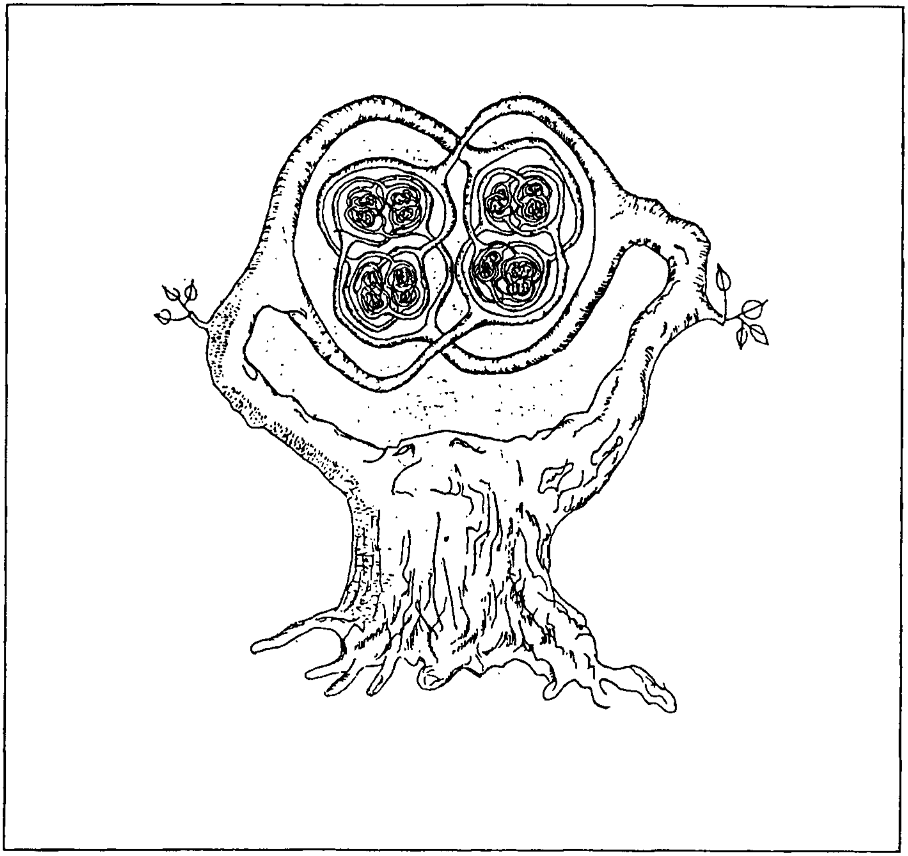

Yggdrasil. The tree of Norse mythology whose branches lead to heaven. Realized as Alexander's horned sphere in this etching by Bill Meyers.

Previous Page: This knot (7_4 in the table) is one of the eight glorious emblems of Tibetan Buddhism. Just as a knot does not exist without reference to its embedding in space, this emblem is a reminder of the interdependence of all things in the phenomenal world.

KNOTS AND LINKS

DALE ROLFSEN

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10 9 8 7 6 5 4 3 2 1 08 07 06 05 04 03

To Amy, Catherine and Gloria

Preface to the AMS Chelsea edition

This book was written as a textbook for graduate students or advanced undergraduates, or for the nonexpert who wants to learn about the mathematical theory of knots. A very basic understanding of algebraic topology is assumed (and outlined in Appendix A).

Since the first edition appeared in 1976, knot theory has been transformed from a rather specialized branch of topology to a very popular, vibrant field of mathematics. The impetus for this change was largely the work of Vaughan Jones, who discovered a new polynomial invariant of knots through his work in operator algebras. This led to astonishing connections between knot theory and physics, and such diverse disciplines as algebraic geometry, Lie theory, statistical mechanics and quantum theory.

Friends have encouraged me to revise *Knots and Links* to include an account of these exciting developments. I decided not to do this for several reasons. First of all, a number of good books on knot theory have appeared since *Knots and Links*, which cover these later developments. Secondly, the present book is already a fairly large tome, and it would be doubled in size if I were to do justice to advances in the field since publication. This didn't seem like a good idea. Finally, I believe this book remains valuable as an introduction to the exciting fields of knot theory and low dimensional topology. For similar reasons, the ''forthcoming entirely new book'' mentioned in the preface to the second printing will very likely never materialize.

Knots and Links would not have existed in the first place, had it not been for Mike Spivak, owner and founder of *Publish or Perish Press*. Spivak has been a faithful friend and constant source of encouragement. It was he who suggested I use my class notes as the basis for a book -- this book was the result. It was also his suggestion that, when the last printing of the book by *Publish or Perish* ran out recently, I seek another publisher. I am extremely pleased that, with the new AMS Chelsea Classics edition, this book will remain available in a high-quality format and at a reasonable price. I'd also like to thank Edward Dunne and Sergei Gelfand and the staff at AMS Books for facilitating this edition.

Finally, I would like to extend my gratitude to a number of friends and colleagues who have pointed out errors in the previous editions. Nathan Dunfield verified all the Alexander polynomials of the knots and links in the tables, and found exactly four errors, which are corrected in this edition (9^2_{29}, 9^2_{55}, 9^2_{57} and 9^2_{59}). Other corrections for this new edition are a matrix entry on page 220, correction of Lemma 8E18, p. 222, and an exercise at the bottom of page 353. These last two were pointed out by Steve Boyer. Thanks also to Jim Bailey, Steve Bleiler, Jim Hoste, Peter Landweber, Olivier Collin and others whose help I may have forgotten. No doubt there are still errors, which I would be glad to hear about. Any future corrections will be posted on the AMS Books website. The url is given on the copyright page.

Dale Rolfsen

(rolfsen@math.ubc.ca)

Vancouver, June, 2003

PREFACE TO THE SECOND PRINTING

This new printing is essentially the same as the original edition, except that I have corrected the errors that I know about. Several colleagues and students have been very helpful in pointing out these errors, and I wish to thank them for their help. Special thanks to Professors Jim Hoste and Peter Landweber for finding lots of them and sending me detailed lists. One of the most embarrassing errors is the duplication in the knot table: 10_{161} and 10_{162} are really the same knot, as K. Perko has pointed out. Also, in the table, the drawing of 10_{144} was wrong.

I didn't make any attempt to update the book with new material. I took the advice of a kind friend who told me not to tamper with a "classic." A lot has happened in knot theory in the decade and a half since this book was written. I will do my best to report that in a forthcoming entirely new book. However, some notable developments really ought to be mentioned. The old conjecture that knots are determined by their complements was recently solved in the affirmative by C. Gordon and J. Leucke. Likewise, we now know the Smith Conjecture to be true, although the Poincaré and Property P conjectures still stand. Equally exciting is the "polynomial fever" rampant for the past five years, inspired by V. Jones' discovery of a new polynomial so powerful that it could distinguish the two trefoils. This breakthrough led to the discovery of plenty of new polynomials, giving us a large new collection of very sharp tools and adding fundamentally to our understanding of that wonder of our natural world: knots.

The best thing that has happened to knot theory, however, is that many more scientists are now interested in it -- not just topologists -- and contributing in their unique ways. Jones led the way by introducing operator algebras and representation theory to the subject. Since then deep contributions have been made by algebraic and differential geometry, and by mathematical physics. Surprising connections with statistical mechanics and quantum field theory are just now being explored and promise to make the end of the 20th century a real golden age for knot theory. Knot theory is not only utilizing ideas from other disciplines, but is beginning to return the favor. Besides stimulating new directions of research in mathematics and physics, ideas of knot theory are being used effectively in such fields as stereochemistry and molecular biology. So knot theory can begin to call itself applied mathematics!

Since the appearance of "Knots and Links," several excellent books on the subject of mathematical knot theory have appeared. Most notable are "Knots," by Burde and Zieschang, and "On knots," by Kauffman. These are highly recommended. Each has an emphasis different from the present work, and the three can be regarded as mutually complementary.

Finally, my sincere thanks go to my publisher, Mike Spivak, for agreeing to put out this new printing, for his patience in making the corrections, and for his realization that it was hopeless to expect my promised new book on knots in the very near future.

Dale Rolfsen

Vancouver, Canada

February 9, 1990

PREFACE

This book began as a course of lectures that I gave at the University of British Columbia in 1973-74. It was a graduate course officially called "Topics in geometric topology." That would probably be a more accurate title for this book than the one it has. My bias in writing it has been to treat knots and links, not as the subject of a theory unto itself, but rather as (1) a source of examples through which various techniques of topology and algebra can be illustrated and (2) a point of view which has real and interesting applications to other branches of topology. Accordingly, this book consists mainly of examples.

The students in that course were graduate level and all had some background in point-set topology and a little algebraic topology. But I think an intelligent undergraduate mathematics student, who is willing to learn algebraic topology as he goes along, should be able to handle the ideas here. As part of my course, the students lectured to each other from Rourke and Sanderson's book [1972] on piecewise-linear topology. So I've used some PL techniques without much explanation, but not to excess.

If you scan through the pages you'll find that there are lots of exercises. Some are routine and some are difficult. My philosophy in teaching the course was to have the students prove things for themselves as much as possible, so the exercises are central to the ideas developed in these notes. Do as many as you can.

I would like to express my thanks to the people who helped me to prepare this manuscript during rather nomadic times for me. They are: Cathy Agnew (Vancouver), Yit-Sin Choo (Vancouver), Cynthia Coddington (Heriot Bay, B. C.), Joanne Congo (Vancouver), Sandra Flint (Cambridge), Judy Gilbertson (Laramie, Wyoming), Carol Samson (Vancouver) and Maria del Carmen Sanchez del Valle (Mexico City). Special thanks are due to Jim Bailey, who took notes in the course on which this book is based, compiled the table which forms appendix C, and helped in many other ways. Also to his friend Ali Roth who drew the knots and links so beautifully. David Gillman gave an excellent series of three lectures on Dehn's Lemma, and I'm grateful to him for writing up the notes for inclusion here as appendix B.

Many friends and mathematicians have given me encouragement and advice, both mathematical and psychological. Among them are Andrew Casson, Francisco Gonzalez-Acuna, Cameron Gordon, Cherry Kearton, Robion Kirby, Raymond Lickorish and Joe Martin, whose own lecture notes were very helpful to me. Finally I want to thank Mary-Ellen Rudin for her advice, which I should have followed sooner: "Don't try to get everything in that book."

TABLE OF CONTENTS

CHAPTER TEN. FOLIATIONS, BRANCHED COVERS, FIBRATIONS AND SO ON

CHAPTER ELEVEN. A HIGHER-DIMENSIONAL SAMPLER

CHAPTER 1. INTRODUCTION

The possibility of a mathematical study of knots was probably recognized first by C. F. Gauss. His investigations of electrodynamics [1833]* included an analytic formulation of linking number, a tool basic to knot theory and other branches of topology.

It appears that the first attempts at classification of knot types were made by a British group some 50 years later. The work of Tait, Kirkman and Little resulted in tabulations of knots of up to "tenfold knottiness" in very beautiful diagrams. Their early methods were combinatorial and somewhat empirical. The unfolding of the subleties of knotting and linking had to await the development of topology and algebraic topology, pioneered by H. Poincaré around the turn of the century. In turn, knot theory provided considerable impetus for developing many important ideas in algebraic topology, group theory and other fields.

Names associated with the development of 'classical' knot theory in the first half of the present century include M. Dehn, J. W. Alexander, W. Burau, O. Schreier, E. Artin, K. Reidemeister, E. R. Van Kampen, H. Seifert, J. H. C. Whitehead, H. Tietze, R. H. Fox. Typifying the general development of the theory and its techniques, the concept of knot as a polygonal curve in 3-space upon which certain 'moves' were permitted became modernised to knot as an equivalence class of embeddings of S^1 in R^3, or even S^n in S^m. Recently, knot theory has attracted renewed interest because of great progress in higher dimensional knot theory, as well as new applications (e.g. to surgery and singularity theory).

* Bracketed dates refer to the bibliography at the end of the book.

A. NOTATION AND DEFINITIONS. Our stock-in-trade will be the following spaces :

$R^n = \{x = (x_1, \ldots, x_n)\}$ = the <u>Euclidean space</u> of real n-tuples with the usual norm $|x| = (\sum x_i^2)^{\frac{1}{2}}$ and metric $d(x,y) = |x - y|$.

D^n = the unit n-<u>ball</u> (or n-<u>disk</u>) of R^n defined by $|x| \leq 1$.

$S^{n-1} = \partial D^n$, the unit (n-1)-<u>sphere</u> $|x| = 1$.

$I = [0, 1]$ the unit interval of $R = R^1$.

The 'natural' inclusions $R^n \subset R^m$, as $\{(x_1, \ldots, x_n, 0, \ldots, 0)\}$, $n<m$, define natural inclusions $D^n \subset D^m$ and $S^{n-1} \subset S^{m-1}$. D^n is sometimes denoted B^n and the letters Q and Z will (usually) stand for the field of rational numbers and the ring of integers. "Map" means continuous function or homomorphism, and "$\cong$" means homeomorphism or isomorphism.*

1. DEFINITION. A subset K of a space X is a <u>knot</u> if K is homeomorphic with a sphere S^p . More generally K is a <u>link</u> if K is homeomorphic with a disjoint union $S^{p_1} \cup \ldots \cup S^{p_r}$ of one or more spheres. Two knots or links K, K' are <u>equivalent</u> if there is a homeomorphism $h : X \longrightarrow X$ such that $h(K) = K'$; in other words $(X, K) \cong (X, K')$.

In the case of links of two or more components we also assign a fixed ordering of the components and require that h respect the orderings. The equivalence class of a knot or link is called its <u>knot type</u> or <u>link type</u>. Unless otherwise stated, we shall always take X to be R^n or S^n .

* depending, of course, on the context.

2. REMARKS. In reading the literature, you should be aware that these definitions are not universally accepted. Some authors consider knots as embeddings $K : S^p \longrightarrow S^n$ rather than subsets. We shall also find this convenient at times and will use the same symbol to denote either the map K or its image $K(S^p)$ in S^n. There are also other (stronger) notions of equivalence which appear in the literature.

map equivalence : K, K' considered as maps and require $h \circ K =$

oriented equivalence : All spaces are endowed with orientations, all of which h is required to preserve,

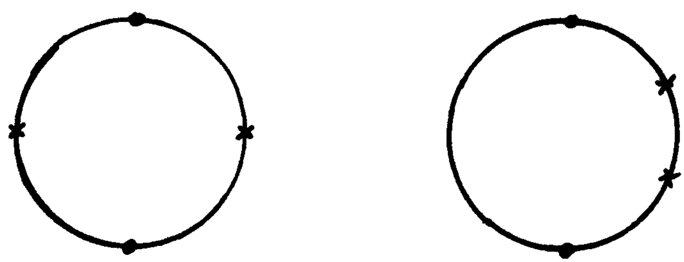
ambient isotopy : require that h be the end map h_1 of an ambient isotopy.

3. DEFINITION. A homotopy $h_t : X \longrightarrow X$ is called an *ambient isotopy* if h_0 = identity and each h_t is a homeomorphism.

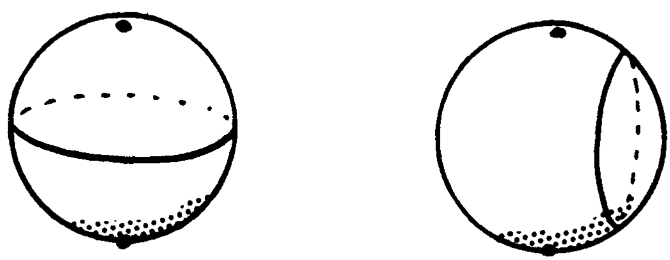
4. REMARK : One may also require everything be piecewise-linear (PL) or C^∞. Our point of view will be to work in the topological category whenever possible. But to avoid pathological complications we will often restrict attention to the PL category. The C^∞ category will be largely ignored, but the reader is advised that C^∞ knot theory is different from PL in several important respects. For example Haefliger [1962] describes an S^3 in S^6 which is topologically and PL, but not differentially, equivalent to the standard one.

B. SOME EXAMPLES OF LINKING. Everyone is familiar with plenty of non-equivalent knots for $p = 1$, $n = 3$, but to prove they are so takes some work. It is easier to detect linking, so we begin with some examples, for a taste of what is to come, which can be handled without special machinery. We need only the well-known theorem that the identity map $id : S^n \longrightarrow S^n$ is not homotopic to a constant map.

1. EXAMPLE : Two inequivalent links of $S^o \cup S^o$ in S^1.

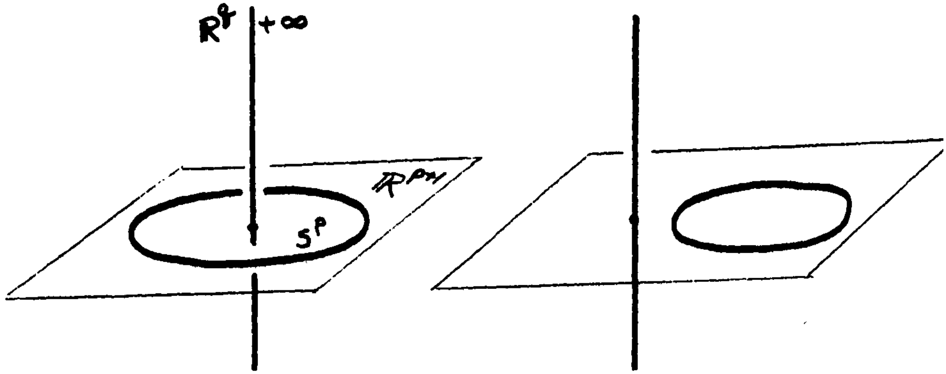


2. EXAMPLE : $S^o \cup S^1$ in S^2



3. EXAMPLE : $S^1 \cup S^1$ in R^3 or $S^3 = R^3 + \infty$

4. EXAMPLE : $S^p \cup S^q$ in S^{p+q+1}



This is a generalization of the first three examples. Consider S^{p+q+1} as $(R^{p+1} \times R^q) + \infty$. For the first link, take S^p to be the unit sphere of $R^{p+1} \times 0$ and S^q to be $(0 \times R^q) + \infty$. For the second link take the same S^q, but let the S^p be a sphere in $R^{p+1} \times 0$ which does not enclose the origin.

Note that in the second link, the S^p can be shrunk to a point in $S^{p+q+1} - S^q$. We show that this is not the case for the first link, hence the links are inequivalent. To see this note that
$S^{p+q+1} - S^q = (R^{p+1} - \{0\}) \times R^q$ and define a retraction

$$r : (R^{p+1} - \{0\}) \times R^q \longrightarrow S^p$$

by the formula $r(x, y) = (\frac{x}{|x|}, 0)$. If there were a homotopy $h_t : S^p \longrightarrow S^{p+q+1} - S^q$ from the inclusion to a constant map, then $r \circ h_t : S^p \longrightarrow S^p$ would be a homotopy between id_{S^p} and a constant map, which is impossible.

The next example is a refinement of this, due to Zeeman [1960]. It is best described using the notion of the <u>join</u> of two spaces.

5. DEFINITION : If X, Y are topological spaces, then their <u>join</u> is the factor space $X * Y = (X \times Y \times I)/\sim$, where $\sim$ is the equivalence relation :

$$(x,y,t) \sim (x',y',t') \iff \begin{cases} t = t' = 0 \text{ and } x = x' \\ \text{or} \\ t = t' = 1 \text{ and } y = y' . \end{cases}$$

Note that X and Y are naturally included in $X * Y$ as the ends $t = 0$ and $t = 1$. As special examples, if Y is a point, we have the <u>cone</u> $C(X) = X * \{pt\}$. Taking Y to be two points $(= S^0)$ we have the <u>suspension</u> $\Sigma(X) = X * S^0$.

6. EXERCISE : Show that $S^p * S^q \cong S^{p+q+1}$, and that example 4 (left hand side) is equivalent to the link of the natural S^p and S^q in the join.

7. EXAMPLE : Suppose $p \geq r \geq q$ are integers such that $\pi_r(S^q) \neq 0$ (that is, some map $f : S^r \to S^q$ is not homotopic to a constant). Then there are nonequivalent links of $S^p \cup S^r$ in S^{p+q+1}. We construct one link such that the r-sphere cannot be shrunk to a point in S^{p+q+1} missing the p-sphere as follows :

Let $i : S^r \to S^p$ be the natural inclusion and define an embedding $e : S^r \to S^{p+q+1} = S^p * S^q$ by $e(x) = (i(x), f(x), \frac{1}{2})$. The link of example 7 is the union of the embedding e and the natural embedding $S^p \to S^p * S^q$.

8. EXERCISE : Prove that e is not homotopic to a constant map in $(S^p * S^q) - S^p$. Complete the example by constructing another link of $S^p \cup S^r$ in S^{p+q+1} for which each component shrinks to a point missing the other (why is this inequivalent to the link described above?).

9. REMARKS : Example 7 can be combined with some esoteric results of homotopy theory to give a good supply of "nontrivial" links. For instance we can link two S^{10}'s nontrivially in S^{21}, S^{20}, S^{19}, S^{18}, S^{16}, S^{15}, S^{14}, and S^{13}. This corresponds to the computations $\pi_{10}(S^{10}) \cong Z$, $\pi_{10}(S^9) \cong Z_2$, $\pi_{10}(S^8) \cong Z_2$, $\pi_{10}(S^7) \cong Z_{24}$, $\pi_{10}(S^5) \cong Z_2$, $\pi_{10}(S^4) \cong Z_{24} \oplus Z_3$, $\pi_{10}(S^3) \cong Z_{15}$, $\pi_{10}(S^2) \cong Z_{15}$.* The gap S^{17} corresponds to $\pi_{10}(S^6) \cong 0$. In fact all (piecewise-linear) links of $S^{10} \cup S^{10}$ in S^{17} are trivial. This follows from a theorem of Zeeman [1961] : If $n \geq 2$ and $c \geq \frac{n}{2} + 2$, then the construction described above provides a one-to-one correspondence between the set $\pi_n(S^{c-1})$ and the set of piecewise-linear equivalence classes of links of $S^n \cup S^n$ in S^{n+c}. Although $\pi_{10}(S^1) \cong 0$ we will see later that two S^{10}'s can be linked in S^{12} in many ways. More generally, two S^p's can be linked in S^{p+2} in infinitely many inequivalent ways.

* Z_n or Z/n denotes the cyclic group of order n. The calculations of homotopy groups are from a table in Toda [1962].

CHAPTER TWO. CODIMENSION ONE AND OTHER MATTERS

As just mentioned, there is a nontrivial theory of linking of two p-spheres ($p>o$) in S^{p+2}, in S^{2p+1}, as well as in many dimensions in between. A fundamental discovery of higher-dimensional knot theory is that, by contrast, there is only one type of knot of S^p in S^q whenever the codimension $q - p$ is greater than two.* The main message of this chapter is that knot theory, as well as link theory, is also more-or-less† trivial in codimension one. Nevertheless, it is a good setting for introducing some useful geometric techniques.

On the other hand, the theory of knots of S^1 in the torus $T^2 = S^1 \times S^1$ is nontrivial. We'll investigate this rather thoroughly for two reasons: (1) knots in T^2 can be completely classified and (2) it is basic to the more interesting theory of surgery in 3-manifolds.

* This is true in the PL category and the topological category for non-pathological embeddings, although false in the differentiable case, as previously noted. Proofs involve considerable machinery from PL topology or engulfing theory. A good treatment may be found in Rourke and Sanderson's book on PL topology [1972].

† The unknown cases correspond to the well-known PL Schönflies problem and the one unsolved case (in S^4) of the topological annulus conjecture.

NOTE TO THE READER : If you are impatient to get on with the traditional theory of knots, it will do you no harm to skip this chapter and refer back to it as the need arises.

A. KNOTS IN THE PLANE.

Two of the early triumphs of topology concern simple closed curves, or knots, in the plane.

1. JORDAN CURVE THEOREM : If J is a simple closed curve in R^2, then $R^2 - J$ has two components, and J is the boundary of each.

2. SCHÖNFLIES THEOREM : With the same hypotheses, the closure of one of the components of $R^2 - J$ is homeomorphic with the unit disk D^2 .

Proofs may be found in any of several standard topology texts. The Schönflies theorem may even be derived from the Riemann mapping theorem of complex analysis [see e.g. Hille, vol. II, Theorems 17.1.1 and 17.5.3]. The Jordan theorem is customarily proved either by an elementary but lengthy point-set theoretical argument, or else as an easy application of homology theory, once the machinery has been established. We will outline later a short proof (that J separates R^2) which is a sort of compromise between these two approaches, using properties of the fundamental group which are readily derived from "first principles".

3. EXERCISE : From the Jordan curve theorem, derive the analogous theorem with S^2 replacing R^2 . Show that the Schönflies theorem implies that the closures of both components of $S^2 - J$ are 2-disks.

4. COROLLARY. Any two knots of S^1 in S^2 (or R^2) are equivalent.

PROOF : Let J_1, J_2 be knots in S^2 and let U_i, V_i denote the components of $S^2 - J_i$ $(i = 1, 2)$. Let $h : J_1 \longrightarrow J_2$ be a homeomorphism. The following lemma permits h to extend to a homeomorphism $h_u : \bar{U}_1 \longrightarrow \bar{U}_2$ and also to $h_v : \bar{V}_1 \longrightarrow \bar{V}_2$. These together give a homeomorphism $S^2 \longrightarrow S^2$ carrying J_1 to J_2.

Proving the result also for R^2 is left to the reader.

5. LEMMA (Alexander) : If $A \cong B \cong D^n$, then any homeomorphism $h : \partial A \longrightarrow \partial B$ extends to a homeomorphism $\bar{h} : A \longrightarrow B$.

PROOF : Without loss of generality, we may assume $A = B = D^n$ (why?). Then in vector notation, if $x \in \partial D^n$, define $\bar{h}(tx) = th(x)$, $0 \leq t \leq 1$.

6. EXERCISE : Let A and B be arcs in R^2 with common endpoints and disjoint interiors. Show that there is an ambient isotopy of R^2, fixed on the endpoints, taking A to B. Moreover, the isotopy may be taken to be fixed on any neighbourhood of the closure of the region bounded by $A \cup B$. What if the interiors of A and B intersect?

Consider now two disjoint simple closed curves J, K in R^2. We may say that J is <u>inside</u> K if it lies in the bounded component U of $R^2 - K$. Clearly this is true if and only if K lies in the unbounded component V of $R^2 - J$. In this case, call $U \cap V$ the <u>region between</u> J and K : it is one of the three components of $R^2 - (J \cup K)$. Similar considerations hold in S^2, except that "inside" cannot be defined without reference to a point "∞" in S^2, suitably chosen.

7. ANNULUS THEOREM: The closure of the region between two disjoint simple closed curves in S^2 (or in R^2 assuming that one is inside the other) is homeomorphic with the annulus $S^1 \times [0,1]$.

7.′ EXERCISE : Prove the annulus theorem, assuming the Schönflies theorem. [Hint: connect the curves with arcs.]

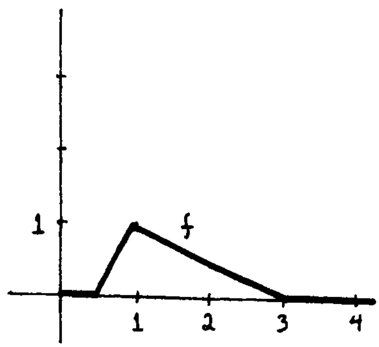
8. LEMMA : Any homeomorphism $h : S^1 \times 0 \longrightarrow S^1 \times 0$ extends to a homeomorphism $\bar{h} : S^1 \times [0,1] \longrightarrow S^1 \times [0,1]$. (PROOF : trivial)

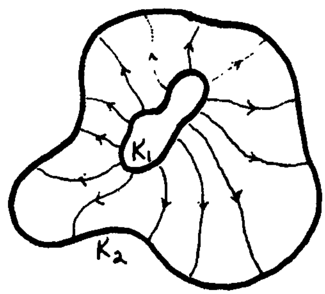
9. COROLLARY : Any two links of $S^1 \cup S^1$ in S^2 are equivalent. In R^2 there are two link types of $S^1 \cup S^1$ according as the first component is inside or outside the second.

PROOF : Let L_1, L_2 be given links in S^2 . Then $S^2 - L_i$ has three components U_i, V_i, W_i where $\overline{U}_i$, $\overline{W}_i$ are disks and $\overline{V}_i$ an annulus. Notation is chosen so that the first component $L_i^{(1)}$ of L_i is $\overline{U}_i \cap \overline{V}_i$. Then a homeomorphism $h : L_1^{(1)} \longrightarrow L_2^{(1)}$ extends to a homeomorphism $\overline{U}_1 \longrightarrow \overline{U}_2$ by lemma 5 , then to a homeomorphism $\overline{U}_1 \cup \overline{V}_1 \longrightarrow \overline{U}_2 \cup \overline{V}_2$ by lemma 8 , and finally to a homeomorphism $S^2 = \overline{U}_1 \cup \overline{V}_1 \cup \overline{W}_1 \longrightarrow \overline{U}_2 \cup \overline{V}_2 \cup \overline{W}_2 = S^2$ taking L_1 to L_2 and respecting the ordering of the components. The R^2 case is left to the reader.

10. COROLLARY : Any two knots in S^2 or R^2 are ambient isotopic.

PROOF : Given two knots, it is a simple matter to find a third knot disjoint from both of them, so it suffices to consider disjoint knots K_1, K_2 in (say) R^2. By Corollary 9, there is a homeomorphism $h : R^2 \longrightarrow R^2$ throwing $K_1 \cup K_2$ onto the circles of radius 1 and 2 centered at the origin.





Let $f : R^1_+ \longrightarrow R^1_+$ be the piecewise-linear function whose graph is sketched here. Define $g_t : R^2 \longrightarrow R^2$ by

$$g_t(x) = (1 + tf(|x|))x .$$

One easily checks that this is an ambient isotopy and g_1 takes the circle of radius 1 to the circle of radius 2. Then $h^{-1}g_t h$ is an ambient isotopy which, when $t = 1$, takes K_1 to K_2. The S^2 case follows from this since the isotopy is fixed near ∞.

11. REMARK : The proof is easily modified to show that we may require the ambient isotopy to be fixed outside any preassigned neighbourhood of the closed annular region between K_1 and K_2 (assuming K_1, K_2 are disjoint).

12. EXERCISE : Sharpen Corollary 9 to replace 'equivalent' by 'ambient isotopic'.

B. THE JORDAN CURVE THEOREM AND THE CHORD THEOREM.

1. EXERCISE : Show that the Jordan curve theorem may be deduced from the following theorem and vice versa.

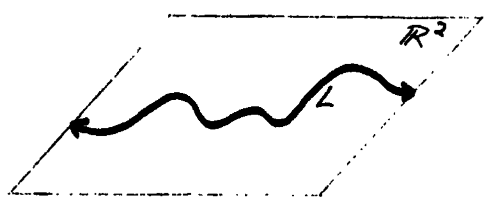
2. THEOREM : If L is a closed subset of R^2 which is homeomorphic with R^1, then $R^2 - L$ has two components and L is the boundary of each.

In the following discussion assume the hypothesis of this theorem; we will prove that $R^2 - L$ is not connected. (The argument is due, I think, to Doyle.) Consider $R^2 \subset R^3$ as the xy-plane of xyz-space.

3. LEMMA : There is a homeomorphism $h : R^3 \longrightarrow R^3$ such that $h(L)$ is the z-axis.

PROOF : Let $f : L \longrightarrow R^1$ be a homeomorphism, and extend to a map $\bar{f} : R^2 \longrightarrow R^1$ by Tietze's extension theorem. Define $g : R^3 \longrightarrow R^3$ by the formula

$$g(x,y,z) = (x,y,z+\bar{f}(x,y)) .$$



Then g is a homeomorphism which sends L to a set intersecting each horizontal plane in a single point. In fact $g(L) \cap \{z = t\}$ has

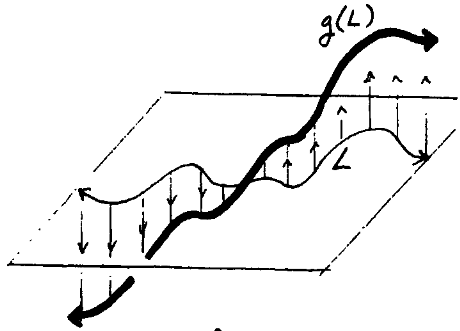
(x,y) coordinates given by $f^{-1}(t)$; we denote

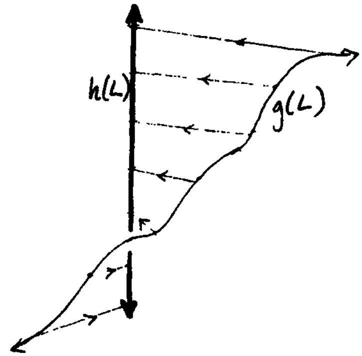
$$f^{-1}(t) = (x(t),\ y(t)).$$

Define $k : R^3 \longrightarrow R^3$ by

$$k(x,y,z) = (x-x(z),\ y-y(z),\ z)\ .$$

Then k is a homeomorphism taking $g(L)$ to the z-axis, so the composite $h = k \circ g$ satisfies the conditions of the lemma.





4. EXERCISE : Show that h is the end of an ambient isotopy of R^3 .

To proceed with the proof that L separates R^2, we need the following special case of Van Kampen's theorem. The interested reader should try to prove it 'from scratch'.

5. SUM THEOREM : Suppose a space X is the union of two simply-connected open sets whose intersection is path-connected. Then X is simply-connected.

Now, <u>suppose</u> that $R^2 - L$ is connected, hence path-connected. Write $R^3 - L$ as $(R^3_+ - L) \cup (R^3_- - L)$, where R^3_+, R^3_- are the half-spaces $z \geq 0$, $z \leq 0$. It is clear that $R^3_+ - L$ and $R^3_- - L$ are simply-connected (why?). The sum theorem then implies that $R^3 - L$ is simply-connected. Since we've shown above that $R^3 - L$ is homeomorphic with R^3 - z-axis, the following easy exercise provides a contradiction which establishes that $R^2 - L$ is not connected, our desired conclusion.

6. EXERCISE : R^3 - z-axis is not simply-connected.

7. EXERCISE : To simplify the discussion we cheated; $R^3_+ - L$ and $R^3_- - L$ are not open in $R^3 - L$, so the sum theorem doesn't apply as stated. Show how to remedy this.

8. EXERCISE : Prove the remaining parts of the Jordan curve theorem.

9. EXERCISE : Any arc in R^3 which lies in a plane is ambient isotopic to a straight line segment.

10. EXERCISE : Any knot of S^1 in R^3 which lies in a plane is unknotted (equivalent to the standard S^1) by an ambient isotopy.

11. EXERCISE : If an embedding $e: S^1 \to R^3$ has only one relative maximum and minimum in, say, the z-direction, then $e(S^1)$ is unknotted.

We close this section with a curious theorem of plane topology. It will find application in the next section.

12. CHORD THEOREM : Suppose X is a path-connected subset of the plane and C is a chord (straight-line segment) with endpoints in X, having length $|C|$. Then for each positive integer n there exists a chord parallel to C, with endpoints in X, and having length $\frac{1}{n}|C|$.

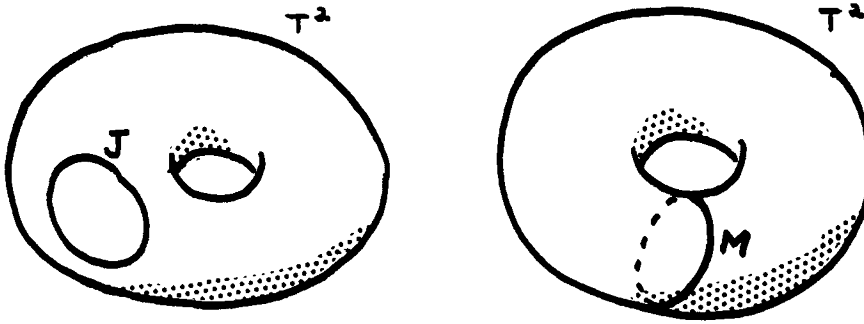
13. EXERCISE : Show that this follows from the more general theorem stated below.

14. EXERCISE : Show that there is always a counterexample to the 'chord theorem' if n is not an integer. [In attempting to draw a counterexample, try holding two pencils at once.]

15. THEOREM : Suppose X is a path-connected subset of R^2 and C is a chord with endpoints in X. Suppose $0 < \alpha < 1$. Then among all chords with endpoints in X and parallel to C, there is either one of length $\alpha|C|$ or one of length $(1 - \alpha)|C|$.

PROOF : Without loss of generality (since path-connectedness implies arc-connectedness in Hausdorff spaces) we may assume X is an arc. We may also suppose that C is the unit interval on the x-axis. For any real number β, let X_β denote $\{(x+\beta, y) \mid (x, y) \in X\}$. The problem then boils down to showing that X cannot be disjoint from both X_α and $X_{1-\alpha}$. Suppose it were. Then $X \cap X_\alpha = X_\alpha \cap X_1 = \emptyset$. Choose points $p \in X_\alpha$ and $q \in X_\alpha$ which have, respectively, maximum and minimum y-coordinates. Let L^+ be the vertical half-line extending upward from p and L^- be the vertical half-line extending downward from q. Let L be the union of L^+, L^-, and the subarc of X_α which connects p and q. Clearly L is a closed subset of R^2 and is homeomorphic with R^1, hence it separates R^2. Moreover one verifies easily (EXERCISE) that X and X_1 must be in different components of $R^2 - L$. But this is absurd, since X and X_1 intersect at the point $(1,0)$, and the theorem is established.

C. KNOTS IN THE TORUS.



These pictures show that there are at least two knot types in $T^2 = S^1 \times S^1$. Since J separates T^2 while M does not, no self-homeomorphism of T^2 can throw J onto M . We will see that these are the only two knot types of S^1 in T^2, up to homeomorphism. Other questions which we consider are :

(a) Which homotopy classes of maps $S^1 \longrightarrow T^2$ are represented by embeddings (i.e. knots) ?

(b) What are the knot types in T^2 up to <u>ambient isotopy</u> ?

(c) What are the self-homeomorphisms of T^2, up to ambient isotopy ?

Since the fundamental group of S^1 is infinite cyclic, the torus has group $\pi_1(T^2) \cong Z \oplus Z$. In fact the π_1 functor carries the canonical diagram of inclusions and projections of $S^1 \times S^1$ to the inclusions and projections of $Z \oplus Z$

$$\begin{tikzcd}[column sep=small] & S^1 \times S^1 \arrow[dl, shift left] \arrow[dr, shift left] & \\ S^1 \times * \arrow[ur, shift left] & & * \times S^1 \arrow[ul, shift left] \end{tikzcd} \quad \overset{\pi_1}{\rightsquigarrow} \quad \begin{tikzcd}[column sep=small] & Z \oplus Z \arrow[dl, shift left] \arrow[dr, shift left] & \\ Z \oplus 0 \arrow[ur, shift left] & & 0 \oplus Z \arrow[ul, shift left] \end{tikzcd}$$

Considering S^1 as the unit complex numbers, any point of T^2 has coordinates $(e^{i\theta}, e^{i\phi})$. We have two standard generators of $\pi_1(T^2)$: the maps

$$e^{i\theta} \longrightarrow (e^{i\theta}, 1) \qquad \text{longitude}$$

$$e^{i\theta} \longrightarrow (1, e^{i\theta}) \qquad \text{meridian}$$

$0 \leq \theta \leq 2\pi$. Any map $f : S^1 \longrightarrow T^2$ may be regarded, once the S^1 is oriented, as a loop representing an element $[f]$ of $\pi_1(T^2)$. Base points are immaterial, since $\pi_1(T^2)$ is abelian -- in fact we could as well use integral homology $H_1(T^2) \cong \pi_1(T^2)$. Then $[f]$ may be written in terms of the longitude-meridian basis $[f] = \langle a, b\rangle$. Thus the longitude has class $\langle 1, 0\rangle$ and the meridian $\langle 0, 1\rangle$.

We can exploit our knowledge of plane topology by considering covering spaces of T^2 . Identify R^2 with the complex numbers C (which may be denoted $z = x + iy$ or $z = re^{i\theta}$) and define maps

$$\begin{tikzcd} C \arrow[rr, "\exp"] \arrow[dr, "p"'] & & C - \{0\} \arrow[dl, "q"] \\ & T^2 & \end{tikzcd}$$

where $$p(x + iy) = (e^{ix}, e^{iy})$$

$$q(re^{i\theta}) = (e^{i \ln r}, e^{i\theta})$$

and exp is the familiar complex exponential function

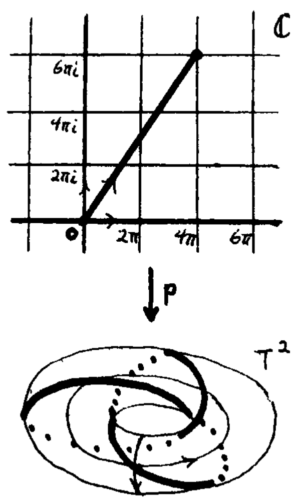
$$\exp(x + iy) = e^x(\cos y + i \sin y) .$$

1. EXERCISE : Verify that the diagram above commutes and that all three maps are covering maps. The universal cover of T^2 is just $p : C \longrightarrow T^2$, and its covering translations are of the form $z \longrightarrow z + 2\pi(m + in)$, $m, n \in Z$, (verifying that $\mathrm{Aut}(C,p) \cong Z \oplus Z$). A loop $\omega : [0,1] \longrightarrow T^2$ is of class $\langle a, b\rangle$ in T^2 iff it lifts to a path $\tilde{\omega}$ in C satisfying $\tilde{\omega}(1) - \tilde{\omega}(0) = 2\pi(a + ib)$. The covering $q : C - \{0\} \longrightarrow T^2$ is the one corresponding to the meridinal* subgroup $0 \oplus Z \subset Z \oplus Z$. Thus loops of class $\langle a, b\rangle$ in T^2 lift to loops in $C - \{0\}$ exactly when $a = 0$.

2. THEOREM : A class $\langle a, b\rangle$ in $\pi_1(T^2)$ is represented by an embedding $S^1 \longrightarrow T^2$ if and only if either $a = b = 0$ or $\text{g.c.d.}(a,b) = 1$.

PROOF : A knot of class $\langle 0, 0\rangle$ is pictured above (J) . To construct a knot of class $\langle a, b\rangle$, consider the map $S^1 \longrightarrow S^1 \times S^1 = T^2$ given by

$$z \longrightarrow (z^a, z^b), \quad z = e^{i\theta} \in S^1 .$$



This is an embedding if (and only if) a, b are relatively prime. [Note that it is just the image, under p, of the (directed) straight line in C from 0 to $2\pi(a + bi)$.]

* the O. E. D. says 'meridional' is the correct spelling.

Now to see that the condition that a and b are coprime is necessary, consider a loop $\omega : [0,1] \longrightarrow T^2$ of class $\langle a, b\rangle \neq \langle 0, 0\rangle$, where g.c.d.$(a,b) = d > 1$. If $\tilde{\omega} : [0,1] \longrightarrow C$ is any lifting (i.e. $\omega = p \circ \tilde{\omega}$) then $\tilde{\omega}(1) - \tilde{\omega}(0) = 2\pi(a + bi)$, the Chord Theorem B12 implies the existence of $s, t \in [0,1]$ such that $\tilde{\omega}(s) - \tilde{\omega}(t) = 2\pi(\frac{a}{d} + \frac{b}{d}i)$. Since $\frac{a}{d}$ and $\frac{b}{d}$ are integers, $\omega(s) = \omega(t)$ and the image of ω cannot be a simple closed curve. This completes the proof.

We now use the other covering space $q : C - \{0\} \longrightarrow T^2$ to study ambient isotopies. Say that a set $X \subset C - \{0\}$ <u>lies in a fundamental region</u> if it has a neighbourhood U, $X \subset U \subset C - \{0\}$, such that $q|U$ is a homeomorphism. Then, for instance, if the support $\operatorname{supp} h = \{z : h(z) \neq z\}$ of a homeomorphism h of $C - \{0\}$ lies in a fundamental region, it induces a homeomorphism h' of T^2 by the formula

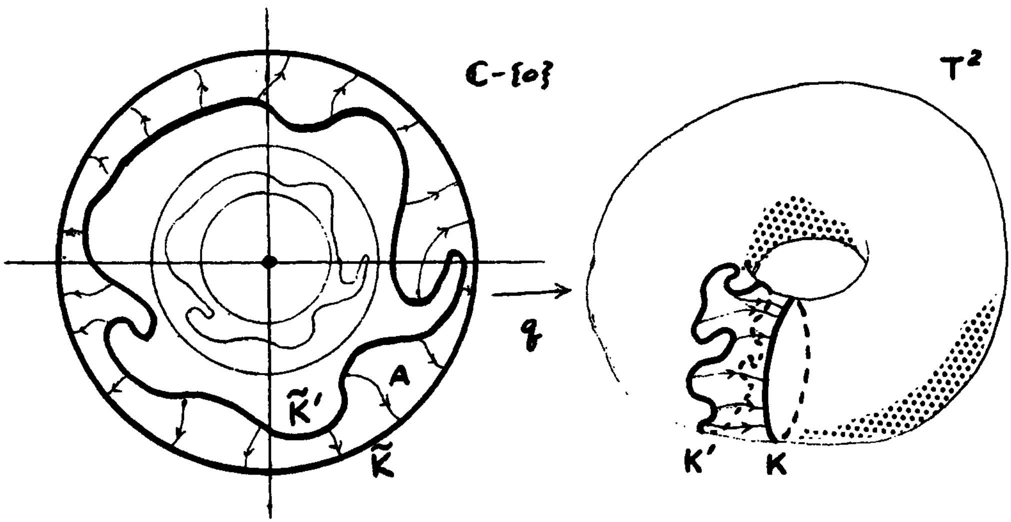
$$h' = \begin{cases} qhq^{-1} & \text{on } q(\operatorname{supp} h) \\ \text{id} & \text{elsewhere .} \end{cases}$$

Our aim is to show that knots which are homotopic are actually ambient isotopic. We shall establish this for the special case of meridinal curves by a series of lemmas, then show that this implies the general case.

3. LEMMA : If K and K' are disjoint knots of class $\langle 0, 1\rangle$, then there is an ambient isotopy of T^2 taking K to K'.

PROOF : There are liftings $\tilde{K}$, $\tilde{K}'$ of K and K' (via q) which cobound an annulus $A \subset C^2 - \{0\}$ such that int A contains no other liftings.

Furthermore, A lies in a fundamental region (why?). Then there is an ambient isotopy of $C - \{0\}$ supported on a sufficiently small neighbourhood of A taking $\tilde{K}$ to $\tilde{K}'$. This projects, as described above, to an isotopy of T^2 which takes K to K' . (See the illustration.)



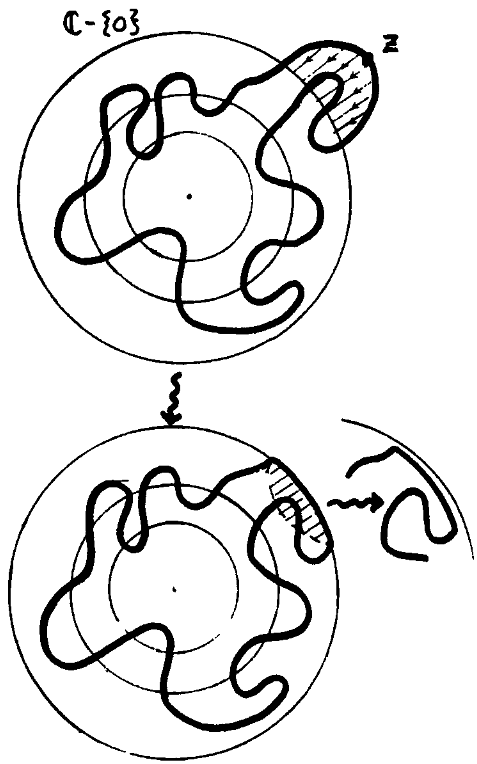
4. EXERCISE : Show that the lemma may be sharpened to provide an ambient isotopy $h_t : T^2 \longrightarrow T^2$, $h_0 = \mathrm{id}$, such that $h_1 \circ K = K'$ as <u>maps</u>.

Two knots J and $K \subset T^2$ are said to be <u>transversal</u> at $p \in J \cap K$ if there is a neighbourhood U of p and homeomorphism $h : U \longrightarrow R^2$ such that $h(U \cap J)$ and $h(U \cap K)$ are perpendicular straight lines.

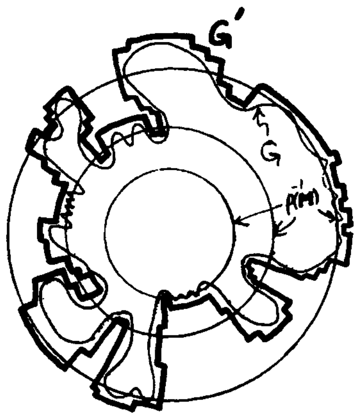
5. LEMMA : Suppose K is a knot of class $<0, \pm 1>$ which intersects the meridian M transversally in a finite number of points. Then K is ambient isotopic to M.

PROOF : Consider a fixed lifting $\tilde{K} \subset C - \{0\}$ of K. Then $\tilde{K} \cap q^{-1}(M)$ is a finite transversal set. It is sufficient to find a finite sequence of ambient isotopies of $C - \{0\}$, supported within fundamental regions, which makes $\tilde{K}$ disjoint from $q^{-1}(M)$. Then these induce an ambient isotopy of T^2 carrying K to a $<0, 1>$ knot disjoint from M and the previous lemma applies.

Now to reduce the intersection of $\tilde{K}$ with $q^{-1}(M)$ consider a point $z \in \tilde{K}$ of maximal norm and consider the component of $\tilde{K} - q^{-1}(M)$ which contains z. This, together with part of $q^{-1}(M)$ bounds a disk which lies in a fundamental region, and we may use this disk to remove two intersections (at least) of $\tilde{K}$ with $q^{-1}(M)$ (using Exercise A6). A finite number of such operations will then reduce the number $|\tilde{K} \cap q^{-1}(M)|$ to zero. Then use lemma 3.



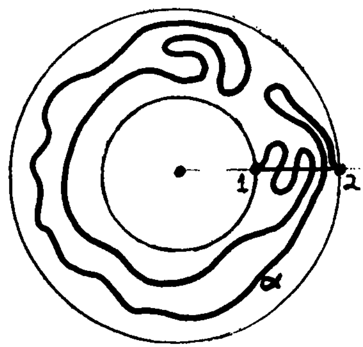
6. EXERCISE : Show that if G is any simple closed curve in $C - \{0\}$, then within any ε-neighbourhood of G lies a simple closed curve G' disjoint from G, homotopic to G, and transversal to all of the curves $q^{-1}(M)$.



7. EXERCISE : Piece together the above information to prove :

8. THEOREM : Any two knots in T^2 of class $<0, \pm 1>$ are ambient isotopic.

9. EXERCISE : Let $A \subset C$ be the annulus $1 \le |z| \le 2$. Let $\alpha : [1,2] \longrightarrow A$ be an embedding such that $\alpha[1,2] \cap \partial A = \{\alpha(1) = 1, \alpha(2) = 2\}$. Use methods of this section to show that if α is homotopic, with fixed endpoints, to the inclusion $j : [1,2] \longrightarrow A$ then there is an ambient isotopy $h_t : A \longrightarrow A$ with $h_0 = \mathrm{id}$, $h_t|\partial A = \mathrm{id}$, and $h_1 \circ \alpha = j$.



10. TWIST HOMEOMORPHISMS. Two basic self-homeomorphisms of T^2 are :

$$h_L(e^{i\theta}, e^{i\phi}) = (e^{i(\theta+\phi)}, e^{i\phi}) \qquad \text{"longitudinal twist"}$$

$$h_M(e^{i\theta}, e^{i\phi}) = (e^{i\theta}, e^{i(\theta+\phi)}) \qquad \text{"meridinal twist"}$$

These and their inverses will be called <u>twists</u>. They induce isomorphisms

h_{L_*} and h_{M_*}, which correspond to 2×2 integral matrices :

$$h_{L_*} \longleftrightarrow \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad h_{M_*} \longleftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} .$$

The matrices operate, however on the right. For example

$$h_{L_*}\langle a,b\rangle = \langle a \; b\rangle \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = \langle a+b,\ b\rangle$$

11. EXERCISE : Find a composite of twists which takes the longitude onto the meridian (hence these knots have the same type). Where does it send the meridian? Is it the same as the map $(z,w) \longrightarrow (w,z)$ of $T^2 \longrightarrow T^2$? Is there a product of twists taking the meridian to itself setwise, but with reversed orientation?

12. LEMMA : For any class $\langle a, b\rangle$ in $\pi_1(T^2)$ there exists a homeomorphism $h : T^2 \longrightarrow T^2$, which may be taken to be a composite of twists, satisfying $h_*\langle a, b\rangle = \langle 0, d\rangle$. Moreover, $d = \pm$ g.c.d.(a, b) unless $a = b = 0$ $(d = 0)$

PROOF : Suppose $|a| \geq |b|$. Then if $b \neq 0$ the Euclidean algorithm provides an integer n such that $a = nb + a'$, where $0 \leq a' < |b|$. Then $h_{L_*}^{-n}\langle a, b\rangle = \langle a', b\rangle$. If, on the other hand $|a| \leq |b|$ and $a \neq 0$, we similarly employ h_M . In any case, one of $|a|$, $|b|$ is strictly reduced, so after finitely many repetitions we obtain a composite carrying $\langle a, b\rangle$ to a class of the form $\langle 0, c\rangle$ or $\langle c, 0\rangle$ and the latter can be changed to the former as in the previous exercise by further twists.

13. THEOREM : For any knot $K \subset T^2$ of class $[K] \neq \langle 0, 0\rangle$, there is a homeomorphism $h : T^2 \longrightarrow T^2$ such that $h(K) = M$, the standard meridian.

PROOF : Let $[K] = \langle a, b\rangle$. Then the homeomorphism of Lemma 12 carries K to a knot K' of class $\langle 0, \pm 1\rangle$. Then by Theorem 8 , we can carry K' onto M by a second homeomorphism.

A knot $K \subset T^2$ which is null-homotopic ($[K] = \langle 0, 0\rangle$) is said to be <u>inessential</u>.

14. EXERCISE : Show that any two inessential knots in T^2 are ambient isotopic. Thus K is inessential $\iff$ K separates T^2 $\iff$ K bounds a disk in T^2.

This exercise and theorem may be summarized by :

15. THEOREM : There are two knot types of S^1 in T^2 : the inessential and the others (essential).

The classification of knots modulo ambient isotopy is more interesting.

16. THEOREM : Two knots $J, K \subset T^2$ are ambient isotopic if and only if $[J] = \pm [K]$.

PROOF : The condition is necessary since an ambient isotopy restricts to a homotopy taking J onto K . If $[J] = \pm[K] \neq \langle 0, 0\rangle$, choose $h : T^2 \longrightarrow T^2$ so that $h_*[J] = \pm h_*[K] = \langle 0, \pm 1\rangle$. Then choose an ambient isotopy $g_t : T^2 \longrightarrow T^2$ such that $g_1 h(J) = h(K)$, applying Theorem 8 . Then $h^{-1} g_t h$ is an ambient isotopy of T^2 with $h^{-1} g_1 h(J) = K$, as required.

D. THE MAPPING CLASS GROUP OF THE TORUS.

The π_1 functor converts a homeomorphism h of T^2 into a group automorphism h_* of $\pi_1(T^2) = Z \oplus Z$. This is a homomorphic map from the group $\mathrm{Aut}(T^2)$ of self-homeomorphisms to the group $GL(2,Z)$ of integral 2×2 matrices, invertible over the integers :

$$\mathrm{Aut}(T^2) \xrightarrow{\ *\ } GL(2,Z) .$$

1. EXERCISE : A 2×2 integral matrix is in $GL(2,Z)$ if and only if its determinant is ± 1. Each such matrix is a product of matrices

$\begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. [Hint : recall row and column operations]

2. LEMMA : The homomorphism (*) above is surjective.

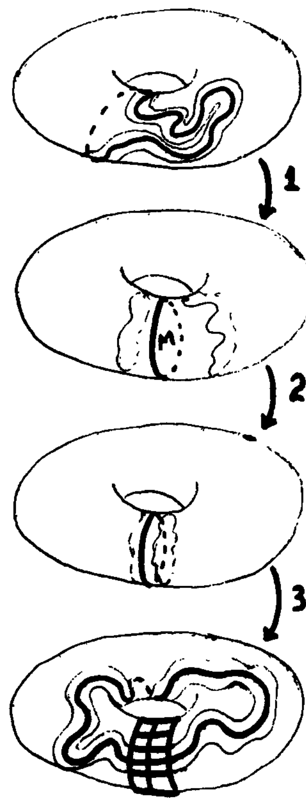
PROOF : By the exercise, $GL(2,Z)$ is generated by h_{L_*}, h_{M_*} and the matrix of the inversion automorphism $(z,w) \longrightarrow (w,z)$ of T^2.

3. LEMMA : The kernel of (*) is exactly the subgroup of homeomorphisms of T^2 which are isotopic to the identity (i.e. the path component of $\mathrm{id} \in \mathrm{Aut}(T^2)$).

These two lemmas establish the main result of this section:

4. THEOREM : The group of self-homeomorphisms of T^2, modulo ambient isotopy, is isomorphic with $GL(2,Z)$. Thus two homeomorphisms of T^2 are ambient isotopic $\Longleftrightarrow$ they have the same matrix $\Longleftrightarrow$ they are homotopic as maps.

5. PROOF OF LEMMA 3 . Since homotopic maps $T^2 \longrightarrow T^2$ induce the same homomorphism $\pi_1(T^2) \longrightarrow \pi_1(T^2)$, it is clear that any automorphism of T^2 isotopic to the identity is in the kernel of (*). Conversely, given an automorphism $h : T^2 \longrightarrow T^2$ with matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, we wish to construct an isotopy from h to the identity. This may be done in stages by a method sometimes called 'handle straightening'. The idea is to modify h by ambient isotopy to make it the identity on successively larger subsets of T^2. To avoid excessive notation, we denote each 'improvement' of h via ambient isotopy by the same letter h. The details justifying the following steps are left as exercises.



Step 1. Since $[h(M)] = \langle 0, 1 \rangle$, C4 and C8 may be employed to change h isotopically so that $h|M$ = identity.

Step 2. Let $N(M)$ be a standard neighbourhood of M in T, h may be modified so that $h|M = \mathrm{id}$ and $h(N(M)) \subset N(M)$.

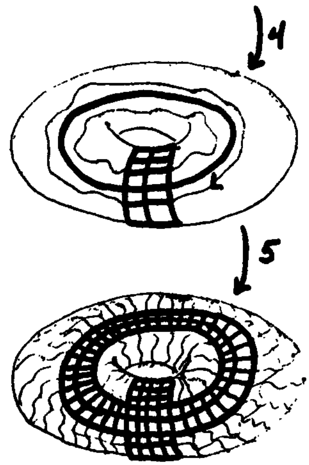
Step 3. Modify h isotopically, using the annulus theorem so that

$h|N(M) = \mathrm{id}$.

Step 4. Using exercise C9 , isotope h so that $h|N(M) \cup L = \mathrm{id}$.

Step 5. Let $N(L)$ be a standard neighbourhood of L in T and, as in steps (2) and (3), move h isotopically so that $h|N(M) \cup N(L) = \text{id}$.

Step 6. The closure of what remains to be straightened is a 2-disk, and h is already the identity on the boundary. Use the following exercise to obtain one more ambient isotopy carrying h to the identity, completing the proof.



6. EXERCISE : If $f : D^2 \longrightarrow D^2$ is a homeomorphism and $f|\partial D^2 = \text{identity}$, then there is an isotopy fixed on ∂D^2 carrying f to the identity.

7. EXERCISE : Show that if K and $K' \subset T^2$ are disjoint knots of classes $\langle a, b\rangle$, $\langle a', b'\rangle$, then $\det\begin{pmatrix} a & b \\ a' & b' \end{pmatrix} = 0$. If they intersect in one point transversally, then $\det\begin{pmatrix} a & b \\ a' & b' \end{pmatrix} = \pm 1$. Show conversely that if $\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$ is a unimodular matrix, there are knots K, K' of types $\langle a, b\rangle$, $\langle a', b'\rangle$ which intersect transversally in one point. Generalize this to argue that $\det\begin{pmatrix} a & b \\ a' & b' \end{pmatrix}$ is the "algebraic intersection" of K with K'.

REMARK : A presentation of the mapping class group of $T^2 \# T^2$ is given in Birman's new book, Braids, Links and Mapping class groups.

E. SOLID TORI. Instead of just the frosting, now consider the whole doughnut. A <u>solid torus</u> is a space homeomorphic with $S^1 \times D^2$. The following exercises concern information about solid tori which will prove useful in the study of classical knots and 3-manifolds. Throughout, V will denote a solid torus; a specified homeomorphism $h : S^1 \times D^2 \to V$ is called a <u>framing</u> of V.

1. EXERCISE : If J is a simple closed curve in ∂V which is essential in ∂V, then the following are equivalent:

(a) J is homologically trivial in V,

(b) J is homotopically trivial in V,

(c) J bounds a disk in V,

(d) for some framing $h : S^1 \times D^2 \to V$, $J = h(1 \times \partial D^2)$.

2. DEFINITION : A simple closed curve in ∂V satisfying the conditions of the exercise is called a <u>meridian</u> of V. A <u>longitude</u> of V is any simple closed curve in ∂V of the form $h(S^1 \times 1)$, for some framing h of V.

3. EXERCISE : If $K \subset \partial V$ is a simple closed curve, the following are equivalent:

(a) K is a longitude of V,

(b) K represents a generator of $H_1(V) \cong \pi_1(V) \cong \mathbb{Z}$,

(c) K intersects some meridian of V (transversally) in a single point.

The following show that "meridian" is an intrinsic part of V, whereas "longitude" involves a choice. They also play different roles in describing self-homeomorphisms of V.

4. EXERCISE : Any two meridians of V are equivalent by an ambient isotopy of V. Any two longitudes are equivalent by a homeomorphism of V; however there are infinitely many ambient isotopy classes of longitudes.

5. EXERCISE : A homeomorphism $f : \partial V \to \partial V$ extends to a self-homeomorphism of V if and only if f takes a meridian to a meridian.

Now consider a solid torus embedded in the 3-sphere, $V \subset S^3$. Let X denote the closure of $S^3 - V$. We assume that V is embedded nicely enough as to make X a manifold,* with boundary $\partial X = \partial V$.

6. EXERCISE : X has integral homology groups as follows:

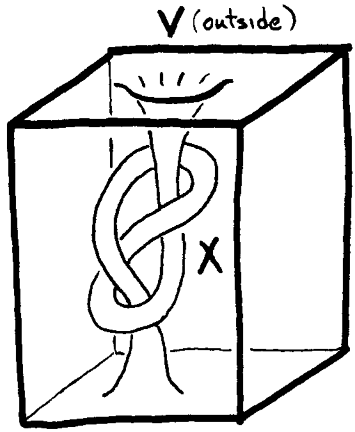
$$H_i(X) \cong \begin{cases} Z & i = 0, 1 \\ 0 & i \geq 2 \end{cases}$$

moreover, any meridian of V represents a generator of $H_1(X)$. [Hint: examine the Mayer-Vietoris sequence of (S^3, V, X)].

* see next section for definition

7. EXERCISE : Up to ambient isotopy of V, there is a unique longitude which is homologically trivial in X. Moreover, there is a framing (also unique up to ambient isotopy) $h : S^1 \times D^2 \to V$ such that $h(S^1 \times 1)$ is homologically trivial in X.

8. DEFINITION : The framing specified in the above exercise is called a <u>preferred framing</u> of a solid torus in S^3.

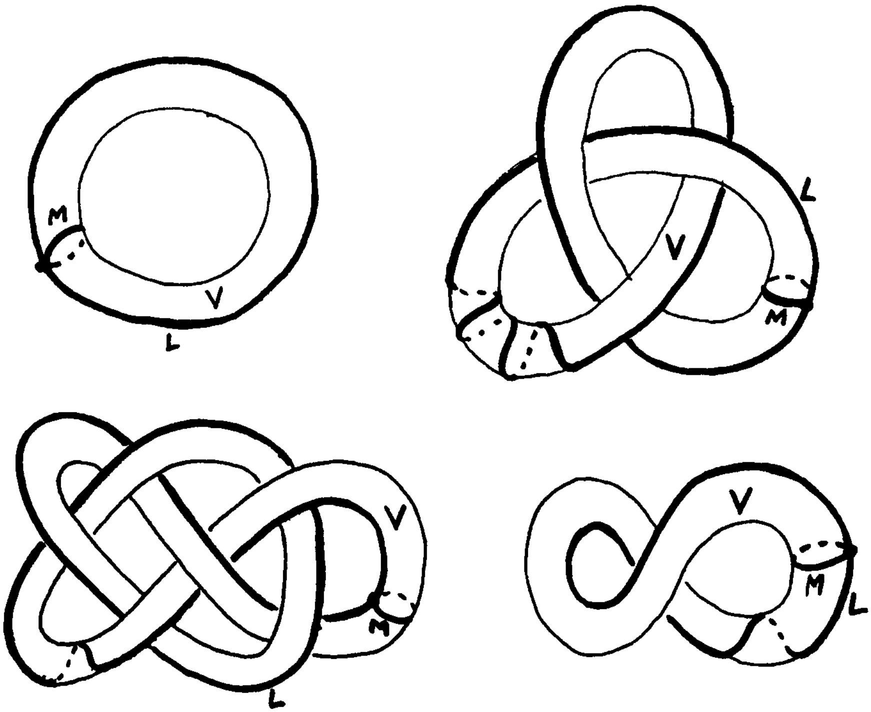


9. REMARK. The space X has the same homology groups as a solid torus and may or may not be one. If not, X is sometimes called a <u>cube-with-knotted-hole</u>.* The homotopy group $\pi_1(X)$ may not be infinite cyclic. We will discuss the other possibilities in the next chapter; these are just the classical knot groups. In chapter four, we will see that $\pi_2(X) \cong \pi_3(X) \cong \ldots \cong 0$ and that $\pi_1(X) \cong Z \Leftrightarrow X$ is a solid torus.

10 QUESTION : Can a torus be embedded in S^3 in such a way that it bounds manifolds on both sides, and neither one is a solid torus? (answered later)

* or <u>knot exterior</u>, by contrast with knot complement, which has no boundary

11. WARNING. The preferred framing may not be the one which looks "obvious". Pictured below are the longitudes and meridians determined by preferred framings for several different embeddings $V \subset S^3$. After we study linking number (section 5D), finding a preferred longitude will be simple, since it is characterized by having linking number zero with the core of V .



F. HIGHER DIMENSIONS. The Jordan curve theorem has a natural generalization to higher dimensions. Closely related to this is the theorem of invariance of domain, fundamental to the study of manifolds. Both are generally credited to L. E. J. Brouwer. Proofs may be found in many standard texts on topology, such as Hurewicz and Wallman, Dimension Theory.

1. BROUWER SEPARATION THEOREM : If K^{n-1} is a topological (n-1)-sphere in R^n, then $R^n - K$ has exactly two components and K is the boundary of each.

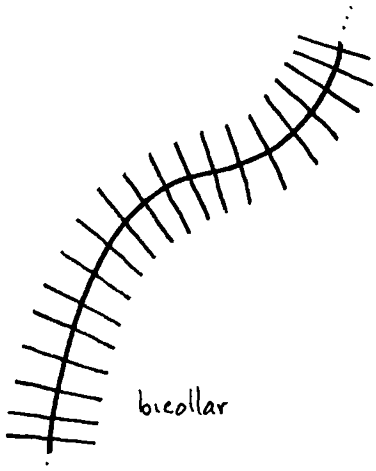
2. INVARIANCE OF DOMAIN THEOREM : If $U \subset R^n$ is open and $h : U \to R^n$ is a continuous 1-1 function, then $h(U)$ is open in R^n.

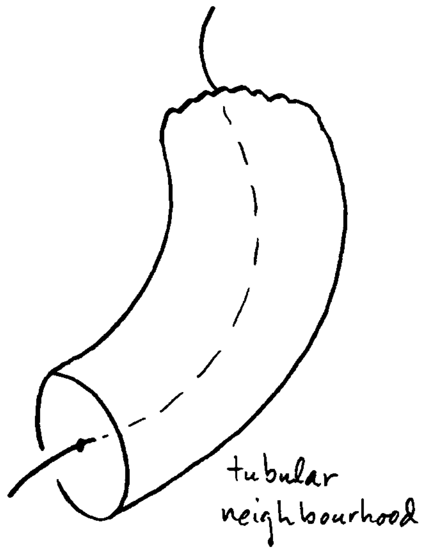
We regard an n-manifold M^n to be a metric space which may be covered by open sets, each of which is homeomorphic with R^n or the half-space R^n_+. Points of M with R^n-like neighbourhoods, are said to be interior points of M ; their union is denoted by Int M or $\mathring{M}$. The boundary of M is then $\text{Bd}M = M - \text{Int } M$, sometimes written ∂M. Invariance of domain implies (EXERCISE) that ∂M is a topological invariant, i.e. any homeomorphism $h : M^n \to N^n$ of n-manifolds takes boundary points to boundary points. We say that a manifold M is closed if it is compact and $\partial M = \phi$ and that it is open if it is non-compact and $\partial M = \phi$.

A verbatim generalization of the Schönflies theorem to dimensions greater than 2 would be false. In the next chapter are constructed "wild" 2-spheres in R^3, first discovered by J. W. Alexander, which are not equivalent to the standard S^2; in fact their complements fail to be simply-connected. The proper generalization requires an additional smoothness assumption to rule out local pathology.

3. GENERALIZED SCHÖNFLIES THEOREM : Suppose K^{n-1} is a bicollared $(n-1)$-sphere in R^n. Then the closure of its bounded complementary domain is homeomorphic with the n-ball B^n.

The proof of this, given by Morton Brown [1960] is to my mind one of the most elegant arguments of geometric topology. [I suggest that the reader do as my class did: read the proof from the original source.] A subset $X \subset Y$ is said to be <u>bicollared</u> (in Y) if there exists an embedding $b : X \times [-1, 1] \to Y$ such that $b(x,0) = x$ when $x \in X$. The map b, or its image, is the bicollar itself. In case X and Y are, respectively, $(n-1)$- and n-manifolds (with empty boundary), the bicollar is a neighbourhood of X in Y, in fact $b(X \times (-1, 1))$ is an open set, by invariance of domain (EXERCISE). This is a good place to define the more general <u>tubular neighbourhood</u> of a submanifold $M^m \subset N^n$ of another manifold N^n (again assume $\partial M = \partial N = \phi$) By this we mean an embedding $t : M \times B^{n-m} \to N$ such that $t(x,0) = x$ whenever $x \in M$. Here B^{n-m} is, as usual, the unit ball of R^{n-m}, centered at 0. For example, a tubular neighbourhood of a knot K^1 in R^3 is a solid torus whose "core" is K.





4. REMARK : Some writers use the term 'product neighbourhood', reserving 'tubular' for the more general disk-bundle neighbourhood.

5. EXERCISE : Prove a generalized Schönflies theorem for S^n . Prove that there is only one bicollared $(n-1)$-dimensional knot type in R^n or S^n .

As in the case $n = 2$ described earlier in this chapter, several powerful results follow from the generalized Schönflies theorem, and the Alexander extension theorem (lemma A5) which allows any homeomorphism between the boundaries of balls to extend to a homeomorphism between the interiors as well. Following are two examples.

