26 02. 2018

Analiză

- Seminar -

Ex 1: ~~Fie (a,b)~~ Fie $a, b \in (0, \infty)$ a.î. $a^x + b^x \geq 2$; $\forall x \in \mathbb{R}$. Să se arate că $a \cdot b = 1$

Fie $f : \mathbb{R} \to \mathbb{R}$, $f(x) = a^x + b^x - 2$

$a^x + b^x \geq 2$; $\forall x \in \mathbb{R} \Leftrightarrow f(x) \geq 0$; $\forall x \in \mathbb{R} \Rightarrow f(x) \geq f(0)$; $\forall x \in \mathbb{R} \Rightarrow$

$f(0) = 0$

$\Leftrightarrow$ 0 este pct de ~~extrem~~ minim global

$\mathring{\mathbb{R}} = \mathbb{R} \Leftrightarrow 0 \in \mathring{\mathbb{R}}$

f derivabilă pe $\mathbb{R}$ (op. cu fcț derivabile $\Rightarrow f$ dvb în 0

~~conform Th~~ Cf. Th. Fermat $\Rightarrow f'(0) = 0$

$f'(x) = a^x \ln a + b^x \ln b$; $\forall x \in \mathbb{R} \Rightarrow f'(0) = \ln a + \ln b = \ln(a \cdot b)$ $\Big\} \Rightarrow$

$\Rightarrow \ln(ab) = 0 \Rightarrow ab = 1$

Ex 2: Fie $f : (0, +\infty) \to \mathbb{R}$ o fcț dvb pe $(0, +\infty)$ a.î.

$\exists \lim\limits_{x \to \infty} f(x) \in \mathbb{R}$, $\exists \lim\limits_{x \to \infty} f'(x)$. Să se arate $\lim\limits_{x \to \infty} f'(x) = 0$

Fie $\lim\limits_{x \to \infty} f(x) = l_1 \in \mathbb{R}$

$\lim\limits_{x \to \infty} f'(x) = l_2$

Fie $x_n = n$, $y_n = n+1$ $\Big|$ $\lim\limits_{n \to \infty} x_n = \lim\limits_{n \to \infty} y_n = +\infty \Rightarrow \lim\limits_{n \to \infty} f(x_n) = \lim\limits_{n \to \infty} f(y_n) = l_1$

$\lim\limits_{n \to \infty} f(y_n) - f(x_n) = l_1 - l_1 = 0 \Rightarrow \lim\limits_{n \to \infty} (f(n+1) - f(n)) = 0$

P.S. /

Fie $f: [n, n+1] \to \mathbb{R}$

$$\left.\begin{array}{l} f \text{ cont pe } [n, n+1] \\ f \text{ deriv pe } (n, n+1) \end{array}\right\} \overset{\text{cf Th Lagr}}{\Longrightarrow} \exists\, c \in (n, n+1) \text{ a.î. } \cancel{f'(c)=0}$$

$$f'(c_n) = \frac{f(n+1) - f(n)}{n+1-n} = \frac{f(n+1) - f(n)}{1}$$

$$\Rightarrow f'(c_n) = f(n+1) - f(n), \ \forall n \in \mathbb{N} \Rightarrow$$

$$\Rightarrow \exists \lim_{n \to \infty} f'(c_n) = 0$$

$$\left.\begin{array}{l} c_n \in (n, n+1) \Rightarrow c_n > n \Rightarrow \lim\limits_{n\to\infty} c_n = \infty; \ \forall n \in \mathbb{N}, n \geq n_0 \\ \lim\limits_{x \to \infty} f'(x) = l_2 \end{array}\right\} \Rightarrow$$

$$\Rightarrow \lim_{n \to \infty} f'(c_n) = l_2$$

$$\lim_{n \to \infty} f'(c_n) = 0$$

$$\Rightarrow l_2 = 0 \Rightarrow \lim_{x \to \infty} f'(x) = 0 \quad \text{qed}$$

Ex 3: Determinați punctele de extrem local ale funcției

$$f: \mathbb{R} \to \mathbb{R}, \ f(x) = \begin{cases} \frac{1}{x} e^{\frac{1}{x}} ; & x < 0 \\ -x ; & x \geq 0 \end{cases}$$

f cont pe $\mathbb{R}^*$ (comp de fct deriv)

$$\lim_{\substack{x \to 0 \\ x > 0}} f(x) = \lim_{\substack{x \to 0 \\ x > 0}} -x = 0 \quad (1)$$

$$\lim_{\substack{x \to 0 \\ x < 0}} f(x) = \lim_{\substack{x \to 0 \\ x < 0}} \frac{1}{x} e^{\frac{1}{x}} \overset{y = \frac{1}{x}}{=} \lim_{y \to -\infty} y e^{y} \overset{z = -y}{=} \lim_{z \to \infty} (-z) e^{-z} = \lim_{z\to\infty} -z \cdot \frac{1}{e^z} =$$

$$= -\lim_{z \to \infty} \frac{z}{e^z} \overset{(L'H)}{=} -\lim_{z \to \infty} \frac{1}{e^z} = -0 = 0 \quad (2)$$

$f(0) = 0$ (3)

(1), (2), (3) $\Rightarrow f$ cont în 0 $\Rightarrow f$ cont pe $\mathbb{R}$

$$\lim_{\substack{x\to 0\\ x<0}} \frac{f(x)-f(0)}{x} = \lim_{\substack{x\to 0\\ x<0}} \frac{\frac{1}{x}e^{\frac{1}{x}}}{x} = \lim_{\substack{x\to 0\\ x<0}} \frac{1}{x^2} e^{\frac{1}{x}} \overset{y=\frac{1}{x}}{=} \lim_{y\to -\infty} y^2 e^{y} \quad z=-y$$

$$= \lim_{z\to+\infty} (-z)^2 e^{-z} = \lim_{z\to+\infty} z^2 \cdot \frac{1}{e^z} = \lim_{z\to\infty} \frac{2z}{e^z} = \lim_{z\to\infty} \frac{2}{e^z} = 0$$

$$\lim_{\substack{x\to 0\\ x>0}} \frac{f(x)-f(0)}{x-0} = \lim_{\substack{x\to 0\\ x>0}} \frac{-x-0}{x} = -1 \neq 0$$

$\Rightarrow$ f nu este deriv în 0

$$f'(x) = \begin{cases} \left(\frac{1}{x}e^{\frac{1}{x}}\right)'; & x<0 \\ (-x)'; & x>0 \end{cases} = \begin{cases} \underbrace{-\frac{1}{x^2}e^{\frac{1}{x}}\left(1+\frac{1}{x}\right)}_{<0}; & x<0 \\ -1; & x>0 \end{cases}$$

$$\left(\frac{1}{x}e^{\frac{1}{x}}\right)' = -\frac{1}{x^2}e^{\frac{1}{x}} + \frac{1}{x}\cdot e^{\frac{1}{x}} \cdot \ln(e)\cdot\left(\frac{1}{x}\right)' = -\frac{1}{x^2}e^{\frac{1}{x}} - \frac{1}{x^3}e^{\frac{1}{x}} =$$

$$= -\frac{1}{x^2}e^{\frac{1}{x}}\left(1+\frac{1}{x}\right)$$

I $x<0$

$f'(x)=0 \Rightarrow 1+\frac{1}{x}=0 \Rightarrow \frac{1}{x}=-1 \Rightarrow x=-1$

II $x>0$

$f'(x)=0 \Rightarrow -1=0$ fals.

