Curs 9

Ecuații liniare pe $\mathbb{R}^n$ cu coeficienți constanți

Structura soluțiilor în cazul general

Lemă $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ $\frac{dx}{dt} = Ax$, $\lambda \in \sigma(A)$ $P_j \in \mathbb{C}^n$

$$\check{\varphi}(t) := e^{\lambda t} \sum_{i=0}^{n-1} P_j t^j$$

Atunci $\check{\varphi}'(t) \equiv A\check{\varphi}(t) \iff \begin{cases} (A-\lambda I_m)^m P_0 = 0 \\ P_j = \frac{1}{j!}(A-\lambda I_m)^j \cdot P_0, \ j=\overline{1,m-1} \end{cases}$

Dem $\check{\varphi}'(t) \equiv A \cdot \check{\varphi}(t)$

$$\lambda \cancel{e^{\lambda t}} \sum_{j=0}^{m-1} P_j t^j + \cancel{e^{\lambda t}} \sum_{j=1} j P_j t^{j-1} \equiv A \cancel{e^{\lambda t}} \sum_{j=0}^{m-1} P_j t^j$$

$$\lambda \sum_{j=0}^{m-1} P_j t^j + \sum_{j=0}^{m-2} (j+1) P_{j+1} t^j \equiv A \sum_{j=0}^{m-1} P_j t^j$$

$j = \overline{0, m-2}$ $\quad \lambda \cdot P_j + (j+1) P_{j+1} = A \cdot P_j$

$j = m-1$ $\quad \lambda \cdot P_{m-1} = A \cdot P_{m-1}$

$j = \overline{0, m-2}$ $\quad P_{j+1} = \frac{1}{j+1}(A - \lambda i_m) \cdot P_j = \frac{1}{(j+1)} \cdot \frac{1}{j} \cdot$

$\cdot (A - \lambda i_m)^2 P_{j-1} = \dots = \frac{1}{(j+1)!}(A - \lambda i_m)^{j+1} \cdot P_0$

$j = m-1$ $\quad \left.\begin{array}{l} (A-\lambda i_m) P_{m-1} = 0 \\ P_{m-1} = \frac{1}{(m-1)!} (A - \lambda i_m)^{m-1} \cdot P_0 \end{array}\right\} \Rightarrow \frac{1}{(m-1)!}(A-\lambda i_m)^m \cdot P_0 = 0$

Teoremă asupra forma canonică Jordan a unei matrice

$\forall A \in M_m(\mathbb{C})$ $\exists C \in M_n(\mathbb{C})$ $\det C \neq 0$ a.î.

celulă

unde $\lambda_r \in \sigma(A)$

$s_{r_i} = \dim J_r \le$ multiplicitatea $(\lambda_r) = m_{\lambda_r}$

$$\sum_{\lambda_r = \lambda} \dim J_r = m_{\lambda_r}$$

$\boxed{\text{Forma echivalentă}}$

$\forall A \in L(\mathbb{R}^n, \mathbb{R}^n)\ \exists\ B_J = \{ b_r^s ; s = \overline{1, s_r}, r = \overline{1, p} \}$

$\subset \mathbb{C}^n$ bază canonică Jordan a.î. (s_r — dim celulei Jordan)

a) $\forall r = \overline{1,p}\ \exists\ \lambda_r \in \sigma(A)$

$$\begin{cases} A \cdot b_r^1 = \lambda_r b_r^1 + b_r^2 \\ A b_r^2 = \lambda_r b_r^2 + b_r^3 \\ \dots \\ A b_r^{s_r - 1} = \lambda_r b_r^{s_r-1} + b_r^{s_r} \\ A b_r^{s_r} = \lambda_r b_r^{s_r} \end{cases}$$

2) $s_r \le m_{\lambda_r}$ $\quad \sum_{\lambda_r = \lambda} s_r = m_\lambda \quad \forall \lambda \in \sigma(A)$

3) Dacă $\lambda_r \in \mathbb{R} \Rightarrow b_r^s \in \mathbb{R}^n$, dacă $\operatorname{Im} \lambda_r > 0$ și $\lambda_r = \overline{\lambda_t}$ atunci $b_r^s = \overline{b_t^s}$

C1) $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ $B_J \subset \mathbb{C}^n$ baza canonică Jordan

Atunci $B_J^{\mathbb{R}^n} = \{ b_r^s ; \lambda_r \in \mathbb{R}, s = \overline{1, s_r} \} \cup$

$\cup \{ \operatorname{Re}(b_r^s), \operatorname{Im}(b_r^s) ; \operatorname{Im} \lambda_r > 0, s = \overline{1, s_r} \} \subset \mathbb{R}^n$ bază

(urma pe $\mathbb{R}^n$ a bazei canonice Jordan)

$$\sum_{\substack{\lambda_r \in \mathbb{R} \\ s=\overline{1,s_r}}} c_r^s b_r^s + \sum_{\Im m\, \lambda_r > 0} \left[c_r^s \,\mathrm{Re}(b_r^s) + k_r^s \,\Im m(b_r^s) \right] = 0$$

$$\overset{?}{\Rightarrow} c_r^s = k_r^s = 0 \quad \forall s, \forall r$$

$$b_r^s = \mathrm{Re}(b_r^s) + i \cdot \Im m(b_r^s)$$

$$\overline{b_r^s} = \mathrm{Re}(b_r^s) - i \cdot \Im m(b_r^s)$$

$$\mathrm{Re}(b_r^s) = \frac{1}{2}(b_r^s + \overline{b_r^s})$$

$$\Im m(b_r^s) = \frac{1}{2i} \cdot (b_r^s - \overline{b_r^s})$$

$$\sum_{\substack{\lambda_r \in \mathbb{R} \\ s=\overline{1,s_r}}} c_r^s b_r^s + \sum_{\substack{\Im m\, \lambda_r > 0 \\ s=\overline{1,s_r}}} c_r^s \cdot \frac{1}{2}(b_r^s + \overline{b_r^s}) + k_r^s \frac{1}{2i}(b_r^s - \overline{b_r^s}) = 0$$

$$\sum_{\substack{\lambda_r \in \mathbb{R} \\ s=\overline{1,s_r}}} c_r^s b_r^s + \sum_{\substack{\Im m\, \lambda_r > 0 \\ s=\overline{1,s_r}}} \left[\left(\frac{c_r^s}{2} + \frac{k_r^s}{2i} \right) b_r^s + \left(\frac{c_r^s}{2} - \frac{k_r^s}{2i} \right) \overline{b_r^s} \right] = 0 \xRightarrow[\text{bază}]{B_J}$$

$$\Rightarrow c_r^s = 0 \quad \lambda_r \in \mathbb{R},\ s = \overline{1,s_r} \qquad \begin{cases} \dfrac{c_r^s}{2} + \dfrac{k_r^s}{2} = 0 \\[2ex] \dfrac{c_r^s}{2} - \dfrac{k_r^s}{2} = 0 \end{cases} \quad \Im m\, \lambda_r > 0,\ s = \overline{1,s_r} \Rightarrow$$

$$\Rightarrow c_r^s = k_r^s = 0 \quad \Im m\, \lambda_r > 0,\ s = \overline{1,s_r}$$

(C2) $A \in L(\mathbb{R}^n, \mathbb{R}^n)$ $B_J \subset \mathbb{C}^n$ baza canonică Jordan

$\lambda \in \sigma(A)$ $B_J^\lambda := \{ b_r^s \mid \lambda_r = \lambda, s = \overline{1,s_r} \}$

Atunci: 1) $\mathrm{card}(B_J^\lambda) = m_\lambda$

2) $(A - \lambda i_m)^{m_\lambda} b_r^s = 0 \quad \forall b_r^s \in B_J^\lambda$

liniar indep.