6. THEOREM : If M^n is a compact manifold which is the union $M = U_1 \bigcup U_2$ of two open sets, each homeomorphic with R^n , then M

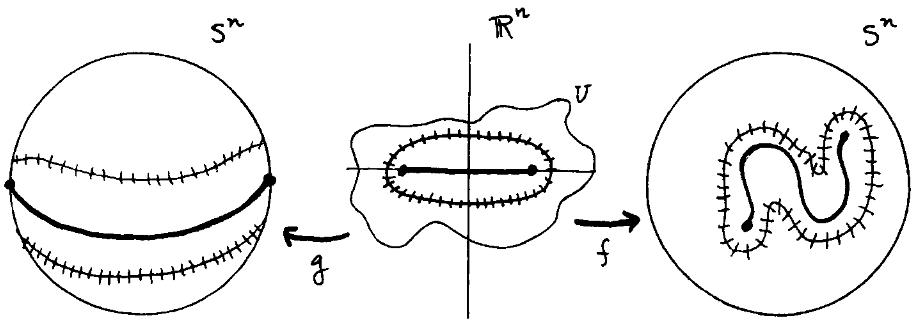
is homeomorphic with S^n .

PROOF : Since $M - U_2$ is a compact set in $U_1 \cong R^n$, there exists an n-ball $B_1 \subset U_1$ which contains $M - U_2$. We may assume ∂B_1 is bicollared in U_1 , hence also in U_2 (why?). By the generalized Schönflies theorem, ∂B_1 bounds a ball B_2 in U_2 . Now B_1 and B_2 have disjoint interiors, and their union is M (why?). Since B_1 is an n-ball, there is a homeomorphism carrying it to, say, the upper hemisphere of S^n . Then use Alexander's lemma to extend this to a homeomorphism from M onto S^n , as desired.

An embedding $f : B^k \to M^n$, M a manifold, is called <u>flat</u> (in the topological, not geometric, sense) if it extends to an embedding $\bar{f} : U \to M$, where U is a neighbourhood of B^k in R^n ($B^k \subset R^n$ in the standard manner). We also say that the subset $f(B^k)$ of M is a <u>flat ball</u>.

7. BASIC UNKNOTTING THEOREM : A knot K^k in S^n $(k<n)$ is equivalent to the trivial knot $S^k \subset S^n$ if and only if K is the boundary of a flat $(k+1)$-ball in S^n .

PROOF : The hypothesis implies the existence of an embedding $f : U \to S^n$ where U is a neighbourhood of the standard B^{k+1} in R^n and $f(\partial B^{k+1}) = K$. Since B^{k+1} has arbitrarily small closed neighbourhoods which are n-balls with bicollared boundary in R^n , we may choose one, say C^n , which, together with a bicollar on the boundary, lies inside U . Now let $g : R^n \to S^n$ be an embedding (the inverse of



the 'stereographic' projection) which takes ∂B^{k+1} to the standard S^k in S^n . Since $g(C^n)$ and $f(C^n)$ are n-balls in S^n with bicollared boundary, the homeomorphism $fg^{-1} : g(C^n) \to f(C^n)$ may be extended to a homeomorphism $h : S^n \to S^n$ by the Alexander lemma (the closures of both $S^n - g(C^n)$ and $S^n - f(C^n)$ are n-balls by exercise 5). Now $h(S^k) = f(\partial B^{k+1}) = K$, so h is the desired equivalence.

8. EXERCISE : Suppose $M^m \subset N^n$ are manifolds and M has <u>nonempty</u> boundary. We want to say that M has a tubular neighbourhood if we can adjoin an open collar $\partial M \times [0, 1)$ to the boundary of M and then find a tubular neighbourhood of $M' = M +$ collar of the form $M' \times B^{n-m}$, all in N . Make this into a precise definition and prove that in case M is a ball, M has a tubular neighbourhood if and only M is flat.

9. REMARK : In the basic unknotting theorem, it would be insufficient to assume the knot bounds a ball whose interior has a tubular neighbourhood. For example section 3I will describe a (wild) knot in S^3 which, though knotted, bounds a 2-disk whose interior is bicollared.

10. EXERCISE : Use the generalized Schönflies theorem to prove a partial generalization of the annulus theorem to higher dimensions: Let K_1^n and K_2^n be disjoint bicollared knots in R^{n+1} or S^{n+1} and let U denote the open region between them. Then U is homeomorphic with an open annulus $S^n \times (0,1)$. Moreover the union of U and either K_1 or K_2 is homeomorphic with $S^n \times [0,1)$.

11. REMARKS : The famous annulus conjecture (that the closure of U in the previous exercise is a closed annulus $S^n \times [0,1]$) was proved only recently by Kirby [1969], except for the case n=3 which remains unsolved.

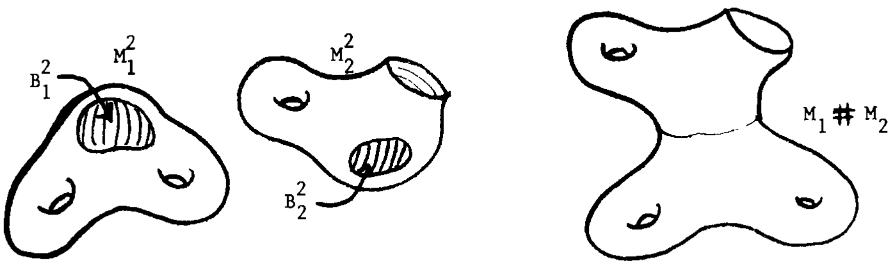
If one wishes to work in the PL category, there is a major stumbling block -- the PL Schönflies problem. Alexander [1928][I] proved that a piecewise-linear 2-sphere in R^3 always bounds a region whose closure is PL homeomorphic with the 3-simplex (PL 3-ball). All attempts to increase the dimensions of this theorem have failed, as of this writing Fortunately a partial PL Schönflies theorem is known, due to Alexander and Newman: If a PL (n-1)-sphere in S^n bounds a PL n-ball on one side, then it also bounds a PL n-ball on the other side. You should verify that this is sufficient to prove PL versions of theorems 5 and 6. For a fuller discussion of the PL Schönflies problem consult Rourke and Sanderson's book [1972].

G. CONNECTED SUMS AND HANDLEBODIES .

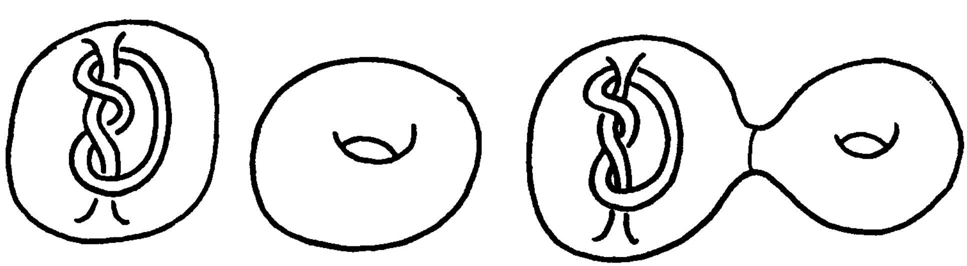
If M_1 and M_2 are n-manifolds, their <u>connected sum</u> $M_1 \# M_2$ is formed by deleting the interiors of n-balls B_i^n in M_i^n and attaching the resulting punctured manifolds $M_i - \mathring{B}_i$ to each other by some homeomorphism $h : \partial B_2 \to \partial B_1$. Thus

$$M_1 \# M_2 = (M_1 - \mathring{B}_1) \bigcup_h (M_2 - \mathring{B}_2)$$

To ensure that this be a manifold the B_i are required to be standard in the sense that B_i is interior to M_i and ∂B_i is bicollared in M_i .

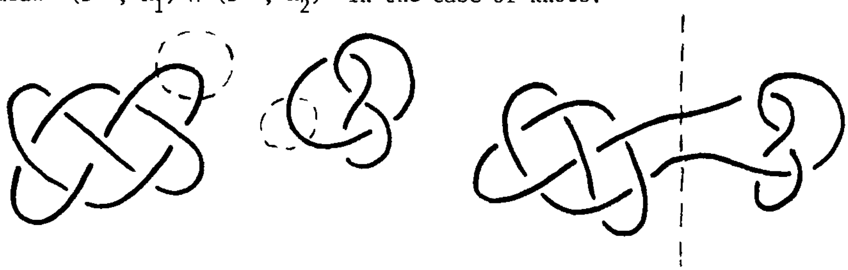


If both M_i have nonempty boundary, one can also construct a <u>boundary connected sum</u> $M_1 \#_\partial M_2$ by identifying standard balls B_i^{n-1} in ∂M_i.



Boundary connected sum of two 3-manifolds

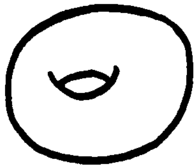
A third type of connected sum is the connected sum of pairs $(M^m_1, N^n_1) \# (M^m_2, N^n_2)$ where N^n_i is a locally-flat submanifold of M^m_i. Locally flat means that each point of N_i has a (closed) neighbourhood U in M_i such that the pair $(U, U \cap N_i)$ is topologically equivalent to the canonical ball pair (B^m, B^n). Such a neighbourhood pair, if also bicollared is called standard. So one removes a standard ball pair $(\mathring{B}^m_i, \mathring{B}^n_i)$ from (M_i, N_i) and sews the resulting pairs by a homeomorphism $h : (\partial B^m_2, \partial B^n_2) \to (\partial B^m_1, \partial B^n_1)$ to form the pair connected sum. In the special case that all the manifolds are spheres, then the connected sum is again a sphere pair and we have the <u>connected sum of knots</u>. By abuse of notation we may write $K_1 \# K_2$ when we really mean $(S^n, K^k_1) \# (S^n, K^k_2)$ in the case of knots.



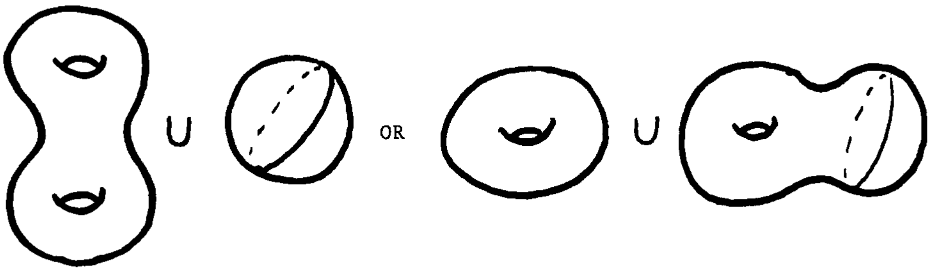
Connected sum of one-dimensional knots in S^3

The following simple example shows that connected sum is in general not a well-defined operation, but depends upon the choice of balls where the connection is to be made, and perhaps also upon the choice of attaching homeomorphism. The square and granny knots (see 3D10 and 3D11, also 8E15) illustrate this ambiguity also for the connected sum of knots.

1. EXAMPLE : To see that connected sum may be ambiguous, consider

M_1 =  M_2 = 

Then, depending upon choice of attaching disks, $M_1 \# M_2$ may be either



Let's investigate conditions under which connected sum will be well-defined. The first matter is the choice of the balls B_i .

2. EXERCISE : Suppose M_1 and M_2 are n-manifolds and $B_i \subset M_i$ are n-disks. If $M_1 - B_1$ is homeomorphic with $M_2 - B_2$, then M_1 is homeomorphic with M_2 .

Of course, we are really interested in the converse. The following implies that the converse is true if $n \neq 4$, the B_i are standard, and the M_i are closed and connected.

3. EXERCISE : Suppose $n \neq 4$ and that B_1^n and B_2^n are standard balls in the closed connected n-manifold M^n . Then there is a homeomorphism $h: M \to M$ such that $h(B_1) = B_2$. Moreover, h is isotopic to the identity. (Hint: Use connectedness to find a finite sequence of Euclidean patches connecting the "center" of B_1 to that of B_2 . Isotopically shrink B_1 until it lies in the first patch; show how to transport the ball from one patch to the next until it lies interior to B_2. Then appeal to the annulus theorem.)

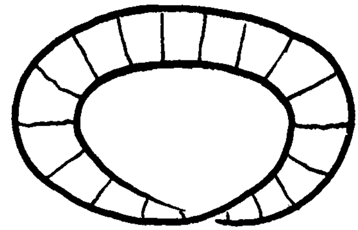
Because, at this writing, the annulus theorem is unsolved for dimension 4 , we don't know if connected sum is well-defined for 4-manifolds, even in the orientable case which we'll discuss now. To investigate the role of the attaching homeomorphism in connected sum it is necessary to discuss orientation. This is a somewhat awkward concept and there are several approaches to it. For a triangulated n-manifold, the idea is to attempt to order the $n+1$ vertices of each n-simplex Δ^n Two orderings are regarded the same or opposite according as they differ by an even or odd permutation. Each face of Δ^n receives an induced ordering in the obvious way, by deletion of the missing vertex. An orientation is a choice of ordering for the vertices of each n-simplex of the n-manifold such that whenever two of them meet in an $(n-1)$-dimensional face, they induce opposite orderings on that face. In the language of simplicial homology this says that there is a nontrivial n-cycle (with integer coefficients). This motivates the following definition for topological manifolds, using singular homology with integral coefficients.

4. DEFINITION : A closed connected n-manifold M^n is <u>orientable</u> if $H_n(M^n) \neq 0$. If the connected compact manifold M^n has nonempty boundary, say it is orientable if $H_n(M,\partial M) \neq 0$.

5. REMARK : As is well-known, the groups above, if nontrivial, are infinite cyclic. A choice of one of the two possible generators of $H_n(M^n)$ is called an orientation, and an orientable manifold together with such a choice is said to be an oriented manifold. By restriction any submanifold (n-dimensional with boundary) of an oriented n-manifold is oriented, i. e. one has a preferred nontrivial element of the n-dimensional relative homology group in dimension n . Furthermore, the boundary of an oriented n-manifold is oriented by choosing the (n-1)-cycle which is the boundary of the preferred relative n-cycle. A homeomorphism (more generally, map) $h: M^n \to N^n$ is said to preserve or reverse orientation, if M and N are oriented n-manifolds, according as the induced homomorphism on the n-th homology carries the preferred generator for M to the preferred generator for N , or to its negative.

Readers for whom this is new and/or confusing may wish to use a more geometrical definition of orientability for 2- and 3-dimensional manifolds. Say M^2 is orientable if it does not contain a Möbius band (see the following example) and say M^3 is orientable if it does not contain the product of a Möbius band with an interval. The definitions are, in fact, equivalent to the homological definition.

6. EXAMPLE : The Möbius band M^2 ⟶ is not orientable. For the homology sequence of the pair $M, \partial M$ is the exact sequence



$$\to \overset{0}{H_2(M)} \to H_2(M,\partial M) \to \overset{Z}{H_1(\partial M)} \to \overset{Z}{H_1(M)} \to H_1(M,\partial M) \to \cdots \quad .$$

This shows that $H_2(M,\partial M)$ is the kernel of the map $Z \to Z$, which by inspection sends a generator to twice a generator. So $H_2(M,\partial M) = 0$ and by our definition M is not orientable.

7. EXERCISE : (Mobius band) $\times$ I is not orientable.

8. EXERCISE : The real projective plane RP^2 is defined to be the quotient space of S^2 in which antipodal points are identified. Show that RP^2 is homeomorphic with a 2-disk with antipodal boundary points identified, and that the punctured manifold $RP^2 - \mathring{B}^2$ is a Möbius band Conclude that RP^2 is not orientable. Likewise the Klein bottle, obtained by sewing together two Möbius bands by a homeomorphism between their two boundaries, is a nonorientable closed 2-manifold. It is the boundary of the manifold of the previous exercise, sometimes called a solid Klein bottle, by analogy with solid torus.

Returning to the question of connected sum, consider two oriented n-manifolds M_1 and M_2 . Deleting standard balls we obtain preferred orientations for the boundaries ∂B_i of the punctured manifolds $M_i - \mathring{B}_i$. Require that the attaching homeomorphism $h : \partial B_2 \to \partial B_1$ be orientation-reversing, so that the boundaries of the preferred relative n-cycles of $H_n(M_i - B_i, \partial B_i)$ will cancel. This makes $M_1 \# M_2$ an oriented n-manifold, which we call the oriented connected sum.

9. EXERCISE : Let $h_1, h_2 : S^1 \to S^1$ be homeomorphisms and give preferred orientations to both the domain and range. Then h_1 and h_2 are isotopic if (and only if) they both preserve orientation or both reverse orientation.

10. EXERCISE : The same for homeomorphisms of S^2.

11. EXERCISE : Oriented connected sum is well-defined for oriented 2- and 3-dimensional closed connected manifolds.

12. REMARK : The closed orientable 2-manifolds have been classified (as S^2 or the connected sum of tori), and as such each enjoys an orientation-reversing homeomorphism. It follows that even unoriented connected sum is well-defined for connected closed 2-manifolds. Orientations are essential, however, in dimension 3 since there are orientable 3-manifolds which do not have orientation-reversing homeomorphisms. (The lens space $L(3,1)$ -- see chapter 9 -- is such an example.)

Oriented connected sum of pairs is similarly defined. One gives preferred orientations to all spaces, and requires that the attaching homeomorphism of the boundaries of the deleted ball pairs reverse orientation of the spheres and subspheres.

13. EXERCISE : Oriented connected sum is well-defined for oriented knots of dimension 1 in the oriented S^3 .

The final topic of this section is (three-dimensional) handlebodies, which are central to chapters 9 and 10 . A <u>handlebody</u> is any space obtained from the 3-ball D^3 (0-handle) by attaching g distinct copies of $D^2 \times [-1,1]$ (1-handles) with homeomorphisms throwing the $2g$ disks $D^2 \times \pm 1$ onto $2g$ disjoint 2-disks on ∂D^3, all to be done in such a way that the resulting 3-manifold is orientable. The integer g is called the genus.

14. EXERCISE : A handlebody of genus g is homeomorphic with a boundary connected sum of g solid tori.

15. EXERCISE : Two handlebodies are homeomorphic if and only if they have the same genus.

16. REMARK : Some authors drop the orientability condition and allow "nonorientable handlebodies" such as the solid Klein bottle.

CHAPTER 3. THE FUNDAMENTAL GROUP

We are now interested in knots and links of codimension two, which includes the classical case of S^1's in R^3 or S^3. The most successful tool by far, in this setting, is the fundamental group of the complement; this chapter discusses various applications of that tool.

A. KNOT AND LINK INVARIANTS.

A <u>knot invariant</u> is a function

$$K \rightsquigarrow f(K)$$

which assigns to each knot K an object $f(K)$ in such a way that knots of the same type are assigned equivalent objects. Similarly for links. One hopes that $f(K)$ is reasonable to calculate and, on the other hand, sensitive enough to solve the problem at hand.

For a one-dimensional link in 3-space, certain <u>numerical invariants</u> have been found useful. We will discuss most of these in subsequent chapters. <u>Linking numbers</u> measure the number of times each pair of components wrap about each other in an algebraic sense; this is a natural generalization of the index of a curve in the complex plane about a point. They are easy to compute, but only give a "first order" description of the nature of the link. <u>Crossing number</u> is the minimal number of simple self-intersections which appear in a planar picture (regular projection) of a link or knot of given type. This simple measure of complexity has been used to order knots in the various existing tables, including the one in Appendix C of this book.

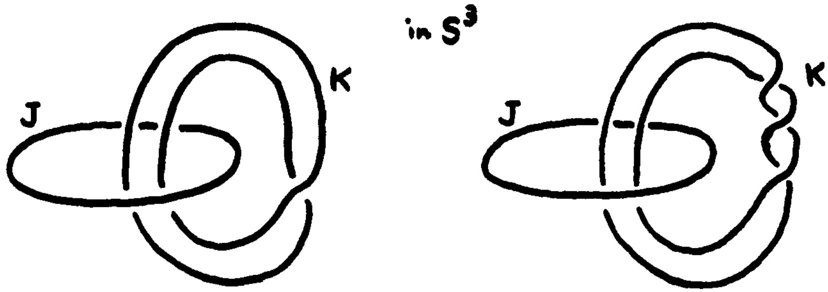
The *minimax number* is the smallest number of local maxima (or minima) of a knot $K : S^1 \longrightarrow R^3$ in a fixed direction, where K ranges over a given knot type. Milnor has studied this in relation to *total curvature* of a knot, in a beautiful little paper [1950]. The *genus* of a link L is the number of handles on a 'minimal surface' in R^3 spanning L, and has generalizations to some higher-dimensional knots and links. *Torsion numbers*, *Minkowski units*, *signature*, *arf invariant* and some other numerical invariants are more difficult to describe without some preliminaries, but also play a role in answering certain questions about knots. We shall also encounter *polynomial invariants* (the Alexander polynomials), *matrix invariants*, *quadratic forms*, etc., which are, in a sense, just glorified numerical invariants.

The *complement* $R^n - L$ of the link L is clearly an invariant, up to homeomorphism type, i.e. a *topological invariant*. Whether this is a complete knot invariant, even for classical knots, remains (at this writing) an open question which has received much attention and partial solution (see section 10I).

CONJECTURE : If two tame knots in R^3 have homeomorphic complements, then the knots have the same type.

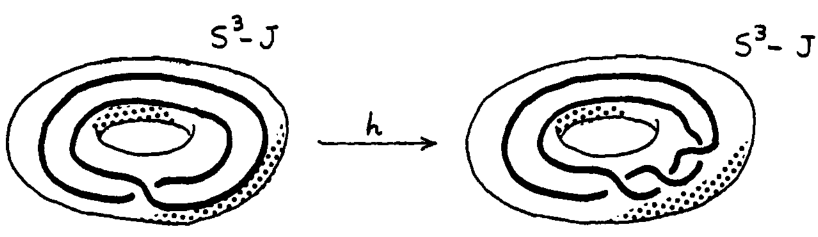
A knot is *tame* if it is equivalent to a polygonal knot. The following example shows that the corresponding conjecture for links is false.

2. EXAMPLE : Two inequivalent links with homeomorphic complements.



Notice that K' is a trefoil  , whereas K is trivial  .

Let us accept for now that these are inequivalent (this will be proved in the next section). Then $J \cup K$ and $J \cup K'$ must be inequivalent links. However, their complements (say in S^3) are homeomorphic! For $S^3 - J$ is homeomorphic with $S^1 \times \text{int } D^2$. Applying the twist homeomorphism $h(z,w) = (z,zw)$, where S^1 and D^2 are suitably identified with subsets of the complex numbers, carries K onto K' by a homeomorphism of $S^3 - J$.



It follows that $S^3 - (J \cup K)$ and $S^3 - (J \cup K')$ are homeomorphic. A similar argument works in R^3 .

Although the complement of a knot or link is usually difficult to describe or characterize topologically, one may derive less sensitive, more computable, invariants from it. Any functor F from topological spaces to, say, an algebraic category becomes a link or knot invariant via the composite

$$L \rightsquigarrow X = R^n - L \rightsquigarrow F(X) .$$
$$\text{or } S^n - L$$

One immediately thinks of homology or cohomology. But these are quite useless, according to Alexander duality.

3. PROPOSITION : The integral homology and cohomology groups of the complement of a link in R^n or S^n are independent of the particular embedding.

For example, with a knot $K^p \subset S^n$, $H_*(S^n - K^p) \cong H_*(S^{n-p-1})$ and $H^*(S^n - K^p) \cong H^*(S^{n-p-1})$. Compare this with exercise 2E6.

On the other hand, the homotopy groups of the complement are often quite good invariants. The fundamental group of the complement, in particular, has been undoubtedly the most useful tool available to knot theorists. The remainder of this chapter discusses some of its applications. Except for pathology, it applies only in codimension two : an easy general position argument shows that a PL link L^k in S^n has simply-connected complement if $n - k \geq 3$.

4. EXERCISE : Prove the remark just made.

B. THE KNOT GROUP. If K^{n-2} is a knot (link) in R^n, the fundamental group $\pi_1(R^n - K)$ of the complement is called, simply, <u>the group of</u> K. Appendix A contains a quick review of π_1 and the basic means for computing fundamental groups, Van Kampen's theorem. We note that the group is the same, up to isomorphism, if we consider the knot in S^n rather than R^n :

1. PROPOSITION : If B is any bounded subset of R^n such that $R^n - B$ is path-connected and $n \geq 3$, then the inclusion induces an isomorphism

$$\pi_1(R^n - B) \xrightarrow[i_*]{\cong} \pi_1(S^n - B) .$$

Proof : Choose any neighbourhood U of ∞ in S^n which misses B and is itself homeomorphic with R^n. Then $U \cap R^n = U - \infty \simeq S^{n-1}$. Thus both U and $R^n \cap U$ are simply-connected and we apply Van Kampen's theorem.

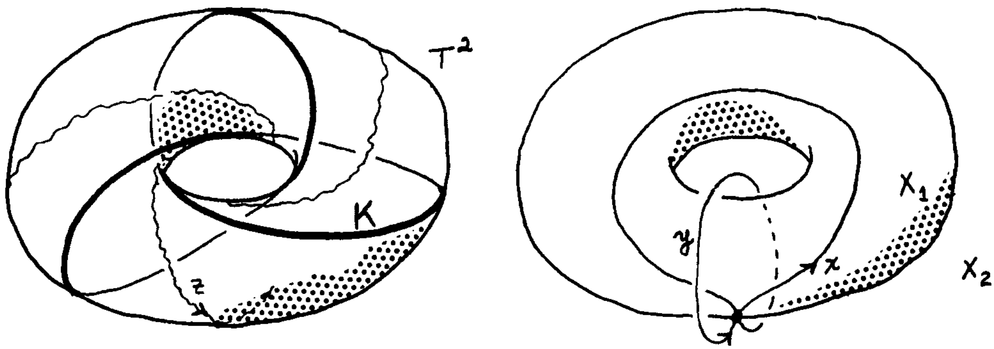
The naturally included $S^{n-2} \subset R^{n-1} \subset R^n \subset R^n + \infty = S^n$ is the <u>trivial knot</u> or <u>unknot</u> of codimension two. This is equivalent to the S^{n-2} of $S^n \cong S^{n-2} * S^1$; see exercise 1B6. Since $(S^{n-2} * S^1) - S^{n-2}$ deformation retracts to S^1, we have :

2. PROPOSITION : The unknot has group $\pi_1(S^n - S^{n-2}) \cong Z$.

We now show that nontrivial knots exist in S^3 .

3. EXAMPLE : The <u>trefoil</u>.

,



This knot has been drawn on the surface of a "standardly embedded" torus T^2 to employ Van Kampen's theorem. Let X_1 and X_2 denote the closed solid tori, as shown, bounded by T^2 but with K removed. Then we have presentations $\pi_1(X_1) = (x; -)$, $\pi_1(X_2) = (y; -)$. Let $X_o = X_1 \cap X_2 = T^2 - K$. This is an annulus, $\pi_1(X_o) \cong Z$ whose generator z equals x^2 in $\pi_1(X_1)$ and y^3 in $\pi_1(X_2)$. Thus by Van Kampen we have

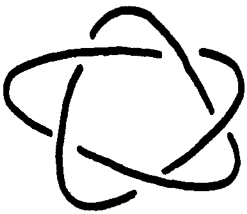
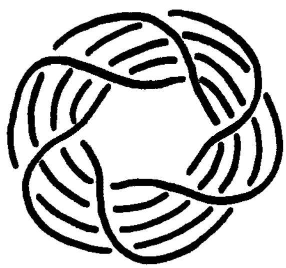
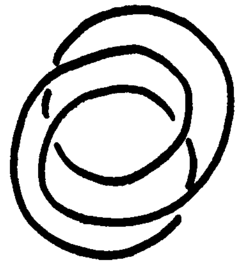
$$\pi_1(S^3 - \text{trefoil}) \cong (x, y; x^2 = y^3) .$$

4. LEMMA : This group is not abelian, hence the trefoil is not of trivial knot type.

PROOF : Let S_3 be the group of permutations of the symbols $\{1, 2, 3\}$; we use the notation of cycles. Let $A = (12)$ and $B = (123) \in S_3$ and map the free group $F(x,y) \to S_3$ by $x \to A$, $y \to B$. Since $A^2 = B^3$ ($= 1$), this induces a homomorphism $h : \pi_1(X) \to S_3$. Since $AB = (13)$ and $BA = (23)$, the image of h is all of S_3, which is nonabelian. Hence $\pi_1(S^3 - \text{trefoil})$ is nonabelian.

Alternative proof : adjoin the relation $x^2 = 1$ to map the knot group onto the nonabelian group $(Z/2) * (Z/3)$.

C. TORUS KNOTS. We generalize the previous example by choosing integers p, q which are relatively prime. The <u>torus knot</u> $T_{p,q}$ of type p, q is the knot which wraps around the standard solid torus T in the longitudinal direction p times and in the meridinal direction q times. Thus the trefoil is $T_{2,3}$. Here are $T_{2,5}$ (the Solomon's seal knot), $T_{5,6}$ and $T_{3,2}$.

A precise way to describe $T_{p,q}$ is to take the torus T to be the $1/2$ level of $S^1 * S^1 \cong S^3$ and let $T_{p,q} : S^1 \to S^3$ be the map

$$\theta \longrightarrow (p\theta,\ q\theta,\ \tfrac{1}{2}) .$$

One computes exactly as above that the group of $T_{p,q}$ is $G_{p,q} = (x,\ y;\ x^p = y^q)$.

We can classify the torus knot types. Note that :

(a) $T_{\pm 1,q}$ and $T_{p,\pm 1}$ are of trivial type.

(b) the type of $T_{p,q}$ is unchanged by changing the sign of p or q, or by interchanging p and q.

Otherwise, all the torus knots are inequivalent by

1. THEOREM (O. SCHREIER) : If $1 < p < q$, then the group $G_{p,q}$ determines the pair p, q.

2. COROLLARY : There exist infinitely many classical knot types.

The following three exercises outline Schreier's argument [1923]. The <u>center</u> of a group is the subgroup consisting of those elements which commute with everything in the group.

3. EXERCISE : Let C be the (infinite) cyclic subgroup of $G = G_{p,q}$ generated by the element x^p $(= y^q)$. Show that C lies in the center of G, that C is normal and that G/C is a free product of the cyclic groups Z/p and Z/q.

4. EXERCISE : Show that the center of the free product of any two nontrivial groups consists of the identity element alone. [Hint : any element of $A * B$ is expressible uniquely as a string of nontrivial elements alternatively from A and from B.] Conclude that C above is exactly the center of G.

5. EXERCISE : Complete the proof of Schreier's theorem.

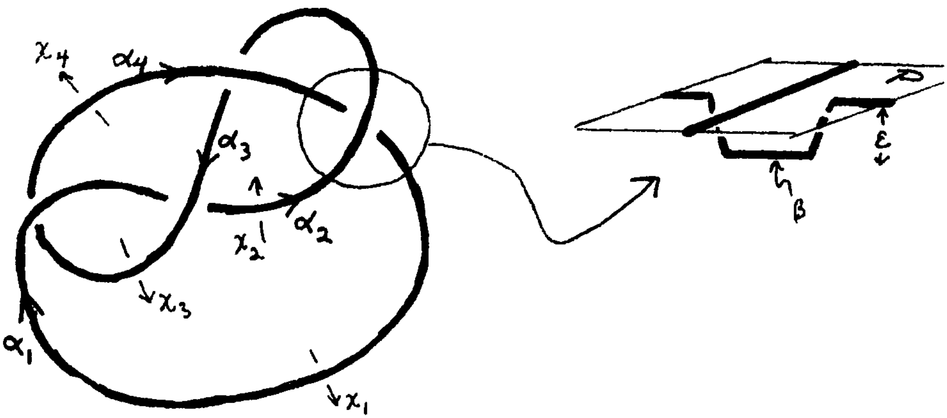
6. NOTES : (1) It is a remarkable fact that the torus knots are the <u>only</u> classical knots whose groups have a nontrivial center. This was proved by Burde and Zieschang [1966].

(2) If one wishes to classify knots by <u>oriented</u> equivalence, he must distinguish between the knots $T_{p,q}$ and $T_{p,-q}$. For example, there are two trefoils :

which are mirror images of each other. Of course, π_1 alone cannot detect the difference. However, Schreier shows how to distinguish between them (and generally between $T_{p,q}$ and $T_{p,-q}$) by studying automorphisms of π_1. The first proof that $T_{3,2}$ and $T_{3,-2}$ are not equivalent by an orientation preserving equivalence of R^3 is due to Dehn [1914]. We shall pursue this question again, in Chapter 8, as an application of the signature invariant.

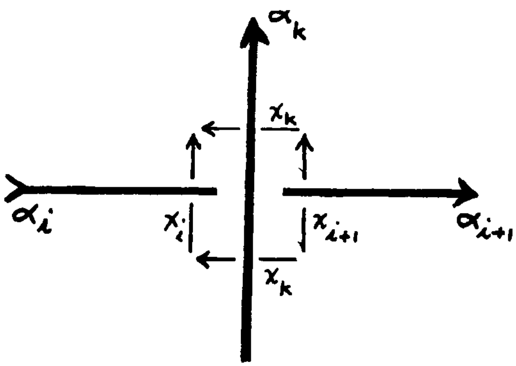
(3) This is a good place to point out that the group $\pi_1(R^3 - K)$ of a tame knot is isomorphic with the group $\pi_1(R^3 - V)$ or $\pi_1(R^3 - \mathring{V})$, where V is a tubular neighbourhood of K. Indeed the complement $R^3 - K$ and the exterior $R^3 - \mathring{V}$ have the same homotopy type by the obvious deformation retraction.

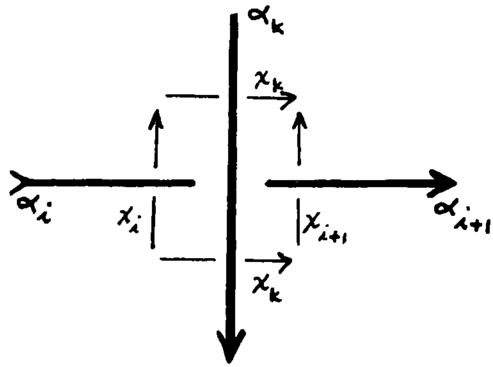
D. THE WIRTINGER PRESENTATION. This section describes a procedure for writing down a presentation of the group of a knot K in R^3, given a 'picture' of the knot. By a picture I mean a finite number of arcs $\alpha_1, \ldots, \alpha_n$ in a plane P (say, the x-y plane). Each α_i is assumed connected to α_{i-1} and α_{i+1} (mod n) by undercrossing arcs exactly as pictured below. The union of these is the knot K.



1. THE ALGORITHM. We assume for convenience that the α_i are oriented (assigned a direction) compatibly with the order of their subscripts. Draw a short arrow labelled x_i passing under each α_i in a right-left direction. This is supposed to represent a loop in $R^3 - K$ as follows. The point $(0, 0, 1) = *$ is taken as basepoint (best imagined as the eye of the viewer), and the loop consists of the oriented triangle from $*$ to the tail of x_i, along x_i to the head, thence back to $*$.

Now at each crossing, there is a certain relation among the x_i's which obviously must hold. The two possibilities are :





r_i : $\quad x_k x_i = x_{i+1} x_k \qquad\qquad x_i x_k = x_k x_{i+1}$

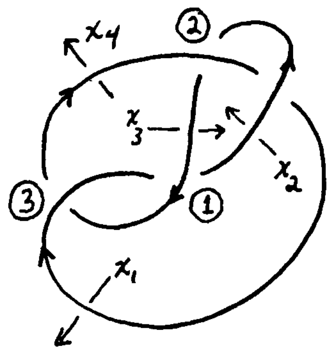
Here α_k is the arc passing over the gap from α_i to α_{i+1} ($k = i$ or $i+1$ is possible). Let r_i denote whichever of the two equations holds. In all, there are exactly n relations $r_1, \ldots, r_n$ which may be read off this way. We will see that these comprise a complete set of relations.

2. THEOREM : The group $\pi_1(R^3 - K)$ is generated by the (homotopy classes of the) x_i and has presentation

$$\pi_1(R^3 - K) = (x_1, \ldots, x_n ; r_1, \ldots, r_n) .$$

Moreover, any one of the r_i may be omitted and the above remains true.

3. EXAMPLE : For the figure-eight knot, we have a presentation with generators x_1, x_2, x_3, x_4 and relations



(1) $x_1x_3 = x_3x_2$

(2) $x_4x_2 = x_3x_4$

(3) $x_3x_1 = x_1x_4$

We may simplify, using (1) and (3) to eliminate $x_2 = x_3^{-1}x_1x_3$ and $x_4 = x_1^{-1}x_3x_1$ and substitute into (2) to obtain the equivalent presentation

$$\pi_1(R^3 - \text{figure-eight}) \cong (x_1, x_3;\ x_1^{-1}x_3x_1x_3^{-1}x_1x_3 = x_3x_1^{-1}x_3x_1) .$$

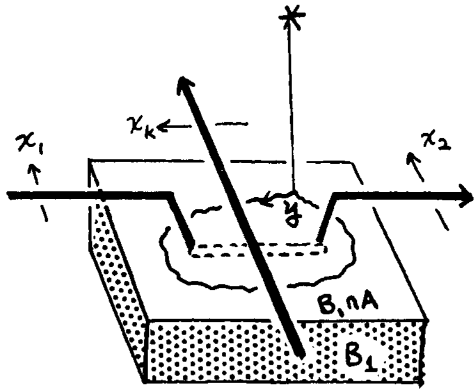
4. EXERCISE : Show combinatorially that the fourth relation $(x_2x_4 = x_1x_2)$ is a consequence of the other three.

5. EXERCISE : Verify that the figure-eight knot is nontrivial. (Try mapping its group onto a nonabelian finite group, as in lemma B4).

6. PROOF OF THEOREM 2 : Recall that K lies in the plane $P = \{z = 0\}$ of R^3, except where it dips down by a distance ε at each crossing. In order to apply Van Kampen's theorem, we dissect $X = R^3 - K$ into $n+2$ pieces $A, B_1, \ldots, B_n$, and C. Let

$$A = \{z \geq -\varepsilon\} - K .$$

The lower boundary of A is the plane $P' = \{z = -\varepsilon\}$ with n line segments $\beta_1, \ldots, \beta_n$ removed. Let B_i be a solid rectangular box whose top fits on P' and surrounds β_i. But we remove β_i itself from B_i, and (in order that B_i contain $*$) adjoin an arc running from the top, straight to $*$, missing K. The B_i may be taken to be disjoint from one another. Finally, let



$$C = \text{the closure of everything below } A \cup B_1 \cup \ldots \cup B_n, \text{ plus an arc to } * .$$

7. EXERCISE : Verify that $\pi_1(A)$ is a free group generated by $x_1, \ldots, x_n$.

Now we investigate the effect of adjoining B_1 to A. B_1 itself is simply-connected, and $B_1 \cap A$ is a rectangle minus β_1, plus the arc to $*$, so $\pi_1(B_1 \cap A)$ is infinite cyclic, with generator y. As is clear from the picture, when y is included in A, it becomes the word $x_k x_1^{-1} x_k^{-1} x_2$ (here we are assuming the crossing is of the first type). Thus, by Van Kampen, $\pi_1(A \cup B_1)$ has generators $x_1, \ldots, x_n$ and the single relation $x_k x_1^{-1} x_k^{-1} x_2 = 1$. This is equivalent to $x_k x_1 = x_2 x_k$, which is r_1. Thus

$$\pi_1(A \cup B_1) = (x_1, \ldots, x_n; r_1) .$$

Similarly, adjoining B_2 , we argue that

$$\pi_1(A \cup B_1 \cup B_2) = (x_1, \ldots, x_n; r_1, r_2) ,$$

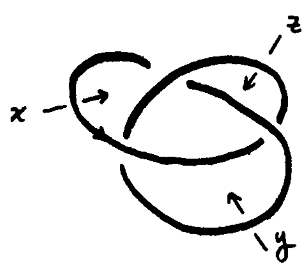
et cetera, so that

$$\pi_1(A \cup B_1 \cup \ldots \cup B_n) = (x_1, \ldots, x_n; r_1, \ldots, r_n) .$$

Finally, adjoining C to $A \cup B_1 \cup \ldots \cup B_n$ has no effect on the fundamental group, since both C and $C \cap (A \cup B_1 \cup \ldots \cup B_n)$ are simply-connected.

This completes the proof, except for the observation that one of the r_i (say r_n) is redundant. To see this, work in $S^3 = R^3 + \infty$. Let $A' = A + \infty$ and $C' = B_n \cup C + \infty$. It is clear that $A' \cup B_1 \cup \ldots \cup B_{n-1} \cup C' = S^3 - K$, $\pi_1(A') = \pi_1(A)$, and adjoining $B_1, \ldots, B_{n-1}$ has the same effect as before. But now we note that $C' \cap (A' \cup B_1 \cup \ldots \cup B_{n-1})$ is simply-connected, being a 2-sphere minus an arc, and so is C' . Thus we reach the same conclusion without adjoining the relation r_n .

8. EXAMPLE : We recompute the group of the trefoil using the Wirtinger method.

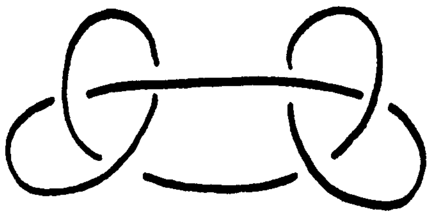


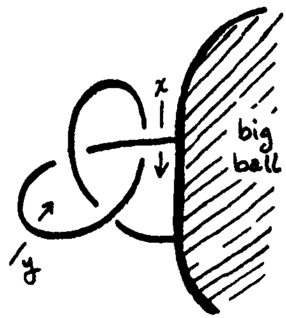
We have generators x, y, z and relations $xz = zy$, $yx = xz$. The second may be used to eliminate $z = x^{-1}yx$, which converts the first relation to $yx = x^{-1}yxy$. Thus we have another presentation for the trefoil group

$$G_{2,3} = (x, y; xyx = yxy).$$

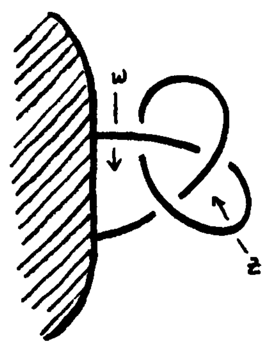
9. EXERCISE : Show directly that this is equivalent to the presentation $(a, b; a^2 = b^3)$.

10. EXAMPLE : The square knot





We may use a short-cut by considering the complement of :



It is clear that this has the homotopy type of the complement of the trefoil. Likewise for the complement of :

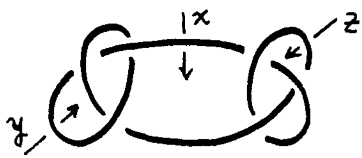
The union of these complements gives the complement of the square knot, so we use Van Kampen's theorem to see that :

Group of the square knot

$$= (x,\ y,\ w,\ z;\ xyx=yxy,\ wzw = zwz,\ x = w)$$

$$= (x,\ y,\ z;\ xyx = yxy,\ xzx = zxz).$$

11. EXAMPLE :



Group of the <u>Granny knot</u>

$$= (x,\ y,\ z;\ xyx = yxy,\ xzx = zxz)$$

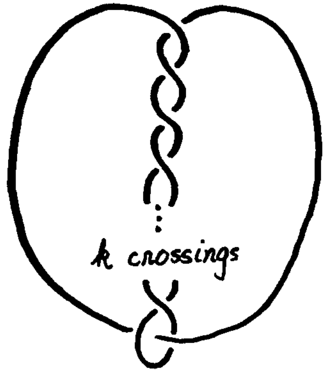
is isomorphic with the group of the square knot.

It happens that, in fact, the square and granny knots are <u>not</u> equivalent although the methods we have discussed so far do not distinguish them. When we do this (see 8E15) we will have established

12. THEOREM : The group of a knot is not a complete knot invariant (that is, $K \rightsquigarrow \pi_1(R^3 - K)$ is not one-to-one).

13. NOTE : The complements of the square and granny knots are actually not homeomorphic, as is shown by Fox [1952] using "peripheral structure" of π_1

14. EXERCISE : Compute a presentation for the group of the knot shown here. Show that it has a presentation with only two generators and one relation.



15. REMARK : Given two knot group presentations, it is often quite difficult to prove that they present non-isomorphic groups. Later we will develop knot invariants which are much more readily compared to distinguish knots and links.

E. REGULAR PROJECTIONS. The usual way of describing a knot is by drawing a picture, as described above. That this algorithm applies to arbitrary tame knots, is the object of the following exercises.

Let K be a polygonal knot in R^3 . Let P be any plane and $p : R^3 \longrightarrow P$ the orthogonal projection. Say that P is regular for K provided every $p^{-1}(x)$, $x \in P$, intersects K in 0, 1 or 2 points and, if 2, neither of them is a vertex of K .

1. EXERCISE : Given any polygonal K and plane P, one can make P regular for K by arbitrarily small perturbations of either P or K .

2 EXERCISE : Let K be a polygonal knot with vertices $v_0, \ldots, v_r$. Then there exists a positive number $\varepsilon = \varepsilon(K)$ such that whenever $v'_0, \ldots, v'_r$ are points in R^3 with $|v_i - v'_i| < \varepsilon$ for all i, the polygon $K' = v'_0 v'_1 \ldots v'_r v'_0$ is also a knot, and is ambient isotopic to K.

3. EXERCISE : If P is regular for K, then K is ambient isotopic to a knot of the type described in the section on the Wirtinger presentation.

4. DEFINITION : The <u>deficiency</u> of a group presentation equals the number of generators minus the number of relations.

5. COROLLARY : Every tame knot group has a finite presentation of deficiency one

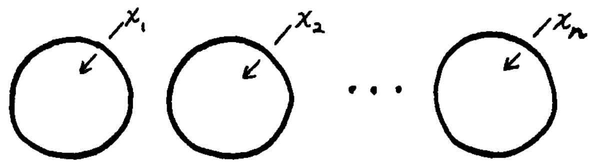
6. EXERCISE : Use the Wirtinger algorithm to prove that the abelianization of any tame knot group is infinite cyclic. (This also follows from exercise 2E6, since π_1 abelianized is H_1 and holds for wild knots as well.)

7. EXERCISE : Show that no (tame) knot group has a presentation with deficiency two.

8. EXERCISE : Every tame knot in R^3 possesses a tubular neighbourhood, which itself is equivalent to a polyhedron in R^3. Components of a tame link in R^3 have disjoint tubular neighbourhoods.

F. COMPUTATIONS FOR LINKS. The argument establishing the Wirtinger presentation theorem adapts in an obvious manner to links.

1. EXAMPLE :

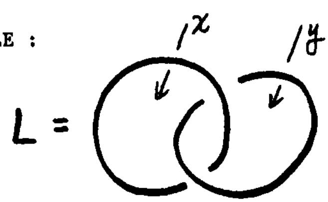


The trivial link (disjoint circles in a plane) of n components has group

$$(x_1, \ldots, x_n; -) = \text{free group of rank } n .$$

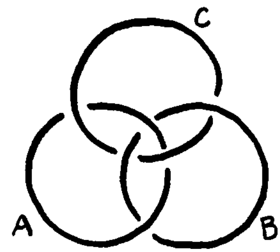
2. EXERCISE : Show that the complement of the trivial link in S^3 has the homotopy type of the wedge of n circles and $n-1$ copies of S^2. Generalize to higher dimensions.

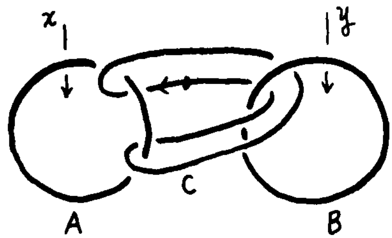
3. EXAMPLE :



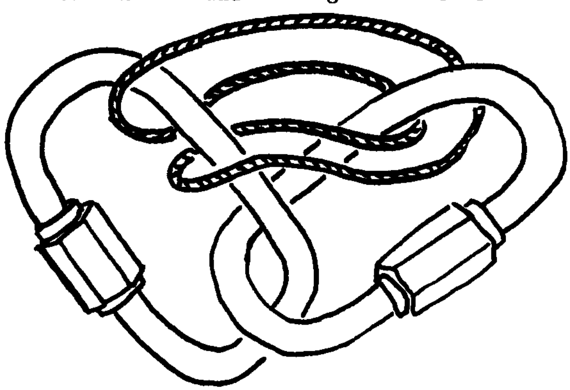
$(x, y; xy = yx)$ = free abelian group of rank 2 . Thus, it is not trivial since the free group of rank 2 is not abelian. Another way to get this presentation is to note that L is the standard link of $S_1^1 \cup S_2^1$ in $S_1^1 * S_2^1 \cong S^3$. There is an obvious deformation retraction of its complement onto the $S^1 \times S^1$ at the 1/2 level of the join, which has fundamental group $Z \oplus Z$.

4. EXAMPLE : The Borromean rings . Note that any two components form a trivial link. Nevertheless, it is not trivial. In fact any component, say C , is a homotopically nontrivial loop in the complement of the other two. For the link is equivalent to this link : And C represents a commutator $xyx^{-1}y^{-1}$ of the two generators of $\pi_1(R^3) - (A \cup B))$. Since this group is not abelian, $C \not\simeq *$.





5. A MAGIC TRICK. Take two rings which can be opened up (mountaineering caribiners would be fine) and arrange them as A and B above. Link a piece of string about them in the manner of C, and ask your audience to undo them.* Now link A and B together without disturbing C .

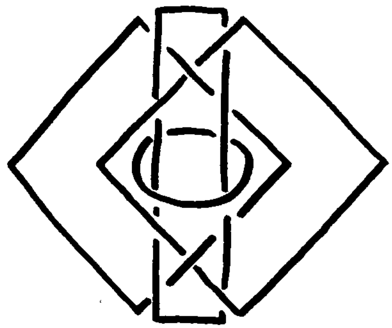


Then C can be slipped off A and B ! Explain.

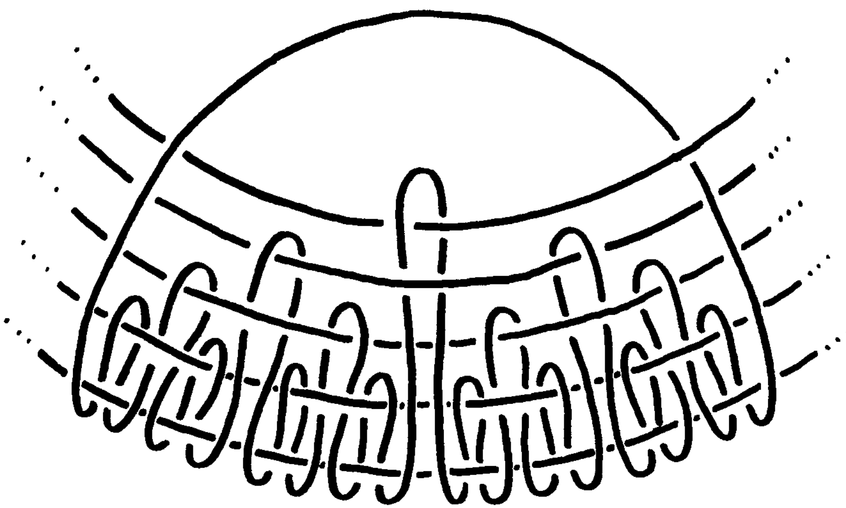
* without opening the rings, of course.

6. DEFINITION : If a link is nontrivial, yet every proper sublink is trivial we say that it has the Brunnian property. This is in honor of Hermann Brunn whose early contribution [1892] to knot theory included the pictures on this page.*

7. EXERCISE : The link of four components pictured at the right is Brunnian.



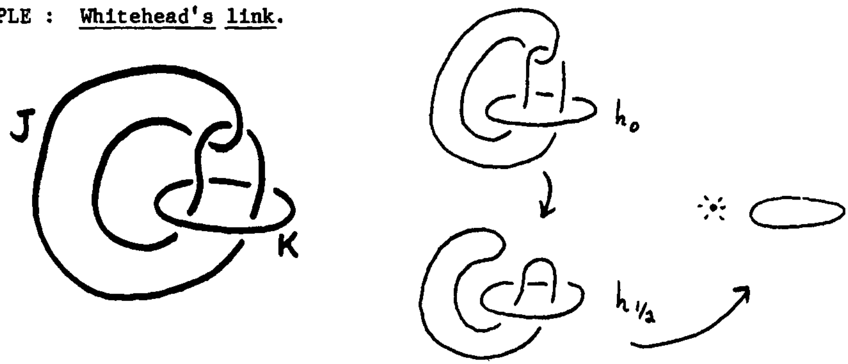
8. EXERCISE : The link pictured below consists of five round circles and a sixth component woven through them as shown. Prove that it is Brunnian and generalize to show there are Brunnian links of arbitrarily many components.



* Note that his version (Fig. 9) of the lower picture is incorrectly drawn.

9. DEFINITION : If $L = L_1 \cup \dots \cup L_n$ is a link with n components, we say that L_i is <u>homotopically unlinked</u> from the remaining components if there exists a homotopy h_t from the embedding L_i to the constant map such that the images of h_t and L_j are disjoint for all $t \in I$, $j \neq i$.

10. EXAMPLE : <u>Whitehead's link</u>.



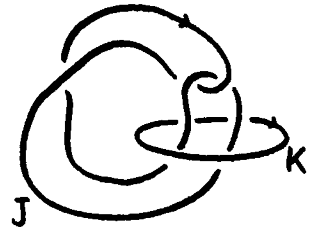
It is clear from the picture that J is homotopically unlinked from K . It is a little harder to see that K is homotopically unlinked from J . In fact, that follows from the symmetric nature of $J \cup K$.

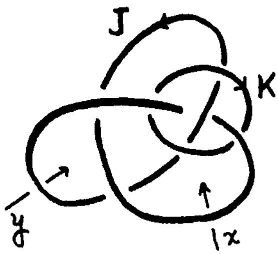
11. EXERCISE : There is a homeomorphism of R^3 interchanging J and K .

12. PROPOSITION : Homotopic linking (for links of two components) is <u>not</u> a symmetric relation.

We demonstrate this by a further example, differing only slightly from the previous one.

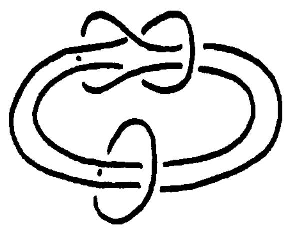
13. EXAMPLE : In the link $J \cup K$ pictured at the right, it is clear that J is homotopically unlinked from K. However, K links J homotopically!



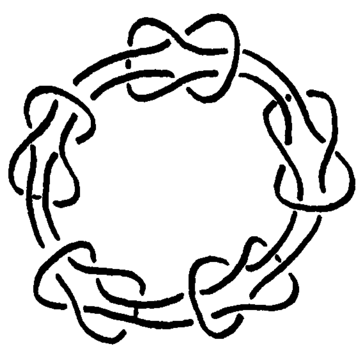


The link is equivalent to the one pictured at the left, and K represents the word xy^{-1} in the Wirtinger presentation of the group of J. But xy^{-1} is not the identity in that group. (Why?)

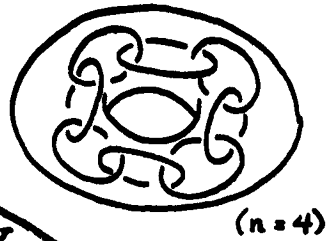
14. EXERCISE : Verify that the link at the right has one component homotopically linked with the other, but not vice-versa.

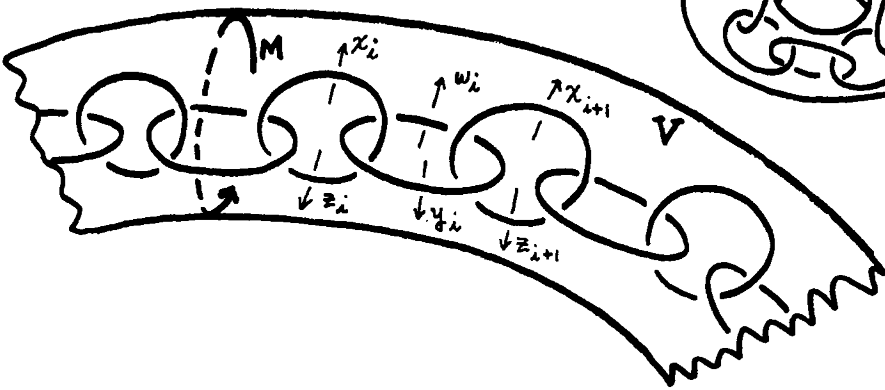


15. EXERCISE : Show that in this link of five components, each component is homotopically unlinked from the others. Show, moreover, that it has the Brunnian property. Does this generalize in the obvious way to any number (≥ 3) of components?



G. CHAINS. (Why is a chain as good as a rope ?)





Consider the above link C of $2n$ unknotted circles arranged in a chain running around the solid torus $V \cong D^2 \times S^1$, standardly embedded in R^3. Wirtinger presentation for $\pi_1(R^3 - C)$ has $4n$ generators x_i, y_i, z_i, w_i $(1 \le i \le n)$ and $4n$ relations $y_i x_i = x_i w_i$, $y_i x_{i+1} = x_{i+1} w_i$, $x_i y_i = y_i z_i$, $x_{i+1} y_i = y_i z_{i+1}$. $(n + 1 \equiv 1)$

1. PROPOSITION : The meridian M of V is not homotopically trivial in $R^3 - C$ or in $V - C$; in fact $[M]$ is of infinite order in $\pi_1(R^3 - C)$.

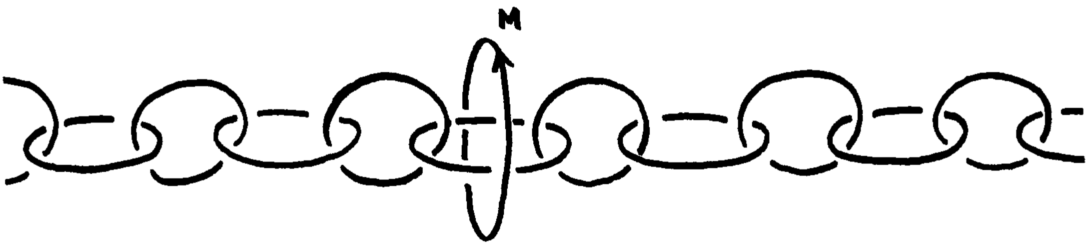
PROOF : The last phrase implies the rest. Consider the map of $\pi_1(R^3 - C)$ onto the free group $F(x,y)$ given by (for all i) :

$$x_i \to x \qquad y_i \to y \qquad z_i \to y^{-1}xy \qquad w_i \to x^{-1}yx .$$

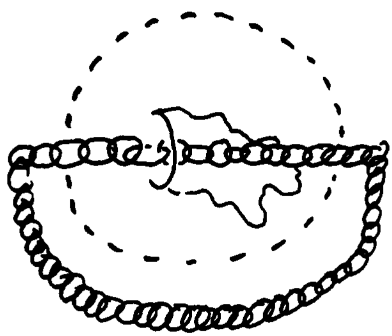
(Check that the relators are in the kernel.) Then the element $[M] = x_1^{-1} z_1$

is mapped to the commutator $x^{-1}y^{-1}xy$, which is of infinite order in $F(x,y)$. It follows that $[M]$ is of infinite order in $\pi_1(R^3 - C)$.

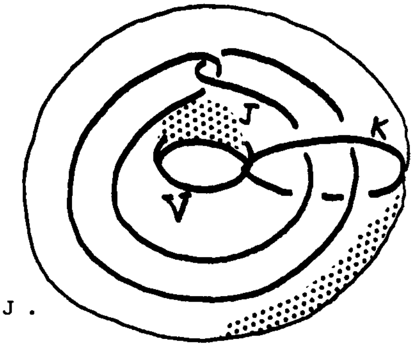
2. COROLLARY : The loop M is not contractible in the complement of the infinite chain pictured below.



PROOF : Suppose there were a homotopy shrinking M to a point, missing the chain. Since the image of the homotopy is a compact set, we may construct, as shown in the sketch at the right, a finite chain of the type discussed above, missing the homotopy and contradict the previous proposition.

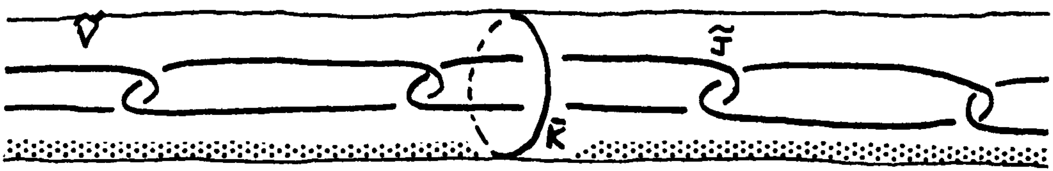


3. WHITEHEAD'S LINK AGAIN. Recall that each component of Whitehead's link is contractible in the complement of the other, in R^3. Now enclose the link $J \cup K$ in a solid torus V as pictured.



4. PROPOSITION : K is <u>not</u> contractible in $V - J$.

PROOF : Let $p : \tilde{V} \longrightarrow V$ be the universal cover, let $\tilde{J} = p^{-1}(J)$ and let

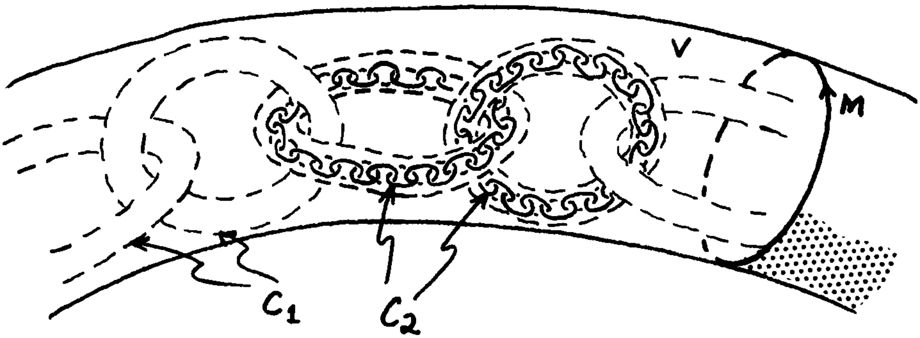


$\bar{K}$ be one component of $p^{-1}(K)$. A homotopy shrinking K in $V - J$ would lift (by the homotopy lifting property) to one which shrinks $\bar{K}$ to a point in $\tilde{V} - \tilde{J}$. But this is impossible by the above corollary, since (after appropriate twists) $\bar{K}$ and $\tilde{J}$ are situated exactly as M and the infinite chain.

5. EXERCISE : Show that neither component of the Whitehead link bounds a disk in R^3 which misses the other component.

6. EXERCISE : Prove a version of proposition 1 for chains in V with an odd number of components, either directly or using a double cover of V.

H. ITERATED CHAINS AND ANTOINE'S NECKLACE. In the previous section we constructed a chain C of $2n$ components in a solid torus V. Now thicken each component of C slightly to form a chain C_1 of $2n$ solid tori in V, where $\pi_1(V - C_1) \cong \pi_1(V - C)$ via inclusion. In each component of C_1, construct a smaller chain of solid tori embedded in that component in exactly the same manner as C_1 in V, as illustrated. Let C_2 denote the union of these smaller solid tori.



Now construct in each component of C_2 a similar chain of solid tori and call their union C_3. Continue this process a countable number of times to obtain a sequence $C_1 \supset C_2 \supset C_3 \supset \ldots$ of compact sets (each C_i being a union of solid tori) taking care that the diameters of the components of C_i tend to zero as $i \to \infty$. A neat way of describing this is to let $f : C_1 \to V$ be a map which is a homeomorphism (simply an enlargement) when restricted to any component of C_1. Then let $C_2 = f^{-1}(C_1) = f^{-2}(V)$, and in general $C_i = f^{-i}(V)$. The intersection

$$A = \bigcap_{i=1}^{\infty} C_i$$

is called Antoine's necklace. It is a nonempty compact subset of R^3.

1. EXERCISE : Show that Antoine's necklace is homeomorphic with the well-known Cantor space obtained by repeatedly deleting middle thirds from the interval $[0,1]$. In fact the same is true of any space obtained from a metric space X by taking the intersection of subsets $X \supset X_1 \supset X_2 \supset \ldots$ such that (1) each X_i is compact and has a finite number of components (2) each component of X_i contains at least two components of X_{i+1} and (3) the components of X_i all have diameter $< \epsilon_i$, for some sequence ϵ_i which converges to zero.

In the remainder of this section we'll investigate the fundamental group of $\mathbb{R}^3 - A$.

2. LEMMA : The inclusion homomorphism $\pi_1(\partial V) \xrightarrow{i_*} \pi_1(V - C_1)$ is injective.

PROOF : Since $\partial V \cong S' \times S'$, $\pi_1(\partial V)$ is a free abelian group with generators λ (longitude) and $\mu = [M]$ (meridian) so a general element of $\pi_1(\partial V)$ is of the form $\lambda^r \mu^s$. Suppose $i_*(\lambda^r \mu^s) = 1$. Since in V itself $\lambda^r \mu^s$ is homotopic with λ^r we conclude that r must be zero. But by proposition 1, μ is of infinite order in $\pi_1(V - C_1)$ so also $s = 0$.

3. COROLLARY : For each component $C_{1,j}$ of C_1 $(j = 1, \ldots, 2n)$ the inclusion $\partial C_{1,j} \subset C_{1,j} - \overset{\circ}{C}_2$ induces injective fundamental group homomorphisms.

4. EXERCISE : Show also that the inclusion homomorphism $\pi_1(\partial C_{1,j}) \to \pi_1(V - \overset{\circ}{C}_1)$ is injective.

Now we are prepared to apply the special case of Van Kampen's theorem (see appendix A) which says that in the usual diagram of inclusion homomorphisms, if the upper two maps are injective,

$$\begin{tikzcd} & \pi_1(X_1 \cap X_2) \arrow[dl] \arrow[dr] & \\ \pi_1(X_1) \arrow[dr] & & \pi_1(X_2) \arrow[dl] \\ & \pi_1(X) & \end{tikzcd}$$

then so are the other two. Then we may apply 3 and 4 above to conclude that $\pi_1(V-\overset{\circ}{C}_1) \to \pi_1((V-\overset{\circ}{C}_1) \cup (\overset{\circ}{C}_{1,1}-C_2))$ is injective. In other words adding one component of $C_1-\overset{\circ}{C}_2$ to $V-C_1$ has simply enlarged the fundamental group.

5. EXERCISE : Investigate the effect of adding each of the remaining components in turn and conclude that the inclusion homomorphism

$$\pi_1(V-C_1) \to \pi_1(V-C_2)$$

is injective. Likewise with $\mathbb{R}^3$ in place of V. What about the problem of base points?

This argument may be applied again and again to show that all these inclusion homomorphisms are injective:

$$\pi_1(V-C_1) \to \pi_1(V-C_2) \to \pi_1(V-C_3) \to \cdots$$
$$\pi_1(R^3-C_1) \to \pi_1(R^3-C_2) \to \pi_1(R^3-C_3) \to \cdots$$

So we may interpret these as inclusions of subgroups. It is also clear that they are proper inclusions, i.e. the groups become larger and larger.

6. PROPOSITION : Each inclusion homomorphism $\pi_1(R^3-C_i) \to \pi_1(R^3-A)$ is injective. (Similarly with V replacing R^3.)

PROOF : Suppose a loop α in R^3-C_i is homotopic to a point in R^3-A. Then the image of the homotopy is a compact set and we may conclude that α shrinks to a point in R^3-C_j for some large enough $j > i$. But $\pi_1(R^3-C_i) \to \pi_1(R^3-C_j)$ is injective and we conclude that α must have been trivial already in R^3-C_i.

7. COROLLARY : The meridian M of V is not homotopically trivial in R^3-A.

Similarly, any loop in R^3-A, being compact, lies in R^3-C_i for large enough i, and we conclude the following.

8. PROPOSITION : The group $\pi_1(R^3-A)$ is the infinite union of the ascending chain of its subgroups $\pi_1(R^3-C_1) \subset \pi_1(R^3-C_2) \subset \cdots$.

9. COROLLARY : The group $\pi_1(R^3-A)$ is not finitely generated.

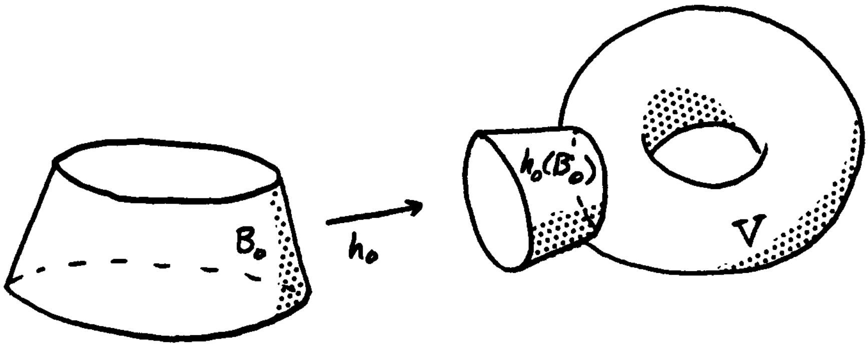
PROOF : If it were, by compactness all the finitely many generators would lie in R^3-C_i for large enough i. Then we could conclude that the naturally included $\pi_1(R^3-C_i)$ is the whole of $\pi_1(R^3-A)$. But this is absurd, since the inclusions of 8 are proper.

The results of this section may be summarised as follows.

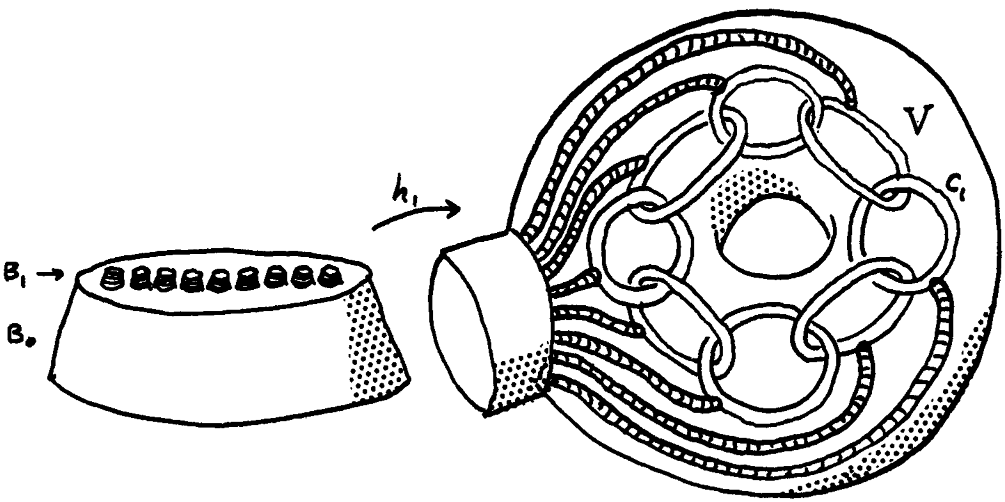
10. THEOREM : There exists a Cantor set A (Antoine's necklace) embedded in R^3 so badly that the fundamental group of its complement is nontrivial, in fact not finitely generated.

11. EXERCISE : Show, by contrast, that $H_1(R^3-A)$ is zero.

I. ANTOINE'S HORNED SPHERE. In a celebrated paper [1924]2 , J.W. Alexander constructed a topological 2-sphere in 3-space whose exterior was not simply-connected. This paper is sandwiched between two other papers by him, one proving that polyhedral 2-spheres are unknotted in R^3 (generalized Schönflies theorem) and the other describing yet another pathological 2-sphere in R^3 using the Antoine necklace construction. Here is the construction (as elaborated by de Rham in his "Lectures on algebraic topology," Tata Institute lecture note series.)

