Lectures on Algebraic Geometry

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Contents

1. Aims & Scope

1.1. Introduction These notes are for myself. For my research I will need a better understanding and intuition for algebraic geometry. Not only will I need to know scheme theoretic algebraic geometry (which comes easier to me), but I will also need a very intuitive understanding of algebraic varieties (as geometric objects). In these notes, I will try to emphasize *examples* and *computations*. This is my way of forcing myself to learn algebraic geometry.

This project will be a difficult and long one. Dear future Alan, please be patient and trust the process. I hope I can make you proud.

1.2. Structure of this paper I will mostly follow the monographs and. For the theory of varieties, I will focus more on . For schemes and other concepts, I will switch to .

2. Affine Varieties

2.1. Affine space The source of all of our affine varieties will come from *affine space*. Later, we generalize object to a ringed space. But locally, our objects will always look like spaces cut out by polynomials in affine space. For the rest of this section let $k = \bar{k}$ be an algebraically closed field. It will do no harm to assume that $k = \mathbb{C}$.

Definition 2.1 (Affine space)**.** We define n-dimensional **affine space** to be the set of all n-tuples with coordinates in k. We denote this by

$$\mathbb{A}_k^n = \{(x_1, \dots, x_n) : x_i \in k \text{ for all } i\}.$$

When the ambient field k is clear, we may not include it in the subscript.

At the moment A^n is just a set. We will soon endow it with a topology, and a sheaf of k-valued functions.

2.1.2. Zariski topology Any polynomial $f \in k[x_1, \ldots, x_n]$ defines a k-valued function on $\mathbb{A}^n_k$. Let $I \subseteq k[x_1, \ldots, x_n]$ be any subset of polynomials. We define the **zero set** or **zero locus** of I to be equal to

$$Z(I) \stackrel{\text{def}}{=} \{x \in \mathbb{A}^n : f(x) = 0 \text{ for all } f \in I\}.$$

When computing the zero locus, we are asking for the common solutions or roots to the polynomials in I.

Exercise 2.3. Show that the zero locus of a set is the same as the zero locus of the ideal generated by that set.

We call any subset $X \subseteq \mathbb{A}^n$ which arises as the zero locus of an ideal a **algebraic set**. Thus, we have a way from going from ideals of $k[x_1, \ldots, x_n]$ to algebraic sets in $\mathbb{A}^n$.

Proposition 2.4.

(1) If $I_1 \subset I_2$, *then* $Z(I_1) \supset Z(U_2)$.
(2) $Z(I_1) \cup Z(I_2) = Z(I_1 I_2)$.
(3) $\bigcap Z(I_\alpha) = Z\left(\sum I_\alpha\right)$.
(4) $\mathbb{A}^n = Z(0)$, $\varnothing = Z(k[x_1, \ldots, x_n])$.

Algebraic sets are closed under arbitrary intersections and finite unions. The empty set and $\mathbb{A}^n$ are both algebraic sets. This means that we can equip $\mathbb{A}^n$ with a the **Zariski topology** where the closed sets are the algebraic sets.

2.1.5. We can go the other way around too. For a set $X \subseteq \mathbb{A}^n$, define the **coordinate ideal** of X to be

$$I(X) \stackrel{\text{def}}{=} \{f \in k[x_1, \ldots, x_n] : f(x) = 0 \text{ for all } x \in X\}.$$

This asks the reverse question: given some subset of $\mathbb{A}^n$, which polynomials vanish over it? A natural question to ask is if these two operations are inverse to each other. If they were, we would get a one-to-one correspondence between algebraic sets and ideals of $k[x_1, \ldots, x_n]$.

$$\begin{tikzcd}
\text{algebraic sets} \arrow[r, bend right, "I(\bullet)"'] & \text{ideals} \arrow[l, bend right, "Z(\bullet)"']
\end{tikzcd}$$

FIGURE 2.6. First algebraic set and ideal "correspondence"

The answer is no! For example, consider the ideal (x^2) in $k[x]$. This corresponds to the algebraic set $\{0\}$ in $\mathbb{A}^1$. The coordinate ideal of $\{0\}$ is equal to

(x) (which is not the same as (x^2)). The Hilbert Nullstellensatz gives us the true correspondence.

Definition 2.7. Let X be a prevariety. We say that X is a **variety** if for every prevariety Y and every two morphisms $f_1, f_2 : Y \to X$, then set $\{p \in Y : f_1(p) = f_2(p)\}$ is closed in Y.

3. Prevarieties and Varieties

4. Projective Varieties

Varieties are algebraic analogs of real and complex manifolds. We have seen that they are constructed by gluing together affine varieties in a compatible way. Affine varieties (over $\mathbb{C}$) are usually not compact over $\mathbb{C}$. We will define the notion of a projective variety which is our analog of a compact manifold. Projective varieties will be closed subset of another proto-typical variety: the projective space.

4.1. Projective space

4.1.1. The projective space $\mathbb{P}^n$ as a set consists of the one-dimensional subspaces of k^{n+1}. In **homogeneous coordinates**, we can represent this as

$$\mathbb{P}^n = \{(a_0 : \ldots : a_n) | a_i \neq 0 \text{ for some } i\}.$$

4.1.2. To understand the geometry of the projective space a little better, let us partition it into subspaces. We look at $(a_0 : \ldots : a_n) \in \mathbb{P}^n$.