$\Rightarrow$ $f'(x)=0 \Leftrightarrow x=-1$

x	$-\infty$	-1	0	$+\infty$
$f'(x)$	$- - - - -$	$0 + + +$	$\vert$	$- - - - - -$
$f(x)$	$0 \searrow$	$-\frac{1}{e} \nearrow$	$0 \searrow$	$-\infty$

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{1}{x} = 0$

$\lim_{x\to\infty} f(x) = -\infty$

Exe 4 S. se a. c. ecuația $x^3 - 3x + 1 = 0$ are soluție reală unică în intervalul $[0, \frac{1}{2}]$. Aproximați soluția reală cu eroare 10^{-2}.

$x^3 + 1 = 3x \Leftrightarrow \frac{x^3+1}{3} = x$

$f : [0, \frac{1}{2}] \to \mathbb{R}, \; f(x) = \frac{x^3+1}{3}$

$f'(x) = \frac{1}{3} \cdot 3x^2 = x^2 \; ; \; x \in [0, \frac{1}{2}]$

$|f'(x)| = |x^2| = x^2 \leq \frac{1}{4} \; ; \; \forall x \in [0, \frac{1}{2}]$

$M = \frac{1}{4} < 1 \Rightarrow f$ contracție pe $[0, \frac{1}{2}] \Rightarrow \exists!\, u \in [0, \frac{1}{2}]$ a.î. $f(u) = u$

Construim u

$x_0 = 0$

$x_{n+1} = f(x_n), \forall n \in \mathbb{N}$

$x_{n+1} = \frac{x_n^3 + 1}{3}, \forall n \in \mathbb{N}$

$\lim_{n \to \infty} x_n = u$

$x_1 = \frac{x_0^3 + 1}{3} = \frac{1}{3}$

$|x_n - u| \leq \frac{M^n}{1-M} |x_1 - x_0|, \forall n \in \mathbb{N}$

$|x_n - u| \leq \frac{(\frac{1}{4})^n}{1 - \frac{1}{4}} \left| \frac{1}{3} - 0 \right|, \forall n \in \mathbb{N}$

$|x_n - u| \leq 4 (\frac{1}{4})^n \cdot \frac{1}{9}, \forall n \in \mathbb{N}$

$|x_n - u| \leq (\frac{1}{4})^{n-1} \cdot \frac{1}{9} \; ; \; \forall n \in \mathbb{N}$

$\left(\frac{1}{4}\right)^{n-1} \frac{1}{9} < 10^{-2} \Leftrightarrow 4^{n-1} \cdot 9 > 100 \Rightarrow n \geq 3 \Rightarrow x_n \simeq u$ cu eroare de 10^{-2}, $\forall n \geq 3$

~~x_3~~ $x_3 \simeq u$

~~$x_3 = f($~~

~~$x_2 = f$~~

$x_3 = (f \circ f \circ f)(x_0) = (f \circ f \circ f)(0) = \cancel{\frac{28}{81}} = \cancel{0{,}34567901223}$

$= 0{,}3380085869$

Ex 5 Fie $f: [0,2] \to \mathbb{R}$ o fct cont pe $[0,2]$ derivabilă de două ori cu $f(0)=0$, $f(1)=1$, $f(2)=2$

Arătați că $\exists c \in (0,2)$ a.î. $f''(c)=0$

$$\left.\begin{array}{l} f: [0,1] \to \mathbb{R} \\ f \text{ continuă pe } [0,1] \\ f \text{ derivabilă pe } (0,1) \end{array}\right\} \overset{\text{Th Lagrange}}{\Longrightarrow} \exists c_1 \in (0,1) \text{ a.î. } f'(c_1) = \frac{f(1)-f(0)}{1-0} = 1$$

$$\left.\begin{array}{l} f: [1,2] \to \mathbb{R} \\ f \text{ cont pe } [1,2] \\ f \text{ dvb. pe } (1,2) \end{array}\right\} \overset{\text{Th Lagr.}}{\Longrightarrow} \exists c_2 \in (1,2) \text{ a.î. } f'(c_2) = \frac{f(2)-f(1)}{2-1} = 1$$

$\Rightarrow 0 < c_1 < 1 < c_2 < 2$

$$\left.\begin{array}{l} f': [c_1, c_2] \to \mathbb{R} \\ f' \text{ cont pe } [c_1, c_2] \\ f' \text{ dvb pe } (c_1, c_2) \\ f'(c_1) = f'(c_2) \end{array}\right\} \overset{\text{Th Rolle}}{\Longrightarrow} \exists c \in (c_1, c_2) \text{ a.î. } (f')'(c) = 0$$

$\Rightarrow \exists c \in (c_1, c_2)$ a.î. $f''(c) = 0$

$(c_1, c_2) \subseteq (0,2)$

$\Rightarrow \exists c \in (0,2)$ a.î. $f''(c) = 0$

12.03.2018

Analiză

- Seminar -

Funcții derivabile

1. Fie $f:(0,+\infty)\to\mathbb{R}$ o fct derivabilă a.î. $\exists$

$\exists \lim_{x\to\infty}(f(x)+xf'(x))=l\in\overline{\mathbb{R}}$. Să se justifice afirmația:

$\exists \lim_{x\to\infty} f(x)=l$

$f(x)+xf'(x)=\dfrac{(x\,f(x))'}{x'}$

Alegem $g,h:(0,+\infty)\to\mathbb{R}$

$g(x)=x\,f(x)$

$h(x)=x$

g,h funcții derivabile

$\lim_{x\to\infty} h(x)=+\infty$

$h'(x)=1\neq 0,\ \forall x\in(0,\infty)$

$\lim_{x\to\infty}\dfrac{g'(x)}{h'(x)}=l \overset{L'H}{\Longrightarrow} \exists \lim_{x\to\infty}\dfrac{g(x)}{h(x)}=\lim_{x\to\infty}\dfrac{g'(x)}{h'(x)}=l \Leftrightarrow$

$\Rightarrow \exists \lim_{x\to\infty}\dfrac{xf(x)}{x}=\lim_{x\to\infty} f(x)=l$ qed

Dacă am avea $l\in\mathbb{R}$, atunci $\lim_{x\to\infty} x f'(x)=\lim_{x\to\infty}(f(x)+xf'(x)-f(x))=$

$=\lim_{x\to\infty}(f'(x)\cdot x+f(x))-\lim_{x\to\infty}(f(x))=l-l=0$

2. Fie $f: \mathbb{R} \to \mathbb{R}$ o funcție derivabilă de 2 ori ~~fie~~ Calc

cu $f(x) \geq 0$; $\forall x \in \mathbb{R}$ și ~~cont~~ $f''(x) \leq 0$. Sarc f este fcț constantă.

~~Dem~~

Pp prin abs. că f nu este fcț ~~cont cos~~ constantă $\Rightarrow$

$\Rightarrow \exists x_0 \in \mathbb{R}$ a.î. $f'(x_0) \neq 0$

$\forall x \in \mathbb{R}, x \neq x_0, \exists c \in \mathbb{R}$ ~~situat~~ situat între x și x_0

a.î.
$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(c)}{2!}(x-x_0)^2$$

$$\frac{f''(c)}{2!}(x-x_0)^2 \leq 0$$

$\Rightarrow f(x) \leq f(x_0) + \frac{f'(x_0)}{\cancel{1!}}(x-x_0)$; $\forall x \in \mathbb{R}, x \neq x_0$

I. $f'(x_0) > 0$

$f(x_0) + f'(x_0)(x-x_0) \geq 0$; $\forall x \in \mathbb{R}, x \neq x_0$

$\underbrace{f(x_0)}_{>0} \cdot x + f(x_0) - f(x_0)x_0 \geq 0$; $\forall x \in \mathbb{R} \setminus \{x_0\} \Rightarrow$

$\Rightarrow x \geq \frac{f(x_0)(x_0-1)}{f'(x_0)}$; $\forall x \in \mathbb{R} \setminus \{x_0\} \Rightarrow$

$\Rightarrow \exists c = \frac{f(x_0)(x_0-1)}{f'(x_0)}$ a.î. $x \geq c$, $\forall x \in \mathbb{R} \setminus \{x_0\}$ fals

II. Analog $x \leq c$; $\forall x \in \mathbb{R} \setminus \{x_0\}$ fals

$\Rightarrow$ absurd $\Rightarrow$

$\Rightarrow$ Pp făcută este falsă $\Rightarrow$ f este fcț constantă.

