[Hartshorne] Algebraic Geometry

Alan Yan

Abstract. To be added.

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This document contains my solutions to the exercises in [1].

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3.2. Schemes

Exercise 1. Let A be a ring, let $X = \operatorname{Spec} A$, let $f \in A$ and let $D(f) \subseteq X$ be the open complement of $V((f))$. Show that the locally ringed space $(D(f), \mathscr{O}_X|_{D(f)})$ is isomorphic to $\operatorname{Spec} A_f$.

Proof. The localization map $\ell : A \to A_f$ induces a bijection between ideals of A not containing f and the ideals of A_f. This bijection gives a one-to-one correspondence between the prime ideals of A not containing f and the prime ideals of A_f. In other words, we have a bijection $\varphi : D(f) \to \operatorname{Spec} A_f$. For any ideal $\mathfrak{a}$ of A not containing f, we have $\mathfrak{p} \supseteq \mathfrak{a}$ if and only if $\varphi(\mathfrak{p}) \supseteq \varphi(\mathfrak{a})$. This proves that φ is a homeomorphism. An isomorphism on the sheaf structure is induced by the natural isomorphism

$$(A_f)_{\varphi(\mathfrak{p})} \to A_{\mathfrak{p}}.$$

$\square$

Exercise 2. Let $(X, \mathscr{O}_X)$ be a scheme, and let $U \subseteq X$ be any open subset. Show that $(U, \mathscr{O}_X|_U)$ is a scheme. We call this the **induced scheme structure** on the open set U, and we refer to $(U, \mathscr{O}_X|_U)$ as an open subscheme of X.

Proof. It suffices to assume X is affine. In this case, the distinguished open sets are affine schemes. Since this forms a basis of the topology, this implies that an open set inherits a scheme structure. □

Exercise 3 (Reduced schemes)**.** A scheme $(X, \mathscr{O}_X)$ is **reduced** if for every open set $U \subseteq X$, the ring $\mathscr{O}_X(U)$ has no nilpotent elements.

(a) Show that $(X, \mathscr{O}_X)$ is reduced if and only if for every $p \in X$, the local ring $\mathscr{O}_{X,p}$ has no nilpotent elements.
(b) Let $(X, \mathscr{O}_X)$ be a scheme. Let $(\mathscr{O}_X)_{\text{red}}$ be the sheaf associated to the presheaf $U \mapsto \mathscr{O}_X(U)_{\text{red}}$, where for any ring A, we denote by A_{red} the quotient of A by its ideal of nilpotent elements. Show that $(X, (\mathscr{O}_X)_{\text{red}})$ is a scheme. We call it the **reduced scheme** of X and denote it X_{red}. Show that there is a morphism of schemes $X_{\text{red}} \to X$ which is a homeomorphism on the underlying topological spaces.
(c) Let $f : X \to Y$ be a morphism of schemes, and assume that X is reduced. Show that there is a unique morphism $g : X \to Y_{\text{red}}$ such that f is obtained by composing g with the natural map $Y_{\text{red}} \to Y$.

Proof. Suppose that $\mathscr{O}_{X,p}$ has no nilpotent elements for all $p \in U$. Let $s \in \mathscr{O}_X(U)$ be nilpotent. Then the germ of s at every $p \in U$ must be zero. From the identity axiom, this implies that $s = 0$. Hence $\mathscr{O}_X(U)$ has no nilpotents. Conversely, suppose that $\mathscr{O}_X(U)$ has no nilpotents for all open $U \subseteq X$. Let $(U, \mathscr{O}_X|_U) \simeq \operatorname{Spec} A$ be an affine open containing $p \in X$ and let p corresponding to the prime ideal $\mathfrak{p}$ in $\operatorname{Spec} A$. Then $\mathscr{O}_{X,p} \simeq A_{\mathfrak{p}}$. From our hypothesis A is reduced. This implies that $A_{\mathfrak{p}}$ is reduced which completes the proof to Part (a).

To construct a morphism $X_{\text{red}} \to X$, we need a continuous map $\varphi : X \to X$ and a sheaf morphism $\varphi^\# : \mathscr{O}_X \to (\mathscr{O}_X)_{\text{red}}$. We can pick $\varphi = \text{id}_X$ and $\varphi^\#$ to be the composition $\mathscr{O}_X \to (\mathscr{O}_X)_{\text{red}}^{\text{pre}} \to (\mathscr{O}_X)_{\text{red}}$. This completes Part (b).