$\boxed{\text{Dem}}$ 1) & 3) imediat din T

2) $A b_r^1 = \lambda b_r^1 + b_r^2 \implies b_r^2 = (A - \lambda i_m) b_r^1$

$A b_r^2 = \lambda b_r^2 + b_r^3 \implies b_r^3 = (A - \lambda i_m) b_r^2 = (A - \lambda i_m$

$\dots$

$A b_r^{s_r - 1} = \lambda b_r^{s_r - 1} + b_r^{s_r}$

$A b_r^{s_r} = \lambda b_r^{s_r}$

$b_r^{s_r} = (A - \lambda i_m) b_r^{s_r - 1} = = (A - \lambda$

$\Downarrow$

$(A - \lambda i_m) b_r^{s_r} = 0$

$\implies (A - \lambda i_m)(A - \lambda i_m)^{s_r - 1} b_r^1 = 0$

$$\left.\begin{array}{l}(A - \lambda i_m)^{s_r} b_r^1 = 0 \\ s_r \leq m_\lambda\end{array}\right\} \implies (A - \lambda i_m)^{m_\lambda} b_r^1 = 0$$

$(A - \lambda i_m)^{m_\lambda} b_r^s = (A - \lambda i_m)^{m_\lambda} \cdot (A - \lambda i_m)^{s-1} b_r^1 =$

$= (A - \lambda i_m)^{m_\lambda + s - 1} b_r^1 = 0$

$\boxed{\text{T (Structura sol în cazul general)}}$

$A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ $\frac{dx}{dt} = A x$

Atunci: 1. $\forall \lambda \in \sigma(A) \cap \mathbb{R}$ $m_\lambda = m \geq 1$ $\exists P_j^{\lambda l} \in \mathbb{R}^n$ a.î.

$\varphi_{\lambda l}(t) = e^{\lambda t} \cdot \sum_{j=0}^{m-1} P_j^{\lambda l} t^j$, $l = \overline{1, m}$

$\{\varphi_{\lambda l}(\cdot)\}_{l = \overline{1, m}} \subset S_A$ liniar independente

2. $\forall \lambda \in \sigma(A)$ $\operatorname{Im} \lambda > 0$ $m_\lambda = m \geq 1$ $\exists P_j^{\lambda l} \in \mathbb{C}^n$ a.

$\varphi_{\lambda l}(t) = \operatorname{Re}\left(e^{\lambda t} \sum_{j=0}^{m-1} P_j^{\lambda l} t^j\right)$, $l = \overline{1, m}$

$\{\varphi_{\lambda l}(\cdot), \varphi_{\overline{\lambda l}}(\cdot)\}_{l=\overline{1,m}} \subset S_A$ liniar independente

3. $\{\varphi_{\lambda l}(\cdot)\}_{\substack{\lambda\in\sigma(A) \\ l=\overline{1,m_\lambda}}} \subset S_A$ sistem fundamental de soluții (bază în spațiul soluțiilor)

$\boxed{\text{Dem}}$ $A\in L(\mathbb{R}^n,\mathbb{R}^n) \to B_J = \{b_r^s,\ s=\overline{1,s_r},\ r=\overline{1,p}\}$

$\subset \mathbb{C}^n$ bază canonică Jordan

1. Fie $\lambda\in\sigma(A)\cap\mathbb{R}$ $\quad m_\lambda = m \geq 1$

Fie $B_J^\lambda = \{b_r^s;\ \lambda_r=\lambda,\ s=\overline{1,s_r}\} \overset{not}{=} \{p_0^{\lambda 1},\dots,p_0^{\lambda m}\}$

liniar indep.

$p_j^{\lambda l} := \frac{1}{j!}(A-\lambda I_n)^j p_0^{\lambda l}$, $j=\overline{1,m-1}$ $\quad l=\overline{1,m}$

$$\left.\begin{aligned} &\text{C2} \Rightarrow (A-\lambda I_n)^m p_0^{\lambda l} = 0,\ l=\overline{1,m} \\ &\text{Fie } \varphi_{\lambda l}(t) = e^{\lambda t}\sum_{j=0}^{m-1} p_j^{\lambda l} t^j \end{aligned}\right\} \xRightarrow{\text{Lema}}$$

$\Rightarrow \varphi'_{\lambda l}(t) = A\cdot\varphi_{\lambda l}(t)$

$\Rightarrow \varphi_{\lambda l}(\cdot)\in S_A \ \forall\ l=\overline{1,m}$

$\{\varphi_{\lambda l}(\cdot)\}_{l=\overline{1,m}} \subset S_A$ liniar independente

$\Updownarrow$ PROP (Sol. liniar independente)

$\{\varphi_{\lambda l}(0)\}_{l=\overline{1,m}} \subset \mathbb{R}^m$ liniar indep

$\|$

$\{p_0^{\lambda l}\}_{l=\overline{1,m}}$ liniar indep

2. $\lambda\in\sigma(A)$ $\quad \operatorname{Im}\lambda>0$ $\quad m_\lambda = m\geq 1$

$P^{\lambda_l}_j := \frac{1}{j!}(A-\lambda i_m)^j P^{\lambda_m}_0 \quad j=\overline{1,m-1}, \; l=\overline{1,m}$

Lemă $\Rightarrow \check{\varphi}'_{\lambda_l}(t) \equiv A\check{\varphi}_{\lambda_l}(t)$

$$\check{\varphi}_{\lambda_l}(t) = \underbrace{Re(\check{\varphi}_{\lambda_l}(t))}_{\varphi_{\lambda_l}(t)} + i\cdot \underbrace{Im(\check{\varphi}_{\lambda_l}(t))}_{\varphi_{\overline{\lambda_l}}(t)} = \varphi_{\lambda_l}(t) + i\cdot\varphi_{\overline{\lambda_l}}$$

$$\varphi'_{\lambda_l}(t) + i\cdot\varphi'_{\overline{\lambda_l}}(t) \equiv A\varphi_{\lambda_l}(t) + i\cdot A\varphi_{\overline{\lambda_l}}(t)$$

$$\Rightarrow \varphi'_{\lambda_l}(t) \equiv A\cdot\varphi_{\lambda_l}(t)$$

$$\varphi'_{\overline{\lambda_l}}(t) \equiv A\cdot\varphi_{\overline{\lambda_l}}(t) \Rightarrow \varphi_{\lambda_l}(\cdot), \varphi_{\overline{\lambda_l}}(\cdot) \in S_A \quad \forall\, l=\overline{1,m}$$

$\{\varphi_{\lambda_l}(\cdot), \varphi_{\overline{\lambda_l}}(\cdot)\}_{l=\overline{1,m}} \subset S_A$ liniar indep.

$\Updownarrow$

PROP (sol liniar independente)

$\{\varphi_{\lambda_l}(0), \varphi_{\overline{\lambda_l}}(0)\}_{l=\overline{1,m}} \subset \mathbb{R}^n$ liniar indep

$$\{Re(P^{\lambda_l}_0), Im(P^{\lambda_l}_0)\}_{l=\overline{1,m}}$$

$\{Re(b^S_n), Im(b^S_n)\}_{l=\overline{1,m}} \subset \mathbb{R}^n$ liniar indep

C1 C2

3. n soluții $\Rightarrow$ E suficient să verificăm doar liniar independența

$\{\varphi_{\lambda\ell}(0)\}_{\substack{\lambda \in \sigma(A) \\ \ell=\overline{1,m_\lambda}}} \subset \mathbb{R}^n$ liniar independente

$\|$

$\{b_r^s, \lambda_r \in \mathbb{R}, s=\overline{1,s_r}\} \cup \{Re(b_r^s), Jm(b_r^s), Jm\lambda_r > 0, s=\overline{1,s_r}\} \subset \mathbb{R}^n$ liniar independente $\Leftarrow$ 1

$\boxed{\text{Algoritm}}$ $\frac{dx}{dt} = Ax$

1. Rezolvăm ec caracteristică $\det(A - \lambda i_n) = 0 \longrightarrow \sigma(A)$

$\sigma(A): (\lambda, m_\lambda)$

$\hookrightarrow$ ordin de multiplicitate a răd. λ

2. Dacă $\lambda \in \sigma(A) \cap \mathbb{R}$, $m_\lambda = 1$ caută $u_\lambda \in \mathbb{R}^n \setminus \{0\}$ a.î. $(A - \lambda i_n) u_\lambda = 0$

Scrie sol $\varphi_\lambda(t) = e^{\lambda t} \cdot u_\lambda$

3. Dacă $\lambda \in \sigma(A) \cap \mathbb{R}$, $m_\lambda = m > 1$

Caută $\{P_0^{\lambda 1}, \dots P_0^{\lambda m}\} \subset Ker(A - \lambda \cdot i_n)^m$ liniar independentă (în $\mathbb{R}^n$)

Scrie $P_j^{\lambda \ell} = \frac{1}{j!}(A - \lambda \cdot i_n)^j P_0^{\lambda \ell}$, $j = \overline{1, m-1}$, $\ell = \overline{1,m}$

Scrie sol $P_{\lambda \ell}(t) = e^{\lambda t} \cdot \sum_{j=0}^{m-1} P_j^{\lambda \ell} t^j$ $\ell = \overline{1,m}$

4. Dacă $\lambda = \alpha + i\beta \in \sigma(A)$, $\beta > 0$, $m_\lambda = 1$.

Caut $u_\lambda \in \mathbb{C}^n \setminus \{0\}$ a.î. $(A - \lambda i_n) u_\lambda = 0$.