3. Calc $\lim\limits_{x\to 0} \dfrac{\sin x - x}{x^3(e^x-1-x)}$

Fie $f, g: \mathbb{R}\to\mathbb{R}$, $f(x) = \sin x$
$g(x) = e^x$

f, g funcții indefinit derivabile pe $\mathbb{R}$

Considerăm g de 2 ori derivabilă pe $\mathbb{R}$
f de 5 ori dvb pe $\mathbb{R}$

Fixăm $x_0 = 0$.

$\forall x \in \mathbb{R}^*, \exists c \in \mathbb{R}$ situat între 0 și x a.î. $\underbrace{\phantom{0\ \text{și}\ x}}_{c\to 0}$

$$g(x) = g(0) + \frac{g'(0)}{1!}(x) + \frac{g''(c)}{2!}x^2$$

$g(0)=1$
$g'(0)=1$
$g''(c)=e^c$

$$g(x) = 1 + x + x^2\cdot\frac{e^c}{2} \Rightarrow e^x = 1+x+x^2\cdot\frac{e^c}{2}$$

$\forall x\in\mathbb{R}, x\neq 0, \exists d\in\mathbb{R}$ între x și 0 a.î.

$$f(x) = f(0) + \frac{f'(0)}{1!}\cdot x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \frac{f^{(5)}(d)}{5!}x^5$$

$$\Rightarrow \sin x = 0 + x + 0 - \frac{1}{6}x^3 + 0 + \frac{\cos d}{5!}x^5 \Rightarrow$$

$$\Rightarrow \sin x = x - \frac{1}{6}x^3 + \frac{\cos d}{120}x^5$$

$$\lim_{\substack{x\to 0\\ d\to 0\\ c\to 0}} \frac{\cancel{x} - \frac{1}{6}x^3 + \frac{\cos d}{120}x^5 - \cancel{x}}{x^3\cdot(\cancel{1}+\cancel{x}+x^2\frac{e^c}{2} - \cancel{1} - \cancel{x})} = \lim_{\substack{x\to 0\\ d\to 0\\ c\to 0}} \frac{\frac{\cos d}{120}x^5 - \frac{1}{6}x^3}{x^5\frac{e^c}{2}} =$$

$$= \lim_{\substack{x\to 0\\ d\to 0}} \frac{\frac{\cos d}{120}x^2 - \frac{1}{6}}{x^2\frac{e^c}{2}} = \lim_{\substack{x\to 0\\ d\to 0}} \frac{\frac{1}{120}\cdot 0 - \frac{1}{6}}{0_+} = \frac{-\frac{1}{6}}{0_+} = -\infty$$

4. Fie $n \in \mathbb{N}$; $n \geq 2$. Să se demonstreze inegalitatea

$$e^x \leq 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^{2n}}{(2n)!}; \forall x \geq 0$$

Fie $f: (-\infty, 0] \to \mathbb{R}$; $f(x) = e^x$

f indefinit derivabilă pe $(-\infty, 0]$

În particular, f este derivabilă de $2n+1$ ori

~~$\forall x \in \mathbb{R}$~~ $\forall x \in (-\infty, 0)$, $\exists c \in (x, 0)$ a.î.

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(2n)}(0)}{(2n)!}x^{2n} + \frac{f^{(2n+1)}(c)}{(2n+1)!}x^{2n+1}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \dots + \frac{1}{(2n)!}x^{2n} + \frac{e^c}{(2n+1)!} \cdot x^{2n+1}$$

$$\frac{e^c}{(2n+1)!} x^{2n+1} < 0$$

$\Rightarrow$

$$e^x < 1 + x + \dots + \frac{1}{(2n+1)!} x^{2n}; \forall x < 0$$

Pt $x = 0$ avem $e^0 \leq 1 + 0 + \dots + 0 \Rightarrow 1 \leq 1$ adv

$\Rightarrow$

$$\Rightarrow e^x \leq 1 + x + \dots + \frac{1}{(2n+1)!} x^{2n}; \forall x \leq 0 \text{ q.e.d.}$$

$$\lim_{\substack{x \to 0 \\ x < 0}} \frac{e^x - 1 - \frac{x}{1!} - \dots - \frac{x^{2n}}{(2n)!}}{x^{2n+1}} = \lim_{\substack{x \to 0 \\ x < 0 \\ c \to 0 \\ c < 0}} \frac{\frac{e^c}{(2n+1)!} \cancel{x^{2n+1}}}{\cancel{x^{2n+1}}} = \lim_{\substack{c \to 0 \\ c < 0}} \frac{e^c}{(2n+1)!} = \frac{1}{(2n+1)!}$$

5. Fie $f:\mathbb{R}\to\mathbb{R}$ o funcție derivabilă a.î. $0\le f'(x)\le f(x)$ și $\forall x\in\mathbb{R}$

$\exists x_0\in\mathbb{R}$ a.î. $f(x_0)=0$. Arătați $f(x)=0;\forall x\in\mathbb{R}$

$f'(x)\ge 0;\forall x\in\mathbb{R} \Rightarrow f \nearrow$ pe $\mathbb{R}$

Pt $x\le x_0$ avem $\left.\begin{array}{l} f(x)\le f(x_0) \\ 0\le f(x)\end{array}\right\} \Rightarrow f(x)=0;\forall x\in(-\infty,x_0]$ (1)

$f'(x)\le f(x);\forall x\in[x_0,\infty) \Rightarrow$

$\Rightarrow f'(x)-f(x)\le 0;\forall x\in[x_0,\infty) \Rightarrow$

$\Rightarrow e^{-x}f'(x)-e^{-x}f(x)\le 0;\forall x\in[x_0,\infty) \Rightarrow$

$\left.\begin{array}{l} \Rightarrow (e^{-x}f(x))'\le 0;\forall x\in[x_0,\infty) \\ \text{Fie } g:[x_0,\infty)\to\mathbb{R};\ g(x)=e^{-x}f(x)\end{array}\right\} \Rightarrow$

$\Rightarrow g'(x)\le 0;\forall x\in[x_0,\infty) \Rightarrow g \searrow$ pe $[x_0,\infty) \Rightarrow$

$\left.\begin{array}{l} \Rightarrow g(x_0)\ge g(x);\forall x\in[x_0,\infty) \\ g(x_0)=e^{-x_0}\underbrace{f(x_0)}_{0}=0\end{array}\right\} \Rightarrow g(x)\le 0;\forall x\in[x_0,\infty) \Rightarrow$

$\left.\begin{array}{l} \Rightarrow \underbrace{e^{-x}}_{>0}f(x)\le 0;\forall x\in[x_0,\infty) \Rightarrow f(x)\le 0;\forall x\in[x_0,\infty) \\ \text{Ip} \Rightarrow f(x)\ge 0;\forall x\in\mathbb{R}\end{array}\right\} \Rightarrow$

$\Rightarrow \left.\begin{array}{l} f(x)=0;\forall x\in[x_0,\infty) \\ (1)\end{array}\right\} \Rightarrow f(x)=0;\forall x\in\mathbb{R}$ qed

Temă: Fie $n\in\mathbb{N};n\ge 2$. Să se dem. că $\sqrt[n]{\frac{2x+y}{3}}\le\frac{2\sqrt[n]{x}+\sqrt[n]{y}}{3}$ $\forall x,y\ge 0$

