[Hartshorne] Algebraic Geometry exercise solutions

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Abstract. To be added.

Contents

1. Introduction

2. Chapter 1: Varieties

3. Chapter 2: Schemes

3.1. Sheaves

3.2. Schemes

Exercise 3.1. Let A be a ring, let $X = \operatorname{Spec} A$, let $f \in A$ and let $D(f) \subseteq X$ be the open complement of $V((f))$. Show that the locally ringed space $(D(f), \mathscr{O}_X|_{D(f)})$ is isomorphic to $\operatorname{Spec} A_f$.

Proof. The localization map $\ell : A \to A_f$ induces a bijection between ideals of A not containing f and the ideals of A_f. This bijection gives a one-to-one correspondence between the prime ideals of A not containing f and the prime ideals of A_f. In other words, we have a bijection $\varphi : D(f) \to \operatorname{Spec} A_f$. For any ideal $\mathfrak{a}$ of A not containing f, we have $\mathfrak{p} \supseteq \mathfrak{a}$ if and only if $\varphi(\mathfrak{p}) \supseteq \varphi(\mathfrak{a})$. This proves that φ is a homeomorphism. An isomorphism on the sheaf structure is induced by the natural isomorphism

$$(A_f)_{\varphi(\mathfrak{p})} \to A_{\mathfrak{p}}.$$

$\square$

Exercise 3.2. Let $(X, \mathscr{O}_X)$ be a scheme, and let $U \subseteq X$ be any open subset. Show that $(U, \mathscr{O}_X|_U)$ is a scheme. We call this the **induced scheme structure** on the open set U, and we refer to $(U, \mathscr{O}_X|_U)$ as an open subscheme of X.

Proof. It suffices to assume X is affine. In this case, the distinguished open sets are affine schemes. Since this forms a basis of the topology, this implies that an open set inherits a scheme structure. □

Exercise 3.3 (Reduced schemes)**.** A scheme $(X, \mathscr{O}_X)$ is **reduced** if for every open set $U \subseteq X$, the ring $\mathscr{O}_X(U)$ has no nilpotent elements.

(a) Show that $(X, \mathscr{O}_X)$ is reduced if and only if for every $p \in X$, the local ring $\mathscr{O}_{X,p}$ has no nilpotent elements.

(b) Let $(X, \mathscr{O}_X)$ be a scheme. Let $(\mathscr{O}_X)_{\text{red}}$ be the sheaf associated to the presheaf $U \mapsto \mathscr{O}_X(U)_{\text{red}}$, where for any ring A, we denote by A_{red} the quotient of A by its ideal of nilpotent elements. Show that $(X, (\mathscr{O}_X)_{\text{red}})$ is a scheme. We call it the **reduced scheme** of X and denote it X_{red}. Show that there is a morphism of schemes $X_{\text{red}} \to X$ which is a homeomorphism on the underlying topological spaces.

(c) Let $f : X \to Y$ be a morphism of schemes, and assume that X is reduced. Show that there is a unique morphism $g : X \to Y_{\text{red}}$ such that f is obtained by composing g with the natural map $Y_{\text{red}} \to Y$.

Proof. Suppose that $\mathscr{O}_{X,p}$ has no nilpotent elements for all $p \in U$. Let $s \in \mathscr{O}_X(U)$ be nilpotent. Then the germ of s at every $p \in U$ must be zero. From the identity axiom, this implies that $s = 0$. Hence $\mathscr{O}_X(U)$ has no nilpotents. Conversely, suppose that $\mathscr{O}_X(U)$ has no nilpotents for all open $U \subseteq X$. Let $(U, \mathscr{O}_X|_U) \simeq \operatorname{Spec} A$ be an affine open containing $p \in X$ and let p corresponding to the prime ideal $\mathfrak{p}$ in Spec A. Then $\mathscr{O}_{X,p} \simeq A_{\mathfrak{p}}$. From our hypothesis A is reduced. This implies that $A_{\mathfrak{p}}$ is reduced which completes the proof to Part (a).

To construct a morphism $X_{\text{red}} \to X$, we need a continuous map $\varphi : X \to X$ and a sheaf morphism $\varphi^\# : \mathscr{O}_X \to (\mathscr{O}_X)_{\text{red}}$. We can pick $\varphi = \mathrm{id}_X$ and $\varphi^\#$ to be the composition $\mathscr{O}_X \to (\mathscr{O}_X)^{\text{pre}}_{\text{red}} \to (\mathscr{O}_X)_{\text{red}}$. This completes Part (b).

For Part (c), the fact that $\mathscr{O}_X$ is reduced implies that the morphism $\mathscr{O}_Y \to \mathscr{O}_X$ can be factored through $(\mathscr{O}_Y)^{\text{pre}}_{\text{red}}$. The rest follows from the universal property of the sheafification. □

Exercise 3.4. Let A be a ring and let $(X, \mathscr{O}_X)$ be a scheme. Given a morphism $f : X \to \operatorname{Spec} A$, we have an associated map on sheaves $f^\# : \mathscr{O}_{\operatorname{Spec} A} \to f_*\mathscr{O}_X$. Taking global sections we obtain a homomorphism $A \to \Gamma(X, \mathscr{O}_X)$. Thus there is a natural map

$$\alpha : \operatorname{Hom}_{\mathsf{Sch}}(X, \operatorname{Spec} A) \longrightarrow \operatorname{Hom}_{\mathsf{Rings}}(A, \Gamma(X, \mathscr{O}_X)).$$

Show that α is bijective.