- When $a_0 \neq 0$, we can always set $a_0 = 1$ and then we get a copy of $\mathbb{A}^n$. The identification is via the map $\alpha : \mathbb{A}^n \hookrightarrow \mathbb{P}^n$ defined by
$$\alpha(x_1, \ldots, x_n) = (1 : x_1 : \ldots : x_n).$$
- When $a_0 = 0$, we get a copy of $\mathbb{P}^{n-1}$. We can consider them as lines in the copy of $\mathbb{A}^n$ from setting $a_0 = 0$. Indeed, if we go along the trajectory $\lambda(x_1, \ldots, x_n)$ in $\mathbb{A}^n$ as $\lambda \to \infty$ (assuming we are in $\mathbb{C}$ or $\mathbb{R}$ so that this makes sense), then we have
$$\begin{aligned}\lim_{\lambda \to \infty} \alpha(\lambda x_1, \ldots, \lambda x_n) &= \lim_{\lambda \to \infty} (\lambda^{-1} : x_1 : \ldots : x_n) \\ &= (0 : x_1 : \ldots : x_n).\end{aligned}$$
Thus, any two distinct lines in $\mathbb{A}^n$ will intersect. If they are not parallel, they will intersect in $\mathbb{A}^n$. If they are parallel, then they will intersect at the point at infinity.

4.1.3. Let $I \subset k[x_0, \ldots, x_n]$ be a homogeneous ideal. This means that the ideal is generated by homogeneous elements or equivalently any polynomial in the ideal has its homogeneous parts in the ideal as well. Then, we define

$$Z_p(I) \stackrel{\text{def}}{=} Z(I) = \{p \in \mathbb{P}^n : f(p) = 0 \text{ for all } f \in I\}.$$

Note that this doesn't make sense a priori. We can make sense of it when $f \in I$ is homogeneous. But since I is homogeneous, we leave it as an exercise that it is correct as is.

Remark 4.4. The p subscript under the $Z_p(I)$ refers to taking the zero locus in projective space. When we take the zero locus in the affine space, we will sometimes write $Z_a(I)$.

Definition 4.5. The **Zariski topology** on $\mathbb{P}^n$ is the topology where the closed sets are given by the (projective) zero locus of homogeneous ideals of $k[x_0, \ldots, x_n]$. Irreducible algebraic sets in $\mathbb{P}^n$ are called **projective varieties**.

Note that we have not yet proved that projective varieties are actually varieties.

4.1.6. We now define a sheaf structure on projective varieties. For X a projective variety, we define

$$I(X) \stackrel{\text{def}}{=} \langle f \in k[x_0, \ldots, x_n] \text{ homogeneous} : f(p) = 0 \text{ for all } p \in X \rangle$$

where the notation means we are taking the ideal generated by the set. This will be a homogeneous ideal in $k[x_0, \ldots, x_n]$.

Definition 4.7. For a projective variety X, we define the **homogeneous coordinate ring** as

$$S(X) \stackrel{\text{def}}{=} k[x_0, \ldots, x_n]/I(X).$$

These elements don't define functions on $\mathbb{P}^n$. We need to take ratios of homogeneous elements of the same degree to get something well-defined.

Definition 4.8. We define the **field of rational functions** to be

$$K(X) \stackrel{\text{def}}{=} \left\{ \frac{f}{g} : f, g \in S(X)^{(d)} \text{ and } g \neq 0 \text{ in } S(X) \right\}.$$

The elements of $K(X)$ give set-theoretic functions on X whenever the denominator is non-zero. To mimic the affine case, we define

$$\mathcal{O}_{X,p} \stackrel{\text{def}}{=} \left\{ \frac{f}{g} \in K(X) : g(p) \neq 0 \right\}, \quad \mathcal{O}_X(U) \stackrel{\text{def}}{=} \bigcap_{p \in U} \mathcal{O}_{X,p}.$$

This gives X the structure of a sheaf of k-valued functions.

4.1.9. We now begin the process to prove that projective varieties are actually varieties. The first step is to prove that it is a prevariety. The idea is that we will take affine charts by intersecting our projective variety with the standard affine charts of projective space.

Proposition 4.10. *Let X be a projective variety. Then $(X, \mathcal{O}_X)$ is a prevariety.*

Proof. Consider the open subset $X_0 \subset X$ given by

$$X_0 = \{(a_0 : \ldots : a_n) \in X : a_0 \neq 0\} = X \cap \mathbb{A}^n_{(0)}.$$

If $X = Z(f_1, \dots, f_r)$ with $f_i \in k[x_0, \dots, x_n]$ homogeneous, let $g_i = f_i(1, x_1, \dots, x_n)$ be the de-homogenization at x_0. Define $Y_0 = Z(g_1, \dots, g_r)$ in $\mathbb{A}^n$. We can construct a map $\varphi_0 : X_0 \to Y_0$ defined by

$$\varphi_0(a_0 : \dots : a_n) = \left(\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0} \right).$$

This map sends elements X_0 to Y_0 because of the homogeneity. For the inverse map, we have

$$\varphi_0^{-1}(x_1, \dots, x_n) = (1 : x_1 : \dots : x_n).$$

We can repeat this for $X_1, \dots, X_n$. These cover X, which implies that X is a prevariety. □

4.1.11. We give one family of examples of morphisms of projective varieties into projective space.

Lemma 4.12. *Let $X \subset \mathbb{P}^n$ be a projective variety (or open subset of a projective variety). Let $f_1, \dots, f_m \in k[x_0, \dots, x_n]$ be homogeneous polyomials of the same degree in the homogeneous coordinates of $\mathbb{P}^n$ and assume that for each $p \in X$, there is some i such that $f_i(p) \neq 0$. Then, the map*

$$f : X \to \mathbb{P}^m, \quad p \in X \mapsto (f_0(p) : f_1(p) : \dots : f_n(p))$$

is a morphism of varieties.

Proof. The conditions are there to make the set map well-defined. To prove that it is a true morphism, consider the map restricted on the canonical affine cover. This reduces a polynomial map which is always a morphism. □

5. Appendix

5.1. Commutative Algebra

5.2. Topology

References

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