Ultra Log-concavity of Basis Partition Sequence

Alan Yan
Senior Thesis Advisor: Professor June Huh

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Abstract

Let $M = (E, \mathcal{I})$ be a matroid of rank r and let $E = A \sqcup B$ be a fixed partition of the ground set. For $0 \leq k \leq r$, let N_k be the number of bases whose intersection with A has size k. We prove that the sequence $\{N_k\}$ is ultra-log-concave.

Contents

1 Background Material

(To be added: Matroids, Mixed Volumes, Mixed Discriminants, Lorentzian Polynomials, Hodge Riemann Relations)

2 Introduction

The question is motivated by the results in [3]. In our setting, we have a matroid $M = (E, \mathcal{B})$ of rank r and size n where $\mathcal{B}$ is the set of bases. Fix a partition of the ground set $E = A \cup B$. From this, we can define the following combinatorial sequence

$$N_k := N_k(A) := \#\{U \in \mathcal{B} : |U \cap A| = k\}.$$

In particular, Stanley uses the Alexandrov-Fenchel inequality to prove the following theorem.

Theorem 2.1. *If M is regular, then $N_k/\binom{r}{k}$ is log-concave.*

Proof. Let $\{v_1, \ldots, v_n\}$ be a unimodular coordinatization of the matroid M where the vectors are in $\mathbb{R}^r$. Consider the zonotopes

$$Z_A := \sum_{x \in A} [0, v_x]$$
$$Z_B := \sum_{x \in B} [0, v_x].$$

Then

$$\mathrm{Vol}_r(\lambda Z_A + \mu Z_B) = \mathrm{Vol}_r\left(\sum_{x \in A}[0, \lambda v_x] + \sum_{x \in B}[0, \mu v_x]\right)$$
$$= \sum_{k=0}^{r} N_k \lambda^k \mu^{r-k}.$$

This implies that

$$N_k = \binom{r}{k} V(Z_A[k], Z_B[r-k]).$$

This proves that $N_k/\binom{r}{k}$ is log-concave. $\square$

Stanley's proof relates the sequence N_k to a sequence of mixed volume. The ultra-log-concave follows from the Alexandrov-Fenchel inequality. The relationship relies on the unimodular coordinazation that exists for regular matroids. It is unclear whether or not the result holds true for general matroids. There are three possible conjectures each of increasing strength.

Theorem 2.2. *Let $M = (E, \mathcal{I})$ be a matroid of rank r and consider a partition*

$$E = S_1 \sqcup S_2 \sqcup \ldots \sqcup S_t$$

of the matroid into t sets. For $(r_1, \ldots, r_t) \in \mathbb{N}_{\geq 0}$ satisfying $r_1 + \ldots + r_t = r$, let $B(r_1, \ldots, r_t)$ denote the number of bases B such that $|B \cap S_i| = r_i$ for all $1 \leq i \leq t$. Let $\{v_1, \ldots, v_n\}$ be a unimodular coordinatization of M where $v_i \in \mathbb{R}^r$. Then

$$B(r_1, \ldots, r_t) = \binom{r}{r_1, \ldots, r_t} \mathsf{D}(A_1[r_1], \ldots, A_t[r_t])$$

where $A_i = X_i X_i^T$ where X_i is the matrix containing as columns the vectors cooresponding to the vectors in S_i.

Proof. This is immediate from the properties of the mixed discriminant. $\square$

Specializing to the case where $t = 2$, we have $N_k = \binom{r}{k}\mathsf{D}(A_1[k], A_2[r-k])$. Thus the equality of

$$\left(\frac{N_k}{\binom{r}{k}}\right)^2 = \frac{N_{k-1}}{\binom{r}{k-1}} \cdot \frac{N_{k+1}}{\binom{r}{k+1}}$$

is really a question on the equality of

$$\mathsf{D}(A_1[k], A_2[r-k])^2 \geq \mathsf{D}(A_1[k-1], A_2[r-k+1]) \cdot \mathsf{D}(A_1[k+1], A_2[r-k-1]).$$

Equality holds if and only if $A_1 = \lambda A_2$ for some real λ.

Conjecture 2.1. *Let M be a matroid of rank r and order n. Then,*

(a) N_k *is log-concave.*

(b) $N_k/\binom{n}{k}$ *is log-concave.*

(c) $N_k/\binom{r}{k}$ *is log-concave.*

The conjectures (a), (b), (c) are listed in order of increasing strength.