For Part (c), the fact that $\mathscr{O}_X$ is reduced implies that the morphism $\mathscr{O}_Y \to \mathscr{O}_X$ can be factored through $(\mathscr{O}_Y)_{\text{red}}^{\text{pre}}$. The rest follows from the universal property of the sheafification. □

Exercise 4. Let A be a ring and let $(X, \mathscr{O}_X)$ be a scheme. Given a morphism $f : X \to \operatorname{Spec} A$, we have an associated map on sheaves $f^\# : \mathscr{O}_{\operatorname{Spec} A} \to f_*\mathscr{O}_X$. Taking global sections we obtain a homomorphism $A \to \Gamma(X, \mathscr{O}_X)$. Thus there is a natural map

$$\alpha : \operatorname{Hom}_{\mathsf{Sch}}(X, \operatorname{Spec} A) \longrightarrow \operatorname{Hom}_{\mathsf{Rings}}(A, \Gamma(X, \mathscr{O}_X)).$$

Show that α is bijective.

Proof. We begin with a ring homomorphism $\phi : A \to \Gamma(X, \mathscr{O}_X)$. For any $p \in X$, we can consider the composition $\phi_p : A \to \Gamma(X, \mathscr{O}_X) \to \mathscr{O}_{X,p}$. This defines a set-map

$f : X \to \operatorname{Spec} A$ defined by $f(p) \stackrel{\text{def}}{=} \phi_p^{-1}(m_{X,p})$. This is continuous because locally this is exactly the map when we restrict to affine open subschemes. For example, if we consider instead

$$A \to \Gamma(X, \mathscr{O}_X) \to \Gamma(\operatorname{Spec} B, \mathscr{O}_X|_{\operatorname{Spec} B}) \simeq B$$

and then composition $A \to B \to B_{\mathfrak{p}}$ for $\mathfrak{p} \in \operatorname{Spec} B$ this is exactly our map $\operatorname{Spec} B \subseteq X \to \operatorname{Spec} A$ and corresponds exactly to the morphism $A \to B$. Since locally it is continuous, it must be continuous on $X \to \operatorname{Spec} A$. From the gluing and identity axiom on affine opens, we see that there exists a unique scheme morphism $X \to \operatorname{Spec} A$ corresponding to the ring morphism $A \to \Gamma(X, \mathscr{O}_X)$. □

Exercise 5. Describe $\operatorname{Spec} \mathbb{Z}$, and show that it is a final object in the category of schemes.

Proof. The points of $\mathbb{Z}$ are (0) and (p) for primes $p \in \mathbb{Z}$. The zero ideal is the generic point. The ideals (p) are closed points. The closed sets in the Zariski topology are the whole space, the empty set, and finite sets of points not containing (0). The open sets are the whole space, the empty set, and infinite number of points including (0). The sections over the whole space is $\mathbb{Z}$, the sections over the empty set is 0, and the sections over the set $\{(p_1), \dots, (p_n)\}^c$ is

$$\mathscr{O}_{\operatorname{Spec} \mathbb{Z}}(\operatorname{Spec} \mathbb{Z} \backslash \{(p_1), \dots, (p_n)\}) \simeq \mathbb{Z}\left[\frac{1}{p_1}, \dots, \frac{1}{p_n}\right]$$

The fact that $\operatorname{Spec} \mathbb{Z}$ is a final object follows from Exercise 4. □

Exercise 6. Describe the spectrum of the zero ring, and show that it is an initial object for the category of schemes. (According to our conventions, all ring homomorphisms must take 1 to 1. Since $0 = 1$ in the zero ring, we see that each ring R admits a unique homomorphism to the zero ring, but that there is no homomorphism from the zero ring to R unless $0 = 1$ in R.)

Proof. The spectrum of the zero ring is empty with sections $\mathscr{O}(\emptyset) = 0$. Since any ring has a unique morphism to the 0 ring, the spectrum of the zero ring is the initial object in the category of schemes. □

Exercise 7. Let X be a scheme. For any $x \in X$, let $\mathscr{O}_x$ be the local ring at x, and m_x its maximal ideal. We define the **residue field** of x on X to be the field $k(x) = \mathscr{O}_x/m_x$. Now let K be any field. Show that to give a morphism of $\operatorname{Spec} K$ to X it is equivalent to give a point $x \in X$ and an inclusion map $k(x) \to K$.

