17.10.2017

Curs 3

$$\frac{dx}{dt} = f(t,x) \quad f(\cdot,\cdot): D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$$

PROP (ec. integrală asociată unei ec. diferențiale)

$f(\cdot,\cdot): D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont, $\frac{dx}{dt} = f(t,x)$

Atunci $\varphi(\cdot): \dot{I} \subseteq \mathbb{R} \to \mathbb{R}^n$ graph$(\cdot) \subset D$ este soluție $\Leftrightarrow$

1. $\varphi(\cdot)$ cont
2. $\varphi(t) = \varphi(t_0) + \int_{t_0}^{t} f(s, \varphi(s))\,ds, \ \forall t, t_0 \in \dot{I}$

T. Peano (Existența locală a sol)

Fie $f(\cdot,\cdot): D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ continuă, $\frac{dx}{dt} = f(t,x)$

Atunci $f(\cdot,\cdot)$ admite E.L pe D ($\forall (t_0, x_0) \in D \ \exists \ \varphi(\cdot): \dot{I}_0 \in \theta(t_0) \to$ $\mathbb{R}^n$ sol cu $\varphi(t_0) = x_0$)

Dem: Fie $(t_0, x_0) \in D$

Planul dem: 1. $\dot{I}_0 = [t_0 - a, t_0 + a]$, $a = ?$

2. $\varphi_m(\cdot): \dot{I}_0 \to \mathbb{R}^n$ șir $\begin{cases} \text{egal mărginit} \\ \text{egal uniform continuu} \end{cases}$

3. [redacted] T. Arzela-Ascoli

$\exists \ \varphi_{m_p}(\cdot) \xrightarrow{u} \varphi(\cdot)$, $\varphi(\cdot): \dot{I}_0 \to \mathbb{R}^n$ continuă

4. $\varphi(\cdot)$ soluția

1. $(t_0, x_0) \in D = \mathring{D} \Rightarrow \exists \ \delta, \rho > 0$ a.î. $\overline{B_\delta}(t_0) \times \overline{B_\rho}(x_0) \subset$

$K := \max \|f(t,x)\|$

$(t,x) \in \overline{B_\delta}(t_0) \times \overline{B_\rho}(x_0)$

$K = 0 \Rightarrow f(t,x) \equiv 0 \Rightarrow \varphi(t) \equiv x_0$

$K > 0 \quad a := \min\{\delta, \frac{\rho}{K}\}$, $\dot{I}_0 = [t_0 - a, t_0 + a]$

$m \geq 1, \quad \varphi_m(\cdot) : \dot{I}_0 \to \mathbb{R}^n$

$$\varphi_m(t) = \begin{cases} x_0, & t \in [t_0 - \frac{a}{m}, t_0 + \frac{a}{m}] \\ x_0 + \int_{t_0}^{t-\frac{a}{m}} f(s, \varphi_m(s))\,ds, & t \in [t_0 + \frac{a}{m}, t_0 + a] \\ x_0 + \int_{t_0}^{t+\frac{a}{m}} f(s, \varphi_m(s))\,ds & t \in [t_0 - a, t_0 - \frac{a}{m}) \end{cases}$$

Arătăm că:

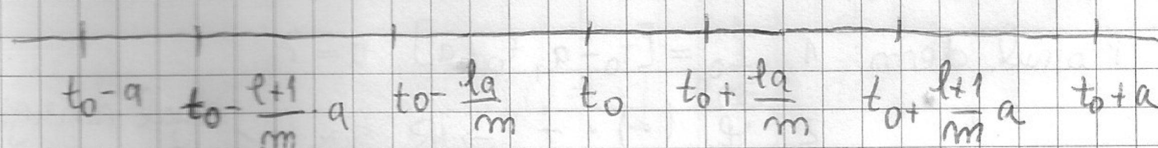
a) $\varphi_m(\cdot)$ bine mărginit

b) $\varphi_m(t) \in \overline{B_\rho}(x_0), \ \forall m \geq 1, \ \forall t \in \dot{I}_0$

c) $\varphi_m(\cdot)$ este egal Lipschitz

$\|\varphi_m(t') - \varphi_m(t'')\| \leq k|t' - t''| \ \forall t', t'' \in \dot{I}_0$

a)



b) inducție relativ la intervalul $[t_0 - \frac{l}{m} \cdot a, t_0 + \frac{l}{m} \cdot a]$

$l = 1$: $\varphi_m(t) \equiv x_0$ ok

$l \mapsto l+1$: Pp că $t \in [t_0 + \frac{l}{m} a, t_0 + \frac{l+1}{m} \cdot a] \Rightarrow$

$\frac{t-a}{m} \in [t_0 + \frac{l-1}{m} \cdot a, t_0 + \frac{l}{m} \cdot a]$

$\Rightarrow \varphi_m(t) = x_0 + \int_{t_0}^{t} f(s, \varphi_m(s))\,ds$ ok

$l = 1 \quad \varphi_m(t) = x_0 \in \overline{B_\rho}(x_0)$

$l \mapsto l+1$ Pp. că $t \in [t_0 + \frac{l}{m} a, t_0 + \frac{l+1}{m} \cdot a]$

$\|\varphi_m(t) - x_0\| = \| x_0 + \int^{t-\frac{a}{m}} \ldots \| \leq$

$$\leq \left| \int_{t_0}^{t-\frac{a}{m}} \| \overbrace{f(s, \underbrace{\varphi_m(s)}_{\in \overline{B_\gamma}(x_0)})}^{\leq K} \| ds \right| \leq K \left| t - \frac{a}{m} - t_0 \right| \leq K \cdot \frac{1}{m} \cdot a \leq K a \leq K \cdot \frac{\gamma}{K}$$

$= \gamma$

$\varphi_m(t) \in \overline{B_\gamma}(x_0)\ \forall m\ \forall t \quad \| \varphi_m(t) \| \leq \| \varphi_m(t) - x_0 \| + \| x_0 \| \leq$
$\leq \gamma + \| x_0 \|,\ \forall m\ \forall t$

c) $t', t'' \in \mathring{I}_0$. De ex, pp că $t', t'' \in [t_0 + \frac{a}{m}, t_0 + a]$

$$\| \varphi_m(t') - \varphi_m(t'') \| = \| \cancel{x_0} + \int_{t_0}^{t'-\frac{a}{m}} f(s, \varphi_m(s)) ds - \cancel{x_0} - \int_{t_0}^{t''-\frac{a}{m}} f(s, \varphi_m(s)) ds \| =$$

$$= \| \int_{t''-\frac{a}{m}}^{t'-\frac{a}{m}} f(s, \varphi_m(s)) ds \| \leq \left| \int_{t''-\frac{a}{m}}^{t'-\frac{a}{m}} \overbrace{\| f(s, \underbrace{\varphi_m(s)}_{\in \overline{B_\gamma}(x_0)}) \|}^{\leq K} ds \right| \leq K |t' - t''|$$

ok.

Analog celelalte cazuri

3. T. Arzela–Ascoli $\Rightarrow \exists\ \varphi_{m_p}(\cdot) \xrightarrow[p \to \infty]{u} \varphi(\cdot),\ \varphi(\cdot) : \mathring{I}_0 \to$
$B_\gamma(x_0)$ continuă

4. $$\varphi_{m_p}(t) = \begin{cases} x_0, & t \in [t_0 - \frac{a}{m_p}, t_0 + \frac{a}{m_p}] \\ x_0 + \int_{t_0}^{t} f(s, \varphi_{m_p}(s)) ds + \int_{t}^{t-\frac{a}{m}} f(s, \varphi_{m_p}(s)) ds, & t \in [t_0 + \frac{a}{m_p}, t_0 + a] \\ x_0 + \int_{t_0}^{t} f(s, \varphi_{m_p}(s)) ds + \int_{t}^{t+\frac{a}{m_p}} f(s, \varphi_{m_p}(s)) ds, & t \in [t_0 - a, t_0 - \frac{a}{m_p}] \end{cases}$$

$$\left.\begin{array}{l} \varphi_{m_p}(\cdot) \xrightarrow{u} \varphi(\cdot) \\ f(\cdot,\cdot) \text{ cont} \end{array}\right] \Rightarrow \int_{t_0}^{t} f(s, \varphi_{m_p}(s))ds \to \int_{t_0}^{t} f(s, \varphi(s)) d$$

$$\left\| \int_{t}^{t-\frac{a}{m_p}} f(s, \varphi_{m_p}(s))ds \right\| \le \left| \int_{t_0}^{t-\frac{a}{m_p}} \underbrace{\| f(s, \varphi_{m_p}(s)) \|}_{\le k} ds \right| \le k \cdot \frac{a}{m}$$

$$\xrightarrow[m_p \to \infty]{} 0$$

$m_p \to \infty$ $\quad \varphi(t) = x_0 + \int_{t_0}^{t} f(s, \varphi(s)) ds, \forall t \in \mathring{I}_0$

$$\left.\begin{array}{l} \varphi(t) = \varphi(t_0) + \int_{t_0}^{t} f(s, \varphi(s)) ds, \forall t \in \mathring{I} \\ \varphi(\cdot) \text{ continuă} \end{array}\right\} \Rightarrow$$

PROP (ec. integrală asociată) $\Rightarrow \varphi(\cdot)$ soluție $\varphi(t_0) = x_0$

<u>Obs</u>

$D = \mathring{D}$, $f(\cdot,\cdot)$ cont

1) $x' = \alpha(t)$, $\alpha(t) = \begin{cases} 1, & t \in \mathbb{Q} \\ 0, & t \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$

nu are E.L în niciun punct

2) $x' = \operatorname{sgn}(x)$, $\operatorname{sgn}(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$

are EUL în fiecare punct.

