Linear Algebraic Groups

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Contents

1	Aims & Scope	1
2	Basic Concepts	1
3	Jordan Decomposition	8
A	Varieties	11

1. Aims & Scope

These are my personal notes on linear algebraic groups. These notes are based on the book [2].

1.1. History

2. Basic Concepts

Definition 2.1. A **linear algebraic group** is an affine algebraic variety equipped with a group structure such that the group operations (multiplication and inversion) are morphisms of varieties. Let $\mu : G \times G \to G$ denote multiplication and $i : G \to G$ denote inversion. The space $G \times G$ is an affine algebraic variety equipped with the Zariski topology.

Example 2.2.

(1) The additive group $G = (k, +)$ corresponds to the zero ideal in $k[x]$. Addition and inversion are given by polynomials. Hence G is an algebraic group with coordinate ring $k[G] = k[x]$. This group is called the **additive group**, denoted $\mathbf{G}_a$.

(2) The multiplicative group $G = (k^\times, \cdot)$ can be identified with the pairs $V(xy - 1) = \{(x, y) \in k^2 : xy = 1\}$. The multiplication operation on $V(xy - 1)$ is given by the polynomials $(x_1, y_1) \cdot (x_2, y_2) = (x_1x_2, y_1y_2)$ and the inverse is given by the polynomial map $(x, y) \mapsto (y, x)$. In this case, $k[G] \simeq k[x, y]/(xy - 1) \simeq k[x^\pm]$. This group is called the **multiplicative group**, denoted $\mathbf{G}_m$.

(3) The **general linear group** $\mathrm{GL}_n \overset{\text{def}}{=} \{A \in k^{n\times n} : \det(A) \neq 0\}$ is an algebraic group. Indeed, GL_n can be identified with the algebraic set

$$\{(A,t) \in k^{n\times n} \times k : \det(A)\cdot t = 1\}.$$

The multiplication map is given by $(A_1,t_1)\cdot(A_2,t_2) = (A_1A_2, t_1t_2)$ is clearly polynomial. The inverse map is given by the polynomial map $(A,t) \mapsto (t\cdot \mathrm{adj}(A), \det A)$ where $\mathrm{adj}(A)$ is the *adjugate matrix*. The coordinate ring is given by

$$k[\mathrm{GL}_n] = k[Y_{ij}, T]/(\det(Y_{ij})T - 1) = k[Y_{ij}]_{\det(Y_{ij})}.$$

Definition 2.3. A map $\varphi : G_1 \to G_2$ of linear algebraic groups is a **morphism of linear algebraic groups** if it is a group homomorphism and also a morphism of varieties. This means that the induced map $\varphi^* : k[G_2] \to k[G_1]$ is a k-algebra homomorphism.

Example 2.4.

(1) Let $G \subseteq \mathrm{GL}_n$ be a closed subgroup. Then the natural embedding $G \hookrightarrow \mathrm{GL}_n$ is a morphism of linear algebraic groups.
(2) The determinant map $\det : \mathrm{GL}_n \to \mathbf{G}_m$ is a group homomorphism and a morphism of varieties. Indeed, the induced map $k[\mathbf{G}_m] \to k[\mathrm{GL}_n]$ is given by

$$k[T] \simeq k[G_2] \longrightarrow k[Y_{ij}]_{\det(Y_{ij})}, \quad T \mapsto \det(Y_{ij}).$$

Proposition 2.5. *Kernels and images of morphisms of algebraic groups are closed.*

Proof. The kernel is closed because it is the continuous pre-image of a closed point. From Proposition A.1, the image contains a (non-empty) open subset of its closure. It remains to show that if G is a linear algebraic group, $H \subseteq G$ is a subgroup, and H contains a (non-empty) open subset of its closure, then H is closed.