26.03.2018

Analiză

– Seminar –

1. Să se det R, A și f pt seria de puteri $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1}$

$x_0 = 0$

$a_0 = 0$

$a_{2n+1} = \frac{1}{2n+1}$

$a_{2n} = 0$

$l = \overline{\lim} \sqrt[n]{|a_n|}$

$\lim_{n\to\infty} \sqrt[2n+1]{|a_{2n+1}|} = \lim_{n\to\infty} \sqrt[2n+1]{a_{2n+1}} = \lim_{n\to\infty} \sqrt[2n+1]{\frac{1}{2n+1}}$

$\lim_{n\to\infty} \frac{\frac{1}{2n+3}}{\frac{1}{2n+1}} = \lim_{n\to\infty} \frac{2n+1}{2n+3} = 1 \Rightarrow \lim_{n\to\infty} \sqrt[2n+1]{\frac{1}{2n+1}} = 1$

$\lim_{n\to\infty} \sqrt[2n]{|a_{2n}|} = 0$

$\Rightarrow l = 1 \Rightarrow$

$R = \begin{cases} +\infty, & l = 0 \\ 0, & l = +\infty \\ \frac{1}{l}, & l = (0,\infty) \end{cases}$

$\Rightarrow R = \frac{1}{l} = 1$

$\left.\begin{array}{l} (x_0 - R, x_0 + R) \subseteq A \subseteq [x_0 - R, x_0 + R] \\ A \subseteq \mathbb{R} \end{array}\right\} \Rightarrow$

$(-1, 1) \subseteq A \subseteq [-1, 1]$

Pt $x = 1$ avem $\sum_{n=0}^{\infty} \frac{1}{2n+1}$ serie divergentă $\Rightarrow 1 \notin A$

$\frac{1}{2n+1} > \frac{1}{2n+2} = \frac{1}{2} \frac{1}{n+1}$

$\sum_{n=0}^{+\infty} \frac{-1}{2n+1}$ ~~conv~~ div $\Rightarrow -1 \notin A$

$A = (-1, 1)$

$f: A = (-1,1) \to \mathbb{R}$

$f(x) = \sum_{n=0}^{+\infty} \frac{x^{2n+1}}{2n+1} \Rightarrow$

$\Rightarrow f(x_0) = a_0$

$f(0) = 0$

f indef. dub pe $(-1,1)$

$f'(x) = \sum_{n=0}^{+\infty} \left(\frac{x^{2n+1}}{2n+1}\right)' = \sum_{n=0}^{+\infty} x^{2n}$

$\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}; \forall x \in (-1,1) \xRightarrow{x \to x^2} \sum_{n=0}^{+\infty} x^{2n} = \frac{1}{1-x^2}; \forall x^2 \in (-1,1)$

$x^2 \in (-1,1) \Leftrightarrow x^2 < 1 \Leftrightarrow x \in (-1,1)$

$\Rightarrow f'(x) = \frac{1}{1-x^2}; \forall x \in (-1,1) \Rightarrow f(x) = \int \frac{1}{1-x^2} dx = -\frac{1}{2} \ln \left|\frac{1-x}{1+x}\right| + C;$
$\forall x \in (-1,1)$

$f(0) = 0 \Rightarrow \frac{1}{2} \ln 1 + C = 0 \Rightarrow f(x) = -\frac{1}{2} \ln \left|\frac{1-x}{1+x}\right|; \forall x \in (-1,1)$

<u>Funcții diferențiabile</u>

2. Studiați diferențiabilitatea funcției $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \sqrt{x^2+y^2}$.
Calculați ~~[illegible]~~ df.

f - continuă pe $\mathbb{R}^2$

$$\frac{\partial f}{\partial x}((x,y)) = \left(\sqrt{x^2+y^2}\right)'_x = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} ; \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

$$x^2+y^2 \neq 0 \Leftrightarrow (x,y) \neq (0,0)$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} ; \forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$$

$$\lim_{t\to 0} \frac{f((0,0)+t(1,0)) - f((0,0))}{t} = \lim_{t\to 0} \frac{f((t,0)) - f((0,0))}{t} = \lim_{t\to 0} \frac{|t|}{t} \; (\nexists) \Rightarrow$$

$$\Rightarrow \nexists \frac{\partial f}{\partial x}(0,0)$$

$$\lim_{t\to 0} \frac{f((0,0)+t(0,1)) - f(0,0)}{t} = \lim_{t\to 0} \frac{|t|}{t} (\nexists) \Rightarrow \nexists \frac{\partial f}{\partial y}(0,0)$$

$$\left.\begin{array}{l} \exists \dfrac{\partial f}{\partial x} ; \dfrac{\partial f}{\partial y} \text{ pe } \underbrace{\mathbb{R}^2 \setminus \{(0,0)\}}_{\text{mulț. deschisă}} \\ \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y} \text{ f.d. cont. pe } \mathbb{R}^2 \setminus \{(0,0)\} \end{array}\right\} \Rightarrow f \text{ este diferențiabilă pe } \mathbb{R}^2 \setminus \{(0,0)\}$$

$$df : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathcal{L}(\mathbb{R}^2, \mathbb{R})$$

$$df(x,y) : \mathbb{R}^2 \to \mathbb{R}$$

$$df(x,y)((u,v)) = u \frac{\partial f}{\partial x}(x,y) + v \cdot \frac{\partial f}{\partial y}(x,y) =$$

$$= u \cdot \frac{x}{\sqrt{x^2+y^2}} + v \frac{y}{\sqrt{x^2+y^2}} ; \forall (u,v) \in \mathbb{R}^2$$

$$df(x,y) = dx \frac{x}{\sqrt{x^2+y^2}} + dy \frac{y}{\sqrt{x^2+y^2}} = \frac{x}{\sqrt{x^2+y^2}} dx + \frac{y}{\sqrt{x^2+y^2}} dy ; \forall x,y \in \mathbb{R}^2 \setminus \{(0,0)\}$$

3. Std. diferențiabilitatea fct. $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

în pct $(0,0)$

Cont în $(0,0)$

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{\substack{x\to 0\\ y\to 0}} \frac{xy}{\sqrt{x^2+y^2}}$$

$$|f(x,y) - 0| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \frac{|xy|}{\sqrt{x^2+y^2}} \le \frac{|xy|}{\sqrt{x^2}} = \frac{|xy|}{|x|} = |y|$$

$\Rightarrow -|y| \le f(x,y) - 0 \le |y|$, $\forall (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$\searrow 0 \swarrow$

$\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 0 = f(0,0) \Rightarrow f$ cont în $(0,0)$

$$\lim_{t\to 0} \frac{f((0,0)+t(1,0)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f((t,0)) - f(0,0)}{t} = \frac{0}{t} = 0 \Rightarrow$$

$\Rightarrow \exists \frac{\partial f}{\partial x}((0,0)) = 0$

$$\lim_{t\to 0} \frac{f((0,0)+t(0,1)) - f(0,0)}{t} = \lim_{t\to 0} \frac{f((0,t)) - f(0,0)}{t} = \lim_{t\to 0} \frac{0-0}{t} = 0 \Rightarrow$$

$\Rightarrow \exists \frac{\partial f}{\partial y}((0,0)) = 0$

~~Tot~~ Temă: 1) Se construiește $T: \mathbb{R}^2 \to \mathbb{R}$, $T(x,y) = x \cdot \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = 0$

2) Se calc. $\lim_{(x,y)\to(0,0)} \frac{|f(x,y) - f(0,0) - T((x,y)-(0,0))|}{\|(x,y)-(0,0)\|}$ și se comp cu 0

