[Hartshorne] Algebraic Geometry

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Abstract. To be added.

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This document contains my solutions to the exercises in [2].

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3.1. Sheaves

3.2. Schemes

Exercise 3.2.1. Let A be a ring, let $X = \operatorname{Spec} A$, let $f \in A$ and let $D(f) \subseteq X$ be the open complement of $V((f))$. Show that the locally ringed space $(D(f), \mathscr{O}_X|_{D(f)})$ is isomorphic to $\operatorname{Spec} A_f$.

Proof. The localization map $\ell : A \to A_f$ induces a bijection between ideals of A not containing f and the ideals of A_f. This bijection gives a one-to-one correspondence between the prime ideals of A not containing f and the prime ideals of A_f. In other words, we have a bijection $\varphi : D(f) \to \operatorname{Spec} A_f$. For any ideal $\mathfrak{a}$ of A not containing f, we have $\mathfrak{p} \supseteq \mathfrak{a}$ if and only if $\varphi(\mathfrak{p}) \supseteq \varphi(\mathfrak{a})$. This proves that φ is a homeomorphism. An isomorphism on the sheaf structure is induced by the natural isomorphism

$$(A_f)_{\varphi(\mathfrak{p})} \to A_{\mathfrak{p}}.$$

□

Exercise 3.2.2. Let $(X, \mathscr{O}_X)$ be a scheme, and let $U \subseteq X$ be any open subset. Show that $(U, \mathscr{O}_X|_U)$ is a scheme. We call this the **induced scheme structure** on the open set U, and we refer to $(U, \mathscr{O}_X|_U)$ as an open subscheme of X.

Proof. It suffices to assume X is affine. In this case, the distinguished open sets are affine schemes. Since this forms a basis of the topology, this implies that an open set inherits a scheme structure. $\square$

Exercise 3.2.3 (Reduced schemes)**.** A scheme $(X, \mathscr{O}_X)$ is **reduced** if for every open set $U \subseteq X$, the ring $\mathscr{O}_X(U)$ has no nilpotent elements.

(a) Show that $(X, \mathscr{O}_X)$ is reduced if and only if for every $p \in X$, the local ring $\mathscr{O}_{X,p}$ has no nilpotent elements.
(b) Let $(X, \mathscr{O}_X)$ be a scheme. Let $(\mathscr{O}_X)_{\text{red}}$ be the sheaf associated to the presheaf $U \mapsto \mathscr{O}_X(U)_{\text{red}}$, where for any ring A, we denote by A_{red} the quotient of A by its ideal of nilpotent elements. Show that $(X, (\mathscr{O}_X)_{\text{red}})$ is a scheme. We call it the **reduced scheme** of X and denote it X_{red}. Show that there is a morphism of schemes $X_{\text{red}} \to X$ which is a homeomorphism on the underlying topological spaces.
(c) Let $f : X \to Y$ be a morphism of schemes, and assume that X is reduced. Show that there is a unique morphism $g : X \to Y_{\text{red}}$ such that f is obtained by composing g with the natural map $Y_{\text{red}} \to Y$.

Proof. Suppose that $\mathscr{O}_{X,p}$ has no nilpotent elements for all $p \in U$. Let $s \in \mathscr{O}_X(U)$ be nilpotent. Then the germ of s at every $p \in U$ must be zero. From the identity axiom, this implies that $s = 0$. Hence $\mathscr{O}_X(U)$ has no nilpotents. Conversely, suppose that $\mathscr{O}_X(U)$ has no nilpotents for all open $U \subseteq X$. Let $(U, \mathscr{O}_X|_U) \simeq \operatorname{Spec} A$ be an affine open containing $p \in X$ and let p corresponding to the prime ideal $\mathfrak{p}$ in $\operatorname{Spec} A$. Then $\mathscr{O}_{X,p} \simeq A_{\mathfrak{p}}$. From our hypothesis A is reduced. This implies that $A_{\mathfrak{p}}$ is reduced which completes the proof to Part (a).

To construct a morphism $X_{\text{red}} \to X$, we need a continuous map $\varphi : X \to X$ and a sheaf morphism $\varphi^{\#} : \mathscr{O}_X \to (\mathscr{O}_X)_{\text{red}}$. We can pick $\varphi = \mathrm{id}_X$ and $\varphi^{\#}$ to be the composition $\mathscr{O}_X \to (\mathscr{O}_X)^{\text{pre}}_{\text{red}} \to (\mathscr{O}_X)_{\text{red}}$. This completes Part (b).

For Part (c), the fact that $\mathscr{O}_X$ is reduced implies that the morphism $\mathscr{O}_Y \to \mathscr{O}_X$ can be factored through $(\mathscr{O}_Y)^{\text{pre}}_{\text{red}}$. The rest follows from the universal property of the sheafification. $\square$

Exercise 3.2.4. Let A be a ring and let $(X, \mathscr{O}_X)$ be a scheme. Given a morphism $f : X \to \operatorname{Spec} A$, we have an associated map on sheaves $f^{\#} : \mathscr{O}_{\operatorname{Spec} A} \to f_* \mathscr{O}_X$. Taking global sections we obtain a homomorphism $A \to \Gamma(X, \mathscr{O}_X)$. Thus there is a natural map

$$\alpha : \mathrm{Hom}_{\mathsf{Sch}}(X, \operatorname{Spec} A) \longrightarrow \mathrm{Hom}_{\mathsf{Rings}}(A, \Gamma(X, \mathscr{O}_X)).$$

Show that α is bijective.

Proof. We begin with a ring homomorphism $\phi : A \to \Gamma(X, \mathscr{O}_X)$. For any $p \in X$, we can consider the composition $\phi_p : A \to \Gamma(X, \mathscr{O}_X) \to \mathscr{O}_{X,p}$. This defines a set-map

$f : X \to \operatorname{Spec} A$ defined by $f(p) \stackrel{\text{def}}{=} \phi_p^{-1}(m_{X,p})$. This is continuous because locally this is exactly the map when we restrict to affine open subschemes. For example, if we consider instead

$$A \to \Gamma(X, \mathscr{O}_X) \to \Gamma(\operatorname{Spec} B, \mathscr{O}_X|_{\operatorname{Spec} B}) \simeq B$$

and then composition $A \to B \to B_{\mathfrak{p}}$ for $\mathfrak{p} \in \operatorname{Spec} B$ this is exactly our map $\operatorname{Spec} B \subseteq X \to \operatorname{Spec} A$ and corresponds exactly to the morphism $A \to B$. Since locally it is continuous, it must be continuous on $X \to \operatorname{Spec} A$. From the gluing and identity axiom on affine opens, we see that there exists a unique scheme morphism $X \to \operatorname{Spec} A$ corresponding to the ring morphism $A \to \Gamma(X, \mathscr{O}_X)$. □

Exercise 3.2.5. Describe $\operatorname{Spec} \mathbb{Z}$, and show that it is a final object in the category of schemes.

Proof. The points of $\mathbb{Z}$ are (0) and (p) for primes $p \in \mathbb{Z}$. The zero ideal is the generic point. The ideals (p) are closed points. The closed sets in the Zariski topology are the whole space, the empty set, and finite sets of points not containing (0). The open sets are the whole space, the empty set, and infinite number of points including (0). The sections over the whole space is $\mathbb{Z}$, the sections over the empty set is 0, and the sections over the set $\{(p_1), \ldots, (p_n)\}^c$ is

$$\mathscr{O}_{\operatorname{Spec} \mathbb{Z}}(\operatorname{Spec} \mathbb{Z} \backslash \{(p_1), \ldots, (p_n)\}) \simeq \mathbb{Z}\left[\frac{1}{p_1}, \ldots, \frac{1}{p_n}\right]$$

The fact that $\operatorname{Spec} \mathbb{Z}$ is a final object follows from Exercise 3.2.4. □

Exercise 3.2.6. Describe the spectrum of the zero ring, and show that it is an initial object for the category of schemes. (According to our conventions, all ring homomorphisms must take 1 to 1. Since $0 = 1$ in the zero ring, we see that each ring R admits a unique homomorphism to the zero ring, but that there is no homomorphism from the zero ring to R unless $0 = 1$ in R.)

Proof. The spectrum of the zero ring is empty with sections $\mathscr{O}(\varnothing) = 0$. Since any ring has a unique morphism to the 0 ring, the spectrum of the zero ring is the initial object in the category of schemes. □

Exercise 3.2.7. Let X be a scheme. For any $x \in X$, let $\mathscr{O}_x$ be the local ring at x, and m_x its maximal ideal. We define the **residue field** of x on X to be the field $k(x) = \mathscr{O}_x / m_x$. Now let K be any field. Show that to give a morphism of $\operatorname{Spec} K$ to X it is equivalent to give a point $x \in X$ and an inclusion map $k(x) \to K$.