Proposition 2.1. *In Conjecture 2.1, (c) $\implies$ (b) $\implies$ (a).*

Proof. Conjecture (c) asserts that

$$\frac{N_k^2}{N_{k-1}N_{k+1}} \geq \frac{k+1}{k} \cdot \frac{r-k+1}{r-k}.$$

Conjecture (b) asserts that

$$\frac{N_k^2}{N_{k-1}N_{k+1}} \geq \frac{k+1}{k} \cdot \frac{n-k+1}{n-k}.$$

Conjecture (a) asserts that

$$\frac{N_k^2}{N_{k-1}N_{k+1}} \geq 1.$$

The proposition follows from the inequality

$$\frac{r-k+1}{r-k} \geq \frac{n-k+1}{n-k} \geq 1.$$

$\square$

Proposition 2.2. *In the case for uniform matroids, Conjecture 2.1(c) is true.*

Proof. Consider the uniform matroid $U_{r,n}$ and partition $|A| = a, |B| = b$ such that $a + b = n$. Then, for any k we have

$$N_k = \binom{a}{k} \cdot \binom{b}{r-k}$$
$$N_{k+1} = \binom{a}{k+1} \cdot \binom{b}{r-k-1}$$
$$N_{k-1} = \binom{a}{k-1} \cdot \binom{b}{r-k+1}.$$

We can compute

$$\begin{aligned}
\frac{N_k^2}{N_{k-1}N_{k+1}} &= \frac{\binom{a}{k}^2}{\binom{a}{k+1}\binom{a}{k-1}} \cdot \frac{\binom{b}{r-k}^2}{\binom{b}{r-k-1}\binom{b}{r-k+1}} \\
&= \frac{k+1}{k} \cdot \frac{a-k+1}{a-k} \cdot \frac{r-k+1}{r-k} \cdot \frac{b-r+k+1}{b-r+k} \\
\frac{\binom{r}{k-1}\binom{r}{k+1}}{\binom{r}{k}^2} &= \frac{k}{k+1} \cdot \frac{r-k}{r-k+1}.
\end{aligned}$$

Thus, we have

$$\frac{(N_k/\binom{r}{k})^2}{(N_{k+1}/\binom{r}{k+1})(N_{k-1}/\binom{r}{k-1})} \geq \frac{a-k+1}{a-k} \cdot \frac{b-r+k+1}{b-r+k} > 1.$$

□

Using the recent methods developed in [2], we can extend Theorem 2.1 without the extra regularity condition.

Theorem 2.3. *Let M be a matroid of rank r (not necessarily regular). Then $N_k/\binom{r}{k}$ is log-concave.*

The proof of Theorem 2.3 uses the Lorentzian property of the basis generating polynomial of a matroid.

Definition 2.1. *Let $M = (E, \mathcal{B})$ be a matroid with bases $\mathcal{B}$. Then, we define the* ***basis generating polynomial*** *of M to be the polynomial in variables $x = \{x_e\}_{e \in E}$ defined by*

$$F_M(x) = \sum_{B \in \mathcal{B}} x_B$$

where for any subset $S \subseteq E$ we define the monomial

$$x_S := \prod_{s \in S} x_s.$$

Proposition 2.3. *Let $M = (E, \mathcal{B})$ be a matroid. Then the basis generating polynomial $F_M(x)$ is Lorentzian.*

Proof. Using the same notation as in [2], it suffices to prove that the support of $F_M(x)$ is M-convex and all partials are Lorentzian. The first follows from the fact that the support consists of all bases of the matroid. To prove that all partials are Lorentzian, we induct on the size of the matroid. Indeed, since all exponents are 1 we have

$$\partial_e F_M(x) = F_{M/\{e\}}(x).$$

This is Lorentzian from the inductive hypothesis since $M/\{e\}$ has one less element. □

Proof of Theorem 2.3. Set all variables indexed by elements in A by u and all variables indexed by elements in B by v. Then the resulting polynomial is

$$\sum_{k=0}^{r} N_k u^k v^{r-k}$$

and this is Lorentzian. This implies that the coefficients are ultra-log-concave. This suffices for the proof. $\square$

Remark. Stanley's method of proof would not have worked with non-regular matroids because there exist basis generating polynomials which cannot be written as the mixed volume polynomial for any set of convex bodies.