Suppose that H contains a non-empty open subset of $\overline{H}$. Then H is open in $\overline{H}$ since it is the union of translates of this open subset. But it is also closed in $\overline{H}$ since it is the complement of its non-trivial cosets. This implies $H = \overline{H}$ and completes the proof. □

Theorem 2.6. *Let G be a linear algebraic group. Then G can be embedded as a closed subgroup into GL_n for some n.*

The proof of this characterization of linear algebraic groups will be postponed. For an example, we can embed $\mathbf{G}_a \to \mathrm{GL}_2$ via the map

$$t \mapsto \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

This is an isomorphism onto a closed subset.

2.1. Examples of Algebraic Groups In this section we introduce important examples of linear algebraic groups.

2.1.7. Upper Triangular Matrices The group of invertible upper triangular matrices and the subgroup of upper triangular matrices with 1's on the diagonal and the group of diagonal invertible matrices are closed subgroups of GL_n (hence algebraic groups).

$$T_n \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in \mathrm{GL}_n \right\}$$

$$U_n \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}_n \right\}$$

$$D_n \stackrel{\text{def}}{=} \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \in \mathrm{GL}_n \right\}$$

Definition 2.8.

(1) A group G is called **nilpotent** if the **descending central series** defined by

$$\mathscr{C}^0 G \stackrel{\text{def}}{=} G, \quad \mathscr{C}^i G \stackrel{\text{def}}{=} [\mathscr{C}^{i-1} G, G] \text{ for } i \geqslant 1$$

eventually reaches 1.

(2) A group G is called **solvable** if the **derived series** defined by

$$G^{(0)} \stackrel{\text{def}}{=} G, \quad G^{(i)} \stackrel{\text{def}}{=} [G^{(i-1)}, G^{(i-1)}] \text{ for } i \geqslant 1$$

eventually reaches 1. The minimum such d for which $G^{(d)} = 1$ is called the **derived length** of G.

For example U_n is nilpotent and T_n is solvable.

2.1.9. The special linear groups The **special linear group** of $n \times n$-matrices of determinant 1 is a closed subgroup of GL_n.

$$\mathrm{SL}_n \stackrel{\text{def}}{=} \{A \in \mathrm{GL}_n : \det(A) = 1\}.$$

This has coordinate ring $k[\mathrm{SL}_n] = k[T_{ij}]/(\det(T_{ij}) - 1)$.

2.1.10. The symplectic groups For $n \geqslant 1$, let $J_{2n} \stackrel{\text{def}}{=} \begin{pmatrix} 0 & K_n \\ -K_n & 0 \end{pmatrix}$ where $K_n \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. The **symplectic group** in dimension $2n$ is the closed subgroup

$$\mathrm{Sp}_{2n} \stackrel{\text{def}}{=} \{A \in \mathrm{GL}_{2n} : A^t J_{2n} A = J_{2n}\}.$$

This is a closed subgroup of GL_{2n}, but the coordinate ring is complicated. The **conformal symplectic group** is also a closed subgroup of GL_{2n} and it is defined by

$$\mathrm{CSp}_{2n} \stackrel{\text{def}}{=} \{A \in \mathrm{GL}_{2n} : A^t J_{2n} A = cJ_{2n} \text{ for some } c \in k^\times\}.$$

Indeed, we have a morphism $\mathrm{GL}_n \to \mathbb{A}_k^{n^2}$ defined by $A \mapsto A^t J_{2n} A$. The group CSp_{2n} is then the pre-image of a closed subvariety isomorphic to $\mathbb{A}_k^1$.

2.1.11. Odd-dimensional Orthogonal Groups Suppose $\mathrm{char}(k) \neq 2$. The **orthogonal group** in odd dimension $2n+1$ is defined by

$$\mathrm{GO}_{2n+1} = \{A \in \mathrm{GL}_{2n+1} : A^t K_{2n+1} A = K_{2n+1}\}.$$

Alternatively, we can consider the quadratic form

$$f(x_1, \ldots, x_{2n+1}) \stackrel{\text{def}}{=} x_1 x_{2n+1} + x_2 x_{2n} + \ldots + x_n x_{n+2} + x_{n+1}^2.$$

The group of isometries

$$\mathrm{GO}_{2n+1} \stackrel{\text{def}}{=} \{A \in \mathrm{GL}_{2n+1} : f(Ax) = f(x) \text{ for all } x \in k^{2n+1}\}$$

is the **odd dimensional orthogonal group** of k. We have the conformal version

$$\mathrm{CO}_{2n+1} \stackrel{\text{def}}{=} \{A \in \mathrm{GL}_{2n+1} : f(Ax) = cf(x) \text{ for all } x \in k^{2n+1} \text{ and some } c \in k^\times\}.$$

This is called the **odd-dimensional conformal orthogonal group**. The **even-dimensional** versions are defined similarly.