Proof. Let $(\phi, \phi^{\#}) : \operatorname{Spec} K \to X$ be a morphism of schemes. Since $\operatorname{Spec} K$ consists of a single point, there is some $p \in X$ such that $\phi(\bullet) = p$. We then have a sheaf morphism

$$\phi^{\#} : \mathscr{O}_X \to \phi_* \mathscr{O}_{\operatorname{Spec} K}.$$

For $U \subseteq X$ not containing p, we have $\phi_* \mathscr{O}_{\operatorname{Spec} K}(U) = 0$ so the map is uniquely determined. When $p \in U$, then we have $\phi^{\#} : \mathscr{O}_X(U) \to K$. For all such U, it factors through $\mathcal{O}_{X,p}$ by taking the induced map on stalks:

$$\begin{tikzcd} \mathcal{O}_X(U) \arrow[r] \arrow[d] & K \arrow[d, equal] \\ \mathcal{O}_{X,p} \arrow[r] & K \end{tikzcd}$$

Since the morphism is a morphism of local rings, it must be the case that the preimage of the zero ideal in K is $m_{X,x}$. Thus, this induces a map $k(x) \to k$ which must be an injection since homomorphisms between fields are always injections. $\square$

Exercise 3.2.8. Let X be a scheme. For any point $x \in X$, we define the **Zariski tangent space** T_x to X at x to be the dual of the $k(x)$-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$. Now assume that X is a scheme over a field k, and let $k[\varepsilon]/\varepsilon^2$ be the **ring of dual numbers** over k. Show that to give a k-morphism of $\operatorname{Spec} k[\varepsilon]/\varepsilon^2$ to X is equivalent to giving a point $x \in X$, **rational over** k (i.e., such that $k(x) = k$), and an element of T_x.

Proof. The dual numbers have only one prime ideal (ε). To get a morphism $\operatorname{Spec} k[\varepsilon]/\varepsilon^2$ to X on the topological level, we need to pick $x \in X$ for the single point to map to. Once we have this point, we need a morphism $\mathcal{O}_{X,x} \to k[\varepsilon]/(\varepsilon^2)$. The maximal ideal must map into the maximal ideal. Hence, this induces a map $k(x) \to k$. But from the condition of being schemes over Spec(k), we have a composition $k \to k(x) \to k$ which is the identity. This proves that x is rational over k. To pick the morphism $\mathcal{O}_{X,x} \to k[\varepsilon]/(\varepsilon^2)$, first note that m_x must map to (ε) and since $\varepsilon^2 = 0$, we know that m_x^2 maps to 0. Thus the locations of m_x are determined by a linear map $m_x/m_x^2 \to (\varepsilon) \simeq k$. This gives us our element of T_x and also determines the values of m_x in the morphism $\mathcal{O}_{X,x} \to k[\varepsilon]/(\varepsilon^2)$. The rest of the map comes from the k-scheme morphism and the fact that $k \to k(x) \to k$ is the identity. $\square$

Exercise 3.2.9. If X is a topological space, and Z can irreducible closed subset of X, a generic point for Z is a point ζ such that $Z = \overline{\{\zeta\}}$. If X is a scheme, show that every irreducible closed subset has a unique generic point.

Proof. First suppose that X is affine. Closed subsets are of the form $V(I)$ where I is a radical ideal. I claim that $V(I)$ is irreducible if and only if I is prime. If I is prime, then it is irreducible because it is the closure of a single point. Now suppose $V(I)$ is irreducible. Suppose that $f, g \in A$ satisfy $fg \in I$. Then $V(I) = (V(I) \cap V(f)) \cup (V(I) \cap V(g))$ which implies that $V(f) \supseteq V(I)$ or $V(g) \supseteq V(I)$ which implies that I is prime. This proves that all irreducible closed subsets are of the form $V(\mathfrak{p})$. Clearly, the unique generic point in this case is $\mathfrak{p}$.

Now let X be a general scheme. Let Z be an irreducible closed subset of X. Let U be an affine open subset of X. Then $Z \cap U$ is an irreducible closed subset of U. From the previous paragrpah, there is a unique point $z \in Z \cap U$ such that the closure of z in U is $Z \cap U$. But this is an open set of Z which is dense since Z is irreducible. So the closure of z in X is the whole of Z. On the other hand, Z has at most 1 generic point. If it had two, if we intersect with an affine open, the affine open must contain both generic points. But we already proved that in affine open there is at most one generic point. This suffices for the proof. □

Exercise 3.2.10. Describe $\mathbb{R}[x]$. How does the topological space compare to $\mathbb{R}$ or $\mathbb{C}$?

Proof. The points are (0), $(x-a)$, and (x^2+k) for $k > 0$. □

Exercise 3.2.11. Let $k = \mathbb{F}_p$ be the finite field with p elements. Describe $\operatorname{Spec} k[x]$. What are the residue fields of its points? How many points are there with a given residue field?

Proof. The points of $\operatorname{Spec} k[x]$ are the prime ideals (0) and (f) where f is an irreducible polynomial of $k[x]$. The prime ideal (0) is the generic point and (f) are closed points. The residue field at (0) is $\mathbb{F}_p(x)$ and the residue field at (f) is $\mathbb{F}_{p^{\deg f}}$.

To figure out the number of points with residue field $\mathbb{F}_{p^d}$, we only need to count the number of monic irreducible polynoimals of degree d in $\mathbb{F}_p[x]$. To count this, we prove two facts:

(1) An irreducible polynomial divides $x^{p^n} - x$ if and only if its degree divides n.
(2) $x^{p^n} - x$ has distinct irreducible factors.

Let f be an irreducible polynomial of degree d with $d|n$. Then $\mathbb{F}_p[x]/(f) \simeq \mathbb{F}_{p^d}$. Since $d|n$, any element in $\mathbb{F}_{p^d}$ is a solution to $x^{p^n} - x = 0$. This implies that $x^{p^n} - x \in (f)$ which implies $f|x^{p^n} - x$. Conversely, suppose that f is an irreducible polynomial dividing $x^{p^n} - x$. We already know that $f|x^{p^d} - x|x^{p^n} - x$ which proves (1). For (2), just take a derivative! □

Exercise 3.2.12 (Glueing Lemma)**.** Let $\{X_i\}$ be a family of schemes (possibly infinite). For each $i \neq j$, suppose given an open subset $U_{ij} \subseteq X_i$, and let it have the induced scheme structure. Suppose also given for each $i \neq j$ an isomorphism of schemes $\varphi_{ij} : U_{ij} \to U_{ji}$ such that

(1) for each i, j, $\varphi_{ji} = \varphi_{ij}^{-1}$, and
(2) for each i, j, k, $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk}$, and $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ on $U_{ij} \cap U_{ik}$.

Then show that there is a scheme X, together with morphisms $\psi_i : X_i \to X$ for each i, such that

(1) ψ_i is an isomorphism of X_i onto an open subscheme of X,
(2) the $\psi_i(X_i)$ cover X,
(3) $\psi_i(U_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$, and
(4) $\psi_i = \psi_j \circ \varphi_{ij}$ on U_{ij}.

We say that X is obtained by **glueing** the schemes X_i along the isomorphisms φ_{ij}.

Proof. We first describe the topological space X. Let $X = \bigsqcup X_i/\sim$ where $x_i \in X_i$ and $x_j \in X_j$ satisfy $x_i \sim x_j$ if and only if $x_i \in U_{ij}$, $x_j \in U_{ji}$, and $x_j = \varphi_{ij}(x_i)$. It is easy to see that this is an equivalence relation.

For every index i, we have a map $\ell_i : X_i \to X$ which is the natural inclusion $X_i \to \bigsqcup X_i$ and then projecting to X. We equip X with the quotient topology. In this topology, $U \subseteq X$ is open if and only if $\ell_i^{-1}(U) \subseteq X_i$ is open for all i. To define the sheaf structure on X, we first define the notation of a family of compatible sections over $U \subseteq X$.

Definition 3.1. A **family of compatible sections** over $U \subseteq X$ is a family of sections $s_i \in \mathscr{O}_{X_i}\left(\ell_i^{-1}(U)\right)$ such that for every $i \neq j$, we have

$$\varphi_{ij}^{\#}\left(s_j|_{\ell_j^{-1}(U)\cap U_{ji}}\right) = s_i|_{\ell_i^{-1}(U)\cap U_{ij}}$$

Now that we have defined families of compatible sections over open subsets of X, we can equip the X with the presheaf of rings defined by

$$\mathscr{O}_X(U) \stackrel{\text{def}}{=} \left\{\left(s_i \in \mathscr{O}_{X_i}(\ell_i^{-1}(U))\right) : \text{ family of compatible sections}\right\}.$$

The restriction maps are component-wise restriction. The gluing axiom and identity axiom follow from the gluing and identity axiom of each $\mathscr{O}_{X_i}$. Thus, we have realized X as a ringed space.

For every i, we have a natural morphism $\psi_i : \left(X_i, \mathscr{O}_{X_i}\right) \to (X, \mathscr{O}_X)$ where the continuous map $\psi_i : X_i \to X$ is the continuous map $\ell_i : X_i \to X$ and the morphism of sheaves

$$\psi_i^{\#} : \mathscr{O}_X \to (\ell_i)_* \mathscr{O}_{X_i}$$
$$\psi_i^{\#}(U) : \mathscr{O}_X(U) \to \mathscr{O}_{X_i}(\ell_i^{-1}(U))$$

which is given by projection. It is clear that $\ell_i(X_i)$ is an open subset of X and ψ_i induces an isomorphism between $(X_i, \mathscr{O}_{X_i})$ and $(\psi_i(X_i), \mathscr{O}_X|_{\psi_i(X_i)})$. This proves that X is a scheme. $\square$

Exercise 3.2.13. A topological space is **quasi-compact** if every open cover has a finite subcover.

(a) Show that a topological space is noetherian if and only if every open subset is quasi-compact.
(b) If X is an affine scheme, show that $\mathrm{sp}(X)$ is quasi-compact, but not in general noetherian. We say that a scheme is **quasi-compact** if $\mathrm{sp}(X)$ is.
(c) If A is a noetherian ring, show that $\mathrm{sp}(\operatorname{Spec} A)$ is a noetherian topological space.
(d) Give an example where $\mathrm{sp}(\operatorname{Spec} A)$ can be noetherian even when A is not.

