Cluster Algebras

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Abstract. Cluster algebras

Contents

1. Aims & Scope

These are my personal notes on cluster algebras. We mainly follow the book by Sergey Fomin, Lauren Williams, and Andrei Zelevinsky.

2. Quivers

Definition 2.1. A **quiver** is a directed graph. An **ice quiver** is a quiver with a partition of the vertices into **frozen** vertices and **mutable** vertices satisfying the following conditions:

(1) There are no edges between frozen vertices.
(2) There are no loops or oriented 2-cycles.

Definition 2.2 (Matrix Mutation)**.** Let Q be an ice quiver. Let $k \in \nu(Q)$ be a mutable vertex of Q. Then we can construct the mutation of Q at k, denoted $\mu_k(Q)$, via the following modifications in order:

(1) For every pair of edges $i \mapsto k \mapsto j$, we add an edge $i \mapsto j$.
(2) Reverse every edge incident to k.
(3) Remove all oriented 2 cycles until there are none.

The next proposition describes some properties of quiver mutation. They are special cases of Proposition 3.4 which are properties of the more general matrix mutation.

Proposition 2.3.

(1) *Mutation is involutive:* $\mu_k(\mu_k(Q)) = Q$.
(2) *Mutation commutes with simultaneous reversal of all arrows.*
(3) *Let* k *and* l *be two mutable vertices which have no arrows between them. Then* $\mu_k(\mu_l(Q)) = \mu_l(\mu_k(Q))$.

3. Matrix Mutation

Definition 3.1. An $n \times n$ matrix $B = (b_{ij})$ with integer entries is called **skew-symmetrizable** if $d_i b_{ij} = -d_j b_{ji}$ for some positive integers $d_1, \dots, d_n$. An **extended skew-symmetrizable matrix** is a $m \times n$ integer matrix with $m \leqslant n$ such that the top $n \times n$ matrix is skew-symmetrizable.

Definition 3.2. Let $B = (b_{ij})$ be an extended skew-symmetrizable $m \times n$ matrix. For $k \in [n]$, the **matrix mutation** in direction k transforms B into $B' = (b'_{ij})$ whose entries are given by

$$b'_{ij} = \begin{cases} -b_{ij} & \text{if } i = k \text{ or } j = k, \\ b_{ij} + b_{ik}b_{kj} & \text{if } b_{ik} > 0 \text{ and } b_{kj} > 0, \\ b_{ij} - b_{ik}b_{kj} & \text{if } b_{ik} < 0 \text{ and } b_{kj} < 0, \\ b_{ij} & \text{otherwise.} \end{cases} \tag{3.3}$$

Proposition 3.4.
(a) *the mutated matrix* $\mu_k(B)$ *is extended skew-symmetrizable with the same choice of* $d_1, \dots, d_n$.
(b) $\mu_k(\mu_k(B)) = B$.
(c) $\mu_k(-B) = -\mu_k(B)$.
(d) $\mu(B^T) = (\mu_k(B))^T$
(e) *if* $b_{ij} = b_{ji} = 0$*, then* $\mu_i(\mu_j(B)) = \mu_j(\mu_i(B))$.

Proof. Let $B' = (b'_{ij})$ be the mutated matrix. If $i = k$ or $j = k$, we have $b'_{ij} = -b_{ij}$ and $b'_{ji} = -b_{ji}$. From this, we deduce the equality

$$d_i b'_{ij} = -d_i b_{ij} = d_j b_{ji} = -d_j b'_{ji}.$$

If $i, j \neq k$ and $b_{ik}, b_{kj} > 0$, then we have $b'_{ij} = b_{ij} + b_{ik}b_{kj}$ and $b'_{ji} = b_{ji} - b_{jk}b_{ki}$. Then,

$$d_i b'_{ij} = d_i b_{ij} + d_i b_{ik} b_{kj} = -d_j b_{ji} + d_j b_{jk} b_{ki} = -d_j b'_{ji}.$$

We get a similar result if $i, j \neq k$ and $b_{ik}, b_{kj} < 0$. Otherwise, we have $b'_{ij} = b_{ij}$ and $b'_{ji} = b_{ji}$. This completes the proof of part (1).