Remark 2.12. Any non-degenerate symmetric bilinear form, non-degenerate skew-symmetric bilinear forms, and quadratic forms lead to the same groups up to conjugacy. The specific versions we chose above give certain natural subgroups a particularly nice shape.

2.1.13. Symmetric Group Let G be a finite group. Then G has a faithful permutation representation $G \hookrightarrow \mathfrak{S}_n$ into a symmetric group. Moreover, we can embed $\mathfrak{S}_n \hookrightarrow \mathrm{GL}_n$ via the permutation representation. Combining these two, we have a closed embedding $G \hookrightarrow \mathrm{GL}_n$ whose image is a closed subgroup. Therefore, any finite group can be considered as a linear algebraic group, with the discrete topology.

2.2. Connectedness We say that a space is **connected** if it cannot be written as the disjoint union of two non-empty proper closed sets. A space is **irreducible** if it cannot be written as the union of two non-empty proper closed sets. Note that irreducible spaces are connected, but not the other way around.

Example 2.14. We give examples of connected linear algebraic groups.

(1) $\mathbf{G}_a$ and $\mathbf{G}_m$ are connected because their coordinate rings are $k[\mathbf{G}_a] = k[x]$ and $k[\mathbf{G}_m] = k[x, x^{-1}]$. These are integral domains, which imply that they are irreducible hence connected.
(2) GL_n is connected because the coordinate ring $k[X_{ij}]_{\det(X_{ij})}$ is an integral domain.

We need the following properties of affine varieties.

Proposition 2.15. *Let X, Y be affine varieties. Then we have:*

(a) A subset Z of X is irreducible if and only if its closure $\overline{Z}$ is irreducible.
(b) If X and Y are irreducible, then $X \times Y$ is irreducible.

Proof. Let C_1 and C_2 be two closed subsets. Then $Z \subseteq C_1 \cup C_2$ if and only if $\overline{Z} \subseteq C_1 \cup C_2$. Part (a) follows immediately. For part (b), it follows from the fact that the tensor product of domains is a domain. □

Proposition 2.16. *Let G be a linear algebraic group.*

(a) The irreducible components of G are pairwise disjoint, so they are the connected components of G.
(b) The irreducible component G° containing $1 \in G$ is a closed normal subgroup of finite index in G.
(c) Any closed subgroup of G of finite index contains G°.

Proof. For part (a), let X and Y be irreducible components of G. Suppose for the sake of contradiction that they intersect. Let $g \in X \cap Y$. Then $g^{-1}X$ and $g^{-1}Y$ are also irreducible components. Thus we can assume that $1 \in X \cap Y$. In this case, $\mu(X \times Y) = XY$ is irreducible since $X \times Y$ is irreducible. But $X \subset XY$ and $Y \subset XY$. Since X and Y are maximal irreducible sets, we have $X = XY = Y$ which completes the proof of (a).

For part (b), let X be the irreducible component containing 1. Then gXg^{-1} is an irreducible component that contains 1. Thus $X = gXg^{-1}$. We also have $X \subseteq X \cdot X$. Since X is maximal irreducible set, this implies $X = X \cdot X$. Also, we have $X = X^{-1}$. Thus X is a normal subgroup of G. The irreducible components are translates of X and they are disjoint. There are only finitely many irreducible components. This implies that X has finite index. Irreducible components are always closed, so it is a closed normal subgroup of finite index in G.