Scrie sol $\varphi_\lambda(t) = R_e(e^{\lambda t}, u_\lambda)$

$\varphi_{\overline{\lambda}}(t) = Jm(e^{\lambda t}, u_\lambda)$

5. Dacă $\lambda = \alpha + i\beta \in \sigma(A)$, $\beta > 0$, $m_\lambda = m \geq 2$

Scrie $P_j^{\lambda\ell} = \frac{1}{j!}(A - \lambda I_m)^j P_0^{\lambda\ell}$, $j = \overline{1, m-1}$, $\ell = \overline{1,m}$

Scrie soluțiile:

$$\varphi_{\lambda\ell}(t) = Re\left(e^{\lambda t} \sum_{j=0}^{m-1} P_j^{\lambda t} t^j\right) \quad \ell = \overline{1,m}$$

$$\varphi_{\overline{\lambda\ell}}(t) = Im\left(e^{\lambda t} \sum_{j=0}^{m-1} P_j^{\lambda\ell} t^j\right)$$

6. Renumerotează $\{\varphi_{\lambda\ell}(\cdot)\}_{\substack{\lambda \in \sigma(A) \\ \ell = \overline{1, m_\lambda}}} =$

$= \{\varphi_1(\cdot), \varphi_2(\cdot), \dots \varphi_n(\cdot)\}$ sistem fundamental de soluții

Scrie soluția generală $\varphi(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$, $c_i \in \mathbb{R}$, $i = \overline{1,n}$

Ecuații diferențiale afine pe $\mathbb{R}^n$

Def $A(\cdot) : I \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n)$, $b(\cdot) : I \to \mathbb{R}^n$ def

$$\frac{dx}{dt} = A(t) \cdot x + b(t)$$

În coordonate ($B \subset \mathbb{R}^n$ bază) $A_B(t) = (a_{ij}(t))_{i,j = \overline{1,n}}$

$$b(t) = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}, \quad \frac{dx_i}{dt} = \sum_{j=1}^{n} a_{ij}(t) x_j + b_i(t), \quad i = \overline{1,n}$$

(sistem de ecuații afine)

Dacă $n = 1$, $L(\mathbb{R}, \mathbb{R}) \simeq \mathbb{R} \to x' = a(t) x + b(t)$,
$a(\cdot), b(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}$ (ecuație afină scalară)

$\downarrow$

Metoda variației constantelor

T(E.U.G)

$$\frac{dx}{dt} = A(t)x + b(t)$$

Atunci $\forall (t_0, x_0) \in \mathring{I} \times \mathbb{R}^n$ $\exists!\ \varphi(\cdot) : \mathring{I} \to \mathbb{R}^n$ soluție cu $\varphi(t_0) = x_0$.

Dem: $f_{A(\cdot), b(\cdot)}(t, x) = A(t) \cdot x + b(t)$

$f_{A(\cdot), b(\cdot)}(\cdot, \cdot)$ cont local Lipschitz $(\mathrm{II}) \xRightarrow[\text{Cauchy-Lipshitz}]{T.}$ E U L pe $\mathring{I} \times \mathbb{R}^n$

$\Longrightarrow$ T. U. G

$$\| f_{A(\cdot), b(\cdot)}(t, x) \| = \| A(t)x + b(t) \| \leq \| A(t) \| \cdot \| x \| + \| b(t) \|$$

adică C. A $\Longrightarrow$ D $\xRightarrow[E.G]{T}$ E. G

$$S_{A(\cdot), b(\cdot)} := \{ \varphi(\cdot) : \mathring{I} \to \mathbb{R}^n ;\ \varphi(\cdot) \text{ soluție } x' = A(t) \cdot x + b(t) \}$$

Seminar 9

Algoritm (cazul general)

$$\frac{dx}{dt} = Ax$$

1. Rezolvă ec. caracteristică

$\det(A - \lambda I_n) = 0 \to \sigma(A) : (\lambda, m_\lambda)$

2. Dacă $\lambda \in \sigma(A) \cap \mathbb{R}$, $m_\lambda = 1$ caută $u_\lambda \in \mathbb{R}^n \setminus \{0\}$

$(A - \lambda I_n) u_\lambda = 0$

Scrie sol $\varphi_\lambda(t) = e^{\lambda t} \cdot u_\lambda$

3. Dacă $\lambda \in \sigma(A) \cap \mathbb{R}$, $m_\lambda = m > 1$

Caută $\{p_0^{\lambda 1}, \ldots, p_0^{\lambda m}\} \in \ker(A - \lambda I_n)^m$ liniar indep (în $\mathbb{R}^n$)

Scrie $p_j^{\lambda \ell} = \frac{1}{j!}(A - \lambda I_n)^j p_0^{\lambda \ell}$, $j = \overline{1, m-1}$, $\ell = \overline{1, m}$

Scrie soluțiile $\varphi_{\lambda \ell}(t) = e^{\lambda t} \sum_{j=0}^{m-1} p_j^{\lambda \ell} t^j$ $\ell = \overline{1, m}$

4. Dacă $\lambda = \alpha + i \cdot \beta \in \sigma(A)$, $\beta > 0$, $m_\lambda = 1$

Caută $u_\lambda \in \mathbb{C}^n \setminus \{0\}$ a.î. $(A - \lambda I_n) u_\lambda = 0$

Scrie sol $\varphi_\lambda(t) = \mathrm{Re}(e^{\lambda t} \cdot u_\lambda)$, $\varphi_{\bar{\lambda}}(t) = \mathrm{Im}(e^{\lambda t}$

5. Dacă $\lambda = \alpha + i\beta \in \sigma(A)$, $\beta > 0$, $m_\lambda = m > 1$

Caută $\{p_0^{\lambda 1}, \ldots, p_0^{\lambda m}\} \subset \ker(A - \lambda I_n)^m$ liniar inde (în $\mathbb{C}^n$)

Scrie $p_j^{\lambda \ell} = \frac{1}{j!}(A - \lambda I_n)^j p_0^{\lambda \ell}$, $j = \overline{1, m-1}$, $\ell =$

Scrie soluțiile $\varphi_{\lambda \ell}(t) = \mathrm{Re}\left(e^{\lambda t} \sum_{j=0}^{m-1} p_j^{\lambda \ell} t^j\right)$ $\ell =$

6 Renumerotează $\{\varphi_{r_\ell}(\cdot)\}_{\substack{\lambda \in \sigma(A) \\ \ell=\overline{1,m}}} = \{\varphi_1(\cdot), \dots \varphi_n(\cdot)\}$

sistem fundamental de soluții
Scrie sol. generală $\varphi(t) = \sum_{i=1}^{n} c_i \varphi_i(t), c_i \in \mathbb{R}$

<u>Ex</u>: Să se determine soluția generală:

1) $\begin{cases} x' = x - y + z \\ y' = x + y - z \\ z' = 2z - y \end{cases}$ $A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix}$

$\det(A - \lambda I_n) = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 0 & -1 & 2-\lambda \end{vmatrix} =$

$= (1-\lambda)^2 \cdot (2-\lambda) - \cancel{1} - \cancel{1} + \cancel{\lambda} + \cancel{2} - \cancel{\lambda}$

$= (1-\lambda)^2 \cdot (2-\lambda)$

$\lambda_1 = 2$
$\lambda_2 = \lambda_3 = 1$

$\lambda = 2 \Rightarrow \overset{?}{u}$ a.î. $(A - 2I_3)u = 0$, unde $u = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

$\begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} -a - b + c = 0 \\ a - b - c = 0 \Rightarrow a = c \\ -b = 0 \Rightarrow b = 0 \end{cases}$

$\Rightarrow \begin{pmatrix} a \\ 0 \\ a \end{pmatrix}, a = 1$

Soluția $\varphi_1(t) = e^{2t} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 1,\ m_\lambda = 2$

$$(A - I_3)^2 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 2 \\ 0 & 0 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$