Funcții local Lipschitz

Local Lipschitz

<u>Def</u>: 1) $g(\cdot): G \subseteq \mathbb{R}^n \to \mathbb{R}^n$ s.n Lipschitz (global)

G dacă $\exists\, L > 0$ a.î. $\| g(x_1) - g(x_2) \| \le L \cdot \| x_1 - x_2 \|$,

3) $g(\cdot): G \subseteq \mathbb{R}^n \to \mathbb{R}^m$ s.n. local Lipschitz în $x_0 \in G$ dacă $\exists\ r > 0,\ \exists\ L > 0$ a.î. $\|g(x_1) - g(x_2)\| \leq L\|x_1 - x_2\|,$ $\forall\ x_1, x_2 \in B_r(x_0) \cap G$

Obs: 1. $g(\cdot)$ Lipschitz $\to$ $g(\cdot)$ local Lipschitz $\Longrightarrow$ $g(\cdot)$ continuă

2. $g(x) = x^2$ este local Lipschitz, nu este global Lipschitz

$g(x) = x^{\frac{1}{3}}$ este continuă, dar nu este local Lipschitz

PROP $g(\cdot): G = \overset{\circ}{G} \subseteq \mathbb{R}^n \to \mathbb{R}^n$. Atunci g este local Lipschitziană $\Longleftrightarrow$ $g(\cdot)|_{G_0}$ este (global) Lipschitz $\forall\ G_0 \subset G$ compact

Dem: "$\Longleftarrow$" Evident

"$\Longrightarrow$"

Pp. că $\exists\ G_0 \subset G$ compact a.î. $g(\cdot)|_{G_0}$ nu este Lipschitz

$\forall\ m \in \mathbb{N}\ \exists\ x_m^1, x_m^2 \in G_0$ a.î. $\|g(x_m^1) - g(x_m^2)\| > m \cdot \|x_m^1 - x_m^2\|$ (1)

$\Longrightarrow \|x_m^1 - x_m^2\| < \frac{1}{m}\|g(x_m^1) - g(x_m^2)\| \leq$

$\leq \frac{1}{m}\left(\|g(x_m^1)\| + \|g(x_m^2)\|\right)$

$g(\cdot)$ local Lipschitz $\Longrightarrow$ $g(\cdot)$ continuă, G_0 compactă,

$K := \max\limits_{x \in G_0} \|g(x)\|$

$x_m^1, x_m^2 \in G_0$ compactă $\Longrightarrow \exists\ x_{m_\ell}^1 \to x^1 \in G_0$

$\exists\ x_{m_\ell}^2 \to x^2 \in G_0$

(*) $\|x_{m_\ell}^1 - x_{m_\ell}^2\| \leq \frac{2K}{m_\ell}$, $m_\ell \to \infty \Longrightarrow \|x^1 - x^2\| \leq 0$

$g(\cdot)$ local Lipschitz în $x_0 \in G \Rightarrow \exists\, r > 0\ \exists\, L > 0$ a.

$\|g(x_1) - g(x_2)\| \le L\|x_1 - x_2\|, \ \forall\, x_1, x_2 \in B_r(x_0)$

$x^1_{m_\ell}, x^2_{m_\ell} \to x_0 \Rightarrow \exists\, \ell_0 \in \mathbb{N},\ \ell \ge \ell_0,$

$$x^1_{m_\ell}, x^2_{m_\ell} \in B_r(x_0)$$

$\|g(x^1_{m_\ell}) - g(x^2_{m_\ell})\| \le L\|x^1_{m_\ell} - x^2_{m_\ell}\|$ (2)

(1), (2) $\Rightarrow m_\ell \|x^1_{m_\ell} - x^2_{m_\ell}\| \le L \cdot \|x^1_{m_\ell} - x^2_{m_\ell}\|,\ m_\ell \to$

<u>PROP</u> $G = \mathring{G} \subseteq \mathbb{R}^n \to \mathbb{R}^n$, derivabilă cu $Dg(\cdot)$ mărgin

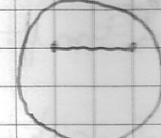
Atunci $g(\cdot)$ local Lipschitziană (pe G).

<u>Dem</u>: T. medie: $\|g(x) - g(y)\| = \sup\limits_{z \in [x,y]} \|Dg(z)\| \cdot \|x-$

$x, y \in G,\ [x,y] \subset G$

$[x,y] = \{x + s(y-x);\ s \in [0,1]\}$

Fie $x_0 \in G = \mathring{G} \Rightarrow \exists\, r > 0$ a.î. $\overline{B_r}(x_0) \subset$

$L := \sup\limits_{z \in \overline{B}_r(x_0)} \|Dg(z)\|$ q.e.d  (Convexitate)

Consecință: $g(\cdot): G = \mathring{G} \subseteq \mathbb{R}^n \to \mathbb{R}^n$, de clasă C^1 atunci $g(\cdot)$ local Lipschitz.

Funcții local Lipschitz în raport cu variabila a doua

$f(\cdot,\cdot): D \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ s.n. local Lipschitz în raport c

$$\|f(t,x_1)-f(t,x_2)\| \le L\cdot\|x_1-x_2\|,\ \forall (t,x_1),(t,x_2)\in D_0$$

<u>PROP</u> Fie $f(\cdot,\cdot): D=\overset{\circ}{D}\subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ continuă.
Atunci $f(\cdot,\cdot)$ local Lipschitz ($\overline{\text{II}}$) (pe D) $\Longleftrightarrow$ $\forall\, \overset{\circ}{D}\subset D$ compactă $\exists\, L>0$ a.î. $\|f(t,x_1)-f(t,x_2)\|\le L\cdot\|x_1-x_2\|$, $\forall (t,x_1),(t,x_2)\in D_0$.

<u>Dem</u>: Ex!

<u>PROP</u> Fie $f(\cdot,\cdot): D=\overset{\circ}{D}\subseteq\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^n$ C^1 ($\overline{\text{II}}$)

$\left[\exists\, D_2 f(t,x)\right.$ şi $(t,x)\to D_2 f(t,x)$ continuă$\left.\right]\Longrightarrow$

deriv. în raport cu variabila a doua $\left(\frac{\partial}{\partial x} f(t,x)\right)$

$f(\cdot,\cdot)$ local Lipschitz ($\overline{\text{II}}$).