Proof. Let $(\phi, \phi^{\#}) : \operatorname{Spec} K \to X$ be a morphism of schemes. Since $\operatorname{Spec} K$ consists of a single point, there is some $p \in X$ such that $\phi(\bullet) = p$. We then have a sheaf morphism

$$\phi^{\#} : \mathscr{O}_X \to \phi_* \mathscr{O}_{\operatorname{Spec} K}.$$

For $U \subseteq X$ not containing p, we have $\phi_* \mathscr{O}_{\operatorname{Spec} K}(U) = 0$ so the map is uniquely determined. When $p \in U$, then we have $\phi^{\#} : \mathscr{O}_X(U) \to K$. For all such U, it factors through $\mathscr{O}_{X,p}$ by taking the induced map on stalks:

$$\begin{tikzcd} \mathscr{O}_X(U) \arrow[r] \arrow[d] & K \arrow[d, equal] \\ \mathscr{O}_{X,p} \arrow[r] & K \end{tikzcd}$$

Since the morphism is a morphism of local rings, it must be the case that the preimage of the zero ideal in K is $m_{X,x}$. Thus, this induces a map $k(x) \to k$ which must be an injection since homomorphisms between fields are always injections. □

Exercise 8. Let X be a scheme. For any point $x \in X$, we define the **Zariski tangent space** T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k, and let $k[\varepsilon]/\varepsilon^2$ be the **ring of dual numbers** over k. Show that to give a k-morphism of $\operatorname{Spec} k[\varepsilon]/\varepsilon^2$ to X is equivalent to giving a point $x \in X$, **rational over** k (i.e., such that $k(x) = k$), and an element of T_x.

Proof. The dual numbers have only one prime ideal (ε). To get a morphism $\operatorname{Spec} k[\varepsilon]/\varepsilon^2$ to X on the topological level, we need to pick $x \in X$ for the single point to map to. Once we have this point, we need a morphism $\mathscr{O}_{X,x} \to k[\varepsilon]/(\varepsilon^2)$. The maximal ideal must map into the maximal ideal. Hence, this induces a map $k(x) \to k$. But from the condition of being schemes over $\operatorname{Spec}(k)$, we have a composition $k \to k(x) \to k$ which is the identity. This proves that x is rational over k. To pick the morphism $\mathscr{O}_{X,x} \to k[\varepsilon]/(\varepsilon^2)$, first note that m_x must map to (ε) and since $\varepsilon^2 = 0$, we know that m_x^2 maps to 0. Thus the locations of m_x are determined by a linear map $m_x/m_x^2 \to (\varepsilon) \simeq k$. This gives us our element of T_x and also determines the values of m_x in the morphism $\mathscr{O}_{X,x} \to k[\varepsilon]/(\varepsilon^2)$. The rest of the map comes from the k-scheme morphism and the fact that $k \to k(x) \to k$ is the identity. □

Exercise 9. If X is a topological space, and Z can irreducible closed subset of X, a generic point for Z is a point ζ such that $Z = \overline{\{\zeta\}}$. If X is a scheme, show that every irreducible closed subset has a unique generic point.

Proof. First suppose that X is affine. Closed subsets are of the form $V(I)$ where I is a radical ideal. I claim that $V(I)$ is irreducible if and only if I is prime. If I is prime, then it is irreducible because it is the closure of a single point. Now suppose $V(I)$ is irreducible. Suppose that $f, g \in A$ satisfy $fg \in I$. Then $V(I) = (V(I) \cap V(f)) \cup (V(I) \cap V(g))$ which implies that $V(f) \supseteq V(I)$ or $V(g) \supseteq V(I)$ which implies that I is prime. This proves that all irreducible closed subsets are of the form $V(\mathfrak{p})$. Clearly, the unique generic point in this case is $\mathfrak{p}$.