For part (c), let H be a closed subgroup of G of finite index. Then $H^0 \subseteq G^0 \subseteq G$. We have

$$[G : H^0] = [G : H][H : H^0] < \infty$$

Thus $G^0 = \bigsqcup gH^0$ for finitely many cosets. But, since it is connected, it must be equal to H^0. Thus $G^0 = H^0 \subseteq H$. □

Punchline: This implies that for linear algebraic groups, the concepts of *connectedness* and *irreducibility* coincide! Thus, we can refer to *connected* and *irreducible* components of G simply as *components*.

Example 2.17.

(1) Let G be a linear algebraic group, H a proper closed subgroup of finite index. Then G is not connected because then we can split up G into a finite number of cosets of H (H contains G^0). For finite algebraic groups, we always have $G^0 = 1$.

(2) Recall the definition of the odd-dimensional orthogonal group (in the case $\mathrm{char}(k) \neq 2$):

$$GO_{2n+1} = \{A \in GL_{2n+1} : A^t K_{2n+1} A = K_{2n+1}\}.$$

The determinant map $\det : GO_{2n+1} \to \mathbf{G}_m$ has image $\{\pm 1\}$ since $-I_{2n+1} \in GO_{2n+1}$. This shows that

$$GO_{2n+1} \simeq \ker(\det) \times \langle -I_{2n+1} \rangle.$$

This is not connected. Similarly, GO_{2n} is not connected because it has a closed subgroup of index 2.

Definition 2.18. For $n \geqslant 2$, the **special orthogonal group** $SO_n \overset{\text{def}}{=} GO_n^\circ$ is the connected component of the identity in GO_n.

The following fact from algebraic geometry allows one to establish the connectedness of some algebraic group.

Proposition 2.19. *Let G be a linear algebraic group and $\varphi_i : Y_i \to G$ be a family of morphisms from irreducible affine varieties Y_i such that $1 \in G_i \overset{\text{def}}{=} \varphi_i(Y_i)$ for all $i \in I$. Then $H \overset{\text{def}}{=} \langle G_i : i \in I \rangle$ is a closed and connected subgroup. Moreover, there exist $n \in \mathbb{N}$ and $(i_1, \dots, i_n) \in I^n$ such that $H = G_{i_1}^{\pm} G_{i_2}^{\pm} \dots G_{i_n}^{\pm}$.*

Proof. See Theorem 2.4.6 in [1]. □

Example 2.20.

(1) Consider the upper and lower triangular groups

$$U_2^+ = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \simeq \mathbf{G}_a, \quad U_2^- = \left\{ \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \right\} \simeq \mathbf{G}_a.$$

Then one can check $SL_2 = \langle U_2^+, U_2^- \rangle$. This show that SL_2 is connected. In general, we can show that SL_n, T_n, U_n, and D_n are all connected.

(2) Centralizers of elements are not necessarily connected, even in a connected group. Let $G = SL_2$ and $\mathrm{char}(k) \neq 2$. Let

$$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in G.$$

The centralizer is

$$C(g) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in k, a^2 = 1 \right\} = H \sqcup \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} H$$

where we have

$$H = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in k \right\}.$$

H is a closed subgroup of index 2, which implies that $C(g)$ is not connected.

Proposition 2.21. *Let H, K be subgroups of a linear algebraic group G where K is closed and connected. Then $[H, K]$ is closed and connected.*

Proof. K is a linear algebraic group since it is closed. For every $h \in H$, we define the morphism $\varphi_h : K \to G$ by $g \mapsto [h, g]$. Then $[H, K] = \langle \varphi_h(K) \rangle$. From the proposition, we have that $[H, K]$ is closed and connected. □

In particular, if G is connected, then $[G, G]$ is also connected.

2.3. Dimension In this section, we discuss the **dimension** of a linear algebraic group.

Definition 2.22. The following are equivalent definitions for dimension.