For part (2), let $B'' = (b''_{ij})$ be the matrix obtained from B after mutating at k twice. When $i = k$ or $j = k$, we have $b''_{ij} = -b'_{ij} = b_{ij}$. When $b_{ik}, b_{kj} > 0$, we have

$$b'_{ij} = b_{ij} + b_{ik}b_{kj}, \quad b'_{ji} = b_{ji} - b_{jk}b_{ki}, \quad b'_{ik} = -b_{ik}, \quad b'_{jk} = -b_{kj}.$$

Then,

$$b''_{ij} = b'_{ij} - b'_{ik}b'_{kj} = b_{ij} + b_{ik}b_{kj} - b_{ik}b_{kj} = b_{ij}.$$

We get a similar result when $b_{ik}, b_{kj} < 0$. In the final case, we get $b''_{ij} = b'_{ij} = b_{ij}$. This completes the proof of part (2).

□

Definition 3.5. Let B be a skew-symmetrizable matrix. The skew-symmetric matrix $S(B) = (s_{ij})$ defined by

$$s_{ij} = \operatorname{sgn}(b_{ij})\sqrt{|b_{ij}b_{ji}|} = \sqrt{\frac{d_i}{d_j}}b_{ij}.$$

From this formulation, we have $S(\mu_k(B)) = \mu_k(S(B))$.

4. Y-Patterns

We want to show that many structural properties of a seed pattern are determined by the mutation class formed by its $n \times n$ exchange matrix and does not depend on the frozen parts.

Given a labeled seed $(\tilde{\mathbf{x}}, \widetilde{B})$ where $\tilde{\mathbf{x}} = (x_1, \dots, x_n, x_{n+1}, \dots, x_m)$ and $\widetilde{B} = (b_{ij})$ is a $m \times n$ extended skew-symmetrizable integer matrix. We define n-tuple $\hat{y} = (\widehat{y_1}, \dots, \widehat{y_n})$ by

$$\hat{y}_j \overset{\text{def}}{=} \prod_{i=1}^{m} x_m^{b_{ij}}.$$

We call these the Y**-variables** of the labeled seed $(\tilde{\mathbf{x}}, \widetilde{B})$. How do these Y-variables behave under mutation? We describe this in the following Theorem. Note that the way they evolve is independent of the extended part of $\widetilde{B}$.

Theorem 4.1. *Let $(\tilde{x}, \widetilde{B})$ and $(\tilde{x}', \widetilde{B}')$ be two labeled seeds related by mutation at k. Let $\hat{\mathbf{y}} = (\hat{y}_1, \dots \hat{y}_n)$ and $\hat{\mathbf{y}}' = (\hat{y}'_1, \dots, \hat{y}'_n)$ be the corresponding Y-variables.*

$$\hat{y}'_j = \begin{cases} \hat{y}_k^{-1} & \text{if } j = k, \\ \hat{y}_j(\hat{y}_k + 1)^{-b_{kj}} & \text{if } j \neq k \text{ and } b_{kj} \leqslant 0, \\ \hat{y}_j(\hat{y}_k^{-1} + 1)^{-b_{kj}} & \text{if } j \neq k \text{ and } b_{kj} \geqslant 0. \end{cases} \tag{4.2}$$

The proof is not difficult! You can just verify the equation in each possible case. Taking this example, we can define the following.

Definition 4.3. A Y**-seed** of a rank n in a field $\mathscr{F}$ is a pair (Y, B) where

- Y is an n-tuple of elements of $\mathscr{F}$;
- B is a skew-symmetrizable $n \times n$ integer matrix.