3 Equality Case

Example 3.1. *Let $A = \{e_1, \ldots, e_n\}$ and $B = \{f_1, \ldots, f_n\}$ such that the bases consist of all two element pairs except $\{e_i, f_i\}$ for all $1 \leq i \leq n$. Then $N_0 = \binom{n}{2}$, $N_1 = n(n-1)$, and $N_2 = \binom{n}{2}$. We have that*

$$\frac{N_0}{\binom{2}{0}} = \frac{N_1}{\binom{2}{1}} = \frac{N_2}{\binom{2}{2}} = \binom{n}{2}.$$

We make the following conjecture.

Conjecture 3.1. *For simple matroids, we have strict inequality in the ultra-log-concavity.*

Using the small matroid database of Gordon Royle and Dillon Mayhew (cite in a more official manner), we have verified the following result.

Theorem 3.1. *Conjecture 3.1 holds for matroids of size at most* 9*.*

We attempt to prove this conjecture for three different cases.

(i) graphic matroids

(ii) unimodular matroids

(iii) general matroids

We can prove the general version of the ultra-log concavity of the matroid counting function. Consider a partition of our matroid $E = E_1 \sqcup E_2 \sqcup \ldots \sqcup E_s$. Let $B(r_1, \ldots, r_s)$ be the number of bases of our matroid that have r_i elements in E_i.

Theorem 3.2. *Let $M = (E, \mathfrak{I})$ be a unimodular matroid of rank n, and let $r_1, \ldots, r_s$ be non-negative integers adding to n. If $r_{s-1} \geq 1$, $r_s \geq 1$, then*

$$B(r_1, \ldots, r_s)^2 \geq B(r_1, \ldots, r_{s-2}, r_{s-1}+1, r_s-1) \cdot B(r_1, \ldots, r_{s-2}, r_{s-1}-1, r_s+1)$$

Proof. Let X be the $n \times |E|$ unimodular matrix representing M. Partition X into $X = \begin{bmatrix} X_1 & \ldots & X_s \end{bmatrix}$ where X_i contains the columns corresponding to the elements in E_i. Since our matrix is unimodular, we have

$$B(r_1, \ldots, r_s) = \frac{1}{r_1! \ldots r_s!} \cdot \sum \det(x_1, \ldots, x_n)^2$$

where in the sum the first r_1 entries are columns in X_1, the next r_2 are columns in X_2, and so on. Let $A_k = X_k X_k^T$. From Proposition **??**(d), we have

$$D(A_1[r_1], \ldots, A_s[r_s]) = \frac{1}{n!} \sum \det(x_1, \ldots, x_n)^2 = \frac{B(r_1, \ldots, r_s)}{\binom{n}{r_1, \ldots, r_s}}.$$

The result now follows from the Alexandrov inequality for mixed discriminants. □

Let's consider the case $s = 2$, as we have been doing. In this case, we have a partition $E = A \sqcup B$, and $N_k = D(A_1[k], A_2[n-k])$ where $A_1 = X_1 X_1^T$ and $A_2 = X_2 X_2^T$ where X_1 is the part of the unimodular matrix corresponding to the elements in A and X_2 is the part of the unimodular matrix corresponding to the elements in B. Let $e_1, \ldots, e_n \in \mathbb{R}^a$ be the rows of X_1 and $f_1, \ldots, f_n \in \mathbb{R}^b$ be the rows of X_2. Then $A_1 = [\langle e_i, e_j \rangle]$ and $A_2 = [\langle f_i, f_j \rangle]$. In the equality case, we must have $\langle e_i, e_j \rangle = \lambda \langle f_i, f_j \rangle$.

Theorem 3.3. *Suppose that* $\frac{N_k}{\binom{r}{k}}^2 = \frac{N_{k-1}}{\binom{r}{k-1}} \frac{N_{k+1}}{\binom{r}{k+1}}$ *for some* k. *Then* $N_i = \lambda^r c \binom{r}{i}$ *for all* $0 \leq i \leq r$ *where* $\lambda \in \mathbb{Q}_{>0}$ *and* $c \in \mathbb{Z}_{>0}$.

Proof. Let $D_i = D(A_1[i], A_2[r-i]) = N_i / \binom{r}{i}$. From the equality case of the Alexandrov inequality for mixed discriminants, we have $A_1 = \lambda A_2$ for some real λ. But then it's an equality case for all things. In particular, we have

$$\frac{D_1}{D_0} = \frac{D_2}{D_1} = \ldots = \frac{D_r}{D_{r-1}} = \lambda$$

for some $\lambda \geq 0$. Thus, for any $k \geq 1$, we have

$$\frac{D_k}{D_0} = \prod_{i=1}^{k} \frac{D_i}{D_{i-1}} = \lambda^r \implies D_k = \lambda^r D_0$$

□

In particular, for equality to hold if suffices to verify that $N_0 / \binom{r}{0} = N_1 / \binom{r}{1}$ or $N_1 = r N_0$.