$P_0 = ?$ a.î. $(A - I_3)^2 \cdot P_0 = 0$, $P_0 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \Rightarrow$

$$\Rightarrow \begin{pmatrix} -1 & -1 & 2 \\ 0 & 0 & 0 \\ -1 & -1 & 2 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow a + b = 2c \Rightarrow c = \frac{a+b}{2}$$

$$\Rightarrow P_0 = \begin{pmatrix} a \\ b \\ \frac{a+b}{2} \end{pmatrix}$$

$$P_{01} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \quad P_{02} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

$$P_j = \frac{1}{j!}(A - I_3)^j P_0,\ j = \underline{1}$$

$$P_{11} = (A - I_3) \cdot P_{01} \qquad P_{12} = (A - I_3) \cdot P_{02}$$

$$P_{11} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$P_{12} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

$$\varphi_2(t) = e^t(P_{01} + t \cdot P_{11})$$

$$\varphi_3(t) = e^t(P_{02} + t \cdot P_{12})$$

$$\varphi_2(t) = e^t \begin{pmatrix} 2 + t \\ t \\ 1 + t \end{pmatrix} \qquad \varphi_3(t) = e^t \begin{pmatrix} -t \\ 2 - t \\ 1 - t \end{pmatrix}$$

2) $\begin{cases} x' = 4x - y \\ y' = 3x + y - z \\ z' = x + z \end{cases}$ $\quad A = \begin{pmatrix} 4 & -1 & 0 \\ 3 & 1 & -1 \\ 1 & 0 & 1 \end{pmatrix}$

$$\det(A - \lambda \cdot I_n) = \begin{vmatrix} 4-\lambda & -1 & 0 \\ 3 & 1-\lambda & -1 \\ 1 & 0 & 1-\lambda \end{vmatrix} =$$

$$= (4-\lambda)(1-\lambda)^2 + 1 + 3(1-\lambda)$$
$$= (1-\lambda)(3 + (4-\lambda)(1-\lambda)) + 1$$
$$= -\lambda^3 + 6\lambda^2 - 12\lambda + 8 = (2-\lambda)^3$$

$\lambda = 2 \quad m_\lambda = 3$

$(A - 2I_3)^3 = ?$

$$(A - 2I_3)^2 = \begin{pmatrix} 2 & -1 & 0 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 1 \end{pmatrix}$$

$$(A - 2I_3)^3 = \begin{pmatrix} 2 & -1 & 0 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$(A - 2I_3)^3 = 0 \Rightarrow (A - 2I_3)^3 \cdot P_0 = 0, \ \forall P_0 \in \mathbb{R}^3$

$P_{01} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad P_{02} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad P_{03} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightarrow$ liniar indep. (baza canonică)

$$P_j = \frac{1}{j!}(A - 2I_3)^j P_0, \ j = 1, 2$$

$$P_{11} = (A - 2I_3)^1 \cdot P_{01} = \begin{pmatrix} 2 & -1 & 0 \\ 3 & -1 & -1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$P_{12} = \begin{pmatrix} -1 \\ -1 \\ \ldots \end{pmatrix} \quad P_{13} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$

$$P_{21} = \frac{1}{2}\cdot(A-\lambda I_3)^2 \cdot P_{01} = \frac{1}{2}\begin{pmatrix}1 & -1 & 1\\ 2 & -2 & 2\\ 1 & -1 & 1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$P_{22} = \frac{1}{2}\begin{pmatrix}-1\\-2\\-1\end{pmatrix} \qquad P_{23} = \frac{1}{2}\begin{pmatrix}1\\2\\1\end{pmatrix}$$

$$\varphi_1(t) = e^{2t}\left(P_{01} + t\cdot P_{11} + t^2\cdot P_{21}\right) \quad \varphi_2(t) = e^{2t}\left(P_{02} + t\cdot P_{12} + t^2 \ldots\right.$$

$$\varphi_3(t) = e^{2t}\left(P_{03} + t\,P_{13} + t^2 P_{23}\right)$$

$$\varphi_1(t) = e^{2t}\begin{pmatrix}1 + 2t + t^2\cdot\frac{1}{2}\\ 3t + t^2\\ t + t^2\cdot\frac{1}{2}\end{pmatrix}$$

$$\varphi_2(t) = e^{2t}\begin{pmatrix}-t - \frac{1}{2}t^2\\ 1 - t - t^2\\ -t^2\cdot\frac{1}{2}\end{pmatrix}$$

$$\varphi_3(t) = e^{2t}\cdot\begin{pmatrix}t^2\cdot\frac{1}{2}\\ -t + t^2\\ -1 - t + \frac{1}{2}\cdot t^2\end{pmatrix}$$

3) $\begin{cases} x' = 2x + y \\ y' = 2y + 4z \\ z' = x - z \end{cases}$

4) $\begin{cases} x' = 2x + y \\ y' = 4y - x \end{cases}$

$A = \begin{pmatrix}1 & 1\\ 2 & -1\end{pmatrix}$ $\quad \det(A - I_2\cdot\lambda) = \begin{vmatrix}2-\lambda & 1\\ 1 & -1-\lambda\end{vmatrix}$

$= (2-\lambda)(-1-\lambda) - 1 = -2 - 2\lambda + \lambda + \lambda^2 - 1$

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \quad \det(A - \lambda \cdot i_2) = \begin{vmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{vmatrix} =$$

$$= (2-\lambda)(4-\lambda) + 1 = (\lambda-3)^2$$

$\lambda_{1,2} = 3$

$\lambda = 3, \quad m_\lambda = 2$

$$(A - 3i_2)^2 = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$(A - 3i_2)^2 \cdot P_0 = 0, \ \forall P_0 \in \mathbb{R}^2$

$P_{01} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad P_{02} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$P_j = \frac{1}{j} (A - 3i_2)^j \cdot P_0, \ j = 1$

$P_{11} = (A - 3i_2) \cdot P_{01} \qquad P_{12} = (A - 3i_3) \cdot P_{02}$

$P_{11} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \quad P_{12} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\varphi_1 = e^{3t} \cdot (P_{01} + t \cdot P_{11}) = \begin{pmatrix} 1-t \\ -t \end{pmatrix} \cdot e^{3t}$

$\varphi_2 = e^{3t} (P_{02} + t \cdot P_{12}) = \begin{pmatrix} t \\ 1+t \end{pmatrix} \cdot e^{3t}$

A doua metodă (recomandată pt. dimensiunea 2)

$y = x' - 2x$

$(x' - 2x)' = 4(x' - 2x) - x \Rightarrow x'' - 2x' = 4x' - 8x - x$

$\Rightarrow x'' - 6x' + 9x = 0$

$\lambda^2 - 6\lambda + 9 = 0 \Rightarrow (\lambda - 3)^2 = 0$

$y(t) = 3 \cdot c_1 \cdot e^{3t} + c_2 \cdot e^{3t} + 3 \cdot c_2 t \cdot e^{3t} - 2 c_1 e^{3t} -$

$- 2 c_2 t e^{3t}$

$= (c_1 + c_2) \cdot e^{3t} + c_2 \cdot t e^{3t}$

5) $\begin{cases} x' = 5x + 3y \\ y' = -3x - y \end{cases}$

6) $\begin{cases} x' = 2y - 3x \\ y' = y - 2x \end{cases}$

Curs 10

Ecuații afine pe $\mathbb{R}^n$

PROP (Varietatea soluțiilor)

Sol. ec. liniare ↓ sol. particulară a ec. afine ↓

$$S_{A(\cdot),b(\cdot)} = S_{A(\cdot)} + \{\varphi_0(\cdot)\}, \quad \varphi_0(\cdot) \in S_{A(\cdot),b(\cdot)}$$

Dem „$\subseteq$" Fie $\varphi_0(\cdot) \in S_{A(\cdot),b(\cdot)} \overset{?}{\Rightarrow} \varphi(\cdot) - \varphi_0(\cdot) \in S_{A(\cdot)}$

$\varphi'(t) \equiv A(t)\varphi(t) + b(t)$

$\varphi_0'(t) \equiv A(t)\varphi_0(t) + b(t)$

$(\varphi-\varphi_0)'(t) \equiv A(t)(\varphi(t) - \varphi_0(t)) \equiv A(t)(\varphi-\varphi_0)(t)$, i.e. $(\varphi-\varphi_0)(\cdot) \in S_{A(\cdot)}$