Lema Bellman - Gronwall

$M\ge 0$, $u(\cdot), v(\cdot): I\subseteq\mathbb{R}\to\mathbb{R}_+$ continue, $t_0\in\overset{\circ}{I}$. Dacă:

$$u(t)\le M+\left|\int_{t_0}^{t} u(s)v(s)\,ds\right|,\ \forall t\in I \text{ atunci}$$

$$u(t)\le M\cdot e^{\left|\int_{t_0}^{t} v(s)\,ds\right|}\quad \forall t\in \overset{\circ}{I}$$

Seminar 3

$\boxed{\text{Ecuații Bernoulli}}$

$$\frac{dx}{dt} = a(t)\,x + b(t)\cdot x^{\alpha}, \quad a(\cdot), b(\cdot): I \subseteq \mathbb{R} \to \mathbb{R} \text{ cont},$$
$$\alpha \in \mathbb{R} \setminus \{0,1\}.$$

$\boxed{\text{Algoritm}}$

1. Cons. ec. lin. asociată: $\frac{d\bar{x}}{dt} = a(t)\cdot \bar{x}$

Scriem sol. generală: $\bar{x}(t) = c\cdot e^{A(t)}$

2. Variația constantelor

Caut sol de forma $x(t) = c(t)\cdot e^{A(t)}$

$x(\cdot)$ sol $\Rightarrow \frac{dc}{dt} = e^{(\alpha-1)A(t)}\cdot b(t)\, c^{\alpha}$ ← ec. var. ~~sep~~

Algoritm
$\Longrightarrow c(t) = \ldots$

$x(t) = \ldots$

Ex: Să se determine sol. generală:

1) $x' = x^2\cdot e^t - 2x$

2) $t x' + x = t^5 x^3 e^t$

3) $t x' = 2t^2\sqrt{x} + 4x$

4) $x' + \frac{x}{t} = \frac{1}{t^2 \cdot x^2}$

1) $x' = -2\cdot x + x^2\cdot e^t$.

$$\frac{dx}{dt} = -2\cdot x + x^2\cdot e^t$$

$$\frac{d\bar{x}}{dt} = -2\cdot \bar{x}$$

~~$\overline{x}(t) = c \cdot e^{-2\cdot\frac{t^2}{2}} \Rightarrow \overline{x}(t) = c \cdot e^{-t^2}$~~

$\overline{x}(t) = c \cdot e^{-2t}$

Revin aici cu ⊛, ⊛ pt a determina sol. finală

Caut sol. de forma $x(t) = c(t) \cdot e^{-2t}$

Pun condiția ca $c(t) \cdot e^{-2t}$ soluție

$(c(t) \cdot e^{-2t})' = (c(t) \cdot e^{-2t})^2 \cdot e^{t} - 2 \cdot (c(t) \cdot e^{-2t})$

$\Leftrightarrow c'(t) e^{-2t} + c(t) \cdot \cancel{(-2) \cdot e^{-2t}} =$

$= c^2(t) e^{-3t} - \cancel{2 c(t) e^{-2t}}$

$\Leftrightarrow c'(t) e^{-2t} = c^2(t) \cdot e^{-3t}$

$\Leftrightarrow c'(t) = c^2(t) e^{-t}$

$\frac{dc}{dt} = c^2 \, e^{-t}$ ecuație cu variabile separabile

$c^2 = 0 \Rightarrow c = 0 \Rightarrow c(t) = 0$ ⊛

$\varphi_1 \equiv 0$ sol staționară

$\frac{dc}{c^2} = e^{-t} dt$

$\int \frac{dc}{c^2} = \int e^{-t} dt \Leftrightarrow -\frac{1}{c} = -e^{-t} + k, k \in \mathbb{R}$

$\Rightarrow c(t) = \frac{1}{e^{-t} + k}, k \in \mathbb{R}$ ⊛

$x_0(t) = 0$

$x_k(t) = e^{-2t} \cdot \frac{1}{e^{-t} + k}, k \in \mathbb{R}$.

2) $t x' + x = t^5 \cdot x^3 \cdot e^t$

$$x' = \frac{1}{t}(t^5 \cdot x^3 \cdot e^t - x)$$

$$x' = t^4 \cdot x^3 \cdot e^t - \frac{x}{t}, \quad t \neq 0, \ t > 0$$

$$\frac{dx}{dt} = \underbrace{t^4 \cdot x^3 \cdot e^t}_{b(t) \cdot x^\alpha} - \underbrace{\frac{x}{t}}_{x \cdot a(t)}$$

$$\overline{x}(t) = -\frac{x}{t}.$$

$$\overline{x}(t) = c \cdot e^{\int -\frac{1}{t} dt} \Rightarrow \overline{x}(t) = c \cdot e^{-\ln t}$$

$$\Rightarrow \overline{x}(t) = \frac{c}{t}$$

Căutăm sol: $x(t) = c(t) \cdot \frac{1}{t}$.

$$\left(c(t) \cdot \frac{1}{t}\right)' = t^4 \cdot \left(c(t) \cdot \frac{1}{t}\right)^3 \cdot e^t - \left(c(t) \cdot \frac{1}{t}\right) \cdot \frac{1}{t}.$$

$$\Leftrightarrow c'(t) \cdot \frac{1}{t} \cancel{- c(t) \cdot \frac{1}{t^2}} = t^4 \cdot c^3(t) \cdot \frac{1}{t^3} \cdot e^t \cancel{- c(t) \cdot \frac{1}{t^2}}$$

$$\Leftrightarrow c'(t) \cdot \frac{1}{t} = t^4 \cdot c^3(t) \cdot \frac{1}{\cancel{t^3}} \cdot e^t$$

$$\Leftrightarrow c'(t) \cdot \frac{1}{t} = t \cdot c^3(t) \cdot e^t$$

$$c'(t) = t^2 \cdot c^3(t) \cdot e^t$$

$c'(t) = c^3(t) \cdot t^2 \cdot e^t \rightarrow$ Ecuație cu variabile separabile.

$$\frac{dc}{dt} = c^3 \cdot t^2 \cdot e^t$$

$c^3 = 0 \Rightarrow c = 0$, $\varphi_0 \equiv 0$ sol. staționară

$c(t) \equiv 0.$

$$dc = t^2 e^t \, dt$$

$$\int \frac{dc}{c^3} = \int t^2 \cdot e^t dt$$

$$-\frac{1}{2} \cdot \frac{1}{c^2} = t^2 e^t - \int 2t e^t dt$$

$$\Leftrightarrow -\frac{1}{2c^2} = t^2 \cdot e^t - 2te^t + \int 2e^t dt$$

$$\Leftrightarrow -\frac{1}{2c^2} = t^2 \cdot e^t - 2t \cdot e^t + 2e^t + k, \; k \in \mathbb{R}$$

$$\Rightarrow c(t) = \pm \frac{1}{\sqrt{-2(t^2 e^t - 2te^t + 2 \cdot e^t + k)}}, \; k \in \mathbb{R}.$$

$$x_0(t) = 0$$

$$x_k(t) = \pm \frac{1}{t\sqrt{(\dots\dots)}}$$

$\boxed{\text{Ecuații Riccati}}$

$$\frac{dx}{dt} = a(t)x^2 + b(t)x + c(t), \quad a(\cdot), b(\cdot), c(\cdot), I \subseteq \mathbb{R} \to \mathbb{R} \text{ cont.}$$

$\exists \, \varphi_0(\cdot)$ soluție

<u>Algoritm</u>: SV $y = x - \varphi_0(t)$ [$\forall \, x(\cdot)$ sol SV def $y(\cdot)$ după regula $y(t) = x(t) - \varphi_0(t)$]

$\Rightarrow$ Ec. Bernoulli în $y \Rightarrow$ Algoritm $\Rightarrow y(t) = \dots$

$\Rightarrow x(t) = y(t) + t = \dots$

<u>Ex</u>: 1) $x' = x^2 - 2xe^t + e^{2t} + e^t$, $\varphi_0(t) = e^t$ sol

2) $x' = x^2 + 6x - 4t + 11$, $\varphi_0(t) = 2t - 3$ sol

3) $x' + x^2 - \frac{2}{t}x + \frac{2}{t^2} = 0$, $\varphi_0(t) = \frac{2}{t}$ sol

1) $\frac{dx}{dt} = x^2 - 2e^t x + e^{2t} + e^t$

$\varphi_0(t) = e^t$ sol

SV: $y = x - e^t$

$\cancel{y(t) = (x - e^t)^2 - 2e^t(x - e^t) + 2e^{2t} + e^t}$

$\cancel{\Leftrightarrow y(t) = x^2 + 2xe^t + e^{2t} - 2e^t x + 2e^{2t} + e^{2t} + e}$

$x(t) = y(t) + e^t$

$(y(t) + e^t)' = (y(t) + e^t)^2 - 2(y(t) + e^t) \cdot e^t + e^{2t} + e^t$

$= y^2(t) + \cancel{e^{2t}} + \cancel{2y(t) e^t} - \cancel{2y(t)} e^t - \cancel{2e^{2t}}$

$+ \cancel{e^{2t}} + e^t$

$(y(t) + e^t)' = y^2(t) + e^t$

$y'(t) + \cancel{e^t} = y^2(t) + \cancel{e^t}$

$y'(t) = y^2(t)$

$\frac{dy}{dt} = y^2 \rightarrow$ Ec. var sep

$y^2 = 0 \Rightarrow y = 0 \Rightarrow y_0(t) = 0$

$\frac{dy}{y^2} = dt \Rightarrow \int \frac{dy}{y^2} = \int dt \Rightarrow -\frac{1}{y} = t + k, k \in$