(1) Let X be an irreducible variety. The coordinate ring $k[X]$ is an integral domain. Let $k(X)$ be the field of fractions of $k[X]$. Then $\dim X \overset{\text{def}}{=} \operatorname{trdeg}_k(k(X))$, the transcendence degree.
(2) The dimension of X equals the maximal length of decreasing chains of prime ideals in $k[X]$.
(3) If X is a reducible affine variety, then we can decompose it into $X = X_1 \cup \ldots \cup X_r$. We define $\dim(X)$ to be the maximum of $\dim(X_i)$.

In particular, for a linear algebra group, we have $\dim G = \dim G^\circ$. Thus $\dim(G) = 0$ if and only if G is finite.

Proposition 2.23. *Let $\varphi : X \to Y$ be a morphism of irreducible varieties with $\varphi(X)$ dense in Y. Then there is a non-empty open subset $U \subseteq Y$ with $U \subseteq \varphi(X)$ such that*

$$\dim \varphi^{-1}(y) = \dim(X) - \dim(Y)$$

Corollary 2.24. *Let $\varphi : G_1 \to G_2$ be a morphism of linear algebraic groups. Then*

$$\dim(\operatorname{im} \varphi) + \dim(\ker \varphi) = \dim(G_1).$$

Proof. Every fiber is a coset of $\ker(\varphi)$. We can consider the map $\varphi : G_1^\circ \to \operatorname{im}(\varphi)^\circ$. We get

$$\dim \ker(\varphi) = \dim G_1 - \dim(\operatorname{im}(\varphi))$$

which suffices for the proof. □

Example 2.25.

(1) $\dim(\mathbf{G}_a) = 1$ since $k[\mathbf{G}_a] = k[x]$ and every prime ideal has height 1.
(2) $\dim(\mathbf{G}_m) = 1$ since $k[\mathbf{G}_m] = k[x, x^{-1}]$, it has the same field of fractions of $\mathbf{G}_a$.
(3) $\dim \mathrm{GL}_n = n^2$ since the field of fractions is just $k(Y_{ij})$.
(4) $\dim \mathrm{SL}_n = n^2 - 1$ from Corollary 2.24.

Proposition 2.26. *If* Y *is a proper closed subset of an irreducible variety* X*, then* $\dim Y < \dim X$.

Proof. The induced map is a quotient map by a prime ideal. This means we can always extend a chain of prime ideals in $k[Y]$ to a chain of prime ideals in $k[X]$. This suffices for the proof. □

3. Jordan Decomposition

3.1. Decomposition of endomorphisms Recall the statement of the usual Jordan decomposition.

Theorem 3.1 (Jordan-Chevalley Decomposition)**.** *Let* V *be a vector space over a nice enough field* k*. Let* $a \in \mathrm{End}(V)$ *be an endomorphism. Then there are unique* $n, s \in \mathrm{End}(V)$ *with* n *nilpotent and* s *semisimple where* $a = n + s$ *and* $ns = sn$*. In this case, there are polynomials* $P, Q \in T \cdot k[T]$ *such that* $s = P(a)$ *and* $n = Q(a)$*.*

This can easily be translated to a multiplicative Jordan decomposition of $\mathrm{GL}(V)$.

Definition 3.2. An endomorphism $u \in \mathrm{End}(V)$ is called **unipotent** if $u - 1$ is nilpotent.

Proposition 3.3. *For* $g \in \mathrm{GL}(V)$*, there exist unique* $s, u \in \mathrm{GL}(V)$ *with* s *simple and* u *unipotent such that* $g = su = us$*.*

Proof. Let $g = s + n$ be an additive Jordan decomposition. Let $u = 1 + s^{-1}n$. Then, we have

$$su = s(1 + s^{-1}n) = s + n = (1 + s^{-1}n)s = us.$$

This gives a multiplicative Jordan decomposition of g. To prove the uniqueness of the multiplicative Jordan decomposition, suppose that we have two $g = s_1u_1$ and $g = s_2u_2$. Then, we have

$$g = s_1 + s_1(u_1 - 1) = s_2 + s_2(u_2 - 1)$$

are two additive Jordan decompositions. From the uniqueness of the additive version, we have $s_1 = s_2$ and $u_1 = u_2$. □

Definition 3.4. In the above, we call s the **semisimple part**, n the **nilpotent part**, and u the **unipotent part**.