Now let X be a general scheme. Let Z be an irreducible closed subset of X. Let U be an affine open subset of X. Then $Z \cap U$ is an irreducible closed subset of U. From the previous paragrpah, there is a unique point $z \in Z \cap U$ such that the closure of z in U is $Z \cap U$. But this is an open set of Z which is dense since Z is irreducible. So the closure of z in X is the whole of Z. On the other hand, Z has at most 1 generic point. If it had two, if we intersect with an affine open, the affine open must contain both generic points. But we already proved that in affine open there is at most one generic point. This suffices for the proof. □

Exercise 10. Describe $\mathbb{R}[x]$. How does the topological space compare to $\mathbb{R}$ or $\mathbb{C}$?

Proof. The points are (0), $(x - a)$, and $(x^2 + k)$ for $k > 0$. □

Exercise 11. Let $k = \mathbb{F}_p$ be the finite field with p elements. Describe $\operatorname{Spec} k[x]$. What are the residue fields of its points? How many points are there with a given residue field?

Proof. The points of $\operatorname{Spec} k[x]$ are the prime ideals (0) and (f) where f is an irreducible polynomial of $k[x]$. The prime ideal (0) is the generic point and (f) are closed points. The residue field at (0) is $\mathbb{F}_p(x)$ and the residue field at (f) is $\mathbb{F}_{p^{\deg f}}$.

To figure out the number of points with residue field $\mathbb{F}_{p^d}$, we only need to count the number of monic irreducible polynoimals of degree d in $\mathbb{F}_p[x]$. To count this, we prove two facts:

(1) An irreducible polynomial divides $x^{p^n} - x$ if and only if its degree divides n.
(2) $x^{p^n} - x$ has distinct irreducible factors.

Let f be an irreducible polynomial of degree d with $d|n$. Then $\mathbb{F}_p[x]/(f) \simeq \mathbb{F}_{p^d}$. Since $d|n$, any element in $\mathbb{F}_{p^d}$ is a solution to $x^{p^n} - x = 0$. This implies that $x^{p^n} - x \in (f)$ which implies $f|x^{p^n} - x$. Conversely, suppose that f is an irreducible polynomial dividing $x^{p^n} - x$. We already know that $f|x^{p^d} - x|x^{p^n} - x$ which proves (1). For (2), just take a derivative! □

Exercise 12 (Glueing Lemma)**.** Let $\{X_i\}$ be a family of schemes (possibly infinite). For each $i \neq j$, suppose given an open subset $U_{ij} \subseteq X_i$, and let it have the induced scheme structure. Suppose also given for each $i \neq j$ an isomorphism of schemes $\varphi_{ij} : U_{ij} \to U_{ji}$ such that

(1) for each i, j, $\varphi_{ji} = \varphi_{ij}^{-1}$, and
(2) for each i, j, k, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$.

Then show that there is a scheme X, together with morphisms $\psi_i : X_i \to X$ for each i, such that

(1) ψ_i is an isomorphism of X_i onto an open subscheme of X,
(2) the $\psi_i(X_i)$ cover X,
(3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$, and
(4) $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}.

We say that X is obtained by **glueing** the schemes X_i along the isomorphisms φ_{ij}.

Proof. We first describe the topological space X. Let $X = \bigsqcup X_i / \sim$ where $x_i \in X_i$ and $x_j \in X_j$ satisfy $x_i \sim x_j$ if and only if $x_i \in U_{ij}$, $x_j \in U_{ji}$, and $x_j = \varphi_{ij}(x_i)$. It is easy to see that this is an equivalence relation.

For every index i, we have a map $\ell_i : X_i \to X$ which is the natural inclusion $X_i \to \bigsqcup X_i$ and then projecting to X. We equip X with the quotient topology. In this topology, $U \subseteq X$ is open if and only if $\ell_i^{-1}(U) \subseteq X_i$ is open for all i. To define the sheaf structure on X, we first define the notation of a family of compatible sections over $U \subseteq X$.

Definition 3.1. A **family of compatible sections** over $U \subseteq X$ is a family of sections $s_i \in \mathscr{O}_{X_i}\left(\ell_i^{-1}(U)\right)$ such that for every $i \neq j$, we have

$$\varphi_{ij}^{\#}\left(s_j|_{\ell_j^{-1}(U)\cap U_{ji}}\right) = s_i|_{\ell_i^{-1}(U)\cap U_{ij}}$$

Now that we have defined families of compatible sections over open subsets of X, we can equip the X with the presheaf of rings defined by

$$\mathscr{O}_X(U) \stackrel{\text{def}}{=} \left\{\left(s_i \in \mathscr{O}_{X_i}(\ell_i^{-1}(U))\right) : \text{ family of compatible sections}\right\}.$$

The restriction maps are component-wise restriction. The gluing axiom and identity axiom follow from the gluing and identity axiom of each $\mathscr{O}_{X_i}$. Thus, we have realized X as a ringed space.