18.04.2018

Analiză

- Seminar -

1. Det pct de extrem local ale fct $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \cancel{x^4 y^4}$
$= x^4 + y^4 - x^2y^2$

<u>Rez</u>: $E = \{x \in D \mid x$ pct de extrem local al $f\}$

1) studiem continuitatea fct f și identif mult $D_1 = \{x \in D \mid f$ nu este cont în $x\}$

2) studiem diferențiabilitatea fct f și identificăm mult $D_2 = \{x \in D \mid f$ nu este diferențiabilă în $x\}$

3) Identificarea punctelor critice ale funcției f și aflarea mult $C = \{x \in D \mid x$ pct critic$\}$

$\frac{\partial f}{\partial x}(x,y) = (x^4+y^4-x^2y^2)'_x = 4x^3 - 2y^2x$ $\quad \forall (x,y) \in \mathbb{R}^2$

$\frac{\partial f}{\partial y}(x,y) = (x^4+y^4-x^2y^2)'_y = 4y^3 - 2yx^2$

$\mathbb{R}^2$ mult deschisă

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ fct cont pe $\mathbb{R}^2$

$\Rightarrow f$ diferențiabilă pe $\mathbb{R}^2 \Rightarrow D_2 = \emptyset$

Egalăm cu 0 toate deriv. parțiale și rezolvăm sistemul

$\begin{cases} 4x^3 - 2y^2x = 0 \\ 4y^3 - 2x^2y = 0 \end{cases} \Rightarrow \begin{cases} 2x(2x^2-y^2) = 0 \\ 2y(2y^2-x^2) = 0 \end{cases} \Rightarrow \begin{cases} 2x(x\sqrt{2}+y)(x\sqrt{2}-y) = 0 \\ 2y(y\sqrt{2}+x)(y\sqrt{2}-x) = 0 \end{cases}$

I $x = 0 \Rightarrow y = 0$

II $y = -x\sqrt{2}$

$-8\sqrt{2}x^3 + 2\sqrt{2}x^3 = 0$

$-6\sqrt{2}x^3 = 0 \Rightarrow x = 0 \Rightarrow y = 0$ $\quad \Rightarrow C = \{(0,0)\}$

III $y = x\sqrt{2}$

$8\sqrt{2}x^3, 2\sqrt{2}x^3 = 0 \Rightarrow x = 0 \Rightarrow y = 0$

4) Stud dif de ordin 2 și $D_3 = \{x \in D \mid f$ nu este dif de 2 ori în $x\}$

$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y) = (4x^3 - 2y^2x)'_x = 12x^2 - 2y^2$

$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}(x,y)\right) = (4y^3 - 2x^2y)'_x = -4xy$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y) = (4x^3 - 2xy^2)'_y = -4xy$$

$$\frac{\partial^2 f}{\partial^2 y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y) = (4y^3 - 2x^2y)'_y = 12y^2 - 2x^2$$

$\mathbb{R}^2$ mulțime deschisă
$\frac{\partial^2 f}{\partial^2 x}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial^2 y}$ cont. pe $\mathbb{R}^2$ $\Big\} \Rightarrow$ f diferențiabilă pe $\mathbb{R}^2 \Rightarrow D_3 = \emptyset$

~~4)~~

5) Aplicarea crit. în pct critice în care funcția este diferențiabilă de 2 ori și stabilirea mulț. $D_4 = \{x \in C \mid$ crit se pronunță în $x\}$
$D_5 = \{x \in C \mid$ crit nu se pron. în $x\}$

$$H_f(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$\left.\begin{array}{l} \Delta_1 = 0 \\ \Delta_2 = 0 \end{array}\right\} \Rightarrow$ nu ne putem pronunța cu crit în $(0,0)$

$D_4 = \emptyset$

$D_5 = \{(0,0)\}$

$E \subseteq D_1 \cup D_2 \cup D_3 \cup D_4 \cup D_5$

6) Dacă $D_5 \neq \emptyset$ sau $D_1 \neq \emptyset$ sau $D_2 \neq \emptyset$ sau $D_3 \neq \emptyset$ se aplică def. pct de extrem local în orice pct al acestor mulț.

$f(x,y) - f(0,0) = x^4 + y^4 - x^2y^2 = x^4 + y^4 - 2x^2y^2 + x^2y^2 = (x^2 - y^2)^2 + x^2y^2 \geq 0;$
$\forall (x,y) \in \mathbb{R}^2 \Rightarrow (0,0)$ este pct. de minim global.

$E = \{(0,0)\}$

Fișa f.16

Fie $f:\mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} , & (x,y) \neq (0,0) \\ 0 , & (x,y) = (0,0) \end{cases}$

Ssa
- f cont în $(0,0)$
- $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$
- f nu e diferențiabilă în $(0,0)$

~~Fie~~

Rez: $T: \mathbb{R}^2 \to \mathbb{R}$, $T((x,y)) = \frac{\partial f}{\partial x}(0,0)\, x + \frac{\partial f}{\partial y}(0,0)\, y = 0; \forall (x,y) \in \mathbb{R}^2$

$$\lim_{(x,y)\to(0,0)} \frac{\| f(x,y) - f(0,0) - T((x,y)-(0,0)) \|}{\| (x,y) - (0,0) \|}$$

$$\lim_{(x,y)\to(0,0)} \frac{\left| \frac{xy}{\sqrt{x^2+y^2}} - 0 - 0 \right|}{\sqrt{x^2+y^2}} = \lim_{(x,y)\to(0,0)} \frac{|xy|}{x^2+y^2}$$

$$\left(\frac{1}{n}, \frac{1}{n}\right) \xrightarrow[n\to\infty]{} (0,0)$$

$$\lim_{n\to\infty} \frac{\left|\frac{1}{n} \cdot \frac{1}{n}\right|}{\frac{1}{n^2} + \frac{1}{n^2}} = \lim_{n\to\infty} \frac{\frac{1}{n^2}}{\frac{2}{n^2}} = \lim_{n\to\infty} \frac{1}{n^2} \cdot \frac{n^2}{2} = \frac{1}{2}$$

$$\left(\frac{1}{2n}, \frac{1}{4n}\right) \xrightarrow[n\to\infty]{} (0,0)$$

$$\lim_{n\to\infty} \frac{\frac{1}{8n^2}}{\frac{5}{16n^2}} = \frac{2}{5}$$

$$\Rightarrow \nexists \lim_{(x,y)\to(0,0)} \frac{|xy|}{x^2+y^2} \Rightarrow$$

$\Rightarrow$ f nu e diferențiabilă în $(0,0)$

Ex 3 : Det pct de extrem local ale fct. $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = x^3 + y^3 - 3x...$

f cont

$D_1 \neq \emptyset$

$$\frac{\partial f}{\partial x}(x,y) = (x^3+y^3+3xy)'_x = 3x^2+3y$$

$$\frac{\partial f}{\partial y}(x,y) = (x^3+y^3+3xy)'_y = 3y^2+3x$$

$\frac{\partial f}{\partial x}$ și $\frac{\partial f}{\partial y}$ cont pe $\mathbb{R}^2$

$\mathbb{R}^2$ mulț deschisă

$\Rightarrow$ f. diferențiabilă pe $\mathbb{R}^2 \Rightarrow D_2 = \emptyset$

$(\mathbb{R}^2)$ $\begin{cases} 3x^2+3y=0 \\ 3y^2+3x=0 \end{cases} \Rightarrow \begin{cases} x^2+y=0 \\ y^2+x=0 \end{cases}$

$C = \{(0,0), (-1,-1)\}$

$$\frac{\partial^2 f}{\partial^2 x}(x,y) = (3x^2+3y)'_x = 6x$$

$$\frac{\partial^2 f}{\partial^2 y}(x,y) = (3y^2+3x)'_y = 6y$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = (3x^2+3y)'_y = 3$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = (3y^2+3x)'_x = 3$$

