unde $v^* = c - A^T w^*$, $u^* = Ax^* - b$

Curs 8 - Exerciții pt. examen 08 Aprilie 2019

① Teorema ecartului complementare

(1) $\inf c^T x$ duala (2) $\sup b^T w$

$Ax \geq b$ $A^T w \leq c$

$x \geq 0$ $w \geq 0$

TEC : x^* e sol. optimă pt. (1) și w^* e sol opt. pt. (2)

$$\Leftrightarrow \begin{cases} (x^*)^T (\overbrace{c - A^T w^*}^{v^*}) = 0 \\ (w^*)^T (\underbrace{Ax^* - b}_{u^*}) = 0 \end{cases} \Leftrightarrow \begin{cases} (x^*)^T v^* = 0 \\ (w^*)^T u^* = 0 \end{cases} \Leftrightarrow \begin{cases} x_j^* v_j^* = 0, \forall j = \overline{1,n} \\ w_i^* u_i^* = 0, \forall i = \overline{1,m} \end{cases}$$

Obs : Dacă $x_j^* > 0 \Rightarrow v_j^* \leq 0$

$v_j^* > 0 \Rightarrow x_j^* = 0$

$w_i^* > 0 \Rightarrow u_i^* = 0$

$u_i > 0 \Rightarrow w_i^* = 0$

Ex 1 $\max x_1 + 2x_2 + 3x_3 + 4x_4$

$x_1 + 2x_2 + 2x_3 + 3x_4 \leq 20 \to w_1$

$2x_1 + x_2 + 3x_3 + 2x_4 \leq 20 \to w_2$

$x_1, \dots, x_4 \geq 0$

duala: $\min 20w_1 + 20w_2$

$w_1 + 2w_2 \geq 1$

$2w_1 + w_2 \geq 2$ → rez. grafic

$2w_1 + 3w_2 \geq 3$ sol. opt $w^* = (1,2 ; 0,2)^T$

$3w_1 + 2w_2 \geq 4$

$w_1, w_2 \geq 0$

Obs : $w_1^* = 1,2 > 0 \Rightarrow$ ec. 1) din pr. primală e satisf. cu =

$w_2^* = 0,2 > 0 \Rightarrow$ ec. 2) — " —

Ecarturile pr. 2

$v_1^* = w_1^* + 2w_2^* - 1 = 1{,}2 + 2 \cdot 0{,}2 - 1 > 0$

$v_2^* = 2w_1^* + w_2^* - 2 = 2 \cdot 1{,}2 + 0{,}2 - 2 > 0$

$v_3^* = 2w_1^* + 3w_2^* - 3 = 2 \cdot 1{,}2 + 3 \cdot 0{,}2 - 3 = 0$

$v_4^* = 3w_1^* + 2w_2^* - 4 = 3 \cdot 1{,}2 + 2 \cdot 0{,}2 - 4 = 0$

(*) Adică $\begin{cases} u_1^* = 20 - (x_1^* + 2x_2^* + 2x_3^* + 3x_4^*) = 0 \\ u_2^* = 20 - (2x_1^* + x_2^* + 3x_3^* + 2x_4^*) = 0 \end{cases}$

$v_1^* > 0 \Rightarrow x_1^* = 0$

$v_2^* > 0 \Rightarrow x_2^* = 0$

$v_3^* = 0$

$v_4^* = 0$

Rez. sist. $\begin{cases} 20 - (x_1^* + 2x_2^* + 2x_3^* + 3x_4^*) = 0 \\ 20 - (2x_1^* + x_2^* + 3x_3^* + 2x_4^*) = 0 \\ x_1^* = 0,\ x_2^* = 0 \end{cases}$

$\Rightarrow \begin{cases} 2x_3^* + 3x_4^* = 20 \\ 3x_3^* + 2x_4^* = 20 \end{cases} \Rightarrow x_3^* = 4,\ x_4^* = 4$