For every i, we have a natural morphism $\psi_i : \left(X_i, \mathscr{O}_{X_i}\right) \to (X, \mathscr{O}_X)$ where the continuous map $\psi_i : X_i \to X$ is the continuous map $\ell_i : X_i \to X$ and the morphism of sheaves

$$\psi_i^{\#} : \mathscr{O}_X \to (\ell_i)_* \mathscr{O}_{X_i}$$
$$\psi_i^{\#}(U) : \mathscr{O}_X(U) \to \mathscr{O}_{X_i}(\ell_i^{-1}(U))$$

which is given by projection. It is clear that $\ell_i(X_i)$ is an open subset of X and ψ_i induces an isomorphism between $(X_i, \mathscr{O}_{X_i})$ and $(\psi_i(X_i), \mathscr{O}_X|_{\psi_i(X_i)})$. This proves that X is a scheme. $\square$

Exercise 13. A topological space is **quasi-compact** if every open cover has a finite subcover.

(a) Show that a topological space is noetherian if and only if every open subset is quasi-compact.
(b) If X is an affine scheme, show that $\operatorname{sp}(X)$ is quasi-compact, but not in general noetherian. We say that a scheme is **quasi-compact** if $\operatorname{sp}(X)$ is.
(c) If A is a noetherian ring, show that $\operatorname{sp}(\operatorname{Spec} A)$ is a noetherian topological space.
(d) Give an example where $\operatorname{sp}(\operatorname{Spec} A)$ can be noetherian even when A is not.

Proof. Suppose that X is a noetherian topological space. This means that it satisfies the ascending chain condition on open sets. Let U be an open set and consider an open cover $\{U_i\}$. If there is no finite subcover, we can find a strictly increasing sequence of open sets contained in U. But this contradicts the ascending chain condition. Conersely, suppose that every open set is quasi-compact. Take an ascending chain of open sets. Their union has a fintie subcover. This proves (a).

Let $X = \operatorname{Spec} A$. It suffices to prove that an open cover by distinguished open sets has a finite subcover. Suppose that

$$\operatorname{Spec} A = \bigcup D(f_i).$$

This means that any prime ideal of A does not contain at least one f_i. Thus, the ideal $I = (f_i)$ is not contained in any prime ideal, which means $1 \in I$. So $1 = \sum c_i f_i$ for some finitely many i. Taking the corresponding f_i we have our finite subcover. Affine schemes are not necessarily noetherian. Consider $A = k[x_1, x_2, \ldots]$ the polynomial ring with infinitely many variables. Then $\bigcup D(x_i)$ is an open set with no finite subcover.

Suppose that A is noetherian. Now consider a descending chain of closed sets $Z(I_1) \supset Z(I_2) \supseteq \ldots$ where the I_k are radical ideals. This corresponds to ascending chain of ideals $I_1 \subseteq I_2 \subseteq \ldots$ which is eventually stationary from noetherianness of A. This proves that $\operatorname{sp}(\operatorname{Spec} A)$ is noetherian.

For (d), consider $A = k[x_1, x_2, \ldots]/(x_1^2, x_2^2, \ldots)$. Any prime ideal of A must contain $x_1, x_2, \ldots$ since they are nilpotent. Thus the only prime ideal is $(x_1, x_2, \ldots)$. Hence $\operatorname{sp}(\operatorname{Spec} A)$ is automatically noetherian. But A is not noetherian because $(x_1, x_2, \ldots)$ is not finitely generated.

Fact 3.2. $\operatorname{Spec} A$ *is a noetherian topological space if and only if* A *satisfies ascending chain condition for radical ideals.*

□

Exercise 14.

(a) Let S be a graded ring. Show that $\operatorname{Proj} S = \varnothing$ if and only if every element of S_+ is nilpotent.