Proof. Suppose that X is a noetherian topological space. This means that it satisfies the ascending chain condition on open sets. Let U be an open set and consider an open cover $\{U_i\}$. If there is no finite subcover, we can find a strictly increasing sequence of open sets contained in U. But this contradicts the ascending chain condition. Conersely, suppose that every open set is quasi-compact. Take an ascending chain of open sets. Their union has a fintie subcover. This proves (a).

Let $X = \operatorname{Spec} A$. It suffices to prove that an open cover by distinguished open sets has a finite subcover. Suppose that

$$\operatorname{Spec} A = \bigcup D(f_i).$$

This means that any prime ideal of A does not contain at least one f_i. Thus, the ideal $I = (f_i)$ is not contained in any prime ideal, which means $1 \in I$. So $1 = \sum c_i f_i$ for some finitely many i. Taking the corresponding f_i we have our finite subcover. Affine schemes are not necessarily noetherian. Consider $A = k[x_1, x_2, \ldots]$ the polynomial ring with infinitely many variables. Then $\bigcup D(x_i)$ is an open set with no finite subcover.

Suppose that A is noetherian. Now consider a descending chain of closed sets $Z(I_1) \supset Z(I_2) \supseteq \ldots$ where the I_k are radical ideals. This corresponds to ascending chain of ideals $I_1 \subseteq I_2 \subseteq \ldots$ which is eventually stationary from noetherianness of A. This proves that $\operatorname{sp}(\operatorname{Spec} A)$ is noetherian.

For (d), consider $A = k[x_1, x_2, \ldots]/(x_1^2, x_2^2, \ldots)$. Any prime ideal of A must contain $x_1, x_2, \ldots$ since they are nilpotent. Thus the only prime ideal is $(x_1, x_2, \ldots)$. Hence $\operatorname{sp}(\operatorname{Spec} A)$ is automatically noetherian. But A is not noetherian because $(x_1, x_2, \ldots)$ is not finitely generated.

Fact 3.2. $\operatorname{Spec} A$ *is a noetherian topological space if and only if* A *satisfies ascending chain condition for radical ideals.*

□

Exercise 3.2.14.

(a) Let S be a graded ring. Show that $\operatorname{Proj} S = \varnothing$ if and only if every element of S_+ is nilpotent.
(b) Let $\varphi : S \to T$ be a graded homomorphism of graded rings. Let $U = \{\mathfrak{p} \in \operatorname{Proj} T : \mathfrak{p} \not\supseteq \varphi(S_+)\}$. Show that U is an open subset of $\operatorname{Proj} T$, and show that φ determines a natural morphism $f : U \to \operatorname{Proj} S$.
(c) The morphism f can be an isomorphism even when φ is not. For example, suppose that $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geqslant d_0$, where d_0 is an integer. Then show that $U = \operatorname{Proj} T$ and the morphism $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is an isomorphism.
(d) Let V be a projective variety with homogeneous coordinate ring S. Show that $t(V) \simeq \operatorname{Proj} S$.

Proof.

(a) Suppose that every element of S_+ is nilpotent. Then S_+ is contained in every prime ideal and $\operatorname{Proj} S = \varnothing$. Conversely, suppose that $\operatorname{Proj} S = \varnothing$. Then any homogeneous prime ideal of S contains S_+. The intersection of all homogeneous prime ideals is the nilradical (exercise for the reader). Hence every element in S_+ is nilpotent.

(b) We have that

$$U^c = \{\mathfrak{p} \in \operatorname{Proj} T : \mathfrak{p} \supseteq \varphi(S_+)\} = V(I)$$

where I is homogeneous ideal generated by $\varphi(S_d)$ for $d \geqslant 1$. This shows that U is open. The natural morphism $f : U \to \operatorname{Proj} S$ takes some $\mathfrak{p} \in U$ and sends it to $\varphi^{-1}(\mathfrak{p})$. This is a prime ideal not containing S_+. It is also homogeneous because it has grading

$$\varphi^{-1}(\mathfrak{p}) = \bigoplus_{d \geqslant 0} \varphi^{-1}(\mathfrak{p}_d).$$

This determines a set map $f : U \to \operatorname{Proj} S$. To prove continuity, let $\mathfrak{a}$ be a homogeneous ideal of S. The pre-image will consist of $V(\mathfrak{b})$ where $\mathfrak{b}$ is the homogeneous ideal of T generated by $\varphi(\mathfrak{a}_d)$ for all $d \geqslant 0$. This proves that $\varphi : U \to \operatorname{Proj} S$ is a continuous map. To define the sheaf morphism, it suffices to define it on a base. We want to define a (local) morphism of sheaves on $\operatorname{Proj} S$.

$$\mathscr{O}_{\operatorname{Proj} S} \to f_* \left(\mathscr{O}_{\operatorname{Proj} T}\Big|_U \right).$$

We only need to define it on the base $D_+(f)$ where $f \in S_+$ is homogeneous and show that it behaves well under restriction. We have

$$S_{(f)} \simeq \mathscr{O}_{\operatorname{Proj} S}(D_+(f)) \longrightarrow \mathscr{O}_{\operatorname{Proj} T}(D_+(\varphi(f))) \simeq T_{(\varphi(f))}$$

where the map is given by the standard localization map $S_f \to T_{\varphi(f)}$ and then taking the degree 0 part. To prove that this is a morphism of locally ringed spaces, we want to show that for every point $\mathfrak{p} \in U$, the induced map

$$S_{(f(\mathfrak{p}))} \simeq \mathscr{O}_{\operatorname{Proj} S, f(\mathfrak{p})} \to \mathscr{O}_{\operatorname{Proj} T, \mathfrak{p}} \simeq T_{(\mathfrak{p})}$$

is a morphism of local rings. But this is clear. There are a few things we should check, for example that $S_{(\mathfrak{p})}$ is actually a local ring! This was not shown in Hartshorne, but seems like a good exercise to show this fact.

(c) Suppose that $\varphi : S_d \to T_d$ is an isomorphism for sufficiently large d. For any $\mathfrak{p} \in \operatorname{Proj} T$, there is some homogeneous $f \in T_+$ not contained in $\mathfrak{p}$. Since $\mathfrak{p}$ is prime, we can take high enough powers of f to get that $\mathfrak{p} \not\supseteq T_d = \varphi(S_d)$ for d sufficiently large. This implies that $U = \operatorname{Proj} T$. To prove that $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is bijective, consider prime ideals $\mathfrak{p}_1, \mathfrak{p}_2 \in \operatorname{Proj} T$ where $\varphi^{-1}(\mathfrak{p}_1) = \varphi^{-1}(\mathfrak{p}_2)$ in $\operatorname{Proj} S$. Let $f \in \mathfrak{p}_1$. Then $f^d \in \mathfrak{p}_2$ for high enough power d. Since $\mathfrak{p}_2$ is prime, this implies $f \in \mathfrak{p}_2$. By symmetry, $\mathfrak{p}_1 = \mathfrak{p}_2$. This proves that f is injective. To prove surjectivity, let $\{t_i\}$ be a family of homogeneous elements generating T_+. This means that $D_+(t_i)$ covers T. By raising the t_i to sufficiently high powers, there exist $\{s_i\}$ in S_+ satisfying $\varphi(s_i) = t_i$ (since $D_+(t_i) = D_+(t_i^d)$). Then $D_+(t_i)$ gets mapped to $D_+(s_i)$ under the map. This also proves that the map is an open map! (Why?) The $D_+(s_i)$ cover $\operatorname{Proj} S$. Indeed, let $\mathfrak{p} \in \operatorname{Proj} S$ be arbitrary. If it contains all of the s_i, then $\varphi^{-1}(\mathfrak{p})$ must contain all of the t_i. But this implies that $\varphi^{-1}(\mathfrak{p})$ contains T_+, which is a contradiction. Thus the map is a bijective open map, which implies that it is a homeomorphism. To give the homeomorphism, we prove that the induced map on sheaves is bijective locally. On $D(s_i)$, it boils down to showing that

$$S_{(s_i)} \simeq T_{(t_i)}$$

via the natural morphism $S_{(s_i)} \to T_{(t_i)}$. To prove injectivity, suppose that $\varphi(x/s_i^N) = \varphi(x)/t_i^N = 0$ in $T_{(t_i)}$. This implies that $t_i^M \varphi(x) = 0$ for some M and taking sufficiently large M, we have $s_i^M x = 0$. This proves injectivity. To prove surjectivity, for y/t_i^N, we can take M sufficiently large again and it is clear that

$$\frac{y}{t_i^N} = \frac{y t_i^M}{t_i^{M+N}}$$

is in the image. This suffices for the proof.

(d) I omit this.

$\square$

Exercise 3.2.15.

(a) Let V be a variety over the algebraically closed field k. Show that a point $p \in t(V)$ is a closed point if and only if its residue field is k.
(b) If $f : X \to Y$ is a morphism of schemes over k, and if $p \in X$ is a point with residue field k, then $f(p) \in Y$ also has residue field k.
(c) Now show that if V, W are any two varieties over k, then the natural map

$$\operatorname{Hom}_{\mathrm{Var}}(V, W) \to \operatorname{Hom}_{\mathrm{Sch}/k}(t(V), t(W))$$

is bijective.