„$\supseteq$" Fie $\varphi_0(\cdot) \in S_{A(\cdot),b(\cdot)}$, și fie $\psi(\cdot) \in S_{A(\cdot)} \overset{?}{\Rightarrow} \psi(\cdot) + \varphi_0(\cdot) \in S_{A(\cdot),b(\cdot)}$

$\varphi_0'(t) \equiv A(t)\varphi_0(t) + b(t)$

$\psi'(t) \equiv A(t)\psi(t)$

$\underbrace{\varphi_0'(t) + \psi'(t)}_{=(\psi+\varphi_0)'(t)} \equiv A(t)(\varphi_0(t) + \psi(t)) + b(t) \equiv A(t)(\psi+\varphi_0)(t) + b(t)$

Corolar Fie $\{\overline{\varphi_1}(\cdot), \dots, \overline{\varphi_n}(\cdot)\} \subset S_{A(\cdot)}$ sistem fundamental de soluții pt. ec. liniară asociată $\frac{d\overline{x}}{dt} = A(t)\overline{x}$ și fie $\varphi_0(\cdot) \in S_{A(\cdot),b(\cdot)}$

Atunci $\varphi(\cdot) \in S_{A(\cdot),b(\cdot)} \Longleftrightarrow \exists\, c_1, \dots, c_n \in \mathbb{R}$ a.î.

$$\varphi(t) = \sum_{i=1}^{n} c_i \overline{\varphi_i}(t) + \varphi_0(t) \quad \text{soluție generală a ec. afine}$$

T (Principiul variației constantelor)

Fie $X(\cdot): \overset{\circ}{I} \to M_n(\mathbb{R})$ matrice fundamentală de soluții pt. ec. liniară asociată $\frac{d\overline{x}}{dt} = A(t)\overline{x}$.

Atunci $\varphi(\cdot) \in S_{\dots} \Longleftrightarrow \exists\, C(\cdot)$ primitivă a funcției

Dem "$\Rightarrow$" $\varphi(\cdot) \in S_{A(\cdot), b(\cdot)}$

Fie $c(t) := X^{-1}(t)\varphi(t) \Rightarrow \varphi(t) = X(t) \cdot c(t)$

$\varphi(t)$ soluție $\Rightarrow \varphi'(t) \equiv A(t) \dots + b(\dots)$

$$\Rightarrow \underbrace{X'(t)}_{\substack{\parallel \\ A(t)\cancel{X(t)}}} c(t) + X(t) c'(t) \equiv A(t)\cancel{X(t)} c(t) + b(t)$$

$$X(t)c'(t) \equiv b(t) \Rightarrow c'(t) = X^{-1}(t) b(t)$$

"$\Leftarrow$" $\varphi(t) \equiv X(t) \cdot c(t) \Rightarrow \varphi'(t) \equiv \underbrace{X'(t)}_{\substack{\parallel \\ A(t)X(t)}} c(t) + X(t) c'(t)$

$$A(t)X(t)c(t) + X(t) \cdot X^{-1}(t) \dots$$

$$\equiv A(t)\varphi(t) + b(t)$$

Corolar Fie $\{\overline{\varphi_1}(\cdot), \dots \overline{\varphi_n}(\cdot)\} \subset S_{A(\cdot)}$ sistem fundam... de soluții pt. ec. liniară asociată $\frac{d\overline{x}}{dt} = A(t)\overline{x}$.

Atunci $\varphi(\cdot) \in S_{A(\cdot), b(\cdot)} \Leftrightarrow \exists\ c(\cdot) = \begin{pmatrix} c_1(\cdot) \\ \vdots \\ c_n(\cdot) \end{pmatrix}$ primitiv... funcției $t \longrightarrow (\mathrm{col}(\overline{\varphi_1}(t), \dots \overline{\varphi_n}(t)))^{-1} \cdot b(t)$ a.î.

$$\varphi(t) = \sum_{i=1}^{n} c_i(t) \cdot \overline{\varphi_i}(t)$$

Dem $\{\overline{\varphi_1}(\cdot), \dots \overline{\varphi_n}(\cdot)\} \subset S_{A(\cdot)}$ sistem fundamental soluții $\Rightarrow X(t) = \mathrm{col}(\overline{\varphi_1}(t), \dots \overline{\varphi_n}(t)),\ t \in \mathring{I}$ matrice fundamentală de soluții $X(t)c(t) = \mathrm{col}(\overline{\varphi_1}(t) \dots \overline{\varphi_n}(t)) \dots$

$$= \sum_{i=1}^{n} c_i(t)\overline{\varphi_i}(t) + T \Rightarrow \text{q.e.d}$$

Algoritm (metoda variației constantelor pt. ec. afine pe...

$$\frac{dx}{dt} = A(t)x + b(t)$$

1 Consideră ec. liniară asociată $\frac{d\overline{x}}{dt} = A(t)\overline{x}$

Scrie soluția generală $\overline{X}(t) = \sum_{i=1}^{n} c_i \overline{\varphi_i}(t)$

2\. (Variația constantelor propriu-zise)

Caută soluții de forma $X(t) = \sum_{i=1}^{n} c_i(t) \cdot \overline{\varphi_i}(t)$

$$X(\cdot) \text{ sol} \Longrightarrow \sum_{i=1}^{n} c_i'(t)\, \overline{\varphi_i}(t) = b(t) \Longrightarrow c_i'(t) = \ldots, \ i = \overline{1,n}$$

$$\Longrightarrow c_i(t) = \ldots \quad i = \overline{1,n}$$

$$\Longrightarrow X(t) = \ldots$$

Ecuații diferențiale liniare de ordin superior

Def: $a_1(\cdot), \ldots a_n(\cdot) : \dot{I} \subseteq \mathbb{R} \to \mathbb{R}$ def $x^{(n)} = \sum_{j=1}^{n} a_j(t)\, x^{(n-j)}$ (1)

Metoda generală de studiu (sistemul canonic asociat)

$$(2)\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = x_3 \\ \ldots \\ \dfrac{dx_{n-1}}{dt} = x_n \\ \dfrac{dx_n}{dt} = \sum_{j=1}^{n} a_j(t)\, x_{n-j+1} \end{cases}$$

$$\hat{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad A(t) = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \ldots & & & & \\ 0 & 0 & 0 & \ldots & 1 \\ a_n(t) & & \ldots & \ldots & a_1(t) \end{pmatrix}$$

Matrice companion

$$\overset{\text{not}}{=} \text{comp}(a_1(t), \ldots a_n(t))$$

$$(2) \quad \frac{d\hat{x}}{dt} = A(t)\,\hat{x}$$

PROP (de echivalență)

$\varphi(\cdot)$ sol a ec(1) $\Longleftrightarrow$ $\hat{\varphi}(\cdot) = (\varphi(\cdot), \ldots \varphi^{(n-1)}(\cdot))$ este soluție a ec. (2)

T (E. U. G)

Fie $a_1(\cdot), \ldots a_n(\cdot) : \dot{I} \subseteq \mathbb{R} \to \mathbb{R}$ cont def $x^{(n)} = \sum_{j=1}^{n} a_j(t)\, x^{(n-j)}$

sol. cu $\varphi(t_0) = x_0$, $\varphi'(t_0) = x_0^1$, ... $\varphi^{(n-1)}(t_0) = x_0^{n-1}$

Dem. Prop. de echivalență + T (E U G) pt. ec. liniare pe $\mathbb{R}^n$ aplicată ec. (2)

$$S_{a_1(\cdot), \dots a_n(\cdot)} := \{\varphi(\cdot) : \dot{I} \to \mathbb{R};\ \varphi(\cdot) \text{ soluție a ec. } x^{(n)} = \sum_{j=1}^{n} a_j(t) x^{(n-j)}\}$$

T (Spațiul soluțiilor)

Fie $a_1(\cdot), \dots a_n(\cdot) : \dot{I} \subseteq \mathbb{R} \to \mathbb{R}$ cont. def. $x^{(n)} = \sum_{j=1}^{n} a_j(t) x^{(n-j)}$

Atunci $S_{a_1(\cdot), \dots a_n(\cdot)} \subset C^{(n)}(\dot{I}, \mathbb{R})$ subsp. vectorial dim $(S_{a_1(\cdot) \dots}$

$A(t) = \text{comp}(a_1(t), \dots a_n(t))$

$T : S_{a_1(\cdot), \dots a_n(\cdot)} \to S_{A(\cdot)}$ $T(\varphi(\cdot)) = \overset{\vee}{\varphi}(\cdot)\ (= \varphi(\cdot), \varphi'(\cdot), \dots$