$\Rightarrow y_k(t) = -\frac{1}{t+k}, k \in \mathbb{R}$

$\Rightarrow x(t) = y(t) + e^t$

$x_0(t) = e^t$

2) $x' = x^2 + 6x - 4t^2 + 11$, $\varphi_0(t) = 2t - 3$ sol

SV: $y(t) = x - 2t + 3$

$x(t) = y(t) + 2t - 3.$

$(y(t) + 2t - 3)' = (y(t) + 2t - 3)^2 + 6(y(t) + 2t - 3) - 4t^2 + 11$

$\Leftrightarrow y'(t) + 2 = (y(t) + 2t - 3)(y(t) + 2t - 3) +$

$+ 6 y(t) + 12t - 18 - 4t^2 + 11$

$\Leftrightarrow y'(t) + \cancel{2} = y^2(t) + 2t\, y(t) - 3\cancel{y(t)} + 2t\cdot y(t) + \cancel{4t^2} - \cancel{6t}$

$- 3\cancel{y(t)} - \cancel{6t} + \cancel{9} + 6\cancel{y(t)} + \cancel{12t} - \cancel{18} - \cancel{4t^2} + \cancel{11}$

$\Leftrightarrow y'(t) = y^2(t) + 4t\, y(t)$

$\frac{dy}{dt} = 4t \cdot y + y^2 \to$ Ec. Bernoulli

$\frac{dy}{dt} = 4t \cdot y$, scriem sol. generală: $\bar{y}(t) = c \cdot e^{\int 4t\, dt} \Rightarrow$

$\bar{y}(t) = c \cdot e^{2t^2} \Leftrightarrow \bar{y}(t) = c \cdot e^{2t^2}$

Peste tot...

Caut sol. de forma $y(t) = c(t) \cdot e^{4t}$ Nu e^{4t} :(

$(c(t) \cdot e^{4t})' = 4t \cdot (c(t) \cdot e^{4t}) + (c(t) \cdot e^{4t})^2$

$c'(t) \cdot e^{4t} + \cancel{4 c(t) e^{4t}} = 4t \cdot \cancel{c(t) e^{4t}} + c^2(t) \cdot e^{8t}$

$c'(t) \cdot e^{4t} = c^2(t) \cdot e^{8(t)}$

$c'(t) = c^2(t) \cdot e^{4(t)}$

$\frac{dc}{dt} = c^2 \cdot e^{4(t)} \Rightarrow$ Ec. var. separabile

$\cancel{\int \frac{dc}{c^2} = \int e^{4t} dt}$

$\cancel{\Leftrightarrow -\frac{1}{c} = \frac{1}{4} e^{4t} + k, k \in \mathbb{R}}$

$\cancel{\Rightarrow c(t) =}$ [illegible] $\cancel{-\frac{1}{\frac{1}{4} e^{4t} + \frac{1}{4} \cdot k}}$

[illegible]

$$\frac{dc}{c^2} = e^{2t^2} dt \Rightarrow \int \frac{dc}{c^2} = \int e^{2t^2} dt$$

$$\Leftrightarrow -\frac{1}{c} = \int_0^t e^{2s^2} ds + k, k \in \mathbb{R}$$

$$c(t) = -\frac{1}{\int_0^t e^{2s^2} ds + k}, k \in \mathbb{R}$$

$y_0(t) = 0$

$y_k(t) = \ldots$

$y(t) = y(t) + 2t - 3$

$x_0(t) = 2t - 3$

$$x_k(t) = -\frac{e^{2t^2}}{\int_0^t e^{2s^2} ds + k}, k \in \mathbb{R}.$$

$\boxed{\text{Ecuații omogene}}$

$$\frac{dx}{dt} = f\left(\frac{x}{t}\right), f(\cdot): D \subseteq \mathbb{R} \to \mathbb{R}$$

Algoritm

SV: $y = \frac{x}{t}$ [$\forall$ $x(\cdot)$ sol s.v def funcție $y(\cdot)$ după regu

$y(t) = \frac{x(t)}{t}$] $\Rightarrow \frac{dy}{dt} = \frac{f(y) - y}{t}$ ec. var. sep $\Rightarrow$ vezi Algorit

Ex: 1) $2t^2x' = t^2 + x^2$

2) $x' = \frac{x+\sqrt{tx}}{t}$

3) $t\cdot x' = x + \sqrt{x^2 - t^2}$

4) $x' = \frac{2tx}{t^2 - x^2}$

1) $2t^2 \cdot x' = t^2 + x^2$, $t^2 \neq 0$, $t > 0 \,|\, :2t^2$

$x' = \frac{1}{2} + \frac{1}{2}\left(\frac{x}{t}\right)^2$

SV $y = \frac{x}{t} \Rightarrow y(t) = \frac{x(t)}{t}$

$x(t) = t\cdot y(t) \Leftrightarrow (t\cdot y(t))' = \frac{1}{2} + \frac{1}{2}\cdot y^2(t)$

$\Leftrightarrow y(t) + t\cdot y'(t) = \frac{1}{2} + \frac{1}{2}\cdot y^2(t)$

$\Leftrightarrow y'(t) = \frac{1}{t}\left(\frac{1}{2} + \frac{1}{2}y^2(t) - y(t)\right) \Rightarrow \frac{dy}{dt} = \frac{(y-1)^2}{2t}$

$(y-1)^2 = 0 \Rightarrow y = 1 \Rightarrow y_0(t) = 1$

$\Rightarrow \frac{dy}{(y-1)^2} = \frac{1}{2}t\,dt \Leftrightarrow \int \frac{dy}{(y-1)^2} = \frac{1}{2}\int t\,dt \Rightarrow \frac{1}{1-y} = \frac{1}{4}t^2 + k$, $k \in \mathbb{R}$

$y(k) = -\frac{1}{\frac{1}{4}t^2 + k} - 1$, $k \in \mathbb{R}$

$x_0(t) = t$

$x_k(t) = t\cdot\left(-\frac{1}{\frac{1}{4}t^2+k} - 1\right)$, $k \in \mathbb{R}$.

2) $x' = \dfrac{x + \sqrt{tx}}{t}$

$\Longleftrightarrow x' = \dfrac{x}{t} + \sqrt{\dfrac{tx}{t^2}} \Longleftrightarrow x' = \dfrac{x}{t} + \sqrt{\dfrac{x}{t}}$, $t \neq 0$, $t >$

SV: $\dfrac{x}{t} = y \Rightarrow y(t) = \dfrac{x(t)}{t} \Rightarrow x(t) = y(t) \cdot t$

$(y(t) \cdot t)' = y(t) + \sqrt{y(t)}$

$y'(t) \cdot t + \cancel{y(t)} = \cancel{y(t)} + \sqrt{y(t)}$

$y'(t) = \dfrac{\sqrt{y(t)}}{t} \Rightarrow y' = \dfrac{\sqrt{y}}{t}$

$\Rightarrow \dfrac{dy}{dt} = \dfrac{\sqrt{y}}{t} \Longleftrightarrow dy\, t = dt \sqrt{y}$

$\dfrac{dy}{\sqrt{y}} = \dfrac{dt}{t}$

$\displaystyle\int \frac{dy}{\sqrt{y}} = \int \frac{dt}{t} \Rightarrow$

$\Rightarrow 2\sqrt{y} = \ln t + c$, $c \in \mathbb{R}$.

$y = \left(\dfrac{\ln t + c}{2}\right)^2$, $c \in \mathbb{R}$.

$x(t) = \left(\dfrac{\ln t + c}{2}\right)^2 \cdot t$, $c \in \mathbb{R}$.

Curs 4

Lema Bellman - Gronwall

Fie $M \geq 0$, $t_0 \in \mathring{I} \subseteq \mathbb{R}$, $u(\cdot), v(\cdot) : \mathring{I} \to \mathbb{R}_+$ continue

Dacă : $u(t) \le M + \left|\int_{t_0}^{t} u(s) v(s) ds\right|, \forall t \in \mathring{I}$ atunci $u(t) \le M \cdot e^{\left|\int_{t_0}^{t} v(s) ds\right|}, \forall t \in \mathring{I}$

Dem: $u(t) \le \underbrace{M + \left|\int_{t_0}^{t} u(s) v(s) ds\right|) \to e^{-\left|\int_{t_0}^{t} v(s) ds\right|}}_{=:\psi(t)} \cdot e^{\left|\int_{t_0}^{t} v(s) ds\right|}$

Arătăm că $\psi(t) \le M, \forall t \Rightarrow$ q.e.d

$\mathring{I}_+ = \{t \in \mathring{I}, t \geq t_0\}$, $\mathring{I}_- = \{t \in \mathring{I}, t < t_0\}$.

