarrow c_i(t) = \dots \quad i = \overline{1,n}$

$\Rightarrow x(t) = \dots$

Scrie sol. generală

TO:

FROM:

## Integrale prime pt. ec. diferențiale în $\mathbb{R}^n$

$\frac{dx}{dt} = f(t,x)$ $\qquad f(\cdot,\cdot) : D \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$

Def: a) $F(\cdot,\cdot) : D_0 \subset D \to \mathbb{R}$ s.n. integrală primă pt. câmpul vectorial ~~f dom~~ $f(\cdot,\cdot)$ (sau pt. ec. $\frac{dx}{dt} = f(t,x)$) dacă $\forall\ \varphi(\cdot)$ soluție cu graph $\varphi(\cdot) \subset D_0$ $\exists\ c_\varphi \in \mathbb{R}$ a.î. $F(t,\varphi(t)) \equiv c$

b) $F(\cdot,\cdot) : D_0 \subset D \to \mathbb{R}^k$ s.n. integrală primă vectorială pt. e.v. $f(\cdot,\cdot)$ (sau pt. ec. $\frac{dx}{dt} = f(t,x)$) dacă $\forall\ \varphi(\cdot)$ soluție cu graph $\varphi(\cdot) \subset D_0$ $\exists\ C_\varphi \in \mathbb{R}^k$ a.î. $F(t,\varphi(t)) \equiv C_\varphi$

Obs1. $F(\cdot,\cdot) = (F_1(\cdot),\dots,F_k(\cdot))$ integr. primă asociată $\Leftrightarrow F_j(\cdot,\cdot)$ int. pr. $\forall\ i = \overline{1,k}$

Obs2: $F(t,x) \equiv c$ integrale prime triviale

Obs3: (Neunicitatea integralelor prime)
$F_1(\cdot,\cdot),\dots,F_k(\cdot,\cdot) : D_0 \to \mathbb{R}$ int. prime. $H : \mathbb{R}^k \to \mathbb{R}$ at. $\tilde{F}(t,x) := H(F_1(t,x),\dots,F_k(t,x))$ este integrală primă

Th1 (Criteriul pt. integrale prime)

Fie $f(\cdot,\cdot) : D = \overset{\circ}{D} \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ cont. $\frac{dx}{dt} = f(t,x)$
Fie $F(\cdot,\cdot) : D_0 = \overset{\circ}{D_0} \subset D \to \mathbb{R}$ derivabilă
Atunci $F(\cdot,\cdot)$ este integrală primă $\Leftrightarrow D_1F(t,x) + D_2F(t,x)\, f(t,x) = 0 \quad \forall (t,x) \in D_0$

Obs: $f(\cdot,\cdot) = (f_1(\cdot,\cdot), f_2(\cdot,\cdot),\dots,f_n(\cdot,\cdot))$
$D_1F(t,x) + DF(t,x)\, f(t,x) = \frac{\partial F}{\partial t}(t,x) + \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}(t,x)\, f_i(t,x)$

Dem: „$\Rightarrow$" Fie $(t_0,x_0) \in D_0 \subset D = \overset{\circ}{D}$, $f(\cdot,\cdot)$ cont. $\Rightarrow$ T. Peano $\Rightarrow \exists\ \varphi(\cdot) : I_0 \in V(t_0) \to \mathbb{R}^n$ sol. a.î. $\varphi(t_0) = x_0$. $\quad I_a$ a.î. graph $\varphi(\cdot) \subset D_0$
$F(\cdot,\cdot)$ int. primă $\Rightarrow \exists\ c \in \mathbb{R}$ a.î. $F(t,\varphi(t)) \equiv c \ \Big|\ \frac{\partial}{\partial t}$ $\quad D_1F(t,\varphi(t)) + D_2F(t,\varphi(t))\cdot\varphi'(t) \equiv 0 \Rightarrow D_1F(t,\varphi(t)) + D_2F(t,\varphi(t)) = 0 \quad \forall t \in I_0$
$t = t_0,\ x_0 = \varphi(t_0)$ : $D_1F(t_0,x_0) + D_2(t_0,x_0)\, f(t_0,x_0) = 0$ O.K.
„$\Leftarrow$" Fie $\varphi(\cdot)$ sol. cu graph $\varphi(\cdot) \subset D_0 \Big| \Rightarrow D_1F(t,\varphi(t)) + D_2F(t,\varphi(t))\, \underbrace{f(t,\varphi(t))}_{\varphi'(t)} \equiv 0$

$D_1F(t,\varphi(t)) + D_2F(t,\varphi(t))\varphi'(t) = 0 \qquad \frac{d}{dt}\big(F(t,\varphi(t))\big) \equiv 0 \Rightarrow \exists\ c \in \mathbb{R}$ a.î. $F(t,\varphi(t)) \equiv c$

Def: $F_1(\cdot,\cdot), F_2(\cdot,\cdot),\dots,F_k(\cdot,\cdot) : D = \overset{\circ}{D_0} \to \mathbb{R}$ derivabile, integrale prime s.n. funcționale independente dacă $\operatorname{rang}\left(\frac{\partial F_i}{\partial x_j}(t,x)\right)_{\substack{i=\overline{1,k}\\ j=\overline{1,n}}} = k\ (\text{maxim}) \le n \quad \forall (t,x) \in D_0$

T2 (Determinarea sol. cu ajutorul integralelor prime):

Fie $f(\cdot,\cdot): D = \overset{\circ}{D} \in \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ cont. $\frac{dx}{dt} = f(t,x)$

Fie $F_1(\cdot,\cdot), \dots, F_n(\cdot,\cdot): D_0 = \overset{\circ}{D_0} \subseteq D \to \mathbb{R}$ (integrale prime funct. independente)

$\left[\det\left(\frac{\partial F_i}{\partial x_j}(t,x)\right)_{\substack{i=\overline{1,n}\\ j=\overline{1,n}}} \neq 0 \ \forall (t,x) \in D_0\right]$. At.