(b) Let $\varphi : S \to T$ be a graded homomorphism of graded rings. Let $U = \{\mathfrak{p} \in \operatorname{Proj} T : \mathfrak{p} \not\supseteq \varphi(S_+)\}$. Show that U is an open subset of $\operatorname{Proj} T$, and show that φ determines a natural morphism $f : U \to \operatorname{Proj} S$.

(c) The morphism f can be an isomorphism even when φ is not. For example, suppose that $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geqslant d_0$, where d_0 is an integer. Then show that $U = \operatorname{Proj} T$ and the morphism $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is an isomorphism.

(d) Let V be a projective variety with homogeneous coordinate ring S. Show that $t(V) \simeq \operatorname{Proj} S$.

Proof. Skipped. I don't want to listen to Proj □

Exercise 15.

(a) Let V be a variety over the algebraically closed field k. Show that a point $p \in t(V)$ is a closed point if and only if its residue field is k.

(b) If $f : X \to Y$ is a morphism of schemes over k, and if $p \in X$ is a point with residue field k, then $f(p) \in Y$ also has residue field k.

(c) Now show that if V, W are any two varieties over k, then the natural map

$$\mathrm{Hom}_{\mathrm{Var}}(V, W) \to \mathrm{Hom}_{\mathrm{Sch}/k}(t(V), t(W))$$

is bijective.

Proof. Skipped. I don't quite understand the functor from varieties to schemes just yet in detail □

Exercise 16. Let X be a scheme, let $f \in \Gamma(X, \mathscr{O}_X)$, and define X_f to be the subset of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal m_x of the local ring $\mathscr{O}_x$.

(a) If $U = \operatorname{Spec} B$ is an open affine subscheme of X, and if $\bar{f} \in B = \Gamma(U, \mathscr{O}_X|_U)$ is the restriction of f, show that $U \cap X_f = D(\bar{f})$. Conclude that X_f is an open subset of X.

(b) Assume that X is quasi-compact. Let $A = \Gamma(X, \mathscr{O}_X)$, and let $a \in A$ be an element whose restriction to X_f is 0. Show that for some $n > 0$, $f^n a = 0$.

(c) Now assume that X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact. Let $b \in \Gamma(X_f, \mathscr{O}_{X_f})$. Show that for some $n > 0$, $f^n b$ is the restriction of an element of A.

(d) With the hypothesis of (c), conclude that $\Gamma(X_f, \mathscr{O}_{X_f}) \simeq A_f$.

Proof. For Part (a), it suffices to prove that for $f \in B$, then open set $D(f) \subseteq \operatorname{Spec} B$ can be characterized as the subset of primes $\mathfrak{p} \in \operatorname{Spec} B$ with $f \notin \mathfrak{p}B_\mathfrak{p}$. Thus, we want to prove that $f \in \mathfrak{p}$ in B if and only if $f \in \mathfrak{p}B_\mathfrak{p}$ in $B_\mathfrak{p}$. One direction is easy. For the other direction, suppose $f \in \mathfrak{p}B_\mathfrak{p}$ in $B_\mathfrak{p}$. Then $f = x/y$ where $x \in \mathfrak{p}$ and $y \notin \mathfrak{p}$. Then $z(yf - x) = 0$ in B for $z \notin \mathfrak{p}$. This implies that $f \in \mathfrak{p}$ in B which suffices for the proof of (a).

For Part (b), it suffices to prove this when X is affine because we can just pick the largest n on a finite cover by affines (which exist since X is quasi-compact). But the restriction of $a \in A$ being zero on X_f implies that $a = 0$ in A_f. This exactly translates to $f^n a = 0$ for some n. This proves (b).

For Part (c), let $U_i = \operatorname{Spec} A_i$ be the finite affine open cover of X. Since $X_f \cap U_i = D_{A_i}(f)$, the section b on $X_f \cap U_i$ comes from some $b_i \in \Gamma(U_i, \mathscr{O}_{U_i})$ and

$$b|_{X_f \cap U_i} = \frac{b_i}{f^N}$$

where we can pick N to be large enough to hold for all affine opens in our finite cover. In particular, we have $f^N b|_{X_f \cap U_i} = b_i|_{X_f \cap U_i}$ for all i. It remains to glue together the $b_i \in \Gamma(U_i, \mathscr{O}_{U_i})$. But this follows because on $U_i \cap U_j$, we can pick M large enough so that $f^M(b_i - b_j) = 0$. We can pick M large enough to hold for all finite pairs i, j. Thus $f^{M+N} b$ comes from a global section.