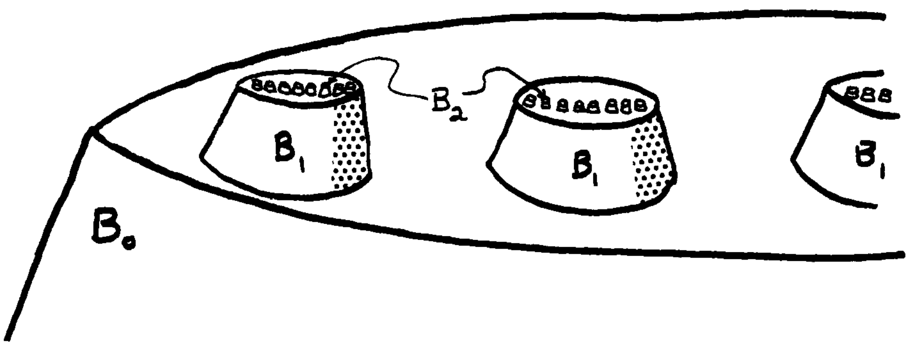


Let B_o be a truncated cone ($\cong B^3$) and embed B_o in R^3 via h_o so that the top of B_o goes onto a disk in ∂V and the rest is exterior to V (V as in the previous section, containing $C_1, C_2, \ldots$).



Now place $2n$ miniature copies of B_o on top (one for each component of C_1) and call their union B_1. Extend h_o to $h_1 : B_o \cup B_1 \to R^3$ which embeds each component of B_1 in $\overline{V - C_1}$ so that the tops go to disks, one in the boundary of each component of C_1.

Since C_2 lies in each component of C_1 exactly as C_1 lies in V, we can imitate the above extension to obtain an extension h_2 of h_1 mapping $(2n)^2$ tiny truncated cones B_2 homeomorphically into $\overline{C_1 - C_2}$, so that their tops go on the boundaries of the C_2's.



Continue this inductively to define embeddings $h_i : B_o \cup \dots \cup B_i \to R^3$. Thus we have an embedding $h : B = \bigcup_1^\infty B_i \to R^3$ which extends to an embedding (check this) $h : \overline{B} \to R^3$. Clearly h takes the Cantor set $\overline{B} - B$ homeomorphically onto the Antoine necklace A. It is easy to see that $\overline{B}$ is homeomorphic with a 3-ball, so $h(\overline{B})$ is a 3-ball in R^3, we call its boundary <u>Antoine's horned sphere</u>.

PROPOSITION : The outer complement of Antoine's horned sphere (i.e. $R^3 - h(\overline{B})$) is not simply-connected.

In fact, a meridian of V, which misses $h(\overline{B})$, cannot be contracted in $R^3 - A$, much less $R^3 - h(\overline{B})$. The same goes for meridians of any component of any C_i, so <u>the group of the outer complement is not even finitely generated</u>.

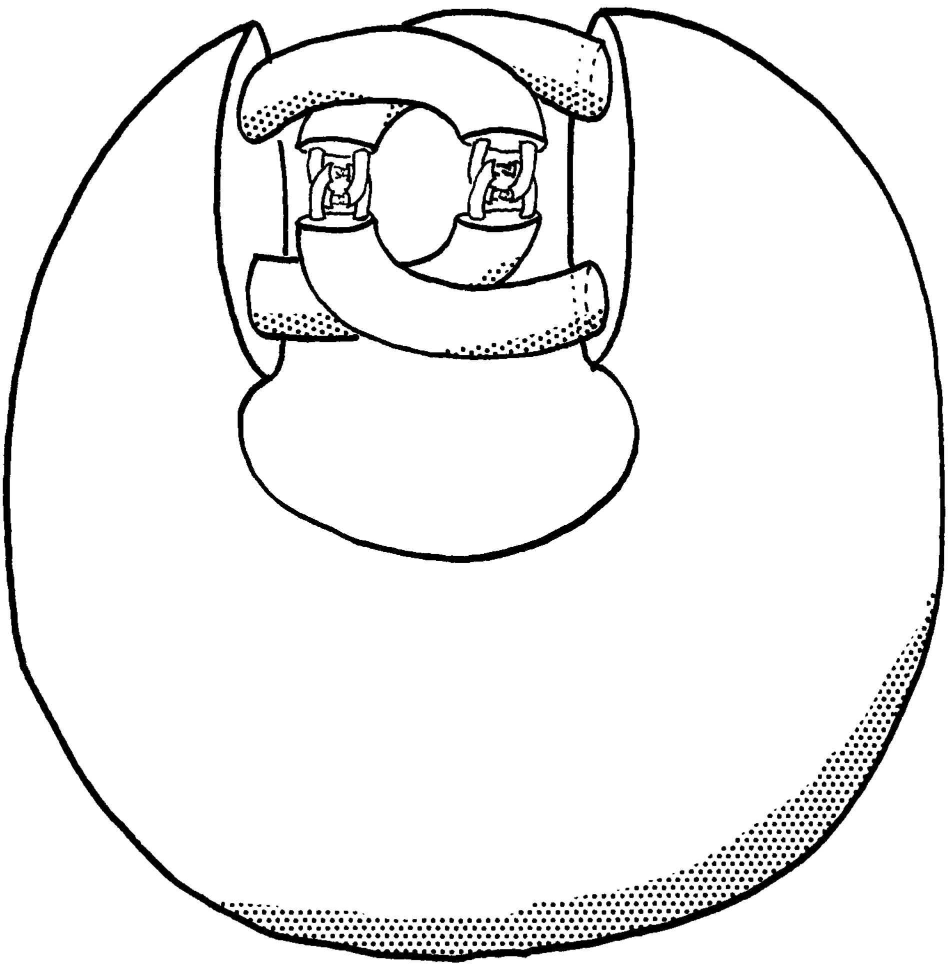
2. THEOREM : There exists a 2-sphere in R^3 which is not the boundary of a 3-ball in R^3.

PROOF : Just turn the horned sphere inside-out by reflecting R^3 about some (round) 2-sphere.

3. PROPOSITION : There exists a wild knot in R^3 whose group is not finitely generated, yet this knot bounds a topological disk in R^3.

PROOF : Just take the image under h of a simple closed curve J in $\partial\overline{B}$ which contains all the points of $\overline{B} - B$. Then $h(J)$ even lies on a 2-sphere in R^3, and divides it into two disks.

4. EXERCISE : Show that this knot actually bounds a disk whose interior is bicollared. (Compare this with the basic unknotting theorem of chapter 2.)

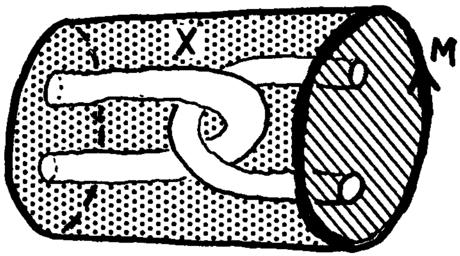


ALEXANDER'S HORNED SPHERE

5. EXERCISE : Verify that the solid object indicated above (a countable union) is homeomorphic with the ball B^3, hence its boundary is a sphere.

6. EXERCISE : The preceding page has a picture of the first four stages of the construction of Alexander's horned sphere.

(a) Give a more precise description of this by expressing its outer complement as a union $V \cup X_1 \cup X_2 \cup \ldots$ of a solid torus V and countably many copies X_i of the space X pictured here (solid cylinder minus two tubes):



(b) Argue that $[M]$ has infinite order in $\pi_1(X)$ and conclude that the inclusion homomorphisms

$$\pi_1(V) \to \pi_1(V \cup X_1) \to \pi_1(V \cup X_1 \cup X_2) \to \ldots$$

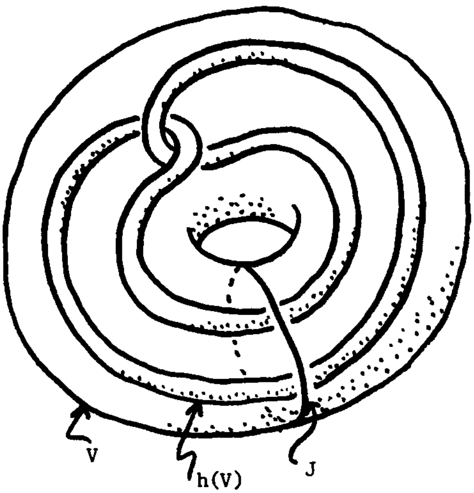
are all injective.

(c) Show that, like Antoine's sphere, its outer complement is not simply-connected. Moreover its fundamental group isn't finitely generated.

7. EXERCISE : Show that the set of bad (= non-locally-flat) points of both horned spheres is a Cantor set. [Hint: a locally flat point $x \in \Sigma^2 \subset R^3$ has neighbourhoods U such that components of $U - (\Sigma \cap U)$ are simply-connected

8. EXERCISE : Show that for Alexander's horned sphere, the complement (in R^3) of the bad points is simply connected. Thus it is inequivalent to the Antoine sphere.

Here's yet another infinite construction in S^3 , which provides an interesting phenomenon in the theory of 3-manifolds, introduced by J. H. C. Whitehead [1935]. Whitehead's manifold is an open 3-manifold which is simply-connected, yet is topologically distinct from Euclidean 3-space. A series of exercises will establish its important properties.



Let $h:S^3 \to S^3$ be a homeomorphism which carries the "standard" solid torus V in S^3 onto the solid torus $h(V) \subset V$ as in the picture at the left. It is clear that such a homeomorphism exists, although an explicit formula is unnecessary. The sequence $V, h(V), h^2(V), h^3(V), \ldots$ is a decreasing sequence of solid tori whose thicknesses we assume convergent to zero. Define

$$X = \bigcap_{i=0}^{\infty} h^i(V)$$

and call $W = S^3 - X$ the Whitehead manifold.

9. EXERCISE : X is compact and connected, but not locally connected. W is a connected open 3-manifold.

10. EXERCISE : There is no bicollared 3-ball in S^3 which contains the meridian of V and misses the set X . Therefore W is not homeomorphic with R^3 .

11. EXERCISE : W is simply-connected. Show, moreover, that $W \times R^1 \cong R^4$ and conclude that W is contractible.

J. APPLICATION OF π_1 TO HIGHER-DIMENSIONAL KNOTS.

Emil Artin [1925] described two methods for constructing knotted 2-spheres in R^4 ; in both case the group of the knot is easy to compute. Another method, drawing level curves, is amply described in Fox's Quick Trip (1962). Further methods will be described in later chapters.

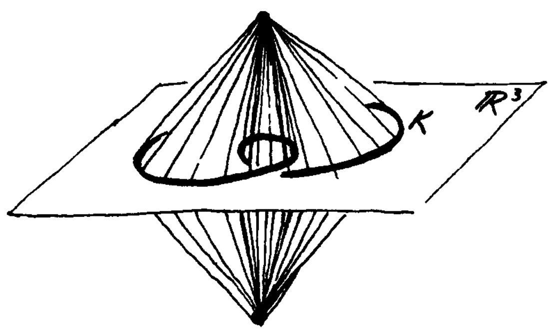
1. SUSPENSION. If K is a knotted 1-sphere in $R^3 = \{x_4 = 0\} \subset R^4$, draw a straight line segment from each point of K to the point $(0, 0, 0, 1)$ and also to $(0, 0, 0, -1)$. The union of these is a 2-sphere in R^4 , called the <u>suspension</u> of K . An equivalent way of constructing the suspension (in S^4) is to simply use the suspension of the map

$$K : S^1 \longrightarrow S^3$$

which yields a map :

$$\begin{array}{ccccc} \Sigma K : & \Sigma S^1 & \longrightarrow & \Sigma S^3 \\ & \| & & \| \\ & S^2 & & S^4 \end{array}$$

defined in the obvious manner.



2. PROPOSITION : The group of ΣK is isomorphic with the group of K , in fact the inclusion map $S^3 - K \to S^4 - \Sigma K$ is a homotopy equivalence.

This is easy, since there is an obvious homeomorphism $S^4 - \Sigma K \cong (S^3 - K) \times (-1, 1)$. Therefore, in fact, the higher homotopy groups are isomorphic, too.

In the same way, a knot of any dimension may be suspended to a knot of the same <u>codimension</u>, but one higher dimension. This may be iterated to arbitrarily high dimensions.

3. THEOREM : For each $n \geq 1$ there are infinitely many inequivalent knots of S^n in S^{n+2} (or R^{n+2}).

Suspension has one major defect; the suspended knot fails to be locally flat at the two 'vertices', unless the original knot is trivial.*

4. DEFINITION : A manifold pair $K^k \subset M^m$ is <u>locally flat</u> at $x \in K$ provided there exists a closed neighborhood N of x in M such that $(N, K \cap N)$ is homeomorphic, as a pair, with the standard ball pair (B^m, B^k).

Related to this is the problem of smoothness. If K were a differentiable knot, one might like to make ΣK differentiable. There is no problem "rounding the corners" at K itself, but there is no reasonable way to define a tangent plane at the vertices. Another shortcoming of suspension is that is does not work for links (why?).

* this is the PL version of the sort of pathology exhibited by the horned spheres -- lack of a product neighbourhood. In chapter 8 we discuss the problem of removal of such 'singularities'.

5. SPINNING. Artin's second method of constructing higher-dimensional knots uses a process of rotation. In R^4, consider the subsets

$$R^3_+ = \{(x_1, x_2, x_3, 0);\ x_3 \geq 0\}\ , \quad \text{which has 'boundary'}$$

$$R^2 = \{(x_1, x_2, 0, 0)\}$$

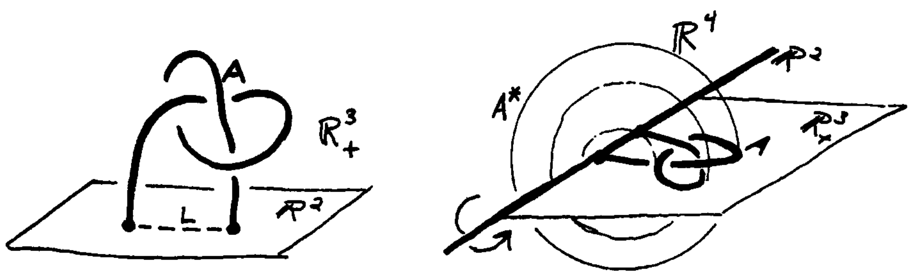
We can spin any point $x = (x_1, x_2, x_3, 0)$ of R^3_+ about R^2 according to the formula

$$x_\theta = (x_1, x_2, x_3\cos\theta, x_3\sin\theta)\ .$$

Define the <u>spin</u>, X^*, of any set $X \subset R^3_+$ to be

$$X^* = \{x_\theta;\ \ x \in X,\ 0 \leq \theta \leq 2\pi\}\ .$$

To obtain a knot in R^4, choose an arc A in R^3_+ with endpoints in R^2. Then A^* is a 2-sphere in R^4, called a <u>spun knot</u>.



The next proposition shows that the group of A^* is isomorphic with $\pi_1(R^3_+ - A)$, which in turn is isomorphic with the group of the knot $A \cup L$ in R^3, where $L \subset R^2$ is the segment joining the endpoints of A.

6. PROPOSITION : The spun knot A^* in R^4 has the same group as the knot $A \cup L$ in R^3 .

Now, if A is PL and intersects R^2 transversally, it is easy to see that A^* is locally flat. A^* may be made either C^∞ or PL .

7. PROPOSITION : There exist infinitely many inequivalent locally flat knots of S^2 in R^4 .

8. PROPOSITION : If X is a path-connected open subset of R^3_+ , such that $X \cap R^2 \neq \emptyset$, then inclusion induces an isomorphism $i_* : \pi_1(X) \xrightarrow{\cong} \pi_1(X^*)$.

PROOF : Clearly X^* is open in R^4 . Using a basepoint p in $X - R^2$, let α be a loop in X^* based at p . By the lemma below we may assume that α misses R^2 (a line and plane intersect in R^4 only by accident). Define a projection $x \longrightarrow \bar{x}$ from X^* onto X by

$$\bar{x} = (x_1, x_2, \sqrt{(x_3^2 + x_4^2)}, 0) .$$

Then α projects to a loop $\bar{\alpha}$ in X .

(a) $\alpha \simeq \bar{\alpha}$ in X^* , so i_* is surjective. For we may define a continuous function $\theta : [0, 1] \to (-\infty, \infty)$ satisfying $\theta(0) = 0$ and $\bar{\alpha}(t)_{\theta(t)} = \alpha(t)$.† Then each point may be rotated through the angle $-\theta(t)$ to provide a homotopy from α to $\bar{\alpha}$. However the endpoint $\alpha(1)$ need not stay fixed, but may describe a loop β rotating p through an angle $2N\pi$. But the fact that p may be connected via a path in X to R^2 shows that $\beta \simeq *$ in X^* . Thus $\bar{\alpha} \simeq \alpha\beta \simeq \alpha$.

† the subscript refers to the definition of x_θ on the previous page.

(b) If γ is a loop in X such that $\gamma \simeq *$ in X^*, then $\gamma \simeq *$ in X; thus i_* is injective. One simply projects the homotopy.

9. LEMMA : If $\alpha : [0, 1] \to U \subset R^n$ is a path in an open subset of Euclidean space, then α is homotopic in U to a path whose image is piecewise-linear.

PROOF : Using compactness, construct a partition $0 = t_0 < t_1 < \ldots < t_r = 1$ and a cover of $\alpha[0, 1]$ by round open balls $B_1, \ldots, B_r$ such that $\alpha[t_{i-1}, t_i] \subset B_i$.

In each B_i, $\alpha|[t_{i-1}, t_i]$ is homotopic to the straight line segment from $\alpha(t_{i-1})$ to $\alpha(t_i)$. Piecing these together provides the result.

K. UNSPLITTABLE LINKS IN 4-SPACE.

1. DEFINITION : A link $L = L_1 \cup L_2$ of two components in R^n is <u>splittable</u> if there are disjoint topological n-balls B_1, B_2 in R^n such that L_i lies interior to B_i $(i = 1, 2)$.

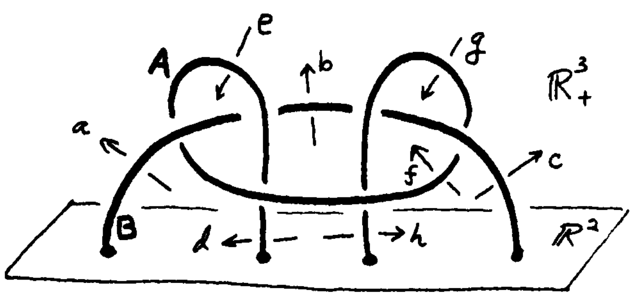
2. PROPOSITION : If a link is splittable, then each component is homotopically unlinked from the other.

Just contract L_1 in B_1 or L_2 in B_2 .

3. REMARK : Let $h : B^n \to R^n$ be an embedding. It is a consequence of the generalized Jordan curve theorem (theorem 2F1) that the topological boundary of $h(B^n)$ in R^n coincides with its 'abstract' boundary $h(S^{n-1})$ and the topological interior is $h(B^n - S^{n-1})$.

4. EXERCISE : If $L = L_1 \cup L_2$ is a splittable link in R^n , then $\pi_1(R^n - L)$ is the free product of the knot groups $\pi_1(R^n - L_1)$ and $\pi_1(R^n - L_2)$.*

5. EXAMPLE (VAN KAMPEN [1928], ZEEMAN [1960]) :



* by a non-canonical isomorphism.

The spun link $A^* \cup B^*$ consists of two unknotted 2-spheres in R^4. It follows that each is homotopically unlinked from the other. That $A^* \cup B^*$ is not splittable follows from the proposition:

6. PROPOSITION : $\pi_1(R^3_+ - (A \cup B))$ is not free.

We will reproduce almost verbatium from [Zeeman 1960] an elegant proof attributed to J.A. Green, which uses the lower central series.

7. DEFINITION : If G is a group, the lower central series of G is defined inductively by $G_0 = G$ and $G_i = [G, G_{i-1}]$, the subgroup consisting of all products of commutators $[g, h] = ghg^{-1}h^{-1}$; $g \in G$, $h \in G_{i-1}$. One checks easily that $G_0 \supset G_1 \supset G_2 \supset \cdots$, each G_i is normal in G and that the quotient group G_i/G_{i+1} is in the center of G/G_{i+1}.

8. LEMMA : If G is free, then $G_\infty = \bigcap_1^\infty G_i$ consists of only the identity element.

A heuristic proof is that the length of nonempty words in G_i must increase with i, so only the empty word (identity) survives in them all. An alternative proof will be given at the end of this section.

9. PROOF OF PROPOSITION 6:* "A moment's thought will convince the reader that the customary method [Wirtinger presentation] of using segments and cross-overs to give generators and relations for $\pi = \pi_1(R^3_+ - (A \cup B))$ is still valid, although we are dealing with curves in the half space R^3_+ instead of loops

* quoted from Zeeman [1960]

in the whole space. Therefore we have generators $a, b, \ldots, h$ corresponding to the oriented segments shown, and relations

(1) $be = ea$ (3) $fc = cg$ (5) $df = fe$

(2) $bg = gc$ (4) $fa = ae$ (6) $hf = fg$.

Let $x = c^{-1}a$ replace the generator a . Relations 5, 6 render the generators d, h redundant. Relations 2, 3, 4 may be used to express b, f, e in terms of c, g, x :

(2) $b = gcg^{-1}$ (3) $f = cgc^{-1}$

(4) $e = a^{-1}fa = x^{-1}c^{-1}fcx = x^{-1}gx$.

The remaining relation 1 becomes

(1) $gcg^{-1}x^{-1}gx = x^{-1}gxcx$, which becomes

$$c[g, x] = [g, x]cx , \quad \text{or}$$

$$[c, [g, x]] = x .$$

We therefore have a presentation of π as

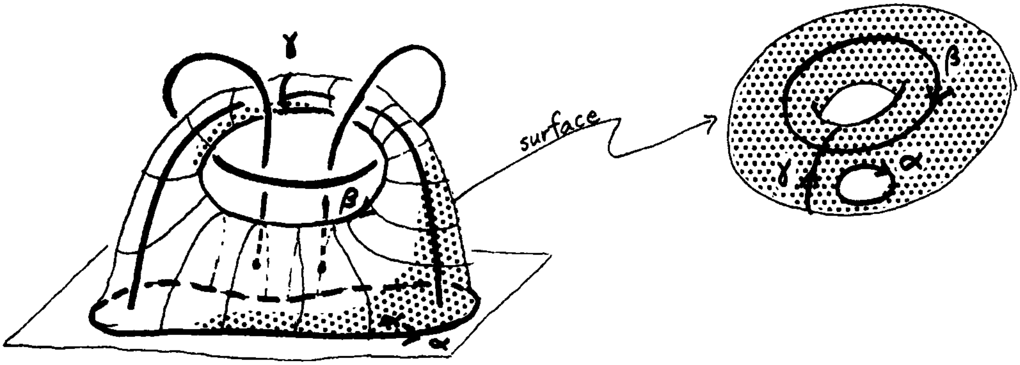
$$\pi = (c, g, x; [c, [g, x]] = x) .$$

From the relation, x is preserved in the lower descending central series. Suppose now that π is free. Then the intersection of the terms of the lower descending central series is the unit element, and so $x = 1$. Therefore the relation $[c, [g, x]] = x$ implies the relation $x = 1$. But this is not so, as is shown by the model* the permutation group on three

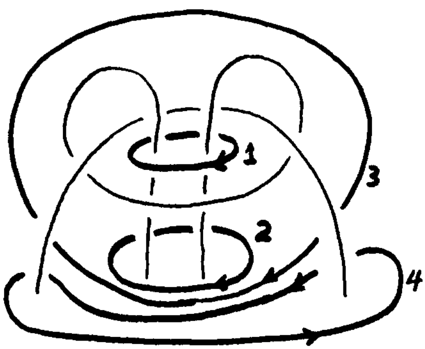
* meaning homomorphic image

elements, taking x of order 3, and $c = g$ of order 2. Therefore π is not free."

10. REMARK : The fact that x is preserved in the lower central series can be seen from the following picture. The curves α and β are both conjugates of x (depending on how the basepoint is chosen). By observing the surface drawn in (punctured torus), it is clear that $\alpha = \beta^{-1}\gamma^{-1}\beta\gamma$ up to conjugation.



Stages in a free homotopy from β to x in $R^3_+ - (A \cup B)$, showing that they are conjugate in $\pi_1(R^3_+ - (A \cup B))$.



11. PROOF OF LEMMA 8 : We wish to show that the intersection of the lower central series of a free group G contains only the identity element. For simplicity, and since that's all we need, suppose G is the free group on two generators x, y. Following an idea of Magnus, consider the ring of formal power series in the non-commuting variables X and Y with integer coefficients:

$$Z[[X,Y]] = \left\{ \sum_{n=0}^{\infty} a_n X^{p_n(1)} Y^{p_n(2)} X^{p_n(3)} \cdots \right\}$$

where $n \to p_n(\ )$ is a fixed bijection from non-negative integers onto <u>finite</u> sequences of non-negative integers. Addition and multiplication are the obvious ones. Now define a multiplicative homomorphism h from G into the units of $Z[[X,Y]]$ by

$$h(x) = 1 + X \qquad h(y) = 1 + Y$$

It is easily checked that h is injective. The image of the commutator $xyx^{-1}y^{-1}$ may be calculated:

$$\begin{aligned} h(xyx^{-1}y^{-1}) &= (1+X)(1+Y)(1-X+X^2-X^3+\cdots)(1-Y+Y^2-Y^3+\cdots) \\ &= 1 + XY - YX + (\text{terms of degree } \geq 3) \end{aligned}$$

It is easy to verify in a similar manner that if v and w are words in x, y and

$$h(v) = 1 + (\text{terms of degree } \geq m)$$

$$h(w) = 1 + (\text{terms of degree } \geq n)$$

then $h(vwv^{-1}w^{-1}) = 1 + (\text{terms of degree } \geq m + n)$.

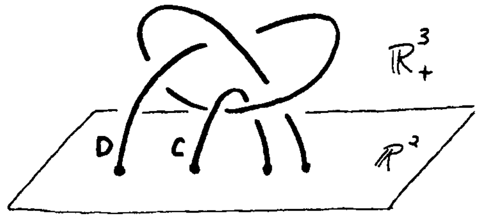
Now an easy induction shows that

$$h(G_n) \subset \{1 + (\text{terms of degree } \geq n + 1)\}$$

It follows that any element common to all the G_n must be in the kernel of h . Hence it is the identity of G , as was to be shown.

12. EXERCISE : Show that the commutator subgroup of the free group on two generators is itself a free group with a countable infinity of generators. [Hint: look at covering spaces.] Conclude that the 2-generator case proved above implies the case of finitely or countably many generators.

13. EXAMPLE (ANDREWS AND CURTIS [1959]) :



This link of two 2-spheres in R^4 is obtained by spinning the arcs C and D pictured here. Thus the link consists of a trivially knotted 2-sphere C^* and a spun trefoil D^*. In $S^4 = R^4 + \infty$, the complement of C^* has the homotopy type of S^1. Thus $\pi_2(S^4 - C^*) = 0$ $[= \pi_2(R^4 - C^*)]$, and we conclude :

14. PROPOSITION : D^* is homotopically unlinked from C^*.

However, Andrews and Curtis show the following, analogous to example F13. We'll defer the proof until section 5E.

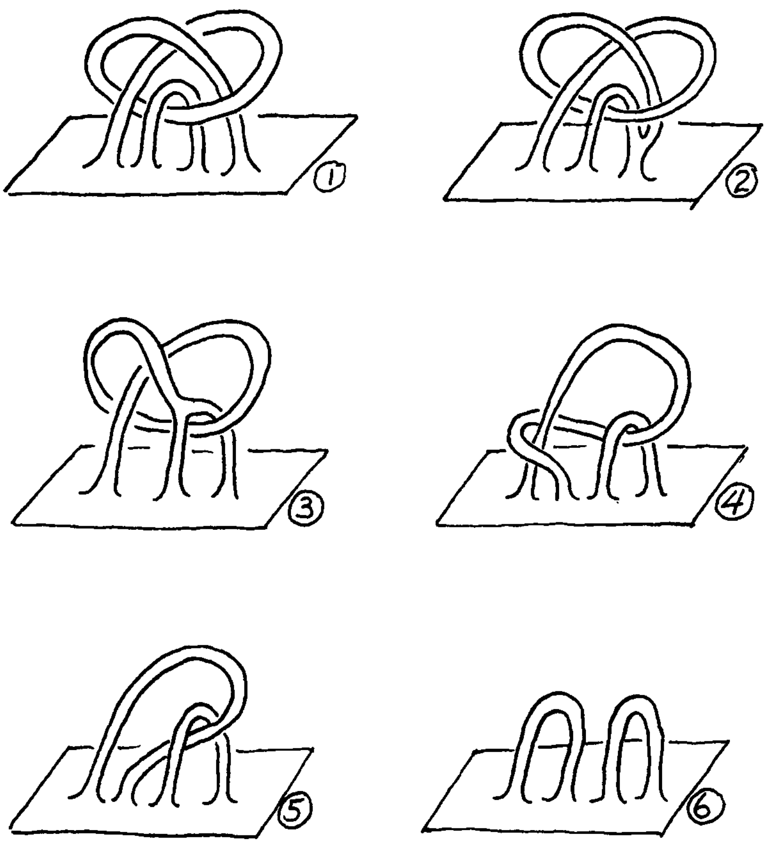
15. PROPOSITION : C^* is homotopically linked with D^*.

16. COROLLARY : $\pi_2(R^4$ - spun trefoil$) \neq 0$.

Thus, unlike suspension, spinning introduces higher homotopy (we will see that $\pi_2(R^3 - K) = 0$ for any classical knot).

17. COROLLARY : $C^* \cup D^*$ is an unsplittable link.

We can show this last assertion in another way. Suppose $C^* \cup D^*$ splits. Then its group would be the free product of the trefoil group and Z, hence not a free group (why?). But in fact $\pi_1(R^4 - (C^* \cup D^*)) \cong \pi_1(R^3_+ - (C \cup D))$ is a free group on two generators, as may be seen from the following homeomorphic deformation of the complement of a thickened $C \cup D$ in R^3_+.



L. GENERALIZED SPINNING. This section describes two ways to generalize the concept of spinning, discussed earlier. One extends it to still higher dimensions, and the other introduces a "twist" while spinning.

The easiest way to describe higher-dimensional spinning that I know is the following, which I learned from Cameron Gordon. Let A^n be an n-ball (or disjoint collection of n-balls) properly embedded in the m-ball B^m, so that $\partial A \subset \partial B$ and $\mathring{A} \subset \mathring{B}$. To define the p-<u>spin</u>, of the pair (B,A), consider the ball D^{p+1}. Then we have a natural inclusion of $A \times D^{p+1}$ in $B \times D^{p+1}$. Taking boundaries we define

$$A^* = \partial(A \times D^{p+1}) \quad \text{and} \quad B^* = \partial(B \times D^{p+1}).$$

In other words, the p-spin of the pair is $(B^*, A^*) = \partial\{(B,A) \times D^{p+1}\}$. It is evident that A^* is a knot (or link) of (n+p)-dimensional spheres in the (m+p)-sphere B^*. If A is locally flat, so is A^* in B^*.

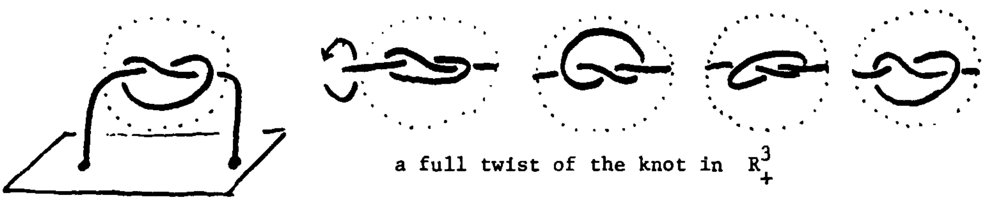
1. EXERCISE: Show that for the case n=1, m=3, and p=1 this construction coincides (up to homeomorphism) with the spinning construction of J5, if the latter is compactified with a point at infinity.

2. EXERCISE : Describe precisely the statement that p-spinning the pair (B,A) consists of sweeping interior points through a p-sphere, while holding the boundary fixed.

3. EXERCISE : Extend the results of sections J and K to higher dimensions. In particular, construct nontrivial knots and links of S^n in S^{n+2}.

4. EXERCISE : Describe a natural inclusion of the pair (B,A) in the pair (B^*,A^*) and show that in the case of codimension 2, this inclusion induces an isomorphism $\pi_1(B - A) = \pi_1(B^* - A^*)$.

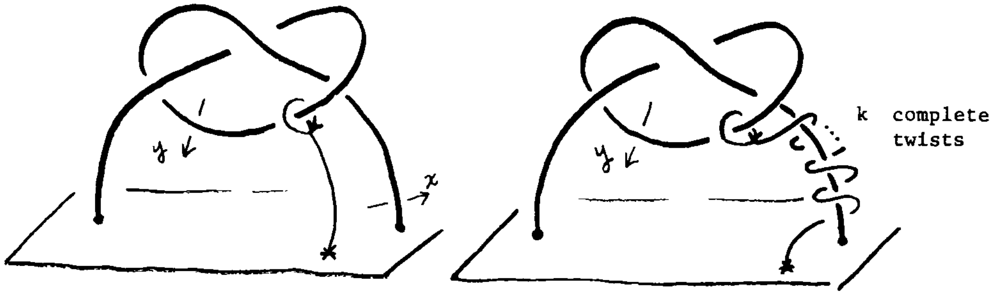
To describe <u>twist-spinning</u> it is easier to revert to our original definition of spinning (see J5). There we considered an arc A in R^3_+ and spun it about the R^2 "axis" into R^4, through an angular variable θ. Consider the twisting motion of A within R^3_+ suggested by the following sequence of pictures. The endpoints are held fixed throughout and in the end, A returns to its original position.



a full twist of the knot in R^3_+

Now as R^3_+ sweeps through the angle θ from 0 to 2π, let A make, say, k complete twists within the R^3_+. By this combined motion in R^4 it is clear that A sweeps out a 2-sphere, which is locally flat if A was locally flat. We'll call this 2-sphere A^*_k.

Let's investigate the knot group of the twist-spun knot A^*_k. In the following picture is drawn a loop in $R^3_+ - A$, which is therefore also a loop in $R^4 - A^*_k$. As R^3_+ sweeps through R^4, let this loop stay within the R^3_+, but we must twist its tail as A twists. This ensures that there is a homotopy within R^4, missing A^*_k, between the loop in the right-hand picture and the original loop on the left.



$\theta = 0$ $\theta = 2\pi$

We have just argued that, using the notation of the pictures, the equation

$$y = x^k y x^{-k}$$

must be a true relation in the knot group $\pi_1(R^4 - A_k^*)$. Regardless of what the knot may be, we can see in this way that the generator x which is a meridian at the "foot" of the arc A has the property that its k^{th} power commutes with all the generators represented by loops in $R_+^3 - A$. The arguments of section J may be adapted to the following.

5. EXERCISE : A presentation for $\pi_1(R^4 - A_k^*)$ may be obtained from a presentation for $\pi_1(R_+^3 - A)$ simply by adjoining additional relations saying that x^k commutes with all the other variables, where x is as described above.

6. EXAMPLE : Consider the 2-twist spin of the trefoil. Using the generators x and y in the above picture, we see that

$$\pi_1(R^4 - A_2^*) = (x,y;\ xyx=yxy,\ x^2y=yx^2).$$

An easy calculation shows that the relations imply $x^2 = y^2$. We can also argue that xy^{-1} has order three in this group. First we calculate $xy^{-1}xy^{-1}xy^{-1} = x(yx^{-2})xy^{-1}xy^{-1} = x^{-2}xyxy^{-1}xy^{-1} = x^{-2}yxyy^{-1}xy^{-1} = x^{-2}yx^2y^{-1}$ But xy^{-1} is itself nontrivial in our group, as may be seen by mapping it onto the permutation group S_3 via $x \to (01)$, $y \to (12)$.

Since the 2-twist spun trefoil has knot group with elements of order three, we have indeed produced a new knot which cannot be produced by ordinary spinning of the trefoil. The trefoil's group has no elements of finite order (why?).

7. EXAMPLE : The 1-twist spin of the trefoil has group, according to our analysis:

$$\pi_1(R^4 - A_1^*) = (x,y;\ xyx=yxy,\ xy=yx).$$

It is easy to see that this group is just infinite cyclic! Likewise for the reverse spin with $k = -1$.

8. REMARK : Twist-spinning is studied in considerable detail in Zeeman [1965] . Among other things he shows that any ± 1-spun knot is in fact unknotted in R^4 ! Here's an exercise for the ambitious reader; if you despair consult Zeeman's article.

9. EXERCISE : The group of the 5-twist spun trefoil is a direct sum $\pi_1(R^4 - A_5^*) \cong Z \oplus G$ of an infinite cyclic group and a group G of order 120 .

CHAPTER FOUR

THREE-DIMENSIONAL PL GEOMETRY

A. THREE THEOREMS OF PAPAKYRIAKOPOULOS.

> "Ein Flächenkomplex C_2 möge ganz in Inneren einer homogenen Mannigfaltigkeit (n>2) liegen. Auf dem C_2 möge die Kurve k ein Elementarflächenstück E_2' begrenzen. Hat E_2' auf seinem Rande keine Singularitäten, dann begrenzt k in der M_n auch ein vollig singularitätenfreies Elementarflächenstück."
>
> M. Dehn [1910]

This is the notorious Dehn's Lemma. Dehn noted that $n = 3$ is the difficult case and proceeded with a lengthy geometric proof which was found, unfortunately, to have serious gaps. It is a testament to the ingenuity of two men, as well as to the hazards of "cut and paste" geometric arguments that (1) Dehn's lemma is true and (2) it took nearly a half century for a correct proof to be found, by Papakyriakopoulos [1957]. In the remainder of this chapter all maps and spaces will be assumed to be piecewise-linear (PL). That is, each space is considered to be a simplicial complex, and each map has the property that, after some subdivision of the domain and range, it sends simplexes linearly onto simplexes. Thanks to theorems of Moise [1952] and Bing [1959] -- every 3-manifold is homeomorphic with a simplicial complex -- and various approximation theorems, the PL theory and the topological case in 3 dimensions coincide except for the local pathology possible in the latter (e.g. horned spheres). E. g. Moise [1954] shows that PL knot types in S^3 are in natural 1-1 correspondence with (tame) topological type

Here, in English, is a statement of Dehn's lemma. A proof is outlined in Appendix B, thanks to guest lectures to my class by David Gillman.

1. DEHN'S LEMMA: Suppose M^3 is a 3-manifold and $f : D^2 \to M^3$ is a map of a disk with no singularities on ∂D^2 (i.e. $x \in \partial D^2$, $x \neq y \in D^2 \Rightarrow f(x) \neq f(y)$) Then there exists an embedding $g : D^2 \to M^3$ with $g(\partial D) = f(\partial D)$.

If M has nonempty boundary and $f(\partial D) \subset \partial M$, we may use a collar of the boundary to push the singularities of f away from the boundary and obtain the following.

2. COROLLARY: Suppose $J \subset \partial M$ is a simple closed curve in the boundary of a 3-manifold M and that J is homotopically trivial in M. Then J bounds a properly embedded disk D in M (i.e. $\partial D \subset \partial M$, $\mathring{D} \subset \mathring{M}$, where $\circ$ indicates interior).

Another theorem due to Papakyriakopoulos is the loop theorem which says that a loop, essential in the boundary of a 3-manifold yet contractible in the manifold itself, may be replaced by a simple closed curve with the same property. We shall only need this in the following combination with Dehn's lemma.

3. DEHN'S LEMMA AND THE LOOP THEOREM: If M is a 3-manifold and the inclusion homomorphism $\pi_1(\partial M) \to \pi_1(M)$ has nontrivial kernel, then

there exists a 2-disk $D \subset M$ such that ∂D lies in ∂M and represents a nontrivial element of $\pi_1(\partial M)$.

The great power of these theorems lies in their deriving a geometric conclusion from an algebraic hypothesis. Here is another of the same type, also due to "Papa."

4. THE SPHERE THEOREM: If M is an orientable 3-manifold with $\pi_2(M) \neq 0$, then there is a 2-sphere S embedded in M which is not contractible in M .

The sphere theorem is even harder than the other two, and I won't attempt to go through a proof. An excellent reference for all of these theorems is Stallings' book [1971]. The next few sections deal with applications of these theorems to knot theory.*

* As this isn't a course in PL topology, we'll take for granted certain standard items -- regular neighbourhoods, general position, transversality and the like. In the next section, for example, we use the fact that a PL disk in S^3 is flat (in the sense of the basic unknotting theorem 2F7). Readers familiar with PL topology might take this as an exercise.

B. THE UNKNOTTING THEOREM. As the square and granny knots show us, the function $K \mapsto \pi_1(S^3-K)$ is not injective. On the other hand, the following application of Dehn's lemma says, loosely speaking, it has no kernel: only the trivial knot has "trivial" (abelian) knot group.

1. UNKNOTTING THEOREM: A tame knot $K \subset S^3$ is trivial if and only if $\pi_1(S^3-K)$ is infinite cyclic.

PROOF: One implication is clear. In light of theorem 2F7, we need only show, assuming K is PL and $\pi_1(S^3-K) \cong Z$, that K bounds a PL disk in S^3. Consider a tubular neighbourhood V of K with a preferred framing $V \cong S^1 \times D^2$, defining meridian and longitude on ∂V (cf. exercise 2E7). Then the longitude L of ∂V, being homologically trivial in $S^3 - \mathring{V} \simeq S^3 - K$, must also be homotopically trivial in $S^3 - \mathring{V}$. Apply Dehn's lemma to the loop L in the boundary ∂V of the manifold $S^3 - \mathring{V}$ and conclude that L bounds a disk in $S^3 - \mathring{V}$. Also there clearly is an annulus in V connecting L and K. The union of this annulus and the disk bounded by L give the desired PL disk in S^3 bounded by K. This proves the theorem, as well as the following.

2. THEOREM : K is knotted if and only if the inclusion homomorphism $\pi_1(\partial V) \to \pi_1(S^3 - \mathring{V})$ is injective.

3. REMARK : Recall that $S^3 - \mathring{V}$ is called the exterior of K (uniqueness of regular neighbourhoods implies that exterior is well-defined). The condition of the theorem above is sometimes referred to as <u>incompressibility</u> of the boundary of the exterior.

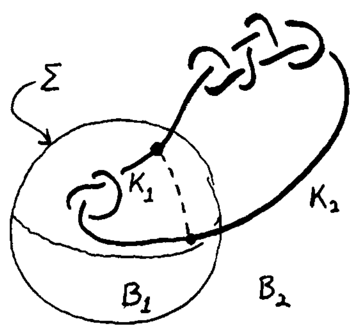
4. COROLLARY: Every knot group (except the trivial one, Z) contains a subgroup isomorphic with $Z \oplus Z$.

5. DEFINITION: The subgroup $i_*(\partial V)$ of $\pi_1(S^3-V) \cong \pi_1(S^3-K)$ is called th peripheral subgroup.

6. REMARK: Fox [1952] used peripheral subgroups to distinguish between sq and granny knots: although their groups are isomorphic, no isomorphis preserves peripheral subgroups.

7. NON-CANCELLATION THEOREM: Suppose a connected sum $K = K_1 \# K_2$ of two tame knots is unknotted. Then both K_1 and K_2 are themselves unknot

PROOF: The definition of connected sum entails a tame 2-sphere Σ dividing S^3 into two balls, B_1 containing K_1 and B_2 containing K_2 ; $K_1 \cap K_2$ is an arc in Σ ; K is the union of K_1 and K_2 , minus the interior of that arc. Now it's easy to see that



$$\pi_1(B_1-K) \cong \pi_1(S^3-K_1) \quad \text{and}$$

$$\pi_1(B_2-K) \cong \pi_1(S^3-K_2) \ .$$

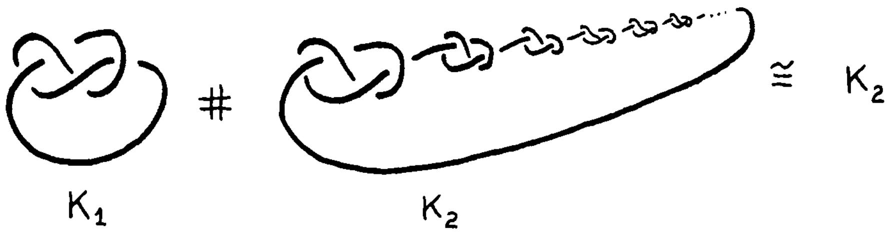
Consider the diagram of inclusion homomorphisms:

$$\begin{tikzcd}
 & \pi_1(\Sigma - K) \arrow[dl] \arrow[dr] & \\
\pi_1(B_1 - K) \arrow[dr] & & \pi_1(B_2 - K) \arrow[dl] \\
 & \pi_1(S^3 - K) &
\end{tikzcd}$$

It is clear that the two upper homomorphisms are injective. Referring to Appendix A , it follows that the other two homomorphisms are also injective. Therefore $\pi_1(S^3-K)$ has two subgroups, one isomorphic with $\pi_1(S^3-K_1)$ and the other with $\pi_1(S^3-K_2)$. Since K is assumed unknotted, we conclude certainly that $\pi_1(S^3-K_1)$ and $\pi_1(S^3-K_2)$ are abelian. Therefore they are infinite cyclic and the previous theorem implies that K_1 and K_2 are unknotted.

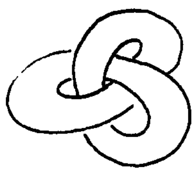
8. REMARK: There are several quite different ways to prove the non-cancellation theorem. One is an easy consequence of the additivity of the genus of knots, which we'll encounter in the next chapter. Another is a clever infinite construction illustrated in Fox's Quick Trip, due to Mazur and dubbed by some 'the Mazur swindle'. (Beware: there is a hidden assumption about PL homeomorphisms of S^3 in that proof.) Still another proof, credited to Conway, is outlined in Martin Gardner's Mathematical Recreations column in the Scientific American.

9. QUESTION: Is the equation $K_1 \# K_2 = K_2$ possible for tame nontrivial knots? It is possible for wild knots, for example:

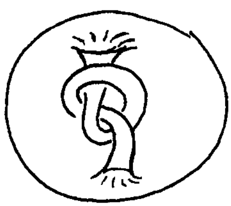


C. KNOTTING OF TORI IN S^3 .

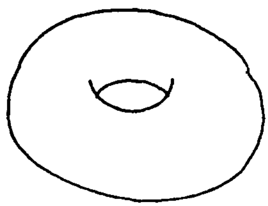
Suppose T is a PL (or tame) torus in S^3 . Then T divides S^3 into two parts: outside (containing ∞) and inside. The inside may be a solid torus and the outside a knot complement, or vice-versa, or both sides may be solid tori.



solid torus inside



outside

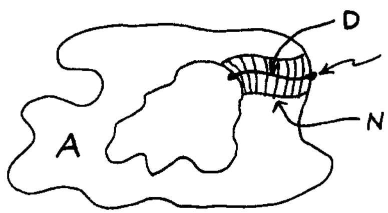


both sides

The question was raised earlier: "Can a tame torus be so badly knotted in S^3 that neither side is a solid torus?" The answer is no. The following theorem is credited to J. W. Alexander.

1. SOLID TORUS THEOREM: Any PL torus in S^3 bounds a solid torus on at least one side.

PROOF: Let A and B denote the two (closed) sides of T, so $S^3 = A \cup B$, $T = A \cap B$. If both inclusion homomorphisms $\pi_1(T) \to \pi_1(A)$ and $\pi_1(T) \to \pi_1(B)$ were injective we would be able to conclude (see Appendix A) that $Z \oplus Z \cong \pi_1(T) \to \pi_1(S^3) \cong 0$ is injective, an obvious absurdity. Choose notation then so that $\pi_1(T) \to \pi_1(A)$ has nontrivial kernel. By Dehn's Lemma and the Loop Theorem, some essential simple closed curve J in T bounds a PL disk D in A. Let N be a bicollar neighbourhood of D in A, so that $N \cap T$ is an annular neighbourhood of J in T. Now the boundary of $A - N$ is just the union of two disks on ∂N and the set $T - N$, which is an annulus (Why?). Thus $A - N$ is bounded by a PL 2-sphere in S^3, and the polyhedral Schönflies theorem implies that $A - N$ is a PL 3-ball. The proof is completed by the following exercise.

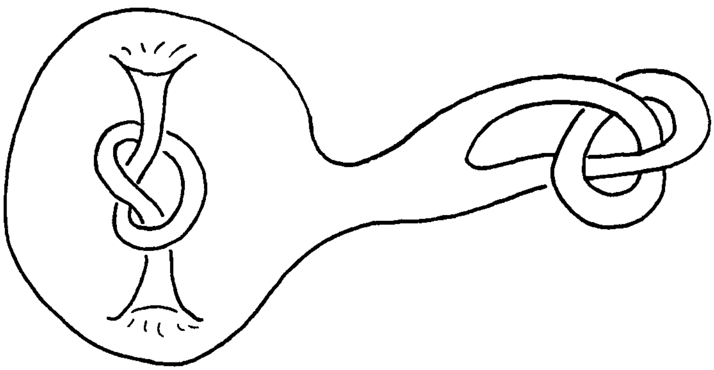


2. EXERCISE: If one attaches a $D^2\times[0,1]$ to a 3-ball by sewing $D^2\times 0$ and $D^2\times 1$ onto disjoint disks on the boundary, the result, if orientable, is homeomorphic with $D^2\times S^1$.

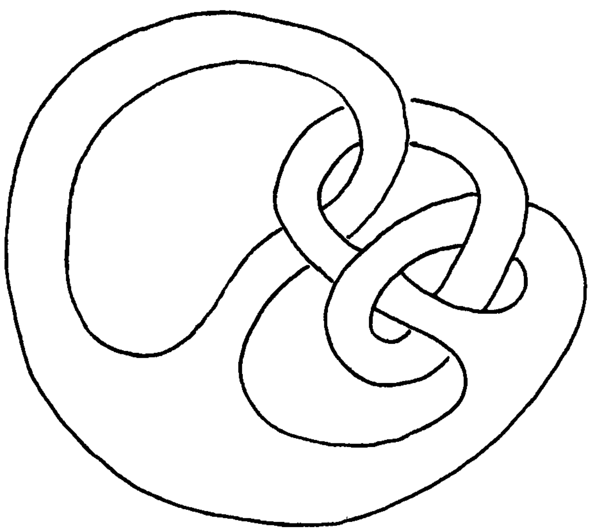
3. REMARK: The above theorem implies that the theory of (tame) knotted tori

in S^3 is equivalent to the theory of tame knots of S^1 in S^3.

4. EXERCISE: Show that the surface pictured below (of genus 2) in S^3 does not bound a handlebody on either side.



5. EXERCISE: This surface bounds handlebodies on both sides!



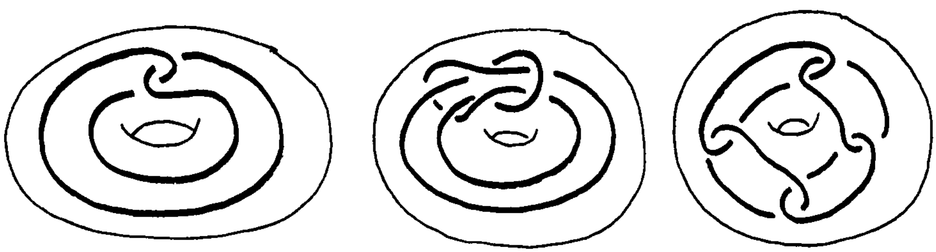
6. EXERCISE: Generalize the solid torus theorem to simply-connected manifolds thus: If M is a closed 3-manifold with $\pi_1(M) = 0$ and $T \subset M$ is a PL torus, then at least one side of T in M has infinite cyclic fundamental group. (You ought to argue first that T does, in fact, separate M.)

D. KNOTS IN SOLID TORI AND COMPANIONSHIP

A disk D in a solid torus $V = S^1 \times D^2$ is called <u>meridinal</u> if its boundary is a nontrivial curve in ∂V (hence a meridian). Let us call a closed subset $X \subset V$ <u>(geometrically)</u> <u>essential</u> in V if X intersects every PL meridinal disk in V.

1. EXERCISE: If X is non-contractible in V, then X is geometrically essential in V. However, X may be essential even when contractible in V.

2. EXERCISE: The following are geometrically essential subsets, though homotopically trivial.



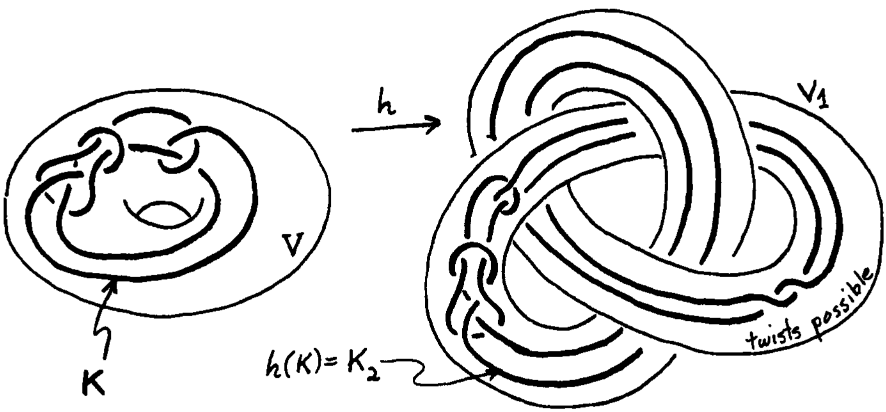
3. EXERCISE: Suppose X is a closed subset interior to V. Then the following are equivalent:

   (a) X is geometrically essential

   (b) there is no PL 3-ball B with $X \subset B \subset V$

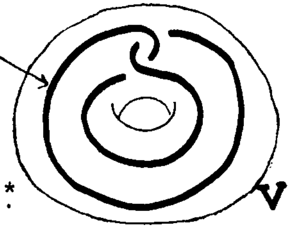
   (c) the inclusion homomorphism $\pi_1(\partial V) \to \pi_1(V-X)$ is injective.

Here is an important method of constructing knots. Let $K \subset V \subset S^3$ be a knot which is <u>geometrically essential</u> in a standardly embedded solid torus in the 3-sphere. Let $K_1 \subset S^3$ be another knot and let V_1 be a tubular neighbourhood of K_1 in S^3. Finally, let $h : V \to V_1$ be a homeomorphism and let K_2 be $h(K)$. We say K_1 is a <u>companion</u> of any knot K_2 constructed (up to knot type) in this manner.



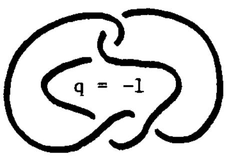
a knot K_2 with trefoil as companion

It is sometimes specified that h be <u>faithful</u>, meaning that h takes the preferred longitude and meridian of V respectively to the preferred longitude and meridian of V_1.



4. EXAMPLE : <u>Doubled knots</u>. Take K to be this knot. If h is faithful, we say that K_2 is an untwisted double of K_1. Otherwise, K_2 is a double with q twists, where $h(\text{longitude}) = \text{longitude} + q(\text{meridian})$.* This construction is due to J. H. C. Whitehead [1937]. See pictures 7B8.

5. EXAMPLE : If K_1 in the previous example is the unknot, then K_2 is called the <u>twist knot</u> with q twists. Some familiar twist knots:



trefoil

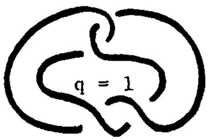
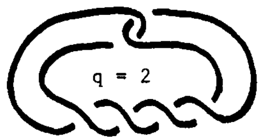


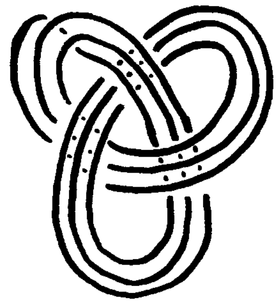
figure-eight



stevedore

6. EXAMPLE : <u>Cable knots</u>. Take K to be the (p,q) torus knot on (or better just under) the boundary of V, and require that h be faithful. Then K_2 is called the (p,q) cable based on its companion K_1, or simply a cable knot. We require $p \neq 0$ (why?), and if $p = \pm 1$, then K_2 is equivalent to K_1.

7. EXERCISE : Determine (p,q) for this cable on the trefoil. [Be careful; it's not obvious!]



8. EXERCISE : Both summands are companions of a composite knot.

* there is some ambiguity in this unless one makes certain conventions for orienting the longitudes and meridians.

It is easy to see that companionship is reflexive and transitive, in other words it is a partial ordering of the class of knot types. The unknot is a companion of every knot. On the other hand, the unknot has no other companions, by the following basic fact.

9. THEOREM : If K_1 is a companion of K_2, then the knot group of K_2 contains a subgroup isomorphic with the knot group of K_1.

PROOF : Referring to the definition of companionship, we have the following diagram of inclusion homomorphisms:

$$\begin{tikzcd} & \pi_1(\partial V_1) \arrow[dl] \arrow[dr] & \\ \pi_1(S^3-\mathring{V}_1) \arrow[dr] & & \pi_1(V_1-K_2) \arrow[dl] \\ & \pi_1(S^3-K_2) & \end{tikzcd}$$

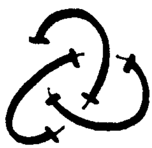
If K_1 is unknotted the theorem is easy; just take the cyclic subgroup of $\pi_1(S^3-K_2)$ generated by a meridian. If K_1 is knotted, then the upper left-hand map of the diagram is injective, by theorem B2. The upper right-hand map is also injective, by exercise 3, so the diagram represents an amalgamated free product. So $\pi_1(S^3-K_1) \cong \pi_1(S^3-\mathring{V}_1) \hookrightarrow \pi_1(S^3-K_2)$ and the theorem is proved.

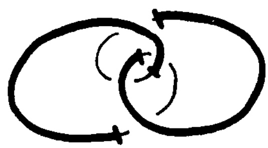
10. COROLLARY : Any knot with a nontrivial companion is itself nontrivial. (Note that this implies the non-cancellation theorem B7.)

11. QUESTION : If K_1 is a companion of K_2 and vice-versa, are they necessarily of the same knot type?

12. THE BRIDGE INDEX: This numerical invariant is introduced here because it was invented by Schubert [1954] largely to deal with companionship. In particular it is used to show that a knot has only finitely many companions. Let K be a tame knot and consider a planar picture (regular projection) of K. An overpass is a subarc of the diagram which contains an overcrossing but no undercrossing points, and the number of maximal overpasses is called the bridge number of the projection. The <u>bridge index</u> $b(K)$ of K is the least bridge number of all planar pictures representing a knot of type K. By convention the bridge index of the unknot is taken to be =1.

13. EXAMPLE: Two pictures of the trefoil, one with bridge number 3, the other 2. The next exercise shows the trefoil is a 2-bridge knot.





14. EXERCISE : $b(K) = 1$ if and only if K is the unknot.

15. EXERCISE : Any 2-bridge knot is prime (not a connected sum of two nontrivial knots).

16. EXERCISE : $b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$. In other words, the bridge index minus one is additive.

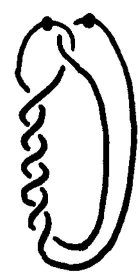
17. EXERCISE : Every nontrivial twist knot is a 2-bridge knot.

18. EXERCISE : If $b(K) = n$, then the knot group of K has a presentation with n generators and $n - 1$ relations.

19. RELATIVE MAXIMA : Closely related to the bridge index is the relative maximum number, or <u>crookedness</u> $\mu(K)$ of a knot, defined by Milnor [1950]. If K is parametrized by the variable vector $\bar{v}(t)$, $t \in S^1$, and $\bar{w}$ is a fixed unit vector in R^3, count the number of relative maxima of the projection function $\bar{w} \cdot \bar{v}(t)$. The minimum such number, over all directions $\bar{w}$ and all K within its knot type, is defined to be $\mu(K)$. Milnor shows that $2\pi\mu(K)$ is the infimum of the total curvature of K, as K varies within its knot type.

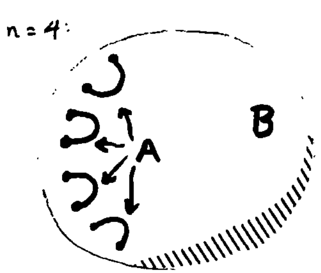
20. EXERCISE : For any tame knot K in R^3, $\mu(K) = b(K)$.

21. REMARK : This is often easier to estimate than is counting bridges. That a twist knot has crookedness at most 2 is easily seen from the picture at the right. The next exercise will be found challenging.



22. EXERCISE : If K_1 is a proper companion of K_2 (meaning they are inequivalent), then $b(K_1) < b(K_2)$.

Here is another interpretation of bridge number. Let A be a collection of n disjoint trivial spanning* arcs in $B = B^3$. Let A' and B' be another such pair. If ∂B and $\partial B'$ are attached by a homeomorphism which preserves setwise the ends of the arcs, $A \cup A'$ becomes a knot or link in the sphere $B \cup B'$.



23. EXERCISE : The minimum n for which the above construction yields a knot (in S^3) of the type of K is equal to the bridge index of K.

* e.g. take $A = (\text{ n points }) \times I$ in $B = D^2 \times I$

E. APPLICATIONS OF THE SPHERE THEOREM. Recall that the sphere theorem states that if some map of S^2 into an orientable 3-manifold is homotopically nontrivial, then one can find an embedding of S^2 with the same property.

1. ASPHERICITY THEOREM (PAPAKYRIAKOPOULOS): If $X = S^3 - K$ is the complement of a knot in S^3, then $\pi_i(X) = 0$, $i = 2,3, \ldots$.

PROOF: First we establish that $\pi_2(X) = 0$. If not, the sphere theorem guarantees a PL sphere S which is non-contractible inside X. But the PL Schönflies theorem says that the closures of both sides of $S^3 - S$, are 3-balls. Since K, being connected, lies entirely on one side of S, the sphere contracts to a point on the other side, hence in X, a contradiction. To establish that $\pi_3(X) = \pi_4(X) = \ldots = 0$, consider the universal covering space U of X, which has the same higher homotopy groups as X. Being an open 3-manifold, its homology groups $H_i(U)$ are zero for $i \geq 3$, as well as $i = 1,2$. By a theorem of Whitehead, then, all its homotopy groups also vanish, and we deduce the conclusion of the theorem.

2. REMARK: The theorem is <u>not</u> true if R^3 replaces S^3 (why?). We saw in the previous chapter that this doesn't generalize to higher dimensions. The example of Andrews and Curtis (see 3K16) shows that the 2-dimensional knot in R^4 obtained by spinning the trefoil has a complement with $\pi_2 \neq 0$*. The asphericity theorem may be restated thus: each classical knot complement is an Eilenberg-MacLane space $K(G,1)$. The Eilenberg-

* this applies to the complement in S^4 as well

MacLane spaces of type $K(G,n)$, $n \geq 1$ and G a given group, are spaces with $\pi_n \cong G$ and $\pi_i = 0$ for $i \neq n$. These are the 'building blocks' of homotopy theory. For example, if two simplicial complexes are Eilenberg-MacLane spaces of the same type, then they are homotopy equivalent.

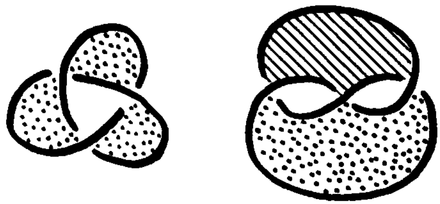
3. COROLLARY: Two tame knots in S^3 have homotopy-equivalent complements if and only if their knot groups are isomorphic.

4. EXERCISE : A link L in S^3 having two components is splittable if and only if $\pi_2(S^3 - L)$ is nontrivial.

CHAPTER 5. SEIFERT SURFACES

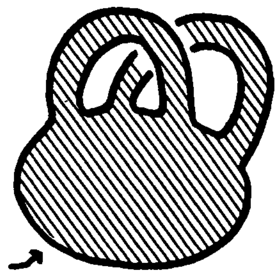
A. SURFACES AND GENUS. A <u>Seifert surface</u> for a knot or link $K^n \subset S^{n+2}$ is a connected, bicollared,* compact manifold $M^{n+1} \subset S^{n+2}$ with $\partial M = K$. For example, the trefoil bounds

a Möbius strip (not bicollared)



a Seifert surface

The Seifert surface pictured on the right side is actually homeomorphic with a punctured torus: T^2 minus an open disk. The skeptic ought to convince himself of this by pushing things around until the surface looks like a punctured torus. Of course the embedding might have to be changed in the process. An intermediate stage might look something like this.



Identifying a surface in the classical case $n=1$ can be made easier if we resort to the the well-known classification of compact 2-manifolds. For a good account of this, see Massey [1967]. Here is a summary. Let M be a compact 2-manifold. Since ∂M is a compact 1-manifold (disjoint union of S^1's) we may construct an associated <u>closed</u> 2-manifold $\hat{M}$ by abstractly sewing disjoint disks onto M, one on each boundary component.

* it follows that M must be orientable.

1. THEOREM (CLASSIFICATION OF 2-MANIFOLDS) : Every closed orientable connected 2-manifold is homeomorphic with one which appears in the table below, and is classified by its genus $g \geq 0$. Two compact connected 2-manifolds with boundary are homeomorphic if and only if they have the same number of boundary components and their associated closed 2-manifolds are homeomorphic.

manifold	S^2	T^2	$T^2 \# T^2$	$T^2 \# \cdots \# T^2$
genus	0	1	2	g
χ =	2	0	-2	2 - 2g

The Euler characteristic $\chi(M)$ may be computed in several ways. From a triangulation of M we can calculate

$$\chi(M) = \sum_{i=0}^{2} (-1)^i (\text{number of } i\text{-simplexes})$$

and from the integral homology groups we have

$$\chi(M) = \sum_{i=0}^{2} (-1)^i (\text{rank of } H_i(M)) = 2 - \text{rank of } H_1(M).$$

The genus of a 2-manifold M with boundary is simply defined to be the genus of its associated closed surface: $g(M) = g(\hat{M})$.

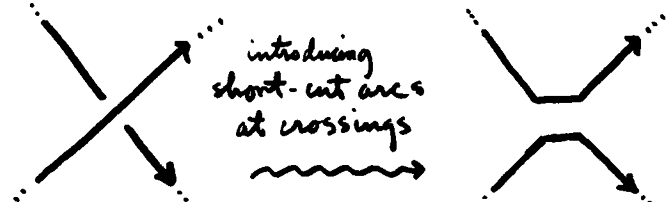
2. EXERCISE : $g(M) = 1 - \dfrac{\chi(M) + b}{2}$ where b = number of boundary components.

3. DEFINITION : The genus of a PL knot or link K^1 in R^3 or S^3 is the least genus of all its PL Seifert surfaces. Write $g(K) = g(M)$ where M is a 'minimal surface.' Genus is clearly a knot invariant.* The following assures that it is always defined.

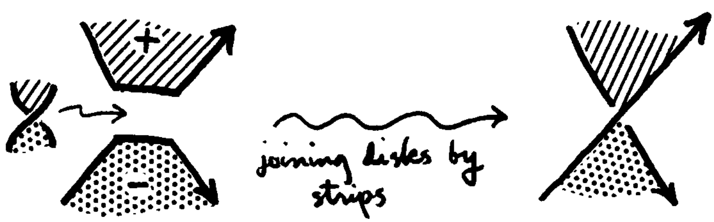
* by Kinoshita [1962] the 'PL' may be dropped without altering the genus.

4. EXISTENCE THEOREM : Every PL knot or link in R^3 or S^3 bounds a PL Seifert surface.

PROOF : Let L denote the link in R^3, assign each component an orientation, and examine a regular projection. Near each crossing point, delete the over- and undercrossings and replace them by 'short-cut' arcs in the projection plane as pictured, minding orientation.

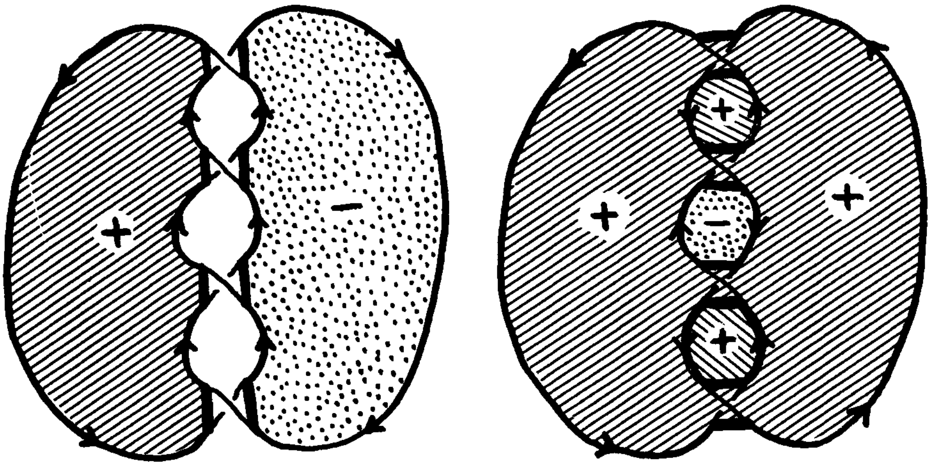


We now have a <u>disjoint</u> collection of simple closed oriented curves in the plane. Each bounds a disk in the plane and, although they may be nested, these disks can be made disjoint by pushing their interiors slightly off the plane, starting with innermost ones and working outward. Moreover these disks have bicollars which may be assigned a "+" and "-" side according to the convention, say, that the oriented boundary runs counterclockwise as seen from the + side. Now connect these disks together at the old crossings with half-twisted strips to form a 2-manifold M whose boundary is the original link L. If L was a knot, M is connected. Otherwise, join components by tubes



5. EXERCISE : Show that the orientation convention ensures that the bicollars on the disks extend compatibly to a bicollar for M.

6. EXAMPLE : For the 2-component link pictured below the algorithm gives two possible Seifert surfaces, depending on one's choice of orientations.



7. EXERCISE : One of these surfaces has genus one; the other has genus zero. The link, therefore, has genus zero.

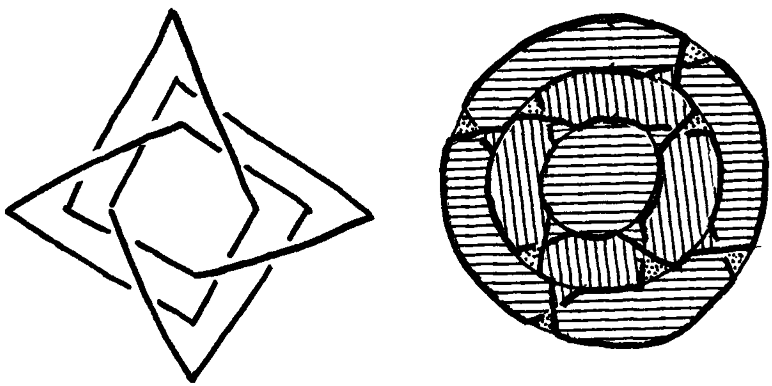
8. EXERCISE: A PL knot has genus zero if and only it is the unknot.

9. EXERCISE: All nontrivial doubled knots (see 4D4) have genus one. Note, however, that they have arbitrarily large bridge indices.

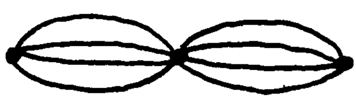
10. EXERCISE : The disjoint curves produced during the course of the algorithmic construction proving the existence theorem may be called 'Seifert circles.' Given a planar picture of a link of n components, let c be the number of crossings and s be the number of Seifert circles. Prove that the Seifert surface constructed has genus:

$$\text{genus} = 1 - \frac{s + n - c}{2} .$$

11. EXAMPLE. Here is the torus knot of type 3,4 and the Seifert surface M as constructed in the proof of Theorem 1 , with three disks connected by eight bands:



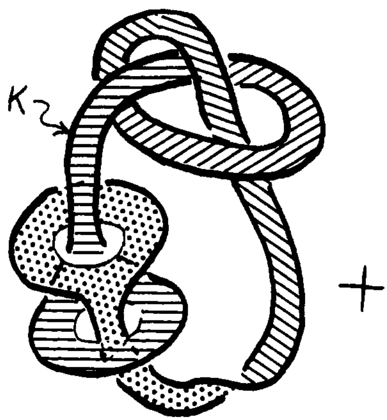
We can compute the genus of M easily. M has the homotopy type of the graph:

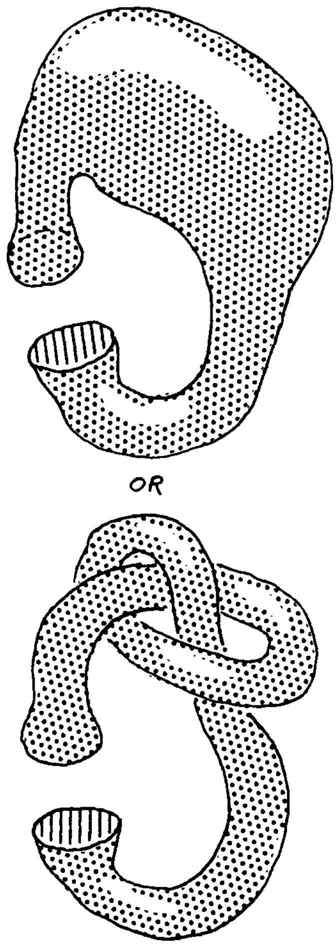


which has Euler characteristic $3 - 8 = -5$. So the genus of M is 3.