We say that two Y-seeds (Y, B) and (Y', B') are related by mutation at k if

- The matrices B and B' are related by mutation at k;

- the n-tuple $Y' = (Y'_1, \dots, Y'_n)$ is obtained from $Y = (Y_1, \dots, Y_n)$ by the rules

$$Y'_j = \begin{cases} Y_k^{-1} & \text{if } j = k, \\ Y_j(Y_k+1)^{-b_{kj}} & \text{if } j \neq k \text{ and } b_{kj} \leqslant 0, \\ Y_j(Y_k^{-1}+1)^{-b_{kj}} & \text{if } j \neq k \text{ and } b_{kj} \geqslant 0. \end{cases} \tag{4.4}$$

A Y**-pattern** is exactly a family $(Y(t), B(t))_{t \in \mathbf{T}_n}$ of Y-seeds related by mutations.

We now relate the matrix mutation with the mutation of Y-seeds. We can re-interpret mutation of the bottom part of the extended exchange matrix as a tropicalized version of Y-seed mutation.

Definition 4.5. A **semifield** is an abelian group $(P, \cdot)$ with an operation of "addition" called $\oplus$ which is commutative, associative, and distributive with respect to the abelian group multiplication.

The main example of a semifield that we will be concerned with is the **tropical semifield** in ℓ indeterminates. We denote this by $\mathrm{Trop}(q_1, \dots, q_\ell)$. This consists of Laurent monomials in $q_1, \dots, q_\ell$ with abelian group structure and $\oplus$ structure given by

$$\left(\prod_{i=1}^{\ell} q_i^{a_i}\right) \oplus \left(\prod_{i=1}^{\ell} q_i^{b_i}\right) \stackrel{\text{def}}{=} \prod_{i=1}^{\ell} q_i^{\min\{a_i, b_i\}},$$
$$\left(\prod_{i=1}^{\ell} q_i^{a_i}\right) \cdot \left(\prod_{i=1}^{\ell} q_i^{b_i}\right) \stackrel{\text{def}}{=} \prod_{i=1}^{\ell} q_i^{a_i+b_i}.$$

Let $\widetilde{B}$ be an $m \times n$ extended exchange matrix. Let $x_{n+1}, \dots, x_m$ be the frozen variables. We can encode the bottom $(m-n) \times n$ matrix with the **coefficient variables** $\mathbf{y} = (y_1, \dots, y_n)$ given by

$$y_j \stackrel{\text{def}}{=} \prod_{i=n+1}^{m} x_i^{b_{ij}} \in \mathrm{Trop}(x_{n+1}, \dots, x_m) \quad \text{for } j \in \{1, \dots, n\}.$$

Thus, $\widetilde{B}$ gives the same information as B and $\mathbf{y}$.

Proposition 4.6. *Let $\widetilde{B} = (b_{ij})$ and $\widetilde{B}'$ be two extended exchange matrices related by a mutation at k, and let $\mathbf{y}$ and $\mathbf{y}'$ be the corresponding coefficient variables. Then*

$$y'_j = \begin{cases} y_k^{-1} & \textit{if } j = k, \\ y_j \cdot (y_k \oplus 1)^{-b_{kj}} & \textit{if } j \neq k \textit{ and } b_{kj} \leqslant 0, \\ y_j \cdot (y_k^{-1} \oplus 1)^{-b_{kj}} & \textit{if } j \neq k \textit{ and } b_{kj} \geqslant 0. \end{cases} \tag{4.7}$$

The set $\mathbb{Q}_{>0}(x_1, \dots, x_m)$ can be considered a semifield by the usual multiplication and addition. This is also the universal semifield: given any semifield K and any $t_1, \dots, t_m \in K$, there is a unique semifield morphism $\mathbb{Q}_{>0}(x_1, \dots, x_m) \to K$ sending $x_i \mapsto t_i$ for $i \in [m]$. This is not hard to show. Using this result, we can get the previous proposition immediately.

Definition 4.8. Let $\mathscr{F}$ be a field of rational functions (over $\mathbb{C}$) in some m variables which include the frozen variables $x_{n+1}, \dots, x_m$. A labeled seed (of geometric type) of rank n is a triple $\Sigma(\mathbf{x}, \mathbf{y}, B)$ consisting of

- a **cluster** $\mathbf{x}$, an n-tuple of elements of $\mathscr{F}$ such that the **extended cluster** $\mathbf{x} \cup \{x_{n+1}, \dots, x_m\}$ freely generates $\mathscr{F}$;
- an **exchange matrix** B, a skew-symmetrizable integer matrix;
- a **coefficient tuple** $\mathbf{y}$, an n-tuple of Laurent monomials in the tropical semifield $\mathrm{Trop}(x_{n+1}, \dots, x_m)$.