Arătăm că $\psi(\cdot)$ este $\downarrow$ pe $\mathring{I}_+$ și $\uparrow$ pe $\mathring{I}_-$, $\psi(t_0)$

$\to \psi(t) = M, \forall t$

De exemplu pt. $t \in \mathring{I}_+$

$\psi(t) = \left(M + \int_{t_0}^{t} u(s) ds\, ds\right) \cdot e^{-\int_{t_0}^{t} v(s) ds}$

$\psi'(t) = u(t) = u(t) \cdot v(t) \cdot e^{-\int_{t_0}^{t} v(s) ds} + \left(M + \int_{t_0}^{t} u(s) v(s) ds\right) \cdot$

$e^{-\int_{t_0}^{t} v(s) ds} \cdot (-v(t)) = \underbrace{v(t)}_{\geq 0} \cdot \underbrace{e^{-\int_{t_0}^{t} v(s) ds}}_{\geq 0} \underbrace{\left[u(t) - \left(M + \int_{t_0}^{t} u(s) v(s) ds\right)\right]}_{\le 0}$

$\le 0, \forall t \Rightarrow \psi(\cdot)$ desc. pe $\mathring{I}_+$

Teorema Cauchy - Lipschitz

Fie $f(\cdot,\cdot) : D = \mathring{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ cont, local Lipschitz (

$\frac{dx}{dt} = f(t, x)$. Atunci $f(\cdot,\cdot)$ admite EUL pe D ($\forall (t_0,$

$\in D, \exists \mathring{I}_0 = [t_0 - a, t_0 + a] \in \mathcal{V}(t_0)$ $\exists! \varphi(\cdot) : \mathring{I}_0 \to \mathbb{R}^n$

soluție cu $\varphi(t_0) = x_0$.

Dem: Fie $(t_0, x_0) \in D = \mathring{D}$, $f(\cdot,\cdot)$ cont $\xRightarrow{\text{T. Peano}}$ $\exists \mathring{I}_0 = [t_0$

$t_0 + a] \in \mathcal{V}(t_0)$ $\exists \varphi(\cdot) : \mathring{I}_0 \to \mathbb{R}^n$ sol cu $\varphi(t_0) = x_0$.

$\varphi(\cdot)$ soluție $\Rightarrow$ Ec. integrală asociată

$$\varphi(t) = \underset{0}{\underset{\parallel}{\varphi(t_0)}} + \int_{t_0}^{t} f(s, \varphi(s))\, ds, \ \forall t \in I_0$$

$\varphi_1(\cdot)$ soluție $\Rightarrow$ —"— $\varphi_1(t) = \underset{0}{\underset{\parallel}{\varphi_1(t_0)}} + \int_{t_0}^{t} f(s, \varphi_1(s))\, ds, \forall t \in I_0$

$$\varphi(t) - \varphi_1(t) = \int_{t_0}^{t} f(s, \varphi(s))\, ds - \int_{t_0}^{t} f(s, \varphi_1(s))\, ds =$$

$$= \int_{t_0}^{t} [f(s, \varphi(s)) - \varphi(s, \varphi_1(s))\, ds$$

$$\underbrace{\| \varphi(t) - \varphi_1(t) \|}_{=u(t)} = \left\| \int_{t_0}^{t} [f(s, \varphi(s)) - f(s, \varphi_1(s))]\, ds \right\| \leq$$

$$\leq \left| \int_{t_0}^{t} \| f(s, \varphi(s)) - f(s, \varphi_1(s)) \|\, ds \right.$$

- Fie $D_0 = \{ (t, \varphi(t)), (t, \varphi_1(t)); t \in I_0 = [t_0 - a, t_0 + a] \} \subset$ D compactă $\left(\begin{array}{l} \varphi(\cdot), \varphi_1(\cdot) \text{ cont} \\ [t_0 - a, t_0 + a] \text{ compact} \end{array} \right)$ imaginea unui int. compact printr-o fct. cont este ~~de~~ tot compact

$f(\cdot,\cdot)$ local Lipschitz (T), $D_0 \subset D$ compact $\Rightarrow f(\cdot)\big|_{D_0}$ este Lipschitz $\Rightarrow \exists L > 0$ a.î. $\| f(t, x_1) - f(t, x_2) \| \leq L \| x_1 - x_2 \|$, $\forall (t, x_1), (t, x_2) \in D_0$.

$$u(t) \leq \left| \int_{t_0}^{t} L \| \varphi(s) - \varphi_1(s) \|\, ds \right| = \left| \int_{t_0}^{t} L u(s)\, ds \right| \ \forall t \overset{\substack{\text{Lema} \\ B \cdot G}}{\Longrightarrow}$$

$$\left. \begin{array}{l} u(t) \leq 0, \ e^{\cdots} = 0, \ \forall t \\ u(t) \geq 0, \ \forall (t) \\ \text{(e normă)} \end{array} \right\} \Rightarrow u(t) \equiv 0 \Rightarrow \text{q.e.d}$$

Obs: $x' = x^{2/3} + a, \ a \in \mathbb{R}$

Ecuații diferențiale de ordin superior
Existența și unicitatea soluțiilor

Def a) $f(\cdot,\cdot): D \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ def. ec. dif. de n,

$$x^{(n)} = f(t, x, x', \dots x^{n-1})$$

b) $\varphi(\cdot): I \subseteq \mathbb{R} \to \mathbb{R}$ s.n. soluție a ec. dacă este de n derivabilă și $\varphi^{(n)}(t) = f(t, \varphi(t), \varphi'(t), \dots, \varphi^{(n-1)}(t)),\ \forall$

Mai general, $F(\cdot,\cdot): D \subseteq \mathbb{R}\times\mathbb{R}^{n+1} \to \mathbb{R}$,

F

$\boxed{\text{Metodă generală de studiu (sistemul canonic asociat)}}$

(1) $x^{(n)} = f(t, x, x', \dots x^{(n-1)})$

(2) $$\begin{cases} \dfrac{dx_1}{dt} = x_2 \\ \dfrac{dx_2}{dt} = x_3 \\ \dots \\ \dfrac{dx_{n-1}}{dt} = x_n \\ \dfrac{dx_n}{dt} = f(t, x_1, \dots x_n) \end{cases}$$

$$\tilde{X} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\tilde{f}(t, (x_1, \dots x_n)) = \begin{pmatrix} x_2 \\ x_3 \\ \vdots \\ f(t, \end{pmatrix}$$

(2) $\dfrac{d\tilde{X}}{dt} = \tilde{f}(t, \tilde{X})$, $\tilde{f}(\cdot,\cdot): D \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$

$\boxed{\text{PROP (de echivalență)}}$

$\varphi(\cdot): I \subseteq \mathbb{R} \to \mathbb{R}$ e soluție a ec. (1) $\Longleftrightarrow$

$\tilde{\varphi}(\cdot) = (\varphi(\cdot), \varphi'(\cdot), \dots \varphi^{(n-1)}(\cdot)): I \to \mathbb{R}^n$ e sol a ec

Dem "$\Longrightarrow$"

$$\varphi'(t) \equiv \varphi'(t)$$

$$\varphi''(t) \equiv \varphi''(t)$$

$\dots$

$$\left(\varphi^{(n-2)}(t)\right)' \equiv \varphi^{(n-1)}(t) \equiv \varphi^{(n-1)}(t)$$

$$\left(\varphi^{(n-1)}(t)\right)' \equiv f\left(t, \varphi(t), \varphi'(t), \dots \varphi^{(n-1)}(t)\right) = \varphi^{(n)}(t)$$

$\Rightarrow \varphi(\cdot)$ sol. a ec (1)

"$\Leftarrow$" Fie $\tilde{\varphi}(\varphi_1(\cdot), \varphi_2(\cdot), \dots \varphi_n(\cdot))$ sol. a ec. (2)

$$\varphi_1'(t) \equiv \varphi_2(t) \Rightarrow \varphi_2(t) = \varphi_1'(t)$$

$$\varphi_2'(t) \equiv \varphi_3(t) \Rightarrow \varphi_3(t) \equiv \varphi_2'(t) \equiv \varphi_1''(t)$$