$\forall (x,y) \in \mathbb{R}^2$

$\mathbb{R}^2$ deschisă

$\frac{\partial^2 f}{\partial^2 x}, \frac{\partial^2 f}{\partial^2 y}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$ fct cont pe $\mathbb{R}^2$

$\Rightarrow$

$\Rightarrow$ f diferențiabilă de 2 ori pe $\mathbb{R}^2 \Rightarrow D_3 = \emptyset$

$H_f(0,0) = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$

$\left.\begin{array}{l} \Delta_1 = 0 \leq 0 \\ \Delta_2 = -9 < 0 \end{array}\right\} \Rightarrow (0,0)$ nu este pct de extrem

$H_f(-1,-1) = \begin{pmatrix} -6 & 3 \\ 3 & -6 \end{pmatrix}$

$\left.\begin{array}{l} \Delta_1 = -6 < 0 \\ \Delta_2 = 27 > 0 \end{array}\right\} \Rightarrow (-1,-1)$ pct de maxim local

$D_4 = \{(-1,-1), (0,0)\}$

$D_5 = \emptyset$

$E \subseteq D_4$

$\Rightarrow E = \{(-1,-1)\}$

- Seminar -

Funcții integrabile Riemann

Ex 1: Studiați integrabilitatea Riemann pt fct $f:[0,1]\to\mathbb{R}$,

$$f(x)=\begin{cases}\dfrac{e^{2x}}{1+e^{x}}, & x\in[0,1]\setminus\{\frac{1}{2^n}\mid n\in\mathbb{N}\}\\ 0, & \exists n\in\mathbb{N} \text{ a.î. } x=\frac{1}{2^n}\end{cases}$$

f cont pe $[0,1]\setminus\{\frac{1}{2^n}\mid n\in\mathbb{N}\}$

$$\lim_{x\to\frac{1}{2^n}} f(x)=\lim_{x\to\frac{1}{2^n}}\frac{e^{2x}}{1+e^{x}}=\frac{e^{\frac{1}{2^{n-1}}}\neq 0}{1+e^{\frac{1}{2^n}}}\neq 0 \Big\} \Rightarrow f \text{ nu este cont în } \tfrac{1}{2^n}, \forall n\in\mathbb{N}$$

$\Rightarrow \{x\in[0,1] / f$ nu este cont$\} = \{\frac{1}{2^n} / n\in\mathbb{N}\}\to$ mulț neglijabilă Lebesgue

$0\le f(x), \forall x\in[0,1]$

$$\left.\begin{array}{l} x\in[0,1]\Rightarrow 2x\in[0,2]\Rightarrow e^{2x}\in[1,e^2]\\ \dfrac{e^{2x}}{1+e^{x}}<e^{2x}\le e^2\Rightarrow f(x)\le e^2; \forall x\in[0,1]\end{array}\right\}\Rightarrow f \text{ mărg} \Rightarrow f\in\mathcal{R}([0,1])$$

Ex 2: Fie $I_n=\int_0^1 (2x-x^2)^n\,dx, n\in\mathbb{N}^*$. Calculați $\lim\limits_{n\to\infty} I_n$

$f_n:[0,1]\to\mathbb{R}$

$f_n(x)=(2x-x^2)$

Fie $x\in[0,1]$

$$\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty}(2x-x^2)^n=\begin{cases}0, & x\in[0,1)\\ 1, & x=1\end{cases}$$

~~$f_n \xrightarrow[{[0,1]}]{s} f$~~ ~~f.s.~~

$2x-x^2=-(x^2-2x)=-(x-1)^2+1=1-(x-1)^2\in[0,1]$

$f_n \xrightarrow[{[0,1]}]{S} f$

$f : [0,1] \to \mathbb{R},\ f(x) = \begin{cases} 0, & x \in [0,1) \\ 1, & x = 1 \end{cases}$

$$|f_n(x)| = |(1-(x-1)^2)^n| = (1-(x-1)^2)^n \le 1; \forall x \in [0,1]; \forall n \in \mathbb{N}^*$$

f_n fct cont pe $[0,1] \Rightarrow f_n \in R([0,1]); \forall n \in \mathbb{N}^*$

f fct crescatoare pe $[0,1] \Rightarrow f \in R([0,1])$

Conform th fct mărginite obținem că

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$$

~~Exercitiu~~

Ex 3: Fie $x_n = \frac{1}{n+\sqrt{1}} + \frac{1}{n+\sqrt{3}} + \dots + \frac{1}{n+\sqrt{n^2-1}}$, $n \in \mathbb{N}^*$. Calc $\lim_{n\to\infty} x_n$

$$x_n = \frac{1}{n}\left(\frac{1}{1+\frac{\sqrt{1}}{n}} + \frac{1}{1+\frac{\sqrt{3}}{n}} + \dots + \frac{1}{1+\frac{\sqrt{n^2-1}}{n}}\right)$$

$$f(x) = \frac{1}{1+x}$$

$f : [0,1] \to \mathbb{R}$

f cont pe $[0,1] \Rightarrow$

$\Rightarrow f \in R([0,1])$

$\Delta_n : x_0 = 0; x_1 = \frac{1}{n}; x_2 = \frac{2}{n}, \dots x_{n-1} = \frac{n-1}{n}, x_n = 1$

$\|\Delta_n\| = \frac{1}{n} \to 0$

$t_\Delta = \left\{ t_1 = \frac{\sqrt{1}}{n}, t_2 = \frac{\sqrt{3}}{n}, t_3 = \frac{\sqrt{8}}{n}, \dots, t_n = \frac{\sqrt{n^2-1}}{n} \right\}$

~~$\sigma_{\Delta_n}(f, t_{\Delta_n}) = f(t_1)$~~

$$\sigma_{\Delta_n}(f, t_{\Delta_n}) = f(t_1)\frac{1}{n} + f(t_2)\frac{1}{n} + \dots + f(t_n)\cdot\frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n} f(t_i) = x_n$$

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} \sigma_{\Delta_n}(f, t_{\Delta_n}) \overset{\|\Delta_n\| \to 0}{=\!=} \int_0^1 f(x)\,dx = \int_0^1 \frac{1}{1+x}\,dx = \ln(x+1)\Big|_0^1 = \ln 2$$

Ex 4 Calc $\lim\limits_{\substack{x\to 0\\ x>0}} \int_x^{3x} \frac{\sin t}{t}\,dt$ și $\lim\limits_{\substack{x\to 0\\ x>0}} \int_x^{2x} \frac{\sin t}{t^2}\,dt$

$f(0,+\infty)\to\mathbb{R},\ f(t)=\frac{\sin t}{t}$

f cont pe $(0,\infty)$

$f\Big|_{[x,3x]} \Rightarrow \exists c\in(x,3x)$ a.î. $\int_x^{3x} f(t)\,dt = (3x-x)f(c) = 2x\frac{\sin c}{c}$

$0 \quad x \quad c \quad 3x$

$\lim\limits_{\substack{x\to 0\\ x>0}} \int_x^{3x} f(t)\,dt = \lim\limits_{\substack{x\to 0\\ x>0\\ c\to 0\\ c>0}} 2x\frac{\sin c}{c} = 0$

Analog $\lim\limits_{\substack{x\to 0\\ x>0}} \int_x^{2x} \underbrace{g(t)}_{\text{integ?}} = x\frac{\sin c}{c^2} = 0$