Theorem 3.4. *If equality holds at some index, then* $N_k = a \binom{r}{k}$ *for some* $a \geq 0$ *and* $c > 0$.

Proof. From the previous theorem, we know that

$$\lambda = \frac{N_0}{\binom{r}{0}} = \ldots = \frac{N_r}{\binom{r}{r}} \implies N_k = \binom{r}{k} \lambda$$

□

This proves that, at least for unimodular matrices, we have $N_k = \lambda \cdot c^k \binom{r}{k}$ in the equality case. We now prove the conjecture for graphic matroids.

Theorem 3.5. *(NOT COMPLETE YET) Suppose that* M *is a graphic matroid. Then Conjecture 2.1 is true.*

Proof. (small issue with proof) □

4 Current Progress

Our problem can be reframed in the following way. Given a matroid M on $[n]$, we can define its **basis generating polynomial** as

$$f_M(x_1, \ldots, x_n) := \sum_{B \in \mathsf{BASE}(M)} \left(\prod_{i \in B} x_i \right).$$

We can then define the polynomial

$$g_M(x_A, x_B) := f_M(x_A, \ldots, x_A, x_B, \ldots, x_B) = \sum_{k=0}^{r} N_k x_A^k x_B^{r-k}.$$

From [?] we have the following technical lemma

Lemma 4.1 (Lemma 4.2 in [?]). *Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial having the HRR$_1$ with respect to l_a with $a \in \mathbb{R}^n$ and let $l_1, l_2 \in S$ be linear forms. If l_1 and l_2 are $\mathbb{R}$-linearly independent in R_f^1 and $(l_1 l_2 f)(a) > 0$, then*

$$\det \begin{pmatrix} (l_1 l_1 f)(a) & (l_1 l_2 f)(a) \\ (l_1 l_2 f)(a) & (l_2 l_2 f(a) \end{pmatrix} < 0.$$

It seems like it would be enough to prove the following things:

- $g(x_A, x_B)$ has HRR$_1$ wrt ∂_A
- ∂_A and ∂_B are linearly independent in R_f^1
- $\partial_A^2 g(1, 0) > 0$.

Given a matroid M, define its truncation TM to be the matroid on the same set with bases being the independent sets of rank $\operatorname{rank}(M) - 1$.

Proposition 4.1. *Let M be a matroid. Then, we have that*

$$g_{TM}(x_A, x_B) = \sum_{k=0}^{r-1} \{(k+1)N_{k+1} + (r-k)N_k\} \, x_A^k x_B^{r-k-1} = (\partial_A + \partial_B) g_M$$

Proof. Combinatorially the formula is clear, but if that is not enough we have that

$$\begin{aligned} g_{TM}(x_A, x_B) &= f_{TM}(x_A, \ldots, x_A, x_B, \ldots, x_B) \\ f_{TM} &= (\partial_1 + \ldots + \partial_n) f_M \\ &= (\partial_1 + \ldots + \partial_n) \sum_{k=0}^{r} \sum_{U \in \mathcal{B}(M) : |U \cap A| = k} x^U \\ &= \sum_{k=0}^{r} \sum_{|U \cap A| = k} (\partial_1 + \ldots + \partial_n) x^U. \end{aligned}$$

Substituting x_A and x_B, we get

$$\begin{aligned} g_{TM}(x_A, x_B) &= \sum_{k=0}^{r} N_k(kx_A^{k-1}x_B^{r-k} + (r-k)x_A^k x_B^{r-k-1}) \\ &= \sum_{k=1}^{r} kN_k x_A^{k-1}x_B^{r-k} + \sum_{k=0}^{r-1}(r-k)N_k x_A^k x_B^{r-k-1} \\ &= \sum_{k=0}^{r-1}(k+1)N_{k+1}x_A^k x_B^{r-k-1} + \sum_{k=0}^{r-1}(r-k)N_k x_A^k x_B^{r-k-1} \\ &= \sum_{k=0}^{r-1}\left\{(k+1)N_{k+1} + (r-k)N_k\right\} x_A^k x_B^{r-k-1}. \end{aligned}$$

This seems less than desirable. $\square$

Perhaps there is another operator to make matroids smaller that preserves simplicity? Or maybe the truncation operator is good enough to imply the desired inequalities.

5 Equality cases for Lorentzian polynomials

(???)

References

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