12. EXERCISE : The torus knot of type p,q has genus $\leq \frac{1}{2}(p-1)(q-1)$.

13. EXAMPLE. Minimal Seifert surfaces may not be unique, i.e. a given knot may posess two inequivalently embedded Seifert surfaces of least genus.





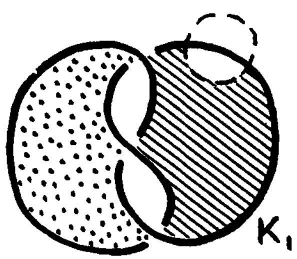
The knot K pictured is nontrivial because it has a trefoil companion (see 4D10). The alternative embeddings of a genus one (hence minimal) Seifert surface appear to be inequivalent. For more information on this phenomenon consult Alford [1970].

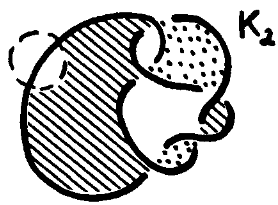
Also see Trotter [1975] .

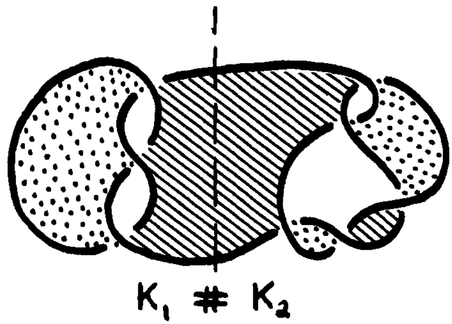
14. THEOREM : The genus of PL knots in R^3 or S^3 is additive:

$$g(K_1 \# K_2) = g(K_1) + g(K_2).$$

PROOF : The inequality $\leqq$ is fairly easy. For, given minimal Seifert surfaces for K_1 and K_2 , one may construct a Seifert surface for the composite knot by taking a "boundary connected sum" as indicated in the following picture. This surface clearly has genus = $g(K_1) + g(K_2)$.





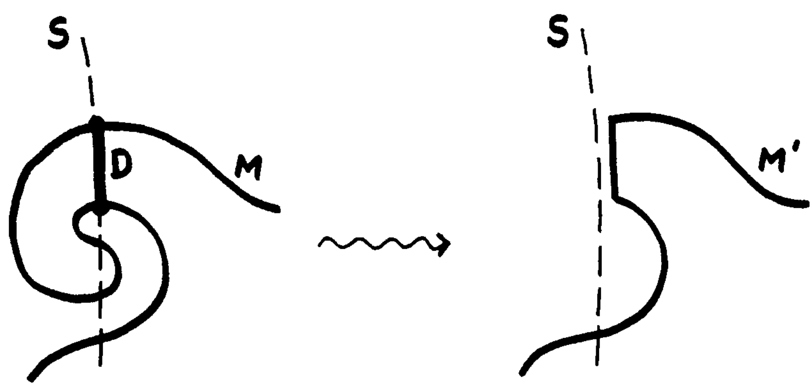


boundary connected sum of two surfaces makes a Seifert surface for $K_1 \# K_2$.

To establish the opposite inequality we'll use a "cut and paste" argument which is typical of three-dimensional techniques. Consider a Seifert surface M for $K_1 \# K_2$ which is of minimal genus. Our task will be to find another minimal Seifert surface M' for $K_1 \# K_2$ which splits up as a boundary connected sum of Seifert surfaces M_1 and M_2 for K_1 and K_2 , respectively. It will follow that $g(K_1) + g(K_2) \leqq g(M_1) + g(M_2) = g(M') = g(K_1 \# K_2)$ and we'll be done.

To find M', consider a PL 2-sphere S in S^3 which splits $K_1 \# K_2$ into its two summands. We may arrange that M and S be transversal and conclude that $M \cap S$ is a compact 1-manifold with exactly two boundary points, where the knot pierces through S. If $M \cap S$ is a single arc, we may take $M' = M$, since the arc splits M into two surfaces, bounded by knots of the type of K_1 and K_2. Otherwise, $M \cap S$ is exactly one arc and a finite number of simple closed curves. We need to remove these latter curves so, using induction, we'll be done if we can construct M' to be a minimal surface for $K_1 \# K_2$ such that $M' \cap S \subset M \cap S$ and has strictly fewer components.

To this end, consider a closed curve $C \subset M \cap S$ which is "innermost on S", meaning that C bounds a disk D in S with $\mathring{D} \cap M = \emptyset$. By the exercise below, C must separate M. Throw away the part of $M - C$ which does not intersect ∂M and replace it by D. Then we may push the resulting 2-manifold away from S near D and call the result M'. Clearly M' can be bicollared, $\partial M' = \partial M$, and M' has strictly fewer components of intersection with S. Moreover, an Euler characteristic argument shows that the "surgery" we've done cannot increase genus, so M' is again a minimal surface.



15. EXERCISE : Suppose M is a PL Seifert surface for $K = \partial M$ in S^3, D is a PL disk in S^3 with $\mathring{D} \cap M = \emptyset$ and $\partial D \subset \mathring{M}$, and further suppose that $M - \partial D$ is connected. Then M is not a minimal surface for K. [Hint: try some cutting and pasting]

16. EXERCISE : Embellish the proof of the additivity theorem to show that every minimal surface for $K_1 \# K_2$ is actually ambient isotopic to a boundary connected sum of minimal surfaces for the K_i. Therefore the property of having unique minimal surfaces is preserved under sums.

17. REMARK : The additivity theorem gives an alternative proof for the non-cancellation theorem (4B7), which may be preferred because it does not require Dehn's Lemma. It is clear that knots of genus one must be prime and that genus may be used to show that any PL knot is a sum of prime knots.

18. EXERCISE : Suppose L is a PL link of two components and each component bounds a PL disk which misses the other component. Then the components of L bound disjoint disks, so L is the trivial link.

19. REMARK : It may be that each component of a link bounds a Seifert surface missing the other, yet they do not bound disjoint Seifert surfaces. Links whose components do bound disjoint Seifert surfaces are called <u>boundary links</u>. A later section of this chapter will be devoted to them.

B. HIGHER DIMENSIONAL SEIFERT SURFACES.

There is also an existence theorem for Seifert surfaces in higher dimensions. Since the proof uses some advanced techniques, although it is conceptually simple, we'll only give a sketch of the proof.

1. THEOREM : (PL or C^∞ category) Any knot or link L^n in S^{n+2} which possesses a tubular (product) neighbourhood bounds a Seifert surface.

SKETCH OF PROOF : Assume $n \geqq 2$. Let $T \subset S^{n+2}$ be a tubular neighbourhood, so that $(T,L) \cong (L \times D^2, L \times 0)$. A map $f\colon T - L \to S^1$ may be defined, corresponding to the map $L \times (D^2 - 0) \to S^1$ given by $(x,y) \to y/|y|$. We wish to extend $f|\partial T$ to a map $F\colon X \to S^1$, where $X = S^{n+2} - \overset{\circ}{T}$ and $\partial T = \partial X$. Obstruction theory says that such an extension is possible if and only if certain elements of the cohomology groups $H^{k+1}(X,\partial X)$, with coefficients in $\pi_k(S^1)$, vanish. But if $k > 1$ the coefficient group is trivial and with $k = 1$ we have integer coefficients and $H^2(X,\partial X) \cong H_n(X) = 0$ by Lefschetz and Alexander dualities. So the obstructions must all vanish and there is a map $F\colon X \to S^1$ extending f; take F to be PL or C^∞ and choose a regular point x in S^1. Then $F^{-1}(x)$ is a bicollared PL or C^∞ submanifold of S^{n+2} of codimension one. After throwing away extraneous components we have the desired Seifert surface. In case L is a link, the surface may be disconnected, but as in the classical case this can be remedied by connecting with tubes, if desired.

2. EXERCISE : The proof given above shows how to obtain a Seifert surface from a map $X \to S^1$ defined on the complement of a link. Show that all Seifert surfaces arise in this way. I. e., given a Seifert surface M for L^n there exists a map $F: S^{n+2} - L^n \to S^1$ and a point x in S^1 such that $M = F^{-1}(x)$ and moreover F^{-1} of a neighbourhood of x is a bicollar on $\mathring{M}$. [Hint: send everything outside a given bicollar of $\mathring{M}$ to a point.]

3. REMARK : Recent work in topological transversality enables the above existence theorem to go through for topological links, with suitable additional hypotheses. That it does not work in certain cases (in dimension four) is pointed out in recent work of Cappell and Shaneson.

C. CONSTRUCTION OF THE CYCLIC COVERINGS OF A KNOT COMPLEMENT USING SEIFERT SURFACES.

There is an important class of covering spaces of a knot complement $X = S^{n+2} - K^n$, which will be used in the next chapter to define certain abelian invariants of K. Readers unfamiliar with covering space theory will find a synopsis in Appendix A.

Seifert surfaces give a convenient means of constructing these covering spaces, in a manner entirely analogous to 'cuts' in the classical theory of Riemann surfaces.

Let M^{n+1} be a Seifert surface for the knot K^n in S^{n+2} and let $N : \mathring{M} \times (-1,1) \to S^{n+2}$ be an open bicollar of the interior $\mathring{M} = M - K$, so $\mathring{M} = N(\mathring{M} \times 0)$. We denote:

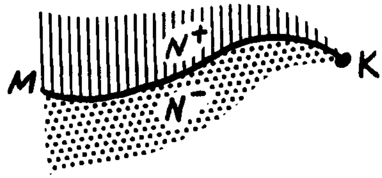
$N = N(\mathring{M} \times (-1,1))$

$N^+ = N(\mathring{M} \times (0,1))$

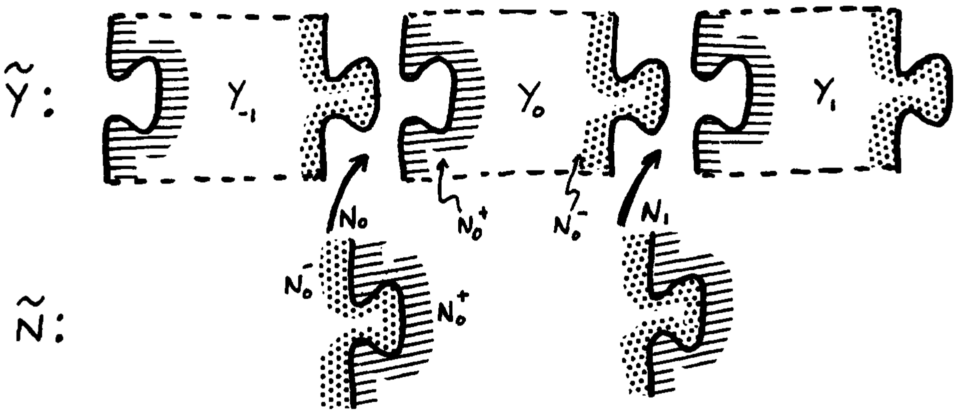
$N^- = N(\mathring{M} \times (-1,0))$

$Y = S^{n+2} - M$

$X = S^{n+2} - K$



Thus we have two triples (N,N^+,N^-) and (Y,N^+,N^-). Form countably many copies of each, denoted (N_i,N_i^+,N_i^-) and (Y_i,N_i^+,N_i^-), $i = 0, \pm 1, \pm 2, \cdots$. Let $\tilde{N} = \bigcup_{i=-\infty}^{\infty} N_i$ and $\tilde{Y} = \bigcup_{i=-\infty}^{\infty} Y_i$ be the disjoint unions. Finally, form an identification space by identifying $N_i^+ \subset Y_i$ with $N_i^+ \subset N_i$ via the identity homeomorphism, and likewise identify each $N_i^- \subset Y_i$ with $N_{i+1}^- \subset N_{i+1}$. Call the resulting space $\tilde{X}$.



EXERCISE. Verify the following facts. $\tilde{X}$ is a path-connected open $(n+2)$-manifold. There is a map $p : \tilde{X} \to X$ which is a regular covering space. There is a covering automorphism $\tau : \tilde{X} \to \tilde{X}$, which takes Y_i to Y_{i+1} and N_i to N_{i+1}, and τ generates the group $\mathrm{Aut}(\tilde{X})$, which is infinite cyclic.

2. DEFINITION. $\tilde{X}$ is called the <u>infinite cyclic</u> cover of X.

3. PROPOSITION. $\tilde{X}$ is the universal abelian cover of X.

4. COROLLARY. $\tilde{X}$ depends (up to covering isomorphism) only on the knot type of K, and not on the choice of Seifert surface or other choices in the above construction.

PROOF OF THE PROPOSITION. We need only the fact that $\mathrm{Aut}(\tilde{X}) \cong Z$. The exact sequence

$$1 \longrightarrow \pi_1(\tilde{X}) \xrightarrow{p_*} \pi_1(X) \longrightarrow \underset{\underset{\textstyle Z}{\textstyle \wr\|}}{\mathrm{Aut}(\tilde{X})} \longrightarrow 1$$

shows that $p_*\pi_1(\tilde{X})$ contains the commutator subgroup C of $\pi_1(X)$. Now the induced map

$$Z \cong \frac{\pi_1(X)}{C} \longrightarrow \mathrm{Aut}(\tilde{X}) \cong Z$$

has kernel $\dfrac{p_*\pi_1(\tilde{X})}{C}$, and (being surjective) must be an isomorphism. Hence $p_*\pi_1(\tilde{X}) \cong C$ and the proposition follows.

5. REMARK. This construction can be streamlined by eliminating reference to the N_i which glue the Y_i together. Let $\bar{Y}_i$ denote the closure of Y_i in $\tilde{X}$.

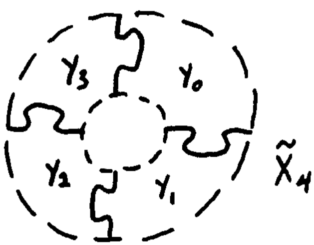
6. EXERCISE : $\bar{Y}_i$ is homeomorphic with $S^{n+2} - (K \cup N)$. This latter space might be called "$S^{n+2} - K$ cut open along M"; it has two boundary components

each homeomorphic with $\overset{\circ}{M}$, and we may regard $\tilde{X}$ as the union of copies of these cut open spaces, suitably sewn together.

A similar construction yields the <u>finite cyclic covering spaces</u> of $X = S^{n+2} - K^n$. Choose a fixed integer $k > 1$ and consider copies (N_i, N_i^+, N_i^-) and (Y_i, N_i^+, N_i^-) as above, $i = 0,1,\dots,k-1$. Let

$$\tilde{Y} = \bigcup_{i=0}^{k-1} Y_i \quad \text{and} \quad \tilde{N} = \bigcup_{i=0}^{k-1} N_i \quad \text{and make}$$

the same identifications as before, except that $N_{k-1}^- \subset Y_{k-1}$ is identified with $N_o^- \subset N_o$. Call the resulting space $\tilde{X}_k$. It is a k-fold 'cyclic' cover of X with $\mathrm{Aut}(\tilde{X}) \cong Z/k$.



7. EXERCISE. Show that $\tilde{X}$ covers each $\tilde{X}_k$, so that $\tilde{X}_k$ may be regarded as a quotient space of $\tilde{X}$.

8. EXERCISE. Prove that $\tilde{X}_k$ corresponds to the kernel of the composite homomorphism

$$\pi_1(X) \to \frac{\pi_1(X)}{C} \cong Z \to Z/k$$

where $C < \pi_1(X)$ is the commutator subgroup and the right-hand map is the canonical projection.

9. COROLLARY. $\tilde{X}_k$ depends only on k and the knot type of K .

D. LINKING NUMBERS.

Let J and K be two disjoint oriented knots in S^3 (or R^3) This section describes eight methods for defining an integer called their <u>linking number</u>, all of which turn out to be equivalent, at least up to sign. Assume J and K are polygonal.

(1) Let [J] be the homology class in $H_1(S^3 - K)$ carried by J . Since $H_1(S^3 - K) \cong Z$, we may choose a generator γ of this group and write $[J] = n\gamma$. Define $\ell k_1(J,K) = n$.

(2) Let M be a PL Seifert surface for K, with bicollar (N,N^+,N^-) of $\mathring{M}$ as in the previous section. Assume (allowing adjustment of J by a homotopy in $S^3 - K$) that J meets M in a finite number of points and at each such point J passes locally (a) from N^- to N^+ or (b) from N^+ to N^-, following its orientation. Weight the intersections of type (a) with + 1 and those of type (b) with − 1 . The sum of these numbers we denote by $\ell k_2(J,K)$. [Note that this seems to depend on M].

(3) Consider a regular projection of $J \cup K$. At each point at which J crosses <u>under</u> K, count

+ 1 for  J K and − 1 for  J K

The sum of these, over all crossings of J under K, is called $\ell k_3(J,K)$.

(4) J is a loop in $S^3 - K$, hence represents an element of $\pi_1(S^3 - K)$ with suitable basepoint. This fundamental group abelianizes to Z, and the loop J is thereby carried to an integer, called $\ell k_4(J,K)$.

(5) $[J]$ and $[K]$ are 1-cycles in S^3. Choose a 2-chain $C \in C_2(S^3;Z)$ such that $[K] = \partial C$. Then the intersection $C \cdot [J]$ is a 0-cycle, well-defined up to homology. Since $H_0(S^3) \cong Z$, $C \cdot [J]$ corresponds to an integer which we call $\ell k_5(J,K)$.

(6) Regard $J, K : S^1 \to R^3$ as maps.

In vector notation, define a map $f : S^1 \times S^1 \to S^2$ by the formula

$$f(u,v) = \frac{K(u) - J(v)}{|K(u) - J(v)|} .$$

If we orient $S^1 \times S^1$ and S^2 then f has a well-defined degree. Let $\ell k_6(J,K) = \deg(f)$.

(7) (Gauss Integral) Define $\ell k_7(J,K)$ to be the integer

$$\frac{1}{4\pi} \int_J \int_K \frac{(x'-x)(dydz'-dzdy') + (y'-y)(dzdx'-dxdz') + (z'-z)(dxdy'-dydx')}{[(x'-x)^2 + (y'-y)^2 + (z'-z)^2]^{3/2}}$$

where (x,y,z) ranges over J and (x',y',z') over K.

(8) Let $p : \tilde{X} \to X$ be the infinite cyclic cover of $X = S^3 - K$ and let τ generate $\mathrm{Aut}(\tilde{X})$. Consider J as a loop in X based at, say, $x \in \mathrm{Im}\, J$. Lift J to a path $\tilde{J}$ in $\tilde{X}$, starting at any $\tilde{x}_0 \in p^{-1}(x)$ and call its terminal point $\tilde{x}_1 \in p^{-1}(x)$. There is a unique integer m such that $\tau^m(\tilde{x}_0) = \tilde{x}_1$. Define $\ell k_8(J,K) = m$.

1. EXERCISE. Identify the choice in each of the above definitions which affects the sign of the linking number.

2. THEOREM. $\ell k_i = \pm \ell k_j$; $i, j = 1, \ldots, 8$.

PROOF: $\ell k_1 = \pm\, \ell k_4$: since the Hurewicz homomorphism $h : \pi_1(S^3-K) \to H_1(S^3-K)$ which carries loops to 1-cycles is just the abelianization map.

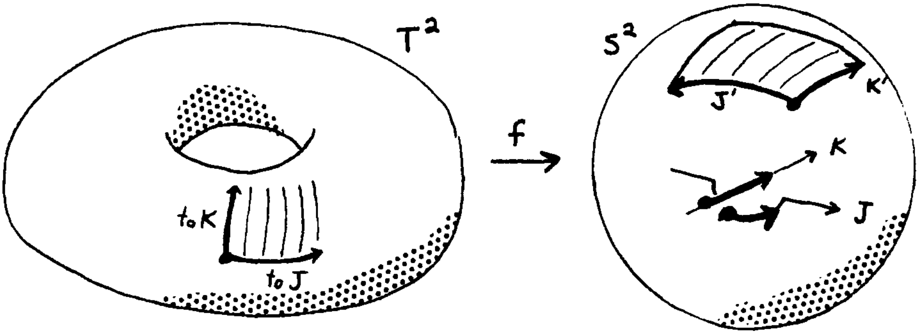
$\ell k_2 = \pm\, \ell k_5$: since we may take the C of (5) to be the 2-cycle carried by M.

$\ell k_2 = \pm\, \ell k_3$: Using the given regular projection of (3) construct a Seifert surface M for K according to the proof of theorem A4, so that J is above M except near the underpasses and intersects M once at each underpass. If the disks are bicollared in such a way that K runs counterclockwise around the boundary as viewed from above, then ± 1 is assigned to the underpasses in the same way in (2) as in (3).

$\ell k_4 = \pm\, \ell k_8$: As described in Appendix A, the τ^m of (8) is just the automorphism τ_J induced by the loop J and the equality follows from the isomorphism $\mathrm{Aut}(\tilde{X}) \cong \pi_1(X)$ / (commutator subgroup).

$\ell k_2 = \pm\, \ell k_8$: Construct $\tilde{X}$ using the M of (2) by the method of the previous section. Choose $\tilde{x}_o \in Y_o \subset \tilde{X}$. Then each intersection of type (a) corresponds to a passage of $\tilde{J}$ from Y_i to Y_{i+1}, while type (b) intersections to the reverse. So $\tilde{J}$ ends up in Y_r, $r = \ell k_2(J,K)$. But if $\tau : \tilde{X} \to \tilde{X}$ is chosen as the shift $Y_i \to Y_{i+1}$ we conclude that $\ell k_8(J,K) = \ell k_2(J,K)$.

$\ell k_3 = \pm\, \ell k_6$: suppose there is a point $z \in S^2$ such that $f^{-1}(z)$ is a finite set and f is a homeomorphism near each point of $f^{-1}(z)$. Then $\deg f$ may be calculated by adding the points, weighted -1 if f locally reverses orientation and $+1$ if f locally preserves orientation. But there is such a point, namely the point $z \in S^2$ directly above the projection plane of (3), corresponding to the viewer's eye: $f^{-1}(z)$ has one element for each crossing of J under K. The picture below shows why the two types of crossings correspond to different orientations.



$\ell k_7 = \pm\, \ell k_6$: see Spivak's Calculus on Manifolds.

This integral (or its negative) is just an analytic way of computing $\deg f$.

3. DEFINITION. Define the <u>linking number</u> $\ell k(J,K)$ to be any of the above.

4. REMARK. The sign ambiguity is usually not a bother, and disappears if one chooses a 'convention' as in (3). Note that definition (6) (and which others?) does not require that J and K be embeddings of S^1, as long as they are disjoint, so the notion of linking number extends to arbitrary disjoint curves in S^3 or R^3 .

5. THEOREM: If there are homotopies $J_t : S^1 \to R^3$ and $K_t : S^1 \to R^3$ so that $\operatorname{Im} J_t$ is disjoint from $\operatorname{Im} K_t$ for each $0 \le t \le 1$, then $\ell k(J_o,K_o) = \ell k(J_1,K_1)$.

PROOF. Using (6) define $f_t(u,v) = \dfrac{K_t(u,v) - J_t(u,v)}{|K_t(u,v) - J_t(u,v)|}$ and we have homotopic maps $f_o, f_1 : S^1 \times S^1 \to S^2$. Hence they have the same degree.

6. THEOREM:

$$\ell k(J,K) = \ell k(K,J)$$

$$\ell k(-J,K) = -\ell k(J,K)$$

where $-J$ is J with the reverse orientation.

7. REMARK : Many of these definitions of linking number may be generalized to define a linking pairing between disjoint cycles of dimensions j and k in S^{j+k+1}. It is bilinear and symmetric or antisymmetric, namely

$$\ell k(J,K+K') = \ell k(J,K) + \ell k(J,K')$$

$$\ell k(J+J',K) = \ell k(J,K) + \ell k(J',K)$$

$$\ell k(xJ,K) = \ell k(J,xK) = x\ell k(J,K)$$

$$\ell k(K,J) = (-1)^{jk+1}\ell k(J,K)$$

A good treatment of this can be found in Seifert-Threlfall [1947]. There is, however no analogous notion of linking number to help us with codimension two link theory, for example, in higher dimensions.

8. EXERCISE : A longitude L of a preferred framing of a tubular neighbourhood of a knot K in S^3 or R^3 is characterized among all possible longitudes by the property $\ell k(L,K) = 0$.

There is a ninth possible definition for the linking number of disjoint PL knots J and K in S^3, which we consider as ∂D^4. Let A and B be 2-chains in D^4, such that $\partial A = J$ and $\partial B = K$. We may assume that A and B intersect transversally in a finite number of points, weighted with a +1 or −1, according to a suitable orientation convention. The weighted sum is the intersection number $A \cdot B$

9. EXERCISE : Devise a suitable orientation convention and prove that $\ell k(J,K) = \pm A\cdot B$. In particular, if two knots J and K bound disjoint PL orientable 2-manifolds in ∂D^4, then $\ell k(J,K) = 0$.

E. BOUNDARY LINKING. Recall that a link $L^n \subset R^{n+2}$ is a <u>boundary link</u> if its components bound disjoint Seifert surfaces. To establish that a link is a boundary link merely requires a construction; to show one is <u>not</u>, may require more cunning. This section discusses some methods, by example, and establishes that both boundary and non-boundary links $L^n \subset R^{n+2}$ exist for all $n \geq 1$. First some crude criteria.

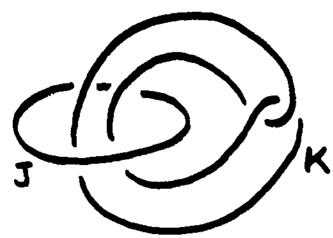
1. PROPOSITION : If any two components of $L^1 \subset S^3$ or R^3 have nonzero linking number, L is <u>not</u> a boundary link.

PROOF : Use definition (2) of linking number.

2. PROPOSITION : If $L^n \subset S^{n+2}$ or R^{n+2} is splittable, then L is a boundary link. (Assuming L is PL or C^∞)

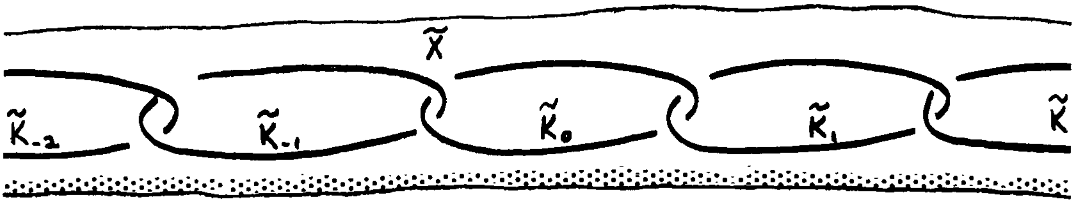
PROOF : Assume the components $L_1,\ldots,L_r$ lie interior to disjoint balls $B_1^{n+2},\ldots,B_r^{n+2}$. Then (EXERCISE) one may construct homeomorphisms $h_i : R^{n+2} \to \text{int } B_i^{n+2}$ which are fixed on L_i. By theorem B1, each L_i bounds a Seifert surface M_i^{n+1}. Then the surfaces $h_i(M_i)$ are again Seifert surfaces for the L_i, and they are disjoint, as required.

3. EXAMPLE : Whitehead's link is not a boundary link. For if M_J and M_K are Seifert surfaces for J and K, respectively, one may construct the universal abelian (= universal) cover



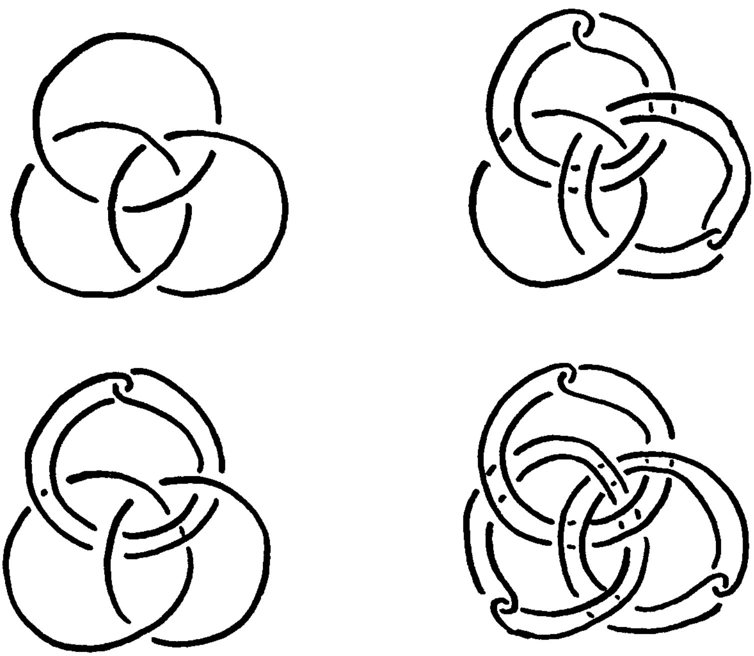
$\tilde{X}$ of $X = S^3 - J$ by cutting along M_J. If $M_J \cap M_K = \phi$, then M_K

lifts to disjoint (why?) Seifert surfaces ... $\tilde{M}_{-1}$, $\tilde{M}_0$, $\tilde{M}_1$ ... for the liftings ... $\tilde{K}_{-1}$, $\tilde{K}_0$, $\tilde{K}_1$ ... of K in $\tilde{X}$. This is impossible,



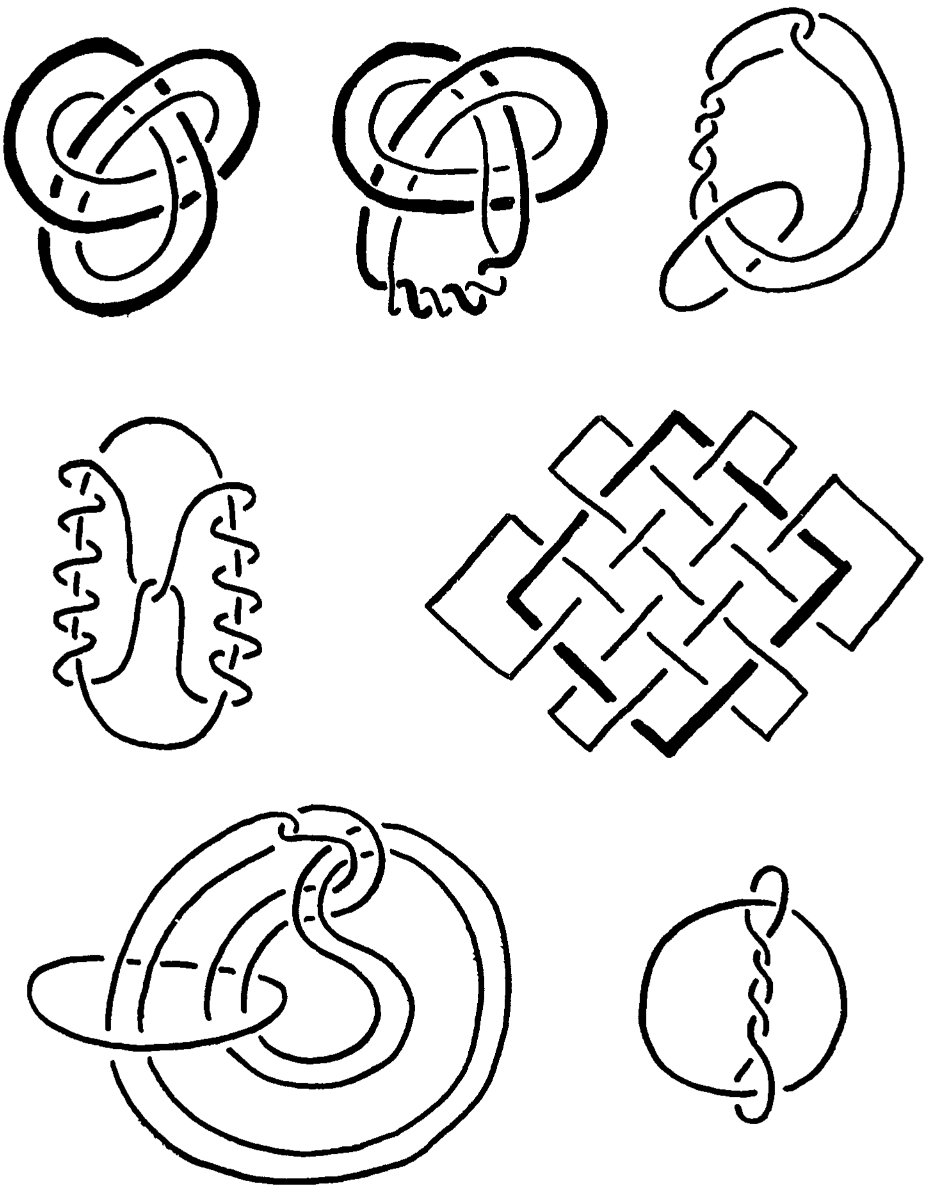
since any two consecutive liftings of K have linking number 1 .

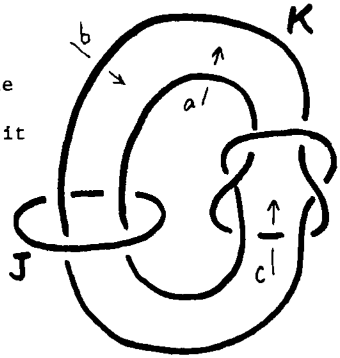
That boundary linking is a fairly subtle property is exhibited by the following variations on the Borromean link.



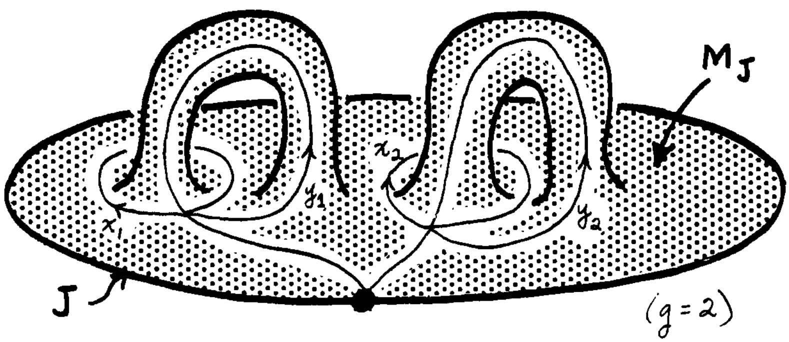
4. EXERCISE : In the above picture, the two links on the right are boundary links. Those on the left are not.

5. EXERCISE : Two of the following are boundary links. Find them and prove that the others are not boundary links.





6. EXAMPLE : The methods of the previous example would fail for this link. We establish that it is <u>not</u> a boundary link by a more algebraic argument. Suppose first that J bounds a Seifert surface M_J missing K (indeed it does!). It is easy to see that J is homotopic within M_J to a product of commutators of certain loops in M_J. (The picture below illustrates M_J in "abstracto"; the homotopy is across the shaded region which is a disk.)



Thus the homotopy class of J satisfies

$$[J] = (x_1 y_1 x_1^{-1} y_1^{-1}) \cdots (x_g y_g x_g^{-1} y_g^{-1}) \quad \text{in} \quad \pi_1(R^3 - K) \ ; \quad g = \text{genus of } M_J \ .$$

Suppose further that M_J is disjoint from a Seifert surface M_K for K. Then we may conclude that $\ell k(x_i, K) = \ell k(y_i, K) = 0$ for all $i \leq g$. Thus x_i , y_i are in the commutator subgroup $C = [\pi, \pi]$ of $\pi = \pi_1(R^3 - K)$

by definition (4) of ℓk. It follows that $[J]$ is in the <u>second</u> commutator subgroup $C' = [C,C]$ of $\pi_1(R^3-K)$. In the next paragraph we'll see that this isn't the case, and this contradiction will establish that such disjoint M_J and M_K cannot exist.

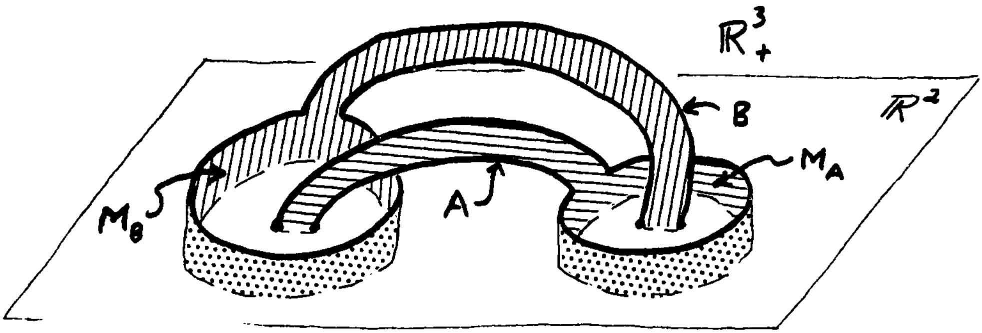
A calculation by the Wirtinger method gives a presentation $\pi = \pi_1(R^3-K) = (a,b,c;\ aca = cac,\ bcb = cbc)$, where a, b, c are the loops pictured. Sending a, b, c to the cycles (01), (02), (12), respectively, of the symmetric group S_3 (permutations of $\{0,1,2\}$) defines a homomorphism $h : \pi \to S_3$ (check this). Moreover $h([J]) = h(ab^{-1}) = (01)(02) = (012)$. Since any homomorphism sends second commutators into second commutators, we can conclude that $[J] \notin [C,C]$ from the following easy exercise.

7. EXERCISE. The commutator subgroup $C_3 = [S_3, S_3]$ is cyclic of order three; the second commutator subgroup $[C_3, C_3]$ consists of the identity element alone.

The argument of the previous example provides a general test for non-boundary linking. That it also works for three or more components; we leave as an EXERCISE:

8. PROPOSITION : If a link L is a boundary link, then each component represents an element in the second commutator subgroup of the fundamental group of the complement of the remaining component(s).

9. EXAMPLE : The spun link of Van Kampen-Zeeman (see 3K5) is a boundary link. (Recall it is unsplittable.) In fact the arcs which are to be spun are equivalent in R^3_+ to those pictured below. In spinning R^3_+ by S^{k-1} , we form the k-dimensional link $A^* \cup B^*$ in R^{k+2} and the surfaces M_A and M_B become disjoint bicollared (k+1)-manifolds M^*_A , M^*_B with $\partial M^*_A = A^*$, $\partial M^*_B = B^*$, as required.



10. EXERCISE : Verify the properties of M^*_A and M^*_B asserted in the last sentence. Can you identify their topological type?

11. EXAMPLE : The spun link of Andrews-Curtis (3K13) is <u>not</u> a boundary link. This includes the more general spinning by S^{n-1} to give n-dimensional links $C^* \cup D^*$ in R^{n+2} . First note that spinning $C \cup D$ by S^0 gives the link which is the same link as that of example 6

The Andrews-Curtis link $C \cup D$ spun by S^0

It is equivalent to the link of example 6.

Thus, for $C^* \cup D^*$ in R^{n+2}, the above link is a cross-section by a hyperplane $R^3 \subset R^{n+2}$; i.e. $J = C^* \cap R^3$, $K = D^* \cap R^3$. If C^* and D^* bound disjoint bicollared $(n+1)$-manifolds M_C and M_D in R^{n+2}, we could adjust them to be transversal to R^3. Then $M_C \cap R^3$ and $M_D \cap R^3$ are disjoint bicollared 2-manifolds in R^3 bounded by J and K, respectively. But we saw in example 6 that this is impossible.

12. PROOF OF PROPOSITION 3K15: Consider again the Andrews-Curtis link $C^* \cup D^*$ of 2-spheres in R^4, obtained by S^1-spinning (3K13). Its intersection with R^3 is the $J \cup K$ pictured above (and example 6 in an equivalent form). We wish to show that C^* is not homotopically trivial in $R^4 - D^*$. Following is essentially the proof given by Andrews and Curtis. Supposing the contrary, we would have a map $f : B^3 \to R^4 - D^*$ which maps $\partial B^3 = S^2$ homeomorphically onto C^*. By approximation we may assume f is transversal to R^3, so $f^{-1}(R^3) = M$ is a bicollared (hence orientable) 2-manifold in B^3, with boundary $\partial M \subset \partial B^3$ carried

by f homeomorphically onto J. As in example 6, ∂M is homotopic in M to a product of commutators of certain loops x_i, y_i in M. This homotopy is carried via f to a homotopy in $R^3 - K$ between J and a similar product of commutators of $f(x_i)$ and $f(y_i)$. Now the $f(x_i)$ and $f(y_i)$ are homotopically trivial in $R^4 - D^*$ (why?). Since the inclusion $R^3 - K \hookrightarrow R^4 - D^*$ is a homology equivalence, we conclude that $f(x_i)$, $f(y_i)$ are homologically trivial in $R^3 - K$, i.e. represent commutators in $\pi_1(R^3-K)$. Thus we have that the homotopy class $[J]$ is in the second commutator subgroup of $\pi_1(R^3-K)$. As argued previously, this is a contradiction.

13. EXERCISE : Generalize proposition 3K15 to higher-dimensional spinning.

14. EXERCISE : Prove the assertion made above that the inclusion homomorphism $H_1(R^3-K) \longrightarrow H_1(R^4-D^*)$ is an isomorphism. Show it is false with π_1 replacing H_1.

15. EXERCISE : Show that there are Brunnian links which are boundary links and there are Brunnian links which are not boundary links.

CHAPTER 6. FINITE CYCLIC COVERINGS AND THE TORSION INVARIANTS

A. TORSION NUMBERS.

In this chapter we study the finite cyclic covering spaces $\tilde{X}_k$ of a knot complement $X = S^{n+2} - K^n$. Unlike the homology of X itself, $H_*(\tilde{X}_k)$ is a useful knot invariant. It may be computed fairly easily.

The classical <u>k^{th} torsion numbers</u> may be defined in terms of $H_1(\tilde{X}_k)$ as follows. Except for wild knots (see exercises 6B8 and 6B9) this group is finitely generated, so by the fundamental theorem of abelian groups,

$$H_1(\tilde{X}_k) = F \oplus Z/m_1 \oplus Z/m_2 \oplus \ldots \oplus Z/m_r \ ,$$

where F is free abelian and Z/m_i is the finite cyclic group of order m_i. The m_i are determined uniquely if we require that each m_i divides m_{i+1}, and these are called the k^{th} torsion numbers of the knot K.

It is clear that the torsion numbers depend only on k and $\pi_1(X)$, since $\pi_1(\tilde{X}_k)$ is the kernel of the composition

$$\pi_1(X) \xrightarrow{\text{Hurewicz}} H_1(X) \xrightarrow{\text{projection}} Z/k$$

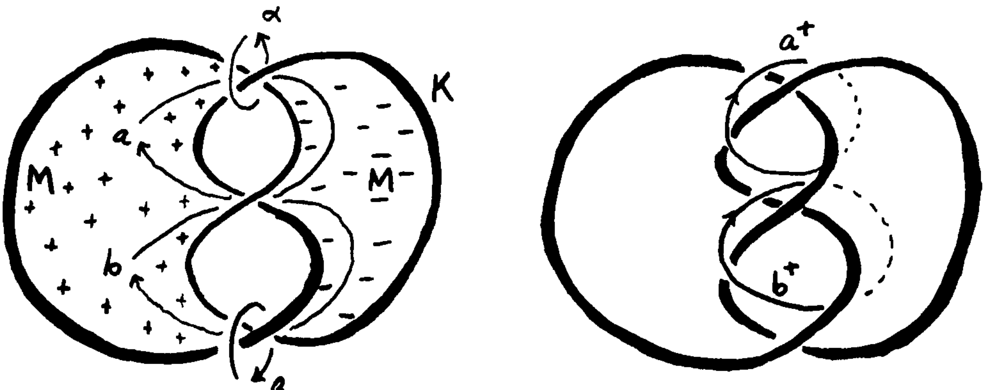
and $H_1(\tilde{X}_k)$ is the abelianization of that. But they are often easier to apply than the knot group (which generally is nonabelian). Besides this algebraic determination, we outline two geometric methods of computing the homology of cyclic covers.

B. CALCULATION USING SEIFERT SURFACES.

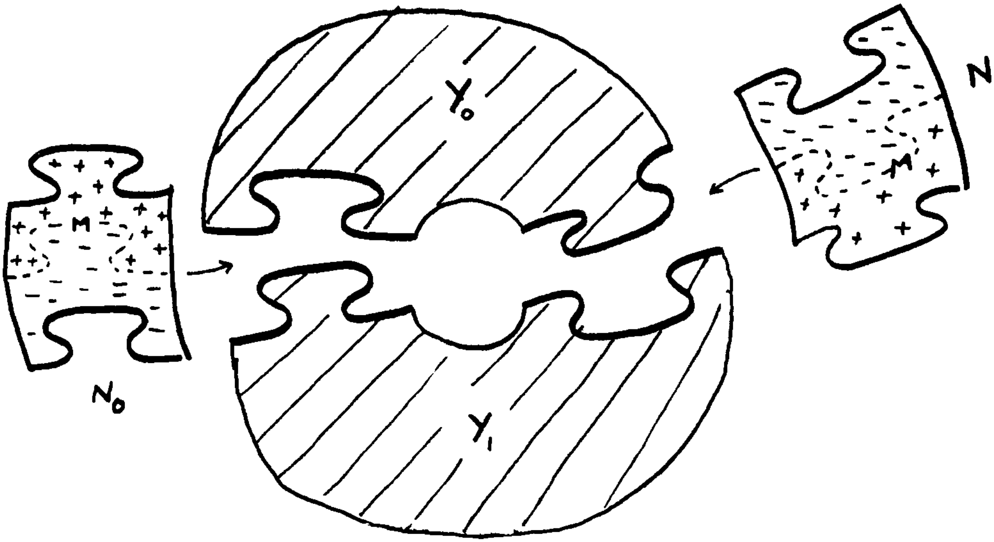
An example will best illustrate the method.

1. EXAMPLE : K = trefoil in S^3. Find the homology of the two-fold cyclic cover $\tilde{X}_2$ of $X = S^3 - K$.

Consider the Seifert surface M pictured



As in the previous chapter, we construct $\tilde{X}_2$ from two copies Y_o, Y_1 of $Y = S^3 - M$ and glue them together by N_o, N_1, each homeomorphic with $\mathring{M} \times (-1,1)$ according to the schematic



the twofold cyclic cover of $X = S^3 - K$

Now $H_1(\mathring{M})$ and $H_1(Y)$ are both free abelian with respective bases a, b and α, β as indicated in the first figure. Pushing a, b off $\mathring{M}$ and into N^+ or N^-, we see from the picture that, in $H_1(Y)$, the following equations hold :

$$a^- = \beta - \alpha \qquad\qquad a^+ = -\alpha$$

$$b^- = -\beta \qquad\qquad b^+ = \alpha - \beta$$

To compute $H_1(\tilde{X}_2)$ note that the subset $Y_o \cup Y_1$ has homology generators $\alpha_o, \beta_o, \alpha_1, \beta_1$ corresponding to α, β. Now putting in N_o introduces relations

$$\beta_1 - \alpha_1 = -\alpha_o \qquad (1) \quad (a_o^- = a_o^+)$$

$$-\beta_1 = \alpha_o - \beta_o \qquad (2) \quad (b_o^- = b_o^+)$$

and adding N_1 introduces relations

$$\beta_o - \alpha_o = -\alpha_1 \qquad (3) \quad (a_1^- = a_1^+)$$

$$-\beta_o = \alpha_1 - \beta_1 \qquad (4) \quad (b_1^- = b_1^+)$$

There is also a nontrivial 1-cycle γ which runs once around $\tilde{X}_2$ and we have the abelian group presentation :

$$H_1(\tilde{X}_2) \cong (\alpha_o, \beta_o, \alpha_1, \beta_1, \gamma; \text{ relations (1)-(4) })$$

Use (2) and (3) to eliminate α_1, β_1 :

$$\begin{aligned} H_1(\tilde{X}_2) &\cong (\alpha_o, \beta_o, \gamma; \ \beta_o = 2\alpha_o, \ \alpha_o = 2\beta_o) \\ &\cong (\alpha_o, \gamma; \ 3\alpha_o = 0) \\ &\cong Z \oplus Z/3 . \end{aligned}$$

2. JUSTIFICATION OF THE CALCULATION.

It is convenient to connect the pieces by a curve $\Gamma \subset \tilde{X}_2$ which lies over a small loop in X linking K once. Let

$$Y' = Y_o \cup Y_1 \cup \Gamma$$

$$N' = N_o \cup N_1 \cup \Gamma$$

and observe that

$$Y' \cup N' = \tilde{X}_2$$

$$Y' \cap N' = N_o^+ \cup N_o^- \cup N_1^+ \cup N_1^- \cup \Gamma$$

Observing that $Y' \cap N'$ is connected, the exact Mayer-Vietoris sequence of reduced homology :

$$\ldots \longrightarrow H_1(N' \cap Y') \xrightarrow{f} H_1(N') \oplus H_1(Y') \longrightarrow H_1(\tilde{X}_2) \longrightarrow 0$$

shows that $H_1(\tilde{X}_2)$ is isomorphic with the cokernel of f. The other groups in the diagram are free abelian, with bases :

$$H_1(N' \cap Y') : \quad a_o^+,\ b_o^+,\ a_o^-,\ b_o^-,\ a_1^+,\ b_1^+,\ a_1^-,\ b_1^-,\ \gamma$$

$$H_1(N') \quad : \quad a_o,\ b_o,\ a_1,\ b_1,\ \gamma'$$

$$H_1(Y') \quad : \quad \alpha_o,\ \beta_o,\ \alpha_1,\ \beta_1,\ \gamma''$$

where $\gamma, \gamma', \gamma''$ are all names for the 1-cycle carried by Γ. Now f is just the sum of the two inclusion-induced homomorphisms. In terms of the bases, f is the map :

$$a_o^+ \longrightarrow (a_o, -\alpha_o) \qquad a_1^+ \longrightarrow (a_1, -\alpha_1)$$

$$b_o^+ \longrightarrow (b_o, \alpha_o - \beta_o) \qquad b_1^+ \longrightarrow (b_1, \alpha_1 - \beta_1)$$

$$a_o^- \longrightarrow (a_o, \beta_1 - \alpha_1) \qquad a_1^- \longrightarrow (a_1, \beta_o - \alpha_o)$$

$$b_o^- \longrightarrow (b_o, -\beta_1) \qquad b_1^- \longrightarrow (b_1, -\beta_o)$$

$$\gamma \longrightarrow (\gamma', \gamma'')$$

The reader should check that calculation of the free abelian group $H_1(N') \oplus H_1(Y')$ of rank 10, modulo the image of f, is equivalent to the calculation in the example above.

3. EXERCISE : Show that for this example $H_p(\tilde{X}_2) = 0$, $p \geq 2$.

4. EXERCISE : Show that for the trefoil :

$$H_1(\tilde{X}_3) \cong Z \oplus Z/2 \oplus Z/2 \qquad H_1(\tilde{X}_6) \cong Z \oplus Z \oplus Z$$

$$H_1(\tilde{X}_4) \cong Z \oplus Z/3 \qquad H_1(\tilde{X}_7) \cong Z$$

$$H_1(\tilde{X}_5) \cong Z \qquad H_1(\tilde{X}_8) \cong Z \oplus Z/3$$

5. EXERCISE : Show that for the figure-eight knot,*

$$H_1(\tilde{X}_2) \cong Z \oplus Z/5$$

$$H_1(\tilde{X}_3) \cong Z \oplus Z/4 \oplus Z/4$$

$$H_1(\tilde{X}_4) \cong Z \oplus Z/3 \oplus Z/15$$

6. EXERCISE : Show in general that $H_1(\tilde{X}_k)$ has a Z summand.

7. EXERCISE : Recall the definition of connected sum of knots in S^3 (section 2G). Show that if $K \cong K' \# K''$ and $\tilde{X}_k$, $\tilde{X}'_k$ and $\tilde{X}''_k$ are their respective k-fold cyclic coverings, then

$$H_1(\tilde{X}_k) \cong Z \oplus A' \oplus A''$$

where

* see Fox's Quick Trip for a longer list

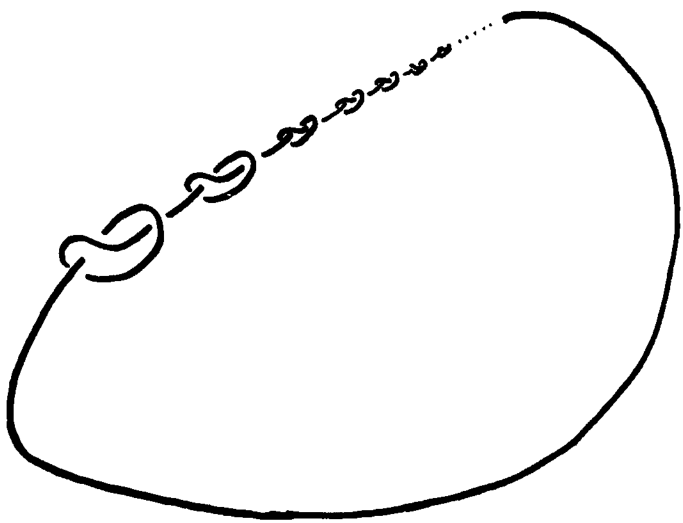
$$H_1(\tilde{X}'_k) \cong Z \oplus A'$$

and

$$H_1(\tilde{X}''_k) \cong Z \oplus A'' .$$

8. EXERCISE : Show that for any tame knot in S^3 (or more generally any PL or smooth knot K^n in S^{n+2} which possesses a tubular neighbourhood), the finite coverings of the complement have finitely-generated homology groups.

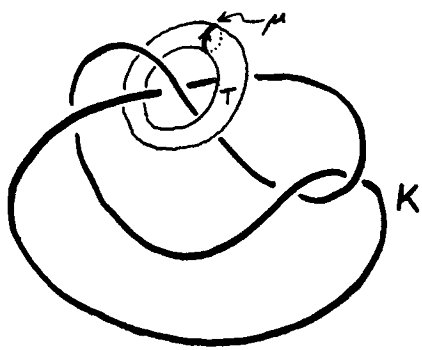
9. EXERCISE : Argue that the following is not a tame knot by showing that $H_1(\tilde{X}_2)$ is not finitely generated.



C. CALCULATION OF $H_*(\tilde{X}_k)$ USING SURGERY IN S^3

First, we describe by example a way of viewing a knot complement which gives a convenient method of visualizing its cyclic covers.

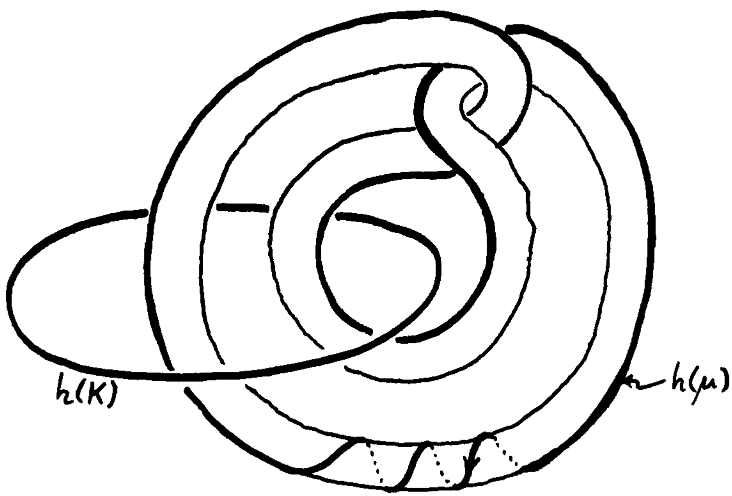
1. EXAMPLE. The figure eight knot in S^3



Consider the unknotted solid torus T pictured and remove its interior from S^3. What remains may be given a twist, defining a homeomorphism $h : S^3 - \mathring{T} \to S^3 - \mathring{T}$ whose image looks like this:

It is important to realize that h does not extend to all of S^3, otherwise we'd have shown that the figure eight knot is trivial, since $h(K)$ is unknotted. This is the same process as described in 3A2.

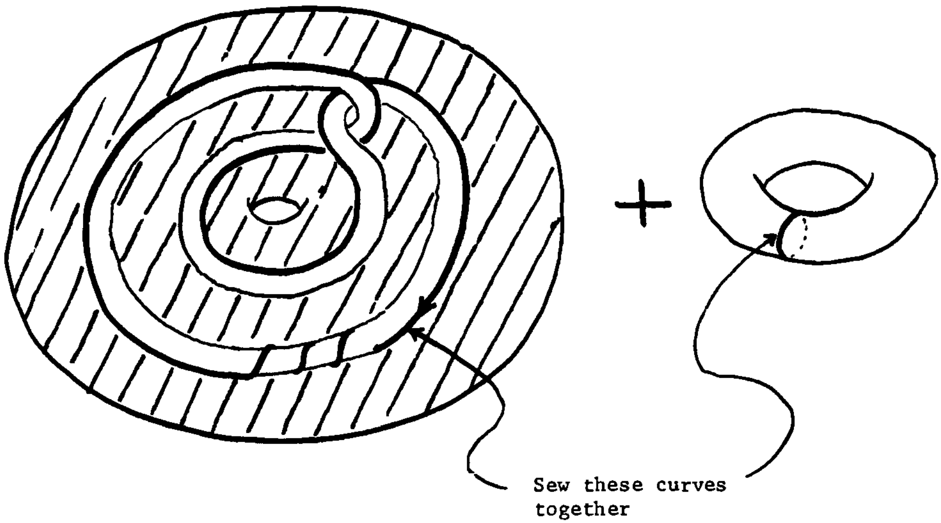
Now change the picture, by a homeomorphism of all of S^3, to:



Your shoelaces may help you see this. Note that $h(\mu)$ has linking number -1 with the core of T, as does μ itself.

So we may conceive of the complement of the figure eight knot as an open solid torus, with an open solid torus removed, and replaced by a solid torus sewn in with meridian running along $h(\mu)$; see next picture.

X = Complement of figure eight knot =

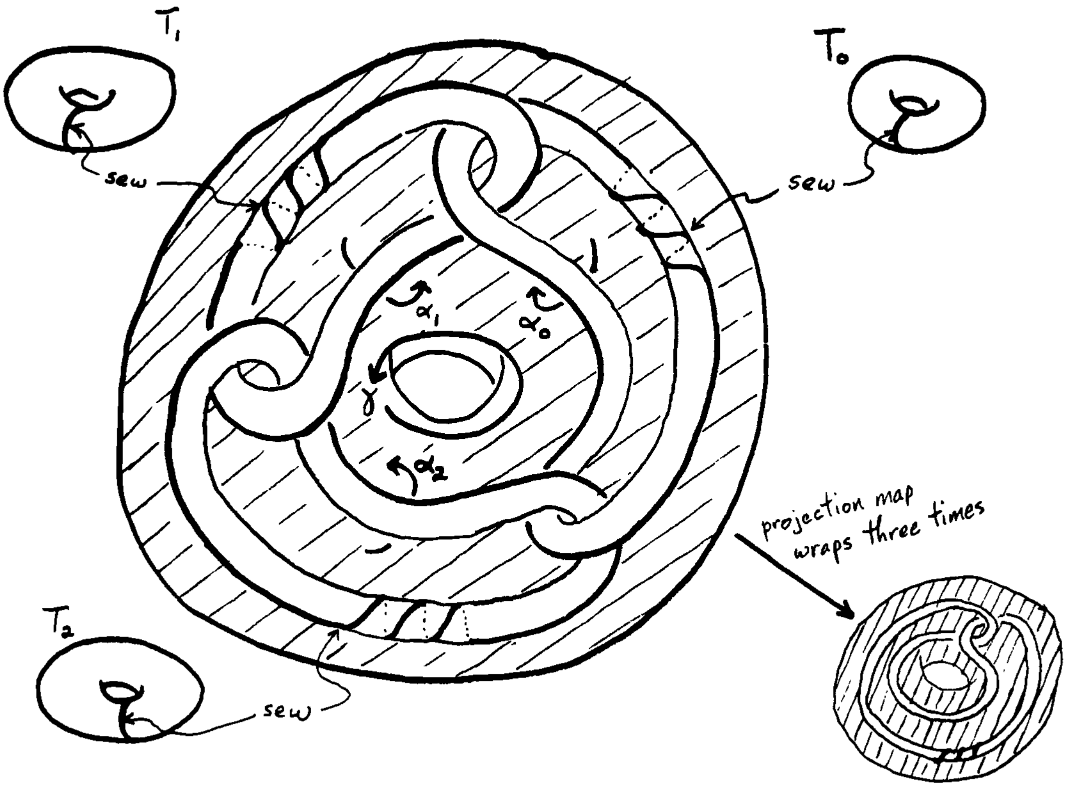


Now it is clear that the k-fold cyclic cover of the complement of the <u>trivial</u> knot is just

$$S^1 \times \mathring{D}^2 \xrightarrow{\ p\ } S^1 \times \mathring{D}^2$$

where p is multiplication by k on the first factor and the identity on the second.

EXAMPLE CONTINUED. Suppose we want to 'see' the 3-fold cyclic cover $\tilde{X}_3$ of X. It is the union of four pieces :



This decomposition enables us to compute $H_1(\tilde{X}_3)$. First, the space in the center of the picture has four unrelated homology generators α_0, α_1, α_2, γ . Imagine sewing in the solid torus T_0 in two stages : first a (thickened) meridinal disk bounded by the curve shown, then the rest of T_0, which is an open ball. The second doesn't affect H_1, the first causes the 'barber-pole' curve around the wormhole to bound and introduces the relation :

$$\alpha_1 + \alpha_2 - 3\alpha_o = 0 \qquad (1)$$

Sewing in T_1 and T_2 likewise gives relations

$$\alpha_2 + \alpha_o - 3\alpha_1 = 0 \qquad (2)$$

$$\alpha_o + \alpha_1 - 3\alpha_2 = 0 \qquad (3)$$

and we have an abelian group presentation

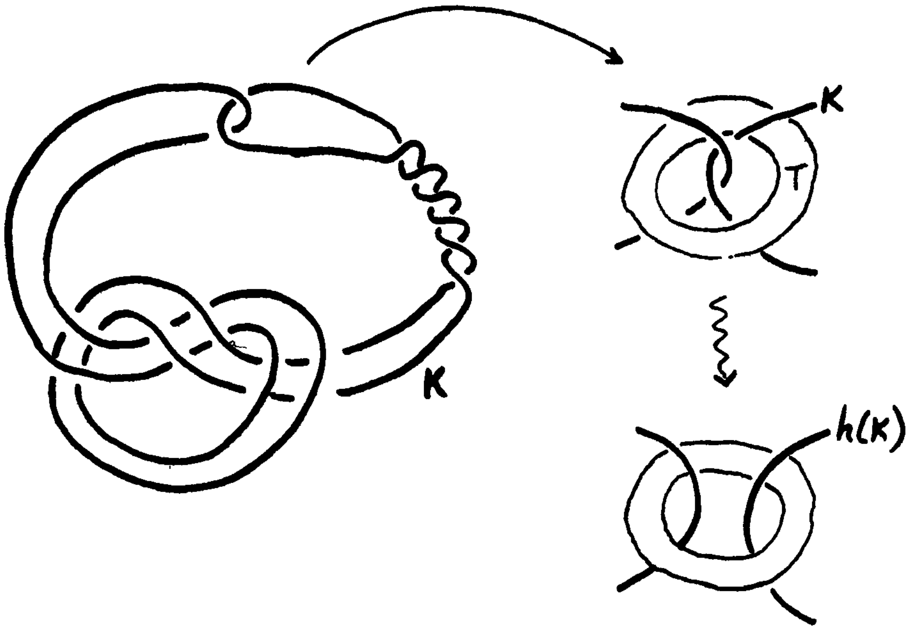
$$H_1(\tilde{X}_3) = (\alpha_o, \alpha_1, \alpha_2, \gamma; \text{ relations (1), (2), (3)})$$

We leave the reader to check that this boils down to

$$H_1(\tilde{X}_3) \cong Z \oplus Z/4 \oplus Z/4 \ .$$

2. EXERCISE : Recompute the homology of the finite cyclic covers of the complement of the trefoil using surgery. (Note that the pictures might look the same as the figure eight case, except that $h(\mu)$ will be different).

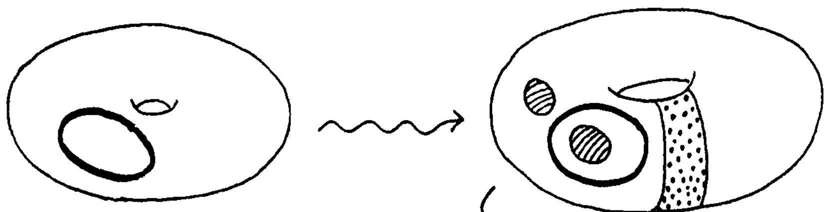
3. EXERCISE : Consider the "doubled" knot K shown below, and remove the solid torus T indicated. Construct the k-fold cyclic cover by the techniques of this section, and show that the any two liftings of T (in the cover of the complement of the trivial knot $h(K)$) are homologically unlinked. Conclude that each $\tilde{X}_k$ has the homology of S^1 just as is the case for the trivial knot.



A knot with trivial torsion numbers : $H_*(\tilde{X}_k) \cong H_*(S^1)$ for all $k = 1, 2, 3, \ldots$.

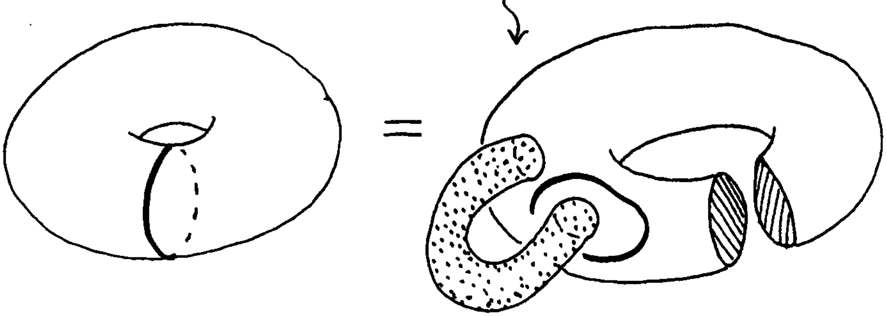
D. SURGERY DESCRIPTION OF KNOTS

The technique used in the last section is quite a useful one. In effect, we may construct a knot (in that case the figure-eight) by starting with the unknot in S^3 and altering its complement by removing solid tori from S^3 (missing the unknot) and replacing them by attaching their boundaries in a different manner. Specifying these solid tori and the attaching homeomorphisms will be called a <u>surgery description</u> of the knot. The alteration of S^3 converts it again into S^3, but the set which used to be unknotted is now knotted in the new version of S^3. By way of analogy, here is a similar procedure by which the knot type of a curve in T^2 is changed by fiddling with its complement.



the unknot in T^2

remove annulus $(S^1 \times D^1)$ and two disks $(D^2 \times S^0)$



what was the unknot is now a nontrivial knot in the new T^2.

replace them in a different way

1. THEOREM: Every tame knot in S^3 has a surgery description.

In other words, given a tame knot K in S^3 there exist pairwise disjoint solid tori $V_1, \ldots, V_r$ in $S^3 - K$ and $W_1, \ldots, W_r$ in $S^3 - T$ (T = the trivial knot) and a homeomorphism $S^3 - (W_1 \cup \ldots \cup W_r) \to S^3 - (V_1 \cup \ldots \cup V_r)$ which throws T onto K. (Why is this an equivalent statement?) The proof of this statement is the content of the next two exercises.

2. EXERCISE: In any regular projection of a knot K in R^3 it is possible to change some overcrossings to undercrossings so that K becomes a trivial knot.

3. EXERCISE: Prove the theorem. Show, moreover, that we may take $V_1 \cup \ldots \cup V_r$ to be a tubular neighbourhood of a trivial link; likewise $W_1 \cup \ldots \cup W_r$. Also, the V_i may be chosen so that their central curves have linking number zero with K; likewise for the W_i and T. Finally, the homeomorphism may be chosen so that the image of a meridian on ∂W_i is the sum of a meridian and a preferred longitude on ∂V_i.

4. REMARK: Don't assume that if one removes solid tori from S^3 and replaces them in any old way, then the result is S^3. In fact the result is usually some other 3-manifold: that's what Chapter 9 is all about. Only certain "surgeries" convert S^3 back to S^3.

CHAPTER 7. INFINITE CYCLIC COVERINGS AND THE ALEXANDER INVARIANT

A. THE ALEXANDER INVARIANT. One of the most useful of all knot invariants is the Alexander polynomial. It also seems to be one of the most mystifying gadgets of knot theory to beginners in the subject. The difficulty lies in the combinatorial approach to defining it, which has been traditional in the literature. One takes a picture of the knot (Alexander [1928]) or a presentation of the knot group (Fox's Quick Trip) and concocts a matrix. Its entries are polynomials, with integer entries, in a formal variable t (which is allowed to have negative exponents, just like Laurent series). Then one computes various minors and looks for a single generator $\Delta(t)$ for the ideal thus determined in the ring of all Laurent polynomials. The mechanics of these and other algorithmic procedures are easy enough to master, but to the neophyte they might seem rather ad hoc and don't shed much light on the geometric significance of the Alexander polynomial. To make matters worse, $\Delta(t)$ does not even exist for some higher-dimensional knots, and there is some disagreement as to what should be the Alexander polynomial of a link. The root of that problem is that the rings in question are not principal ideal domains.

Alexander himself recognized that $\Delta(t)$ can be regarded as a shorthand description of the first homology $H_1(\tilde{X})$ of the infinite cyclic cover of the knot's complement. In fact the whole collection of homology groups $H_*(\tilde{X})$ has a natural geometrically-defined module structure (so does the cohomology, for that matter), which carries more information than $\Delta(t)$ alone. I think that the conceptual difficulties mentioned above can be overcome by studying the module $H_*(\tilde{X})$ itself, which has come to be known as the <u>Alexander invariant</u>. The purpose of this chapter is to describe various methods for calculating it. We will study the Alexander polynomial and its properties in the next chapter.

1. DEFINITION : The symbol Λ will denote the ring of (finite) Laurent polynomials with integer coefficients. A typical element has the form

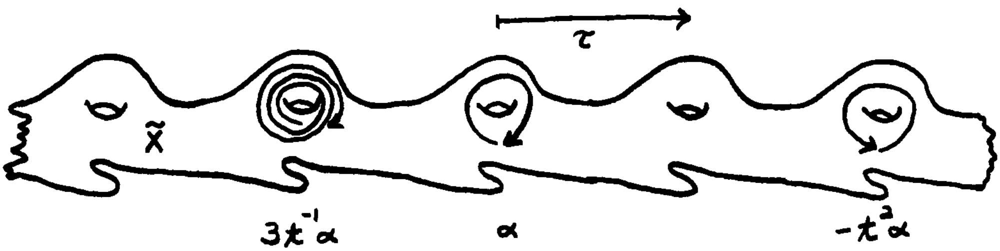
$$c_{-r}t^{-r} + \cdots + c_o + c_1 t + \cdots + c_s t^s$$

with the c_i integers. Addition and multiplication are as usual with polynomials. Other notations for Λ are $Z[t, t^{-1}]$ or $Z[J]$. The latter symbol denotes the "group ring" where J is the infinite cyclic group written multiplicatively. The units (invertible elements) of Λ are easily seen to be just the monomials $\pm t^i$.