Fie $g,h: (0,+\infty)\to$

$g(x)=\frac{\sin x}{x}$

$h(x)=\frac{1}{x}$

g,h cont pe $(0,\infty)\to$ au prop Darboux

$g\Big|_{[x,3x]}$ și $h\Big|_{[x,3x]} \Rightarrow \exists c\in(x,3x)$ a.î. $\int_x^{3x} g(t)h(t)\,dt =$

$h(t)\geq 0,\ \forall t\in[x,3x]$ $\Big| = g(c)\int_x^{3x} h(t)\,dt = \frac{\sin c}{c}\int_x^{3x}\frac{1}{t}\,dt = \frac{\sin c}{c}\ln t\Big|_x^{3x} =$

$= \frac{\sin c}{c}\ln 3$

$\lim\limits_{\substack{x\to 0\\ x>0}} \int_x^{3x} g(t)h(t)\,dt = \lim\limits_{\substack{c\to 0\\ c>0}} \frac{\sin c}{c}\ln 3 = \ln 3$

Ex5: Det fct cont $f: \mathbb{R} \to \mathbb{R}$ care verifică egalitatea

$$f(x) = x^2 + \int_0^x e^{t-x} f(t)\, dt \,;\ \forall t \in \mathbb{R}$$

$$f(0) = 0 + 0 = 0$$

$g: \mathbb{R} \to \mathbb{R}$, $g(x) = \int_0^x e^{t-x} f(t)\, dt$

g este dvb pe $\mathbb{R}$

$f(x) = x^2 + g(x)$, $\forall x \in \mathbb{R}$

f dvb pe $\mathbb{R}$

$f'(x) = 2x + g'(x)$, $\forall x \in \mathbb{R}$

$f'(x) = 2x + f(x)$; $\forall x \in \mathbb{R}$

$f'(x) - f(x) = 2x$, $\forall x \in \mathbb{R} \mid e^{-x} \Rightarrow$

$$\Rightarrow e^{-x}(f'(x) - f(x)) = 2x e^{-x} \Rightarrow (e^{-x} f(x))' = 2x e^{-x} \Rightarrow \int 2x e^{-x} dx = e^{-x} f(x) + C$$

$$\int 2x e^{-x} dx = \int 2x (-e^{-x})' dx = -2x e^{-x} - \int (2x)' (-e^{-x})\, dx =$$

$$\cancel{= -2x e^{-x} + 2\int e^{-x} dx} = -2x e^{-x} - 2e^{-x} = e^{-x} f(x) + C \,;\ \forall x \in \mathbb{R}$$

$$e^{-x}(-2x - 2) = e^{-x} f(x) + C$$

$\cancel{(-1)(-2)} \Rightarrow$ $C = -2$

$$e^{-x}(-2x - 2) = e^{-x} f(x) - 2$$

$$\frac{e^{-x}(-2x - 2) + 2}{e^{-x}} = f(x)$$

$$f(x) = -2x - 2 + \frac{2}{e^{-x}} = -2x - 2 + 2e^{x}, \forall x \in \mathbb{R}$$

Ex 6 : Fie $f: [0,1] \to \mathbb{R}$ o fct cont. Calc $I = \int_0^1 \frac{f(x)}{f(x)+f(1-x)} dx$

$x = 1-t$

$dx = -dt$

$$I = \int_1^0 \frac{f(1-t)}{f(1-t)+f(t)}(-1)dt = \underbrace{\int_0^1 \frac{f(1-t)}{f(1-t)+f(t)} dt}_{J}$$

$I = J$

$$I + J = \int_0^1 \left(\frac{f(1-x)}{f(1-x)+f(x)} + \frac{f(x)}{f(x)+f(1-x)} \right) dx = \int_0^1 \frac{f(x)+f(1-x)}{f(1-x)+f(x)} dx = x \Big|_0^1 = 1 \Rightarrow$$

$\Rightarrow I = \frac{1}{2}$

21.05.2018

Analiză
- Seminar -

Exc 1: Studiați natura ~~acestor~~ următoarelor integrale improprii

$\int_0^1 \frac{1}{x^3}dx$ și $\int_{0+0}^1 \frac{e^{\frac{1}{x}}}{x^3}dx$.

Rez: $f,g:(0,1] \to \mathbb{R}$, $f(x)=\frac{1}{x^3}$, $g(x)=\frac{e^{\frac{1}{x}}}{x^3}$

f,g funcții continue pe $(0,1] \Rightarrow f,g \in \mathcal{R}_{loc}((0,1])$

$f(x)>0$
$g(x)>0$ $\quad \forall x\in(0,1]$

$$\int_{0+0}^1 \frac{1}{x^3}dx = \int_{0+0}^1 f(x)\,dx = \lim_{\substack{x\to 0\\ x>0}} \int_x^1 f(t)dt = \lim_{\substack{x\to 0\\ x>0}} \int_x^1 \frac{1}{t^3}dt =$$

$$= \lim_{\substack{x\to 0\\ x>0}} \frac{t^{-2}}{-2}\Big|_x^1 = \lim_{\substack{x\to 0\\ x>0}} -\frac{1}{2t^2}\Big|_x^1 = \cancel{\lim \frac{1}{2}} -\frac{1}{2} + \lim_{\substack{x\to 0\\ x>0}} \frac{1}{2x^2} = \infty \Rightarrow$$

$\Rightarrow$ Integrala improprie $\int_{0+0}^1 \frac{1}{x^3}dx$ este divergentă

$$\left.\begin{aligned} &\lim_{\substack{x\to 0\\ x>0}} \frac{g(x)}{f(x)} = \lim_{\substack{x\to 0\\ x>0}} \frac{\frac{e^{\frac{1}{x}}}{x^3}}{\frac{1}{x^3}} = \infty \\ &\int_{0+0}^1 f(x)dx \text{ - divergentă}\end{aligned}\right\} \Rightarrow \int_{0+0}^1 g(x)dx \text{ divergentă}$$

Exc 2: Studiați natura integralei improprii $\int_{0+0}^1 \frac{\ln(1+x)}{x}dx$

$f:(0,1] \to \mathbb{R}$, $f(x)=\frac{\ln(1+x)}{x}$

f cont pe $(0,1] \Rightarrow f\in\mathcal{R}_{loc}((0,1])$

$f(x)>0,\ \forall x\in(0,1]$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{\substack{x\to 0\\ x>0}} \frac{\ln(1+x)}{x} \cdot \frac{1}{\frac{1}{x}} = \lim_{\substack{x\to 0\\ x>0}} \ln(1+x) = 0$$

$\int_{0+0}^{1} g(x)dx = \int_{0+0}^{1} \frac{1}{x} dx = \ln x \Big|_{0+0}^{1} = \ln 1 - \lim_{\substack{x \to 0 \\ x>0}} \ln x = +\infty$ - nu ne putem pronunța

$g(x) = \frac{1}{x^2}$; $\lim_{\substack{x \to 0 \\ x>0}} \frac{f(x)}{g(x)} = \lim_{\substack{x \to 0 \\ x>0}} f(x) = \lim_{\substack{x \to 0 \\ x>0}} \frac{\ln(1+x)}{x} \overset{L'H}{=} \lim_{\substack{x \to 0 \\ x>0}} \frac{1}{1+x} = 1$

$\int g(x)dx$ ... $\Rightarrow \int_{0-0}^{1} g(x)dx$ este convergentă

Exc 3: Studiați natura integralei improprii $\int_{1}^{+\infty} \frac{\ln(1+x)}{x} dx$

$f : [1, +\infty) \to \mathbb{R}$, $f(x) = \frac{\ln(1+x)}{x}$

f cont pe $[1, \infty) \Rightarrow f \in R_{loc}([1, \infty))$

$f(x) > 0$; $\forall x \in [1, \infty)$

$\int \ln(1+x) dx = \int (x+1)' \ln(x+1) dx = (x+1)\ln(x+1) - \int (x+1) \frac{1}{x+1} dx =$