This contains the information as a labeled seed $(\widetilde{\mathbf{x}}, \widetilde{B})$! We know how B and $\mathbf{y}$ change under mutation. The matrix B mutates via the usual rules of matrix mutation. The coefficient tuple $\mathbf{y}$ mutates based on the tropical Y-seed mutation rule. How does the cluster mutate? This is answered by the next proposition.

Proposition 4.9. *Let $(\mathbf{x}, \mathbf{y}, B)$ and $(\mathbf{x}', \mathbf{y}', B')$ be two seeds related by a mutation at k. Then $(\mathbf{x}', \mathbf{y}', B')$ are obtained from $(\mathbf{x}, \mathbf{y}, B)$ as follows:*

- $B' = \mu_k(B)$;
- $\mathbf{y}'$ *is given by the tropical Y-seed mutation;*
- $\mathbf{x}' = (\mathbf{x} - \{x_k\}) \cup \{x_k'\}$ *where x_k' is defined by the exchange relation*

$$x_k x_k' = \frac{y_k}{y_k \oplus 1} \prod_{b_{ik} > 0} x_i^{b_{ik}} + \frac{1}{y_k \oplus 1} \prod_{b_{ik} < 0} x_i^{-b_{ik}}. \tag{4.10}$$

This is also not hard to see! We can just translate the usual exchange relation into this language.

Remark 4.11 (A few notes)**.** We can fix the frozen variables in the very beginning because these never change in the mutation process. When mutating the extended exchange matrix $\widetilde{B}$, the exchange matrix mutation only occurs with data from the exchange matrix. That is, we do not need the extended part at all to deal with it.

5. Folding

In this section, we introduce a way to construct a quotient exchange matrix, and under certain conditions using equivariant mutation dynamics to construct a quotient version of seeds and patterns.

Definition 5.1. Let Q be an ice quiver. Let $1, 2, \dots, m$ be the vertices where $1, \dots, n$ are mutable and $n+1, \dots, m$ are frozen. Let G be a group acting on the vertices. We say that Q is **G**-admissible if

(1) If $i \sim j$, then i is mutable if and only if j is mutable.
(2) for any i and j, and any $g \in G$, we have $b_{ij} = b_{g(i)g(j)}$.
(3) for mutable indices $i \sim i'$, we have $b_{ii'} = 0$.
(4) for any $i \sim i'$, and any mutable j, we have $b_{ij} b_{i'j} \geqslant 0$.

If G acts on the vertices of a quiver, the idea of the quotient object will be to identify all G-orbits with each other. Condition (1) just says that mutable vertices will all group with each other and frozen ones will group with each other. That way, the frozen vertices will remain frozen in any mutation process. The second condition allows us to define

$$b_{IJ}^{G} \stackrel{\text{def}}{=} \sum_{i \in I} b_{ij} \quad \text{for } j \in J \tag{5.2}$$

in a well-defined way. This is our new exchange matrix. Condition 3 means we won't have loops. Condition 4 means that we won't have oriented 2 cycles.

When Q is G-admissible with matrix $\widetilde{B}$, let $\widetilde{B}^G = (b_{IJ}^G)$. Then $\widetilde{B}^G$ is an extended skew-symmetrizable matrix. (Not hard to check).

Conditions (2) and (4) implies that $b_{IJ}^G > 0$ if and only if $b_{ij} > 0$ for all $i \in I$ and $j \in J$ if and only if for some $i \in I$ and $j \in J$. From Condtion (3), we can mutate all elements in the same orbit without worrying about the order in which we do it.

Lemma 5.3. *Let Q be a G-admissible quiver, with $\widetilde{B} = \widetilde{B}(Q)$. Let K be a mutable G-orbit such that $\mu_K(Q)$ is also G-admissible. Then*

$$\left(\mu_K(\widetilde{B})\right)^G = \mu_K(\widetilde{B}^G).$$

Now that we need the extra condition that $\mu_K(Q)$ is G-admissible as this is not always the case. We are done with folding quivers. Now we want to see if we extend this to seeds. We need some extra conditions to do so.