Now consider a knot $K^n \subset S^{n+2}$. Its complement X has an infinite cyclic covering space $\tilde{X}$; we are concerned with its integral homology $H_*(\tilde{X})$ which may be considered as a collection of groups or one big graded group, according to taste. We need to describe the Λ-module structure of $H_*(\tilde{X})$. It combines the two different actions of the infinite cyclic group, one as the coefficient group and the other induced by the group of covering translations. Choose a generator $\tau: \tilde{X} \to \tilde{X}$ of the group of covering translations; two choices are possible. Now define the product of an element $p(t)$ of Λ with an element α of $H_i(\tilde{X})$ by the formula

$$p(t)\alpha = c_{-r}\tau_*^{-r}\alpha + \cdots + c_o\alpha + c_1\tau_*\alpha + \cdots + c_s\tau_*^s\alpha$$

where $p(t) = c_{-r}t^{-r} + \cdots + c_o + c_1 t + \cdots + c_s t^s$ and $\tau_*: H_i(\tilde{X}) \to H_i(\tilde{X})$ is the homology isomorphism induced by τ. Thus $p(t)\alpha$ is again an element of the i^{th} homology group of $\tilde{X}$. The choice of τ is of course fixed throughout. We can eliminate the ambiguity if K and S^{n+2} are assigned fixed orientations and we choose the τ corresponding to a loop in X which has linking number $+1$ with K. We'll see that for classical knots, a special symmetry property ensures that the choice doesn't matter anyway. Here's a pictorial description of, for example, the one-dimensional homology element $(3t^{-1} + 1 - t^2)\alpha$:



2. EXERCISE : Verify that this Λ-multiplication satisfies all the formulas which make $H_i(\tilde{X})$ a (left unitary) Λ-module. Show that equivalent knots have isomorphic Alexander invariants (as Λ-modules) in each dimension, modulo appropriate choices of τ.

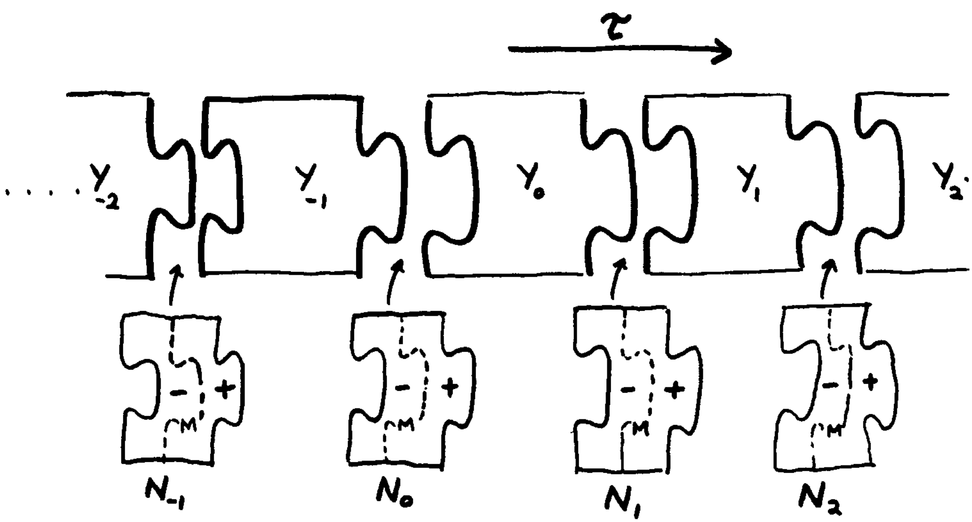
3. EXERCISE : The trivial knot $S^n \subset S^{n+2}$ has Alexander invariant: $H_i(\tilde{X}) \cong 0$, all $i > 0$, and $H_o(X) \cong \Lambda/(t-1)$, the quotient of Λ by the ideal generated by $t-1$.

We'll now turn to some techniques for computing the Alexander invariant of a knot. I think specific examples are the best way to illustrate the general methods.

B. SEIFERT SURFACES AGAIN

We saw in Chapter 5 how to construct $\tilde{X}_k$ and $\tilde{X}$ from a Seifert surface for K. The two methods outlined in Chapter 6 for computing $H_*(\tilde{X}_k)$ also adapt to $H_*(\tilde{X})$.

1. EXAMPLE. Compute the Alexander invariant of the <u>trefoil</u> K. Construct $\tilde{X}$ from copies of $Y = S^3 - M$ and $N \cong \mathring{M} \times (-1, 1)$, where M is the Seifert surface of Example 6B1. Schematically $\tilde{X}$ is



Now $H_1(\bigcup_{-\infty}^{\infty} Y_i)$ is an infinitely generated free abelian group, generated by $\{t^i\alpha, t^i\beta\}$ $(i \in Z)$, where α, β are the 1-homology generators of Y_o. As a Λ-module, $H_1(\bigcup_{-\infty}^{\infty} Y_i)$ is free on two generators α, β .

Recall from example 6B1 that we had (with $\alpha, \beta \in H_1(Y)$ $a, b \in H_1(\mathring{M})$) :

$$\beta - \alpha \xleftarrow{-} a \xrightarrow{+} -\alpha$$

$$-\beta \xleftarrow{-} b \xrightarrow{+} \alpha - \beta .$$

This means that when we attach Y_o to Y_1 with N_1 in $\tilde{X}$ we introduce the relations in $H_1(\tilde{X})$:

$$\beta - \alpha = -t\alpha \qquad (1)$$

$$-\beta = t(\alpha - \beta) \qquad (2)$$

In general, adding N_i gives the relations

$$t^{i-1}(\beta - \alpha) = -t^i\alpha \qquad (1_i)$$

$$-t^{i-1}\beta = t^i(\alpha - \beta) . \qquad (2_i)$$

Thus as an abelian group we have the infinite presentation

$$H_1(\tilde{X}) = (\{t^i\alpha, t^i\beta\} ; \text{relations } (1_i), (2_i)), \quad i \in Z$$

but as a Λ-module we have the presentation

$$H_1(\tilde{X}) \cong (\alpha, \beta \text{ ; relations (1), (2)}) .$$

Note that all the relations (1_i), (2_i) are equivalent to (1), (2) since each t^i is a unit in Λ .

Relation (1) can be used to eliminate $\beta = \alpha - t\alpha$ and we obtain Λ- module presentations

$$\begin{aligned} H_1(\tilde{X}) &\cong (\alpha \ ; -(\alpha - t\alpha) = t(\alpha - (\alpha - t\alpha))) \\ &\cong (\alpha \ ; 0 = t^2\alpha - t\alpha + \alpha) \\ &\cong (\alpha \ ; (t^2 - t + 1)\alpha = 0) \\ &\cong \Lambda/(t^2 - t + 1) . \end{aligned}$$

Thus the 1-dimensional Alexander invariant of the trefoil is the cyclic* Λ-module of order $t^2 - t + 1$.

2. EXERCISE. Verify this calculation by means of a Mayer-Vietoris sequence. Note that the maps of the sequence are also Λ-module maps since τ_* commutes with the inclusion homomorphisms and homology boundary maps.

3. EXERCISE. Show that, as a group, $H_1(\tilde{X})$ is free abelian on two generators

4. REMARK. Not all knots have $H_1(\tilde{X})$ finitely generated as a group. For more on that question, see 10H6.

5. EXERCISE. For the above example, $H_o(\tilde{X}) \cong \Lambda/(t-1)$, $H_k(\tilde{X}) = 0$, $k \geq 2$.

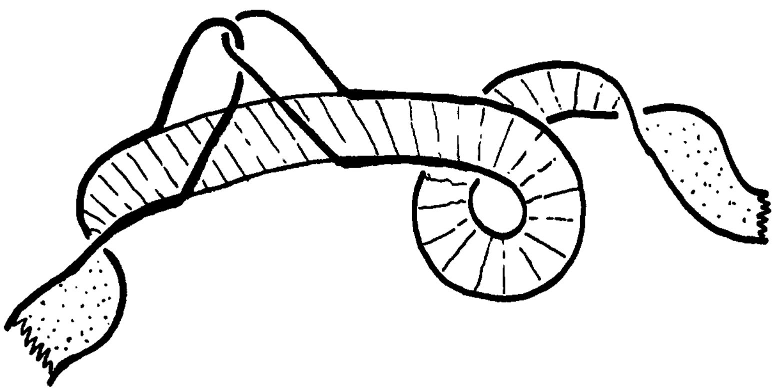
* meaning one-generator

6. EXERCISE. By similar techniques calculate that the Alexander invariants of the figure eight are :

$$H_1(\tilde{X}) \cong \Lambda/(t^2 - 3t + 1)$$

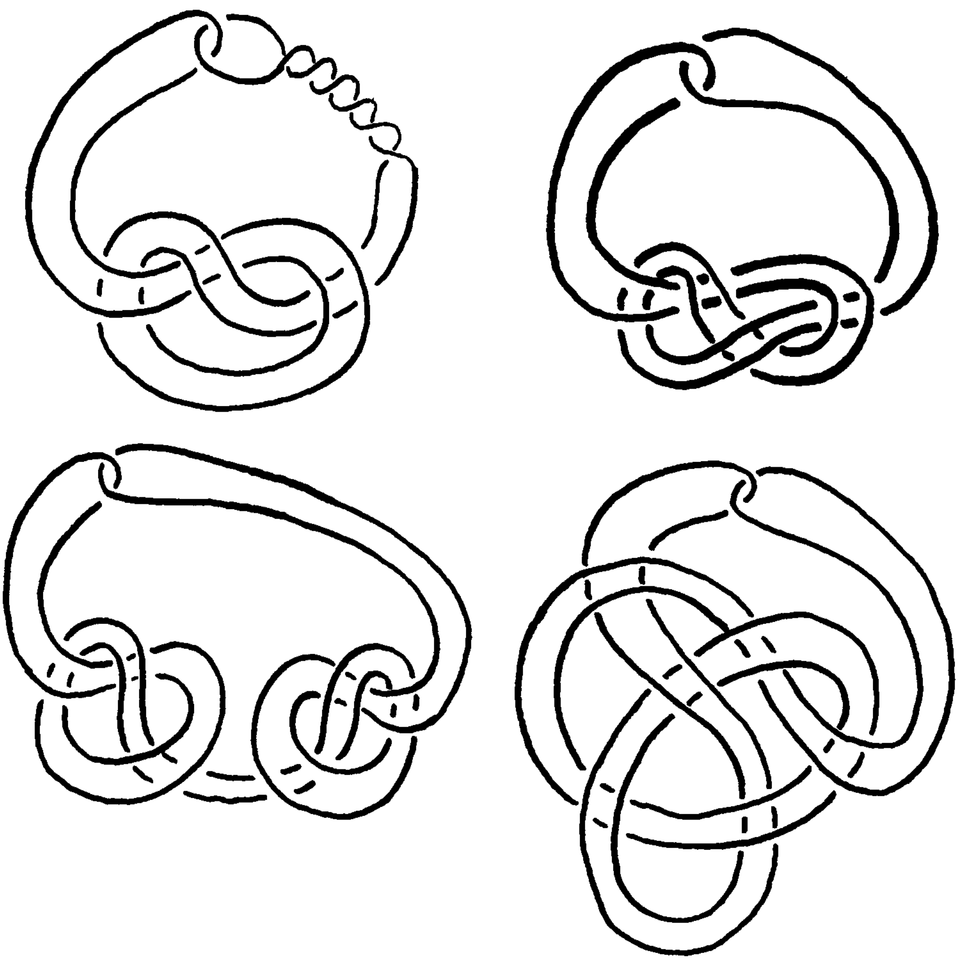
$$H_p(\tilde{X}) \cong 0, \quad p \geq 2 .$$

Consider the <u>doubled knots</u> described in 4D4. Picture the knot K as lying along the boundary components of an annulus embedded in S^3, except in one portion where it hooks itself and doubles back as shown. Define the <u>twisting number</u> of K to be the linking number of the two boundary curves of the annulus, both oriented as to run around the annulus in the same direction. (this is just the q of 4D4)



7. EXERCISE. Prove that any doubled knot has Alexander invariant $H_1(\tilde{X}) \cong \Lambda/(mt^2 + (1 - 2m)t + m)$, $0 \cong H_2(\tilde{X}) \cong H_3(\tilde{X}) \cong \ldots$, where m is its twisting number. [Hint : extend the annulus to a Seifert surface.]

8. EXAMPLE. Knots with trivial Alexander invariant. According to the exercise, a doubled knot with twisting number = 0 has $H_1(\tilde{X}) \cong \Lambda/(t) \cong \Lambda/(1) \cong 0$. For example the following knots all have trivial Alexander invariant, and one must turn to the knot group or other methods to distinguish these knots from each other or from the trivial knot. In fact it may be shown that K determines the knot type of the centerline of its associated annulus.

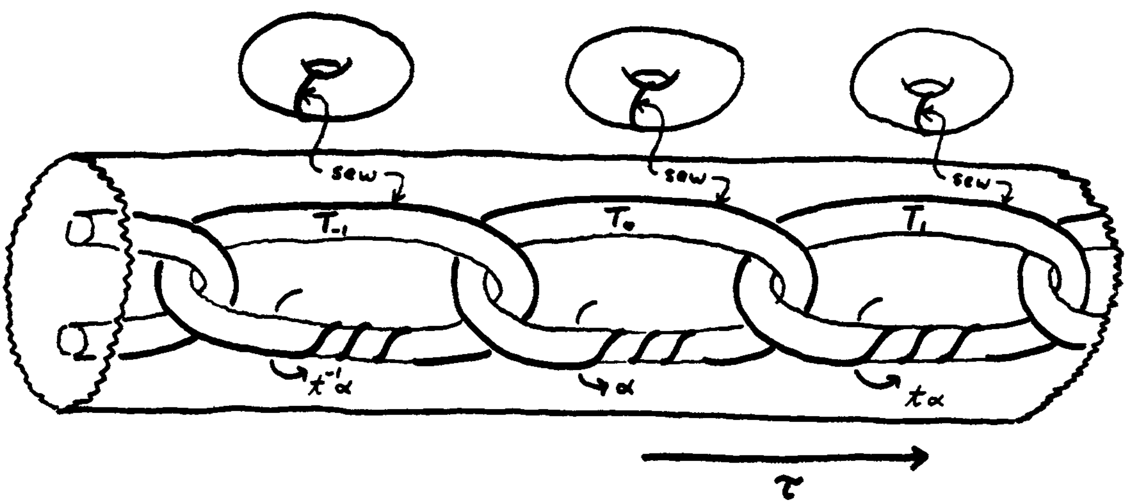


9. EXERCISE. Verify that the torsion invariants $H_1(\tilde{X}_k)$ of these knots are also the same as those of the trivial knot.

C. SURGERY AGAIN.

The revisions to the methods of 6C necessary to compute the homology of the infinite cyclic cover should be obvious.

1. EXAMPLE. Find the Alexander invariant of the figure eight knot. The infinite cyclic cover of the complement may be constructed thus (cf. Example 6C1). The dark curves drawn on the boundaries of the T_i show where to sew the meridians when the solid tori are removed and sewn back into the cover $B^2 \times R^1$ of the complement of the unknot.

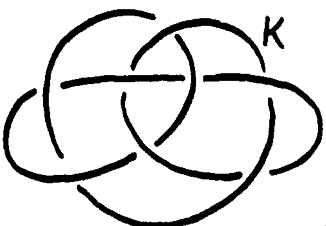


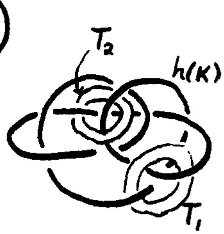
The 1-dimensional homology of the complement of the solid tori $\ldots T_{-1}, T_o, T_1, \ldots$ is freely generated (as a Λ-module) by α as shown. Sewing in T_1 adds the relation $t^2\alpha - 3t\alpha + \alpha = 0$. Sewing in the others give relations which are equivalent (over Λ) to this one, so

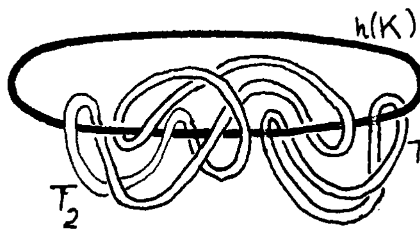
$$H_1(\tilde{X}) \cong (\alpha \; ; \; (t^2 - 3t + 1)\alpha = 0)$$

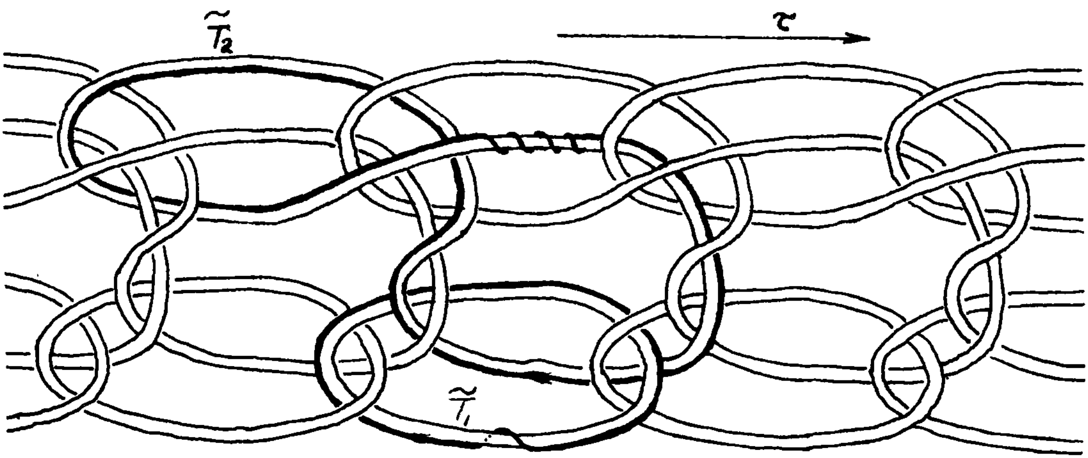
$$\cong \Lambda/(t^2 - 3t + 1) \; .$$

2. EXAMPLE. The knot 8_{16}. Find its Alexander invariants. Since two crossings must be changed to untie K, we remove two solid tori T_1, T_2.









A presentation for $H_1(\tilde{X})$ has generators α, β and relations :

$$(-t^{-1} + 1 - t)\alpha + (t^{-1} - 1)\beta = 0$$

$$(t - 1)\alpha + (t^{-2} - 3t^{-1} + 5 - 3t^{1} + t^{2})\beta = 0$$

Adding the two yields :

$$-t^{-1}\alpha + (t^{-2} - 2t^{-1} + 4 - 3t + t^{2})\beta = 0 \qquad \text{or}$$

$$\alpha = (t^{-1} - 2 + 4t - 3t^{2} + t^{3})\beta$$

which we use to eliminate α. Thus

$$\begin{aligned} H_1(\tilde{X}) &= (\beta \ ; \ (-t^{-1} + 1 - t)(t^{3} - 3t^{2} + 4t - 2 + t^{-1})\beta + (t^{-1} - 1)\beta = 0) \\ &\cong (\beta \ ; \ (-t^{-2} + 4t^{-1} - 8 + 9t - 8t^{2} + 4t^{3} - t^{4})\beta = 0) \\ &\cong \Lambda/(1 - 4t + 8t^{2} - 9t^{3} + 8t^{4} - 4t^{5} + t^{6}) \end{aligned}$$

As before, $H_2(\tilde{X}) \cong H_3(\tilde{X}) \cong \ldots \cong 0$.

3. EXERCISE : We haven't verified that the spaces $\tilde{X}$ and $\tilde{X}_k$ constructed in this section and in 6C are in fact the universal abelian and the k-fold cyclic covers of the knot complement X . Do this by showing that they are indeed covering spaces and that $\tilde{X}$ corresponds to the kernel of

$$\pi_1(X) \xrightarrow{\text{abelianization}} Z$$

whereas $\tilde{X}_k$ corresponds to the kernel of the composite

$$\pi_1(X) \xrightarrow{\text{abelianization}} Z \xrightarrow{\text{projection}} Z/k$$

4. EXERCISE : Show that the only interesting Alexander invariant for classical knots is in dimension one. That is, if X is the complement of a tame knot in S^3 and $\tilde{X}$ is the infinite cyclic cover, then $H_i(\tilde{X})$ is always zero when $i>1$ and $H_o(\tilde{X})$ is always the cyclic module of order $1 - t$. [One approach to this might be to use a surgery description of a knot, as described in 6D, to show that any such $\tilde{X}$ can be constructed by performing surgeries on $B^2 \times R^1$.]

I think the following theorem is originally due to Seifert.* The proof given here is due to Levine, and is a nice illustration of the use of surgery to construct knots with desired properties. A similar application, in higher dimensions, will be given in chapter 11.

5. THEOREM : Let $p(t)$ be any Laurent polynomial satisfying:

(i) $p(1) = \pm 1$ (notice this is the sum of the coefficients)

(ii) $p(t) = p(t^{-1})$ (the coefficients are symmetric)

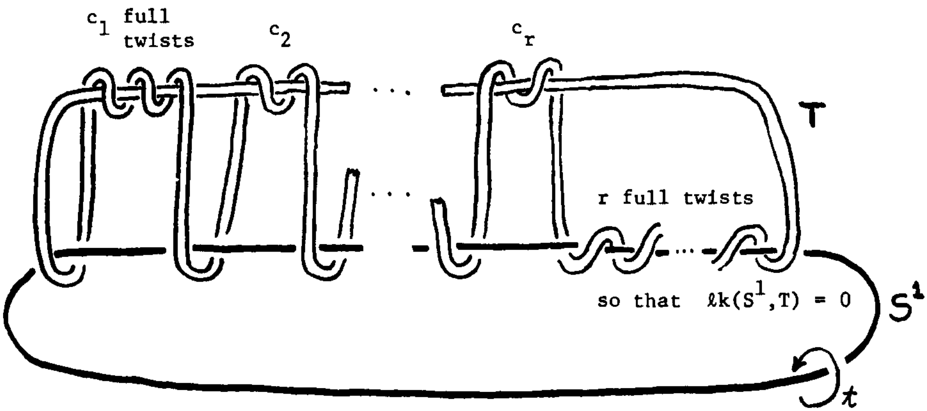
Then there exists a tame knot in S^3 whose 1-dimensional Alexander invariant is $H_1(\tilde{X}) \cong \Lambda/(p(t))$.

PROOF : Without loss of generality $p(1) = 1$ and we may write out

$$p(t) = c_r t^{-r} + \cdots + c_1 t^{-1} + c_o + c_1 t + \cdots + c_r t^r$$

Arrange a solid torus T in the complement of the unknot S^1 in S^3 as shown in the following picture. Let J be a simple closed curve on ∂T which is a longitude and has linking number +1 with the centreline of T (thus it's not a preferred longitude). Noting that T itself is

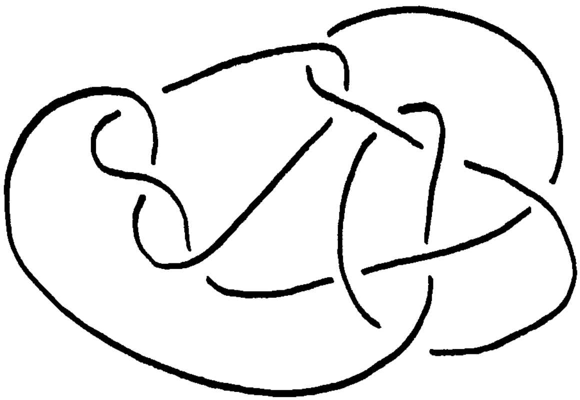
* see Seifert [1934], page 589, Satz 6

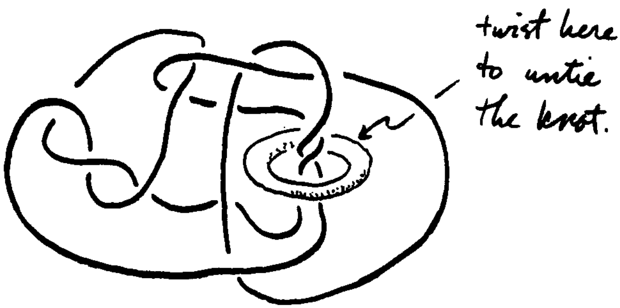


(in this example, $c_1 = 3$, $c_2 = 2$, ... , $c_r = -2$)

unknotted, we may perform a surgery by removing the interior of T from S^3 and sewing a solid torus back to the resulting space, attaching boundaries in such a way that a meridian is sewn onto J. The new space is again S^3 but the S^1, previously unknotted, is embedded in the new S^3 as a knot, which we'll call K. To visualise K explicitly we could straighten out T by a homeomorphism of S^3 (the S^1 will then look tangled). Then give the complement of T a twist, turning the S^1 into a picture of K. The claim is that K is a knot with Alexander invariant as prescribed in the conclusion of the theorem. Noting that T and S^1 have linking number zero, we can construct the infinite cyclic cover $\tilde{X}$ associated with K by removing the solid tori $\ldots, \tilde{T}_{-1}, \tilde{T}_o, \tilde{T}_1, \ldots$ which lie above T in the cover $\mathring{D}^2 \times R^1$ of $S^3 - S^1$ and sewing them back so that meridians are attached to the lifts $\ldots, \tilde{J}_{-1}, \tilde{J}_o, \tilde{J}_1, \ldots$ of J. We leave the resulting calculation of $H_1(\tilde{X})$ as an EXERCISE, noting that with suitable linking convention in $\mathring{D}^2 \times R^1$, $\tilde{J}_o$ has linking number $c_{\pm i}$ with the centreline of $\tilde{T}_i$ (for $i=0$ use $c_o = 1 - 2c_1 - \cdots - 2c_r$).

6. EXERCISE : The following knot was discovered by Conway [1970]* in enumerating knots with eleven crossings. Show that its Alexander polynomial is trivial by considering surgery on the solid torus pictured.





* see also Montesinos [1975].

D. COMPUTING THE ALEXANDER INVARIANT $H_1(\tilde{X})$ FROM $\pi_1(X)$.

Let $K^n \subset S^{n+2}$ be a knot with complement X and universal abelian cover $p : \tilde{X} \longrightarrow X$. Recall that $p_* : \pi_1(\tilde{X}) \longrightarrow \pi_1(X)$ is an isomorphism onto the commutator subgroup $C = [G, G]$ of $G = \pi_1(X)$. This induces the group isomorphism

$$\overline{p_*} : H_1(\tilde{X}) \longrightarrow C/[C, C].$$

This section describes how to define a Λ-module structure on $C/[C, C]$ in a purely algebraic manner, which makes the above a Λ-isomorphism. This leads to a method for computing a Λ-module presentation for the one-dimensional Alexander invariant, given a presentation for the knot group.

1. DEFINITION. Suppose $c \in C$. Let $x \in G$ be an element which is sent to a generator of $G/C \cong Z$ under abelianization. Define

$$t\{c\} = \{xcx^{-1}\}$$

where $\{\ \}$ denotes the coset in $C/[C, C]$.

2. EXERCISE. Verify that $t : C/[C, C] \longrightarrow C/[C, C]$ is well-defined. (That is, if c and $d \in C$ are congruent $\mathrm{mod}[C, C]$ and x and $y \in G$ are congruent $\mathrm{mod}[G, G]$, then xcx^{-1} and ydy^{-1} are elements of C and $(xcx^{-1})(ydy^{-1})^{-1} \in [C, C]$.)

3. EXERCISE. Show that the following diagram commutes:

$$\begin{array}{ccc} H_1(\tilde{X}) & \xrightarrow{\overline{p}_*} & C/[C, C] \\ \downarrow{\scriptstyle\tau_*} & & \downarrow{\scriptstyle t} \\ H_1(\tilde{X}) & \xrightarrow{\overline{p}_*} & C/[C, C] \end{array}$$

provided the proper choice of generator of G/C is made (of the two alternatives) in the definition of t.

Thus the natural $Z[t, t^{-1}] = \Lambda$-module structure on $C/[C, C]$ implicit from the definition by linearity makes $\overline{p}_*$ a Λ-isomorphism.

We now describe a method for computing a presentation for the one-dimensional Alexander invariant from a presentation of $\pi_1(X)$. It should be realized that the higher-dimensional Alexander invariants do <u>not</u> depend simply on $\pi_1(X)$; see section G of this chapter for examples.

4. EXERCISE. If a knot group $G \cong \pi_1(X)$ is finitely presentable, then it has a presentation of the form

$$G \cong (x, a_1, \ldots, a_p;\ r_1, \ldots, r_q)$$

where $x \longrightarrow 1$ under the abelianization $G \longrightarrow G/C = Z$ and $a_i \in C$ (i.e. $a_i \longrightarrow 0$). For tame knots of dimension 1 in S^3 we may take $p = q$.

5. EXERCISE. Notation being as above, the commutator subgroup C of G is generated by all words of the form :

$$x^k a_i^{\pm 1} x^{-k} .$$

Furthermore, each r_i is equivalent (in fact conjugate) to a word r_i' which is a product of words of that form. [Hint : check that the exponents of x in r_i must add to zero.]

So we may obtain a Λ-module presentation of $C/[C, C]$ by taking generators $\alpha_1, \ldots, \alpha_p$ (the images of $a_1, \ldots, a_p$ under abelianization) and formally rewriting r_i' additively, substituting

$$\pm t^k \alpha_i \quad \text{for} \quad x^k a_i^{\pm 1} x^{-k}.$$

6. EXAMPLE. The <u>trefoil</u> knot group has presentation

$$G \cong (x, y; yxyx^{-1}y^{-1}x^{-1})$$

where $x \longrightarrow 1$, $y \longrightarrow 1$ under abelianization. Let $a = yx^{-1}$, so

$$G \cong (x, a; ax^2ax^{-1}a^{-1}x^{-1})$$

$$\cong (x, a; a(x^2ax^{-2})(xa^{-1}x^{-1})) .$$

Thus a Λ-module presentation for $C/[C, C]$ is

$$C/[C, C] \underset{\Lambda}{\cong} (\alpha; \alpha + t^2\alpha - t\alpha)$$

which checks with the calculation of B1 : $H_1(\tilde{X}) \cong \Lambda/(t^2-t+1)$

7. EXAMPLE. Fox's 'Quick trip' example 12, on pages 136-7 describes, by 'level curves', a knotted S^2 in S^4 whose group has presentation :

$$G \cong (x,\ a;\ xa^2 = ax,\ a^2x = xa)$$

$$\cong (x,\ a;\ xa^2x^{-1} = a,\ a^2 = xax^{-1})\ .$$

This has the form described in exercise four , so we have

$$C/[C,\ C] \underset{\Lambda}{\cong} (\alpha;\ 2t\alpha = \alpha,\ 2\alpha = t\alpha)$$

Thus the one-dimensional Alexander invariant

$$H_1(\tilde{X}) \cong \Lambda/(2t - 1,\ 2 - t)$$

is a cyclic Λ-module whose order ideal is non-principal. This is an example of a knot which has no Alexander polynomial.

8. EXERCISE. Following is an outline of a calculation of the Alexander invariant of the <u>torus knot</u> of type p, q with p and q coprime positive integers. Fill in the details.

(i) The knot group has presentation $G \cong (u,\ v;\ u^p = v^q)$ where $u \longmapsto q$, $v \longmapsto p$ under abelianization.

(ii) Choose integers r, s satisfying $pr + qs = 1$, $r > 0$, $s < 0$. Let $x = u^s v^r$, $a = ux^{-q}$, $b = vx^{-p}$ to obtain the presentation with $x \longrightarrow 1$, $a \longrightarrow 0$, $b \longrightarrow 0$:

$$G \cong (x,\ a,\ b\ ;\ (ax^q)^p = (bx^p)^q,\ x = (ax^q)^s(bx^p)^r\ ).$$

(iii) $C/[C, C]$ has a Λ-module presentation with generators α, β and relations :

$$(t^q + t^{2q} + \dots + t^{pq})\alpha = (t^p + t^{2p} + \dots + t^{qp})\beta$$

$$(t^q + t^{2q} + \dots + t^{(-s)q})\alpha = (t^p + t^{2p} + \dots + t^{rp})\beta$$

(iv) $H_1(\tilde{X}) \cong \Lambda/(\Delta(t))$ where

$$\Delta(t) = \frac{(1 - t)(1 - t^{pq})}{(1 - t^p)(1 - t^q)}$$

9. REMARK. You should verify in the calculation above that $\Delta(t)$ is, in fact a polynomial. For example, the torus knot of type 4, 7 has

$$\Delta(t) = \frac{(1 - t)(1 - t^{28})}{(1 - t^4)(1 - t^7)}$$

$$= 1 - t + t^4 - t^5 + t^7 - t^9 + t^{11} - t^{13} + t^{14} - t^{17} + t^{18} .$$

10. EXERCISE. Show that the polynomials which arise for torus knots of types p,q and r,s are distinct unless $\{|p|, |q|\} = \{|r|, |s|\}$. [Hint: which roots of unity are roots of $\Delta(t)$?] Use this to give another proof of Schreier's theorem (3C1).

E. ADDITIVITY OF THE ALEXANDER INVARIANT.

1. THEOREM : Let $K^n = K_1^n \# K_2^n$ be a composite knot in S^{n+2} and let X, X_1 and X_2 denote their respective knot complements. Then their Alexander invariants are connected by the Λ-isomorphisms:

$$H_i(\tilde{X}) \cong H_i(\tilde{X}_1) \oplus H_i(\tilde{X}_2) \qquad \text{for all } i > 0,$$

assuming appropriate choices of generators $\tau: \tilde{X}_i \to \tilde{X}_i$ of the covering translation group, determining the Λ-action. (Orientations would take care of this.)

PROOF : By definition of knot sum there is an S^{n+1} in S^{n+2} which intersects K in an S^{n-1}, unknotted in S^{n+1}, and separates the complement of K into two parts which (by abuse of notation) we may call X_1 and X_2 since they have the same homotopy type of the complements of K_1 and K_2. Thus we may write $X = X_1 \cup X_2$ and the intersection $X_1 \cap X_2 = S^{n+1} - S^{n-1}$ is a trivial knot complement which we call W for short. Likewise we may write $\tilde{X} = \tilde{X}_1 \cup \tilde{X}_2$ and $\tilde{W} = \tilde{X}_1 \cap \tilde{X}_2$ where $\tilde{}$ means universal abelian cover in each case. Now examine the exact Mayer-Vietoris sequence (the maps are easily checked to be Λ-module homomorphisms):

$$\cdots \longrightarrow H_i(\tilde{W}) \longrightarrow H_i(\tilde{X}_1) \oplus H_i(\tilde{X}_2) \longrightarrow H_i(\tilde{X}) \longrightarrow H_{i-1}(\tilde{W}) \longrightarrow \cdots$$

But W is a homotopy S^1 and so $\tilde{W}$ is contractible. Therefore $H_k(\tilde{W}) = 0$ when $k > 0$ and the result for $i > 1$ follows immediately.

2. EXERCISE : Show that the middle arrow in the above sequence is also an isomorphism when $i = 1$ by examining the whole tail of the sequence. This will complete the proof.

F. HIGHER-DIMENSIONAL EXAMPLES: PLUMBING.

This section describes some codimension two knots which may be distinguished by their higher-dimensional Alexander invariants. Reversing the usual procedure, we construct a Seifert surface first and then take its boundary for our knot.

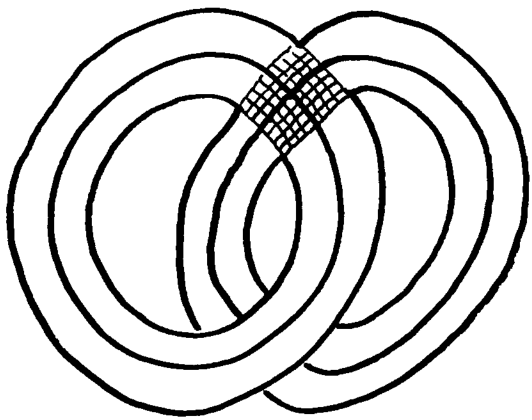
1. DEFINITION. The manifold obtained by plumbing a p-sphere and a q-sphere is defined as the disjoint union of $S^p \times D^q$ and $D^p \times S^q$ with their common $D^p \times D^q$ identified via the identity homeomorphism.* Here $D^p \subset S^p$ and $D^q \subset S^q$ are regarded as inclusions of hemispheres. Write

$$P^{p+q} = (S^p \times D^q) \bigcup_{D^p \times D^q} (D^p \times S^q).$$

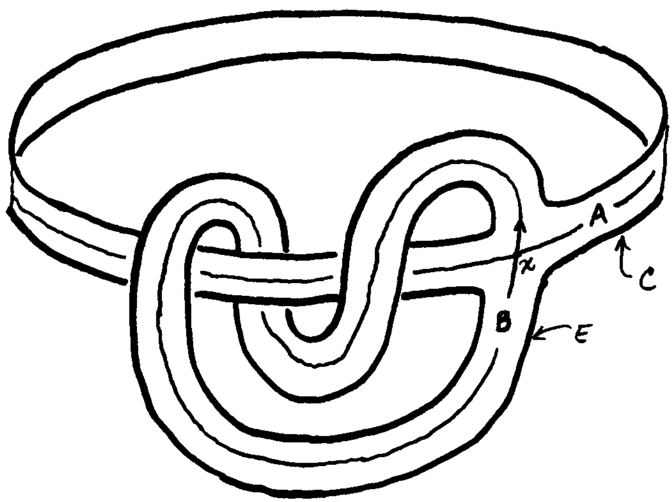
2. EXERCISE. There is a natural inclusion of the wedge, or one-point union, $S^p \vee S^q \subset P$; and this is a homotopy equivalence (in fact a strong deformation retract). The boundary ∂P is a sphere of dimension $p + q - 1$. [Hint: consider $S^p \times S^q$]

* actually this is just a special case of a more general type of plumbing whereby disk bundles are fused together.

3. EXAMPLE : Plumbing of two 1-spheres.



Another embedding in 3-space with knotted boundary:



4. EXAMPLE. A knotted S^2 in S^4.

Imitate the above picture to construct a plumbing of S^2 and S^1 in S^4. Let A be the standard S^2 in S^4, bounding a standard disk $D^3 \subset S^4$. Now D has a bicollar neighborhood $N \cong D^3 \times [-1,1]$ which restricts to a bicollar (but not neighborhood) $C \cong A \times [-1,1]$. Then construct an oriented simple closed curve B in S^4 which runs straight through N in two places along a $[-1,1]$ fibre in the direction $-1 \to 1$: once through the interior of D^3 and once through a point x of A. This extends to a bicollared $E \cong D^2 \times B$ in S^4 such that $C \cup E$ is, in a natural way, the plumbing of A and B. Let $M = C \cup E$ and $K = \partial M$.

5. EXERCISE. Describe explicitly such a B and E and show that M may be bicollared (so that M is a Seifert surface for K).

We now compute the Alexander invariant of the above example. If we specify the positive side of the bicollar to be the one which intersects D near x, we have:

$$\left.\begin{aligned} lk(a,b^+) &= 2 = lk(a^-,b) \\ lk(a^+,b) &= 1 = lk(a,b^-) \end{aligned}\right\} (1)$$

where a is the 2-cycle in S^4 carried by A (and suitably oriented), b is the 1-cycle carried by B, and lk is the linking pairing* of 2-cycles and 1-cycles in S^4. The infinite cyclic cover $\tilde{X}$ of $X = S^4 - K$

* some fixed orientation of S^4 is assumed

has fundamental region $Y = S^4 - M$. By Alexander duality:

$$H_1(Y) = Z \text{ generated by } \alpha^1 \text{ (dual to } a^2)$$
$$H_2(Y) = Z \text{ generated by } \beta^2 \text{ (dual to } b^1)$$

In gluing the regions together via copies of $\mathring{M}$ we have homology maps

$$\alpha \xleftarrow{-} b \xrightarrow{+} 2t\alpha \quad \text{in dimension } 1$$
$$2\beta \xleftarrow{-} a \xrightarrow{+} t\beta \quad \text{in dimension } 2$$

And we compute the Alexander invariants of K to be

$$H_1(\tilde{X}) \cong \Lambda/(2t - 1)$$
$$H_2(\tilde{X}) \cong \Lambda/(t - 2) \cong \Lambda/(2t^{-1} - 1)$$

6. EXERCISE. Show by plumbing that if $\Delta(t)$ is any polynomial of degree 1 satisfying $\Delta(1) = \pm 1$, then there is a knotted S^2 in S^4 with Alexander invariants

$$H_1(\tilde{X}) = \Lambda/(\Delta(t)) \qquad H_2(\tilde{X}) = \Lambda/(\Delta(t^{-1})).$$

7. EXERCISE. Generalize this construction to obtain a bicollared embedding $f:P \to S^{p+q+1}$ of the plumbed space

$$P = S^p \times D^q \bigcup_{D^p \times D^q} D^p \times S^q$$

into the $(p + q + 1)$-sphere so that the formulas (1) hold, where a and b are p- and q- cycles carried by $f(S^p \times 0)$ and $f(0 \times S^q)$, respectively. Let $K = f(\partial P)$. This is a knotted $(p + q - 1)$-sphere in the $(p + q + 1)$-sphere. Verify that the Alexander invariants of K are:

Case 1: $(p \neq q)$ $\quad H_p(\tilde{X}) \cong \Lambda/(2t - 1)$

$\qquad\qquad\qquad\quad H_q(\tilde{X}) \cong \Lambda/(t - 2)$

Case 2: $(p = q)$ $\quad H_p(\tilde{X}) \cong \Lambda/(2t - 1) \oplus \Lambda/(t - 2)$

8. REMARK: In case $p = q$, we must also worry about the self-linking numbers: $\mathrm{lk}(a,a^+) = \mathrm{lk}(a^{-1},a)$ and $\mathrm{lk}(b,b^+) = \mathrm{lk}(b^-,b)$. The above calculation assumes the construction is such that both numbers are zero. The interested reader might show that these may be made whatever one chooses, in fact we may manufacture any matrix of linking numbers of the form:

$$\begin{array}{cc} & \begin{array}{cc} a^+ & b^+ \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} x & z \\ z\pm 1 & y \end{pmatrix} \end{array}.$$

G. NONTRIVIAL KNOTS WITH GROUP Z.

1. THEOREM: If $n \geq 3$, there exist nontrivial knots K^n in S^{n+2} with infinite cyclic knot group: $\pi_1(S^{n+2} - K^n) \cong Z$.

PROOF. Choose any $p \geq 2$, $q \geq 2$ such that $n = p + q - 1$ and consider the knot $K = f(\partial P)$ constructed by plumbing S^p and S^q as in the last exercise of the previous section. K has a Seifert surface $f(P)$ which has the homotopy type of $S^p \vee S^q$, a simply-connected space. Further, since $(n + 2) - p = q + 1 \geq 3$ and likewise $(n + 2) - q \geq 3$ any loop in S^{n+2} shrinks missing the $S^p \vee S^q$ (by general position) and therefore shrinks missing $f(P)$ and so $\pi_1(S^{n+2} - f(P)) \cong 1$. Using the construction of $\tilde{X}$ outlined in Chapter 5 , we see that $\pi_1(\tilde{X}) \cong \underline{1}$, so the infinite cyclic cover of $X = S^{n+2} - K^n$ is also the universal cover. It follows that $\pi_1(X) \cong Z$; the Alexander invariant shows that K is nontrivial.

2. COROLLARY: For knots of dimension ≥ 3, the group does not determine the Alexander invariant.

3. EXERCISE: Show that, in fact, for each $n \geq 3$ there are infinitely many inequivalent knots satisfying the theorem.

4. QUESTION: Is there a knotted 2-sphere in S^4 with infinite cyclic group?

5. EXAMPLE: A knotted K^4 in S^6 with $\pi_1(S^6 - K^4) \cong Z$ and $\pi_2(S^6 - K^4) \cong$ the additive group of dyadic rationals!

Plumb S^2 and S^3 in the manner we have discussed to obtain a knot $K^4 = f(\partial P^{2+3})$ with

$$\pi_1(\tilde{X}) \cong 1$$

$$H_2(\tilde{X}) \cong \Lambda\ /(2t - 1)$$

By the Hurewicz theorem, $\pi_2(\tilde{X}) \cong H_2(\tilde{X})$ as groups, and $\pi_2(\tilde{X}) \cong \pi_2(X)$ by general covering space theory $(X = S^6 - K^4)$. But the Λ-module presentation

$$(\alpha; (2t - 1)\alpha = 0)$$

corresponds to the infinite group presentation

$$(\{\alpha_i\}; \{2\alpha_{i+1} = \alpha_i\}) \quad i \in Z.$$

Sending α_i to 2^{-i} induces an isomorphism of this with the group of rationals of the form $n \cdot 2^i$, $(n, i \in Z)$.

6. EXERCISE: Compute $\pi_2(S^5 - K^3)$ where K^3 is the knot plumbed from S^2 and S^2 in S^5 as in the previous section.

7. EXERCISE : Recall that a tame knot in S^3 is unknotted if and only if its group is infinite cyclic. Find a <u>wild</u> nontrivial knot in S^3 whose group, nevertheless, is infinite cyclic.

H. HIGHER-DIMENSIONAL KNOTS WITH SPECIFIED POLYNOMIAL.

1. THEOREM : Let $\Delta(t)$ be any polynomial satisfying $\Delta(1) = \pm 1$. Then there exists a knotted 2-sphere K^2 in S^4 whose Alexander invariant is :

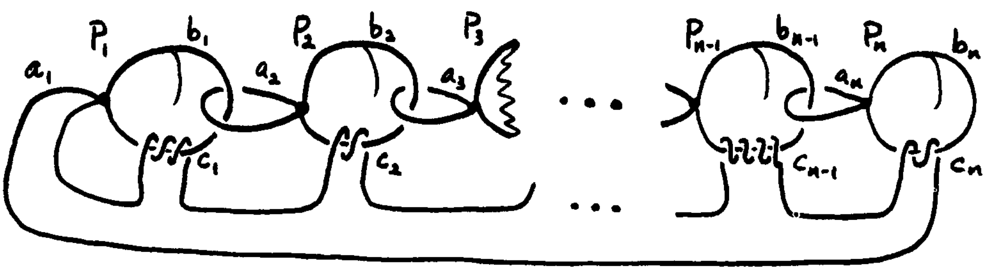
$$H_1(\tilde{X}) \cong \Lambda/(\Delta(t))$$

$$H_2(\tilde{X}) \cong \Lambda/(\Delta(t^{-1})) .$$

PROOF : Suppose $\Delta(1) = -1$ and expand in powers of $t - 1$.

$$\Delta(t) = -1 + c_1(t-1) + c_2(t-1)^2 + \ldots + c_r(t-1)^r .$$

Arrange r disjoint copies $P_1^3, \ldots, P_r^3$ of plumbings of S^1 and S^2 in S^4 so that their S^1's and S^2's link according to the schematic :



The P_i are oriented in such a way that $lk(a_i^+, b_i) = 0$ while $lk(a_i^-, b_i) = 1$, where a_i, b_i are the 1- and 2-cycles carried by the natural $S^1 \vee S^2$ in P_i . We further require that $lk(a_{i+1}, b_i) = 1$,

$i = 1, \ldots, r-1$, and $\mathrm{lk}(a_1, b_j) = c_j$, $j > 1$ while $\mathrm{lk}(a_1^+, b_1) = c_1$ and $\mathrm{lk}(a_1^-, b_1) = c_1 + 1$. Thus if α_i^1 , β_i^1 are elements of $H_*(S^4 - \cup P_i)$ dual to a_i^1, b_i^2 , we have

$$(*)\quad \left\{\begin{array}{ccccc} (1+c_1)\beta_1 + c_2\beta_2 + \cdots + c_r\beta_r & \xleftarrow{-} & a_1 & \xrightarrow{+} & c_1\beta_1 + c_2\beta_2 + \cdots + c_r\beta_r \\ \beta_1 + \beta_2 & \xleftarrow{-} & a_2 & \xrightarrow{+} & \beta_1 \\ & & \vdots & & \\ \beta_{r-1} + \beta_r & \xleftarrow{-} & a_r & \xrightarrow{+} & \beta_{r-1} \end{array}\right.$$

Connect now each P_i to P_{i+1} $(i = 1, \ldots, r-1)$ by a tube between their boundaries. That is, $T_i \cong [0,1] \times D^2$ is situated in S^4 so that $T_i \cap P_i \cong 0 \times D^2$ and lies in ∂P_i, $T_i \cap P_{i+1} \cong 1 \times D^2$ lies in ∂P_{i+1} , the T_i are otherwise disjoint from the P_j's and each other, and they are oriented compatibly with the P_i . (This is called a boundary connected sum). Let $M^3 = (\cup P_i) \cup (\cup T_i)$. Then M is connected and has the homotopy type of an r-fold wedge of S^1's, wedged with an r-fold wedge of S^2's. Further ∂M is a 2-sphere, being a connected sum of 2-spheres. Finally, the equations $(*)$ still hold in $H_1(S^4 - M)$.

Let $K = \partial M$. To compute the Alexander invariant, construct the infinite cyclic cover $\tilde{X}$ of $X = S^4 - K$ as usual by cutting along M . Then $H_1(\tilde{X})$ has generators $\beta_1, \ldots, \beta_r$ and relations

$$\begin{aligned} (1 + c_1)\beta_1 + c_2\beta_2 + \cdots + c_r\beta_r &= t(c_1\beta_1 + c_2\beta_2 + \cdots + c_r\beta_r) \\ \beta_1 + \beta_2 &= t\beta_1 \\ &\vdots \end{aligned}$$

$$\beta_{r-1} + \beta_r = t\beta_r .$$

Equivalently,

$$-1 + (t-1)c_1\beta_1 + (t-1)c_2\beta_2 + \dots + (t-1)c_r\beta_r = 0$$

$$\beta_2 = (t-1)\beta_1$$
$$\vdots$$
$$\beta_r = (t-1)\beta_{r-1} .$$

Thus $\beta_2 = (t-1)\beta_1$, $\beta_3 = (t-1)^2\beta_1$, ... can be used to eliminate all generators save β_1, and the remaining relation becomes $\Delta(t)\beta_1 = 0$ as required. The calculation of $H_2(\tilde{X})$ is similar.

2. EXERCISE : Show that if $\Delta_1(t), \dots, \Delta_k(t)$ is any sequence of polynomials satisfying $\Delta_i(1) = \pm 1$, then there is a K^2 in S^4 with

$$H_1(\tilde{X}) = \Lambda/(\Delta_1(t)) \oplus \dots \oplus \Lambda/(\Delta_k(t))$$

$$H_2(\tilde{X}) \cong \Lambda/(\Delta_1(t^{-1})) \oplus \dots \oplus \Lambda/(\Delta_k(t^{-1})) .$$

3. EXERCISE : Show that if $\Delta_1(t), \Delta_2(t)$ both satisfy $\Delta_i(t) = \pm 1$, then there is a knot K^3 in S^5 with

$$H_1(\tilde{X}) \cong \Lambda/(\Delta_1(t))$$

$$H_2(\tilde{X}) \cong \Lambda/(\Delta_2(t)) \oplus \Lambda/(\Delta_2(t^{-1}))$$

$$H_3(\tilde{X}) \cong \Lambda/(\Delta_1(t^{-1})) .$$

4. EXERCISE : Extend this construction to higher dimensions as generally as you can.

I. ALEXANDER INVARIANTS OF LINKS. In the case of links $L^n \subset S^{n+2}$ of k components, $k > 1$, the universal abelian cover is not an infinite cyclic cover. Rather its group of covering translations is a free abelian group on k generators. To see this, note that the abelianization map is the same as the Hurewicz map $\pi_1(X) \to H_1(X)$ and the one-dimensional homology of $X = S^{n+2} - L$ is isomorphic with Z^k. As with knots we combine the action of Z as coefficients and Z^k as covering automorphisms by imposing a module structure on the homology of the universal abelian cover $\tilde{X}$. The ring we must use is the ring of Laurent polynomials in k commuting variables, with integer coefficients. We'll denote the variables by $x_1, \ldots, x_k$ or $x, y, \ldots$ and call the ring Λ_k. This is just the group ring $Z[Z^k]$. Let's agree to order and orient the components $L_1, \ldots, L_k$ of L (and fix an orientation of S^{n+2}), so we have canonically-defined generators of the Z^k action on $\tilde{X}$. That is, x_i corresponds to a meridian m_i of L_i which satisfies $\ell k(m_i, L_j) = \delta_{i,j}$. The reader can easily supply a formal definition of the Λ_k -module structure on $H_*(\tilde{X})$. Notice that the Hurewicz map $\pi_1(X) \to Z^k$ may be given explicitly by

$$\omega \longmapsto (\ \ell k(\omega, L_1), \ \ldots \ , \ell k(\omega, L_k)\ )$$

and a loop ω in X lifts to a loop in $\tilde{X}$ exactly when it is in the kernel.

1. EXAMPLE :

For this link $X \cong S^1 \times S^1 \times (0, 1)$ and the universal cover is also the universal abelian cover

$$\tilde{X} \cong R^1 \times R^1 \times (0, 1) \longrightarrow S^1 \times S^1 \times (0, 1) \cong X.$$

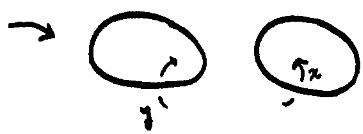
Since the total space $\tilde{X}$ is contractible the Alexander invariant of the link pictured is $H_i(\tilde{X}) \cong 0$ for all $i > 0$.

2. EXERCISE : Show that for the above example, $H_o(\tilde{X})$ is isomorphic with Λ_2 modulo the non-principal ideal generated by $x - 1$ and $y - 1$. More generally for any link L^n in S^{n+2} of k components the zero-dimensional Alexander invariant depends only on k and is

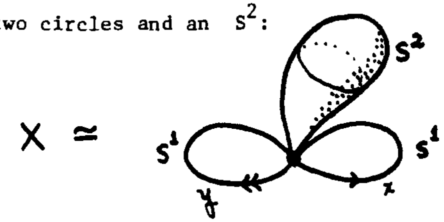
$$H_o(\tilde{X}) \cong \Lambda_k / (x_1-1, \ldots , x_k-1)$$

So we can safely ignore dimension zero.

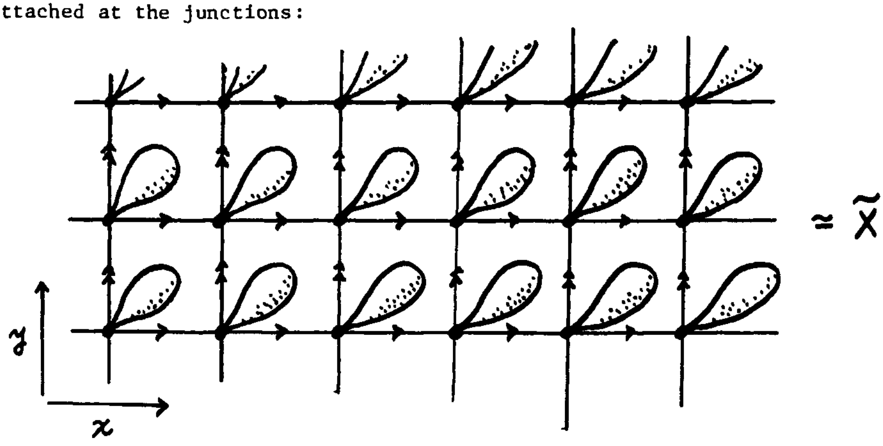
3. EXAMPLE : The trivial link of two components:



The complement X has the homotopy type of a wedge of two circles and an S^2:



The universal abelian cover then looks like an infinite net with balloons attached at the junctions:



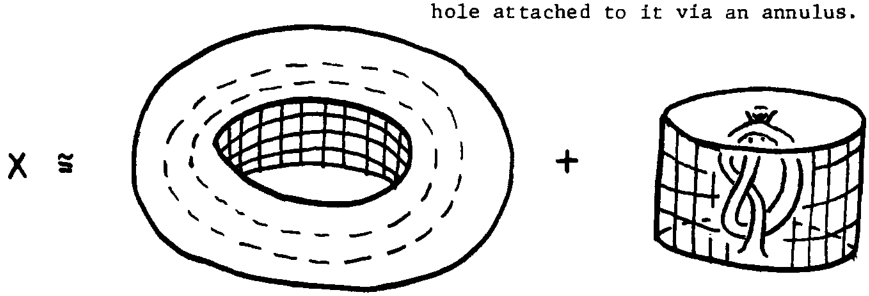
Although the homology of $\tilde{X}$ is infinitely generated as a group, in dimensions 1 and 2, each of $H_1(\tilde{X})$ and $H_2(\tilde{X})$ has a single generator as a Λ_2 -module. The H_2 generator is just one of the balloons; the H_1 generator may be taken to be a loop running around one of the little squares of the net. In fact both these Alexander invariants are free cyclic Λ_2 -modules:

$$H_1(\tilde{X}) \cong H_2(\tilde{X}) \cong \Lambda_2 .$$

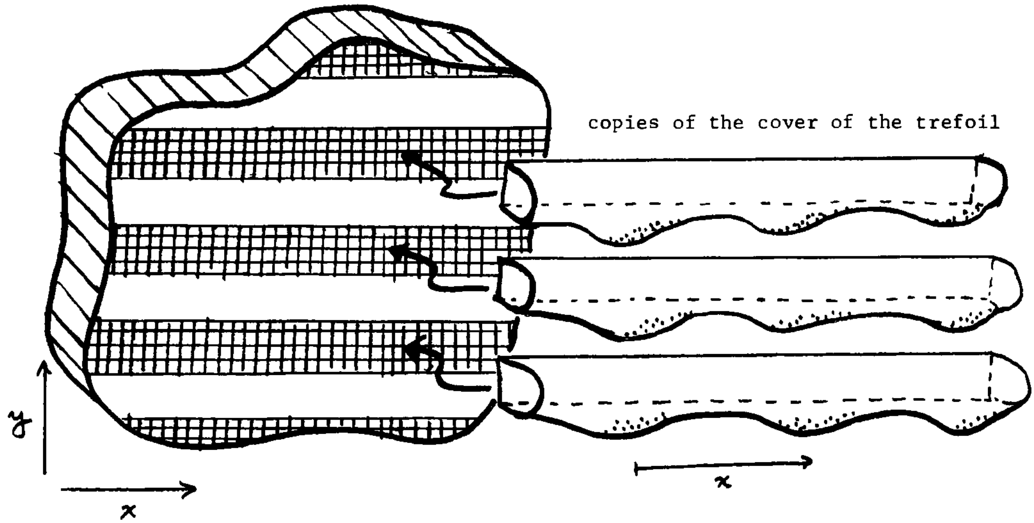
Notice that the Alexander invariants of the trivial link are, in fact, more complicated than those of the first example.

4. EXAMPLE:

The complement of this link (or of a tubular neighbourhood of the link) looks like the complement of Example 1, with a cube-with-knotted-hole attached to it via an annulus.



Then we may picture $\tilde{X}$ as an $R^1 \times R^1 \times [0, 1]$ with countably many copies of the infinite cyclic cover of a trefoil complement attached via strips which run in the x-direction as indicated in the following drawing.



$\tilde{X}$ as a union of $R^1 \times R^1 \times [0, 1]$ with countably many infinite cyclic covers.

Since in the infinite cyclic cover of the trefoil complement every 1 - cycle is annihilated by the polynomial $1 - x + x^2$, the same is seen to be true for every 1 - cycle in $\tilde{X}$. In fact a straightforward Mayer-Vietoris argument shows that the Alexander invariant of this example is:

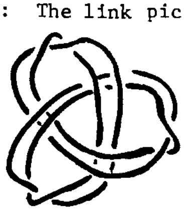
$$H_1(\tilde{X}) \cong \Lambda_2 / (1-x+x^2)$$
$$H_i(\tilde{X}) \cong 0 \qquad \text{when } i > 1.$$

5. EXERCISE : The Alexander invariant of →

is $H_1(\tilde{X}) \cong \Lambda_2 / (1-x+x^2) \oplus \Lambda_2 / (1-3y+y^2)$

$H_1(\tilde{X}) \cong 0, \quad i > 1.$

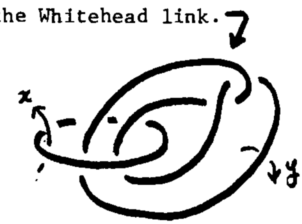
6. EXERCISE : The link pictured below has one-dimensional Alexander invariant:

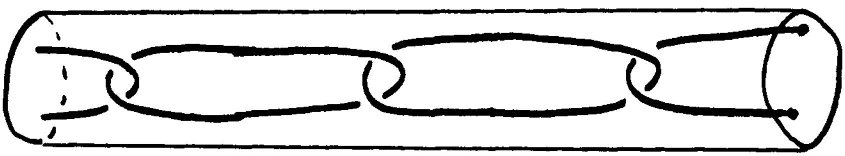


$$H_1(\tilde{X}) \cong \Lambda_2 \oplus \Lambda_2 / (1 - xy + x^2y^2).$$

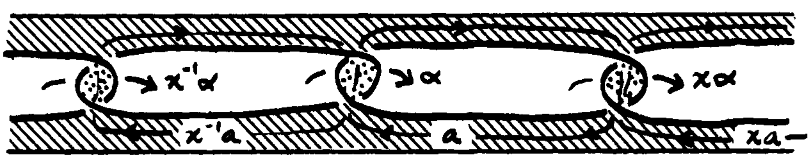
Another trick in studying $\tilde{X}$ is to examine intermediate covers of the link complement. Here is an illustration.

7. EXAMPLE : Calculate the Alexander invariant of the Whitehead link. Consider the cover $\hat{X} \to X$ obtained by unwrapping about one component. This is an infinite cyclic cover and can actually be drawn as a tube with an infinite link removed.





There is also a covering map $\tilde{X} \to \hat{X}$ and it is also an infinite cyclic cover. We can construct $\tilde{X}$ from $\hat{X}$ by cutting along a surface like:



and thus compute the homology of $\tilde{X}$ just as in the case of knot covers. We see that the inclusions of the surface to both sides are homologically

$$\alpha - x\alpha \xleftarrow{\ -\ } a \xmapsto{\ +\ } \alpha - x\alpha$$

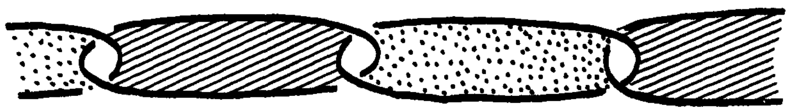
and compute that $H_1(\tilde{X})$ has the generator α subject to the relation

$$\alpha - x\alpha = y(\alpha - x\alpha)$$

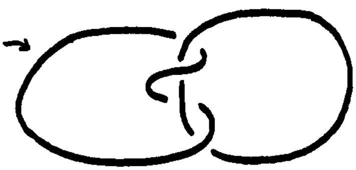
which is equivalent to $(1 - x)(1 - y)\alpha = 0$.

So the one-dimensional Alexander invariant is $H_1(\tilde{X}) \cong \Lambda_2 / ((1-x)(1-y))$

8. REMARK : In this example it was important that the "Seifert" surface used to cut open $\hat{X}$ be invariant under the x translation. Moreover, the orientation (or assignment of + and - sides of a bicollar) should be translation invariant to construct the correct cover $\tilde{X}$. For example, the following surface wouldn't do for the purpose:



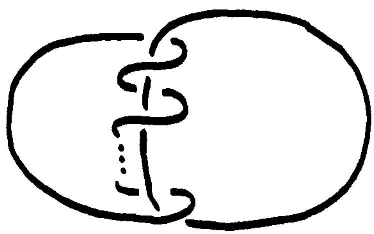
9. EXERCISE : The Alexander invariant of



is, in dimension one:

$H_1(\tilde{X}) \cong \Lambda_2 / (1 + xy)$

More generally for the simple link of two components of linking number r

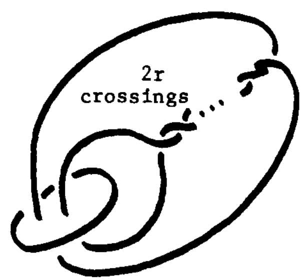


the Alexander invariant is $H_1(\tilde{X}) \cong \Lambda_2 / (p(x,y))$ where

$$p(x,y) = 1 + xy + x^2y^2 + \cdots + x^{r-1}y^{r-1} = \frac{1 - x^ry^r}{1 - xy}$$

10. EXERCISE : For the link at the right the one-dimensional Alexander invariant is

$H_1(\tilde{X}) \cong \Lambda_2 / (r(1-x)(1-y))$.



11. EXERCISE : The link pictured at the right has Alexander invariant:

$$H_1(\tilde{X}) \cong \Lambda_2 / (1 + x^3y)$$

12. REMARK : As pointed out way back in 3A2, the link of the previous exercise and the first link of exercise 9 have homeomorphic complements. It follows that they have homeomorphic universal abelian covers, even by a homeomorphism which respects the covering automorphisms. Therefore $H_1(\tilde{X})$, as a group with $Z \oplus Z$ acting on it, is the same in both cases. Yet the calculation of these, as Λ_2 -modules, comes out differently in the two examples! I'll leave the reader to ponder this mystery, with the hint that $Z \oplus Z$ has numerous possible pairs of generators and that there is an automorphism of Λ , carrying y to x^2y , which enters the picture I think it's rather remarkable that the Alexander invariant can distinguish links which even have homeomorphic complements.

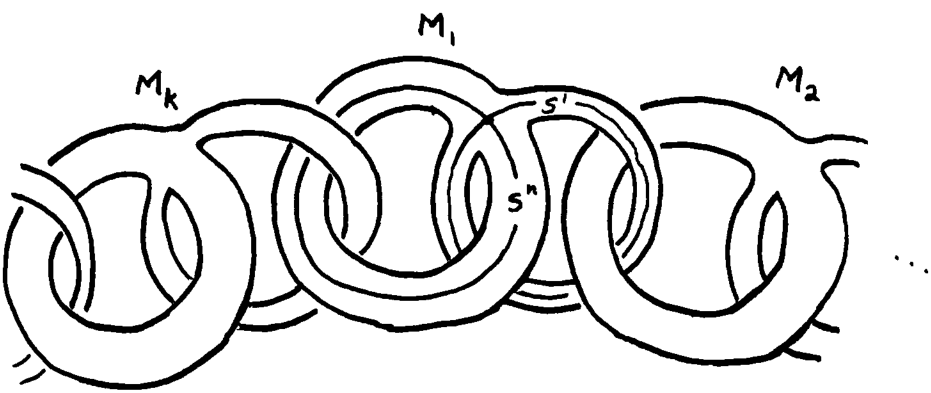
In the case of a boundary link $L = L_1 \cup \cdots \cup L_k$ in S^3 we can construct the universal abelian cover $\tilde{X}$ of $X = S^3 - L$ using the disjoint Seifert surfaces $M_1 \cup \cdots \cup M_k$ for the components. We can cut S^3 open along all the surfaces simultaneously, take one copy of the resulting space for each element of Z^k, and glue them together in a manner which should be obvious. You might use this technique to supply a proof of the following theorem of Gutiérrez [1974].

13. THEOREM : The one-dimensional Alexander invariant of a boundary link in S^3 has a free summand of rank $k - 1$, where k is the number of components. Thus $H_1(\tilde{X}) \cong \Lambda_k^{k-1} \oplus$ (other stuff).

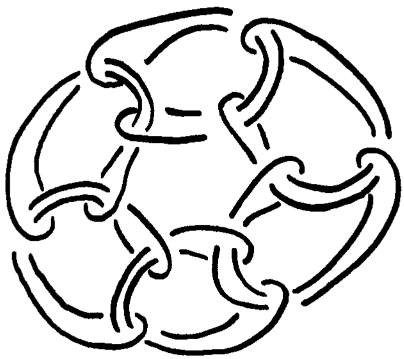
J. BRUNNIAN LINKS IN HIGHER DIMENSIONS. As an application of the Alexander invariant we will prove the following.

1. THEOREM : For each $n \geqq 1$ and $k \geqq 2$ there exists a link L^n in S^{n+2} which has k components and is Brunnian. In other words, L is non-trivial, but any proper sublink is a trivial link.

The example will be concocted by plumbing of S^1's and S^n's as described in section F of this chapter. It will be a boundary link; in fact, as in F, we'll construct the Seifert surfaces first and then take their boundaries as the components of the link. For convenience, we'll do the construction in R^{n+2}. Arrange k copies of S^n in some cyclic fashion in the hyperplane R^{n+1}, so they are unlinked and unknotted Plumb an S^1 onto each of these in such a way that it links once around the next copy. This gives k plumbed manifolds $M_1^{n+1}, \ldots, M_k^{n+1}$ which are disjoint but intertwined. We may picture them as lying inside R^{n+1} except at "crossings" where the S^1 part gets pushed up or down into the $n+2$ dimension. This diagram should explain the embeddings of the M_i.



For the link of the theorem take $L = \partial M_1 \cup \cdots \cup \partial M_k$. By F2, this is indeed a union of n-spheres. Here's a picture of L in case n=1, k=6.



To verify that L is a nontrivial link, let's compute its n-dimensional Alexander invariant. Construct $\tilde{X}$ by cutting S^{n+2} open along the M_i and glueing together copies, one for each element of Z^k, by their cut-open surfaces. Now $S^{n+2} - (M_1 \cup \cdots \cup M_k)$ has k free homology generators in dimension n . Call them $\alpha_1, \ldots, \alpha_k$ where α_i is dual to the S^1 of M_i . Each M_i also has an n-dimensional homology generator, represented by the S^n of the plumbing. Call it a_i . Now pushing a_i off M_i gives:

$$\alpha_{i-1} - \alpha_i \xleftarrow{-} a_i \xrightarrow{+} \alpha_{i-1}$$

which gives a relation in $H_n(\tilde{X})$, as a Λ_k -module:

$$\alpha_{i-1} - \alpha_i = x_i\alpha_{i-1}$$

or

$$\alpha_i = (1 - x_i)\alpha_{i-1}$$

Subscripts are, of course, considered modulo k .

Therefore $H_n(X)$ has generators $\alpha_1, \dots, \alpha_k$ and relations $\alpha_1 = (1-x_1)\alpha_k$, $\alpha_2 = (1-x_2)\alpha_1$, $\dots$, $\alpha_k = (1-x_k)\alpha_{k-1}$. (In case $n = 1$, there is more to it). We can eliminate all but one generator and obtain the Λ_k - module presentation:

$$H_n(\tilde{X}) \cong (\alpha_1 ; \ (1-x_1)(1-x_2)\cdots(1-x_k)\alpha_1 = \alpha_1) .$$

Thus we conclude that the n-dimensional Alexander invariant of L is the cyclic Λ_k -module of order $(1-x_1)(1-x_2)\cdots(1-x_k) - 1$. Again if $n = 1$ there is more to it, but in any case we've shown that L is a nontrivial link.

2. EXERCISE : Complete the proof of Theorem 1 by showing that removing any component of L makes it a trivial link: the components bound nice disjoint $(n+1)$-disks.

3. EXERCISE : In this particular example, each component of L is homotopically unlinked from the others. (The case $n=1$ is the hardest

4. EXERCISE : Each component of any Brunnian link L^n in S^{n+2}, $n > 1$, is homotopically unlinked from the others. [Hint: the higher homotopy of a wedge of circles is trivial.]

CHAPTER 8. MATRIX INVARIANTS

It should be obvious that many of the calculations of the last two chapters can be streamlined by using matrices. The first part of this chapter will show how to derive all of the torsion invariants and the Alexander invariant, at least for classical knots, from just one matrix. This remarkable (Seifert) matrix even produces some new knot invariants; an example is the signature of a knot. Later in this chapter we'll examine an important equivalence relation on knots, called concordance, which is weaker than the usual notion of equivalence. With oriented knot sum as the operation, the concordance classes of knots actually form an abelian group. Signature, which turns out to be a concordance invariant, helps us to examine this group, to detect non-slice knots, to distinguish between the left and right- handed trefoils, and other related matters.