$\dots$

$$\varphi_n'(t) \equiv \varphi_n(t) \Rightarrow \varphi_n(t) \equiv \varphi_{n-1}'(t) \equiv \left(\varphi_1^{(n-2)}(t)\right)' = \varphi_1^{(n-1)}(t)$$

$$\varphi_n'(t) \equiv f(t, \varphi_1(t), \dots \varphi_n(t))$$

$$\varphi_1^{(n)}(t) = \left(\varphi_1^{(n-1)}(t)\right)' \equiv f\left(t, \varphi_1(t), \varphi_1'(t), \dots \varphi_1^{(n-1)}(t)\right),$$

$\varphi_1(\cdot)$ sol. a ec. (1)

Problema Cauchy

Problema Cauchy pt. ec (2). Se dau $\begin{cases} \tilde{f}(\cdot,\cdot) \to \dfrac{d\tilde{x}}{dt} = \tilde{f}(t, x) \\ (t_0, \tilde{x}_0) \in D \text{ - condiție inițială} \end{cases}$

Se caută $\tilde{\varphi}(\cdot) : I \subseteq \mathbb{R} \to \mathbb{R}^n$ sol. a ec. (2) cu $\tilde{\varphi}(t_0), \tilde{x}_0$

$\tilde{x}_0 \in \mathbb{R}^n, \tilde{x}_0 = (x_0, x_0', \dots x_0^{n-1})$

Prop. de echivalență $\tilde{\varphi}(\cdot) = (\varphi(\cdot), \varphi'(\cdot), \dots \varphi^{n-1}(\cdot))$ cu $\varphi(\cdot)$ sol a ec. (1) : $x^{(n)} = f(t, x, x', \dots x^{(n-1)})$

$\Rightarrow \varphi(\cdot)$ sol. a ec (1) cu $\varphi(t_0) = x_0, \varphi'(t_0) = x_0', \dots \varphi^{(n-1)}(t_0) = x_0^{n-1}$

Problema Cauchy pt. ec (1)

Se dau $\begin{cases} f(\cdot,\cdot) \to x^{(n)} = f(t, x, x', \dots x^{(n-1)}) \\ (t_0, x_0, x_0^1, \dots x_0^{n-1}) \in D \text{ condiție inițială} \end{cases}$

Se caută $\varphi_0(\cdot): I \subseteq \mathbb{R} \to \mathbb{R}^n$ sol. a ec. cu $\varphi(t_0) = x_0$,

$\varphi'(t_0) = x_0^1, \dots \varphi^{(n-1)}(t_0) = x_0^{(n-1)}$

În acest caz spunem că $\varphi(\cdot)$ este sol. a pb. Cauchy $(f, t_0, x_0, x_0^1, \dots x_0^{(n-1)})$

T. Peano (pt. ec. de ordin superior)

Fie $f(\cdot,\cdot): D = \overset{\circ}{D} \subseteq \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ cont. def. $x^{(n)} = f(t, x, x'$ (1)

Atunci $f(\cdot,\cdot)$ admite E. L pe D $(\forall (t_0, (x_0, x_0^1, \dots x_0^{n-1}))$

$\exists \varphi(\cdot): I_0 \in \vartheta(t_0) \to \mathbb{R}$ soluție a ec. $\varphi(t_0) = x_0$, $\varphi'(t_0) = x$

$\varphi^{(n-1)}(t_0) = x_0^{n-1})$

Dem: Fie $(t_0, (x_0, x_0^1, \dots x_0^{n-1})) \in D$, $\tilde{x}_0 := (x_0, x_0^1, \dots x_0^{n-}$

$(t_0, \tilde{x}_0) \in D$

(2) $\frac{d\tilde{x}}{dt} = \tilde{f}(t, \tilde{x})$, $\tilde{x} = (x_1, \dots x_m)$, $\tilde{f}(t, (x_1, \dots x_m))$

$= (x_2, x_3, \dots x_m, f(t, x_1, \dots x_m))$

$(t_0, \tilde{x}_0) \in D = \overset{\circ}{D}$, $\tilde{f}(\cdot,\cdot)$ cont $(f(\cdot,\cdot)$ cont$)$ $\xRightarrow[\text{pt. (2)}]{\text{T. Peano}} \exists$

$I_0 \in \vartheta(t_0) \to \mathbb{R}^n$ sol. a ec. (2) cu $\tilde{\varphi}(t_0) = \tilde{x}_0$

Prop. de echivalență

$\tilde{\varphi}(\cdot) = (\varphi(\cdot), \varphi'(\cdot), \dots \varphi^{(n-1)}(\cdot))$ cu $\varphi(\cdot)$ sol. a ec (1)

$\tilde{\varphi}(t_0) = \tilde{x}_0 \Leftrightarrow \varphi(t_0) = x_0, \varphi'(t_0) = x_0^1, \dots \varphi^{(n-1)}(t_0) = x$

q.e.d

T. Cauchy-Lipschitz (pt ec. de ordin superior)

Fie $f(\cdot,\cdot) : D = \mathring{D} \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ cont, local Lipschitz (II),

$x^{(n)} = f(t, x, x', \dots x^{(n-1)})$ (1)

Atunci $\forall (t_0, (x_0, x_0^1, \dots x_0^{n-1})) \in D$ $\exists!$ $\varphi(\cdot) : I_0 \in \mathcal{V}(t_0) \to \mathbb{R}$ sol cu $\varphi(t_0) = x_0$, $\varphi'(t_0) = x_0^1$, $\dots$ $\varphi^{(n-1)}(t_0) = x_0^{n-1}$

<u>Dem</u>: Fie $(t_0, (x_0, x_0^1, \dots x_0^{n-1})) \in D$,

$$\tilde{x}_0 := (x, x_0^1, \dots x_0^{n-1})$$

(2) $\dfrac{d\tilde{x}}{dt} = \tilde{f}(t, \tilde{x})$, $\tilde{x} = (x_1, x_2, \dots x_n)$, $\tilde{f}(t, (x, \dots x_n)) =$ $(x_2, x_3, \dots x_n, f(t, x_1, \dots x_n))$

$f(\cdot,\cdot)$ cont, local Lipschitz (II) $\Rightarrow \tilde{f}(\cdot,\cdot)$ cont. local Lip (II)
$D = \mathring{D}$

$\Rightarrow$ T. Cauchy-Lipschitz pt. ec. (2)

$\Rightarrow$ pt $(t_0, \tilde{x}_0) \in D$ $\exists!$ $\tilde{\varphi}(\cdot) : I_0 \in \mathcal{V}(t_0) \to \mathbb{R}^n$ sol. a ec. (2), $\tilde{\varphi}(t_0) = \tilde{x}_0$.

Prop de echivalență $\tilde{\varphi}(\cdot) = (\varphi(\cdot), \varphi'(\cdot), \dots \varphi^{(n-1)}(\cdot))$ cu $\varphi(\cdot)$ sol. a ec. (1)

$\tilde{\varphi}(t_0) = \tilde{x}_0 \Leftrightarrow \varphi(t_0) = x_0$, $\varphi'(t_0) = x_0^1$, $\dots$ $\varphi^{(n-1)}(t_0) = x_0^{n-1}$ $\to$ q.e.d.