For Part (d), first consider the restriction map $A \to \Gamma(X_f, \mathscr{O}_{X_f})$. To induce a map $A_f \to \Gamma(X_f, \mathscr{O}_{X_f})$ we want to prove that $f|_{X_f}$ is invertible. Let $f_i \in \Gamma(U_i \cap$

$X_f, \mathscr{O}_{U_i \cap X_f})$ be an element on the affine open which is the inverse of $f|_{U_i \cap X - f}$. That is, we have $f|_{U_i \cap X_f} f_i = 1$ on $U_i \cap X_f$. To prove that we can glue the f_i together, we look at $f_i - f_j$ on $U_i \cap U_j \cap X_f$ and prove that it is zero. But this follows because on stalks f is invertible, which means that the stalk of $f_i - f_j$ is 0 on all points in $U_i \cap U_j \cap X_f$. This implies that we have a well-defined map

$$A_f \longrightarrow \Gamma\left(X_f, \mathscr{O}_{X_f}\right)$$

From (b), we know that this is injective. This is surjective from (c). This suffices for the proof. □

Exercise 17. *A Criterion for Affineness*.

(a) Let $f : X \to Y$ be a morphism of schemes, and suppose that Y can be covered by open subsets U_i, such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.

(b) A scheme X is affine if and only if there is a finite set of elements

$$f_1, \ldots, f_r \in A = \Gamma(X, \mathscr{O}_X),$$

such that the open subsets X_{f_i} are affine, and $f_1, \ldots, f_r$ generate the unit ideal in A.

Proof. We solve part (a). Since $f^{-1}(U_i) \to U_i$ is an isomorphism, on the level of topological spaces we have homeomorphisms $f^{-1}(U_i) \to U_i$. This implies that $f : X \to Y$ is a homeomorphism (standard topological fact). It remains to show that the map on sheaves

$$f^{\#} : \mathscr{O}_Y \to f_* \mathscr{O}_X$$

is an isomorphism. We know that

$$f^{\#}|_{U_i} : \mathscr{O}_Y|_{U_i} \to f_* \mathscr{O}_X|_{U_i}$$

is an isomorphism for all U_i. But this implies that the induced map on stalks are isomorphisms which implies that $f^{\#}$ is an isomorphism. This proves (a).

We solve part (b). If X is affine, then we can let $f_1 = 1$ and this trivially generates all of A while generating the unit ideal. Conversely, suppose that we have a finite set of elements $f_1, \ldots, f_r \in \Gamma(X, \mathscr{O}_X)$ such that X_{f_i} are affine and $1 = (f_1, \ldots, f_r)$. From the previous exercise, we know that $X_{f_i} \simeq \operatorname{Spec} A_{f_i}$. Since f_i generate A, we know that the X_{f_i} cover X. From exercise 4, the canonical isomorphism $A \to \Gamma(X, \mathscr{O}_X)$ gives us a natural scheme morphism $X \to \operatorname{Spec} A$. This maps X_f to $D(f)$ and is an isomorphism (you can see this by reviewing the construction in Exercise 4). This implies that X and $\operatorname{Spec} A$ are isomorphic. □

Exercise 18. In this exercise, we compare some properties of a ring homomorphism to the induced morphism of the spectra of the rings.

(a) Let A be a ring, $X = \operatorname{Spec} A$, and $f \in A$. Show that f is nilpotent if and only if $D(f)$ is empty.

(b) Let $\varphi : A \to B$ be a homomorphism of rings, and let $f : Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$ be the induced morphism of affine schemes. Show that φ is injective if and only if the map of sheaves $f^\# : \mathscr{O}_X \to f_*\mathscr{O}_Y$ is injective. Show furthermore in that case f is *dominant*, i.e., $f(Y)$ is dense in X.

(c) With the same notation, show that if φ is surjective, then f is a homeomorphism of Y onto a closed subset of X, and $f^\# : \mathscr{O}_X \to f_*\mathscr{O}_Y$ is surjective.

(d) Prove the converse to (c), namely, if $f : Y \to X$ is a homeomorphism onto a closed subset, and $f^\# : \mathscr{O}_X \to f_*\mathscr{O}_Y$ is surjective, then φ is surjective.