T. liniară și bijectivă $\dim(S_{A(\cdot)}) = n \Rightarrow$ q.e.d

Def: $\{\varphi_1(\cdot), \dots \varphi_n(\cdot)\} \subset S_{a_1(\cdot), \dots a_n(\cdot)}$ bază s.n. sistem fundamental de soluții. Dacă $\{\varphi_1(\cdot), \dots \varphi_n(\cdot)\} \subset S_{a_1(\cdot), \dots a_n(\cdot)}$ sistem fundamental de soluții $\varphi(\cdot) = S_{a_1(\cdot), \dots a_n(\cdot)}$ $\Leftrightarrow$

$$\exists\ c_1, \dots c_n \in \mathbb{R} \text{ a.î. } \underbrace{\varphi(t) = \sum_{i=1}^{n} c_i \varphi_i(t)}_{\text{soluția generală}}$$

Ecuații liniare de ordin superior cu coeficienți constanți

Def: $a_1, \dots a_n \in \mathbb{R}$ def $x^{(n)} = \sum_{j=1}^{n} a_j x^{(n-j)}$

Caz particular $a_j(t) \equiv a_j \in \mathbb{R}$, $j = \overline{1,n}$

Sistem canonic asociat:

$$(2)\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \vdots \\ \dfrac{dx_{n-1}}{dt} = x_n \\ \dfrac{dx_n}{dt} = \sum_{j=1}^{n} a_j x_{n-j+1} \end{cases} \qquad \hat{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad A = comp(a_1, \dots a_n) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ a_n & a_{n-1} & \dots & & a_1 \end{pmatrix}$$

$$(2)\ \frac{d\hat{X}}{dt} = A\hat{X}$$

<u>PROP</u> $\underset{\text{spectrul}}{\sigma}(comp(a_1, \dots a_n)) = \{\lambda \in \mathbb{C};\ \lambda^n = \sum_{j=1}^{n} a_j \lambda^{n-j}\} =: \sigma(a_1 \dots$

<u>Def</u>: $\underset{\text{cvasi-polinoame}}{CP}(\sigma(a_1, \dots a_n)) = \{\varphi(t) = \sum_{j=1}^{n} e^{\lambda_j t} P_j(t) + \sum_{j=l+1}^{k} e^{\alpha_j t} \cdot$

$\cdot (P_j(t) \cos\beta_j t + Q_j(t) \sin\beta_j t)$, unde

$\sigma(a_1, \dots a_n) = \{\lambda_1, \dots \lambda_l \in \mathbb{R},\ \lambda_{l+1} = \alpha_{l+1} + i \cdot \beta_{l+1}\ \overline{\lambda}_{l+1}, \dots$

$\lambda_k = \alpha_k + i \cdot \beta_k,\ \overline{\lambda_k} \to$ cu ordinele de multiplicitate $m_1, \dots m$

și $P_j(t), Q_j(t)$ sunt polinoame de grad $\leq m_j - 1$

T(structura soluțiilor)

$$S_{a_1 \dots a_n} = CP(\sigma(a_1, \dots a_n))$$

<u>Dem</u>: $\underset{\substack{\downarrow \\ \dim = n}}{S_{a_1 \dots a_n}} \underset{\substack{\uparrow \\ \text{prop. de} \\ \text{echivalență}}}{\cong} S_A \underset{\substack{\uparrow \\ T(\text{struct. sol})}}{\subseteq} \underset{\substack{\downarrow \\ \dim = n}}{CP(\sigma(a_1, \dots a_n))}$

<u>Algoritm</u> $\quad x^{(n)} = \sum_{j=1}^{n} a_j x^{(n-j)} \qquad a_1, \dots a_n \in \mathbb{R}$

1. Rezolvă ec. caracteristică $\lambda^n = \sum^{n} a_j \lambda^{n-j} \to \sigma(a_1, \dots a_n)$

2. Pt. $\lambda \in \sigma(a_1, \dots a_n)$ scrie soluțiile

$$\varphi_\lambda^j(t) = \begin{cases} t^{j-1} e^{\lambda t}, & \lambda \in \mathbb{R}, \ j = \overline{1, m_\lambda} \\ t^{j-1} e^{\alpha t} \cos \beta t & \lambda = \alpha + i\beta \\ t^{j-1} e^{\alpha t} \sin \beta t & \beta > 0 \end{cases} \quad j = \overline{1, m_\lambda}$$

3. Renumerotarea: $\{\varphi_\lambda^j(\cdot)\}_{\substack{\lambda \in \sigma(a_1, \dots a_n) \\ j = \overline{1, m_\lambda}}} =$

$= \{\varphi_1(\cdot), \dots \varphi_n(\cdot)\}$ sistem fundamental de soluții

Scrie sol. generală $x(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$, $c_i \in \mathbb{R}$ $i = \overline{1, n}$

Ecuații afine de ordin superior

Def: $a_1(\cdot), \dots a_n(\cdot), b(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}$ def $x^{(n)} = \sum_{j=1}^{n} a_j(t) x^{(n-}$

Sistemul canonic asociat

$$(2) \begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = x_3 \\ \vdots \\ \dfrac{dx_{n-1}}{dt} = x_n \\ \dfrac{dx_n}{dt} = \sum_{j=1}^{n} a_j(t) x_{n-j+1} + b(t) \end{cases}$$

$\hat{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $A(t) = \mathrm{comp}(a_1(t), \dots a_n(t))$

$\hat{b}(t) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b(t) \end{pmatrix}$

$(2) \ \dfrac{d\hat{x}}{dt} = A(t)\hat{x} + \hat{b}(t)$

PROP (de echivalență)

$\varphi(\cdot)$ este soluție a ec. (1) $\Longleftrightarrow$ $\hat{\varphi}(\cdot) = (\varphi(\cdot), \dots \varphi^{(n-1)}(\cdot))$

Sol. a ec. (2)

T(EUG)

Fie $a_1(\cdot), \dots a_n(\cdot), b(\cdot) : \dot{I} \to \mathbb{R}$ continue def. $x^{(n)} = \sum_{j=1}^{n} a_j(t) x^{(n-j)} + b(t)$

Atunci $\forall (t_0, (x_0, x_0^1, \dots x_0^{n-1})) \in \dot{I} \times \mathbb{R}^n \ \exists! \ \varphi(\cdot) : \dot{I} \to \mathbb{R}$ soluție

$\varphi(t_0) = x_0, \ \varphi'(t_0) = x_0^1, \dots \ \varphi^{(n-1)}(t_0) = x_0^{n-1}$

<u>Dem</u>: Prop. de echivalență + T(EUG) pt. ec. afine pe $\mathbb{R}^n$ aplicată ec. (2)

$$S_{a_1(\cdot), \dots a_n(\cdot) b(\cdot)} = \left\{ \varphi(\cdot) : \dot{I} \to \mathbb{R}, \ \varphi(\cdot) \text{ sol a ec. } x^{(n)} = \sum_{j=1}^{n} a_j(t) x^{(n-j)} + b(t) \right\}$$

PROP (varietatea soluțiilor)

$$S_{a_1(\cdot), \dots a_n(\cdot) b(\cdot)} = S_{a_1(\cdot), \dots a_n(\cdot)} + \{ \varphi_0(\cdot) \}, \ \varphi_0(\cdot) \in S_{a_1(\cdot) \dots a_n(\cdot) b(\cdot)}$$

<u>Dem</u> „$\subseteq$" Fixăm $\varphi_0(\cdot) \in S_{a_1(\cdot) \dots a_n(\cdot) b(\cdot)}$

Fie $\varphi(\cdot) \in S_{a_1(\cdot), \dots a_n(\cdot) b(\cdot)} \overset{?}{\Longrightarrow} \varphi(\cdot) - \varphi_0(\cdot) \in S_{a_1(\cdot) \dots a_n(\cdot)}$

$$\varphi^{(n)}(t) = \sum_{j=1}^{n} a_j(t) \varphi^{(n-j)}(t) + b(t)$$

$$\varphi_0^{(n)}(t) = \sum_{j=1}^{n} a_j(t) \varphi_0^{(n-j)}(t) + b(t)$$

$$(\varphi - \varphi_0)^{(n)}(t) = \sum_{j=1}^{n} a_j(t) (\varphi - \varphi_0)^{(n-j)}(t) \quad \varphi - \varphi_0 \in S_{a_1(\cdot) \dots a_n(\cdot)}$$

