# Junior Seminar Notes

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# 1 The Initial Conditions

We first describe the initial markov chain.

**Definition 1** (Markov Chain I). *We have a Markov chain $x_0, x_1, \dots$ where $x_0$ is some initial vector in $\mathbb{R}^n$. Notation-wise, let $x_k = (x_{k,1}, \dots, x_{k,n})$. The Markov chain progresses as follows: given $x_k$ pick two distinct coordinates $I$ and $J$ uniformly at random, and replace $x_{k,I}$ and $x_{k,J}$ by their average $\frac{x_{k,I}+x_{k,J}}{2}$.*

We have a Markov chain $x_0, x_1, \dots$ where $x_0$ is some initial vector in $\mathbb{R}^n$. Notation-wise, let $x_k = (x_{k,1}, \dots, x_{k,n})$. The Markov chain progresses as follows: given $x_k$ pick two distinct coordinates $I$ and $J$ uniformly at random, and replace $x_{k,I}$ and $x_{k,J}$ by their average $\frac{x_{k,I}+x_{k,J}}{2}$. We want to study the rate of convergence of this Markov chain to its average $(\overline{x}_0, \dots, \overline{x}_0)$ where

$$\bar{x_0} = \frac{x_{0,1} + \ldots + x_{0,n}}{n}.$$

We have two ways to measure the distance from the average:

$$T(k) = \sum_{i=1}^{n} |x_{k,i} - \overline{x}_0|$$
$$S(k) = \sum_{i=1}^{n} |x_{k,i} - \overline{x}_0|^2.$$

We have an exact formula for the expected $L^2$ norm. Without loss of generality, we can assume $\bar{x_0} = 0$.

**Proposition 1** (Proposition 2.1 in the paper). *For any $k \geq 0$, we have*

$$\mathbb{E}(S(k+1)|\mathcal{F}_k) = \left(1 - \frac{1}{n-1}\right) S(k).$$

*Proof.* The calculation is done in the paper. $\square$

To avoid issues of dependence in the discrete time setting, we approximate Markov chain I with the following continuous time Markov chain.

**Definition 2** (Markov Chain II). *We have a Markov chain $\{x_t\}_{t\geq 0}$. Give each pair of coordinates an independent Poisson clock with rate $\binom{n}{2}^{-1}$. When a clock rings, replace the corresponding pair of coordinates with their average.*

With this new Markov chain, we introduce the same measurements from the average.

$$T'(k) := \sum_{i=1}^{n} |x_{t,i} - \overline{x}_0|$$
$$S'(k) := \sum_{i=1}^{n} |x_{t,i} - \overline{x}_0|^2.$$

We get a similar formula for the expected $L^2$ distance.

**Proposition 2.** *For any $t \geq s \geq 0$, we have that*

$$\mathbb{E}[S'(t)|\mathcal{F}_s] = \exp\left(-\frac{t-s}{n-1}\right) S'(s).$$

*Proof.* Let $E_k$ be the event that there are $k$ rings between times $s$ and $t$. Then

$$\begin{aligned}
\mathbb{E}[S'(t)|\mathcal{F}_s] &= \sum_{k=0}^{\infty} \mathbb{E}\left[S'(t)|\mathcal{F}_s \text{ and } k \text{ rings}\right] \cdot \Pr[E_k] \\
&= \sum_{k=0}^{\infty} \left(1 - \frac{1}{n-1}\right)^k S'(s) \cdot e^{-(t-s)} \cdot \frac{(t-s)^k}{k!} \\
&= \exp\left((t-s)\left(1 - \frac{1}{n-1}\right)\right) \cdot e^{-(t-s)} S'(s) \\
&= \exp\left(-\frac{t-s}{n-1}\right) \cdot S'(s).
\end{aligned}$$

This suffices for the proof. $\square$

From here on out, we study the process starting from $x_0 = (1 - 1/n, -1/n, \ldots, -1/n)$. The main theorem that the paper proves is the following.

**Theorem 1** (Theorem 2.3 in the paper)**.** *Take $\Phi : \mathbb{R} \to [0, 1]$ as the cdf of the standard normal distribution. For any $a \in \mathbb{R}$, we have $T'(n(\log_2(n) + a\sqrt{\log_2(n)})/2) \xrightarrow{\mathbb{P}} 2\Phi(-a)$ in probability as $n \to \infty$.*

In the discrete case, we have the following theorem.

**Theorem 2** (Theorem 1.2 in the paper)**.** *For any $a \in \mathbb{R}$, we have that*

$$T(\lfloor n(\log_2(n) + a\sqrt{\log_2(n)})/2) \xrightarrow{\mathbb{P}} 2\Phi(-a).$$

**Proposition 3.** *Theorem 1 and Theorem 2 are equivalent.*

*Proof.* TODO $\square$