Definition 5.4. Let G be a group acting on the set of indices $\{1, \dots, m\}$ so that every $g \in G$ maps the set $\{1, \dots, n\}$ to itself. Let m^G denote the number of orbits of this action. Let $\mathscr{F}$ be the field isomorphic to the field of rational functions in m independent variables. Let $\mathscr{F}^G$ be the field isomorphism to the field of rational functions in m^G independent variables. Let $\mathscr{F}_{sf}$ and $\mathscr{F}_{sf}^Q$ be the subtraction-free semifield versions of these fields. That is, if $\mathscr{F} = \mathbb{C}(x_1, \dots, x_m)$ we let $\mathscr{F}_{sf} = \mathbb{Q}_{>0}(x_1, \dots, x_m)$. Let $\psi : \mathscr{F}_{sf} \to \mathscr{F}_{sf}^G$ be a surjective semifield homomorphism.

Let Q be a quiver as above. A seed $\Sigma = (\tilde{\mathbf{x}}, \widetilde{B}(Q))$ in $\mathscr{F}$, with the extended cluster $\tilde{\mathbf{x}} = (x_i)$, is called (G, ψ)**-admissible** if

- Q is G-admissible,
- for any $i \sim i'$, we have $\psi(x_i) = \psi(x_{i'})$.

We define the **folded seed** $\Sigma^G = (\tilde{\mathbf{x}}^G, \widetilde{B}^G)$ in $\mathscr{F}_{sf}^G \subset \mathscr{F}^G$ whose extended exchange matrix $\widetilde{B}^G$ is given as before, and the extended cluster $\tilde{\mathbf{x}} = (x_I)$ has m^G elements x_I indexed by G-orbits and defined by $x_I = \psi(x_i)$ for $i \in I$. Since ψ is surjective, the x_I generate $\mathscr{F}^G$ and are algebraically independent!

Lemma 5.5. *Let $\Sigma = (\tilde{\mathbf{x}}, \widetilde{B})$ be a (G, ψ)-admissible seed. Let K be a mutable G-orbit. If the quiver $\mu_K(Q)$ is G-admissible, then the seed $\mu_K(\Sigma)$ is (G, ψ)-admissible, and moreover $(\mu_K(\Sigma))^G = \mu_K(\Sigma^G)$.*

Definition 5.6. Let G be a group acting on the vertex set of a quiver Q. We say that Q is **globally foldable** with respect to G if Q is G-admissible and moreover for any sequence of mutable G-orbits $J_1, \ldots, J_k$, the quiver $\mu_{J_K} \circ \ldots \circ \mu_{J_1}(Q)$ is G-admissible.

Corollary 5.7. *Let Q be a quiver which is globally foldable with respect to the group G. Let $\Sigma = (\tilde{\mathbf{x}}, \widetilde{B})$ be a seed in the field $\mathscr{F}$ of rational functions freely generated by an extended cluster $\tilde{\mathbf{x}} = (x_i)$. Let $\tilde{\mathbf{x}}^G = (x_I)$ be a collection of formal variables labeled by the G-orbits I, and let $\mathscr{F}^G$ denote the field of rational functions in those variables. Define the surjective homomorphism*

$$\psi : \mathscr{F}_{sf} \to \mathscr{F}_{sf}^G$$

mapping $x_i \mapsto x_I$ so that Σ is a (G, ψ) admissible seed. Then for any mutable G orbits $J_1, \ldots, J_k$ the seed $\mu_{J_k} \circ \ldots \circ \mu_{J_1}(\Sigma)$ is (G, ψ)-admissible, and moreover the folded seeds $((\mu_{J_k} \circ \ldots \circ \mu_{J_1})(\Sigma))^G$ forms a seed pattern in $\mathscr{F}^G$ with initial exchange matrix $(\widetilde{B}(Q))^G$.

References