For each $x \in G$, consider the right multiplication map $r_x : G \to G$ given by $r_x(g) = gx$. This is a morphism of affine algebraic varieties. This corresponds to a k-algebra morphism $\rho_x : k[G] \to k[G]$ given by

$$\rho_x(f)(g) \stackrel{\text{def}}{=} f(gx), \quad \text{for all } f \in k[G] \text{ and } g \in G.$$

This gives $k[G]$ an (abstract) left G-action. We have the following "local" result:

Proposition 3.5. *Let G be a linear algebraic group and V a finite-dimensional subspace of $k[G]$. Then there exists a finite-dimensional G-invariant subspace X of $k[G]$ containing V. In particular, $k[G]$ is the union of finite dimensional G-invariant subspaces. Moreover, the restriction to any such finite-dimensional subspace X affords a morphism of algebraic groups $\rho : G \to GL(X)$.*

Proof. It suffices to prove this result for $V = \mathbb{C}\langle f \rangle$ a one-dimensional subspace of $k[G]$. The multiplication map gives a k-algebra morphism

$$\mu^* : k[G] \to k[G \times G] = k[G] \otimes k[G].$$

Let $\mu^*(f) = \sum f_i \otimes g_i$ where the sum is finite. Then $\rho_x(f) = \sum_i g_i(x) f_i$ from Equation 3.6.

$$\begin{tikzcd} G \arrow[r, "g \mapsto (g,x)"] \arrow[rr, bend right, "r_x"'] & G \times G \arrow[r, "\mu"] & G \end{tikzcd}$$

(3.6)

$$\begin{tikzcd} k[G] & k[G] \otimes k[G] \arrow[l, "f \otimes g \mapsto g(x) f"'] & k[G] \arrow[l, "\mu^*"'] \arrow[ll, bend left, "\rho_x"] \end{tikzcd}$$

This implies that $\{\rho_x(f)\}$ is contained in the finite dimension vector space spanned by $\{f_i\}$. Thus $G \cdot V$ is G-invariant and is finite-dimensional. The co-ordinates of ρ_x are clearly polynomial. Thus $g \mapsto \rho_x|_X$ gives a morphism of linear algebraic groups $G \mapsto GL(X)$. $\square$

Let V be a k-vector space. We say $x \in End(V)$ is **locally finite** if V is the union of finite-dimensional x-stable subspaces. We say x is **locally semisimple**, respectively **locally nilpotent**, if its restriction to any finite-dimensional x-stable subspace is semisimple, respectively nilpotent.

Any locally finite endomorphism of V has a Jordan decomposition. We also know that ρ_x is a locally finite endomorphism of $k[G]$ for all $x \in G$. This allows us to prove the following theorem.

Theorem 3.7 (Jordan decomposition)**.** *Let G be a linear algebraic group.*

(a) *For any embedding ρ of G into some $GL(V)$ and for any $g \in G$, there exist unique $g_s, g_u \in G$ such that $g = g_s g_u = g_u g_s$ where $\rho(g_s)$ is semisimple and $\rho(g_u)$ is unipotent.*
(b) *The decomposition $g = g_s g_u = g_u g_s$ is independent of the chosen embedding.*
(c) *Let $\varphi : G_1 \to G_2$ be a morphism of algebraic groups. Then $\varphi(g_s) = \varphi(g)_s$ and $\varphi(g_u) = \varphi(g)_u$.*