„$\supseteq$" Fie $\psi(\cdot) \in S_{a_1(\cdot), \dots a_n(\cdot)}$ $\varphi_0(\cdot) \in S_{a_1, \dots a_n(\cdot) b(\cdot)}$ $\overset{?}{\Longrightarrow}$

$\psi(\cdot) + \varphi_0(\cdot) \in S_{a_1(\cdot) a_2(\cdot), \dots a_n b(\cdot)}$

$$\psi^{(n)}(t) = \sum_{j=1}^{n} a_j(t) \psi^{(n-j)}(t)$$

$$\varphi_0^{(n)}(t) = \sum_{j=1}^{n} a_j(t) \varphi_0^{(n-j)}(t) + b(t)$$

Corolar Dacă $\{\overline{\varphi_1(\cdot)}, \dots \overline{\varphi_n(\cdot)}\}$ sistem fundamental de soluții pt. ec. lin. asociată $x^{(n)} = \sum_{j=1}^{n} a_j(t) \overline{x}^{(n-j)}$ și $\varphi_0(\cdot)$ $\in S_{a_1(\cdot), \dots a_n(\cdot) b(\cdot)}$ atunci $\varphi(\cdot) \in S_{a_1(\cdot), \dots a_n(\cdot) b(\cdot)}$ $\Longleftrightarrow$

$$\exists\, c_1 \dots c_n \in \mathbb{R} \text{ a.î. } \varphi(t) = \underbrace{\sum_{i=1}^{n} c_i \overline{\varphi_i}(t) + \varphi_0(t)}_{\text{sol. generală}}$$

Seminar 10

$\boxed{\text{Ecuații afine pe } \mathbb{R}^n \text{ - Algoritm}}$

(Metoda variației constantelor)

$$\left.\begin{array}{l} A(\cdot): \overset{\circ}{I} \subseteq \mathbb{R} \to L(\mathbb{R}^n, \mathbb{R}^n) \\ b(\cdot): \overset{\circ}{I} \to \mathbb{R}^n \end{array}\right| \text{cont}$$

1. Considerăm ec. liniară asociată $\frac{d\overline{x}}{dt} = A(t) \cdot \overline{x}$

Determină $\{\overline{\varphi_1}(\cdot), \dots \overline{\varphi_n}(\cdot)\}$ sistem fundamental de solu

Obs: Dacă $A(t) \equiv A \in L(\mathbb{R}^n, \mathbb{R}^n) \to$ vezi Algoritm

Scrie sol. generală $\overline{x}(t) = \sum_{i=1}^{n} c_i \overline{\varphi_i}(t)$

2. Caută sol. de forma $x(t) = \sum_{i=1}^{n} c_i(t) \overline{\varphi_i}(t)$

$$x(\cdot) \text{ sol} \Longrightarrow \sum_{i=1}^{n} c_i'(t) \overline{\varphi_i}(t) = b(t) \Longrightarrow c_i'(t) = \dots \quad i = \overline{1,}$$

$$\Longrightarrow c_i(t) = \dots \quad i = \overline{1,}$$

$$\Longrightarrow x(t) = \dots$$

$\boxed{\text{Ecuații liniare de ordin superior cu coeficienți constanți - Alg}}$

$$x^{(n)} = \sum_{j=1}^{n} a_j x^{(n-j)} \quad a_j \in \mathbb{R}, \; j = \overline{1,n}$$

1. Rezolvă ec. caracteristică: $\lambda^n = \sum_{j=1}^{n} a_j \lambda^{n-j} \to \sigma(a_1, \dots a_n)$:

2. Pt $\lambda \in \sigma(a_1, \dots a_n)$ scrie sol

$$\varphi_\lambda^j(t) = \begin{cases} t^{j-1} e^{\lambda t} & \lambda \in \mathbb{R}, j = \overline{1, m_\lambda} \\ t^{j-1} e^{\alpha t} \cos \beta t & \\ t^{j-1} e^{\alpha t} \sin \beta t & \lambda = \alpha + i\beta, \beta > 0, j = \overline{1, m_\lambda} \end{cases}$$

3. Renumerotează $\{\varphi_\lambda^j(\cdot)\}_{\substack{\lambda \in \sigma(a_1, \dots a_n) \\ j = \overline{1, m_\lambda}}} = \{\varphi_1(\cdot), \dots \varphi_n(\cdot)\}$ sistem fundamental de soluții. Scrie sol generală $x(t) = \sum_{i=1}^{n} c_i \varphi_i(t)$, $c_i \in \mathbb{R}, i = \overline{1, n}$

Ex: Să se determine soluția generală

1) $\begin{cases} x' = 2x - y + e^{-t} \\ y' = 3y - 2x + e^{-t} \end{cases}$

$\begin{cases} \bar{x}' = 2\bar{x} - \bar{y} \\ \bar{y}' = 3\bar{y} - 2\bar{x} \end{cases}$ $\leftarrow$ $\bar{y} = 2\bar{x} - \bar{x}'$

$\bar{y}' = 3(2\bar{x} - \bar{x}') - 2\bar{x}$

$\bar{y}' = 6\bar{x} - 3\bar{x}' - 2\bar{x}$

$\bar{y}' = 4\bar{x} - 3\bar{x}'$

$(2\bar{x} - \bar{x}')' = 4\bar{x} - 3\bar{x}'$

$2\bar{x}' - \bar{x}'' = 4\bar{x} - 3\bar{x}'$

$-\bar{x}'' = -5\bar{x}' + 4\bar{x}$

Ar fi trebuit să înlocuiesc direct

$-\lambda^2 = -5\lambda + 4 \iff \lambda^2 - 5\lambda + 4$

$\iff (\lambda - 1)(\lambda - 4) = 0$

$\lambda_1 = 1, \ \lambda_2 = 4$

$\Rightarrow \bar{x}(t) = c_1 e^t + c_2 e^{4t}, \ c_1, c_2 \in \mathbb{R}$

$\bar{y}(t) = -(c_1 e^t + c_2 e^{4t})' + 2(c_1 e^t + c_2 e^{4t})$

$x(t) = c_1 e^t + c_2 e^{4t}$

$$\begin{cases} y(t) = c_1(t) e^t - 2 c_2(t) e^{4t} \\ x(t) = c_1(t) e^t + c_2(t) e^{4t} \end{cases}$$

$(c_1(t) \cdot e^t + c_2(t) e^{4t})' = 2(c_1(t) e^t + c_2(t) e^{4t}) -$

$- c_1(t) e^t + 2c_2(t) e^{4t} + e^{-t}$

$c_1'(t) \cdot e^t + c_1(t) e^t + c_2'(t) e^{4t} + 4c_2(t) e^{4t} =$

$= 2c_1(t) \cdot e^t + c_2(t) e^{4t} - c_1(t) e^t + 2c_2(t) e^{4t} + e^{-t}$

$(c_1(t) e^t - 2 c_2(t) e^{4t})' = 3(c_1(t) e^t - 2c_2(t) e^{4t})$

$-2(c_1(t) e^t + c_2(t) e^{4t}) + e^{-t}$

$\Leftrightarrow c_1'(t) \cdot e^t + c_1(t) e^t - 2c_2'(t) e^{4t} - 8c_2(t) \cdot e^{4t}$

$= 3c_1(t) e^t - 6 c_2(t) e^{4t} - 2 c_1(t) e^t - 2c_2(t) e^{4t}$

$\Leftrightarrow c_1'(t) e^t + \cancel{c_1(t) e^t} - 2c_2'(t) e^{4t} - \cancel{8 c_2(t) e^{4t}}$

$= \cancel{c_1(t) e^t} - \cancel{8c_2(t) e^{4t}} + e^{-t}$

$$\begin{cases} c_1'(t) e^t + c_2'(t) e^{4t} = e^{-t} \\ c_1'(t) e^t - 2c_2'(t) e^{4t} = e^{-t} \end{cases}$$

$3c_2'(t) = 0 \Rightarrow c_2'(t) = 0$

$c_1'(t) e^t = e^{-t} \Rightarrow c_1'(t) = e^{-2t}$

$c_2(t) = k_2, \; k_2 \in \mathbb{R}$

$c_1(t) = -\frac{1}{2} e^{-2t} + k_1, \; k_1 \in \mathbb{R}$

$$\begin{cases} x(t) = (-\frac{1}{2} e^{-2t} + k_1) e^t + k_2 e^{4t}, \; k_1, k_2 \in \mathbb{R} \\ y(t) = (-\frac{1}{2} e^{-2t} + k_1) e^t - 2k_2 e^{4t}, \; k_1, k_2 \in \mathbb{R} \end{cases}$$

