

#### 4.4 Linear programming duality

Write the dual of the following linear program

$$\begin{aligned}
 \min \quad & 2x_2 + x_3 - 3x_4 \\
 & x_1 - x_2 + 2x_4 \geq 2 \\
 & 2x_2 + x_3 = 4 \\
 & 2x_1 - x_3 + x_4 \leq 1 \\
 & x_1 \geq 0 \quad x_2 \geq 0 \\
 & x_3, x_4 \text{ unrestricted}
 \end{aligned}$$

#### 4.5 Dual of the transportation problem

A company owns  $m$  production factories, and produces goods for  $n$  warehouses. Depot  $i$ ,  $i \in \{1, \dots, m\}$ , has a production capacity of  $b_i$  units, while the warehouse  $j$ ,  $j \in \{1, \dots, n\}$ , requires  $d_j$  units. Let  $c_{ij}$  be the unit transportation cost from factory  $i$  to warehouse  $j$ .

Consider the problem of determining a transportation plan that minimized the total cost, satisfying the warehouse demands and the production limits. Let  $x_{ij}$  be the quantity of product transported from  $i$  to  $j$ . We consider the following formulation

$$\begin{aligned}
 (\mathbf{P}_1) \quad & \min \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij} \\
 & - \sum_{j=1}^n x_{ij} \geq -b_i \quad i \in \{1, \dots, m\} \\
 & \sum_{i=1}^m x_{ij} \geq d_j \quad j \in \{1, \dots, n\} \\
 & x_{ij} \geq 0 \quad i \in \{1, \dots, m\} \quad j \in \{1, \dots, n\}
 \end{aligned}$$

Assume, without loss of generality, that the total production capacity satisfies  $\sum_{j=1}^n d_j = \sum_{i=1}^m b_i$ . Indeed, if  $\sum_{j=1}^n d_j < \sum_{i=1}^m b_i$ , we can introduce a dummy warehouse  $(n+1)$  with zero transportation costs, with demand  $\sum_{j=1}^n d_j - \sum_{i=1}^m b_i$ .

Determine the linear programming dual of  $P_1$ , providing an economical interpretation.

## 4.6 Complementary slackness

Given the problem

$$\begin{array}{llll} (\mathbf{P}_2) & \max & 2x_1 & + \quad x_2 \\ & & x_1 & + \quad 2x_2 \leq 14 \\ & & 2x_1 & - \quad x_2 \leq 10 \\ & & x_1 & - \quad x_2 \leq 3 \\ & & & x_1, x_2 \geq 0 \end{array}$$

- a) write its dual
- b) show that  $\underline{x} = (\frac{20}{3}, \frac{11}{3})$  is a feasible solution
- c) show that  $\underline{x}$  is also an optimal solution, by means of the complementary slackness theorem
- d) determine an optimal solution to its dual.

## SOLUTION

## 4.4 Linear programming duality

The dual reads

$$\begin{aligned}
 \max \quad & 2y_1 + 4y_2 + y_3 \\
 & y_1 + 2y_3 \leq 0 \\
 & -y_1 + 2y_2 \leq 2 \\
 & y_2 - y_3 = 1 \\
 & 2y_1 + y_3 = -3 \\
 & y_1 \geq 0 \\
 & y_2 \text{ unrestricted} \\
 & y_3 \leq 0
 \end{aligned}$$

## 4.5 Dual of the transportation problem

Let  $u_i, v_j$  be the dual variables of the two classes of constraints. Since for each column of the constraint matrix of the primal problem only two entries are nonzero, taking value -1 and +1 for, respectively, rows corresponding to the first and second class of constraints, the dual reads

$$\begin{aligned}
 (\mathbf{D}_1) \quad \max \quad & - \sum_{i=1}^m b_i u_i + \sum_{j=1}^n d_j v_j \\
 & v_j - u_i \leq c_{ij} \quad i \in \{1, \dots, m\} \quad j \in \{1, \dots, n\} \\
 & u_i \geq 0, v_j \geq 0 \quad i \in \{1, \dots, m\} \quad j \in \{1, \dots, n\}
 \end{aligned}$$

Economical interpretation. We suppose that the production company (company A) hires a logistics company to handle the transportation business (company B). This company buys all the products from the different factories, paying a unit cost of  $u_i$  Euro for factory  $i$ . Then, it sells the products to the warehouses at a unit cost of  $v_j$  Euro, per warehouse  $j$ . The objective of company B amounts to maximizing the profit, i.e.,

$$\max - \sum_{i=1}^m b_i u_i + \sum_{j=1}^n d_j v_j$$

Company B has to decide the prices  $u_i, v_j$  such that it is not cheaper, for company A, to handle the transportation business without involving B. Suppose that, for a pair  $i, j$ , the prices are such that  $v_j - u_i > c_{ij}$ . In such case, company A would prefer to transport the product by itself, being cheaper than hiring company B. Therefore, for each pair  $i, j$ , the prices must satisfy the condition  $v_j - u_i \leq c_{ij}$ .

#### 4.6 Complementary slackness

a) The dual reads

$$\begin{aligned}
 (\mathbf{D}_2) \quad & \min \quad 14y_1 + 10y_2 + 3y_3 \\
 & y_1 + 2y_2 + y_3 \geq 2 \\
 & 2y_1 - y_2 - y_3 \geq 1 \\
 & y_1, y_2, y_3 \geq 0
 \end{aligned}$$

b)  $\bar{x} = (\frac{20}{3}, \frac{11}{3})$  is feasible, as it satisfies the constraints of  $(P_2)$ .

c-d) By the complementary slackness theorem, if  $\bar{x} = (x_1, x_2)$  is feasible for the primal and  $\bar{y} = (y_1, y_2, y_3)$  is feasible for the dual, and they satisfy

$$y_i(a_i^T \bar{x} - b_i) = 0 \quad \forall i$$

$$(c_j - \bar{y}^T A_j)x_j = 0 \quad \forall j$$

then they are optimal for the respective problems. The complementary slackness conditions for the problem at hand read

$$y_1(x_1 + 2x_2 - 14) = 0$$

$$y_2(2x_1 - x_2 - 10) = 0$$

$$y_3(x_1 - x_2 - 3) = 0$$

$$x_1(y_1 + 2y_2 + y_3 - 2) = 0$$

$$x_2(2y_1 - y_2 - y_3 - 1) = 0$$

We obtain  $\bar{y}$ , by substituting for  $\bar{x}$ . Since  $\bar{x} = (\frac{20}{3}, \frac{11}{3})$  satisfies as equations the first and third constraints, but not the second one, we deduce  $y_2 = 0$ . Since  $x_1 > 0$  and  $x_2 > 0$ , we obtain

$$y_1 + 2y_2 + y_3 - 2 = 0$$

$$2y_1 - y_2 - y_3 - 1 = 0$$

$$y_2 = 0$$

By solving the system, we obtain  $\bar{y} = (1, 0, 1)$ , which satisfies the dual constraints of  $D_2$ . Since  $\bar{x}$  is primal feasible and  $\bar{y}$  is dual feasible, the primal/dual pair  $(\bar{x}, \bar{y})$  satisfies the complementary slackness constraints and, therefore,  $\bar{x}$  is an optimal solution to the primal and  $\bar{y}$  is an optimal solution to the dual.

To double check, verify that the two solutions have the same objective function value in the respective problems, i.e., verify that  $\bar{c}^T \bar{x} = \bar{y}^T \bar{b}$ .