Proof. Skipped. I don't quite understand the functor from varieties to schemes just yet in detail $\square$

Exercise 3.2.16. Let X be a scheme, let $f \in \Gamma(X, \mathcal{O}_X)$, and define X_f to be the subset of points $x \in X$ such that the stalk f_x of f at x is not contained in the maximal ideal $\mathfrak{m}_x$ of the local ring $\mathcal{O}_x$.

(a) If $U = \operatorname{Spec} B$ is an open affine subscheme of X, and if $\bar{f} \in B = \Gamma(U, \mathcal{O}_X|_U)$ is the restriction of f, show that $U \cap X_f = D(\bar{f})$. Conclude that X_f is an open subset of X.
(b) Assume that X is quasi-compact. Let $A = \Gamma(X, \mathcal{O}_X)$, and let $a \in A$ be an element whose restriction to X_f is 0. Show that for some $n > 0$, $f^n a = 0$.
(c) Now assume that X has a finite cover by open affines U_i such that each intersection $U_i \cap U_j$ is quasi-compact. Let $b \in \Gamma(X_f, \mathcal{O}_{X_f})$. Show that for some $n > 0$, $f^n b$ is the restriction of an element of A.

(d) With the hypothesis of (c), conclude that $\Gamma(X_f, \mathscr{O}_{X_f}) \simeq A_f$.

Proof. For Part (a), it suffices to prove that for $f \in B$, then open set $D(f) \subseteq \operatorname{Spec} B$ can be characterized as the subset of primes $\mathfrak{p} \in \operatorname{Spec} B$ with $f \notin \mathfrak{p}B_{\mathfrak{p}}$. Thus, we want to prove that $f \in \mathfrak{p}$ in B if and only if $f \in \mathfrak{p}B_{\mathfrak{p}}$ in $B_{\mathfrak{p}}$. One direction is easy. For the other direction, suppose $f \in \mathfrak{p}B_{\mathfrak{p}}$ in $B_{\mathfrak{p}}$. Then $f = x/y$ where $x \in \mathfrak{p}$ and $y \notin \mathfrak{p}$. Then $z(yf - x) = 0$ in B for $z \notin \mathfrak{p}$. This implies that $f \in \mathfrak{p}$ in B which suffices for the proof of (a).

For Part (b), it suffices to prove this when X is affine because we can just pick the largest n on a finite cover by affines (which exist since X is quasi-compact). But the restriction of $a \in A$ being zero on X_f implies that $a = 0$ in A_f. This exactly translates to $f^n a = 0$ for some n. This proves (b).

For Part (c), let $U_i = \operatorname{Spec} A_i$ be the finite affine open cover of X. Since $X_f \cap U_i = D_{A_i}(f)$, the section b on $X_f \cap U_i$ comes from some $b_i \in \Gamma(U_i, \mathscr{O}_{U_i})$ and

$$b|_{X_f \cap U_i} = \frac{b_i}{f^N}$$

where we can pick N to be large enough to hold for all affine opens in our finite cover. In particular, we have $f^N b|_{X_f \cap U_i} = b_i|_{X_f \cap U_i}$ for all i. It remains to glue together the $b_i \in \Gamma(U_i, \mathscr{O}_{U_i})$. But this follows because on $U_i \cap U_j$, we can pick M large enough so that $f^M(b_i - b_j) = 0$. We can pick M large enough to hold for all finite pairs i, j. Thus $f^{M+N} b$ comes from a global section.

For Part (d), first consider the restriction map $A \to \Gamma(X_f, \mathscr{O}_{X_f})$. To induce a map $A_f \to \Gamma(X_f, \mathscr{O}_{X_f})$ we want to prove that $f|_{X_f}$ is invertible. Let $f_i \in \Gamma(U_i \cap X_f, \mathscr{O}_{U_i \cap X_f})$ be an element on the affine open which is the inverse of $f|_{U_i \cap X-f}$. That is, we have $f|_{U_i \cap X_f} f_i = 1$ on $U_i \cap X_f$. To prove that we can glue the f_i together, we look at $f_i - f_j$ on $U_i \cap U_j \cap X_f$ and prove that it is zero. But this follows because on stalks f is invertible, which means that the stalk of $f_i - f_j$ is 0 on all points in $U_i \cap U_j \cap X_f$. This implies that we have a well-defined map

$$A_f \longrightarrow \Gamma\left(X_f, \mathscr{O}_{X_f}\right)$$

From (b), we know that this is injective. This is surjective from (c). This suffices for the proof. ☐

Exercise 3.2.17. *A Criterion for Affineness.*

(a) Let $f : X \to Y$ be a morphism of schemes, and suppose that Y can be covered by open subsets U_i, such that for each i, the induced map $f^{-1}(U_i) \to U_i$ is an isomorphism. Then f is an isomorphism.

(b) A scheme X is affine if and only if there is a finite set of elements

$$f_1, \ldots, f_r \in A = \Gamma(X, \mathscr{O}_X),$$

such that the open subsets X_{f_i} are affine, and $f_1, \ldots, f_r$ generate the unit ideal in A.

Proof. We solve part (a). Since $f^{-1}(U_i) \to U_i$ is an isomorphism, on the level of topological spaces we have homeomorphisms $f^{-1}(U_i) \to U_i$. This implies that

$f : X \to Y$ is a homeomorphism (standard topological fact). It remains to show that the map on sheaves
$$f^{\#} : \mathscr{O}_Y \to f_*\mathscr{O}_X$$
is an isomorphism. We know that
$$f^{\#}|_{U_i} : \mathscr{O}_Y|_{U_i} \to f_*\mathscr{O}_X|_{U_i}$$
is an isomorphism for all U_i. But this implies that the induced map on stalks are isomorphisms which implies that $f^{\#}$ is an isomorphism. This proves (a).

We solve part (b). If X is affine, then we can let $f_1 = 1$ and this trivially generates all of A while generating the unit ideal. Conversely, suppose that we have a finite set of elements $f_1, \dots, f_r \in \Gamma(X, \mathscr{O}_X)$ such that X_{f_i} are affine and $1 = (f_1, \dots, f_r)$. From the previous exercise, we know that $X_{f_i} \simeq \operatorname{Spec} A_{f_i}$. Since f_i generate A, we know that the X_{f_i} cover X. From exercise 4, the canonical isomorphism $A \to \Gamma(X, \mathscr{O}_X)$ gives us a natural scheme morphism $X \to \operatorname{Spec} A$. This maps X_f to $D(f)$ and is an isomorphism (you can see this by reviewing the construction in Exercise 4). This implies that X and $\operatorname{Spec} A$ are isomorphic. □

Exercise 3.2.18. In this exercise, we compare some properties of a ring homomorphism to the induced morphism of the spectra of the rings.

(a) Let A be a ring, $X = \operatorname{Spec} A$, and $f \in A$. Show that f is nilpotent if and only if $D(f)$ is empty.

(b) Let $\varphi : A \to B$ be a homomorphism of rings, and let $f : Y = \operatorname{Spec} B \to X = \operatorname{Spec} A$ be the induced morphism of affine schemes. Show that φ is injective if and only if the map of sheaves $f^{\#} : \mathscr{O}_X \to f_*\mathscr{O}_Y$ is injective. Show furthermore in that case f is *dominant*, i.e., $f(Y)$ is dense in X.

(c) With the same notation, show that if φ is surjective, then f is a homeomorphism of Y onto a closed subset of X, and $f^{\#} : \mathscr{O}_X \to f_*\mathscr{O}_Y$ is surjective.

(d) Prove the converse to (c), namely, if $f : Y \to X$ is a homeomorphism onto a closed subset, and $f^{\#} : \mathscr{O}_X \to f_*\mathscr{O}_Y$ is surjective, then φ is surjective.

Proof. This problem has four parts.

(a) This part follows from the fact that the intersection of all prime ideals is exactly the nilpotent elements.

(b) Suppose that $f^{\#} : \mathscr{O}_X \to f_*\mathscr{O}_Y$ is injective. Then the induced map on the global sections
$$f^{\#}(\operatorname{Spec} A) : \Gamma(\operatorname{Spec} A, \mathscr{O}_{\operatorname{Spec} A}) \to \Gamma(\operatorname{Spec} B, \mathscr{O}_{\operatorname{Spec} B})$$
is exactly our original ring homomorphism $\varphi : A \to B$. This implies that φ is injective.

Conversely, suppose that $\varphi : A \to B$ is injective. To prove that $f^{\#} : \mathscr{O}_X \to f_*\mathscr{O}_Y$ is injective, it is enough to prove that it is injective on distinguished open sets of $\operatorname{Spec} A$. Consider an arbitrary $f \in A$. The induced map on sections of $D(f)$ becomes the localization map

$$f^{\#}(D(f)) : A_f \simeq \mathscr{O}_{\operatorname{Spec} A}(D(f)) \to \mathscr{O}_{\operatorname{Spec} B}(D(\varphi(f))) \simeq B_{\varphi(f)}$$

which is injective. To prove that $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is dominant, let $D(f)$ be an open set not intersection $f(\operatorname{Spec} B)$. But then $\varphi(f)$ is contained in all prime ideal of B. This implies that $\varphi(f)$ is nilpotent. Since φ is injective, we know that f is nilpotent. So $D(f)$ is empty which suffices for the proof.