Continuitatea sol. locale în raport cu datele inițiale și parametrii

$\dfrac{dx}{dt} = f(t, x)$, $f(\cdot,\cdot) : D \subseteq \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$ admite EUL pe D

$\forall (\tau, \xi) \in D$ $\exists!$ $\varphi_{\tau,\xi}(\cdot) : I_{\tau,\xi} \in \mathcal{V}(\tau) \to \mathbb{R}^n$

tau csi

Def Sn. curentul local asociat câmpului vectorial $f(\cdot,\cdot)$ în $(t_0,x_0)\in D$ funcția $\alpha(\cdot,\cdot,\cdot): I_1\times I_0\times G_0 \in \mathcal{V}(t_0,t_0,x_0)\to\mathbb{R}^n$ cu prop. că $\forall(\tau,\xi)\in I_0\times G_0$

$\alpha(\cdot,\tau,\xi)$ sol a pb Cauchy (f,τ,ξ)

Def: s.n câmp vectorial parametrizat $f(\cdot,\cdot,\cdot): D\subseteq \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^n$ a.î. $\forall\lambda\in pr_3 D$ $f(\cdot,\cdot,\lambda)$ c.v

a treia componentă

def $\dfrac{dx}{dt}=f(t,x,\lambda)$

→ familie parametrizată de ec. diferențiale

Def: S.n curent local parametrizat asociat c.v.p $f(\cdot,\cdot,\cdot)$ $(t_0,x_0,\lambda_0)\in D$ funcția $\alpha(\cdot,\cdot,\cdot,\cdot): I_1\times I_0\times G_0\times\Lambda_0$ $\in\mathcal{V}(t_0,t_0,x_0,\lambda_0)$ a.î. $\forall(\tau,\xi,\lambda)\in I_0\times G_0\times\Lambda_0$,

$\alpha(\cdot,\tau,\xi,\lambda): I_0\to\mathbb{R}^n$ e sol. Cauchy $(f(\cdot,\cdot,\lambda),\tau,$

T. (E.U și continuitatea curentului local parametrizat)

Fie $f(\cdot,\cdot,\cdot): D=\mathring{D}\subseteq\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^k\to\mathbb{R}^n$ cont, local Lipschit

(I) $\dfrac{dx}{dt}=f(t,x,\lambda)$. Atunci $\forall(t_0,x_0,\lambda_0)\in D$ $\exists!$

$\alpha(\cdot,\cdot,\cdot,\cdot): I_1\times I_0\times G_0\times\Lambda_0\in\mathcal{V}(t_0,t_0,x_0,\lambda_0)\to\mathbb{R}^n$ continuă. curent local parametrizat

Dem (Schiță): Fie $(t_0,x_0,\lambda_0)\in D$.

1. $E := I_1\times I_0\times G_0\times\Lambda_0 = ?$

$(t_0,x_0,\lambda_0)\in\mathring{D}\Rightarrow$

$\exists\, \vartheta, \gamma, \eta > 0 \quad \overline{B_\vartheta (t_0)} \times \overline{B_\gamma (x_0)} \times \overline{B_\eta (\lambda_0)} \in D$

$\overset{\|}{}$ D_0 compactă

$K := \max\limits_{(t, x, \lambda) \in D_0} \| f(t, x, \lambda) \|$

$K = 0 \Rightarrow \alpha(t, \vartheta, \xi, \lambda) = \xi$

$K > 0 \quad a := \min \left\{ \vartheta, \dfrac{\gamma}{4K} \right\} \quad r := \dfrac{\gamma}{2}$

$E = \overline{B_a(t_0)} \times \overline{B_\vartheta(t_0)} \times \overline{B_r(x_0)} \times \overline{B_\eta(\lambda_0)}$

$\left. \begin{array}{l} f(\cdot,\cdot,\cdot) \text{ local Lipschitz (II)} \\ \quad D_0 - \text{compactă} \end{array} \right\} \Rightarrow \exists\, L > 0$ a.î.

$\| f(t, x_0, \lambda) - f(t, x_2, \lambda) \| \le L \| x_1 - x_2 \|, \ \forall (t, x_1, \lambda), (t, x_2, \lambda) \in D_0$

2. $\alpha_m(\cdot,\cdot,\cdot,\cdot) : E \to \mathbb{R}^n$ şirul aproximaţiilor succesive ale lui Picard

$\alpha_0(t, \sigma, \xi, \lambda) = \xi$

$\alpha_m(t, \sigma, \xi) = \xi + \int_\sigma^t f(s, \alpha_m(s, \vartheta, \xi, \lambda), \lambda)\, ds,$

$m \geq 1$

a) $\alpha_m(\cdot,\cdot,\cdot,\cdot)$ continuă $\forall\, m$ (ind. după m)

b) $\alpha_m(t, \sigma, \xi, \lambda) \in \overline{B_\gamma(x_0)}, \ \forall\, m, \ \forall (t, \vartheta, \xi, \lambda) \in E$

c) $\alpha_m(\cdot,\cdot,\cdot,\cdot)$ şir uniform Cauchy $\Rightarrow \alpha_m(\cdot,\cdot,\cdot,\cdot) \xrightarrow[n \to \infty]{u} \alpha(\cdot,\cdot,\cdot,\cdot)$

$\alpha(\cdot,\cdot,\cdot,\cdot) : E \to \mathbb{R}^n$ cont

3. $\alpha(\cdot,\cdot,\cdot,\cdot)$ curent local param.

4. Unicitatea ($\sim$ dem T. Cauchy-Lipschitz)

Seminar 4

$\boxed{\text{Ec. de ordin superior care admit reducerea ordinului}}$

1) $F(t, x^{(k)}, x^{(k+1)}, \dots x^{(n)}) = 0,\ k \geq 1$ → Nu apare x, doar derivatele sale.

SV $y = x^{(k)} \to F(t, y, y', \dots y^{(n-k)}) = 0$.

2) $F(t, \frac{x'}{x}, \frac{x''}{x}, \dots \frac{x^{(n)}}{x}) = 0$ → Ec. omogenă

SV $y = \frac{x'}{x} \Rightarrow G(t, y, y', \dots y^{(n-1)}) = 0$

3) $F(x, x', \dots x^{(n)}) = 0$ (Autonomă)

SV $x' = y(x) \Rightarrow G(x, y, y', \dots y^{(n-1)}) = 0$.

4) Ec. Euler

$F(x, t \cdot x', t^2 x'', \dots t^n x^{(n)}) = 0$

S.V $|t| = e^s \Rightarrow G(y, y', \dots y^{(n)}) = 0$.

Ecuații liniare de ordinul al doilea cu coef. constanți

$x'' + a \cdot x' + b = 0,\ a, b \in \mathbb{R}$

Ec. caracteristică: $\lambda^2 + a \cdot \lambda + b = 0 < \begin{matrix} \lambda_1 \\ \lambda_2 \end{matrix}$

Dacă $\lambda_1, \lambda_2 \in \mathbb{R},\ \lambda_1 \neq \lambda_2 \Rightarrow$ sol generală este

$$x(t) = c_1 \cdot e^{\lambda_1 t} + c_2 \cdot e^{\lambda_2 t},\ c1, c2 \in \mathbb{R}$$

Dacă $\lambda_1 = \lambda_2 = \lambda \in \mathbb{R} \Rightarrow$ sol. generală $x(t) = C_1 \cdot e^{\lambda t} + C_2 \cdot e^{\lambda t} \cdot t$

Dacă $\lambda_1 = \alpha + i \cdot \beta$
$\lambda_2 = \alpha - i \cdot \beta,\ \beta > 0$
sol. generală $\Rightarrow x(t) = C_1 \cdot e^{\alpha t} \cos \beta t + C_2 \cdot e^{\alpha t} \sin \beta t$

Ex Să se determine sol. generală

1) $x''' + x'' = 0$

SV: $y = x'' \Rightarrow y' = x'''$

$$y' + y = 0 \Rightarrow y' = -y \Rightarrow y(t) = c \cdot e^{-t}, \ c \in \mathbb{R}$$

$x'' = c \cdot e^{-t}$

$$x'(t) = \int c \cdot e^{-t} dt = -c \cdot e^{-t} + c_1, \ c_1 \in \mathbb{R}$$

$$x(t) = \int (-c \cdot e^{-t} + c_1) dt = c \cdot e^{-t} + c_1 \cdot t + c_2,$$

$c, c_1, c_2 \in \mathbb{R}$

b) $t \cdot x'' + x' + t = 0$

c) $t \cdot x \cdot x'' + t \cdot (x')^2 - x \cdot x' = 0 \quad | x^2$

$x(t) \equiv 0$ verifică ec.
Căutăm sol care sunt diferite

$$\frac{t \cdot \cancel{x} \cdot x''}{x^{\cancel{2}}} + \frac{t \cdot (x')^2}{x^2} - \frac{\cancel{x} \cdot x'}{x^{\cancel{2}}} = 0$$

$$\frac{t x''}{x} + t\left(\frac{x'}{x}\right)^2 - \frac{x'}{x} = 0 \quad \text{ec. omogenă}$$

$$y = \frac{x'}{x} \Rightarrow y(t) = \frac{x'(t)}{x(t)}$$

$$x'(t) = y(t) \cdot x(t)$$

$$x''(t) = y'(t) \cdot x(t) + y(t) \cdot x'(t)$$

$$\Leftrightarrow \frac{x''(t)}{x(t)} = y'(t) + \frac{(x'(t))^2}{x^2(t)} \Leftrightarrow \frac{x''(t)}{x(t)} = y'(t) + y^2(t)$$

$$t(y' + y^2) + t \cdot y^2 - y = 0$$

$$y' = \frac{-2ty^2 + y}{t}, \ t \neq 0$$

$$\frac{dy}{dt} = \frac{y}{t} - 2y^2$$

Ec. liniară asoc. $\bar{y}' = \frac{\bar{y}}{t} \Rightarrow \bar{y}(t) = c \cdot e^{\int \frac{1}{t} dt} = c \cdot t$

„Variația constantelor"

$(c(t)\cdot t)' = \frac{c(t)\cdot \cancel{t}}{\cancel{t}} - 2c^2(t)\cdot t^2$

$c'(t)\cdot t + \cancel{c(t)} = \cancel{c(t)} - 2c^2(t)\cdot t^2$

$c'(t) = -2c^2(t) \rightarrow$ ec. var. separabile.