Proof. For part (a), the endomorphism $\rho_g : k[G] \to k[G]$ is locally finite, invertible linear transformation. There is a Jordan decomposition $\rho_g = (\rho_g)_s(\rho_g)_u$. Since ρ_g is an algebra morphism, we can deduce that $(\rho_g)_s$ and $(\rho_g)_u$ are also algebra morphisms. Then, we maps

$$f \mapsto (\rho_g)_s(f)(e), \quad f \mapsto (\rho_g)_u(f)(e)$$

are algebra homomorphism $k[G] \to k$. That is, they correspond to points $g_s, g_u \in k[G]$. In other words, for all $f \in k[G]$ we have $(\rho_g)_s(f)(e) = g_s$ and $(\rho_g)_u(f)(e) = g_u$. We have that

$$((\rho_g)_s f)(x) = (\ell(x^{-1})(\rho_g)_s f)(e) = ((\rho_g)_s \ell(x^{-1})f)(e) = \ell(x^{-1})f(g_s) = \rho_{g_s} f(x).$$

Thus, we have $(\rho_g)_s = \rho_{g_s}$ and similarly $(\rho_g)_u = \rho_{g_u}$. We have $\rho_g = \rho_{g_1}\rho_{g_2} = \rho_{g_1 g_2}$ and $\rho_g = \rho_{g_2}\rho_{g_1} = \rho_{g_2 g_1}$. Thus $g = g_u g_s = g_s g_u$. □

Definition 3.8. Let G be a linear algebraic group. The decomposition $g = g_u g_s = g_s g_u$ is called the **Jordan decomposition** of $g \in G$, and g is called semisimple, respectively unipotent, if $g = g_s$, respectively $g = g_u$. We write G_u for the unipotent elements and G_s for the semisimple elements.

Example 3.9.

(1) The additive group $\mathbf{G}_a$ is a unipotent group (and more generally so is U_n). This is because we can embed it into

$$\mathbf{G}_a = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} : a \in k \right\}.$$

(2) $\mathbf{G}_m$ and D_n consist of only semisimple elements.

In general G_u is closed, but G_s is badly behaved. For example, lets look at $G = GL_2$ and $g \in G \backslash G_s$. Then g is conjugate to $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ for some $a \in k^\times$. We can define $\varphi : \mathbf{G}_m \to GL_2$ by $\varphi(t) = \begin{pmatrix} at & 1 \\ 0 & at^{-1} \end{pmatrix}$. Then φ is a morphism and $\varphi(t)$ is semisimple whenever $t \neq \pm 1$. This means $\varphi(\mathbf{G}_m \backslash \{\pm 1\}) \subseteq G_s$. We have

$$\varphi(\mathbf{G}_m) = \varphi\left(\overline{\mathbf{G}_m \backslash \{\pm 1\}}\right) \subseteq \overline{G_s}$$

Thus all non-semisimple elements lie in $\overline{G_s}$ so $\overline{G_s} = G$.

3.2. Unipotent groups

Proposition 3.10. *Let $G \leqslant GL_n$ be a unipotent group. Then there exists $g \in GL_n$ such that $g^{-1}Gg \leqslant U_n$.*

Proof. Let $V = k^n$. We induct on n. For $n = 1$, the only unipotent element in GL_1 is 1. This case is clear. Now suppose that $n > 1$. Suppose that there is a G-invariant proper subspace $0 \neq W \subseteq V$. By picking an appropriate basis, we have

$$G \leqslant \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}.$$

The G-invariance induces natural homomorphisms $\varphi : G \to GL(W)$, $\Phi : G \to GL(V/W)$. By induction, up to a change of basis, we have

$$G \leqslant \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} = U_n$$

as claimed.

If V is a irreducible representation of G, then the elements of g generate the full endomorphism algebra $\mathrm{End}(V)$. Note that $\mathrm{tr}((g-1)h) = \mathrm{tr}(gh) - \mathrm{tr}(h) = 0$ for all $g, h \in G$ since the trace of unipotent elements is n. This implies that $\mathrm{tr}((g-1)x) = 0$ for all $x \in \mathrm{End}(V)$. This can only happen if $g = 1$. But this implies $G = 1$, contradicting irreducibility of G acting on V. □

A. Varieties

Proposition A.1. *Let $\varphi : X \to Y$ be a morphism of varieties. Then $\varphi(X)$ contains a non-empty open subset of $\overline{\varphi(X)}$.*

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