Alternatively, we can look at the stalk of $f^{\#} : \mathscr{O}_{\operatorname{Spec} A} \to f_* \mathscr{O}_{\operatorname{Spec} B}$. As we pointed out before, the induced map on $D(f)$ is

$$f^{\#}(D(f)) : A_f \simeq \mathscr{O}_{\operatorname{Spec} A}(D(f) \to \mathscr{O}_{\operatorname{Spec} B}(D(\varphi(f))) \simeq B_{\varphi(f)} \simeq B \otimes_A A_f$$

where we view B as an A-module via the morphism $\varphi : A \to B$. Since colimits commute with tensor products, the induced map on stalks is

$$f^{\#}_{\mathfrak{p}} : A \otimes_A A_{\mathfrak{p}} \to B \otimes_A A_{\mathfrak{p}}.$$

Since localization of modules is an exact functor, the injectivity or surjectivity of φ implies the injectivity or surjectivity of $f^{\#}$.

(c) Since φ is surjective, from the previous argument we know $f^{\#}$ is surjective. We only need to show that $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is a closed embedding. We can assume that the ring homomorphism is of the form $A \to A/I$ where I is an ideal. Then the map $f : \operatorname{Spec}(A/I) \to \operatorname{Spec} A$ is a homeomorphism on $V(I)$. Moreover, we have $V_{A/I}(\mathfrak{p}) = V_A(f(\mathfrak{p}))$ which proves that they are homeomorphic.

(d) This follows from the previous observation about the induced map on stalks. The hypothesis that $f : Y \to X$ is a homeomorphism onto a closed subset is not needed.

□

Exercise 3.2.19. Let A be a ring. Show that the following conditions are equivalent:

(i) $\operatorname{Spec} A$ is disconnected;

(ii) there exist nonzero elements $e_1, e_2 \in A$ such that $e_1 e_2 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$, and $e_1 + e_2 = 1$. (These elements are called orthogonal idempotents).

(iii) A is isomorphic to a direct product $A_1 \times A_2$ of non-zero rings.

Proof. Suppose $\operatorname{Spec} A$ is disconnected. Then we can write $\operatorname{Spec} A = V(I_1) \sqcup V(I_2)$ where $V(I_1)$ and $V(I_2)$ are clopen. Then

$$A \simeq \mathscr{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \simeq \mathscr{O}_{\operatorname{Spec} A}(V(I_1)) \times \mathscr{O}_{\operatorname{Spec} A}(V(I_2)).$$

This implies (iii). (iii) clearly implies (ii). (ii) clearly implies (iii) with the isomorphism $A \simeq e_1 A \times e_2 A$. (iii) clearly implies (i). This suffices for the proof. □

3.3. First Properties of Schemes

Exercise 3.3.1. Show that a morphism $f : X \to Y$ is locally of finite type if and only if for *every* affine open subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by open affine open subsets $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra.

Proof. The original definition for a morphism being locally of finite type is that there *exists* a covering of Y by affine open subsets whose preimages can be covered by affine open subsets where the corresponding ring are finitely generated as algebras. Clearly, if this holds for every affine open subset, then the morphism is locally of finite type.

To prove the other direction, we only need to prove the following two facts required in the application of the **affine communication lemma**. Let P be the property of an affine open subsets $\operatorname{Spec} B \hookrightarrow Y$ that $f^{-1}(\operatorname{Spec} B)$ can be covered by affine open subsets $U_j = \operatorname{Spec} A_j$ where each A_j is a finitely generated B-algebra.

(i) if $\operatorname{Spec} A \hookrightarrow Y$ has property P, then $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A \hookrightarrow Y$ has property P for any $f \in A$.

(ii) if $(f_1, \dots, f_n) = A$, and $\operatorname{Spec} A_{f_i} \hookrightarrow Y$ has property P for all i, then $\operatorname{Spec} A \hookrightarrow Y$ has property P too.

For (i), suppose that $\operatorname{Spec} A \hookrightarrow Y$ has property P. This means that we can cover the pre-image of $\operatorname{Spec} A$ with affine open subsets $\operatorname{Spec} B$ such that $\operatorname{Spec} B \to \operatorname{Spec} A$ is induced by $A \to B$ where B is a finitely generated A-algebra. The subset $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$ consists of the prime ideals avoiding f. The part of the pre-image in $\operatorname{Spec} B$ will then be $\operatorname{Spec} B_{\varphi(f)}$, the prime ideals of B avoiding $\varphi(f)$. The induced map on affine schemes comes from the localization $A_f \to B_{\varphi(f)}$. Then $B_{\varphi(f)}$ is still finitely generated since we just add $1/\varphi(f)$ to our list of generators. This proves (i).

For (ii), let $(f_1, \dots, f_n) = A$ so that $A = \bigcup D(f_i)$ and $\operatorname{Spec} A_{f_i}$ all have property P. But this is clear. The cover of the pre-images of $\operatorname{Spec} A_{f_i}$ will be a cover of the pre-image of $\operatorname{Spec} A$ by affine open subsets. Moreover, we have

$$\operatorname{Spec} B \to \operatorname{Spec} A_{f_i} \to \operatorname{Spec} A$$

induced by $A \to A_{f_i} \to B$ where B is a finitely generated A_{f_i} algebra. But then B would be a finitely generated A algebra as well.

□

Exercise 3.3.2. A morphism $f : X \to Y$ of schemes is **quasi-compact** if there is a cover of Y by open affines V_i such that $f^{-1}(V_i)$ is quasi-compact for each i. Show that f is quasi-compact if and only if for every open affine subset $V \subseteq Y$, $f^{-1}(V)$ is quasi-compact.

Proof. Recall the two hypotheses of the affine communication lemma.

(i) if $\operatorname{Spec} A \hookrightarrow Y$ has property P, then $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A \hookrightarrow Y$ has property P for any $f \in A$.

(ii) if $(f_1, \dots, f_n) = A$, and $\operatorname{Spec} A_{f_i} \hookrightarrow Y$ has property P for all i, then $\operatorname{Spec} A \hookrightarrow Y$ has property P too.

For (i), since the pre-image of $\operatorname{Spec} A$ is quasi-compact, we can cover it with finitely many affine open subsets $\operatorname{Spec} B_i$. Each $\operatorname{Spec} B_i \to \operatorname{Spec} A$ is induced by some ring morphism $\varphi_i : A \to B_i$. The pre-image of $\operatorname{Spec} A_{f_i}$ in each affine open covering the pre-image will then be $\operatorname{Spec} (B_i)_{\varphi_i(f_i)}$. Since the pre-image of $\operatorname{Spec} A_{f_i}$ is covered by finitely many open affines, it is quasi-compact.

For (ii), this is clear because $\operatorname{Spec} A = \bigcup \operatorname{Spec} A_{f_i}$ and the union of a finite number of quasicompact sets is quasicompact. □

Exercise 3.3.3.

(a) Show that a morphism $f : X \to Y$ is of finite type if and only if it is locally of finite type and quasi-compact.

(b) Conclude from this that f is of finite type if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ can be covered by a finite number of open affines $U_j = \operatorname{Spec} A_j$, where each A_j is a finitely generated B-algebra.

(c) Show that if f is of finite type, then for every open affine subset $V = \operatorname{Spec} B \subseteq Y$, and for every open affine subset $U = \operatorname{Spec} A \subseteq f^{-1}(V)$, A is a finitely generated B-algebra.

Proof. Suppose that $f : X \to Y$ is locally of finite type and quasi-compact. Then from Exercise 3.3.2 it must be of finite type. Now suppose that $f : X \to Y$ is of finite type. Then it is locally of finite type. Moreover it is quasi-compact since the cover in the definition of finite type has pre-images which are quasi-compact (as finite unions of affine schemes). This proves (a).

(b) follows from (a) and Exercise 3.3.1 and Exercise 3.3.2.

We want to prove that if $f : X \to \operatorname{Spec} B$ is a morphism of finite type, then any affine open $\operatorname{Spec} A \hookrightarrow X$ is a finitely generated B-algebra. Let property P be exactly this property. We want to prove the conditions of the affine communication lemma:

(i) if $\operatorname{Spec} A \hookrightarrow Y$ has property P, then $\operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A \hookrightarrow Y$ has property P for any $f \in A$.

(ii) if $(f_1, \dots, f_n) = A$, and $\operatorname{Spec} A_{f_i} \hookrightarrow Y$ has property P for all i, then $\operatorname{Spec} A \hookrightarrow Y$ has property P too.

Part (i) follows from the fact that if A is a finitely generated B algebra, then A_f will also be finitely generated by the finite generators and $1/f$. It is enough to prove the following algebra fact:

Fact 3.3. *Let $\varphi : B \to A$ be a ring morphism. Let $(f_1, \dots, f_n) = A$ such that A_{f_i} are finitely generated B-algebras. Then A is a finitely generated B-algebra.*

Proof. Pick $x_1, \dots, x_m \in A$ such that $x_1, \dots, x_m, 1/f_i$ generate A_{f_i} as a B-algebra. Let $a_1, \dots, a_n \in A$ such that $\sum a_i f_i = 1$. I claim that

$$x_1, \dots, x_m, a_1, \dots, a_n, f_1, \dots, f_n$$

generate A as a B-algebra. For any $x \in A$, we can write

$$f_i^N x = P_i(x_1, \ldots, x_m, f_i)$$

for some $P_i \in B[x_1, \ldots, x_{m+1}]$ for all i. By exponentiating $\sum a_i f_i = 1$ to the power of N, there are some $Q_i \in B[a_1, \ldots, a_m]$ such that $\sum Q_i f_i^N = 1$. This gives us the result. □

□

Exercise 3.3.4. Show that a morphism $f : X \to Y$ is finite if and only if for every open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ is affine, equal to $\operatorname{Spec} A$, where A is a finite B-module.