$\frac{dc}{dt} = -2c^2\cdot t \Rightarrow c^2 = 0 \Rightarrow c = 0 \Rightarrow c(t) \equiv 0$

Sol. staționară

$-\frac{dc}{c^2} = 2t\, dt \Rightarrow \int -\frac{dc}{c^2} = \int 2t\, dt \Leftrightarrow$

$\Leftrightarrow \frac{1}{c} = t^2 + k,\ k\in\mathbb{R} \Rightarrow c(t) = \frac{1}{t^2+k},\ k\in\mathbb{R}.$

$y_0(t) = 0$

$y_k(t) = \frac{t}{t^2+k},\ k\in\mathbb{R}$

$y = \frac{x'}{x} \Rightarrow \frac{x'}{x} = 0 \Rightarrow x' = 0 \Rightarrow x_0(t) = c_1,\ c_1\in\mathbb{R}.$

$\frac{x'}{x} = \frac{t}{t^2+k} \Rightarrow x' = \frac{t}{t^2+k}\cdot k. \rightarrow$ Ec. liniară cu sol

$x(t) = c_2\cdot e^{\int\frac{t}{t^2+k}dt} = c_2\cdot e^{\frac{1}{2}\int\frac{2t}{t^2+k}dt} =$

$= c_2 e^{\frac{1}{2}\ln|t^2+k|} = c_2\sqrt{t^2+k},\ c_1, c_2, k\in\mathbb{R}.$

d) $t^2 x x'' - (x - t\cdot x')^2 = 0.$

e) $x x'' + 1 = (x')^2$

$x' = y(x)$

Se caută o fct. y a.î. $x'(t) = y(x(t)),\ \forall t \in$ domeniul lui x

$x''(t) = y'(x(t))\cdot x'(t) = y'(x(t))\cdot y(x(t))$

$x \cdot y'(x) \cdot y(x) + 1 = y^2(x)$

$y'(x) = (y^2(x) - 1) \cdot \frac{1}{x \cdot y(x)}$

$\frac{dy}{dx} = \frac{y^2-1}{x \cdot y} \Rightarrow \frac{y^2-1}{y} = 0 \Leftrightarrow y^2 = 1 \Rightarrow y = \pm 1$

$y_1(x) \equiv 1, \; y_2(x) \equiv -1$

$\frac{y}{y^2-1} dy = \frac{dx}{x} \Leftrightarrow \frac{1}{2}\int \frac{2y}{y^2-1} dy = \int \frac{dx}{x} \Leftrightarrow$

$\frac{1}{2} \ln|y^2-1| = \ln|x| + \underset{\ln k, \; k \in \mathbb{R}_+^*}{\overset{\parallel}{c}}, \; c \in \mathbb{R}$

$\Leftrightarrow \sqrt{|y^2-1|} = k \cdot |x|, \; k \in \mathbb{R}_+^*$

$\Leftrightarrow |y^2-1| = k^2 \cdot x^2 \Rightarrow y^2 - 1 = c_1 x^2, \; c_1 \in \mathbb{R}^*$

$\Rightarrow y(x) = \pm \sqrt{c_1 x^2 + 1}, \; c_1 \in \mathbb{R}^*$

$x' = 1 \Rightarrow x_1(t) \equiv t + c_2, \; c_2 \in \mathbb{R}$

$x' = -1 \Rightarrow x_2(t) \equiv -t + c_3, \; c_3 \in \mathbb{R}$

$x' = \pm\sqrt{c_1 x^2 + 1} \rightarrow$ Ec. var. separabilă

f) $x \cdot x'' + (x')^2 = 0$

g) $t^2 x'' - 4t x' + 6x = 0$

Ec. Euler

S.V: $|t| = e^s \Rightarrow t = e^s, \; t > 0$

$t = -e^s, \; t < 0$

Pp. $t > 0$: $t = e^s$

$t^2 \cdot x''(t) - 4t x'(t) + 6x(t) = 0$

$e^{2s} x''(e^s) - 4e^s x'(e^s) + 6x(e^s) = 0$

$$\boxed{y(s) = x(e^s) \iff x(t) = y(\ln t)}$$

$x(t) \neq x(e^s) \Rightarrow$ ambiguitate

$$e^{2s}\, x''(e^s) - 4e^{2}\, x'(e^s) + 6\, x(e^s) = 0.$$

$$y'(s) = x'(e^s)\cdot e^s \Rightarrow x'(e^s) = \frac{y'(s)}{e^s} = y'(s)\cdot e^{-s}$$

$$x''(e^s) = y''(s)\cdot e^{-s} - y'(s)\cdot e^{-s}$$

$$x''(e^s) = y''(s)\cdot e^{-2s} - y'(s)\cdot e^{-2s}$$

$$\cancel{e^{2s}}\left(y''(s)\cdot \cancel{e^{-2s}} - y'(s)\cdot \cancel{e^{-2s}}\right) - 4\cdot \cancel{e^{s}}\cdot y'(s)\cdot \cancel{e^{-s}} + 6\cdot y(s) =$$

$$y''(s) - 5y'(s) + 6y(s) = 0.$$

$$\lambda^2 - 5\lambda + 6 = 0.$$

$$\Delta = 25 - 24 = 1.$$

$$\lambda_1 = \frac{5-1}{2} = 2,\quad \lambda_2 = \frac{5+1}{2} = 3.$$

$$y(t) = c_1\cdot e^{2t} + c_2\cdot e^{3t},\quad c_1, c_2 \in \mathbb{R}.$$

$$x(t) = y(\ln t) \Rightarrow x(t) = c_1\cdot e^{2\ln t} + c_2\cdot e^{3\ln t}$$

$$\Rightarrow x(t) = c_1\cdot t^2 + c_2\cdot t^3,\quad c_1, c_2 \in \mathbb{R}$$

h) $t^2 x'' + t\cdot x' + x = 0.$
Ec. Euler

$|t| = e^s$, pp. $t < 0$

SV $t = -e^s$

$$t^2\cdot x''(t) + t\cdot x'(t) + x(t) = 0.$$

$$e^{2s}\cdot x''(-e^s) - e^s\, x'(-e^s) + x(-e^s) = 0.$$

$$\Rightarrow x'(-e^{s}) = \frac{-y'(s)}{e^{s}} \Rightarrow x'(-e^{s}) = -e^{-s}\cdot y'(s)$$

$$x''(-e^{s}) = (-e^{-s}\cdot y'(s))'$$

$$\Leftrightarrow x''(-e^{s}) = (-e^{-2s}\cdot y''(s) + (e^{-s}\cdot y'(s))\cdot e^{-s}$$

$$\Leftrightarrow x''(s) = -y''(s)\cdot e^{-2s} + y'(s)\cdot e^{-2s}$$

$$\cancel{e^{2s}}\cdot(-y''(s)\cdot \cancel{e^{-2s}} + y'(s)\cdot \cancel{e^{-2s}}) - \cancel{e^{s}}\cdot(-\cancel{e^{-s}}\cdot y'(s)) + y(s) = 0$$

$\Leftrightarrow$ $-$ [illegible]

$$y^2 + 1 = 0$$

$$\lambda_{1,2} = \pm i$$ [illegible]

[illegible]

$$\Rightarrow y(s) = C_1\cdot e^{0\cdot s}\cdot\cos 1 s + C_2\cdot e^{0\cdot s}\sin 1\cdot s$$

$$y(s) = C_1\cos s + C_2\cdot \sin s,\quad C_1, C_2 \in \mathbb{R}$$

$$x(t) = C_1\cdot\cos(\ln t) + C_2\cdot\sin(\ln t),\quad C_1, C_2 \in \mathbb{R}$$

i) $t^2 x'' + 5t\cdot x' - 5x = 0$

j) $t^2 x'' - t\cdot x' - 2x = 0$