$= (x+1)\ln(x+1) - x = G$

$G(x)\Big|_1^{\infty} = \lim_{x \to \infty} G(x) - G(1) = \lim_{x \to \infty} \left((x+1)\ln(x+1) - x\right) - 2\ln 2 + 2 =$

$= \lim_{x \to \infty} x\left(\left(\frac{1}{x} + 1\right)\ln(x+1) - 1\right) - \cancel{G(1)}\ 2\ln 2 + 2 = +\infty$

$x+1 \geq 2 \Rightarrow \ln(x+1) \geq \ln 2 \ / \frac{1}{x} \Rightarrow \frac{\ln(x+1)}{x} \geq \frac{\ln 2}{x} \Rightarrow f(x) \geq g(x)$; $\forall x \in [1, \infty)$

$\int_1^{+\infty} g(x)dx = \int_1^{+\infty} \frac{\ln 2}{x} dx = \ln 2 \ln x \Big|_1^{\infty} = \ln 2 \left(\lim_{x \to \infty} \ln x - 0\right) = \infty \Rightarrow$

$\Rightarrow \int_1^{+\infty} g(x)dx$ e divergentă

$f(x) \geq g(x)$; $\forall x \in [1, \infty)$

$\int_1^{+\infty} f(x)dx$ divergentă

Ex 4: Calculați, folosind funcțiile Γ și B, următoarele integrale improprii: a) $\int_0^{+\infty} x^4 e^{-2x^2} dx$

b) $\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx$

c) $\int_0^{+\infty} \frac{x^n}{(1+x)^n} dx$, $n \geq 2$

$$\Gamma(p) = \int_{0+0}^{+\infty} x^{p-1} e^{-x} dx$$

a) $t = 2x^2 \Leftrightarrow x^2 = \frac{t}{2} \underset{x\geq 0}{\Rightarrow} x = \sqrt{\frac{t}{2}}$

$$dx = \left(\sqrt{\tfrac{t}{2}}\right)' dt = \frac{1}{2\sqrt{\frac{t}{2}}} \left(\tfrac{t}{2}\right)' dt = \frac{1}{4\sqrt{\frac{t}{2}}} dt = \frac{1}{2\sqrt{2} \cdot \sqrt{t}} dt$$

$x = 0 \Rightarrow t = 0$

$x \to \infty \Rightarrow t \to \infty$

$$I = \int_0^{\infty} \left(\sqrt{\tfrac{t}{2}}\right)^4 e^{-t} \frac{1}{2\sqrt{2}\sqrt{t}} dt = \int_0^{\infty} \frac{t^2}{4} e^{-t} \frac{1}{2\sqrt{2}\sqrt{t}} dt = \frac{1}{8\sqrt{2}} \int_0^{\infty} t\sqrt{t}\, e^{-t} dt =$$

$$= \frac{1}{8\sqrt{2}} \int_0^{\infty} t^{\frac{3}{2}} e^{-t} dt$$

$p - 1 = \frac{3}{2} \Rightarrow p = \frac{5}{2} \Rightarrow I = \frac{1}{8\sqrt{2}} \Gamma\left(\frac{3}{2}+1\right) = \frac{1}{8\sqrt{2}} \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}$

~~Ex 5 Calc. folosind fct. Γ și B, urmat. integrale improprii~~

b) $I = \int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx$

$$B(p,q) = 2\int_0^{\frac{\pi}{2}} \sin^{2p-1} x \cos^{2q-1} x \, dx$$

$2p - 1 = 2n \Rightarrow p = \frac{2n+1}{2}$

$2q - 1 = 0 \Rightarrow q = \frac{1}{2}$

$$I = \frac{1}{2} B\left(\frac{2n+1}{2}, \frac{1}{2}\right) = \frac{1}{2} \frac{\Gamma\left(\frac{2n+1}{2}\right) \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma(n+1)} = \frac{1}{2} \frac{\Gamma\left(\frac{2n+1}{2}\right)\sqrt{\pi}}{n!}$$

$$\Gamma\left(\frac{2n+1}{2}\right) = \Gamma\left(\frac{2n-1}{2}+1\right) = \dots = \frac{2n-1}{2} \cdot \frac{2n-3}{2} \cdot \dots \cdot \frac{3}{2} \cdot \frac{1}{2} \cdot \Gamma\left(\frac{1}{2}\right) = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^n} \sqrt{\pi}$$

$$I = \frac{1}{2} \frac{\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^n}}{n^2} \pi = \frac{1 \cdot 3 \dots (2n-1)\pi}{2^{n+1} n!}$$

c) $I = \int_0^\infty \frac{x^n}{(1+x)^n} dx,\ n \geq 2$

$$B(p,q) = \int_{0+0}^{+\infty} \frac{x^{p-1}}{(1+x)^{p+q}} dx$$

$p-1 = n \Rightarrow p = n+1$

$p+q = 2n \Rightarrow q = n-1$

$$I = B(n+1, n-1) = \frac{\Gamma(n+1)\,\Gamma(n-1)}{\Gamma(2n)} = \frac{n!\,(n-2)!}{(2n-1)!}$$

Exc 5 : Calculate $\int_0^2 x\sqrt[3]{8-x^3}\,dx$

S.V. $x = 2\sqrt[3]{t}$

$dx = \frac{2}{3} t^{-\frac{2}{3}} dt$

$$I = \int_0^1 2\sqrt[3]{t}\,\sqrt[3]{8-8t}\cdot\frac{2}{3} t^{-\frac{2}{3}} dt$$

$$I = \frac{8}{3}\int_0^1 t^{\frac{1}{3}-\frac{2}{3}} \sqrt[3]{1-t}$$

$$I = \frac{8}{3}\int_0^1 t^{-\frac{1}{3}} (1-t)^{\frac{1}{3}} dt$$

$\left.\begin{array}{l} p-1 = -\frac{1}{3} \Rightarrow p = \frac{2}{3} \\ q-1 = \frac{1}{3} \Rightarrow q = \frac{4}{3} \end{array}\right\} \Rightarrow I = \frac{8}{3} B\left(\frac{2}{3}, \frac{4}{3}\right) = \frac{8}{3} \frac{\Gamma\left(\frac{2}{3}\right)\cdot\Gamma\left(\frac{4}{3}\right)}{\Gamma(2)} = \frac{8}{3}\Gamma\left(\frac{2}{3}\right)\cdot\Gamma\left(\frac{4}{3}\right)$

$$I = \frac{8}{3}\Gamma\left(1-\frac{1}{3}\right)\cdot\Gamma\left(1+\frac{1}{3}\right) = \frac{8}{3}\Gamma\left(1-\frac{1}{3}\right)\cdot\Gamma\left(\frac{1}{3}\right)\cdot\frac{1}{3} = \frac{8}{9}\Gamma\left(1-\frac{1}{3}\right)\Gamma\left(\frac{1}{3}\right) =$$

$$= \frac{8}{9}\frac{\pi}{\sin\frac{\pi}{3}} = \frac{8}{9}\cdot\frac{2\pi}{\sqrt{3}} = \frac{16\pi}{9\sqrt{3}}$$

Exc 6 Calculate $\int_0^\infty \sqrt{x^{11}}\, e^{-x}\, dx$

$$I = \int_0^\infty x^{\frac{11}{2}} e^{-x}\, dx = \Gamma\left(\tfrac{13}{2}\right) = \frac{11\cdot 9\cdot 7\cdot 5\cdot 3\cdot 1}{2^6}\sqrt{\pi} \qquad \Gamma(p) = \int_0^\infty x^{p-1} e^{-x}\, dx$$

$p+1 = \frac{11}{2} \Rightarrow p = \frac{9}{2}$