Proof. It is clear from the definition of a finite morphism that if for every open affine subset $V = \operatorname{Spec} B$ of Y, $f^{-1}(V)$ is affine and equal to $\operatorname{Spec} A$, where A is a finite B-module, then $f : X \to Y$ is a finite morphism. The other direction is the harder part. We will apply the affine communication lemma.

Let P be the property of affine open subsets $\operatorname{Spec} A \subseteq Y$ that the pre-image $\operatorname{Spec} B$ is affine and the map gives a finite module A module structure to B. It suffices to prove the following two conditions in the affine communication lemma:

(i) if $\operatorname{Spec} A \hookrightarrow Y$ has property P, then $\operatorname{Spec} A_f \hookrightarrow Y$ has property P for any $f \in A$.

(ii) if $(f_1, \ldots, f_n) = A$, and $\operatorname{Spec} A_{f_i} \hookrightarrow Y$ has property P for all i, then $\operatorname{Spec} A \hookrightarrow Y$ has property P too.

Part (i) corresponds to the algebraic fact that if $\varphi : A \to B$ is finite, then $\varphi_f : A_f \to B_{\varphi(f)}$ is also finite. For Part (ii), it is enough to assume that our morphism is $\varphi : X \to \operatorname{Spec} A$, and the pre-images of $\operatorname{Spec} A_{f_i}$ are $\operatorname{Spec} B_i$. We want to prove that X is affine and the induced map on global sections gives a finite module structure.

To prove that X is affine, we will use Exercise 3.2.17, the criterion for affineness. Consider the induced map on the global sections

$$\varphi^\# : A \simeq \mathscr{O}_{\operatorname{Spec} A}(\operatorname{Spec} A) \longrightarrow \mathscr{O}_X(X).$$

Then $\varphi^\#(f_i)$ generates the unit ideal in $\mathscr{O}_X(X)$ since the f_i generate the unit ideal in A. To complete the prove of affineness, it is enough to show that

$$X_{\varphi^\#(f_i)} = \operatorname{Spec} B_i = \varphi^{-1}(\operatorname{Spec} A_{f_i}).$$

For any $x \in X$, we have a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathscr{O}_X(X) \arrow[d] \\ \mathscr{O}_{\operatorname{Spec} A, \varphi(x)} \arrow[r] & \mathscr{O}_{X,x} \end{tikzcd}$$

where the bottom map is a morphism of local rings. Suppose that $x \in X_{\varphi^\#(f_i)}$. The tracking the trajectory of f_i from the top left, horizontal, and then vertical we see that it does not land in the maximal ideal. But this implies that if we start from f_i and then go vertical, it doesn't land in the maximal ideal of $\mathscr{O}_{\operatorname{Spec} A, \varphi(x)}$. In other words, we have $\varphi(x) \in \operatorname{Spec} A_{f_i}$ or $x \in \varphi^{-1}(\operatorname{Spec} A_{f_i})$. Now, if $x \notin X_{\varphi^\#(f_i)}$, then by the same reasoning f_i lands in the maximal ideal and we have $x \notin \varphi^{-1}(\operatorname{Spec} A_{f_i})$. This implies that $X_{\varphi^\#(f_i)} = \operatorname{Spec} B_i$ and its affine! This implies that X is affine from Exercise 3.2.17.

Then the fact that this gives a finite module structure follows from Part (i). This suffices for the proof. □

Exercise 3.3.5. A morphism $f : X \to Y$ is **quasi-finite** if for every point $y \in Y$, $f^{-1}(y)$ is a finite set.

(a) Show that a finite morphism is quasi-finite.
(b) Show that a finite morphism is **closed**, i.e., the image of any closed subset is closed.
(c) Show by example that a surjective, finite-type, quasi-finite morphism need not be finite.

Proof. For part (a), it suffices to work over $f : \operatorname{Spec} B \to \operatorname{Spec} A$ where B is a finite A-module. We want to prove that this map has finite fibers. The fiber over $\mathfrak{p} \in \operatorname{Spec} A$ has scheme structure given by

$$\begin{tikzcd} \operatorname{Spec}\ (B \otimes_A k(\mathfrak{p})) \arrow[r] \arrow[d] & \operatorname{Spec} k(\mathfrak{p}) \arrow[d] \\ \operatorname{Spec} B \arrow[r] & \operatorname{Spec} A. \end{tikzcd}$$

Since B is a finite A-module, $B \otimes_A k(\mathfrak{p})$ is a finite-dimensional algebra over a field. Thus it is an Artinian ring and has finitely many prime ideals. This implies that finite morphisms have finite fibers, i.e., are quasi-finite.

For part (b), it is enough to look locally and again we want to prove that $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is closed when $A \to B$ is finite. We want to prove that the image of $\operatorname{Spec}(B/I)$ for any ideal I under f is closed in $\operatorname{Spec} A$. In other words, it is enough to prove that

$$\operatorname{Spec}(B/I) \to \operatorname{Spec} B \to \operatorname{Spec} A$$

has closed image. Since $A \to B \to B/I$ is finite we can prove the more general statement that $A \to B$ finite implies that the image of $\operatorname{Spec} B \to \operatorname{Spec} A$ is closed. The induced map $A/I \to B$ where I is the kernel is also finite. We have $\operatorname{Spec} B \to \operatorname{Spec} A$ factors through

$$\operatorname{Spec} B \to \operatorname{Spec}(A/I) \to \operatorname{Spec} A$$

where the second morphism is a closed immersion. Thus, it is enough to prove that $\operatorname{Spec} B \to \operatorname{Spec}(A/I)$ has closed image. We can prove the more general statement that $A \subseteq B$ is a finite extension then $\operatorname{Spec} B \to \operatorname{Spec} A$ has closed image. But this follows from Theorem 5.10 in [1].

For part (c), consider the natural morphism from the affine line with doubled origin to the affine line. □

Exercise 3.3.6. Let X be an integral scheme. Show that the local ring $\mathscr{O}_\xi$ of the generic point ξ of X is a field. It is called the **function field** of X, and is denoted by $K(X)$. Show also that if $U = \operatorname{Spec} A$ is any open affine subset of X, then $K(X)$ is isomorphic to the quotient field of A.

Proof. Let $U = \operatorname{Spec} A$ is any open affine subset of X. Since X is integral, A is an integral domain. Moreover, we have

$$\mathscr{O}_\xi \simeq \mathscr{O}_{\operatorname{Spec} A,(0)} = S^{-1}A = K(A).$$

where $S = A \backslash \{0\}$. □

Exercise 3.3.7. A morphism $f : X \to Y$, with Y irreducible, is **generically finite** if $f^{-1}(\eta)$ is a finite set, where η is the generic point of Y. A morphism $f : X \to Y$ is **dominant** if $f(X)$ is dense in Y. Now let $f : X \to Y$ be dominant, generically finite morphsim of finite type of integral schemes. Show that there is an open subset $U \subseteq Y$ such that the induced morphism $f^{-1}(U) \to U$ is finite.

Proof. Let $\operatorname{Spec} A$ be an open affine subscheme of Y. Then we have a morphism $f : f^{-1}(\operatorname{Spec} A) \to \operatorname{Spec} A$. In this setting, A is an integral domain, f is still dominant, f is still generically finite, f is still finite type and, and $f^{-1}(\operatorname{Spec} A)$ is still integral. Let $\operatorname{Spec} B \subset f^{-1}(\operatorname{Spec} A)$ be an affine open subscheme. The morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ factors as

$$\operatorname{Spec} B \to \operatorname{Spec} A/I \to \operatorname{Spec} A$$

where I is the kernel of $A \to B$. From dominance, I consists of nilpotents and hence is the 0 ideal since A is integral. Thus we have $A \hookrightarrow B$ is an extension of rings. This also induces a field extension $\operatorname{Frac}(A) \hookrightarrow \operatorname{Frac}(B)$. From finite type, we have $B = A[x_1, \dots, x_n]$. From generically finite, we have

$$\operatorname{Spec}(A[x_1, \dots, x_n] \otimes_A \operatorname{Frac}(A)) = \operatorname{Spec} \operatorname{Frac}(A)[x_1, \dots, x_n]$$

is finite. The field of fractions of $\operatorname{Frac}(A)[x_1, \dots, x_n]$ is $\operatorname{Frac}(B)$. Since algebraic and finitely generated extensions are finite, it is enough to prove that $\operatorname{Frac}(A) \subseteq \operatorname{Frac}(B)$ is algebraic. Suppose we have a transcendental element α which we can pick to be in $\operatorname{Frac}(A)[x_1, \dots, x_n]$. Then by taking irreducible polynomials, we get infinitely many prime ideals of $\operatorname{Frac}(A)[x_1, \dots, x_n]$. This contradicts the generic finiteness condition. Thus we must have $K(Y) \to \operatorname{Frac}(A) \subseteq \operatorname{Frac}(B) = K(X)$ is a finite field extension.

For any $x_i \in B = A[x_1, \dots, x_n]$, it satisfies some minimal polynomial when we consider it over the extension $\operatorname{Frac}(A)[x_1, \dots, x_n]$. This is algebraic, over $\operatorname{Frac}(A)$.