2) $\begin{cases} x' = x - y + 2\sin t \\ y' = 2x - y \end{cases}$

$\begin{cases} \bar{x}' = \bar{x} - \bar{y} \Rightarrow \bar{y} = \bar{x} - \bar{x}' \\ \bar{y}' = 2\bar{x} - \bar{y} \end{cases}$

$(\bar{x} - \bar{x}')' = 2\bar{x} - \bar{x} + \bar{x}'$

$\cancel{\bar{x}'} - \bar{x}'' = \bar{x} + \cancel{\bar{x}'} \Rightarrow \bar{x}'' + \bar{x} = 0.$

$\lambda^2 + 1 = 0 \Rightarrow \lambda = \pm 1$

Fie $\lambda = i$

$\Rightarrow \bar{\varphi}_1(t) = e^{0t} \cdot \cos 1t = \cos t$

$\bar{\varphi}_2(t) = e^{0t} \sin 1t = \sin t$

$\Rightarrow \bar{x}(t) = c_1 \cos t + c_2 \sin t$

$\Rightarrow \bar{y}(t) = c_1 \cos t + c_2 \sin t + c_1 \sin t - c_2 \cos t$

$\bar{y}(t) = c_1(\cos t + \sin t) + c_2(\sin t - \cos t)$

$c_1, c_2 \in \mathbb{R}$

$\begin{cases} x(t) = c_1(t) \cdot \cos t + c_2(t) \cancel{\cos t}\ \sin t \\ y(t) = c_1(t)(\cos t + \sin t) + c_2(t) \cdot (-\cos t + \sin t) \end{cases}$

$(c_1(t) \cos t + c_2(t) \sin t)' = \cancel{c_1(t)\cos t} + c_2(t) \sin t -$

$- \cancel{c_1(t)} \cos t - c_1(t) \sin t + c_2(t) \cos t - \cancel{c_2(t)} \sin t + 2\sin t$

$c_1'(t) \cdot \cos t - \cancel{c_1(t)} \sin t + c_2'(t) \sin t + \cancel{c_2(t)} \cos t =$

$- \cancel{c_1(t)\sin t} + \cancel{c_2(t)} \cos t + 2\sin t$

$(c_1(t)(\cos t + \sin t) + c_2(t)(\sin t - \cos t))'$

$= 2(c_1(t)\cos t + c_2(t)\sin t) - c_1(t)(\cos t + \sin t) -$

$- c_2(t)(\sin t - \cos t)$

$\Leftrightarrow c_1'(t)(\cos t + \sin t) + c_1(t)(-\sin t + \cos t)$

$+ c_2'(t)(\sin t - \cos t) + c_2(t)(\cos t + \sin t) =$

$= 2c_1(t)\cos t + 2c_2(t)\sin t - c_1(t)\cos t - c_1(t)\cdot \sin$

$- c_2(t)\sin t + c_2(t)\cos t$

$\Leftrightarrow c_1'(t)(\cos t + \sin t) - c_1(t)\sin t + \cancel{c_1(t)}\cos t$

$+ c_2'(t)(\sin t - \cos t) + \cancel{c_2(t)}\cos t + \cancel{c_2(t)}\sin t$

$= \cancel{c_1(t)}\cos t + \cancel{c_2(t)}\sin t - \cancel{c_1(t)}\sin t + \cancel{c_2(t)}\cos$

$\Leftrightarrow c_1'(t)(\cos t + \sin t) + c_2'(t)(\sin t - \cos t) = 0.$

$$\begin{cases} c_1'(t)\cos t + c_2'(t)\sin t = 2\sin t \\ c_1'(t)(\cos t + \sin t) + c_2'(t)(\sin t - \cos t) = 0. \end{cases}$$

$$\begin{cases} c_1'(t)\cos t + c_2'(t)\sin t = 2\sin t & \Big|\ \cos t \,/\, \sin \\ c_1'(t)\sin t - c_2'(t)\cos t = -2\sin t & \Big|\ \sin t \,/\, \cos \end{cases}$$

$$\begin{cases} c_1'(t)\cos^2 t + c_2'(t)\sin t\cos t = 2\sin t\cos t \\ c_1'(t)\sin^2 t - c_2'(t)\cos t\cdot\sin t = -2\sin^2 t \end{cases}$$

$c_1'(t)(\sin^2 t + \cos^2 t) = 2\sin t\cos t - 2\sin^2 t$

$c_1'(t) = 2\sin t\cos t - 2\sin^2 t$

$c_1'(t) = -2\sin^2 t + 2\sin t\cos t$

$$\Rightarrow c_1(t) = \int (2\sin t\cos t - 2\sin^2 t)\,dt$$

$$= \int (\sin 2t - 1 + \cos 2t)\,dt$$

$$= -\frac{1}{2}\cos 2t - t + \frac{1}{2}\sin 2t + k_1,\ k_1 \in \mathbb{R}$$

$$\Rightarrow c_2(t) = \int (2\sin^2 t + 2\sin t\cos t)\,dt$$

$$= \int (1 - \cos 2t + \sin 2t)\,dt$$

$$= t - \frac{1}{2}\sin 2t - \frac{1}{2}\cos 2t + k_2,\ k_2 \in \mathbb{R}.$$

3) $\begin{cases} x' = 2y - x + 1 \\ y' = 3y - 2x \end{cases}$

4) $x^{IV} + x'' = 0$

$\lambda^4 + \lambda^2 = 0 \Leftrightarrow \lambda^2(\lambda^2 + 1) = 0$
 $\lambda_1 = 0,\ \lambda_2 = 0$
 $\lambda_3 = i,\ \lambda_4 = -i$

1 răd reală dublă, 2 răd complexe conjugate.

$\lambda = 0,\ m_\lambda = 2$

$\varphi_1(t) = e^{0t} \Rightarrow \varphi_1(t) = 1.$

$\varphi_2(t) = t \cdot e^{0t} \Rightarrow \varphi_2(t) = t$

$\lambda = i$

$\varphi_3(t) = e^{0t}\cos 1t = \cos t$

$\varphi_4(t) = e^{0t}\sin 1t = \sin t$

$$\Rightarrow x(t) = c_1\varphi_1(t) + c_2\varphi_2(t) + c_3\varphi_3(t) + c_4\varphi_4(t)$$

$$= c_1 + c_2 t + c_3\cos t + c_4\sin t$$

$c_1, c_2, c_3, c_4 \in \mathbb{R}$

5) $x''' - 3x'' + 3x' - x = 0.$

$x(0) = 1,\ x'(0) = 0,\ x''(0) = -1.$

$\lambda^3 - 3\lambda^2 + 3\lambda - 1 = 0.$

$(\lambda - 1)^3 = 0, \quad \lambda = 1, \ m_\lambda = 3$

$$\varphi_1(t) = e^{1t} = e^t$$

$$\varphi_2(t) = t \cdot e^t$$

$$\varphi_3(t) = t^2 \cdot e^t$$

$\Rightarrow x(t) = c_1 e^t + c_2 e^t \cdot t + c_3 e^t \cdot t^2$

$c_1, c_2, c_3 \in \mathbb{R}.$

$x(0) = c_1 \cdot e^0 = 1 \Rightarrow c_1 = 1$

$x'(t) = c_1 e^t + c_2 e^t + c_2 t e^t + 2c_3 t e^t + c_3 t$

$\quad = e^t(c_1 + c_2) + e^t \cdot t(2c_3 + c_2) + e^t \cdot t^2.$

$\overset{t=0}{\Rightarrow} e^0(1 + c_2) = 0$

$$c_2 = -1.$$

$x''(t) = e^t(c_1 + c_2) + e^t(c_2 + c_3) + e^t t(c_2 + c_3$

$\quad 2c_3 t e^t + e^t t^2 c^3$

$\quad = e^t(c_1 + 2c_2 + c_3) + e^t \cdot t(c_2 + 3c_3) + e$

$\overset{t=0}{\Rightarrow} e^0(1 - 2 + 2c_3) = -1$

$$c_3 = 0$$

$x(t) = e^t - t e^t \cancel{- t^2 e^t}$