Proof. This problem has four parts.

(a) This part follows from the fact that the intersection of all prime ideals is exactly the nilpotent elements.

(b) Suppose that $f^\# : \mathscr{O}_X \to f_*\mathscr{O}_Y$ is injective. Then the induced map on the global sections

$$f^\# (\operatorname{Spec} A) : \Gamma(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A}) \to \Gamma(\operatorname{Spec} B, \mathscr{O}_{\operatorname{Spec} B})$$

is exactly our original ring homomorphism $\varphi : A \to B$. This implies that φ is injective.

Conversely, suppose that $\varphi : A \to B$ is injective. To prove that $f^\# : \mathscr{O}_X \to f_*\mathscr{O}_Y$ is injective, it is enough to prove that it is injective on distinguished open sets of $\operatorname{Spec} A$. Consider an arbitrary $f \in A$. The induced map on sections of $D(f)$ becomes the localization map

$$f^\#(D(f)) : A_f \simeq \mathscr{O}_{\operatorname{Spec} A}(D(f)) \to \mathscr{O}_{\operatorname{Spec} B}(D(\varphi(f))) \simeq B_{\varphi(f)}$$

which is injective. To prove that $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is dominant, let $D(f)$ be an open set not intersection $f(\operatorname{Spec} B)$. But then $\varphi(f)$ is contained in all prime ideal of B. This implies that $\varphi(f)$ is nilpotent. Since φ is injective, we know that f is nilpotent. So $D(f)$ is empty which suffices for the proof.

Alternatively, we can look at the stalk of $f^\# : \mathscr{O}_{\operatorname{Spec} A} \to f_*\mathscr{O}_{\operatorname{Spec} B}$. As we pointed out before, the induced map on $D(f)$ is

$$f^\#(D(f)) : A_f \simeq \mathscr{O}_{\operatorname{Spec} A}(D(f) \to \mathscr{O}_{\operatorname{Spec} B}(D(\varphi(f))) \simeq B_{\varphi(f)} \simeq B \otimes_A A_f$$

where we view B as an A-module via the morphism $\varphi : A \to B$. Since colimits commute with tensor products, the induced map on stalks is

$$f^\#_{\mathfrak{p}} : A \otimes_A A_{\mathfrak{p}} \to B \otimes_A A_{\mathfrak{p}}.$$

Since localization of modules is an exact functor, the injectivity or surjectivity of φ implies the injectivity or surjectivity of $f^\#$.

(c) Since φ is surjective, from the previous argument we know $f^\#$ is surjective. We only need to show that $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is a closed embedding. We can assume that the ring homomorphism is of the form $A \to A/I$ where I is an ideal. Then the map $f : \operatorname{Spec}(A/I) \to \operatorname{Spec} A$ is a homeomorphism on $V(I)$. Moreover, we have $V_{A/I}(\mathfrak{p}) = V_A(f(\mathfrak{p}))$ which proves that they are homeomorphic.

(d) This follows from the previous observation about the induced map on stalks. The hypothesis that $f : Y \to X$ is a homeomorphism onto a closed subset is not needed.

$\square$

Exercise 19. Let A be a ring. Show that the following conditions are equivalent:

(i) $\operatorname{Spec} A$ is disconnected;
(ii) there exist nonzero elements $e_1, e_2 \in A$ such that $e_1e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$, and $e_1 + e_2 = 1$. (These elements are called orthogonal idempotents).
(iii) A is isomorphic to a direct product $A_1 \times A_2$ of non-zero rings.

Proof. Suppose $\operatorname{Spec} A$ is disconnected. Then we can write $\operatorname{Spec} A = V(I_1) \sqcup V(I_2)$ where $V(I_1)$ and $V(I_2)$ are clopen. Then

$$A \simeq \mathscr{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \simeq \mathscr{O}_{\operatorname{Spec} A}(V(I_1)) \times \mathscr{O}_{\operatorname{Spec} A}(V(I_2)).$$

This implies (iii). (iii) clearly implies (ii). (ii) clearly implies (iii) with the isomorphism $A \simeq e_1A \times e_2A$. (iii) clearly implies (i). This suffices for the proof. $\square$

References

[1] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, Springer, 1977. ←1

Department of Mathematics, Harvard University, Cambridge, MA 02138
Email address: alanyan@math.harvard.edu