$x^* = (0, 0, 4, 4)^T$ sol. opt. pt. pr. 1)

Tabelul simplex

$\inf c^T x$

$Ax = b$

$x \geq 0$

B baza primal admisibilă

$B^{-1} b \geq 0$, B, R

$x = (B^{-1} b, \theta)$

$\Delta_j = c_j - c_B^T B^{-1} b = 0$, $j \in B$

≥ 0, $j \in R$

V.B	V.V.B	$\overset{c_1}{y^1}$	$\overset{c_2}{y^2}$ ...	$\overset{c_n}{y^n}$
$\bar{x}^B$	$B^{-1}b$	$Y = B^{-1}A$		
	z	r_1	r_2 ---	r_n

Ex: $\inf \; x_1 + 2x_2 + x_3$

$x_1 + x_2 + x_3 = 5$

$x_2 - 2x_3 = 2$

$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix}$

$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

$B^{-1}b = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$r_1 = r_2 = 0 \qquad r_3 = 1 - (1 \;\; 2)\begin{pmatrix} 3 \\ 2 \end{pmatrix} = -6$

val. var. bazei

var. bazei ← VB

VB	VVB	y^1	y^2	y^3
x_1	3	1	0	3
x_2	2	0	1	-2
	$z = 7$	$r_1 = 0$	$r_2 = 0$	$r_3 = -6$

$x_1 + x_2 = 5 - x_3$

$x_2 = 2 + 2x_3$

$Y = B^{-1}A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} - \begin{pmatrix} x_3 \\ -2x_2 \end{pmatrix}$

$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = B^{-1}\begin{pmatrix} 5 \\ 2 \end{pmatrix} - B^{-1}\begin{pmatrix} 1 \\ -2 \end{pmatrix} x_3$

$z = x_1 + 2x_2 + x_3 = 3 \cdot 1 + 2 \cdot 2 + 1 \cdot 0 = 7$

$r_3 = -6 < 0$ schimbăm baza - intră x_3

- iese $\min\left\{\frac{5}{3}\right\} \Rightarrow$ iese x_1

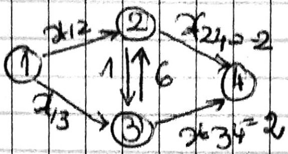
$\tilde{B} = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$ $\qquad B^{-1} = \frac{1}{-3}\begin{pmatrix} -2 & -1 \\ -1 & -1 \end{pmatrix}$

$x_1 = 0 \qquad \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = B^{-1}\begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad \cancel{\tilde{B}^T = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}} \qquad \tilde{B}^T = \begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix}$

$\begin{pmatrix} x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} 3 \\ 2 \end{pmatrix} =$

__Aplicatie__ : Problema fluxului de cost minimum

__Ex 1__ :



Grafuri orientate : $\Gamma = \{N, A\}$ → N - noduri

A - arce

15. Aprilie 2019

Curs 9

Optimizare în rețele

Graf orientat $G = (N, A)$, $|N| = n$

flux $x = (x_{ij})$ $i, j \in N$

$y = (y_i)$ $i \in N$, $y_i = \sum_{\{j \in N \mid (i,j) \in A\}} x_{ij} - \sum_{\{j \in N \mid (j,i) \in A\}} x_{ji}$

conservarea fluxului : $y_i = 0$, $\forall i \in N$, $i \neq 1, n$

pp. 1 sursă, n destinație

$0 \le x_{ij} \le k_{ij}$, $\forall (i,j) \in A$

k_{ij} - capacitatea arcului (i,j)

$f = \sum_{j \in N} x_{1j} - \sum_{j \in N} x_{j1}$ = valoarea fluxului x

$-f = \sum_{j \in N} x_{nj} - \sum_{j \in N} x_{jn}$

__Obs__ 1) $x_{ji} = -x_{ij}$, $\forall i, j \in N$

antisimetrie

2) $x_{ij} = 0$ dc $(i, j) \notin A$