Thus, after clearing denominators we have a polynomial relation of x_i with coefficients in A. Let ℓ_i be the leading coefficient of this polynomial and let $a = \prod \ell_i$. Then, localization gives us the restricted map

$$\operatorname{Spec} B \supset \operatorname{Spec}(B_a) \to \operatorname{Spec}(A_a) \subset \operatorname{Spec} A$$

where the corresponding ring morphism $A_a \to B_a$ is finite. By covering $f^{-1}(\operatorname{Spec} A)$ by finitely many $\operatorname{Spec} B$ (we can do this because f is a morphism of finite type so pre-image of affines are always quasi-compact), we can shrink A so that $f^{-1}(\operatorname{Spec} A)$ is covered by $\operatorname{Spec} B_i$ such that $\operatorname{Spec} B_i \to \operatorname{Spec} A$ come from finite morphisms $A \to B_i$.

Let us review what we have so far. We have found an affine open $\operatorname{Spec} A \subseteq Y$ where the map $f : f^{-1}(\operatorname{Spec} A) \to \operatorname{Spec} A$ has the property that the domain $f^{-1}(\operatorname{Spec} A)$ is covered by finitely many open affine subschemes $\operatorname{Spec} B_i$ where the $\operatorname{Spec} B_i \to \operatorname{Spec} A$ is induced by a finite morphism $A \hookrightarrow B_i$. Moreover, the induced map $K(A) \hookrightarrow K(B_i)$ of quotient fields is algebraic. We use the following fact:

Fact 3.4. *Let $A \subseteq B$ be an extension of integral domains. If $K(A) \hookrightarrow K(B)$ is algebraic, then any non-empty prime ideal of B contains a non-trivial element of A.*

The proof is not difficult. Let $x \in \mathfrak{p}$ be an element of a prime ideal of B. There is some polynomial $p \in A[t]$ with $p(x) = 0$ and non-zero constant term (since B is an integral domain). The constant term of p is an element of A in $\mathfrak{p}$.

Let $U_i = \operatorname{Spec} B_i \subseteq f^{-1}(\operatorname{Spec} A)$. The finite intersection $W = \bigcap U_i$ is a non-empty open subscheme since X is irreducible. This means that there are ideals $\mathfrak{b}_i \subseteq B_i$ such that

$$\operatorname{Spec} B_i \supseteq V(\mathfrak{b}_i) = U_i \backslash W$$

by definition of the Zariski topology. From the Fact, there are non-zero $f_i \in \mathfrak{b}_i \cap A$. We have

$$D(f_i) \subseteq W \text{ in } \operatorname{Spec} B_i.$$

Consider the open subscheme $\operatorname{Spec} A_{f_1 \ldots f_r} \subseteq \operatorname{Spec} A$. The pre-image is exactly $\bigcap X_{f_i}$ where we view f_i as elements in $\Gamma(X, \mathscr{O}_X)$ and $X = f^{-1}(\operatorname{Spec} A)$. This is also contained in W. So, for all j, we have that

$$f^{-1}(\operatorname{Spec} A_{f_1 \ldots f_r}) = \operatorname{Spec} B_j \cap \bigcap X_{f_i} = \operatorname{Spec} \left(B_j\right)_{f_1 \ldots f_r}$$

where we view $f_i \in A \subset B_j$. Thus, $f^{-1}(\operatorname{Spec} A_{f_1 \ldots f_r})$ is affine and the induced map is finite. This suffices for the proof. □

Exercise 3.3.8 (Normalization)**.** A scheme is **normal** if all of its local rings are integrally closed domains. Let X be an integral scheme. For each open affine subset $U = \operatorname{Spec} A$ of X, let $\widetilde{A}$ be the integral closure of A in its quotient field, and let $\widetilde{U} = \operatorname{Spec} \widetilde{A}$. Show that one can glue the schemes $\widetilde{U}$ to obtain a normal integral scheme $\widetilde{X}$, called the **normalization** of X. Show also that there is a morphism $\widetilde{X} \to X$, having the following universal property: for every normal integral scheme

Z, and for every dominant morphism $f : Z \to X$, f factors uniquely through $\widetilde{X}$. If X is of finite type over a field k, then the morphism $\widetilde{X} \to X$ is a finite morphism.

Proof. I first give the main idea behind the construction. Given an open affine subscheme $\operatorname{Spec} A \subseteq X$ we have a copy of $\operatorname{Spec} \widetilde{A}$. Since X is already a scheme, we already having gluing data for all open affine subschemes of X to form X. To upgrade this data to gluing data between the $\operatorname{Spec} \widetilde{A}$, we will simply use the fact that **localization commutes with algebraic closures**. Specifically, we lay out this idea in the following fact:

Fact 3.5. *Let A be an integral domain. Let $S \subseteq A$ be a multiplicative subset. Then we have $\widetilde{S^{-1}A} = S^{-1}\widetilde{A}$ where we can view the equality in the quotient field K.*

We can pick simultaneously distinguished affine opens in $\operatorname{Spec} A \cap \operatorname{Spec} B$ with isomorphisms $\operatorname{Spec} A_f \simeq \operatorname{Spec} B_g$. By taking integral closures, this gives an isomorphism $\operatorname{Spec} \widetilde{A}_f \simeq \operatorname{Spec} \widetilde{B}_g$. This provides the gluing data to get $\widetilde{X}$.

The morphism $\widetilde{X} \to X$ in the affine case just comes from the natural inclusion $A \subseteq \widetilde{A}$. We can define it in this way for each affine open subscheme. The compatibility of these maps again follows from the fact that localization commutes with algebraic closures.

For the universal property, you can verify it on affine schemes. It just boils down to injectivity! The domininant map between affine schemes is algebraically equivalent to injectivitity. Since $A \hookrightarrow \widetilde{A}$ is injective this gives the right notion.

We want to show that last result: if X is of finite type over a field k, then the morphism $\widetilde{X} \to X$ is a finite morphism. We can cover X with affine opens which are finitely generated k-algebras. Take one of these, $\operatorname{Spec} A$. The pre-image is $\operatorname{Spec} \widetilde{A}$. We want to prove that $A \subset \widetilde{A}$ is a finite extension. But this follows from a well-known theorem by Noether:

Fact 3.6. *Let A be a finitely generated k-algebra and let $\widetilde{A}$ be the integral closure. Then $\widetilde{A}$ is a finite A-module.*

□

Exercise 3.3.9 (The Topological Space of a Product)**.** Recall that in the category of varieties, the Zariski topology on the product of two varieties is not equal to the product topology. Now we see that in the category of schemes, the underlying point set of a product of schemes is not even the product set!

(a) Let k be a field, and let $\mathbf{A}^1_k = \operatorname{Spec} k[x]$ be the affine line over k. Show that $\mathbf{A}^1_k \times_{\operatorname{Spec} k} \mathbf{A}^1_k \simeq \mathbf{A}^2_k$, and show that the underlying point set of the product is not the product of the underlying point sets of the factors (even if k is algebraically closed).

(b) Let k be a field, let s and t be indeterminates over k. Then $\operatorname{Spec} k(s)$, $\operatorname{Spec} k(t)$, and $\operatorname{Spec} k$ are all one-point spaces. Describe the product scheme $\operatorname{Spec} k(s) \times_{\operatorname{Spec} k} \operatorname{Spec} k(t)$.

Proof. For part (a), the fibered product over $\operatorname{Spec} k$ is equal to

$$\mathbf{A}^1_k \times_{\operatorname{Spec} k} \mathbf{A}^1_k = \operatorname{Spec}(k[x] \otimes_k k[y]) = \mathbf{A}^2_k.$$

The points of the product of the sets $\mathbf{A}^1_k \times \mathbf{A}^1_k$ correspond to $(k \cup \{*\})^2$ where $*$ corresponds to the generic point. For $\mathbf{A}^2_k$, the points are irreducible polynomials. There are a lot more points in this set.

For part (b), it is $\operatorname{Spec} k(s,t)$ which is also a single point. □

Exercise 3.3.10 (Fibers of a morphism)**.**

(a) If $f : X \to Y$ is a morphism, and $y \in Y$ a point, show that $\operatorname{spa}(X_y)$ is homeomorphic to $f^{-1}(y)$ with the induced topology.

(b) Let $X = \operatorname{Spec} k[s,t]/(s-t^2)$, let $Y = \operatorname{Spec} k[s]$, and let $f : X \to Y$ be the morphism defined by sending $s \to s$. If $y \in Y$ is the point $a \in k$ with $a \neq 0$, show that the fibre X_y consists of two points, with residue field k. If $y \in Y$ corresponds to $0 \in k$, show that the fibre X_y is a nonreduced one-point scheme. If η is the generic point of Y, show that X_η is a one-point scheme, whose residue field is an extension of degree two of the residue field of η. (Assume k is algebraically clsoed).

Proof. For part (a), we first prove it in the case where X and Y are affine schemes. Suppose that $f : \operatorname{Spec} B \to \operatorname{Spec} A$ can be represented as the ring map $\varphi : A \to B$. For $\mathfrak{p} \in \operatorname{Spec} A$, we want to compute the fibered product along the map $\operatorname{Spec} k(\mathfrak{p}) \to \operatorname{Spec} A$. We will do it in parts according to the composition of maps

$$A \to A_\mathfrak{p} \to k(\mathfrak{p}).$$

Consider the commutative diagram

$$\begin{tikzcd}
\text{Spec } B_\mathfrak{p}/\mathfrak{p}B_\mathfrak{p} \arrow[d, equal] & \text{Spec } B_\mathfrak{p} \arrow[d, equal] & \\
\text{Spec } (B_\mathfrak{p} \otimes_{A_\mathfrak{p}} A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}) \arrow[r] \arrow[d] & \text{Spec } (B \otimes_A A_\mathfrak{p}) \arrow[r] \arrow[d] & \text{Spec } B \arrow[d] \\
\text{Spec } (A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}) \arrow[r] & \text{Spec } A_\mathfrak{p} \arrow[r] & \text{Spec } A
\end{tikzcd}$$

For the first Cartesian square, we are taking the fibered product of $\operatorname{Spec} B \to \operatorname{Spec} A$ along the morphism $\operatorname{Spec} A_\mathfrak{p} \to \operatorname{Spec} A$. The result is the spectrum

$$\operatorname{Spec}(B \otimes_A A_\mathfrak{p}) = \operatorname{Spec} B_\mathfrak{p}.$$

The prime ideals of $B_\mathfrak{p}$ are exactly the prime ideals in $\operatorname{Spec} B$ which do not contain $\varphi(A \backslash \mathfrak{p})$. The morphism $\operatorname{Spec} B_\mathfrak{p} \to \operatorname{Spec} A_\mathfrak{p}$ is the same map $\operatorname{Spec} B \to \operatorname{Spec} A$ after a change of domain and codomain.