Many of the techniques have generalizations to links and to higher dimensions and other categories. For simplicity, however, we'll restrict attention in this chapter to knots of S^1 in S^3 and assume all spaces and maps to be piecewise-linear. We consider S^3 with a fixed orientation.

A. SEIFERT FORMS AND MATRICES. If K is a given knot in S^3, choose a Seifert surface M^2 in S^3 for K and also choose a particular bicollar $\mathring{M} \times [-1,1]$ in $S^3 - K$. If $x \in H_1(\mathring{M})$ is represented by a 1-cycle (which we'll also call x) in $\mathring{M}$, let x^+ denote the homology cycle carried by $x \times 1$ in the bicollar. Similarly let x^- denote $x \times -1$.

Since 1-cycles with disjoint carriers in S^3 have a well-defined linking number we make the following definition.

1. DEFINITION : The function $f : H_1(\mathring{M}) \times H_1(\mathring{M}) \longrightarrow Z$ defined by

$$f(x,y) = \ell k(x,y^+)$$

is called a *Seifert form* for K . It clearly depends upon the choice of M and choice of a bicollar. If we further choose a basis $e_1, \dots, e_{2g}$ for $H_1(\mathring{M})$ as a Z-module (so every element is expressible uniquely as $n_1e_1 + \cdots + n_{2g}e_{2g}$, n_i integers) define the associated *Seifert matrix* $V = (v_{i,j})$ to be the $2g$ by $2g$ integral matrix with entries

$$v_{i,j} = \ell k(e_i, e_j^+) .$$

2. EXAMPLE : In example 6B1 , using the basis a, b , we see that the right-hand trefoil has a Seifert matrix

$$V = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

3. EXERCISE : A Seifert form f is a bilinear form on the $\mathbb{Z}$ -module $H_1(\mathring{M})$ that is, Z-linear in each variable when the other is held fixed. With a chosen basis $\{e_i\}$, f may be computed from V by the equation

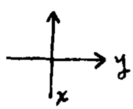
$$f(x,y) = XVY^T = \sum_{i,j} v_{ij}x_iy_j .$$

where $X = (x_1, \dots, x_{2g})$ is the row vector associated with $x = x_1e_1 + \cdots + x_{2g}e_{2g}$ and T denotes the transpose of a matrix.

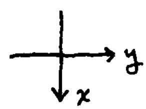
4. EXERCISE : Verify that $\ell k(x,y^+) = \ell k(x^-,y^+) = \ell k(x^-,y)$.

5. EXERCISE : Two $2g$ by $2g$ matrices V and W with integer entries correspond to the same Seifert form (with appropriate choices of bases) if and only if they are <u>congruent</u> over Z , that is $W = PVP^T$ for some matrix P which is invertible over Z .

Here is another bilinear form definable on the homology of a surface. If x and $y \in H_1(\mathring{M})$, we may choose representative 1-cycles which intersect transversally. With an appropriate weighting of the intersections by ± 1, according to orientation conventions, their sum is the <u>intersection number</u> $\iota(x,y)$, sometimes denoted $x \cdot y$. For example, one convention would be to view an intersection from the + side of a bicollar and count +1 for



and -1 for



6. EXERCISE : The function $\iota : H_1(\mathring{M}) \times H_1(\mathring{M}) \longrightarrow Z$ is a well-defined bilinear form defined for any bicollared surface M^2 in S^3. It is antisymmetric: $-\iota(x,y) = \iota(y,x)$. Given a choice of basis, its matrix $\mathcal{A}$ then satisfies $-\mathcal{A} = \mathcal{A}^T$. If M is a Seifert surface for a knot (that is, $\partial M \cong S^1$) then any matrix for the intersection form has determinant ± 1. [Hint: there exists a basis for which $\mathcal{A}$ is zero except for blocks along the diagonal of the form $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$].

7. EXERCISE : The Seifert form and intersection form of a fixed bicollared surface are connected by the formula $\iota(x,y) = f(x,y) - f(y,x)$.

An important consequence of these two exercises follows.

8. COROLLARY : Any Seifert matrix for a knot K in S^3 satisfies

$$\det(V - V^T) = \pm 1.$$

To repeat, the Seifert matrix is <u>not</u> a knot invariant, although it may be used to distinguish between different Seifert surfaces for a given knot (see Trotter [1975]). Before attempting to derive the familiar "abelian" invariants from V, it is necessary to discuss the connection between matrices and modules.

B. PRESENTATION MATRICES : In this section, let A be a commutative ring with unit and consider a finitely-presented module M over A.

$$M \cong (\alpha_1, \cdots , \alpha_r ; \rho_1 , \ldots , \rho_s)$$

Each relator is a linear combination of the generators

$$\rho_i = a_{i1}\alpha_1 + \cdots + a_{ir}\alpha_r \qquad (a_{ij} \in A)$$

and we define the s by r matrix $P = (a_{i,j})$ as a <u>presentation matrix</u> for M corresponding to the given presentation. That is, the rows of P are the coefficients of the relators relative to the generators. Knowing P is the same as knowing the specific presentation to which it corresponds So P determines M up to A-isomorphism.

Here is another way of looking at P. By multiplication on the right, P may be regarded as an A-linear map from A^s into A^r. M is then determined as the cokernel of this map, so it fits into an exact sequence:

$$0 \longrightarrow A^s \xrightarrow{P} A^r \longrightarrow M \longrightarrow 0$$

It is easy to see that the module presented by a given matrix P is unchanged, up to A-isomorphism, by any of the following operations on P:

(1) interchange two rows or two columns

(2) add to any row an A-linear combination of other rows

(3) add to any column an A-linear combination of other columns

(4) multiply a row or column by a unit of A

(5) replace P with the matrix

$$\begin{array}{c|ccc} 1 & ? & ? & ? \\ \hline 0 & & & \\ 0 & & P & \\ \vdots & & & \end{array}$$

(6) the reverse of (5).

(7) adjoin a new row which is an A-linear combination of rows of P

(8) delete a row which is an A-linear combination of other rows.

Operations (1) - (4) correspond to changes of basis, operation (5) to introducing a new generator and a relation defining it in terms of the old generators, (6) to removing a redundant generator, and (7) and (8) to adding or removing redundant relations. A sort of converse also holds.

1. PROPOSITION : Two matrices, with entries in A, present isomorphic A-modules if and only if one can be transformed into the other by a finite number of applications of operations (1) - (8).

2. EXERCISE : Prove this proposition. [Hint: if $(\alpha_1, \dots ; \rho_1, \dots)$ and $(\beta_1, \dots ; \sigma_1, \dots)$ present modules isomorphic via h, consider a third presentation with generators $\alpha_1, \dots, \beta_1, \dots$ and relators $\rho_1, \dots, \sigma_1, \dots$ plus relations expressing $h(\alpha_i)$ in terms of the β_j. If you get stuck, consult Zassenhaus, Theory of Groups, chapter III.]

3. DEFINITION : If M is a module over A which has an $s \times r$ presentation matrix P, the ideal of A generated by all $r \times r$ minors is called the order ideal of M. In case $s < r$ take it to be the ideal containing only zero. One easily checks using the above proposition that the definition does not depend on the particular choice of P. Similarly one may define a whole string of elementary ideals, $\mathcal{E}_i$ generated by all $(r-i) \times (r-i)$ minors. We will only concern ourselves with the order ideal $\mathcal{E}_o$.

4. EXERCISE : In the special case $A = Z$, we are dealing with abelian groups G. Show that G is a infinite group if and only if its order ideal is the zero ideal. If G is a finite abelian group, its order ideal is the principal ideal generated by the number $|G|$, the order in the usual sense.

5. REMARK : In case M has a square presentation matrix P, the order ideal is principal and is generated by $\det(P)$.

C. ALEXANDER MATRICES AND ALEXANDER POLYNOMIALS . Recall that the Alexander invariant $H_*(\tilde{X})$ is the homology of the infinite cyclic cover of the complement of a knot, considered as a module over Λ, the ring of integral Laurent polynomials. In the case of classical tame knots it is finitely presentable (why?) and only H_1 plays a significant rôle.

1. DEFINITION : Any presentation matrix for the Alexander invariant $H_1(\tilde{X})$ of a knot K is called an Alexander matrix for K.* The associated order ideal in Λ is called the Alexander ideal of K, and if this is principal, any generator of the Alexander ideal is called the Alexander polynomial.

2. EXERCISE : For any knot K^n in S^{n+2} whose complement has the homotopy type of a finite complex, the Alexander invariant is finitely generated and therefore finitely presentable. [Hint: Λ is Noetherian.]

Thus for PL or C^∞ knots an Alexander matrix always exists. Although not strictly a knot invariant, we may obtain a knot invariant by considering its equivalence class under transformations of the type (1) to (8) described in the previous section. The Alexander ideal is also defined for all these knots, and is a strict knot invariant. The Alexander polynomial, when it exists, is a knot invariant which is well-defined only up to multiplication by units of Λ, i. e. monomials $\pm t^n$. That classical knots always have an Alexander polynomial follows from the main result of this section.

* this terminology is not completely standard. See remark 16.

3. THEOREM : If V is a Seifert matrix for a tame knot K in S^3 then $V^T - tV$ is an Alexander matrix for K. So is $V - tV^T$.

By tV we mean of course that every entry of V is multiplied by the variable t, producing a matrix over Λ. Before giving a proof, which is not difficult, here are some consequences.

4. COROLLARY : The Alexander invariant of a tame knot in S^3 has a square presentation matrix, so its Alexander ideal is principal and it has an Alexander polynomial $\Delta(t)$.

5. COROLLARY : For tame knots in S^3, $\Delta(t) = \det(V^T - tV)$.

6. REMARK : Corollary 4 could have been deduced long before. One could establish a presentation of $H_1(\tilde{X})$ with an equal number of generators and relations by using any surgery description (compare sections 6D and 7C). Alternatively, one could use a Wirtinger presentation, of deficiency one, and the methods of section 7D.

7. COROLLARY : The Alexander polynomial of a tame knot in S^3 satisfies:

$$\Delta(t) \doteq \Delta(t^{-1})$$
$$\Delta(1) = \pm 1$$

8. REMARK : By combining this with 7C5 we see that these two conditions actually characterize the class of Laurent polynomials which arise as Alexander polynomials of tame knots in S^3. The symbol $\doteq$ means 'equal up to multiplication by units of Λ.'

9. EXERCISE : Prove corollary 7. Argue that the first condition (symmetry) may not hold for higher-dimensional knots. Show that, regardless of dimension, if a knot has an Alexander polynomial then it satisfies $|\Delta(t)| = 1$. [Hint: if $A(t)$ is a presentation matrix for $H_1(\tilde{X})$, show that $A(1)$ is a presentation matrix for $H_1(S^{n+2})$].

10. EXERCISE : Define the <u>degree</u> of a Laurent polynomial to be the difference between the highest and lowest exponents at which nonzero coefficients occur. Prove the following inequality connecting the genus of a knot in S^3 and the degree of its Alexander polynomial:

$$\text{degree}(\Delta(t)) \leqq 2g(K)$$

11. EXERCISE : The genus of a torus knot of type p,q is exactly $\frac{(p-1)(q-1)}{2}$

12. EXERCISE : Establish the following relation between the number of crossings in a projection of a knot and its Alexander polynomial's degree:

$$\text{crossing number} \geqq \text{degree}(\Delta) + 1$$

and show that equality is achieved for $2,n$ torus knots (n crossings).

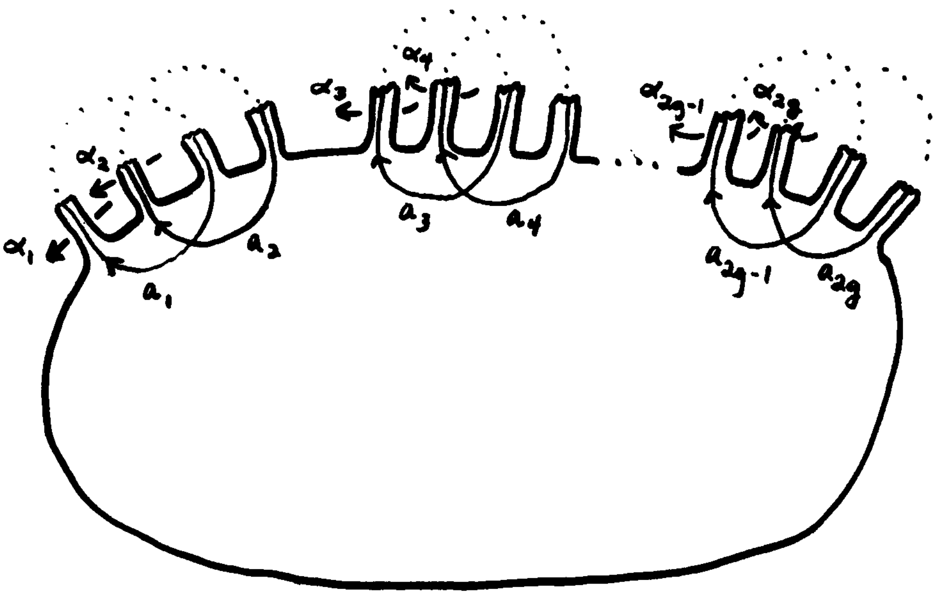
13. EXERCISE : Give an alternative proof of the symmetry of Alexander polynomials for knots in S^3 by using surgery techniques. Show that, in fact, a tame knot has an Alexander matrix $A(t)$ which is "hermitian" with respect to the "conjugation" $p(t) \to p(t^{-1})$ defined on Λ :

$$A(t^{-1})^T = A(t)$$

The following lemma will be useful in proving theorem 3, and in the next section. Although it follows from Alexander duality, a geometric proof will do no harm.

14. LEMMA : Let M be a Seifert surface for a knot in S^3 and let $a_1, \ldots, a_{2g}$ be a basis for $H_1(\overset{\circ}{M})$. Then there is a basis $\alpha_1, \ldots, \alpha_{2g}$ for $H_1(S^3 - M)$ which is dual to $\{a_i\}$ with respect to the linking pairing. That is, $\ell k(a_i, \alpha_j) = \delta_{i,j}$.

PROOF : It suffices to prove the lemma for any one particular basis (why?). Now M is, abstractly, homeomorphic to a disk with $2g$ strips attached. Knowing this, one can easily construct an isotopy of S^3 so that M in fact looks like a flat disk with $2g$ thin strips* attached. The following picture indicates how to choose the α_i .



* They may of course be horribly twisted and intertwined.

15. PROOF OF THEOREM 3 : Let M be a bicollared Seifert surface for K and let V be the Seifert matrix corresponding to some basis $a_1, \dots, a_{2g}$ for $H_1(\mathring{M})$. Its entries are $v_{i,j} = \ell k(a_i, a_j^+)$. Referring to section 5C, the infinite cyclic cover $\tilde{X}$ may be obtained by cutting open $X = S^3 - K$ along M . Let $\alpha_1, \dots, \alpha_{2g}$ be the basis for $H_1(S^3 - M)$ which is dual to $\{a_i\}$, as in lemma 14. Note that the coefficients of any element $\alpha = \Sigma\, c_j\alpha_j$ may be recovered by $c_j = \ell k(\alpha, a_j)$. As in section 7B, we see that the homology $H_1(\tilde{X})$ can be presented, as a Λ-module, with generators $\alpha_1, \dots, \alpha_{2g}$ and relations

$$a_i^- = ta_i^+ , \qquad i = 1, \dots, 2g.$$

Writing out in terms of the α_j these relations become

$$\sum_j \ell k(a_i^-, a_j)\alpha_j = t[\sum_j \ell k(a_i^+, a_j)\alpha_j]$$

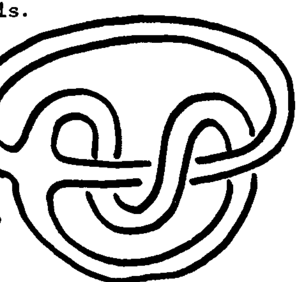
Since $\ell k(a_i^-, a_j) = \ell k(a_i, a_j^+) = v_{i,j}$ the relations may be rewritten as

$$\sum_j (v_{i,j} - tv_{j,i})\alpha_j = 0 , \qquad i = i, \dots, 2g.$$

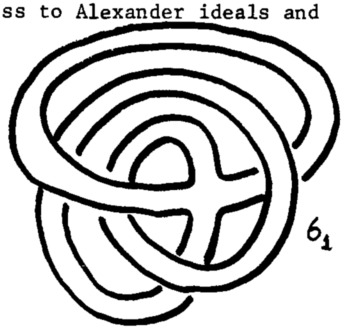
The relation matrix corresponding to this is precisely $V - tV^T$. Note that if one interchanges the + and − sides of the bicollar on M , the new Seifert matrix is just the transpose of the old one. From this we may conclude that $V^T - tV$ is also a presentation matrix for $H_1(\tilde{X})$.

16. REMARK : Our terminology differs somewhat from Fox's Quick Trip and similar sources. What Fox calls the Alexander matrix is a certain matrix over Λ obtained by means of the free calculus from a presentation of the knot group. It turns out to be a presentation matrix for the modul $H_1(\tilde{X}) \oplus \Lambda$. Therefore the n^{th} elementary ideal of our Alexander matrix equals the $n+1^{th}$ elementary ideal of Fox's.

17. EXAMPLE : The two knots pictured below as boundaries of genus one Seifert surfaces are 9_{46} and 6_1 , respectively, in the knot table. They are of interest in that their Alexander invariants are different, although the distinction is lost when we pass to Alexander ideals and polynomials.



9_{46}



6_1

$$V = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} \qquad V = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$

$$V^T - tV = \begin{bmatrix} 0 & 1-2t \\ 2-t & 0 \end{bmatrix} \qquad V^T - tV = \begin{bmatrix} 0 & 1-2t \\ 2-t & t-1 \end{bmatrix}$$

$$\Delta(t) = 2 - 5t + 2t^2 \qquad \Delta(t) = 2 - 5t + 2t^2$$

18. EXERCISE : The Alexander invariant of 6_1 is a cyclic Λ-module, while that of 9_{46} necessarily has at least two generators.

D. THE TORSION INVARIANTS. Recall that the torsion invariants of a knot K in S^3 are computed from the homology $H_1(\tilde{X}_k)$ of the finite cyclic covers of the complement of K. It is convenient to work with the k-fold <u>branched</u> cyclic covers Σ_k associated with K, which are defined in section 10B. All that need concern us here is the equation

$$H_1(\tilde{X}_k) \cong H_1(\Sigma_k) \oplus Z .$$

1. THEOREM : The 2-fold branched cover has homology $H_1(\Sigma_2)$ presented as an abelian group by the matrix $V + V^T$, where V is any Seifert matrix for K.

PROOF : Using the notation of C15 above, we can construct the 2-fold cover $\tilde{X}_2$ by pasting together two copies of $S^3 - M$. Then, as in section 6B, we compute the homology of $\tilde{X}_2$ (except for the cyclic infinite summand) by taking generators $\alpha_1^0, \ldots, \alpha_{2g}^0, \alpha_1^1, \ldots, \alpha_{2g}^1$ for the homology of the two copies of $S^3 - M$. The pasting gives relations which we may compute as in C15:

$$\sum_j v_{i,j}\alpha_j^0 = \sum_j v_{j,i}\alpha_j^1 \quad\text{and}\quad \sum_j v_{i,j}\alpha_j^1 = \sum_j v_{j,i}\alpha_j^0 , \quad i = 1, \ldots , 2g.$$

Write α^0 for the row vector $[\alpha_1^0, \ldots , \alpha_{2g}^0]$ and α^1 similarly. Then these relations may be abbreviated by the matrix equations

$$\alpha^0 V = \alpha^1 V^T \quad\text{and}\quad \alpha^1 V = \alpha^0 V^T.$$

Subtracting yields $\alpha^1(V - V^T) = \alpha^0(V^T - V)$, and since $V - V^T$ is invertible, by A8, we conclude that $\alpha^1 = -\alpha^0$. This allows us to eliminate half the generators, so that $H_1(\Sigma_2)$ has generators α^0 and relations $\alpha^0(V + V^T) = 0$. The theorem follows.

2. EXERCISE : Show that if $\Delta(t)$ is an Alexander polynomial, then $\Delta(-1)$ is an odd integer. [Hint: compute modulo 2 and use $\Delta(1) = \pm 1$.]

3. COROLLARY : The group $H_1(\Sigma_2)$ is finite and its order is given by $|\Delta(-1)|$, which is always odd.

4. DEFINITION : The number $|\Delta(-1)|$ is called the <u>determinant</u> of the knot K.

Now let's investigate the branched cyclic covers of order greater than 2 . Just as in Theorem 1, we can obtain a presentation for $H_1(\Sigma_k)$, as an abelian group, with generators $\alpha_1^0, \ldots, \alpha_{2g}^0, \alpha_1^1, \ldots \alpha_{2g}^1, \ldots, \alpha_1^{k-1}, \ldots, \alpha_{2g}^{k-1}$ and relations, in matrix form:

$$\alpha^0 V^T = \alpha^1 V, \quad \alpha^1 V^T = \alpha^2 V, \quad \ldots \quad , \quad \alpha^{k-1} V^T = \alpha^0 V \ .$$

If it happens that V is invertible over the integers (in other words, if $|\det V| = 1$) we can eliminate the generators other than, say, α^0 and conclude the following.

5. THEOREM : If a knot K in S^3 has an invertible Seifert matrix V, then

$$(V^T V^{-1})^k - I \qquad (I = \text{identity matrix})$$

is a presentation matrix for $H_1(\Sigma_k)$, as an abelian group.

6. EXERCISE : Show by example that a Seifert matrix may not be invertible.

7. EXAMPLE : According to A2 the right-hand trefoil has a Seifert matrix

$$V = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \quad \text{so} \quad V^{-1} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad V^T V^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

One can then verify the calculations of 6B1 and 6B4 to compute the torsion invariants of the trefoil. The usual procedure in dealing with a presentation matrix is to make it upper triangular or diagonal by row and column operations, in order to recognise the group which it presents. Notice that $V^T V^{-1}$ has order six. This shows two things. First, $H_1(\Sigma_6)$ has a 2 by 2 presentation matrix consisting of all zeros, so this group is $Z \oplus Z$. Second, it implies that the torsion invariants of the trefoil are periodic: $H_1(\Sigma_{k+6}) \cong H_1(\Sigma_k)$ whenever $k \geq 0$.

By the discussion preceding theorem 5, whether the Seifert matrix V is invertible or not, $H_1(\Sigma_k)$ has presentation matrix of the form shown at the right, with $2gk$ rows and columns. In calculations this is usually much more awkward than the matrix of theorem 5 (see also exercise 10). But a few interesting conclusions can be made by studying this big matrix.

$$\begin{bmatrix} V^T & -V & 0 & 0 & \cdots & 0 & 0 \\ 0 & V^T & -V & 0 & \cdots & 0 & 0 \\ 0 & 0 & V^T & -V & \cdots & 0 & 0 \\ & \cdots & & & \ddots & & \cdots \\ 0 & 0 & 0 & 0 & \cdots & V^T & -V \\ -V & 0 & 0 & 0 & \cdots & 0 & V^T \end{bmatrix}$$

As noted by Gordon [1971] when k is odd, say $k = 2r + 1$, one can put the first r rows last and obtain a presentation matrix for $H_1(\Sigma_k)$ which is skew-symmetric. That is, its transpose equals its negative. Here it is for the case $k = 7$.

$$\begin{bmatrix} 0 & 0 & 0 & V^T & -V & 0 & 0 \\ 0 & 0 & 0 & 0 & V^T & -V & 0 \\ 0 & 0 & 0 & 0 & 0 & V^T & -V \\ -V & 0 & 0 & 0 & 0 & 0 & V^T \\ V^T & -V & 0 & 0 & 0 & 0 & 0 \\ 0 & V^T & -V & 0 & 0 & 0 & 0 \\ 0 & 0 & V^T & -V & 0 & 0 & 0 \end{bmatrix}$$

Now it is a well-known fact of matrix theory that a skew-symmetric matrix over the integers may be diagonalized by row and column operations. What's more, if the number of rows and columns is even, the diagonal elements occur in pairs. (See MacDuffee [1933], or one may verify it directly, noting that the characteristic polynomial is an even function.) We have just shown that if k is odd, $H_1(\Sigma_k)$ has a presentation matrix of the form: $\operatorname{diag}(d_1, d_1, d_2, d_2, \ldots, d_{gk}, d_{gk})$. From this follows a theorem of Plans [1953].

8. THEOREM : If k is odd, then the homology of the k-fold cyclic branched cover of any tame knot in S^3 is a direct double: $H_1(\Sigma_k) \cong G \oplus G$.

9. EXERCISE : Show that even if V is non-invertible, one may obtain a presentation matrix for the k^{th} torsion invariant of a knot, with only 2g rows and columns as follows. Let $\Gamma = V(V^T - V)$, and let I denote the identity matrix of $2g$ rows and columns. Then $\Gamma^k - (\Gamma - I)^k$ presents $H_1(\Sigma_k)$

10. EXERCISE : Show that the knots of example C17, which have the same Alexander polynomial, may be distinguished by their 2-fold branched covers.

11. EXERCISE : Find the determinant of the p,q torus knot. [L'Hôpital's rule might help.]

E. SIGNATURE AND SLICE KNOTS. The matrix $V + V^T$ has already seen service as a presentation matrix for $H_1(\Sigma_2)$. But notice also that it is a symmetric matrix, and corresponds to the symmetrized Seifert form $f(x,y) + f(y,x)$. Although the entries of $V + V^T$ are integers, we may consider them as lying in the field of rationals, reals, even the complex numbers. We can then take advantage of two basic theorems of linear algebra.

1. SPECTRAL THEOREM : Any symmetric matrix S over a field F, in which $1+1\neq 0$ is congruent over F to a diagonal matrix. That is, for some invertible matrix P with entries in F , PSP^T is a diagonal matrix.

2. SYLVESTER'S THEOREM : Two diagonal matrices over the reals are congruent (over R) if and only if they have the same number of positive entries, the same number of negative entries, and the same number of zeros.

Therefore each symmetric matrix over the reals is congruent to exactly one of the canonical forms:

$$\begin{bmatrix} \overbrace{1 \ \ddots \ 1}^{p} & & \text{\Large 0} \\ & \overbrace{-1 \ \ddots \ -1}^{n} & \\ \text{\Large 0} & & \overbrace{0 \ \ddots \ 0}^{z} \end{bmatrix}$$

3. DEFINITION: The number $\sigma = p - n$ is called the <u>signature</u> of the matrix, or of the symmetric bilinear form which the matrix represents.

Clearly two non-singular (i. e. det $\neq 0$) symmetric matrices of the same size are congruent over R if and only if they have the same signature. Notice that, by D2, $V + V^T$ is non-singular for a knot.

4. DEFINITION : If K is a knot in S^3 with Seifert matrix V, define the <u>signature</u> of K to be $\sigma(K) = \sigma(V + V^T)$.

Since a change of basis for $H_1(\overset{\circ}{M})$ changes V , and hence $V + V^T$, only by a congruence, the definition of signature depends only on the choice of Seifert surface M for K . Note also that if we reverse the choice of $+$ and $-$ sides of a bicollar the Seifert matrix changes from V to V^T , and the symmetrized matrix $V + V^T$ is unchanged. We will show that signature is dependent only on K , up to orientation-preserving homeomorphisms of S^3 , but until then we'll interpret it as being defined with respect to a fixed Seifert surface M and may write $\sigma_M(K)$ to emphasize this dependence.

5. PROPOSITION : If $r: S^3 \to S^3$ is an orientation-reversing homeomorphism, then for any knot K , $\sigma(rK) = -\sigma(K)$.

PROOF : Since linking numbers change sign throughout, if V is a Seifert matrix corresponding to the surface M , then $-V$ is the matrix corresponding to rM . It follows that a diagonalization of the symmetrized Seifert matrix for rK can be obtained from one for K simply by changing all signs.

6. EXERCISE : The signature of a knot is always an even number.

7. PROPOSITION : Signature is additive: $\sigma(K_1 \# K_2) = \sigma(K_1) + \sigma(K_2)$

PROOF : We interpret this, for the moment, as asserting that $\sigma_{M_1 \# M_2}(K_1 \# K_2) = \sigma_{M_1}(K_1) + \sigma_{M_2}(K_2)$, where $M_1 \# M_2$ is the boundary connected sum of Seifert surfaces M_i for K_i. But $H_1(M_1 \# M_2)$ is, in a natural way, isomorphic with the direct sum of $H_1(M_1)$ and $H_1(M_2)$ Moreover the corresponding Seifert matrices V_1 and V_2 yield the Seifert matrix

$$V = V_1 \oplus V_2 \overset{\text{def}}{=} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix} \quad \text{(block sum of matrices)}$$

for $K_1 \# K_2$. It is clear that $V + V^T$ can be diagonalized by diagonalizing $V_1 + V_1^T$ and $V_2 + V_2^T$ seperately and taking the block sum of these diagonal matrices. Then the proposition follows easily.

8. DEFINITION : A knot K in $S^3 = \partial D^4$ is a <u>slice knot</u> if it bounds a disk Δ^2 in D^4 which has a tubular neighbourhood $\Delta^2 \times D^2$ whose intersection with S^3 is a tubular neighbourhood $K \times D^2$ for K .

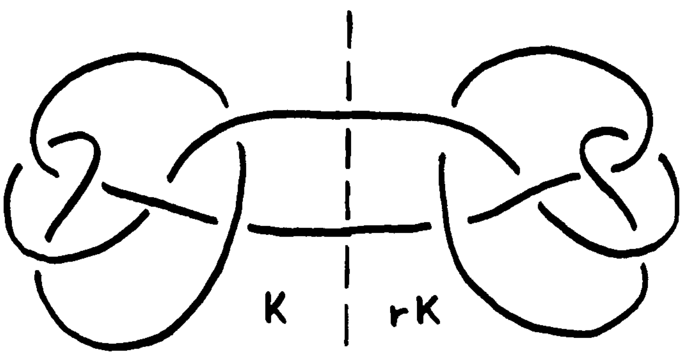
9. EXERCISE : Every knot in S^3 bounds a disk in D^4 . This points out that the existence of the tubular neighbourhood is the essential part of the definition. That is, Δ is required to be locally flat.

10. THEOREM : If K is a slice knot, then $\sigma(K) = 0$. This is true regardless of the Seifert surface used to define the signature.

A later part of this section will be devoted to proving this, but first let's use the theorem to verify that signature is well-defined. If we view knots in R^3 instead of S^3, then being a slice knot corresponds to bounding a locally flat disk in R^4_+, the "upper half" of 4-space. Suppose K is a knot which is symmetric with respect to a plane, say R^2, in R^3. Then K is clearly a slice knot, because we can spin it through R^4_+ about the axis R^2 to produce the desired locally flat disk. (See 3J5).

11. LEMMA : If K is any tame knot in S^3, and $r: S^3 \to S^3$ an orientation-reversing homeomorphism, then $K \# rK$ is a slice knot.

PROOF : This follows because $K \# rK$ may be situated in R^3 so as to be symmetric with respect to a plane, as in the following picture.



12. COROLLARY : Signature is well-defined. Thus, if M_1 and M_2 are Seifert surfaces for the same knot K, then $\sigma_{M_1}(K) = \sigma_{M_2}(K)$.

PROOF : $\sigma_{M_1}(K) - \sigma_{M_2}(K) = \sigma_{M_1}(K) + \sigma_{rM_2}(rK) = \sigma_{M_1 \# rM_2}(K \# rK) = 0$.

13. EXAMPLE : For the right-handed trefoil: $V = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$

$$V + V^T = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \text{, congruent to } \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} = \begin{bmatrix} -3/2 & 0 \\ 0 & -2 \end{bmatrix}$$

(Notice that a congruence may be effected by performing a row operation and then the corresponding column operation. In fact these moves generate all matrices congruent to a given square matrix).

Thus the trefoil has signature -2. Therefore it is not a slice knot.

14. EXAMPLE : The left-handed trefoil has signature $+2$. Consequently, it is not equivalent to the right-handed trefoil by an orientation-preserving homeomorphism.

15. EXAMPLE : The square knot is the sum of a left- and right-handed trefoil. Its signature is zero and it is a slice knot. There are two granny knots, up to orientation-preserving equivalence. One is the sum of two right-handed trefoils and has signature -4 . The other is the sum of two left-handed trefoils and has signature $+4$. We have finally verified that the square and granny knots are inequivalent (even allowing orientation-reversing homeomorphisms of S^3), a distinction which could not be made by studying the knot groups alone. (See 3D11). Granny knots are not slice knots.

Let's now turn to proving theorem 10. If we are going to regard $V + V^T$ as a matrix over the rationals, in order to diagonalize and determine its signature, we can consider it as representing a symmetric bilinear form on the vector space Q^{2g}.† A symmetric bilinear form (which is the same thing as an inner product) on Q^n corresponds to symmetric matrix, say A, by the usual formula

$$< X, Y > \; = \; XAY^T$$

It is nonsingular if $\det(A) \neq 0$ or, what amounts to the same thing (why? for any $X \neq 0$ there exists Y such that $< X, Y > \neq 0$. Recall that $V + V^T$ is nonsingular if V is a Seifert form for a knot (this is not always the case for links, if we define Seifert form in the same way). If the signature of a nonsingular inner product is zero, then Q^{2g} splits into orthogonal subspaces such that it is positive definite on one subspace, negative definite on the other, and both subspaces have dimension g.* An easy dimension argument establishes the following.

16. LEMMA : If a nonsingular inner product on Q^{2g} vanishes on a subspace of dimension g, or equivalently if it has a matrix (under some choice of basis) of the form

$$A = \begin{bmatrix} B & C \\ C^T & 0 \end{bmatrix}, \quad B \text{ and } C \text{ square matrices,}$$

then its signature is zero. That is, $\sigma(A) = 0$.

† we could use the reals as well.

* the converse also is true.

We need two more technical lemmas. The proof of the first will be left as an EXERCISE. Its proof might require a little obstruction theory and might resemble the proof of theorem 5B1. Recall everything is PL

17. LEMMA : If a slice knot K in S^3 bounds a locally flat disk Δ in D^4 and also bounds a Seifert surface M in S^3, then there exists a bicollared 3-manifold W in D^4 such that $W \cap S^3 = M$ and $\partial W = M \cup \Delta$.

18. LEMMA : If W is a closed connected orientable 3-manifold such that ∂W is a connected 2-manifold of genus g, then there exists a basis for $H_1(\partial W)$ represented by 1-cycles, half of which bound rational 2-chains in W.

PROOF : In this argument we will use rational coefficients, so that the homology groups are vector spaces. We recall that for any linear map $h : G \longrightarrow H$ of vector spaces we have

$$\mathrm{rank}(G) = \mathrm{rank}(\mathrm{kernel}\ h) + \mathrm{rank}(\mathrm{image}\ h) .$$

Now consider the exact homology sequence of the pair $(W, \partial W)$. Since the maps $H_3(W,\partial W) \to H_2(\partial W)$ and $H_0(\partial W) \to H_0(W)$ are isomorphisms $Q \longrightarrow Q$, we have a short exact sequence

$$0 \to H_2W \to H_2(W,\partial W) \to H_1\partial W \to H_1W \to H_1(W,\partial W) \to 0 .$$

Now we can invoke the **Poincaré-Lefschetz duality isomorphism**

$H_1(W,\partial W) \cong H^2W$ along with the fact that $\mathrm{rank}(H^2W) = \mathrm{rank}(H_2W)$ to conclude that H_2W and $H_1(W,\partial W)$ have the same rank (see, for example, Greenberg [1967] pages 136 and 186). Similarly $H_2(W,\partial W)$ and H_1W have the same rank. I'll leave it to the reader to piece these bits of information together and complete the proof of the lemma.

We are now prepared to prove the following, which by lemma 16, easily implies theorem 10.

19. PROPOSITION : If K is a slice knot in S^3 and M is any Seifert surface for K, then there exists a basis for $H_1(\mathring{M})$ such that the associated Seifert matrix has the form

$$V = \begin{bmatrix} B & C \\ D & 0 \end{bmatrix}, \quad B, C \text{ and } D \text{ square integral matrices.}$$

PROOF : If K bounds a locally flat disk Δ, choose $W^3 \subset D^4$ as in lemma 17. It is easily verified that the inclusion homomorphism $H_1\mathring{M} \to H_1(M \cup \Delta) = H_1\partial W$ is an isomorphism and so by lemma 18 there is a basis $a_1, \ldots, a_{2g}$ for $H_1\mathring{M}$ such that $a_{g+1}, \ldots, a_{2g}$ bound 2-chains in W. It follows that if $g+1 \leq i,j \leq 2g$, then a_i and a_j^+ bound disjoint 2-chains in D^4, by pushing out in the bicollar the one bounded by a_j^+. Referring to section 5D, $\ell k(a_i, a_j^+) = 0$ and V has the required form.

Following is an important Corollary to this proposition.

20. THEOREM : The Alexander polynomial of a slice knot in S^3 has the form

$$\Delta(t) = p(t)p(t^{-1})$$

where $p(t)$ is a polynomial with integer coefficients.

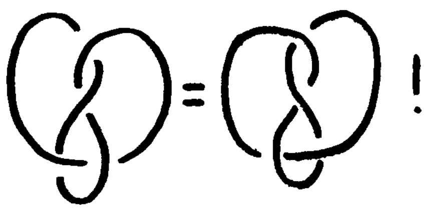
PROOF : Using C5 and the Seifert matrix promised by Proposition 19,

$$\Delta(t) = \det(V^T - tV) = \det\begin{bmatrix} B^T - tB & D^T - tC \\ C^T - tD & 0 \end{bmatrix} = \det(C^T - tD)\det(D^T - tC)$$

$$= (-t)^g \det(C^T - tD)\det(C^T - t^{-1}D) .$$

21. COROLLARY : The determinant $|\Delta(-1)|$ of a slice knot is a square integer

22. EXAMPLE : The figure-eight knot K has signature zero. This could, of course, be computed directly, but it may also be deduced from proposition 5 and the fact that K is equivalent,



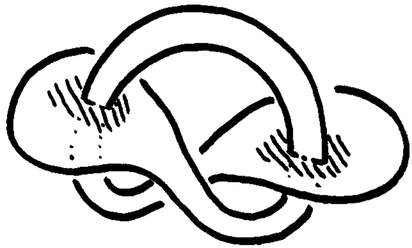
by an orientation preserving homeomorphism, to its reflection rK (you ought to convince yourself of this). In other words, the figure-eight is amphicheiral. Although the signature doesn't detect this, the figure eight is <u>not</u> a slice knot. For its Alexander polynomial $\Delta(t) = 1 - 3t + t^2$ cannot be of the form $p(t)p(t^{-1})$ since $\Delta(-1) = 5$.

23. EXERCISE : The 2-fold branched cover of a slice knot has $|H_1(\Sigma_2)| =$ an odd square. Is it true that, in fact $H_1(\Sigma_2) \cong G \oplus G$?

24. EXERCISE : Show that the polynomials of degree ≤ 2 which are Alexander polynomials of slice knots must be, up to units of Λ , among the list

$$\Delta(t) = 1, \quad 2t^2 - 5t + 2, \quad 6t^2 - 13t + 6, \quad 12t^2 - 25t + 12, \quad \ldots$$
$$= nt^2 - (2n+1)t + n \quad \text{where} \quad n = k(k-1),\ k = 1, 2, 3, 4, \ldots \ .$$

25. EXAMPLE : The stevedore's knot 6_1 has $\Delta(t) = 2t^2 - 5t + 2$. See example C17. Here is another picture of it ⟶



Notice that it bounds a singular disk, with two arcs of self-intersection. If the cross-hatched parts are pushed off S^3 slightly into D^4 , then these self-intersections disappear and we see that 6_1 is a slice knot.

26. DEFINITION : Suppose the knot K is the boundary $K = f(S^1)$ of a singular disk $f:D^2 \to S^3$ which has the property that each component of self-intersection is an arc $A \subset f(D^2)$ for which $f^{-1}(A)$ is two arcs in D^2, one of which is interior. Then K is called a <u>ribbon knot</u>.

27. PROPOSITION : Ribbon knots are slice knots.

28. CONJECTURE : Every slice knot is a ribbon knot.

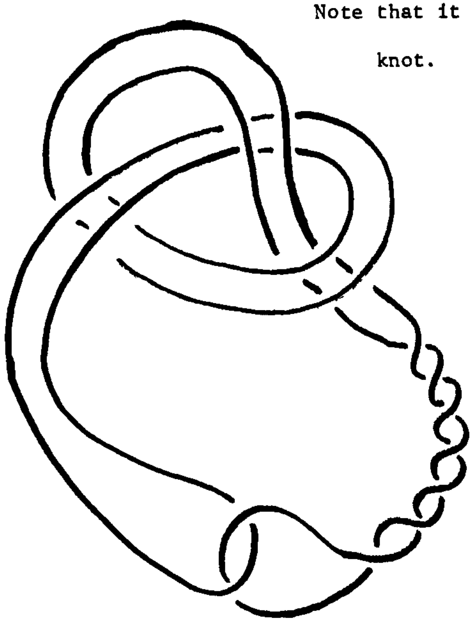
29. EXERCISE : The other knot of example C17 , namely 9_{46} , is a ribbon knot.

30. EXERCISE : The square knot is a ribbon knot. So is every knot of the form $K \# rK$, r = reflection.

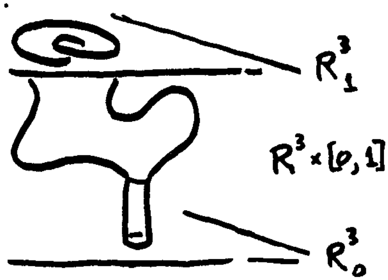
31. EXERCISE : If K is a slice knot in S^3 and K' is the double of K with zero twists, then K' is also a slice knot.

32. REMARK : According to 7B7 and 24, a doubled knot can be a slice knot only when its twisting number is a non-negative triangular number $n = k(k-1)$. Thus among the twist knots (doubles of the unknot) the only candidates for slice knots are the ones with twisting number $n = 0, 2, 6, 12, 20, 30, \ldots$. For $n = 0$ or 2 we have the unknot and the stevedore, which are slice. Recent work of Casson and Gordon [1975] shows these two are the only twist knots which are slice knots.

33. EXERCISE : (Casson) The following is a slice knot, in fact a ribbon knot. Note that it is a double of a non-slice knot.



F. CONCORDANCE. A <u>concordance</u>† between knots K_0 and K_1 in S^3 is a locally flat* cylinder $C \cong S^1 \times [0,1]$ embedded in $S^3 \times [0,1]$ in such a way that the ends $S^1 \times \{i\}$ are embedded in $S^3 \times \{i\}$ as $K_i \times \{i\}$. If we consider knots in R^3, this is the same as a locally flat tube embedded in the slab of R^4 defined by $0 \leqq x_4 \leqq 1$. Remember that we are still working in the PL category. Two knots connected by a concordance are said to be <u>concordant</u>; it is easy to see that this is an equivalence relation on the class of all knots. It is strictly weaker than ordinary (oriented) equivalence.



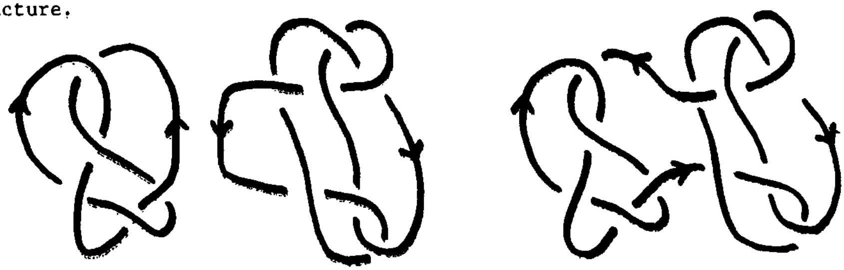
1. EXERCISE : As the picture suggests, a knot is concordant to the unknot if and only if it is a slice knot.

Recall that knots form a commutative semigroup under the connected sum operation #. To be precise we consider <u>oriented knot types</u>, in which each knot is endowed with a preferred orientation and equivalences are required to preserve orientation both of the knot and of S^3. For a concordance between oriented knots we require that they be homologous (rather than opposite) in the tube. Finally, we require

† also known as knot cobordism.

* take this to mean that C has a tubular neighbourhood $C \times D^2$ which intersects $S^3 \times \{i\}$ in tubular neighbourhoods of $K_i \times \{i\}$, $i = 0, 1$.

connected sum to respect orientation, as suggested by the following picture.*



2. THEOREM : The set $\mathcal{C}_1$ of concordance classes of oriented knots of S^1 in S^3 is an abelian group under the addition defined by:

$$[J] + [K] = [J \# K] \qquad [\ ] = \text{class in } \mathcal{C}_1 .$$

The following sequence of lemmas establish the proof.

3. LEMMA : If J_0 and K_0 are concordant with J_1 and K_1, respectively, then $J_0 \# K_0$ is concordant with $J_1 \# K_1$. Therefore $+$ is well-defined

PROOF : This is easily established from the following exercise by "adding" the concordances.

4. EXERCISE : Any concordance may be assumed to be straight on an arc. That is, after orientation-preserving homeomorphism of $S^3 \times [0,1]$, there is an arc $A \subset S^1$ such that the subset $A \times [0,1]$ of $C = S^1 \times [0,1]$ is embedded in $S^3 \times [0,1]$ as the product of an inclusion $A \subset S^3$ and the identity on $[0,1]$.

* a precise definition is easy to formalize.

5. LEMMA : The operation $+$ is associative and commutative. The zero element is the class [unknot] and the inverse of a class $[K]$ is the class $[r\overline{K}]$ where $r: S^3 \to S^3$ is a reflection and $\overline{}$ denotes the reverse orientation.

PROOF : The first part follows from commutativity and associativity of $\#$, and the rest is easily checked using 1 and E11.

6. THEOREM : Concordant knots have the same signature. In fact signature induces a homomorphism of abelian groups

$$\sigma : \mathcal{C}_1 \longrightarrow Z$$

whose image is the set of even integers.

PROOF : Firstly, $[K_0] = [K_1]$ if and only if $K_0 \# r\overline{K}_1$ is a slice knot. So $\sigma(K_0) - \sigma(K_1) = \sigma(K_0) + \sigma(rK_1) = \sigma(K_0) + \sigma(r\overline{K}_1) = \sigma(K_0 \# rK_1)$ Additivity follows from E7. By the example of sums of trefoils, every even number is the signature of some knot.

7. COROLLARY : If $\sigma(K) \neq 0$, then $[K]$ has infinite order in $\mathcal{C}_1$.

8. EXAMPLE : The figure-eight knot 4_1 represents an element of order 2 in $\mathcal{C}_1$. By E22, $[4_1] \neq 0$, but because 4_1 is oriented-equivalent with $r\overline{4_1}$, $2[4_1] = [4_1] + [r\overline{4_1}] = [4_1 \# r\overline{4_1}] = 0$.

The notion of concordance is convenient in the study of singularities* (non-locally flat points) of, say, a PL submanifold M^2 of a PL manifold W^4. If $p \in M$ is such a point (they are necessarily isolated), the singularity type at p is the class $\kappa_p \in \mathcal{C}_1$ determined by taking a PL ball $B^4 \subset W^4$ such that $B \cap M$ is a proper 2-ball in B and letting $\kappa_p = [\partial B \cap M]$, where $\partial B \cap M$ is the knot in the 3-sphere ∂B.† It is straightforward to verify that κ_p is well-defined and that the following is true.

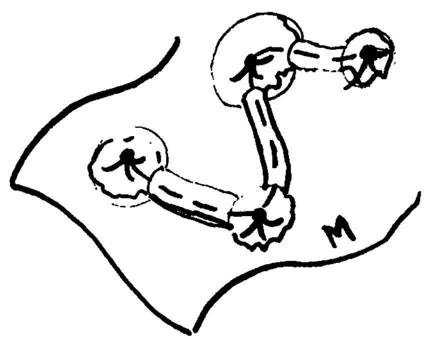
9. PROPOSITION : For a singular point p of a submanifold $M^2 \subset W^4$, the following are equivalent:

(i) $\kappa_p = 0$ in $\mathcal{C}_1$,

(ii) with B^4 as above, $\partial B \cap M$ is a slice knot in ∂B,

(iii) the singularity is removable in the sense that in arbitrarily small neighbourhoods of p, we may revise M to be locally flat.

Here are two connections between the geometry of the situation and the group structure of $\mathcal{C}_1$. The proof of the first is strongly suggested by this picture ⟶ For further details see Fox-Milnor [1966]. We'll leave them as interesting exercises.



* not to be confused with singularities of a map in the sense of not being one-to-one.

† one may also want to keep track of orientation.

10. EXERCISE : Let M^2 be a PL submanifold of W^4 and let $B^4 \subset W^4$ be a ball such that $B \cap M$ is a proper sub-disk which encloses the singularities $p_1, \ldots, p_r$ in its interior (and has none on the boundary). Then the knot $\partial B \cap M$ in ∂B^4 has concordance class equal to $\kappa_{p_1} + \cdots + \kappa_{p_r}$.

11. EXERCISE : There exists a PL 2-sphere embedded in S^4 with exactly the set $\kappa_1, \ldots, \kappa_r$ of singularity types if and only if $\kappa_1 + \cdots + \kappa_r = 0$.

12. REMARKS : The group $\mathcal{C}_1$ is not finitely generated. One proof of this is in Tristram [1969]. In brief, he concocts from V a hermitian form over C^{2g} for each prime p. Diagonalizing this over the complex numbers defines a p-signature σ_p. Each σ_p is, like σ (which equals his σ_2) a homomorphism from $\mathcal{C}_1$ onto the group of even integers. Moreover, $\lim_{i\to\infty} \sigma_{p_i}(K) = \sigma(K)$ where $\{p_i\}$ is the sequence of odd primes in ascending order. By example they are shown to be independent: for each i there exists a knot with $\sigma_{p_i} = 2$, $\sigma_{p_j} = 0$ if $j \neq i$. If Z^∞ is the abelian group of infinite sequences of integers which are eventually constant, then the map $[K] \to \{\frac{1}{2}\sigma_{p_i}(K)\}$ is a homomorphism of $\mathcal{C}_1$ onto Z^∞ and it follows that no finite set of knots generates $\mathcal{C}_1$. A similar proof may be found in Milnor [1968][1], using another family of signatures σ_θ. Milnor also describes the Seifert pairing in terms of cohomology and cup products, which is in some ways a slicker way to do it.

For links one can also define a Seifert form and a signature. The symmetrized Seifert form may be singular, so one splits it into a nonsingular part and a null part, using the former to define signature and the latter to define nullity. Both of these turn out to be invariants of link concordance (disjoint locally flat tubes connecting the links in $S^3 \times [0,1]$). See Tristram [1969] as well as Murasugi [1965].

As the notation $\mathcal{C}_1$ suggests there is a concordance group $\mathcal{C}_n$ for knots of S^n in S^{n+2}, defined in much the same way as $\mathcal{C}_1$. By Kervaire [1965], $\mathcal{C}_n = 0$ whenever n is even. In other words all even-dimensional knots are slice knots. For n odd and greater than one, there is a periodicity $\mathcal{C}_n \cong \mathcal{C}_{n+4}$. Levine [1969] has studied these higher-dimensional concordance groups and describes them algebraically in terms of groups of equivalence classes of matrices. A key observation is that in higher dimensions, a knot is slice if and only if it has a Seifert matrix (defined on the middle homology of a Seifert surface) of the form shown in E16. This is not true for classical knots, see Casson and Gordon [1975]. One can also use generalized Seifert forms to compute Alexander invariants for higher-dimensional knots and prove a duality property which generalizes classical symmetry $\Delta(t^{-1}) \doteq \Delta(t)$. See, for example, Levine [1966].

13. EXERCISE : Linking number is an invariant of concordance of links.

CHAPTER 9 . 3-MANIFOLDS AND SURGERY ON LINKS

A. INTRODUCTION. Up to this point we have been, by and large, concerned with knots and links in their own right. Although knot theory per se is by no means a closed book, many of its more interesting developments in the last decade or two have been in the directions of application of knot-theoretical techniques to other branches of mathematics, or at least other parts of topology. For example, the study of singularities in algebraic geometry has been enriched by the application of knot methods. Milnor's book [1968][2] is a highly recommended introduction to this. Another important application knot theory is in the study of manifolds. In this chapter we will focus attention on three-dimensional manifolds. A technique dating back to Dehn -- now known as surgery -- has become a powerful tool in constructing, and proving theorems about, 3-manifolds. Unless otherwise stated, all closed 3-manifolds M^3 under consideration will be assumed connected and orientable, in other words, $H_1(M) \cong H_3(M) \cong Z$.

B. LENS SPACES. We already know from the Alexander trick that if one attaches two 3-balls together via any homeomorphism of their boundaries, the resulting space is S^3. Consider the analogous construction using two solid tori V_1 and V_2 . If $h : \partial V_2 \to \partial V_1$ is a homeomorphism we may form the space

$$M^3 = V_1 \bigcup_h V_2$$

which is the result of identifying each $x \in \partial V_2$ with $h(x) \in \partial V_1$ in the disjoint union of V_1 and V_2 .

1. EXERCISE : M^3 is a closed connected orientable 3-manifold which depends, up to homeomorphism, only upon the homotopy class of $h(m_2)$ in ∂V_1 , where m_2 is a meridian of V_2 . [see section 2E].

2. DEFINITION : Choosing fixed longitude and meridian generators ℓ_1 and m_1 for $\pi_1(\partial V_1)$, we may write

$$h_*(m_2) = p\ell_1 + qm_1$$

where p and q are coprime integers. The resulting M^3 is called the <u>lens space</u> of type (p,q) and denoted traditionally by

$$M^3 = L(p,q).$$

In other words a 3-manifold is a lens space if and only if it contains a solid torus, the closure of whose complement is also a solid torus. Some writers don't count S^3 and $S^2 \times S^1$ as lens spaces.

3. EXERCISE : Establish the following homeomorphisms:

$$L(1,q) \cong S^3$$
$$L(0,1) \cong S^2 \times S^1$$
$$L(2,1) \cong RP^3 \text{ (= } S^3 \text{ with antipodal points identified)}$$

4. EXERCISE : Show that $L(p,q) \cong L(p,-q) \cong L(-p,q) \cong L(-p,-q) \cong L(p,q+kp)$ for each integer k .

Thus we adopt the convention that $0 < q < p$. This exhausts the list of lens spaces which are not the 'degenerate' ones: S^3 and $S^2 \times S^1$.

5. EXERCISE : Show that the fundamental group of $L(p,q)$ is the finite cyclic group Z/p.

So two lens spaces $L(p,q)$ and $L(p',q')$ are definitely not homeomorphic -- nor even of the same homotopy type -- unless $p = p'$.

6. EXERCISE : Show that $L(p,q) \cong L(p,q')$ if $\pm qq' \equiv 1 \pmod p$.
[Hint: Regard $V_2 \cup_{h^{-1}} V_1$] .

7. REMARK : Among the list $L(p,1), \ldots, L(p,p-1)$ of all lens spaces with group Z_p (recall that q must be prime to p) there are duplications, up to homeomorphism, according to the exercise. Actually lens spaces have been completely classified. According to Brody [1960]* we have that $L(p,q)$ and $L(p,q')$ are

$$\text{homeomorphic} \iff \pm q' \equiv q^{\pm 1} \pmod p .$$

They are, according to Whitehead [1941], of the

$$\text{same homotopy type} \iff \pm qq' \text{ is a quadratic residue, mod } p .$$

This means that $\pm qq' \equiv m^2 \pmod p$ for some m. Thus for example $L(7,1)$ and $L(7,2)$ are 3-manifolds of the same homotopy type which are not homeomorphic.

A theorem of Fermat and Euler states that if p is a prime congruent to 3 modulo 4, then for any q, exactly one of $\pm q$ is a quadratic residue mod p. For all other primes p either both or neither of $\pm q$ is a quadratic residue. Thus given $p = 3, 7, 11, \ldots$

* there are earlier proofs, but Brody's actually uses knot theory.

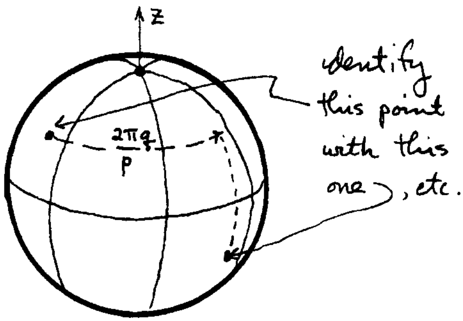
there is only one homotopy type of lens spaces $L(p,q)$. For $p = 5$, 13, ... there are two homotopy types. What if p isn't prime?

8. OTHER DESCRIPTIONS : There are other ways of describing $L(p,q)$. We list three.

(I) one may regard $L(p,q)$ as the result of removing a tubular neighbourhood of a trivial knot in S^3 and sewing it back so that a meridian lies on a curve which has linking number p with the trivial knot and runs q times along it (i.e. a (q,p) torus knot). Note that the roles of 'meridian' and 'longitude' become reversed if we speak relative to the neighbourhood of the trivial knot, rather than relative to its complementary solid torus.

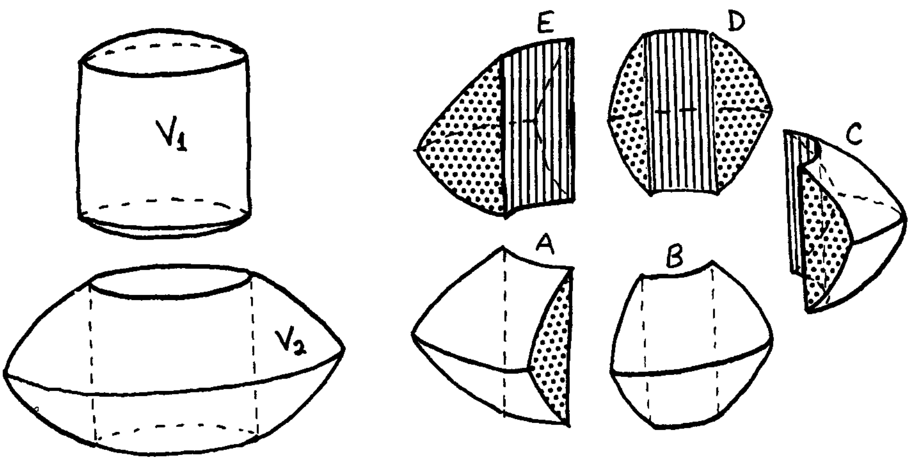
(II) Consider the unit ball B^3 in R^3. Then identify each point on the upper half $(z \geq 0)$ of ∂B^3 with its image under counter-clockwise rotation by an angle of $2\pi q/p$ about the z-axis, followed by reflection in the $x - y$ plane, as in the following picture.

To construct $L(p,q)$

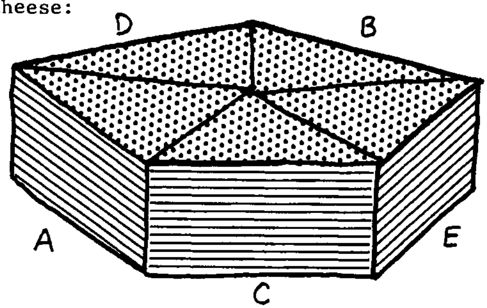


To verify that this identification space is $L(p,q)$ and for reasons which will become apparent later, we depict the B^3 as a lens-shaped solid with edge on the circle $x^2 + y^2 = 1$ and edge-angle $2\pi/p$. Let V_1 be the part of the space inside the cylinder $x^2 + y^2 \leq 1/4$ (with identifications) and let V_2 be the closure of what remains.

It is clear that V_1 is a solid cylinder with ends identified (with a $2\pi q/p$ twist); thus it's a solid torus. That V_2 is also a solid torus can be seen by chopping it into pieces thus (shown for the case $p = 5$, $q = 2$) :



and re-assembling according to the prescribed identifications, like wedges of a big cheese:



We still must identify the parts of V_2 which were separated by the dissection. But this is just sewing the top and bottom of the cheese together (with a twist) to form a solid torus. Now a meridinal curve of V_2 is just one which runs around the perimeter of the big cheese. Thus on V_1 (before identifying top and bottom) it consists of p vertical lines equally spaced on the boundary of the cylinder $x^2 + y^2 = 1/4$. Since top and bottom become twisted by q/p full turns, the meridinal curve of V_2 becomes a p, q curve on V_2 and our identification space is indeed $L(p,q)$.

(III) Here is another description of $L(p,q)$; the most concise of all. Consider S^3 as the unit sphere of C^2 , complex 2-space. Let $\tau : S^3 \to S^3$ be the homeomorphism given by

$$\tau(z_0,z_1) = (z_0w,\ z_1w^q) ,$$

where $w = \exp(2\pi i/p)$ is the principal p^{th} root of unity. Then τ is periodic of period p (thus generates a Z/p-action on S^3 .) Consider the orbit space of this action; that is, we identify points

x, y of S^3 if $x = \tau^k(y)$ for some k .

9. EXERCISE : Verify that the orbit space is homeomorphic with $L(p,q)$. [Hint: Show that after stereographic projection, the lens-shaped region of the discussion above may be taken as a fundamental region of the action.]

Another view of this is that S^3 is the universal covering space of $L(p,q)$ and that τ is a generator of the cyclic group of covering translations. This checks with our calculation that the fundamental group of $L(p,q)$ is Z/p (and thus so is the covering translation group of the universal cover).

C. HEEGAARD DIAGRAMS . We now generalize the construction of the previous section. Recall that a <u>handlebody</u> of genus g is the result of attaching g disjoint "1-handles" $D^2 \times [-1,1]$ to a 3-ball B^3 by sewing the parts $D^2 \times \{\pm 1\}$ to $2g$ disjoint disks on ∂B^3 in such a way that the result is an orientable 3-manifold with boundary. Two handlebodies of the same genus are homeomorphic (and vice versa). The boundary of a handlebody of genus g is a closed orientable 2-manifold of genus g , as genus was defined previously. Let H_1 and H_2 be handlebodies of the same genus, g , and let $h : \partial H_2 \to \partial H_1$ be a homeomorphism. Then form the identification space

$$M^3 = H_1 \bigcup_h H_2$$

as before. Again it is easy to see that M^3 is a closed orientable

3-manifold. The triple (H_1, H_2, h) is called a Heegaard diagram* of genus g for the manifold M.

1. EXERCISE : Given any positive integer g, show that S^3 has a Heegaard diagram of genus g.

The genus of a 3-manifold M is the smallest genus of all the Heegaard diagrams which yield M (up to homeomorphism). Thus we have:

$$\text{genus }(M) = 0 \quad \Longleftrightarrow \quad M \cong S^3$$

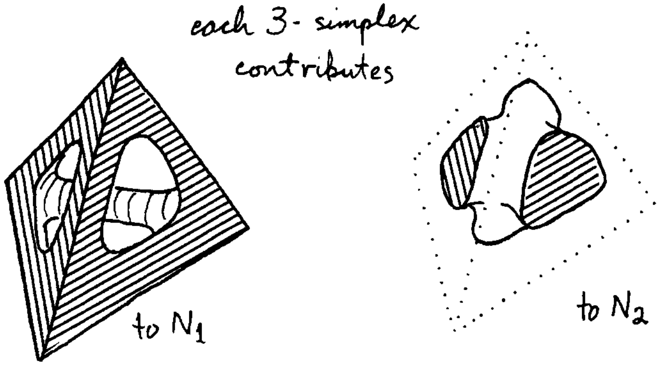
$$\text{genus }(M) = 1 \quad \Longleftrightarrow \quad M \text{ is a lens space or } S^2 \times S^1$$

2. THEOREM : Every closed orientable connected PL 3-manifold has a Heegaard diagram, and hence a well-defined genus.

OUTLINE OF PROOF : Consider a triangulation of the 3-manifold M, and let M_1 be the 1-skeleton. In a second barycentric subdivision let $\hat{M}_1$ be the dual 1-skeleton; that is, the union of all new vertices and edges which do not lie on a simplex which meets M_1. Let N_1 and $\hat{N}_1$ be the simplicial neighbourhoods of M_1 and $\hat{M}_1$, with respect to the subdivision. It is easy to see that N_1 and $\hat{N}_1$ are 3-manifolds which meet in a common boundary and whose union is M. We are done once we show that they are handlebodies. To this end, note that M_1, being a graph, contains a maximal tree which contains all the vertices. An easy induction shows that the simplicial neighbourhood of this tree

* or Heegaard splitting

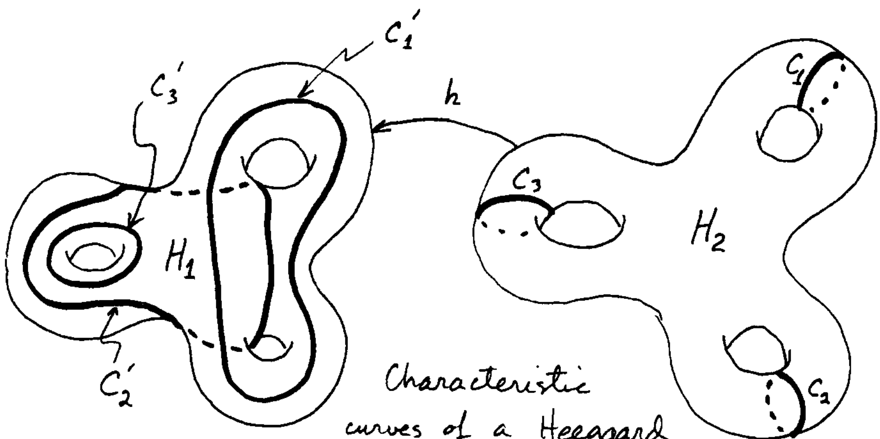
is a 3-ball. Putting in the simplicial neighbourhood of each remaining 1-simplex of M_1 just adds a 1-handle to this ball, and after adding them all we have exactly N_1. Since M is orientable, N_1 must also be orientable, so N_1 is a handlebody. Ditto for $\hat{N}_1$.



3. REMARK : The PL assumption is unnecessary by the basic theorem due to Moise and Bing: Every 3-manifold may be triangulated.

Here is a way of viewing a Heegaard diagram which is sometimes useful. Let $D_1,\ldots,D_g$ be the disks of H_2 corresponding to the centers $D_i \times \{0\}$ of the 1-handles $D_i^2 \times [-1,1]$. (Note that given an abstract handlebody, there may be many ways to choose such a collection.) Then each 1-handle of H_2 is just a collar on its central disk and we have that H_2 - (open collars of the D_i) is a 3-ball. Let C_i be the boundary of D_i and let $C_i' = h(C_i)$ the image on ∂H_1 of C_i under the attaching homeomorphism $h : \partial H_2 \to \partial H_1$ Call these the characteristic curves of the diagram.

4. EXERCISE : The manifold $M = H_1 \underset{h}{\cup} H_2$ is determined (up to homeomorphism) by the collection of curves $C_1' \cup \ldots \cup C_g'$ on ∂H_1. Moreover, if another diagram $N = H_1 \underset{j}{\cup} H_2$ of the same genus has characteristic curves which may be thrown onto $C_1' \cup \ldots \cup C_g'$ (in any order) by a homeomorphism of H_1, then M and N are homeomorphic.



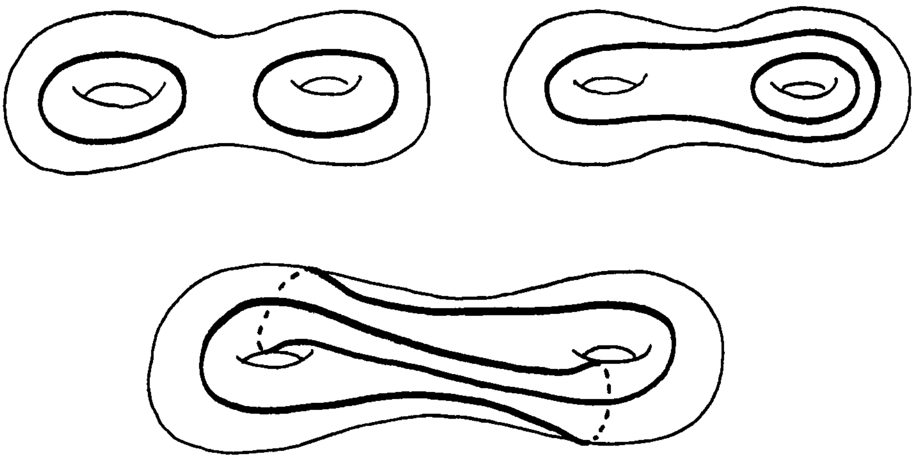
Characteristic curves of a Heegaard diagram.

A consequence of this observation is that some problems of 3-manifold theory might boil down to link theory in 2-manifolds. The fundamental group of M may be presented by taking g generators—one for each handle of H_1 -- and taking the g relations $[C_i'] = 1$, where $[C_i']$ is the class of C_i' in $\pi_1(H_1)$ expressed in terms of the generators.

5. EXERCISE : Verify this last sentence.

6. EXERCISE : Give necessary and sufficient conditions that a given collection of curves on the boundary of a handlebody be characteristic curves for a Heegaard diagram.

7. EXAMPLE : Here are some diagrams which yield simply-connected 3-manifolds (just H_1 and the characteristic curves are shown)



8. EXERCISE : Verify that, in fact, all three diagrams yield S^3 .

9. EXERCISE : Find, for each positive integer g , a 3-manifold of genus

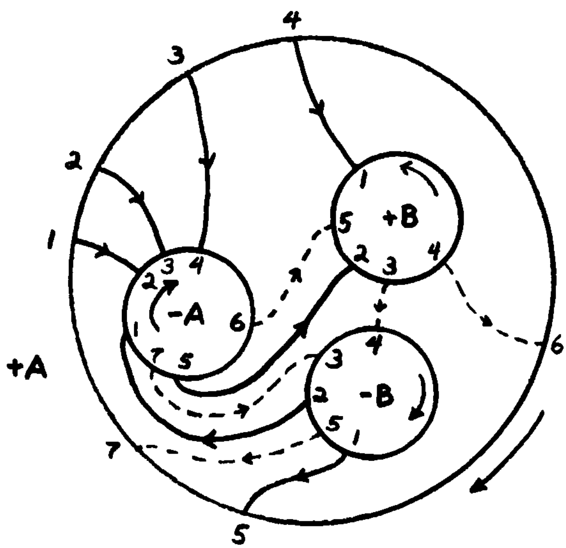
10. EXERCISE : The genus of $S^1 \times S^1 \times S^1$ is three.

D. THE POINCARÉ CONJECTURE, HOMOLOGY SPHERES AND DEHN'S CONSTRUCTION. The Poincaré conjecture states that if a closed 3-manifold is connected and simply connected, then it is homeomorphic with S^3. It is probably the most famous unsolved problem of topology. Its higher-dimensional analogue (is every closed n-manifold of the homotopy type of S^n necessarily homeomorphic with S^n ?) is solved affirmatively for $n \geq 5$, and of course $n \leq 2$, but has resisted repeated attacks in dimensions 3 and 4.*

Poincaré originally conjectured [1900] that any closed 3-manifold with the same <u>homology</u> groups as S^3 is actually homeomorphic with S^3. But in [1904] he published a counterexample (which appears below) and rephrased the conjecture in terms of <u>homotopy</u>. We'll say that a closed connected 3-manifold M^3 is a <u>homology sphere</u> if $H_1(M) = 0$ and a <u>homotopy sphere</u> if $\pi_1(M) = 0$. Clearly homotopy spheres are homology spheres. Using Poincaré duality one can show that for homology spheres all the homology groups coincide with those of S^3, and likewise for homotopy. In this section we'll encounter three methods for constructing homology spheres which aren't S^3, including the surgery technique introduced by Dehn [1910] which will be studied in greater detail later in this (and the next) chapter.

* this is true also in the PL category. In the C^∞ category if one demands a diffeomorphism with S^n it is definitely false for $n \geq 7$. The theory of exotic spheres deals with such counterexamples.