Proof. We begin with a ring homomorphism $\phi : A \to \Gamma(X, \mathscr{O}_X)$. For any $p \in X$, we can consider the composition $\phi_p : A \to \Gamma(X, \mathscr{O}_X) \to \mathscr{O}_{X,p}$. This defines a set-map $f : X \to \operatorname{Spec} A$ defined by $f(p) \overset{\text{def}}{=} \phi_p^{-1}(m_{X,p})$. This is continuous because locally this is exactly the map when we restrict to affine open subschemes. For example, if we consider instead

$$A \to \Gamma(X, \mathscr{O}_X) \to \Gamma(\operatorname{Spec} B, \mathscr{O}_X|_{\operatorname{Spec} B}) \simeq B$$

and then composition $A \to B \to B_{\mathfrak{p}}$ for $\mathfrak{p} \in \operatorname{Spec} B$ this is exactly our map $\operatorname{Spec} B \subseteq X \to \operatorname{Spec} A$ and corresponds exactly to the morphism $A \to B$. Since locally it is continuous, it must be continuous on $X \to \operatorname{Spec} A$. From the gluing and identity axiom on affine opens, we see that there exists a unique scheme morphism $X \to \operatorname{Spec} A$ corresponding to the ring morphism $A \to \Gamma(X, \mathscr{O}_X)$. □

Exercise 3.5. Describe $\operatorname{Spec} \mathbb{Z}$, and show that it is a final object in the category of schemes.

Proof. The points of $\mathbb{Z}$ are (0) and (p) for primes $p \in \mathbb{Z}$. The zero ideal is the generic point. The ideals (p) are closed points. The closed sets in the Zariski topology are the whole space, the empty set, and finite sets of points not containing (0). The open sets are the whole space, the empty set, and infinite number of points including (0). The sections over the whole space is $\mathbb{Z}$, the sections over the empty set is 0, and the sections over the set $\{(p_1), \dots, (p_n)\}^c$ is

$$\mathscr{O}_{\operatorname{Spec} \mathbb{Z}}(\operatorname{Spec} \mathbb{Z} \backslash \{(p_1), \dots, (p_n)\}) \simeq \mathbb{Z}\left[\frac{1}{p_1}, \dots, \frac{1}{p_n}\right]$$

The fact that $\operatorname{Spec} \mathbb{Z}$ is a final object follows from Exercise 3.4. □

Exercise 3.6. Describe the spectrum of the zero ring, and show that it is an initial object for the category of schemes. (According to our conventions, all ring homomorphisms must take 1 to 1. Since $0 = 1$ in the zero ring, we see that each ring R admits a unique homomorphism to the zero ring, but that there is no homomorphism from the zero ring to R unless $0 = 1$ in R.)

Proof. The spectrum of the zero ring is empty with sections $\mathscr{O}(\varnothing) = 0$. Since any ring has a unique morphism to the 0 ring, the spectrum of the zero ring is the initial object in the category of schemes. □

Exercise 3.7. Let X be a scheme. For any $x \in X$, let $\mathscr{O}_x$ be the local ring at x, and $\mathfrak{m}_x$ its maximal ideal. We define the **residue field** of x on X to be the field $k(x) = \mathscr{O}_x/\mathfrak{m}_x$. Now let K be any field. Show that to give a morphism of $\operatorname{Spec} K$ to X it is equivalent to give a point $x \in X$ and an inclusion map $k(x) \to K$.

Proof. Let $(\phi, \phi^\#) : \operatorname{Spec} K \to X$ be a morphism of schemes. Since $\operatorname{Spec} K$ consists of a single point, there is some $p \in X$ such that $\phi(\bullet) = p$. We then have a sheaf morphism

$$\phi^\# : \mathscr{O}_X \to \phi_* \mathscr{O}_{\operatorname{Spec} K}.$$

For $U \subseteq X$ not containing p, we have $\phi_* \mathscr{O}_{\operatorname{Spec} K}(U) = 0$ so the map is uniquely determined. When $p \in U$, then we have $\phi^\# : \mathscr{O}_X(U) \to K$. For all such U, it factors through $\mathscr{O}_{X,p}$ by taking the induced map on stalks:

$$\begin{tikzcd} \mathscr{O}_X(U) \arrow[r] \arrow[d] & K \arrow[d, equal] \\ \mathscr{O}_{X,p} \arrow[r] & K \end{tikzcd}$$

Since the morphism is a morphism of local rings, it must be the case that the preimage of the zero ideal in K is $\mathfrak{m}_{X,x}$. Thus, this induces a map $k(x) \to k$ which must be an injection since homomorphisms between fields are always injections. $\square$

Exercise 3.8. Let X be a scheme. For any point $x \in X$, we define the **Zariski tangent space** T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k, and let $k[\varepsilon]/\varepsilon^2$ be the **ring of dual numbers** over k. Show that to give a k-morphism of $\operatorname{Spec} k[\varepsilon]/\varepsilon^2$ to X is equivalent to giving a point $x \in X$, **rational over** k (i.e., such that $k(x) = k$), and an element of T_x.

References

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