In the second Cartesian square, we are taking the fibered product with the closed immersion

$$\operatorname{Spec}(k(\mathfrak{p})) = \operatorname{Spec}(A_\mathfrak{p}/\mathfrak{p}A_\mathfrak{p}) \to \operatorname{Spec} A_\mathfrak{p}.$$

The fibered product ends up being

$$\operatorname{Spec}(B_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} A_{\mathfrak{p}}/\mathfrak{p}A_{\mathfrak{p}}) = \operatorname{Spec} B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}}.$$

This is the prime ideals of B containing $\varphi(A\backslash\mathfrak{p})$ and not containing $\varphi(\mathfrak{p})$. This exactly corresponds to the prime ideals of $\mathfrak{q} \in \operatorname{Spec} B$ for which $\varphi^{-1}(\mathfrak{q}) = \mathfrak{p}$. It is also clear that top horiziontal maps are inclusions and thus order-preserving. This proves that $\mathrm{spa}(X_y)$ is homeomorphic to $f^{-1}(y)$. □

Exercise 3.3.11 (Closed Subschemes)**.**

(a) Closed immersions are stable under base extension: if $f : Y \to X$ is a closed immersion, and if $X' \to X$ is any morphism, then $f' : Y \times_X X' \to X'$ is also a closed immersion.

(b) If Y is a closed subscheme of an affine scheme $X = \operatorname{Spec} A$, then Y is also affine, and in fact, Y is the closed subscheme determined by a suitable ideal $\mathfrak{a} \subseteq A$ as the image of the closed immersion $\operatorname{Spec} A/\mathfrak{a} \to \operatorname{Spec} A$.

(c) Let Y be a closed subset of a scheme X, and give Y the reduced induced subscheme structure. If Y' is any other closed subscheme of X with the same underlying topological space, show that the closed immersion $Y \to X$ factors through Y'. We express this property by saying that the reduced induced structure is the smallest subscheme structure on a closed subset.

(d) Let $f : Z \to X$ be a morphism. Then there is a unique closed subscheme Y of X with the following property: the morphism f factors through Y, and if Y' is any other closed subscheme of X through which f factors, then $Y \to X$ factors through Y' also. We call Y the **scheme-theoretic image** of f. If Z is a reduced scheme, then Y is just the reduced induced structure on the closure of the image $f(Z)$.

Proof. For part (1), it suffices to prove this for affine schemes. This follows from the well-known fact that $M \otimes_A A/I \simeq M/IM$.

For part (2), we will first prove that if $\operatorname{Spec} B \to \operatorname{Spec} A$ is a closed subscheme, then the corresponding ring homomorphism $A \to B$ is surjective. In otherworse, affine closed subschemes of affine schemes are of the form $\operatorname{Spec}(A/I) \to \operatorname{Spec} A$ induced by the quotient ring homomorphism $A \to A/I$. From $\operatorname{Spec} B \to \operatorname{Spec} A$ being a closed immersion, the map on sheaves is surjective. This implies that there is some open cover, which we can pick to be distinguished $D(f_i)$ of $\operatorname{Spec} A$, such that the maps

$$A_{f_i} \to \mathscr{O}_{\operatorname{Spec} A}(D(f_i)) \to \mathscr{O}_{\operatorname{Spec} B}(D(\varphi(f_i))) = B_{\varphi(f_i)}$$

are surjective where $\varphi : A \to B$ is the corresponding ring homomorphism. Since the $D(f_i)$ cover $\operatorname{Spec} A$, we also know that $(f_1, \dots, f_r) = 1$ (we can pick a finite number of them). So we want to prove the following algebraic fact: let $(f_1, \dots, f_r) = 1$ and $\varphi : A \to B$ be a ring map such that $A_{f_i} \to B_{\varphi(f_i)}$ are surjective. Then $A \to B$ is surjective. For $b \in B$, there is a large enough N and $a_i \in A$

so that
$$\frac{\varphi(a_i)}{\varphi(f_i)^N} = b \text{ in } B_{\varphi(f_i)}.$$
For large enough M, we have that
$$\varphi(f_i)^M \left(\varphi(a_i) - b\varphi(f_i)^N\right) = 0 \text{ in } B \text{ for all } i.$$
By absorbing the f_i^M into a_i, we have $b\varphi(f_i^{M+N}) = \varphi(a_i)$ in B for all i. There is some $c_i \in A$ such that $\sum c_i f_i^{M+N} = 1$. Thus, we have
$$\varphi\left(\sum c_i a_i\right) = b$$
which implies surjectivity. Thus, any affine closed subscheme of $\operatorname{Spec} A$ is of the form $\operatorname{Spec} A/I \to \operatorname{Spec} A$.

Now let $Y \to \operatorname{Spec} A$ be any closed subscheme. Let $\operatorname{Spec} B_i \subseteq Y$ be an affine open cover of Y. We have open sets $U_i \subseteq X$ such that
$$\operatorname{Spec} B_i = Y \cap U_i.$$
We can refine U_i with $D_{\operatorname{Spec} A}(g_{ij}) \subseteq U_i$. Our map $f : Y \to \operatorname{Spec} A$ we can restrict to
$$f : \operatorname{Spec} B_i \to \operatorname{Spec} A.$$
The pre-image of $D(g_{ij})$ is affine too since it consists of all prime ideals of B avoiding $\varphi(g_{ij})$. Since the pre-image of U_i under $Y \to X$ is $\operatorname{Spec} B_i$, the pre-image of $D(g_{ij})$ under the original map is exactly this affine open. Thus, we have some open cover of X by distinguished affine opens whose pre-images (intersection with Y) are also affine. Moreover we can pick finitely many of them from quasi-compactness. Relabel them as $D(f_1), \ldots, D(f_r)$ and we have $(f_1, \ldots, f_r) = 1$ in A. It is not hard to see that the pre-image of $D(f_i)$ is exactly Y_{f_i} and is affine where we consider the image of f_i to global sections of Y. Since $f_1, \ldots, f_r$ generate unit ideal, the image of these generators must also generate the unit ideal of the global sections of Y. From Exercise III.2.17, this implies that Y is affine. We have already taken care of this case in the beginning.

For part (3), we get the idea on affine opens. It is not hard to see that they glue in a compatible way. In the affine case, algebraically we want to show that if I is radical, J is an ideal so that $\sqrt{J} = I$, then $A \to A/I$ factors uniquely through A/J. But this is trivial because $J \subseteq I$.

I'll skip part (4) and maybe do it at a later date □

Exercise 3.3.12 (Closed Subschemes of $\operatorname{Proj} S$)**.**

(a) Let $\varphi : S \to T$ be a surjective homomorphism of graded rings, preserving degrees. Show that the open set U of (Exercise 3.2.14) is equal to $\operatorname{Proj} T$, and the morphism $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is a closed immersion.

(b) If $I \subseteq S$ is a homogeneous ideal, take $T = S/I$ and let Y be the closed subscheme of $X = \operatorname{Proj} S$ defined as the image of the closed immersion $\operatorname{Proj} S/I \to X$. Show that different homogeneous ideals can give rise to the same closed subscheme. For example, let d_0 be an integer, and let $I' = \bigoplus_{d \geqslant d_0} I_d$. Show that I and I' determine the same closed subscheme.

Proof. Let $\varphi : S \to T$ be a surjective graded homomorphism of graded rings. Then $\varphi(S_+) = T_+$. Thus, we have

$$U = \{\mathfrak{p} \in \operatorname{Proj} T : \mathfrak{p} \not\supseteq \varphi(S_+)\} = \{\mathfrak{p} \in \operatorname{Proj} T : \mathfrak{p} \not\supseteq T_+\} = \operatorname{Proj} T.$$

To prove that $f : \operatorname{Proj} T \to \operatorname{Proj} S$ is a closed immersion, it suffices to find an open cover U_i of $\operatorname{Proj} T$ with open sets V_i of $\operatorname{Proj} S$ such that $f(U_i) \subseteq V_i$ and $f : U_i \to V_i$ is a closed immersion. We can pick $D_{\operatorname{Proj} T}(f)$ for all homogeneous $f \in T$. Let $t \in T$ be arbitrary, from surjectivity we have $s \in S$ such that $\varphi(s) = t$. The induced map

$$f : \operatorname{Spec}(T_{(t)}) \simeq D_{\operatorname{Proj} T}(t) \to D_{\operatorname{Proj} S}(s) \simeq \operatorname{Spec}(S_{(s)})$$

where the map is the localization and taking degree 0 part. This is surjective which means we have a closed immersion! This completes the first part.

$\square$

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