1. EXAMPLE : The <u>Poincaré</u> <u>manifold</u>. Poincaré constructed this non-simply-connected 3-manifold by means of a Heegaard diagram. In the previous section we saw that a 3-manifold may be unambiguously described by means of n disjoint simple closed curves on the boundary surface of a genus n handlebody. If we cut the handlebody open at the handles its boundary can be flattened onto the surface of a 2-sphere, leaving 2n holes. Or we can lay it out on the plane as a disk with 2n-1 holes. The characteristic curves of a Heegaard diagram can then be drawn as a collection of arcs in this punctured disk, having their endpoints in the boundary. (Such a picture is, I think, the original meaning of "Heegaard diagram"). Here is Poincare's diagram, of genus two.

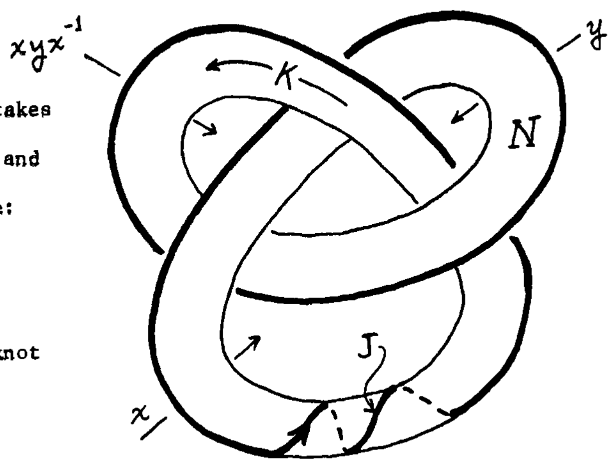


The boundary curves +A and -A are to be identified in the sense indicated by arrows, likewise +B and -B, so that points labelled by the same number are matched up. The dashed curves describe one of the characteristic curves, while the solid curves define the other one.

2. EXERCISE : Verify that there actually is a homeomorphism $h: \partial H_2 \to \partial H_1$ of genus two handlebodies with these characteristic curves. Calculate a presentation of the fundamental group of the Poincaré manifold $P^3 = H_1 \bigcup_h H_2$ and show that $\pi_1(P^3)$ is nontrivial.

Actually we'll establish the assertions of this exercise in a rather indirect way. The following construction, which appears in Dehn [1910], is an alternative method for constructing homology spheres which aren't homotopy spheres. After describing this manifold and calculating its fundamental group, we'll demonstrate geometrically that it is the same manifold as that described by Poincaré. The advantage of Dehn's method is that it immediately suggests a more general technique for constructing plenty of homology spheres. There is even some hope that it can be used to construct a counterexample to the Poincaré conjecture. (See the section on Property P at the end of this chapter).

3. EXAMPLE : Dehn's construction of a homology sphere. Let N be a tubular neighbourhood of a right-handed trefoil K and let J be the curve on ∂N pictured here.

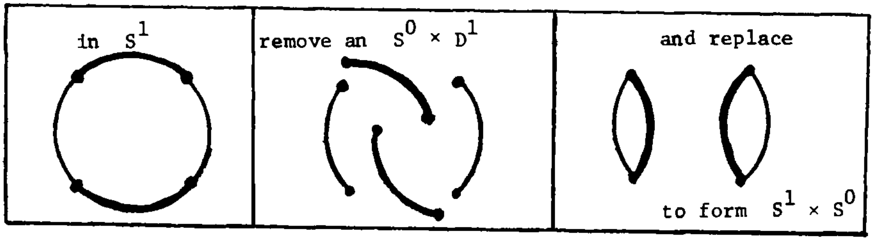


Now consider a homeomorphism $h: \partial(S^1 \times D^2) \to \partial N$ which takes a meridian $* \times S^1$ onto J and form the identification space:

$$Q^3 = (S^3 - \mathring{N}) \bigcup_h (S^1 \times D^2)$$

sewing a solid torus to the knot exterior via h .

4. AN ASIDE: Due to limitations on our spatial perception, it is difficult to visualise the manifold Q obtained from S^3 by removing a solid torus and replacing it differently. It is completely analogous to the following operation whereby S^1 is converted into another 1-manifold, $S^1 \times S^0$, by removing an $S^0 \times D^1$ and replacing it in another way:



3. EXAMPLE (CONTINUED) : It is easy to verify that Q^3 constructed above is a connected, closed, orientable 3-manifold. Its homology groups may easily be calculated using a Mayer-Vietoris sequence [EXERCISE] and will be found to coincide with those of S^3. Computing $\pi_1(Q^3)$ is more interesting. Think of forming Q by first adjoining a meridinal disk $* \times D^2$ of $S^1 \times D^2$ to the knot exterior $X = S^3 - \overset{\circ}{N}$, then adding a collar neighbourhood of the disk, and finally sewing on what remains of $S^1 \times D^2$ -- a 3-ball. Only the first of these additions affects the fundamental group. So $\pi_1(Q^3)$ is isomorphic with the knot group $\pi_1(X)$ with the additional relation $[J] = 1$. As in 3D8 we have

$$\pi_1(X) \;=\; (\ x,\ y;\ xyx = yxy\ )$$

and from the picture we read off $[J] = yx(xyx^{-1})x^{-2} = yx^2yx^{-3}$, and so

$$\pi_1(Q^3) \cong (\, x,\ y;\ xyx = yxy,\ yx^2y = x^3 \,)\ .$$

Make the substitution $z = xy$:

$$\pi_1(Q^3) \cong (\, x,\ z;\ zx = x^{-1}z^2,\ x^{-1}zxz = x^3 \,) \qquad \text{or}$$

$$\pi_1(Q^3) \cong (\, x,\ z;\ (zx)^2 = z^3 = x^5 \,)\ .$$

This presentation is well-known to group theorists. It clearly becomes trivial upon abelianization, which is to be expected since $H_1(Q^3) = 0$. But the following argument shows that it's not itself a trivial group. Consider the regular icosahedron in 3-space, having 12 vertices, 30 edges and 20 triangular faces. Its group of rigid motions (reflections being ruled out) or symmetries has order 60 and may be generated by

$\alpha = 120^\circ$ rotation about an axis through the center of a face

and $\beta = 72^\circ$ rotation about an axis through a vertex on that face.

If we send $x \to \beta$ and $z \to \alpha$ it is easy to see geometrically that the relations in the presentation above are respected. So $\pi_1(Q^3)$ maps homomorphically onto a group of order 60, hence is nontrivial.

5. REMARK : With a little more analysis one can show that the homomorphism above is two-to-one, hence $\pi_1(Q^3)$ has order 120. This group is commonly called the binary icosahedral group and is an example of a finite perfect* group. The icosahedral group of symmetries itself has presentation $(\, \alpha,\ \beta;\ (\alpha\beta)^2 = \alpha^3 = \beta^5 = 1 \,)$.

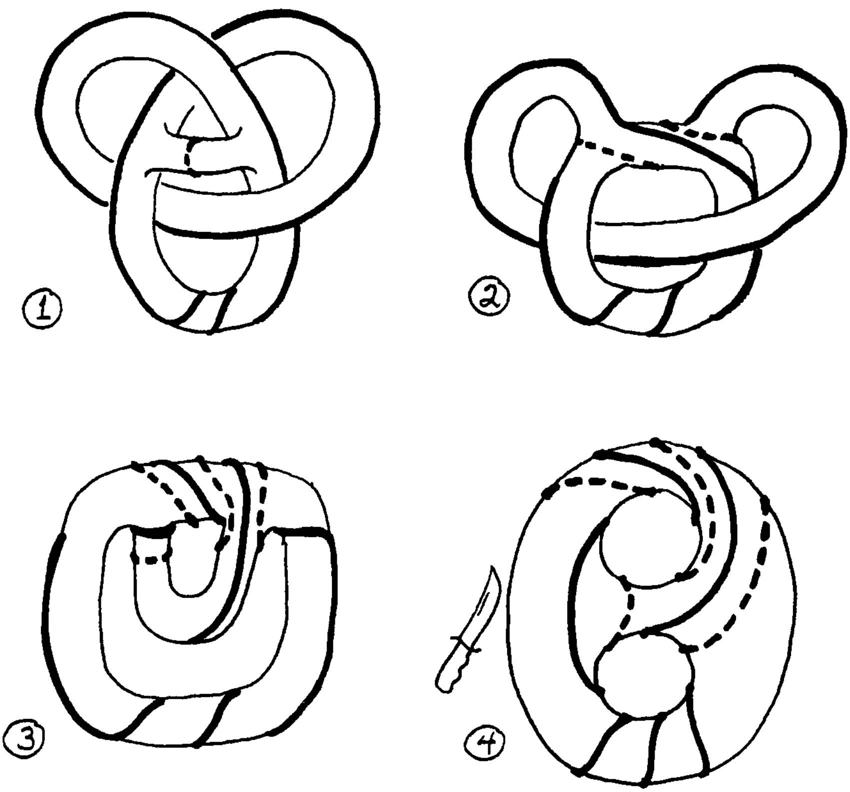
* meaning that the commutator subgroup is the whole group. It may be realized isomorphically as the matrix group $SL(2, Z/5)$.

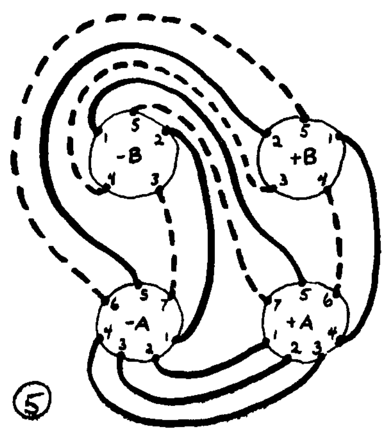
6. EXERCISE : The icosahedral group is isomorphic with the alternating group A_5 of even permutations of five symbols. The full icosahedral group (reflections allowed) of symmetries is isomorphic with S_5, the full symmetric group. Instead of the icosahedron, we could as well use a regular dodecahedron (twelve pentagonal faces). Although they both have order 120, the binary icosahedral group is not isomorphic with S_5. [Hint: S_5/A_5 is abelian.]

7. PROPOSITION : The examples above, of Poincaré and Dehn, are homeomorphic. That is, $P^3 \cong Q^3$.

PROOF: The argument is contained in the series of pictures on the next page. Starting with Dehn's example, we drill out a tube from the trefoil's exterior and add it to the $S^1 \times D^2$, making the latter into a handlebody of genus two which is to be attached to the outer part of (1) by the characteristic curves shown. The isotopic deformation (1) - (4) shows that this outer part is also a handlebody. (Since the deformation is done equivariantly with respect to 180° rotation about the vertical, the back of the figure looks exactly like the front). Again the solid and dashed curves of (4) are directions for sewing a handlebody onto the handlebody on the <u>outside</u> of the surface pictured. Cutting this surface open and flattening with the inside toward us gives us the picture (5) in the plane (plus a point at ∞). Moving the point at infinity to be in the hole labelled $+A$ gives us exactly the Heegaard diagram of Poincaré's example.

8. REMARK : The Poincaré manifold is also known in the literature as the <u>icosahedral space</u> or the (spherical) <u>dodecahedral space</u>. (See also 11E).





Demonstration of the equivalence of the constructions of Dehn and Poincaré, producing a homology sphere with fundamental group of order 120.

9. EXAMPLE : Here is yet another construction of non-simply-connected homology spheres -- in fact the fundamental group is necessarily infinite. Instead of attaching a solid torus to a knot exterior, we may sew together two knot exteriors along their boundaries. Specifically let X_1 and X_2 be complements of tubular neighbourhoods of <u>nontrivial</u> knots in S^3. Let ℓ_i and m_i be longitude-meridian pairs on ∂X_i determined by the preferred framings of the neighbourhoods and choose a homeomorphism $h: \partial X_2 \to \partial X_1$ such that $h(\ell_2) = m_1$ and $h(m_2) = \ell_1$. Consider

$$M^3 = X_1 \bigcup_h X_2 .$$

Since the X_i look homologically like solid tori it is easy to see that M^3 is a homology sphere.

10. EXERCISE : In this type of example $\pi_1(M^3)$ contains subgroups isomorphic with the knot groups $\pi_1(X_i)$, hence is infinite.

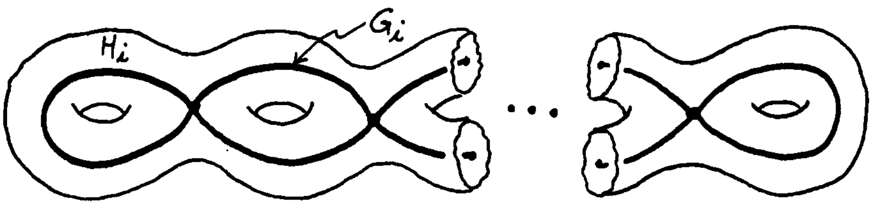
11. EXERCISE : There are infinitely many homotopy types of homology spheres.

E. A THEOREM OF BING. This section presents a partial positive result in the direction of the Poincaré conjecture, whose assumption of simple connectivity may be phrased, "every simple closed curve shrinks to a point." Bing's theorem uses a stronger geometric hypothesis. Since the proof depends strongly on knot-theoretical techniques, and is somewhat involved, we'll cover it in some detail. The version given here was shown to me by David Gillman, but differs only slightly from the original of Bing [1958].*

,

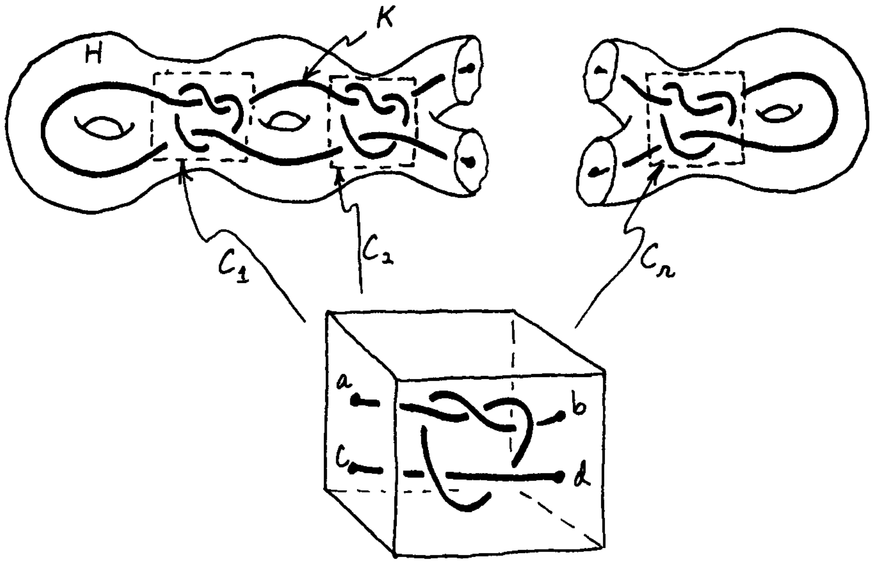
1. THEOREM : If M^3 is a closed orientable connected 3-manifold such that every simple closed curve in M lies interior to a ball in M, then M is homeomorphic with S^3.

PROOF : Although there are ways around this, we'll interpret the hypothesis as saying that everything is PL. Let $M = H_1 \cup_h H_2$ be a Heegaard diagram for M and let G_i be a graph which is a "spine" of H_i as pictured below $(i=1,2)$.



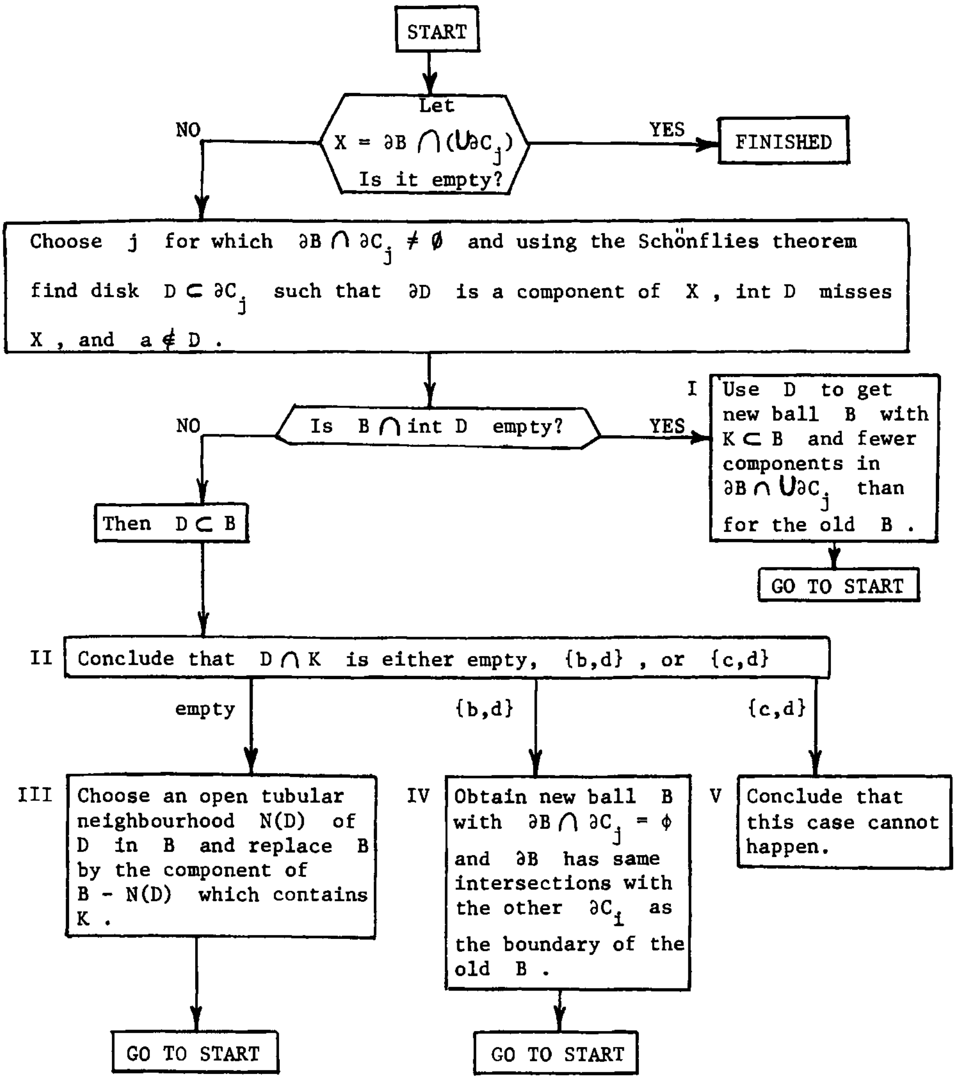
The important property of G_i is that any neighbourhood U of G_i may be expanded by a self-homeomorphism f of M so that it engulfs all of H_i, that is $f(U) \supset H_i$. Then we will be finished if we can show that both G_1 and G_2 lie interior to balls in M. For, after expanding we can make one ball contain H_1 and the other contain H_2, so that M is a union of two 3-balls. Then Theorem 2F6 would imply that $M \cong S^3$.

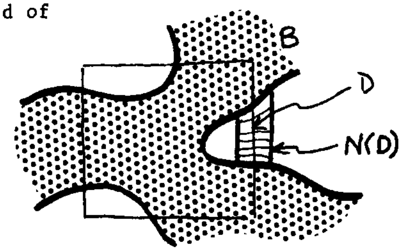
Concentrating on one side, say $i = 1$ (and dropping the subscript for convenience), replace the spine G by a simple closed curve K in H which coincides with G outside cubes $C_1,\ldots,C_r$ $(r+1 = \text{genus } H)$ which enclose the crossing points of G. Inside each C_j, K consists of a pair of arcs, knotted and linked as shown.



Then by hypothesis, K lies interior to a 3-ball $B \subset M$. There is no reason to believe that G also is interior to B, as we desire, so we must modify B to do just that. The idea is to remove the intersections of ∂B with all the boundaries of the C_j (we may assume by general position that this intersection is already just a finite union of simple closed curves) and preserve the fact that $K \subset \text{int } B$. Then it is clear that $K \cup \bigcup C_i \subset \text{int } B$ and hence $G \subset \text{int } B$ and we'd be done.

This modification of B is somewhat complicated; the idea may best be illustrated by the following "flow chart". The parts of the argument labelled by Roman numerals shall be explained more fully; filling in details is left as an exercise.


START
Let $X = \partial B \cap (\bigcup \partial C_j)$ Is it empty?
NO
YES
FINISHED
Choose j for which $\partial B \cap \partial C_j \neq \emptyset$ and using the Schönflies theorem find disk $D \subset \partial C_j$ such that ∂D is a component of X, int D misses X, and $a \notin D$.
Is $B \cap \operatorname{int} D$ empty?
NO
YES
I
Use D to get new ball B with $K \subset B$ and fewer components in $\partial B \cap \bigcup \partial C_j$ than for the old B.
GO TO START
Then $D \subset B$
II
Conclude that $D \cap K$ is either empty, $\{b,d\}$, or $\{c,d\}$
empty
$\{b,d\}$
$\{c,d\}$
III
Choose an open tubular neighbourhood $N(D)$ of D in B and replace B by the component of $B - N(D)$ which contains K.
IV
Obtain new ball B with $\partial B \cap \partial C_j = \phi$ and ∂B has same intersections with the other ∂C_i as the boundary of the old B.
V
Conclude that this case cannot happen.
GO TO START
GO TO START

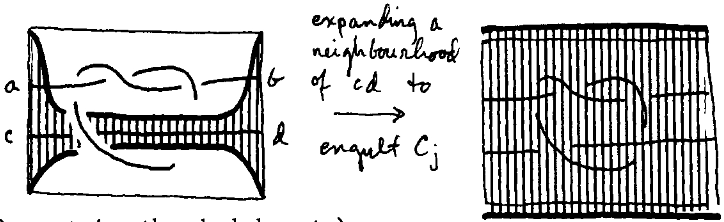


(I) Let $N(D)$ be a regular neighbourhood of D in $M - \overset{\circ}{B}$. Then $Q = B \cup \overline{N(D)}$ is homeomorphic with $S^2 \times I$ by Exercise 2 below. One boundary component of Q definitely intersects the interior of C_j. If it doesn't stick out, it bounds a 3-ball in C_j which may be adjoined to Q to form a new 3-ball which we call B again. In the contrary case there is an arc A in Q, connecting the two components of ∂Q and missing $\bigcup \partial C_i$ (and missing K as well). Then let the new B be Q minus a tubular neighbourhood of A; B is a 3-ball by Exercise 3. In either case we have a new 3-ball $B \supset K$ with strictly fewer components of intersection with $\bigcup \partial C_i$ than previously.

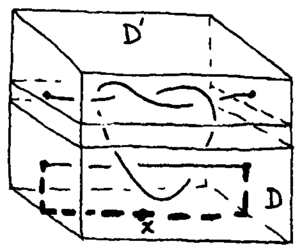
(II) Consider the linking number defined in the 3-ball B. Clearly $\ell k(\partial D, K) = 0$, since $K \subset \text{int } B$ and $\partial D \subset \partial B$. Since D is a Seifert surface for ∂D, we may compute this linking number by inspecting $K \cap D$, which consists of at most $\{b,c,d\}$, and reach the stated conclusion.

(III) That both components of $B - N(D)$ are 3-balls may be deduced from the polyhedral Schönflies theorem. Note that at least one component of X is removed in this way.

(IV) We may conclude that, after an isotopy of M fixed off a small neighbourhood of C_j, D is exactly the right-hand face of C_j and that B contains the left-hand face of C_j as well. Since B contains the arc cd which runs straight through C_j, a neighbourhood of the union of these faces and arc may be expanded, by a homeomorphism g of M fixed except near C_j, to engulf C_j. Let $g(B)$ be the new B.



(B contains the shaded part)



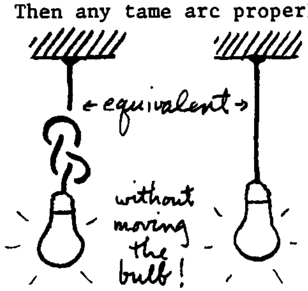
(V) In this case we may assume that D is precisely the bottom half of ∂C_j and that $B \cap \partial C_j$ contains another disk D' with $\{a,b\} \subset D'$. Let Y be the component of $B \cap C_j$ which contains the arc cd; it also contains an arc cxd on ∂C_j. Since D separates $cd - \{c,d\}$ from ab in B, Y does not intersect either ab or D'. Clearly, the curve $cdxc$ is contractible in B, i.e. there is a map $f : D^2 \to B$ so that $f(\partial D^2) = cdxc$. One can then define a map $g : D^2 \to C_j - ab$ so that $g = f$ on $f^{-1}(Y)$ and $g(D^2 - f^{-1}(Y)) \subset \partial C_j - \text{int } D'$, using Tietze's extension theorem. This implies that $cdxc$ is contractible in $C_j - ab$. This is clearly impossible, yielding the desired contradiction.

In each of the steps I, III, IV the number of components of X is reduced by at least one, so after finitely many loops through the diagram we obtain a ball B with $\partial B \cap \partial C_j = \phi$ and the proof of Bing's theorem is complete. Below are the two exercises mentioned above, which also have some interest in their own right.

2. EXERCISE : Let $h : S^1 \times D^1 \to S^2$ be any embedding. Then $D^3 \bigcup_h (D^2 \times D^1)$ is homeomorphic with $S^2 \times I$. (In other words, there is only one way to attach a 2-handle to a 3-ball.)

3. EXERCISE : Let $H \subset S^2 \times I$ be a subset homeomorphic with $D^2 \times I$ such that $H \cap (S^2 \times \{0\}) \cong D^2 \times \{0\}$ and $H \cap (S^2 \times \{1\}) \cong D^2 \times \{1\}$. Then the closure of $(S^2 \times I) - H$ is homeomorphic with the 3-ball. (In other words, there is only one way to remove a spanning 1-handle from $S^2 \times I$. This is related to the following, sometimes called the Light Bulb theorem, which says that one cannot really tie a knot in a light bulb chord (the region of the room outside the bulb considered as $S^2 \times I$).

4. EXERCISE : Let $* \in S^2$ be a basepoint. Then any tame arc properly embedded in $S^2 \times I$ which connects $\{*\} \times \{0\}$ to $\{*\} \times \{1\}$ is equivalent to $\{*\} \times I$, by an ambient isotopy keeping $S^2 \times \{0,1\}$ fixed.



F. SURGERY ON 3-MANIFOLDS. We wish now to generalize the construction that has already been encountered in example D3 and sections 6C,D and 7C.

Suppose the following data are given:

(a) a 3-manifold M, perhaps with boundary,

(b) a link $L = L_1 \cup \ldots \cup L_n$ of simple closed curves in the interior

$\mathring{M}$ of M,

(c) disjoint closed tubular neighbourhoods N_i of the L_i in $\mathring{M}$,

(d) a specified simple closed curve J_i in each ∂N_i.

We may then construct the 3-manifold

$$M' = (M - (\mathring{N}_1 \cup \cdots \cup \mathring{N}_n)) \bigcup_h (N_1 \cup \cdots \cup N_n) ,$$

where h is a union of homeomorphisms $h_i : \partial N_i \to \partial N_i \subset M$, each of which take a meridian curve μ_i of N_i onto the specified J_i.

1. EXERCISE : The homeomorphism type of M' does not depend on choice of h, that is M' is well-defined by conditions (a) - (d).

2. DEFINITION : The 3-manifold M' is said to be the result of a <u>Dehn surgery on</u> M <u>along the link</u> L <u>with surgery instructions</u> (c) and (d)

By Theorem 2C16, we need only specify the homotopy class of the J_i in ∂N_i. Also, assuming the theorem on uniqueness of regular neighbourhoods, the choice of (c) is irrelevant. The most important special case will be $M = S^3$ or R^3, and in this setting the surgery instructions can be expressed simply by assigning a rational number r_i (possibly $= \infty$) to each component L_i of the link as described in the next section.

G. SURGERY INSTRUCTIONS IN R^3 (OR S^3). Each component L_i of an <u>oriented</u> link L in R^3 has a preferred framing (see 2E8) for a tubular neighbourhood N_i in which the longitude λ_i is oriented in the same way as L_i and the meridian μ_i has linking number $= +1$ with L_i. Then we may write the curve J_i in terms of this basis:

$$h_*(\mu_i) = [J_i] = a_i\lambda_i + b_i\mu_i$$

with ambiguity of a $\pm$ sign depending on how one wishes to orient J. Notice that $b_i = \ell k(L_i, J_i)$. The ambiguity disappears if we take the <u>ratio</u>

$$r_i = b_i/a_i$$

We'll call r_i the <u>surgery coefficient</u> associated with the component L_i. If $a_i = 0$, then $b_i = \pm 1$ and we write $r_i = \infty$.

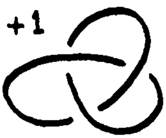
The choice of orientation of the L_i is also irrelevant to the definition of the r_i. However, reversing the orientation of R^3 or S^3 changes the signs of all the r_i, since linking number changes sign. Therefore we'll <u>assume</u> R^3 <u>or</u> S^3 <u>always endowed with a fixed orientation</u>, corresponding with the "right-hand" rule for computing linking number (ℓk_3 as defined in section 5D).

Thus any link L in S^3 with rational numbers attached to its components determines a surgery which yields unambiguously a closed oriented 3-manifold. We shall see in a later section that all closed oriented 3-manifolds arise in this way. In practice we describe the manifold by drawing a picture of L and writing the surgery coefficients near their respective components.

1. EXAMPLE : Take L = the trivial knot. Then the surgery coefficient $r = b/a$ determines the surgery manifold $M^3 \cong L(b,a)$, a lens space. In particular

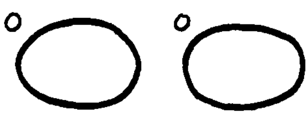
$$M \cong S^2 \times S^1 \quad \text{if} \quad r = 0$$
$$M \cong S^3 \quad \text{if} \quad r = \pm 1 \ , \ \pm 1/2 \ , \ \pm 1/3 \ , \ \ldots \ , \ \infty \ .$$

2. EXAMPLE : Dehn's construction of the Poincaré manifold (section D) is the surgery on a right-handed trefoil using coefficient $+1$.



yields the Poincaré manifold (dodecahedral space)

yields the lens space $L(3,4) = L(3,1)$

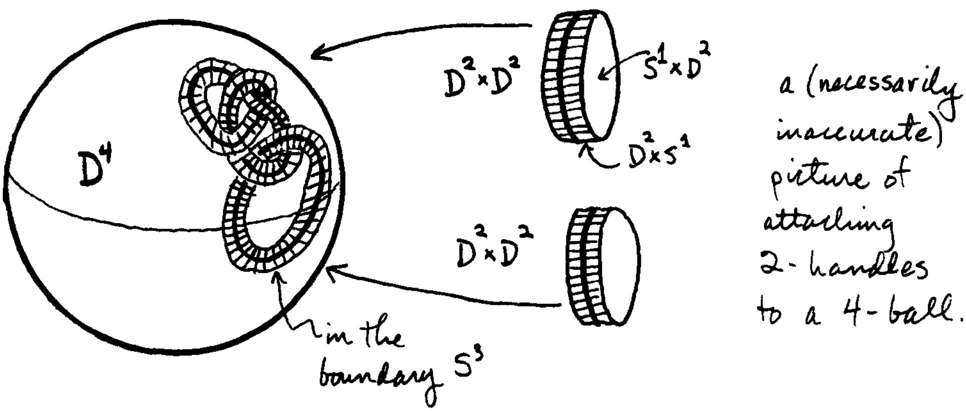


yields the connected sum $S^2 \times S^1 \# S^2 \times S^1$

3. EXERCISE : If some bicollared 2-sphere separates the components of a link in S^3, then surgery on the link yields a 3-manifold which is a connected sum of the manifolds resulting from surgery on the two sublinks.

4. EXERCISE : Show that the surgery manifold determined by any link $L = L_1 \cup \ldots \cup L_n$ in S^3 with coefficients $r_1,\ldots,r_n$ is unchanged by erasing all components of L which have coefficient ∞.

Some, but not all, Dehn-surgeries on S^3 may be viewed as the result on the boundary of attaching 2-handles to the 4-ball. This means that 2-handles $D^2 \times D^2$ are attached to D^4 via attaching maps $f_i : D^2 \times S^1 \to \partial D^4$ (embeddings with disjoint images). The boundary of the resulting 4 manifold is S^3 minus the tubes $f_i(D^2 \times S^1)$ with the copies of $S^1 \times D^2$ added via $f_i | S^1 \times S^1$.

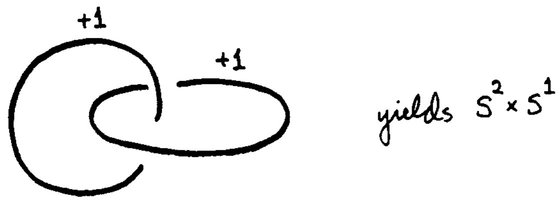


5. EXERCISE : A Dehn surgery corresponds to a handle surgery as described above exactly when all the surgery coefficients are (finite) integers.

This is one reason for my choice of notation for surgery coefficients. Its being an integer indicates a handle surgery (Lickorish calls it 'honest' surgery) and corresponds to the 'framing index' describing the attaching map.

6. EXERCISE : A Dehn surgery on a <u>knot</u> yields a homology sphere exactly when r_1 is the reciprocal of an integer (including the case $r_1 = \infty$, resulting in S^3) .

7. EXAMPLE :



The space X complementary to a tubular neighbourhood, in S^3 of the link pictured is homeomorphic with $S^1 \times S^1 \times I \longleftrightarrow X$ so that

$h(S^1 \times * \times 0)$ = meridian of first component

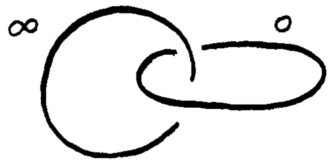
$h(S^1 \times * \times 1)$ = longitude of second component

$h(* \times S^1 \times 0)$ = longitude of first component

$h(* \times S^1 \times 1)$ = meridian of second component.

Both surgery instruction curves are then $\langle 1, 1 \rangle$ curves in the corresponding coordinates of $H_1(X) \cong Z \oplus Z$. Choose a homeomorphism g of $S^1 \times S^1 \times I$ onto itself taking a $\langle 1, 1 \rangle$ curve onto a $\langle 0, 1 \rangle$ curve; for example choose one with homology matrix $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. Revising

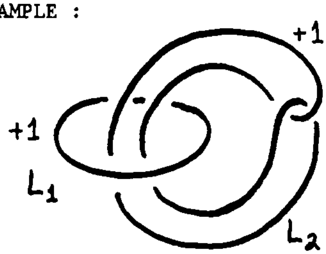
X by the homeomorphism hgh^{-1}, we have that our surgery manifold also has surgery description:



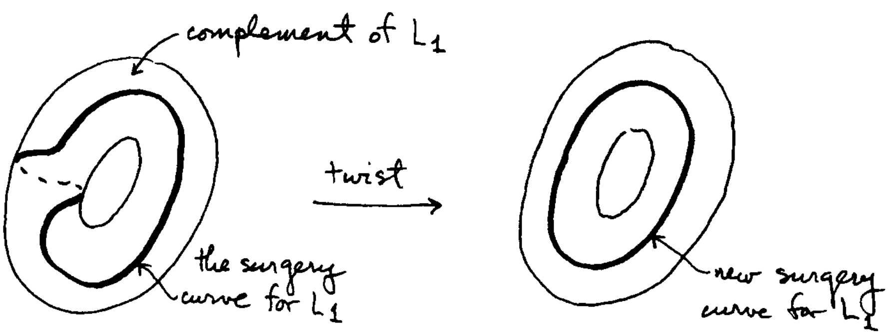
or just 

, which we know is $S^2 \times S^1$.

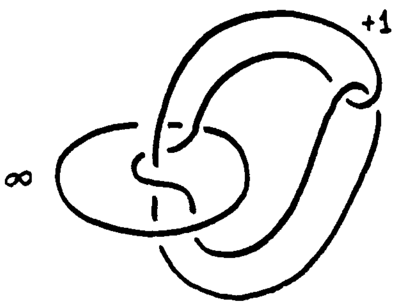
8. EXAMPLE :



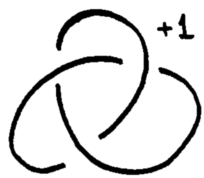
Here we look just at the complement of L_1. Being a solid torus, we can give it a meridinal twist as indicated below.



This changes L_2 and we have the picture



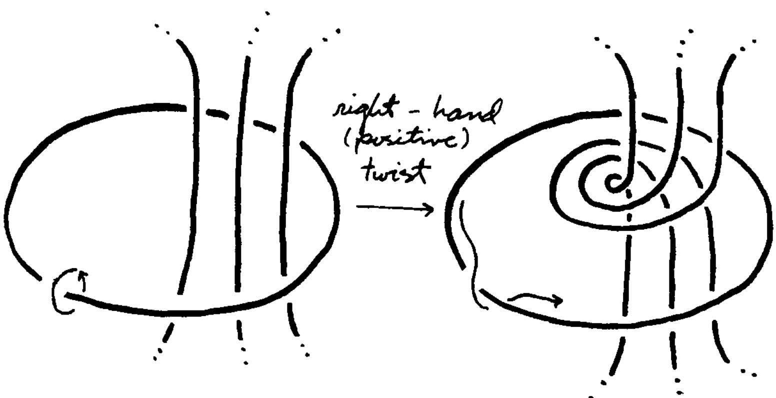
or simply



describing the same manifold. Thus the surgery manifold of this example is precisely the Poincaré manifold.

H. MODIFICATION OF SURGERY INSTRUCTIONS. We have seen, in the last two examples, a general technique useful in modifying (and possibly simplifying) a surgery presentation of a manifold. Namely, one constructs a homeomorphism of the complement of the link onto the complement of another, possibly different link. The images of the surgery instruction curves for the first link give surgery instruction curves for the second link, and the resulting surgery manifolds are then homeomorphic.

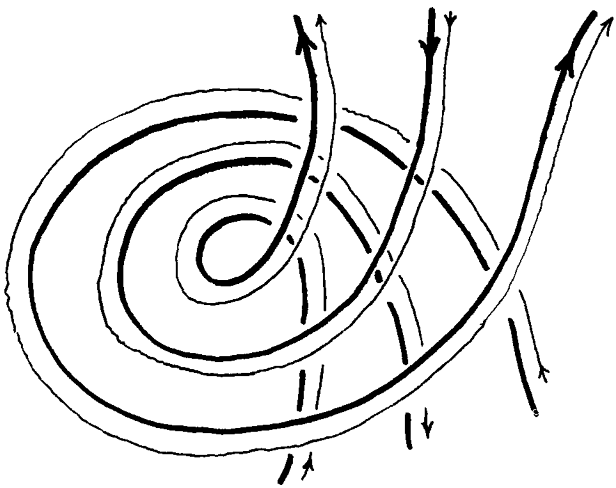
Twisting about one unknotted component. Suppose one component say L_1, of a link $L = L_1 \cup \ldots \cup L_n$ (with surgery coefficients $r_1,\ldots,r_n$) is unknotted. Then the complement of a tubular neighbourhood of L_1 is a solid torus, which we may revise with a meridinal twist. This alters the rest of the link as follows: Let D be a disk bounded by L_1 and such that the other components pass through in straight segments (for convenience, assume D is round and flat). Then each straight piece is replaced by a helix which screws through a collar of D in the right-hand sense.



The new link $L' = L_1 \cup L_2' \cup \ldots \cup L_n'$ gives the same surgery manifold provided we pay attention to the corresponding revision of surgery coefficients. First look at $r_1 = b_1/a_1$. The twist takes longitude to longitude and meridian to meridian + longitude (regardless of the orientation of L_1). Thus the new coefficient is

$$r_1' = b_1/(a_1+b_1) = \frac{1}{1+\frac{1}{r_1}}$$

This formula is consistent with: $r_1' = 0$ if $r_1 = 0$, $r_1' = \infty$ if $r_1 = -1$ and $r_1' = 1$ if $r_1 = \infty$. The surgery coefficients of the other components behave differently. Investigate a meridian on the tubular neighbourhood of, say, L_2 . Its image under this homeomorphism is again a meridian. A longitude, however, picks up meridinal components as it is twisted around; as illustrated.



To compute how many, picture the disk as horizontal and let u be the number of bits of L_2 passing through D in an upward direction and d be the number passing through downward. If λ_2 is a longitude of L_2 , we wish to compute the linking number of λ_2' with L_2' . By viewing on edge and assigning ± 1 to undercrossings, as in section 5D , we have that every upward bit of λ_2' contributes $u - d$ while every downward bit contributes $d - u$. All the other crossings of λ_2' under L_2' are old crossings of λ_2 under L_2 , which cancel, since

λ_2 is a (preferred) longitude. The total linking number of λ_2' with L_2' is thus

$$\ell k(\lambda_2', L_2') = u(u-d) + d(d-u) = (u-d)^2 .$$

But $u - d$ is just $\pm \ell k(L_2, L_1)$.

1. EXERCISE : Complete this analysis to show that the new surgery coefficient of L_2' is

$$r_2' = r_2 + [\ell k(L_2, L_1)]^2$$

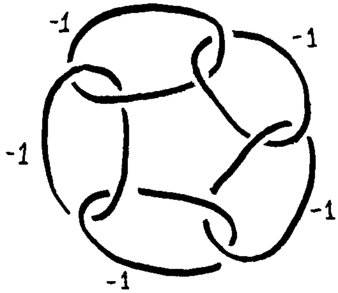
and similarly for r_3' , etc.

2. PROPOSITION : Here are the formulas for the new surgery coefficients, to be assigned after performing t right-hand twists about an unknotted component L_i of a link L ($t < 0$ for left-hand twists) :

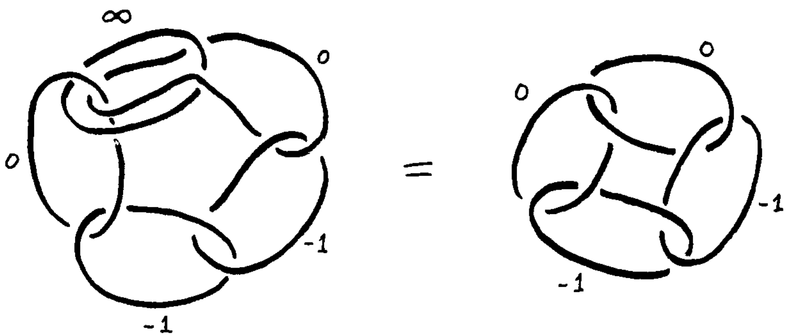
component of the twist: $\displaystyle r_i' = \frac{1}{t + \frac{1}{r_i}}$

other components: $r_j' = r_j + t(\ell k(L_i, L_j))^2$

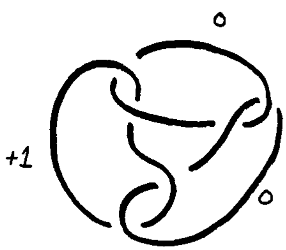
3. EXAMPLE : Each of the following diagrams yields the same manifold when surgery is performed on S^3 by the given instructions.



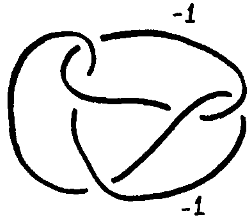
Twist about the topmost component once in the right-hand sense (t=+1)



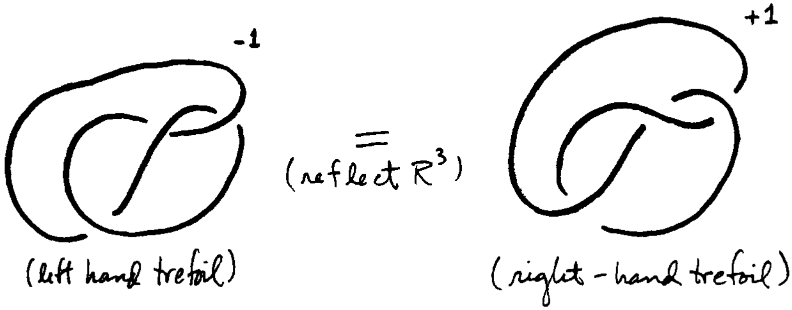
similarly twist about the lowermost component and rub it out.



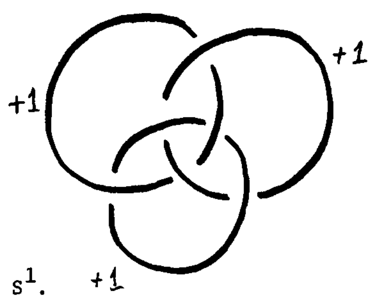
Do a left -handed twist about the leftmost component



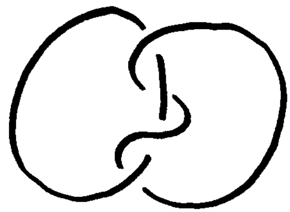
A right-hand twist about the upper right component yields our old friend, the Poincaré manifold!



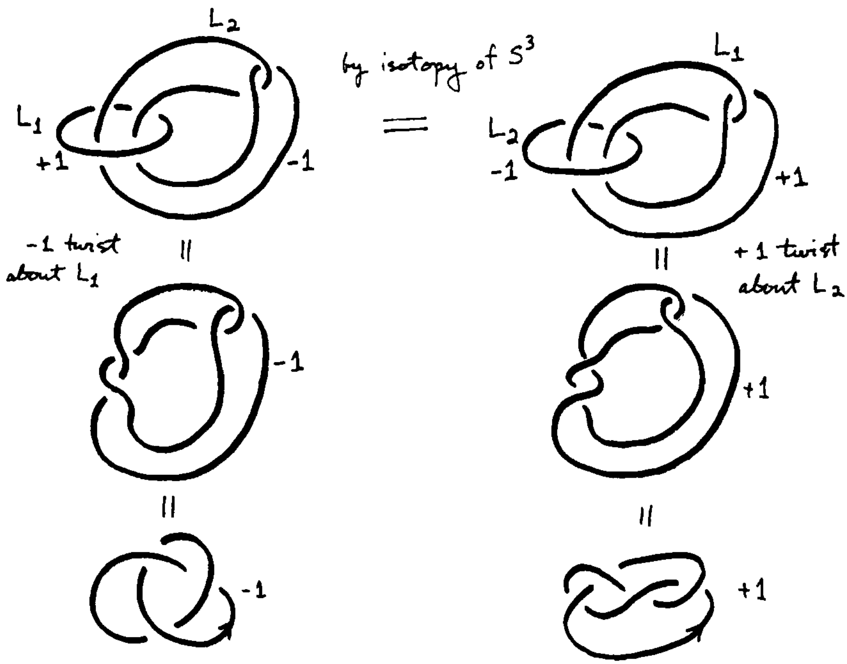
4. EXERCISE : Show that the Poincaré manifold again results from surgery on the Borromean link according to this diagram. Thus this manifold has a symmetry of order 3, as well as one of order 5. Find another of order 2. In each case the fixed point set is an S^1.



5. EXERCISE : A certain surgery may be performed on this link to obtain the Poincaré manifold. Find the appropriate surgery coefficients.



6. EXAMPLE :

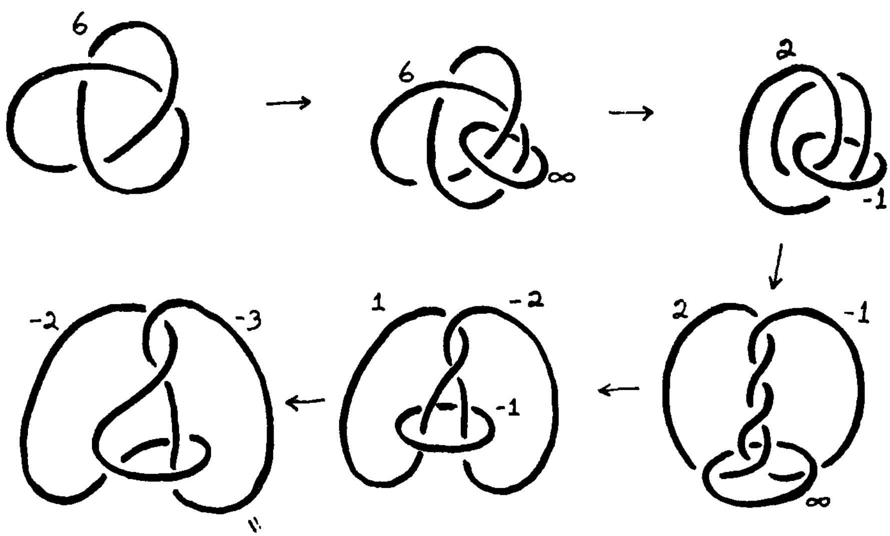


In this example we see that a certain 3-manifold (in fact a homology sphere) results from surgery on two different knots: a +1 surgery on the figure-eight knot as well as a -1 surgery on the right-handed trefoil.

7. EXERCISE : Show that the manifold described by surgery in the previous example has fundamental group presented by $(x, z; (zx)^2 = z^3 = x^7)$. This is an <u>infinite</u> perfect group, as may be seen by mapping it onto a certain group of motions of the hyperbolic plane (see Coxeter-Moser [1957]). The same manifold results from -1 surgery on the figure-eight knot. Why?

8. EXERCISE : Use surgery to give another proof that the trefoil is not amphicheiral (no orientation-preserving homeomorphism of S^3 throws the right-hand trefoil onto the left-hand trefoil).

9. EXAMPLE : The following shows that surgery on a single knot -- even a prime knot -- can yield a 3-manifold which is a nontrivial connected sum. Note the trick of introducing new components with coefficient ∞ .



$L(2,1) \# L(3,1)$

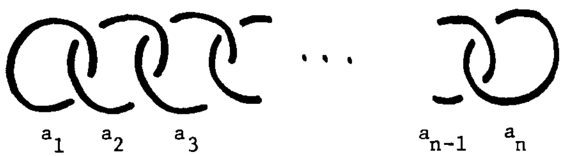
10. EXERCISE : Show that 

is a surgery description of a certain lens space.

11. EXERCISE : The connected sum $L(p,q) \# L(q,p)$ results from a certain surgery on the torus knot of type p,q .

12. REMARK : L. Moser [1971] has determined all 3-manifolds which result from Dehn surgery on torus knots. They are: lens spaces, connected sums of two lens spaces, and the so-called Seifert manifolds with three exceptional fibres. These latter are completely classified (as are the lens spaces) and in particular are not simply-connected. So only trivial surgery (coefficient ∞) on a torus knot yields a simply-connected 3-manifold.

13. EXERCISE : If $a_1, a_2, \ldots, a_n$ are integers then the surgery description



yields the lens space $L(p,q)$ where the coprime integers p and q are determined by the continued fraction decomposition

$$\frac{p}{q} = a_n - \cfrac{1}{a_{n-1} - \cfrac{1}{\ddots - \cfrac{1}{a_2 - \cfrac{1}{a_1}}}}$$

[Hint: starting at one end, try to eliminate components by twisting].

Since every rational has a continued fraction of the form described, we see that any lens space may be obtained by 'honest' surgery (integer coefficients) on a link, as well as Dehn surgery on the unknot. A similar trick can be used to convert any Dehn surgery picture into one with only integer coefficients [EXERCISE]. As special cases of Exercise 13, it is amusing to note that if all $a_n = 0$ then the surgery manifold is S^3 or $S^2 \times S^1$ according as n is even or odd. If all $a_n = 1$ the result is $S^2 \times S^1$ if $n \equiv 2 \bmod 3$ and S^3 otherwise.

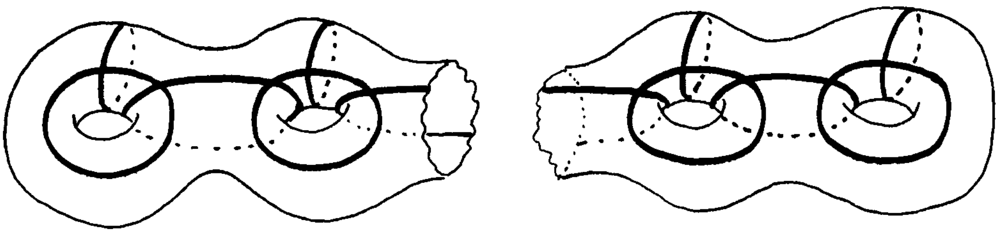
I. THE FUNDAMENTAL THEOREM. We'll complete this chapter with a proof of what I call the fundamental theorem of surgery on 3-manifolds.

1. THEOREM : Every closed, orientable, connected 3-manifold may be obtained by surgery on a link in S^3 . Moreover, one may always find such a surgery presentation in which the surgery coefficients are all ± 1 and the individual components of the link are unknotted.

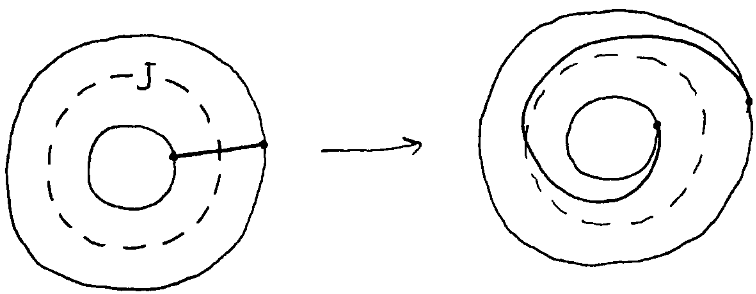
The proof given here is due to Lickorish [1962]; another quite different proof was discovered independently by Wallace [1960]. Actually we will omit the one complicated step; I suggest the interested reader consult directly the beautiful elementary proof in Lickorish [1962] of the following theorem. (He also assumes the homeomorphism is PL--which is all we need--but the theorem is true in the generality stated here.)

2. THE LICKORISH TWIST THEOREM : Let M^2 be a closed orientable 2-manifold of genus g . Then every orientation-preserving homeomorphism of M is isotopic to a product of twist homeomorphisms along the $3g - 1$

curves pictured.



3. DEFINITION : A <u>twist homeomorphism along a curve</u> J in a 2-manifold is defined to be the identity outside an annular neighbourhood of J and inside the neighbourhood it looks like



Thus there are essentially two twists possible, depending on choices of orientation: both are allowed and one is the inverse of the other.

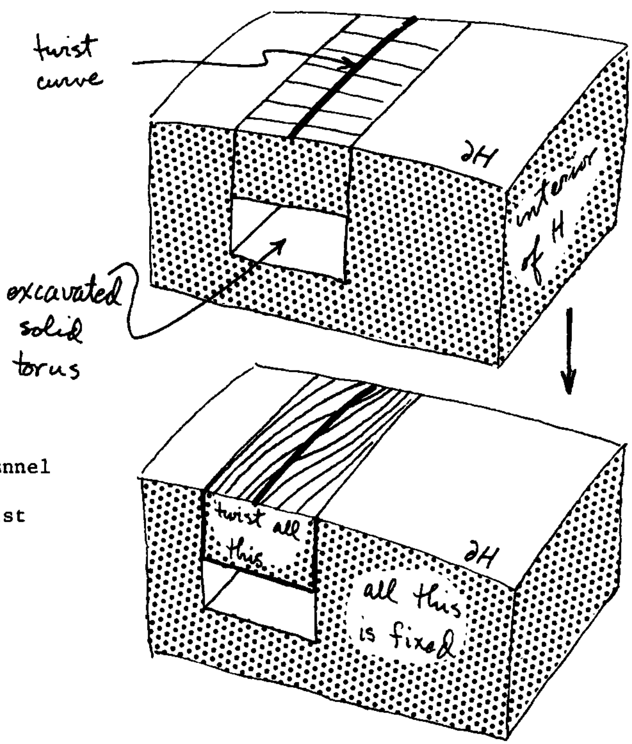
Recall that we've already proved the twist theorem in the case $g = 1$. Another way of stating the twist theorem is that the mapping class group (homeomorphisms mod isotopy) is generated by a specific finite set of elements; $3g$ of them if we include, say, a reflection to get orientation-reversing homeomorphisms. The proof of the fundamental theorem will depend on the following, which is a consequence of the twist theorem. (The handlebodies are 3-dimensional.)

4. LEMMA : Suppose H and H' are handlebodies of the same genus and let $f: \partial H \to \partial H'$ be any homeomorphism. Then there exist disjoint solid tori $V_1, \ldots, V_r$ in H and $V_1', \ldots, V_r'$ in H' such that f extends to a homeomorphism $\bar{f}: H - (\mathring{V}_1 \cup \cdots \cup \mathring{V}_r) \to H - (\mathring{V}_1' \cup \cdots \cup \mathring{V}_r')$.

PROOF : Since H and H' are homeomorphic, we may assume that $H = H'$. Moreover we may assume that $f: \partial H \to \partial H$ preserves orientation. Now any homeomorphism of ∂H which is isotopic to the identity, easily extends to a homeomorphism of all of H to itself, moving only a collar of the boundary. So by theorem 2, we need only consider the case $f = \tau_r \cdots \tau_2 \tau_1$, a composite of twists along some or all of the $3g-1$ "canonical" curves. Now τ_1 is the identity off an annular neighbourhood A of its twisting curve in ∂H . Imagine a tunnel excavated from H just under this annulus, within a collar of ∂H (see the picture on the following page). This tunnel is a solid torus; call it V_1 . The region between V_1 and A is a copy of $A \times I$, which may be twisted by $\tau_1 \times \mathrm{id}$. Now τ_1 may be extended by this map, together with the identity elsewhere on $H - \mathring{V}_1$. Call this extension $\bar{\tau}_1 : H - \mathring{V}_1 \to H - \mathring{V}_1$. Similarly τ_2 may be extended to a homeomorphism $\bar{\tau}_2 : H - \mathring{V}_2 \to H - \mathring{V}_2$. By excavating slightly deeper than before, if necessary, it can be arranged that V_2 misses V_1 and that $\bar{\tau}_1$ is the identity on V_2. Inductively, define in this way a collection of disjoint solid torus tunnels $V_1, \ldots, V_r$ and extensions $\bar{\tau}_i : H - \mathring{V}_i \to H - \mathring{V}_i$ so that $\bar{\tau}_i$ is fixed on V_j whenever $i < j$. Define $\bar{f}$ to be the composite $\bar{f} = \bar{\tau}_1 \cdots \bar{\tau}_2 \bar{\tau}_1$, restricted to $H - (\mathring{V}_1 \cup \cdots \cup \mathring{V}_r)$. The solid tori to be deleted to get the range of $\bar{f}$ are $V_r' = V_r$ and $V_i' = \tau_r \cdots \tau_{i+1}(V_i)$ for $i < r$. Then

$$\bar{f} : H - (\mathring{V}_1 \cup \cdots \cup \mathring{V}_r) \to H - (\mathring{V}_1' \cup \cdots \cup \mathring{V}_r')$$

is the required extension of f.



Excavating a tunnel to extend a twist homeomorphism.

5. PROOF OF THE FUNDAMENTAL THEOREM : Given a closed, connected, orientable M^3 one may choose Heegaard decompositions of the same genus:

$$S^3 = H_1 \cup_g H_2 \qquad \text{and} \qquad M^3 = H_1' \cup_{g'} H_2'$$

where $g: \partial H_2 \to \partial H_1$ and $g': \partial H_2' \to \partial H_1'$ are homeomorphisms attaching the handlebodies. Since all handlebodies of a given genus are homeomorphic, choose any homeomorphism $h: H_1 \to H_1'$. By lemma 4, the homeomorphism $(g')^{-1}hg : \partial H_2 \to \partial H_2'$ extends to a homeomorphism of $H_2 - (V_1 \cup \cdots \cup V_r)$ onto $H_2' - (V_1' \cup \cdots \cup V_r')$, the deleted sets being disjoint solid tori. This extends h to a homeomorphism

$$h : S^3 - (V_1 \cup \cdots \cup V_r) \to M^3 - (V_1' \cup \cdots \cup V_r').$$

From the proof of lemma 4, we see that h carries ∂V_i to $\partial V_i'$ and that the preimage of a meridian of V_i' is a meridian $\pm$ longitude of V_i. In other words, M is the result of a ± 1 surgery on S^3 on the solid tori $V_1, \ldots, V_r$ and the theorem is proved.

6. COROLLARY : Every closed orientable 3-manifold is the boundary of some orientable 4-manifold, in fact a simply-connected one.

PROOF : We may assume the given 3-manifold is also connected, so by the fundamental theorem it results from S^3 by surgery with coefficients ± 1. In section G we saw that this is the boundary of B^4 with 2-handles attached, a 4-manifold which is clearly orientable. This 4-manifold has the homotopy type of a wedge of 2-spheres and is therefore simply-connected

7. REMARK : The results of the last three sections may be summarised as follows. A link in S^3 with a rational (or ∞) coefficient assigned to each component describes, by Dehn surgery, a closed connected orientable 3-manifold. All such 3-manifolds arise in this way. We discussed two sorts of modifications which may be performed on such surgery descriptions:

(1) Introduce or delete a component with coefficient ∞ ,

(2) Find an unknotted component and twist its complement. Change the coefficients according to the formulas of H2.

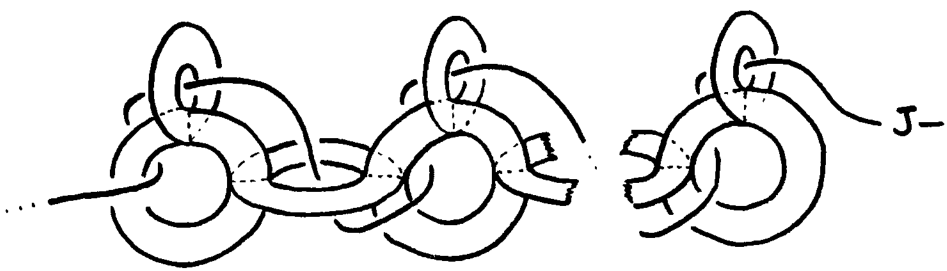
Say that two surgery descriptions are <u>equivalent</u> if one can be transformed into the other by a finite sequence of modifications (1) and (2). Thus equivalent surgery descriptions yield homeomorphic 3-manifolds.

A very recent result of Kirby [1976] implies a converse to this last statement. That is, if two surgery descriptions yield the same 3-manifold, then they are equivalent in the sense just defined. (One may require that the component of (1) be unknotted and this remains true). In a sense, then, this reduces the theory of 3-manifolds (at least closed, orientable ones) to the theory of links via the one-to-one correspondence:

Links with coefficients, modulo equivalence	$\longleftrightarrow$	Closed, connected, orientable 3-manifolds, modulo homeomorphism

What consequences this may have to manifold theory remain to be seen.

8. REMARK : In the next chapter, we'll make use of the fact that the surgery link required to produce any M^3 may be chosen from a special class of links in S^3 . In the proof of the fundamental theorom we may as well have taken the handlebody H_2 to be "canonically embedded" in S^3, H_1 the closure of its complement, and the identity on their common boundary as the attaching homeomorphism. Since the surgery solid tori in S^3 are obtained by burrowing out tunnels in H_2 directly under the $3g-1$ curves pictured in theorem 2, we may require that the surgery curves (the cores of these solid tori) belong to the following class. Consider the $3g-1$ annuli in $S^3 = R^3 + \infty$ pictured below. Each is round and planar, and the intersections are orthogonal. The special class consists of any finite number of round circles, each of which lies on, and is concentric with, one of these annuli. They must, of course, be disjoint. To obtain 3-manifolds of arbitrarily high genus, we must allow the number of annuli to be arbitrarily large, or alternatively to run off to the right without termination.



The extra curve drawn shows in particular that one may obtain any M^3 by surgery on a link with the property that each of its components wraps once, in a monotone sense, around some fixed unknot J in S^3 .

J. KNOTS WITH PROPERTY P. If there is a counterexample to the Poincaré conjecture it can be obtained from S^3 by surgery along a link, according to the fundamental theorem. It is natural to look first at surgery on a knot, and this has motivated the definition of a property which I wish had a better name.

1. DEFINITION : A tame knot K in S^3 has <u>property P</u> if every nontrivial surgery along K yields a non-simply-connected 3-manifold.

If ℓ and m denote elements of $\pi_1(S^3 - K)$ corresponding to a preferred longitude and a meridian of a tubular neighbourhood of K, this may be stated in more purely algebraic terms.

2. EXERCISE : K has property P if and only if its knot group is never trivialised by adjoining a relation of the form $m = \ell^a$, $a \neq 0$.

3. EXAMPLE : The unknot does <u>not</u> have property P.

At this writing this is the only known example. Despite considerable attention by knot theorists, the following conjecture of Bing and Martin [1971] and González-Acuña [1970] is still unanswered.

4. CONJECTURE : Every nontrivial knot has property P.

5. EXERCISE : If the knot $K \subset S^3$ has property P, then (a) one cannot construct a counterexample to the Poincaré conjecture by surgery along K, and (b) any PL embedding of the exterior (S^3 minus tubular neighbourhood of K) into S^3 extends to a PL homeomorphism $S^3 \to S^3$. Conversely, any knot satisfying both (a) and (b) has property P.

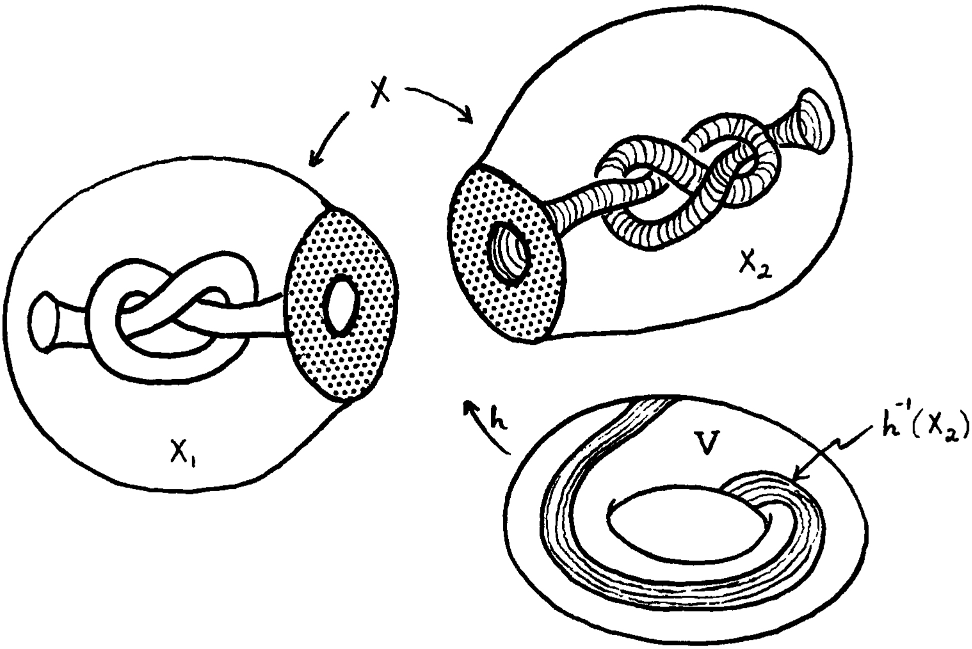
6. REMARK : A theorem of C.H. Edwards [1964] states that two compact 3-manifolds with boundary are homeomorphic if and only if their interiors are homeomorphic

7. EXERCISE : Use this remark to prove that if two tame knots in S^3 have homeomorphic complements and one of them has property P, then the knots are equivalent.

Thus the property P conjecture implies the conjecture that tame knots are determined by their complements. Which knots are known to have property P? The following theorem shows that it is a large class indeed; only prime knots can fail to have property P. The proof uses the fact (exercise 4C6) that any tame torus in a simply-connected closed 3-manifold bounds, on at least one side, a submanifold with infinite cyclic fundamental group.

8. THEOREM : Any composite $K = K_1 \# K_2$ of nontrivial tame knots has property P.

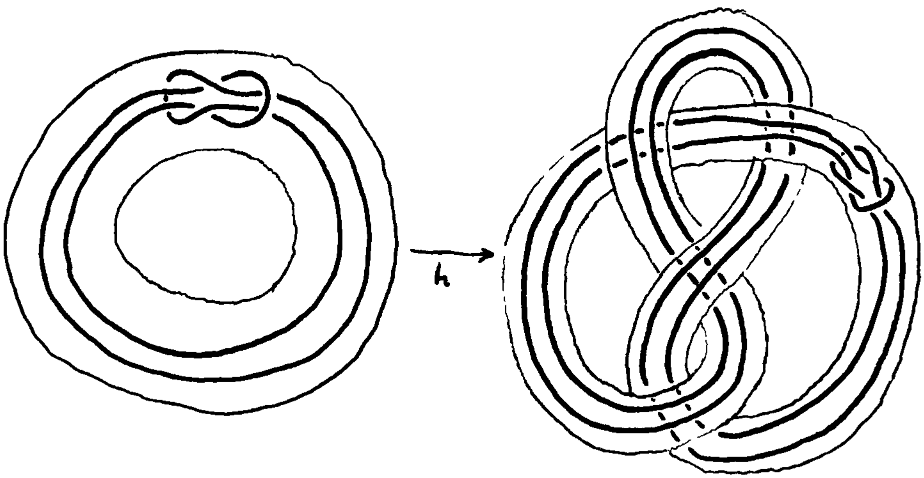
PROOF : As indicated in the drawing below, the exterior X of K can be regarded as the union of the exteriors X_1 and X_2 of K_1 and K_2, joined along an annulus. A nontrivial surgery along K corresponds to sewing a solid



torus V to X via a homeomorphism $h:\partial V \to \partial X$ for which the annulus $h^{-1}(X_2)$ is homotopically nontrivial in V. We need to show that the manifold $M = X \cup_h V$ is not simply-connected. But consider the torus ∂X_1 in M. On one side it bounds X_1, whose fundamental group isn't $\mathbb{Z}$, since K_1 is knotted. On the other side it bounds the set $X_2 \cup V$, adjoined by $h|h^{-1}(X_2)$. It is easy to check that its fundamental group, being a free product with amalgamation, contains $\pi_1(X_2)$ as a subgroup and therefore also cannot be infinite cyclic. It follows that M is not simply-connected.

9. EXERCISE : Suppose $K \subset S^3$ is a knot with property P and that V is an unknotted solid torus in S^3 with $K \subset \text{int } V$. Suppose $W \subset S^3$ is a knotted tame solid torus in S^3 and $h:V \to W$ is a faithful homeomorphism (preserves preferred longitudes). Then h(K) has property P.

10. EXAMPLE : This shows, for example that knots formed as in Whitehead's doubles, but based instead on the following construction, have property P.



11. REMARK : Many other knots are known to have property P. Seifert [1933] proved that nontrivial <u>torus knots</u> have property P, a fact rediscovered by Hempel [1964]. In fact Moser [1971] has identified all the manifolds arising from surgery on torus knots (remark H12) . It follows that all (p,q) - <u>cable knots</u> have property P, if $|p| \geq 2, |q| \geq 2$, using the above exercise. All nontrivial <u>doubled knots</u> also have property P (González-Acuña and Bing-Martin). Some of their methods involve detailed and difficult analysis of the knot group.

CHAPTER 10

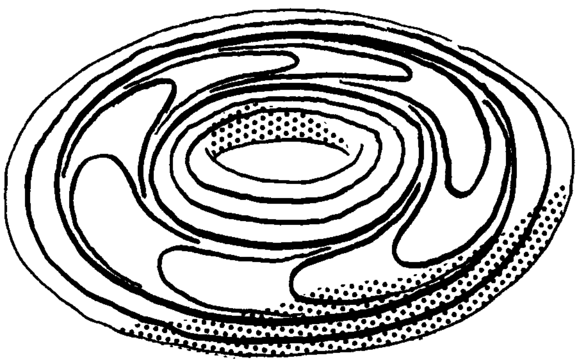
FOLIATIONS, BRANCHED COVERS, FIBRATIONS AND SO ON.

This chapter deals with various topological structures which may be associated with a manifold; we'll concentrate mainly on dimension three. Much of this ties in quite beautifully with knot and link theory. Of special significance are the several existence theorems, all of which are applications of the surgery theory covered in the previous chapter.