$\varphi(\cdot): I \subset \mathbb{R} \to \mathbb{R}^n$, graph $\varphi(\cdot) \subset D_0$ este sol. $\Leftrightarrow \exists c_i \in \mathbb{R}$ a.î. $F_i(t,\varphi(t)) \equiv c_i, \forall i = \overline{1,n}$

Dem: „$\Rightarrow$" Evident din def.

„$\Leftarrow$" Fie $\varphi(\cdot): I \to \mathbb{R}^n$, graph $\varphi(\cdot) \subset D_0$, $\exists c_i \dots$

1. Se arată $\varphi(\cdot)$ este derivabilă [cu Th. de funcții implicită]

2. Arătăm că $\varphi(\cdot)$ verifică ecuația

Fie $F(\cdot,\cdot) = (F_1(\cdot,\cdot), F_2(\cdot,\cdot), \dots, F_n(\cdot,\cdot))$ $\quad c = (c_1, \dots c_n)$

$\Rightarrow F(t,\varphi(t)) \equiv c \quad | \frac{d}{dt}$

$D_1 F(t,\varphi(t)) + D_2 F(t,\varphi(t)) \cdot \varphi'(t) \equiv c$

$F_1, \dots, F_n$ funct. indep $\Rightarrow \det\left(\frac{\partial F_i}{\partial F_j}\right)_{i,j=\overline{1,n}} \neq 0 \ \forall (t,x) \in D_0 \Rightarrow D_2 F(t,x)$ inversabilă

$\varphi'(t) = -\left(D_2 F(t,\varphi(t))\right)^{-1} D_1 F(t,\varphi(t)) \overset{?}{\equiv} f(t,\varphi(t))$ (1)

$F_i(\cdot,\cdot)$ int. prime $\overset{T1}{\Longrightarrow} D_1 F_i(t,x) + D_2 F_i(t,x) \equiv 0 \ i = \overline{1,n}$

$(D_2 F(t,x))^{-1} \Big| \ D_1 F(t,x) + D_2 F(t,x) f(t,x) \equiv 0$

$\Rightarrow f(t,x) \equiv -(D_2 F(t,x))^{-1} D_1 F(t,x)$

$x = \varphi(t) \quad f(t,\varphi(t)) \equiv -\left(D_2 F(t,\varphi(t))\right)^{-1} D_1 F(t,\varphi(t))$ (2)

(1) & (2) $\Rightarrow$ q.e.d.

Obs: $F_1(\cdot,\cdot), \dots, F_n(\cdot,\cdot)$ integrale prime funct. indep., $F_i(t,x) = c_i$, $c_i \in \mathbb{R}, i = \overline{1,n}$ sol. generală sub forma implicită

Obs: (Algoritm) [Reducerea ordinului cu ajutorul integralelor prime]

$\left\| \frac{dx}{dt} = f(t,x) \quad f(\cdot,\cdot): D \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n, \ f(\cdot,\cdot) = (f_1(\cdot,\cdot), \dots f_n(\cdot,\cdot)) \right.$

$\left\| \frac{dx_i}{dt} \quad i = \overline{1,n} \right.$

$F_1(\cdot,\cdot), \dots, F_k(\cdot,\cdot) \subset D_0 = \overset{\circ}{D_0} \to \mathbb{R}^n$, $k < n$ integrale prime funct. indep.

P.$\det\left(\frac{\partial F_i}{\partial x_j}(t,x)\right)_{i,j=\overline{1,k}} \neq 0 \ \forall (t,x) \in D_0$

TO:

FROM:

Pasul 1 : Rezolvă sistemul algebric în necunoscutele $x_1, \dots, x_k$ următor :

$$\begin{cases} F_1(t, x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_n) = c_1 \\ \dots \\ F_k(t, x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_n) = c_k \end{cases} \Rightarrow x_j = \varphi_j(t, x_{k+1}, \dots, x_n, c_1, \dots c_k),\ j = \overline{1,n}$$

Pasul 2 : Integrează sistemul de ec.

$$\begin{cases} \frac{dx_{k+1}}{dt} = f_{k+1}(t, \varphi_1(t, x_{k+1}, \dots, x_n, c_1, \dots, c_k), \dots, \varphi_k(t, x_{k+1}, \dots x_n, c_1, \dots c_k), x_{k+1}, \dots, x_n) \\ \frac{dx_n}{dt} = f_n(t, \varphi_1(t, x_{k+1}, \dots, x_n, c_1, \dots c_k), \dots, \varphi_k(t, x_{k+1}, \dots, x_n, c_1, \dots c_k), x_{k+1}, \dots, x_n) \end{cases}$$

Th 3 (Existența integralelor prime)

Fie $f(\cdot,\cdot) : D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont. $C^1$ (II) $\frac{dx}{dt} = f(t, x)$.

At. $\forall (t_0, x_0) \in D$ $\exists F_1(\cdot,\cdot), \dots, F_n(\cdot,\cdot) : D_0 \in V(t_0, x_0) \to \mathbb{R}^n$ $C^1$ integrale prime funcțional indep.

Mai mult, dacă $\tilde{F}(\cdot,\cdot) : D_0 \to \mathbb{R}$ este integrală primă at. $\exists H(\cdot) : \mathbb{R}^n \to \mathbb{R}$ a.î. $\tilde{F}(t, x) = H(F_1(t, x), \dots, F_n(t, x))$