A. FOLIATIONS. Let M^n denote an n-manifold and let $\mathcal{F} = \{F_\alpha\}$ denote a partition of M into disjoint path-connected subsets. Then the family $\mathcal{F}$ is said to be a <u>foliation of</u> M <u>of codimension</u> c $(0<c<n)$ if there exists a covering of M by open sets U, each equipped with a homeomorphism $h : U \to R^n$ (or R^n_+) which throws each nonempty component of $F_\alpha \cap U$ onto a parallel translate of the standard hyperplane R^{n-c} in R^n. Each F_α is called a <u>leaf</u>, and is not necessarily closed or compact.

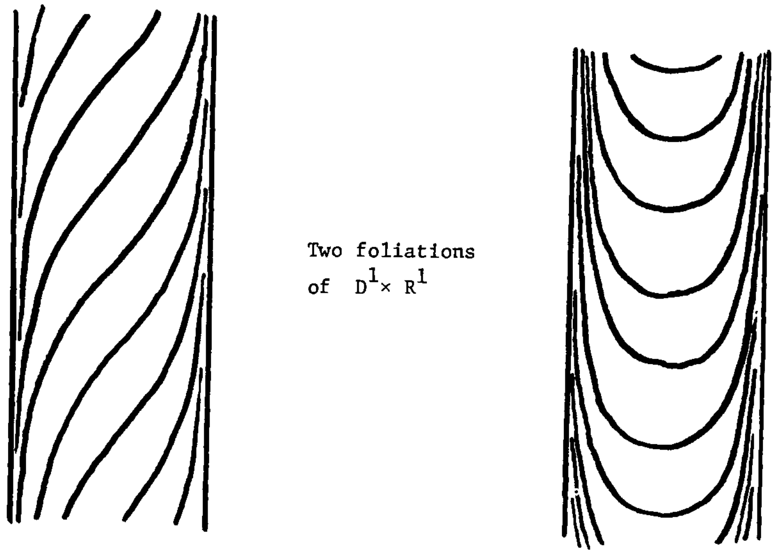
1. EXAMPLE: The torus $T^2 = S^1 \times S^1$ may be foliated in codimension 1 in many ways: by longitudinal leaves, meridinal leaves, or leaves which are p,q-curves (i.e. the images under the universal cover map

$R^1 \times R^1 \to S^1 \times S^1$ of lines in the plane of slope q/p. Here the leaves are all compact. A foliation with no compact leaves is obtained by taking the images of all lines in $R^1 \times R^1$ with a fixed irrational slope, in fact each leaf is dense in T^2. Here is still another (cf. the next two examples):



2. EXERCISE: Two foliations on M are <u>equivalent</u> if a homeomorphism $M \to M$ throws one family onto the other. Which of the above foliations of T^2 are equivalent?

3. EXAMPLE: Here are two foliations of $R^1 \times D^1$. If we realize this space as the set $\pi \leq x \leq \pi$ in the xy-plane, one may be described as the family of graphs $y = \tan x + K$ $(-\infty<K<\infty)$ and the other as the graphs of $y = \sec x + K$. In both foliations we also include the vertical lines $x = \pm\pi$ as leaves.

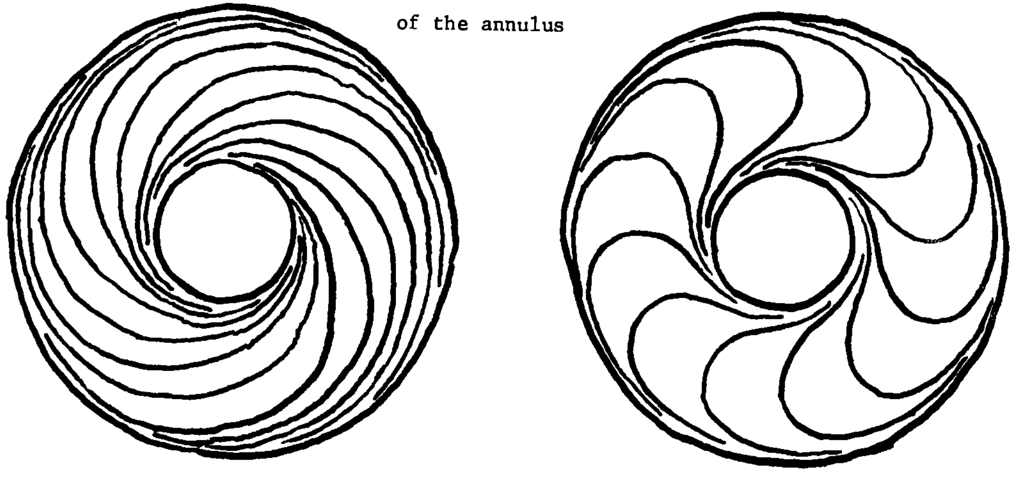


Two foliations
of $D^1 \times R^1$

4. EXERCISE: Are these foliations equivalent?

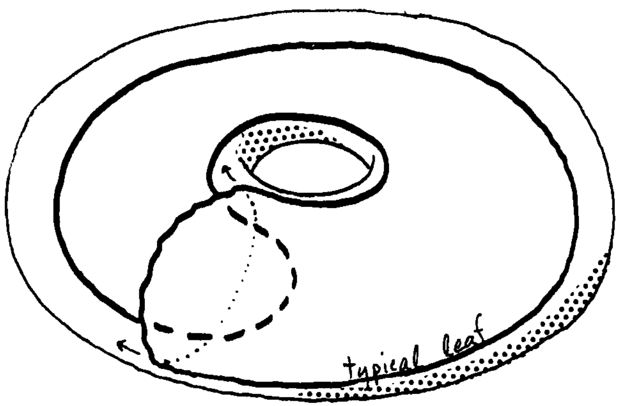
5. EXAMPLE: Since both the above families are invariant under translation in the y-direction, they project, under the universal covering map $D^1 \times R^1 \to D^1 \times S^1$ to foliations of the annulus. Both have two compact leaves.

Two foliations
of the annulus



6. EXERCISE: Leaves of a foliation of M^n of codimension c are not necessarily submanifolds. However they are one-to-one continuous images of connected manifolds of dimension $n - c$. In case $c = 1$, each component of ∂M is a leaf of any foliation.

7. EXAMPLE: Since the second foliation of example 3 is symmetric about the y-axis, we may rotate each leaf about the y-axis in xyz-space to foliate $R^1 \times D^2$ by surfaces of rotation. Again, the covering map $R^1 \times D^2 \to S^1 \times D^2$ induces a foliation, called the Reeb foliation of the solid torus. All leaves are topologically planes, save the boundary $S^1 \times S^1$.



Reeb foliation of the solid torus.

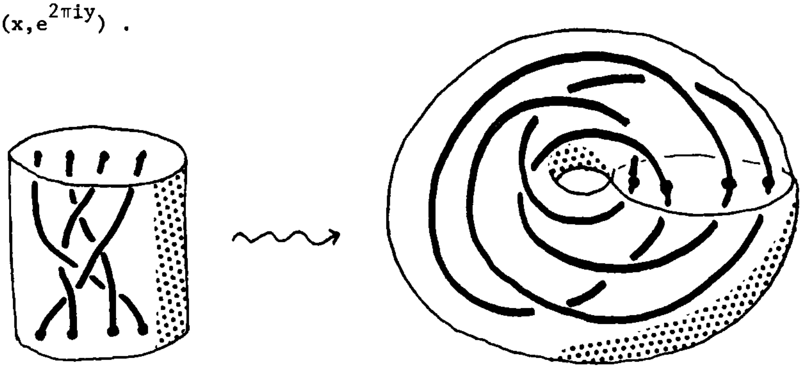
8. EXAMPLE: The Reeb foliation of S^3 is constructed by considering S^3 as the union of two solid tori, with common boundary. Give each one a Reeb foliation, as above, to construct a codimension one foliation. There is one compact leaf. In exactly the same way, we may foliate any

lens space.

9. EXERCISE: Construct infinitely many inequivalent codimension one foliations of S^3. Uncountably many?

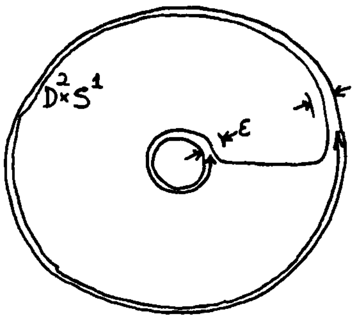
10. EXERCISE: Construct a codimension 2 foliation on S^3. [Hint: try foliating $S^1 \times D^2$ so that the leaves on the boundary are $\langle 1,1 \rangle$ curves]. Does every lens space have a codimension 2 foliation?

11. EXAMPLE: A codimension 1 foliation of the complement of (a neighbourhood of) a braided link in $D^2 \times S^1$. This example is of interest mainly because it will be used later to foliate arbitrary closed orientable 3-manifolds. By a <u>braided link</u> in $D^2 \times S^1$ we mean one which is the image of a braid (finite collection of strings with no interior maxima or minima) in $D^2 \times I$ under the map $D^2 \times I \to D^2 \times S^1$ which sends (x,y) to $(x, e^{2\pi i y})$.

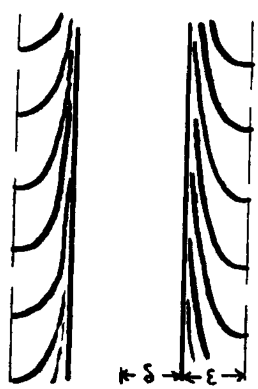


A braided link in a solid torus

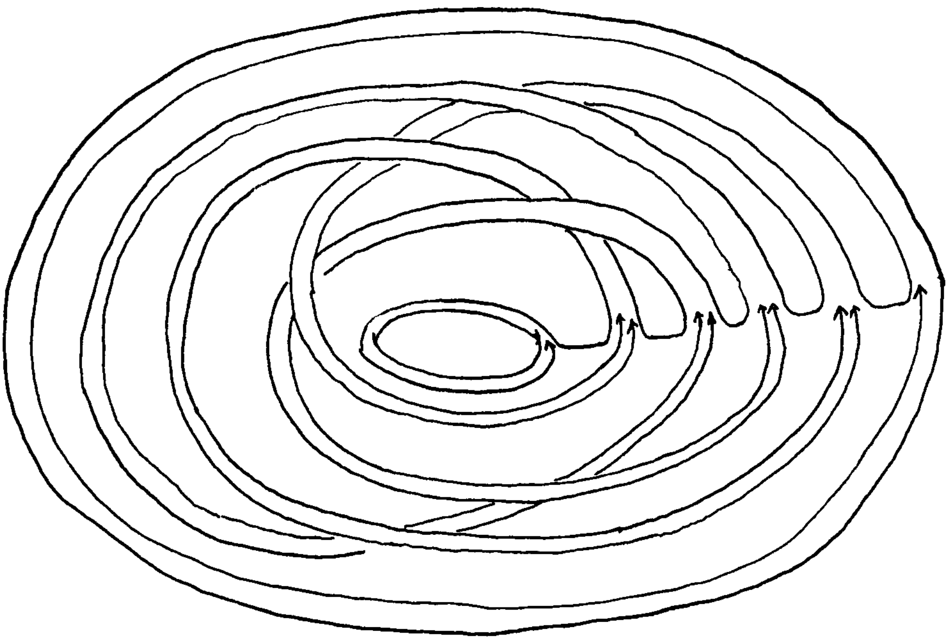
Let $L \subset D^2 \times S^1$ be such a link and let N be an open neighbourhood of L which, in order to aid visualization, we'll suppose to intersect every meridinal disk $D^2 \times x$ in, say, r disjoint little round disks of radius δ. Now choose a very small positive number ε and give $D^2 \times S^1$ a Reeb-like foliation whose leaves outside an ε-neighbourhood of the boundary, coincide with the meridinal disks.



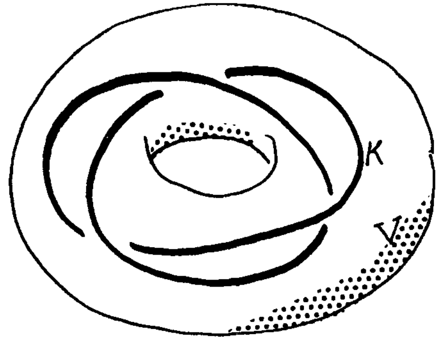
Now we revise this foliation near N as follows. Having had the foresight to make ε so small that the little δ-disks are still disjoint when increased to have radius $\delta + \varepsilon$, consider the strings back in $D^2 \times I$ and fill up the tubes consisting of points of horizontal distance between δ and $\delta + \varepsilon$ from each string with families of surfaces looking like this in cross-section, after the string is straightened out. That is, trumpet-like surfaces of revolution, each of which meets every horizontal level in a round circle, emerging at circles of radius $\delta + \varepsilon$ and asymptotic to the tube of radius δ (which is considered part of the family). Now,



back in $D^2 \times S^1$, we see that these trumpets fit together perfectly to foliate the part within ε (in the D^2-direction) of N. Moreover, if we throw away all the little $\delta + \varepsilon$-disks of the Reeb-like foliation, the leaves match up with the trumpets to foliate $D^2 \times S^1 - N$. The only compact leaves are the boundary $S^1 \times S^1$ and the components of ∂N. A typical noncompact leaf might look like this.



12. EXAMPLE: A foliation of the Poincaré space P^3. We have P^3 (see example 9D3) as the result of a $+1$ surgery along the right-handed trefoil K, which we situate in S^3 as braided in a standardly-embedded solid torus V, pictured. We are supposed to remove a neighbourhood N of K and sew it



back differently; we may assume N is nice as in the previous example. Now foliate the closure of $S^3 - V$, a solid torus, with a Reeb foliation, foliate $V - N$ as in the previous example and finally give N itself a Reeb foliation. These piece together to foliate P^3. There are two compact leaves.

13. EXERCISE: Recognize the braided link pictured in example 11 and use it to foliate P^3 in an essentially different way, with three compact leaves. Try also the Borromean rings. (compare exercise 9H4).

14. REMARK: S^2 has no codimension 1 foliations. In fact, the only connected closed 2-manifolds which admit such foliations are those of Euler characteristic zero: the torus and the Klein bottle. This makes the following existence theorem all the more surprising.

15. THEOREM: Every closed orientable 3-manifold M^3 admits a codimension 1 foliation.

PROOF: The idea is to mimic the method of the previous example. All this requires is that M^3 (which may be assumed connected) may be obtained from S^3 by surgery along a link which is braided in some standardly embedded solid torus in S^3. But this can be arranged. In remark 9I8 it was pointed out that the surgery link may be taken so that its components are braided in the solid torus complementary to a neighbourhood of an unknotted curve, called J there.

B. BRANCHED COVERINGS . In complex analysis, particularly the theory of analytic continuation, it is standard practice to construct 2-manifolds (Riemann surfaces) and maps from them onto the Gaussian sphere $C + \infty$ which are covering maps when certain branch points are deleted. We wish to generalize this process to higher dimensions, and exploit it as a method of constructing 3-manifolds.

1. DEFINITION : Let M^n and N^n be compact manifolds with proper submanifolds $A^{n-2} \subset M$ and $B^{n-2} \subset N$. Then a continuous function $f: M \to N$ is said to be a <u>branched covering</u> with <u>branch sets</u> A (upstairs) and B (downstairs) if

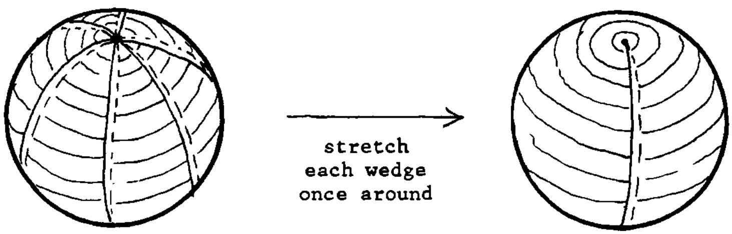
(1) components of preimages of open sets of N are a basis for the topology of M, and

(2) $f(A) = B$, $f(M-A) = N-B$, and $N-B$ is exactly the set of points in N which are evenly covered, i. e. have neighbourhoods U such that f sends each component of $f^{-1}(U)$ homeomorphically onto U .

The restriction $f: M-A \to N-B$ is clearly a covering space, called the <u>associated unbranched covering</u>. By compactness of M, it is finite-sheeted. Each branch point $a \in A$ has a <u>branching index</u> k, meaning that f is k-to-one near a , and this number is constant on components of A .

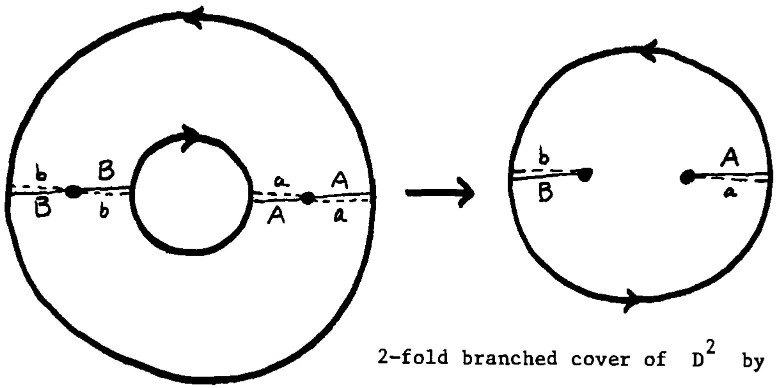
2. REMARK : A more general definition of branched covering is given by Fox [1957], for locally-connected Hausdorff spaces. The definition is, accordingly, much more complicated, involving the notions of "spread", "completions" and so on. Rather than confuse our discussion with these details, let me just mention that Fox proves M is completely determined by specifying N, the branch set B, and a finite unbranched covering of $N-B$. In general, however, the total space M thus determined may not always be a manifold.

3. EXAMPLE : The prototype of branched coverings is the complex map $z \to z^k$ by which the complex unit disk covers itself k times, branched with index k at the origin. A similar map is $z \to z^k/|z^{k-1}|$ which, moreover, preserves norm. The same map extends to a branched cover of 2-spheres $C + \infty \to C + \infty$, branched over two points, each of index k.



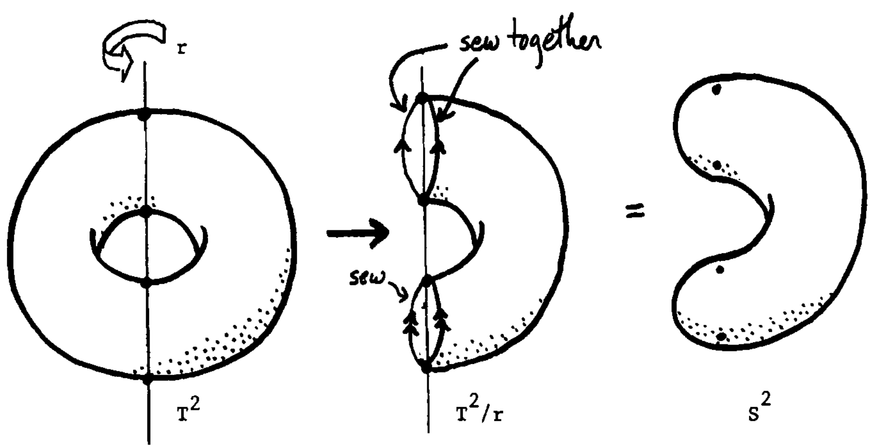
4. EXAMPLE : A branched cover $S^n \to S^n$ branched over an unknotted S^{n-2} both upstairs and down may be obtained by suspending the unbranched k-fold cover $S^1 \to S^1$ $n-1$ times. For n=2, this is just the previous example.

5. EXAMPLE : Consider the special case of a 2-fold branched cover $S^2 \to S^2$ a described in example 3. If we remove an open disk downstairs, whose closure misses the branch points, and remove the two disks upstairs which map onto it, then what remains is a branched cover of a disk by an annulus $S^1 \times I \to D^2$, with two branch points upstairs and down. An alternative method of describing this is by means of "cuts" as in the following picture.

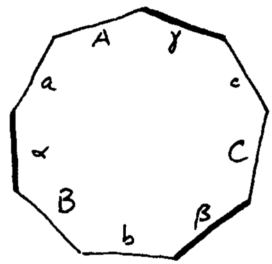


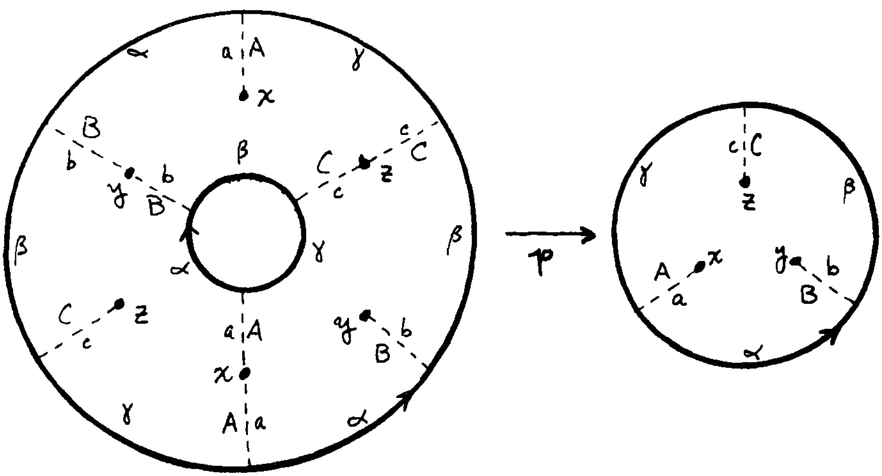
2-fold branched cover of D^2 by $S^1 \times I$

6. EXAMPLE : Notice that the torus T^2 may be expressed as the union of two annuli sewn together along their boundaries. Similarly S^2 is the union of two disks. Using two copies of the previous example, we can piece them together to produce a 2-fold branched cover $T^2 \to S^2$, with four branch points both upstairs and down. Another way of describing this map is to consider the involution $r:T^2 \to T^2$ corresponding to 180° rotation about the vertical axis in the figure below. The orbit space T^2/r is a 2-sphere (as suggested in the picture) and the natural projection $T^2 \to T^2/r$ is the branched covering.



7. EXAMPLE: Here is a fancier branched cover of a 2-disk by an annulus; the branch set is 3 points downstairs and 6 points upstairs, and the associated unbranched cover is 3-fold. Consider the 9-sided figure shown at the right. We form a disk by sewing A to a, B to b, C to c and also form an annulus by sewing three of them together, with identifications as shown. Each copy of the nonagon in the annulus is mapped onto the disk in a way which should be clear from the picture.





The branch points are shown as heavy dots, labelled x,y,z . Note that each downstairs branch point has two points in its preimage, one of branch index 2 (locally 2 to 1), one of index 1 . Also note that one boundary component of the annulus is mapped homeomorphically upon the boundary of the disk, whereas the other component covers it twice.

8. EXERCISE: Use this example to construct a 3-fold branched cover of S^2 by itself with 4 branch points downstairs and 8 upstairs. Given the number of folds of the cover and the number of branch points downstairs in a branched cover of S^2 by itself, find a formula for the number of branch points upstairs. [Hint: consider Euler numbers.]

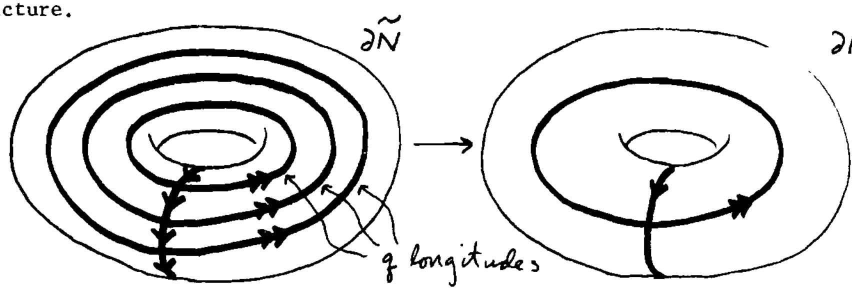
9. EXERCISE: Is there a branched cover of the disk by itself with one branch point downstairs and several upstairs?

10. EXERCISE : Show that every closed connected orientable 2-manifold is a branched cover of the 2-sphere.

11. QUESTION : Can a non-orientable 2-manifold branch cover an orientable one? What about vice-versa?

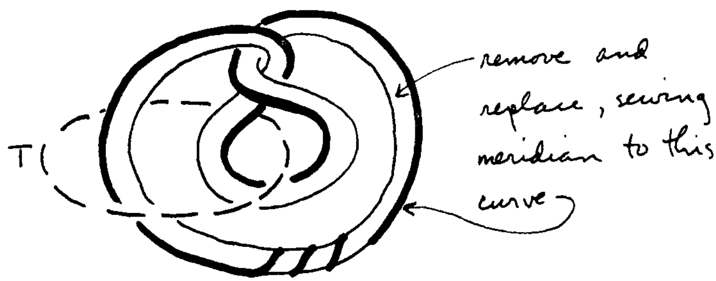
C. CYCLIC BRANCHED COVERS OF S^3. We've just seen that S^3 is a q-fold branched cover of itself with branch set the standard S^1 (the unknot) both upstairs and down. There are many other branched covers of S^3 with downstairs branch set a knot or link--in fact we'll see later on that every 3-manifold (closed, connected, orientable) branch covers S^3. This is a good way to construct 3-manifolds. First we'll consider the cyclic branched covers, branched over a knot. These are by no means the only ones, but they are easiest to visualize.

In chapter 6 we considered the finite q-fold cyclic covers of $S^3 - K$, K a given knot (and used their homology as knot invariants). These open 3-manifolds may be compactified to become closed 3-manifolds and the covering map extended to a branched cover as follows. Consider an open tubular neighbourhood N of K and construct the q-fold cyclic unbranched cover of $S^3 - N$. The boundary ∂N is a torus and its preimage in the cover is easily seen to be a torus. Moreover, the preferred longitude on ∂N is covered by q disjoint loops (why?) whereas the meridian is covered q times by a single curve upstairs. Here's a picture.



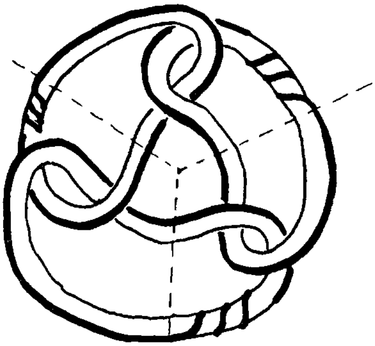
Now attach a solid torus $S^1 \times D^2$ to this unbranched cover, boundary to boundary, in such a way that a meridian $* \times S^1$ matches with the preimage of a meridian in $\partial\tilde{N}$. This union forms a closed connected orientable 3-manifold M^3 upstairs. We may extend the covering map to a branched covering $M^3 \to S^3$ by sending $D^2 \times S^1$ onto $N \cong D^2 \times S^1$ with the product of the maps $z \to z^q/|z^{q-1}|$ on D^2 and identity on S^1. The branch set downstairs is the given knot in S^3, upstairs it is some knot in our newly-formed M^3. We'll denote M^3 by Σ_q.

A particularly convenient way to visualize cyclic branched covers of S^3 over K is by using a surgery description of K, cf. section 6D. In this setting we start with a trivial knot T in S^3 and perform certain surgeries on solid tori (missing T) which changes S^3 into S^3 again, but in which T has the knot type of the given K. Recall, for example, the surgery description of the figure-eight knot (6C1)

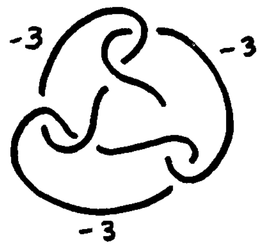


Now consider the q-fold cyclic cover $p : S^3 \to S^3$. Pulling the solid tori, as well as the surgery instruction curves, back via p^{-1}, we obtain another set of solid tori and surgery instructions in S^3. (In practice this is easy to draw if we view T on end as a straight line in R^3).

1. EXAMPLE : The 3-fold branched cyclic covering of S^3, branched over the figure-eight knot has surgery instructions:



or simply



Here's another point of view which can be quite useful for visualizing branched coverings of a knot or link. Let's consider first a 2-bridge knot K ($b(K) = 2$, see section 4D). This means that S^3 may be regarded as the union, along their boundary, of two 3-balls B_1 and B_2, such that the set $A_i = K \cap B_i$ is, for each i, two arcs properly embedded in B_i in an unknotted, unlinked manner. Now the k-fold branched cyclic cover $f\colon \Sigma_k \to S^3$ branched over K is clearly the union of the sets $f^{-1}(B_1)$ and $f^{-1}(B_2)$. But each of these is a branched cover of a ball, branched over two trivial arcs downstairs. It's a simple matter, then, to figure out that each $f^{-1}(B_i)$ must be a handlebody, so this procedure actually gives us a Heegaard splitting of Σ_k. For $k = 2$, we can take example B5, crossed with an interval, to see that the two-fold covering of a 3-ball branched over two trivial arcs is a solid torus. So:

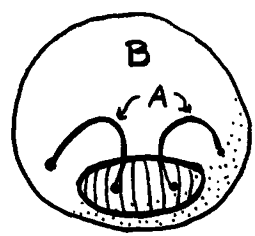
2. PROPOSITION : The 2-fold branched cyclic cover of S^3 branched over a 2-bridge knot (or link) is a lens space.

(To properly define the k-fold branched cyclic covering of a link $L = L_1 \cup \cdots \cup L_r$ one must orient the components and then take the unbranched covering of $S^3 - L$ associated with the kernel of the map $\pi_1(S^3 - L) \to Z/k$ defined by $\alpha \to \sum_i \ell k(\alpha, L_i)$ modulo k).†

3. EXERCISE : Show that the k-fold branched cyclic cover of S^3, branched over a 2-bridge knot has genus strictly less than k. More generally, find an upper bound for the genus of a k-fold branched cyclic covering over a b-bridge knot or link.

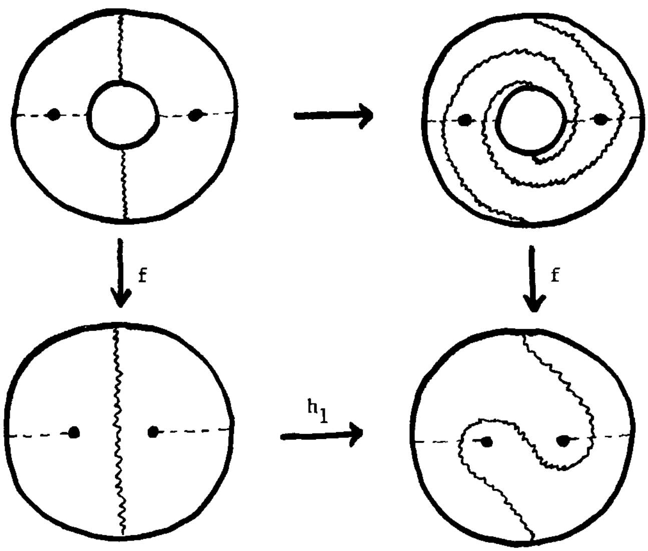
Now let's look at how the $f^{-1}(B_i)$ of the previous discussion should be sewn together to form Σ_k. Take $k = 2$ for simplicity. Let A be two unknotted, unlinked arcs in the 3-ball B, and take two copies (B_i, A_i) of (B, A). The 2-fold branched cover of B branched over A is, as we've seen, a solid torus. If we attach B_2 to B_1 by the identity map $\partial B_2 \to \partial B_1$ we get S^3, and the union of A_1 and A_2 is the trivial link of two components. Upstairs the solid tori $f^{-1}(B_i)$ are attached by the identity map on their boundaries, so their union is $S^2 \times S^1$. So we conclude that the 2-fold branched covering of S^3 branched over the link  is $S^2 \times S^1$.

Consider now the homeomorphism $h_1 \colon \partial B \to \partial B$ which is the identity outside the shaded disk but inside this disk twists things in such a way that two of the branch points become interchanged, as in the following drawing. The 2-fold cover of the disk, branched

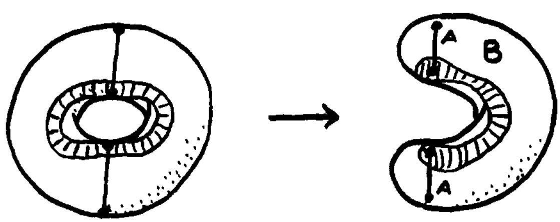


† if $k = 2$ the orientations are obviously irrelevant.

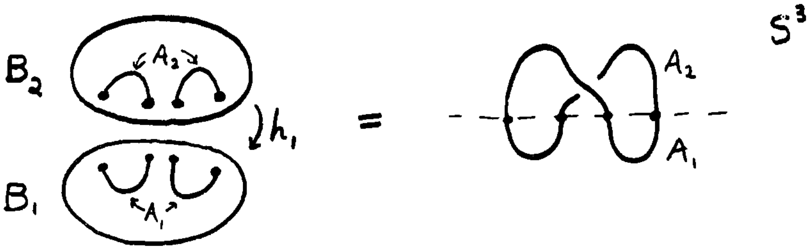
over the two points is an annulus (by B5) and the picture should make it clear that the homeomorphism h_1 of disks lifts to a twist homeomorphism of the annuli.



By the following representation of the 2-fold branched cover of B by a solid torus, it is clear that $h_1 : \partial B \to \partial B$ is covered by a longitudinal twist of the solid torus.

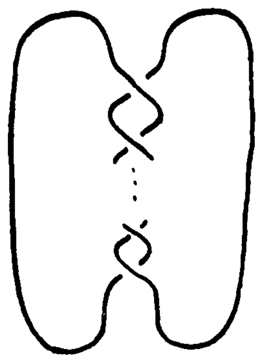


We can actually visualise $B_1 \bigcup_{h_1} B_2 = S^3$ as in the following picture.



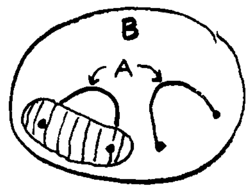
The branch set $A_1 \bigcup_{h_1} A_2$ is the unknot. Upstairs, we have Σ_2 expressed as two solid tori sewn together along their boundaries by a longitudinal twist, with matrix (in longitude-meridian coordinates) $\longrightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ We've just shown that the branched cover Σ_2 associated with the unknot is the lens space $L(1,1) \cong S^3$. Although this is something we already knew, more twisting will give us new information

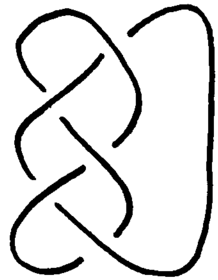
4. EXAMPLE : The 2-fold branched covering of S^3 branched over the torus link of type $p,2$ is the lens space $L(p,1)$. For downstairs we apply h_1 p times. Then the solid tori are attached by a homeomorphism which has matrix:



$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^p = \begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix}$$

By similar considerations, a twist h_2 with support a disk in ∂B shown here (shaded) $\longrightarrow$ but in an 'anticlockwise' direction, corresponds upstairs to a meridinal twist. Now we can determine Σ_2 for any 2-bridge knot.

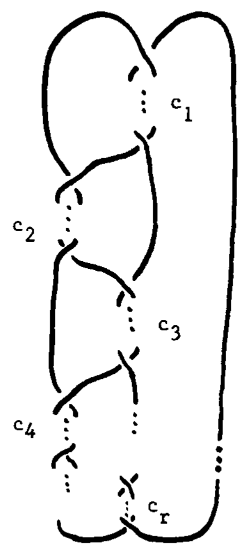




5. EXAMPLE : The 2-fold cover of S^3 branched over Whitehead's link is the lens space $L(8,5)$. The picture shows we should attach B_2 to B_1 by the composite $h_1h_2h_1h_2h_1$. So the solid tori of Σ_2 are attached by a homeomorphism with matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 8 & 5 \end{bmatrix}.$$

(Although the symmetry of this example doesn't show it, the matrices are in the reverse order of the h_i).

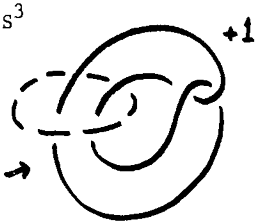


6. EXERCISE : Every 2-bridge knot or link may be put in the form shown here, where the c_i indicate number of crossings and are negative if the sense of the crossings is reversed. The 2-fold branched covering of S^3 associated with this knot is the lens space $L(p,q)$, where p and q are computed from the continued fraction:

$$\frac{p}{q} = c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{}{\ddots + \cfrac{1}{c_r}}}}$$

7. REMARK : According to Conway [1970] this actually sets up a one-to-one correspondence between 2-bridge knots (up to knot type) and the lens spaces up to homeomorphism.

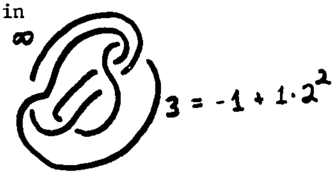
D. CYCLIC COVERINGS OF S^3 BRANCHED OVER THE TREFOIL (a lengthy example). As an illustration of the use of surgery descriptions, we'll look here at the first few cyclic branched coverings Σ_k of S^3 branched over the trefoil. Besides, they turn out to be an interesting bunch of 3-manifolds. We'll use this surgery description of the trefoil



The homology $H_1(\Sigma_k)$ has already been determined as $Z \oplus Z$, 0, $Z/3$, or $Z/2 \oplus Z/2$ when $k \equiv 0, \pm 1, \pm 2$, or $3 \pmod 6$. The surgery method can be used to determine their fundamental groups. Considerable simplification of our task can be made by using the techniques of section 9H for changing surgery instructions for a given manifold.

<u>2-fold</u>: This has a surgery diagram

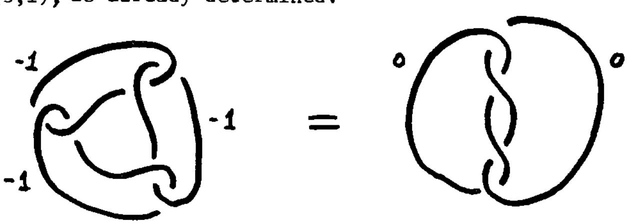
(Why -1 coefficients, rather than $+1$?), This can be simplified by a twist about one component (as in section 9H) to:



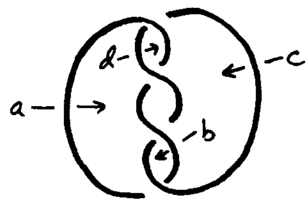
Or simply:

Thus the 2-fold cover of S^3 branched over the trefoil is the lens space $L(3,1)$, as already determined.

<u>3-fold</u>:



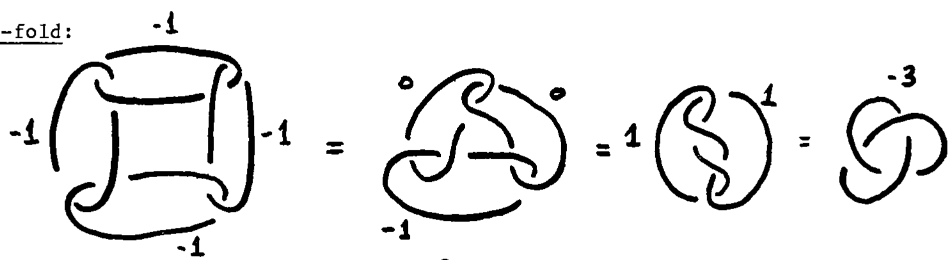
We can't expect to reduce this surgery diagram to one component, because the homology has two generators. One might conjecture from the homology that what we have is the connected sum of two projective spaces. This is wrong; we'll see from computing the fundamental group that this is a 3-manifold we've not encountered before. By the Wirtinger presentation, the complement of the link has group $(a,b,c,d;\ ad=ca,\ bd=da,\ bc=db)$. Sewing in the two solid tori according to the surgery instructions adds the



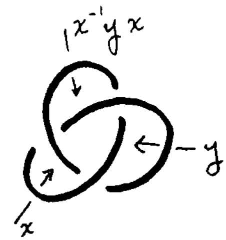
relations $cd = 1$ and $ab = 1$ so the group of our 3-fold branched cover works out to: $(a,c;\ ac^{-1}=ca,\ a^{-1}c^{-1}=c^{-1}a,\ a^{-1}c=c^{-1}a^{-1})$ or equivalently $(a,c;\ a^2=c^2=(ac)^2)$.

This is a <u>finite</u> group of order 8 called the <u>quaternion group</u> (being isomorphic via $a \to i$, $c \to j$ to the multiplicative subgroup $\{\pm 1, \pm i, \pm j, \pm k\}$ of the quaternions). Coxeter and Moser [1957] point out on page 8 that it's the smallest nonabelian group all of whose subgroups are normal. Note that this manifold not only has the 3-fold symmetry as the branched cover, but also a nice 2-fold symmetry.

<u>4-fold</u>:

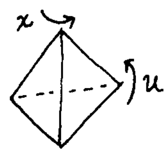


Thus the four-fold cyclic cover of S^3 branched over the right-hand trefoil is also the result of a -3 surgery on the left-hand trefoil (or, taking the mirror image, a +3 surgery on the right-hand trefoil)! Again this is a new manifold to add to our growing list. Computing its fundamental group from the knot group we get generators x,y and relations $xyx = yxy$ from the knot group and $xyx^{-1}yx = 1$ from the surgery. If we substitute $u = xy$ to replace y we get the following presentation for the fundamental group of our manifold:



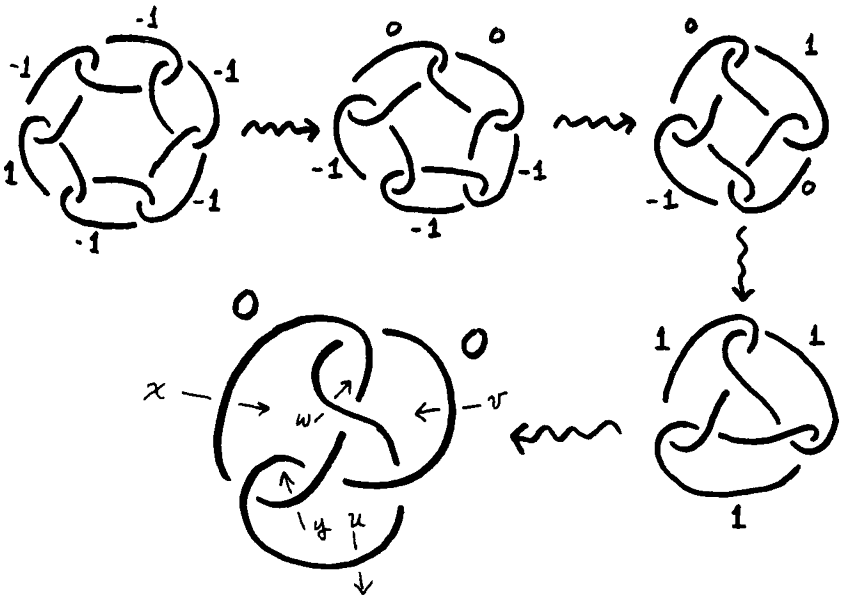
$$(x,u;\ x^3=u^3=(ux)^2)$$

Coxeter and Moser [1957] tell us in their table (p.134) that this is a group of order 24 , called the binary tetrahedral group. By sending x and u to 120° rotations about two different vertices of a tetrahedron, we obtain a homomorphism of the group presented above onto the group of symmetries of the tetrahedron (CHECK THIS) which has order 12 (this is the same as the alternating subgroup of S_4.) Then one shows that the homomorphism is two to one. An interesting feature of this group is that it has no subgroup of order 12 .



5-fold: We've already done this example (see 9H3) . It is the Poincaré manifold, with π_1 the binary icosahedral group of order 120 .

6-fold: The 6-fold branched cyclic covering of S^3 over the trefoil may be obtained by surgery on Whitehead's link, according to the following reductions of surgery instructions:



It has the same homology as $S^1{\times}S^2 \# S^1{\times}S^2$ (obtainable from a 0-surgery on the trivial link of two components), but must be a different manifold The following argument shows that its fundamental group cannot even be a nontrivial free product. From the Wirtinger presentation get (the last two relations coming from the surgery):

$$\pi_1(\Sigma_6) \cong (x,y,u,v,w;\ vx=xw,\ yw=wx,\ vy=yu,\ yu=ux,\ w^{-1}u=1,\ yvx^{-1}v^{-1}=1)$$

$$\cong (x,u;\ [[x,u],u] = [[x,u],x] = 1)\ ,$$

where $[x,u]$ denotes the commutator $xux^{-1}u^{-1}$. Suppose this were a free product. Since the relations imply that $[x,u]$ commutes with everything, and the centre of any nontrivial free product contains only the identity element [EXERCISE], we conclude that $[x,u] = 1$. But this means the group is abelian, a contradiction.

1. EXERCISE: Verify that the group above is not abelian.

7-fold:

2. EXERCISE: The 7-fold branched cyclic cover of S^3 over the trefoil is the homology sphere already encountered in example 9H6 ; the result of a +1 or -1 surgery on a figure-eight knot, as well as a -1 surgery on the right-handed trefoil.

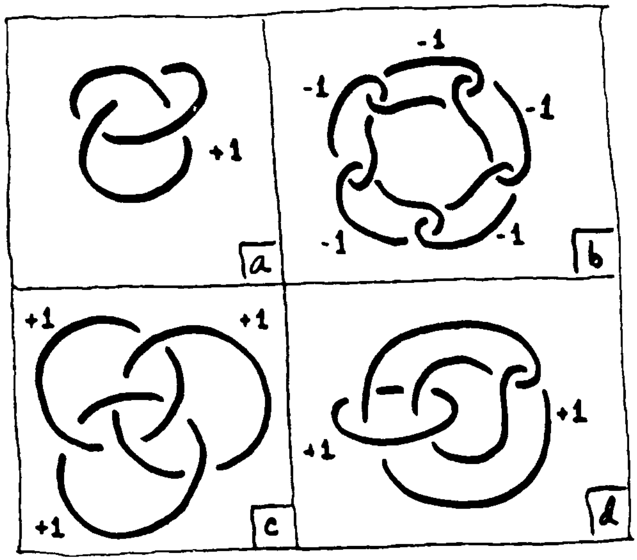
8-fold:

3. EXERCISE: This branched cyclic cover over the trefoil can be obtained from a +3 or -3 surgery on the figure-eight knot.

4. EXERCISE: The $(6n\pm 1)$-fold covers of S^3 branched over the trefoil are all homology spheres since H_1 is trivial. Show that they are all distinct, and in fact have different homotopy types.

E. THE UBIQUITOUS POINCARÉ HOMOLOGY SPHERE

We've already seen that the Poincaré manifold P^3 can be obtained from any of the following surgery instructions in S^3 (see the previous chapter).



1. THEOREM: P^3 is a cyclic branched cover of S^3 in each of the following ways:

5-fold, branched over the 2,3 torus knot in S^3 ,

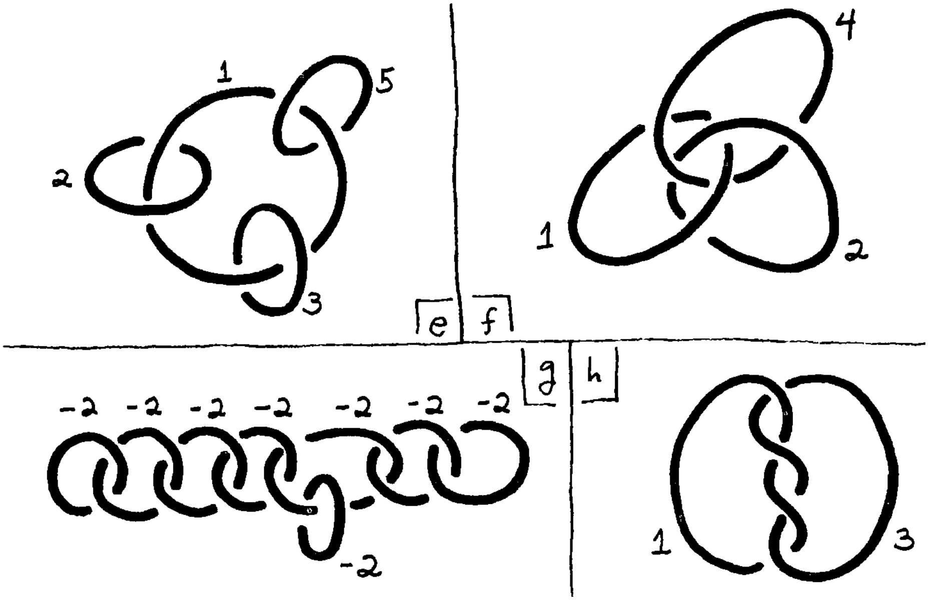
3-fold, branched over the 2,5 torus knot in S^3 ,

2-fold, branched over the 3,5 torus knot in S^3 .

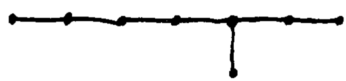
The first part was just verified (using picture (b)).

2. EXERCISE: Use pictures (c) and (d) to prove the remainder of the theorem. [Hint: use a more symmetrical form of (d)].

3. EXERCISE: Show that the following are also surgery presentations of P^3 , using the methods of the previous chapter.



4. REMARKS : The surgery description (e) or (f) may be used to verify that P^3 is a Seifert manifold with exceptional orbits of type (5,1), (3,1) and (2,1). For details consult Orlik [1972] or Seifert [1932]. Algebraic geometers will recognize (g), which may be used to show that P^3 is the boundary of a certain 4-manifold of index 8, constructed by plumbing together copies of a 2-sphere's tangent disk bundle according to the "E_8-diagram":



This procedure was first exploited, I think, by Hirzebruch. Here are some other well-known descriptions of P^3. P^3 is homeomorphic with the set of points (u,v,w) in complex 3-space which satisfy the simultaneous equations (see Milnor [1975]):

$$u^2 + v^3 + w^5 = 0 \qquad \text{and} \qquad |u|^2 + |v|^2 + |w|^2 = 1 .$$

P^3 may be constructed from a regular dodecahedron by identifying each boundary point with the point on the opposite face rotated 36° about the axis perpendicular to the faces, in a clockwise sense. A final description is to consider S^3 as the group of unit quaternions. If we regard R^3 embedded in the quaternions as the subspace spanned by i, j, and k, then S^3 acts on R^3 by conjugation $s(r) = srs^{-1}$. This defines a homomorphism of S^3 onto the group $SO(3)$ of rigid motions of R^3 fixing the origin. The kernel consists of the quaternions +1 and −1, so this homomorphism is a 2-fold covering map and S^3 is the universal cover of $SO(3)$. (Alternatively we could consider S^3 as the group $SU(2)$ of unitary 2 by 2 complex matrices.) Now the group I of rigid motions of a regular icosahedron (or dodecahedron) centered at the origin is, in an obvious way, a subgroup of $SO(3)$ of order 60. It lifts, via the double covering to a subgroup I^* of S^3, the binary icosahedral group of order 120, and we have a commutative diagram of homomorphisms shown at the right. The coset space S^3/I^* is homeomorphic with the Poincaré manifold P^3.† This displays S^3 as the universal cover of P^3 having 120 sheets.

$$\begin{tikzcd} I^* \arrow[r, hook] \arrow[d] & S^3 \arrow[d] \\ I \arrow[r, hook] & SO(3) \end{tikzcd}$$

5. EXERCISE : In the previous chapter we constructed homology spheres by sewing together two cubes-with-knotted-hole along their boundary. Show that we can never obtain P^3 in this way.

6. EXERCISE : If p, q, r are integers ≥ 2 show that the manifold

$$M(p,q,r) = \{(u,v,w)\in C^3 \mid u^p + v^q + w^r = 0 \text{ and } |u|^2 + |v|^2 + |w|^2 = 1\}$$

is the p-fold branched cyclic covering of the q,r torus link.† Thus, as in theorem 1, we may permute the roles of p, q, and r.

† see Milnor [1975], which also discusses fundamental groups of $M(p,q,r)$.

F. OTHER BRANCHED COVERINGS OF S^3. As mentioned earlier, a branched covering of S^3 is determined by its branch link L in S^3 and a specified finite-sheeted (unbranched) covering of $S^3 - L$. So far we have discussed only the cyclic ones, but there are plenty of others.

1. EXAMPLE : A six-fold branched cover $S^3 \to S^3$ with downstairs branch set a trefoil may be obtained by composing the 2-fold cyclic branched covering $L(3,1) \to S^3$ branched over the trefoil (see section D) with the 3-fold unbranched universal covering $S^3 \to L(3,1)$ of the lens space, as shown in the diagram.

$$\begin{tikzcd}
S^3 \arrow[rr, dashed, "\text{6-fold branched cover}"] \arrow[dr, swap, "\text{3-fold cover}"] & & S^3 \\
& L(3,1) \arrow[ur, swap, "\text{2-fold branched cover}"] &
\end{tikzcd}$$

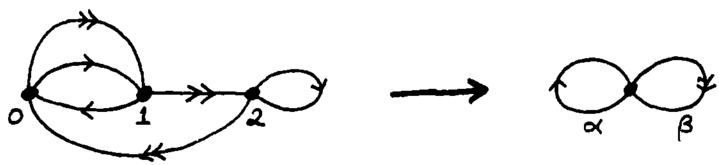
Since for the trefoil we calculated that $H_1(\Sigma_6) \cong Z \oplus Z$, what we have here is definitely not a cyclic branched covering.

2. EXERCISE : There are 3 components, each of branching index 2, in the upstairs branch link in S^3 for the previous example. Draw the branch link. Find a 600-fold branched noncyclic covering $S^3 \to S^3$ branched over the trefoil.

3. EXERCISE : If L is a tame link in S^3 and $p : C \to S^3 - L$ is any finite-sheeted covering space, then there exists a 3-manifold M^3 and an inclusion $C \subset M$ such that p extends to a branched covering $\bar{p} : M^3 \to S^3$. The branch set downstairs is L , or possibly a sublink of L because some components may be evenly covered by $\bar{p}$. [Hint: any finite covering of a torus is a torus. A covering map $\partial V \to \partial V$, $V = S^1 \times D^2$, extends to a branched cover $V \to V$ with branch sets $S^1 \times 0$ iff a meridian is sent to a multiple of a meridian.]

To list the k-fold unbranched covers of $S^3 - L$ we could try to determine all subgroups of $\pi_1(S^3 - L)$ of index = k , or rather all conjugacy classes of them, but the following is often more convenient when k is finite. Let $\tilde{Y} \to Y$ denote an unbranched covering with exactly k sheets.† Choose a basepoint * in Y and label the preimages of * by the symbols 0, 1, ... , k-1 . Now any loop ω in Y based at * has k distinct liftings to paths in $\tilde{Y}$. By running from the tail to the head of each path we obtain a permutation σ_ω of the symbols 0, ... , k-1 . It is straightforward to verify that this correspondence induces a homomorphism $\sigma : \pi_1(Y) \to S_k$, called a representation of $\pi_1(Y)$ in the symmetric group S_k . Except for relabelling of the lifts of the basepoint, σ is well-determined by $\tilde{Y} \to Y$. That is, the covering determines a representation up to inner automorphism of S_k . Note that connectedness of $\tilde{Y}$ implies that σ is transitive, meaning that given i, j in {0, ... , k-1} one may find an α in $\pi_1(Y)$ such that $\sigma(\alpha)(i) = j$. Note also that a loop ω in Y based at * lifts to a loop (not just a path) in $\tilde{Y}$ based, say, at 0 if and only if $\sigma_\omega(0) = 0$. So conversely, given a transitive representation σ of $\pi_1(Y)$ in S_k we can reconstruct $\tilde{Y} \to Y$ by taking the covering corresponding with the subgroup $\{ \alpha \in \pi_1(Y) \mid \sigma(\alpha)(0) = 0 \}$. As Fox (Quick trip) points out, this is not necessarily a normal subgroup of $\pi_1(Y)$ and is not the kernel of σ (but it does contain the kernel).

4. EXAMPLE : Consider a wedge of two circles with π_1 free on α, β. This is the 3-fold covering associated with the representation $\alpha \to (01)$ $\beta \to (012)$:



† assume $\tilde{Y}$ path-connected and that Y has a universal covering.

5. EXERCISE : Using the correspondence discussed in the preceding page, show that k-sheeted covering spaces of Y, up to equivalence of covering spaces, are in a one-to-one correspondence with the transitive representations of $\pi_1(Y)$ in S_k, up to inner automorphism of S_k. [Hint: it may be useful to prove a lemma that a transitive representation is completely determined by its effect on the symbol 0.]

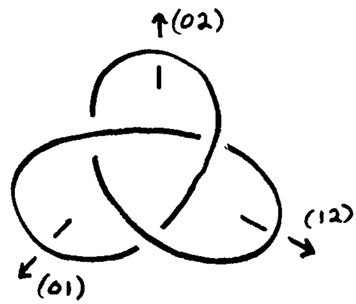
To list all k-fold branched covers of S^3 with a given link L as branch set one might proceed as follows. Obtain a finite presentation of $\pi_1(S^3 - L)$, always possible if L is tame. Assign to each generator a permutation of $\{0, \ldots, k-1\}$. Verify that the relations are respected by this assignment, so a representation is defined. Check whether it is transitive. Clearly, given $\underline{k}$, there are only finitely many possibilities and this is a finite algorithm for finding them.

6. EXERCISE : There is exactly one 2-fold branched cover of S^3 with a given knot or link as branch set, the cyclic one.

7. EXAMPLE : List the 3-fold unbranched coverings of the complement of the trefoil in S^3. Using the presentation $(a, b; a^2 = b^3)$ of its fundamental group, we check the possibilities for transitive representations σ in S_3. Since the identity permutation is the only element of S_3 which is both a square and a cube, $\sigma(b)$ must be either the identity or a permutation of order 3. Transitivity rules out the first possibility so up to inner automorphism the only possibility is $\sigma(b) = (012)$. Similar considerations leave two possibilities for a. Either $\sigma(a) =$ identity or $\sigma(a) = (01)$. So S^3 - (trefoil) has exactly two different 3-fold coverings. In terms of the Wirtinger presentation $(\ x,\ y;\ xyx = yxy\ )$ with generators represented by meridians, the two possibilities translate via the substitutions $x=a^{-1}b$, $y=ba^{-1}$ to

$$\begin{cases} x \to (012) \\ y \to (012) \end{cases} \quad \text{or} \quad \begin{cases} x \to (02) \\ y \to (12) \end{cases}$$

The first representation corresponds to the cyclic covering. The other one determines an <u>irregular</u> covering, corresponding to a non-normal subgroup of the knot group (why?). We'll shortly describe an explicit construction of the associated branched irregular covering of S^3. In practice one can specify the covering like this.



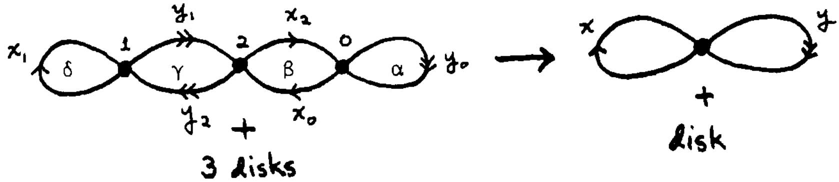
8. EXERCISE : Show that there are three 4-fold coverings of S^3 - (trefoil).

If a branched cover M^3 of S^3 has been specified by a link L in S^3 and a transitive representation of its group in S_k , there is a straightforward method, due to Reidemeister, for obtaining a presentation for $\pi_1(M)$ from a presentation of $\pi_1(S^3 - L)$. An example should illustrate the general technique.

9. EXAMPLE : Compute the fundamental group of the 3-fold irregular branched cover of S^3 over the trefoil, determined by the picture above. Since

$\pi_1(S^3 - \text{trefoil}) \cong (\, x,\ y;\ xyxy^{-1}x^{-1}y^{-1}\,)$, we may represent the base for our purposes as a wedge of two circles, together with a disk with its boundary attached to the loop $xyxy^{-1}x^{-1}y^{-1}$. The unbranched cover corresponding to $x \to (02)$, $y \to (12)$ has a 1-skeleton as pictured. The disk lifts to three disks whose boundaries are attached to the 1-skeleton by the three liftings of the word $xyxy^{-1}x^{-1}y^{-1}$, namely $x_0y_2x_1y_2^{-1}x_0^{-1}y_0^{-1}$, $x_1y_1x_2y_0^{-1}x_2^{-1}y_1^{-1}$ and $x_2y_0x_0y_1^{-1}x_1^{-1}y_2^{-1}$.

unbranched covering $\qquad\qquad\qquad\qquad$ S^3 - trefoil ~



The unbranched covering has fundamental group (taking 0 as base point) generated by $\alpha = y_0$, $\beta = x_0x_2$, $\gamma = x_0y_2y_1x_0^{-1}$ and $\delta = x_0y_2x_1y_2^{-1}x_0^{-1}$.

Attaching the three disks gives relations:

$\delta\alpha^{-1} = 1$

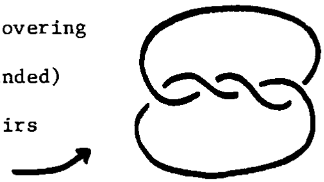
$\delta\gamma\beta\alpha^{-1}\beta^{-1}\gamma^{-1} = 1$

$\beta\alpha\gamma^{-1}\delta^{-1} = 1$

Using the first relation to eliminate δ and the third to eliminate β , we find that the irregular 3-fold unbranched cover of S^3 - trefoil has fundamental group with presentation $(\,\alpha,\gamma\ ;\ \alpha\gamma\alpha\gamma = \gamma\alpha\gamma\alpha\,)$. Now in the <u>branched</u> cover, the loops covering a meridian (say y) also bound disks, so we must adjoin the relations $\alpha = 1$ and $\gamma = 1$. This obviously kills the group and we conclude that this 3-fold branched cover is <u>simply-connected</u>!

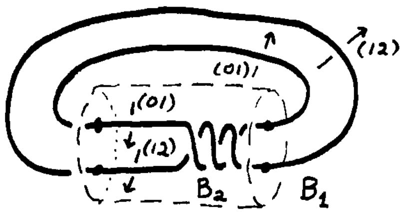
Have we just constructed a counterexample to the Poincaré conjecture? Unfortunately, we haven't, although if such a counterexample exists it must be obtainable as a 3-fold irregular branched covering of S^3 branched over some knot (see section G).

10. PROPOSITION: The irregular 3-fold branched covering of S^3 with downstairs branch set a (right-handed) trefoil is homeomorphic with S^3. The upstairs branch link is the 2,4 torus link shown here

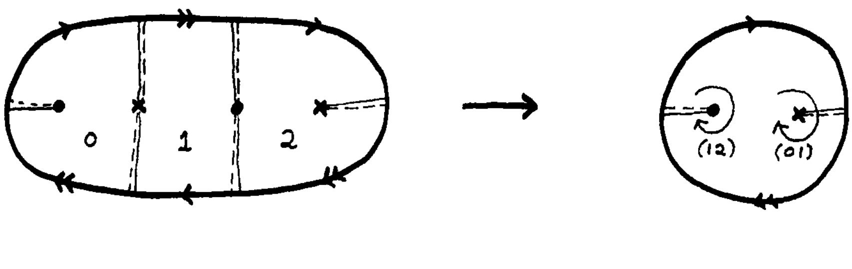


with one component of branch index 2 and the other of index one.

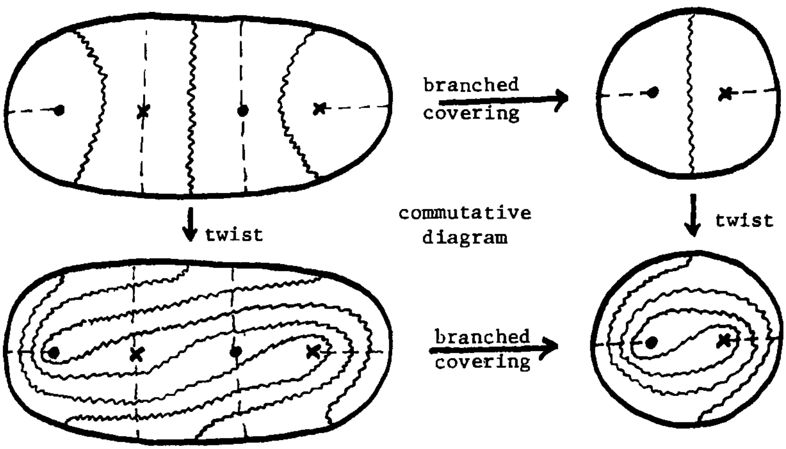
PROOF: We can actually construct the branched covering in question by methods similar to those in the discussion between C3 and C4 of this chapter. Visualize the trefoil as a 2-bridge knot as in the picture at the right. The irregular 3-fold branched cover $f: M^3 \to S^3$ is the union of $f^{-1}(B_1)$ and $f^{-1}(B_2)$, where



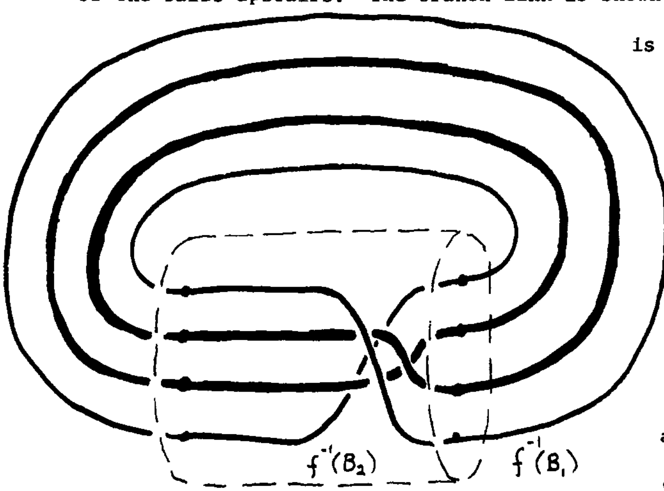
B_i are balls as indicated in the picture. Each of these is the irregular covering of a ball branched over two unknotted unlinked spanning arcs, such that a meridian of one arc is represented as the permutation (01) and the other as (12). But this is equivalent to the following branched cover $D^2 \to D^2$ (indicated by cuts) multiplied by an interval. Since M

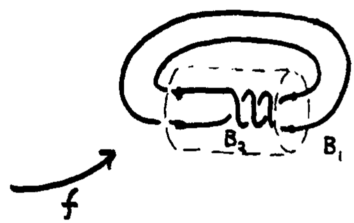


is therefore the union of two 3-balls, we conclude that $M \cong S^3$. To visualise the branch set upstairs, we look at a twist of the base D^2 in the above picture which is fixed on the boundary, tapering to a 3π revolution on most of the interior so the branch points are interchanged. The following diagram shows how this twist is covered by a twist upstairs.



This shows how the attachment of B_2 to B_1 is covered by the attachment of the balls upstairs. The branch link is shown in the picture below, and is easily seen to be a 2,4 torus link. The darker component has branch index 2.





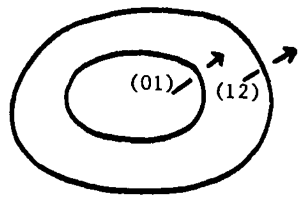
a 3-fold irregular branched covering of S^3 by S^3.

G. ARBITRARY 3-MANIFOLDS AS BRANCHED COVERINGS OF S^3.

J. W. Alexander [1920] stated, and proved rather sketchily, that every connected closed orientable 3-manifold may be constructed as a branched covering of S^3. Recently this has been sharpened so that one may require that the associated unbranched covering be at most 3-fold* and that the downstairs branch set be connected, i. e. a knot. This improvement is due independently to H. Hilden [1974] and J. Montesinos [1974], using rather different methods. The proof given below is Montesinos'.

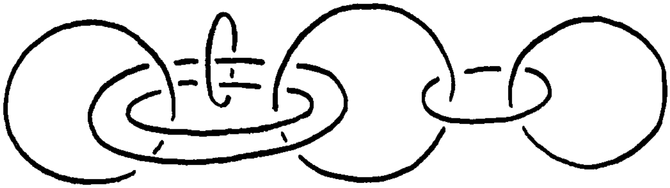
1. THEOREM : Every closed, connected, orientable 3-manifold is a 3-fold irregular branched covering of S^3 with downstairs branch set a knot.

Let us consider the specific 3-fold irregular branched covering of S^3 by itself which corresponds to the represented link shown here.



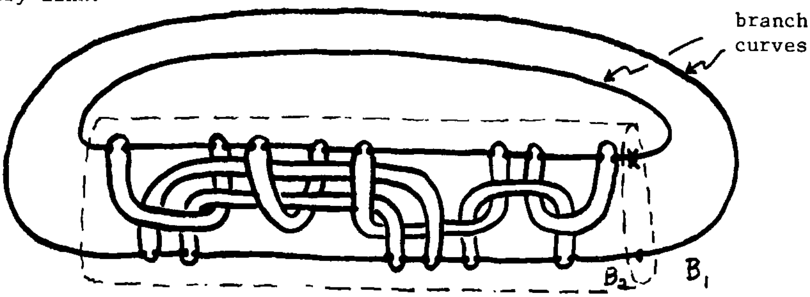
This may be constructed explicitly, exactly like the construction of proposition F10 above, except that the balls B_1 and B_2 are attached by the identity on their boundaries (the twist is omitted). The upstairs branch set is a trivial link of four components. Two are sent homeomorphically onto one branch curve, one with branching index two; the other two are sent similarly to the other downstairs branch curve. Let $f : S^3 \to S^3$ denote this branched covering. By way of illustration, here's how it is used to prove G1. A typical surgery instruction for a given M^3, according to the proof of the fundamental theorem of surgery, might look something like this, with +1 or -1 assigned to each component.

* this is best possible. For example $S^1 \times S^1 \times S^1$ is not a 2-fold, or any cyclic, branched cover of S^3. See Hirsch and Newmann [1975]



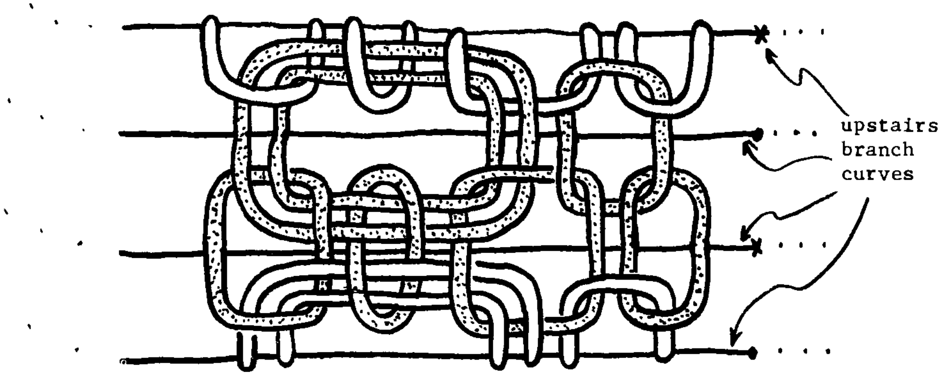
typical surgery curves in S^3

Now, in the downstairs S^3 of the branched covering consider the following collection of disjoint 3-balls, one for each component of the surgery link.

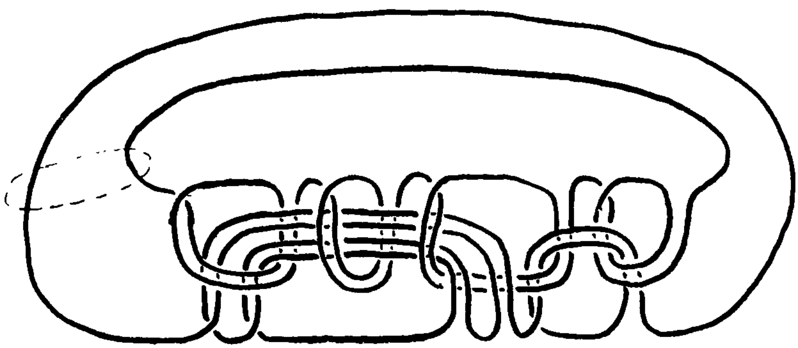


These lift, via f^{-1}, to the following collection of balls and solid tori in the upstairs S^3.

in the upstairs S^3

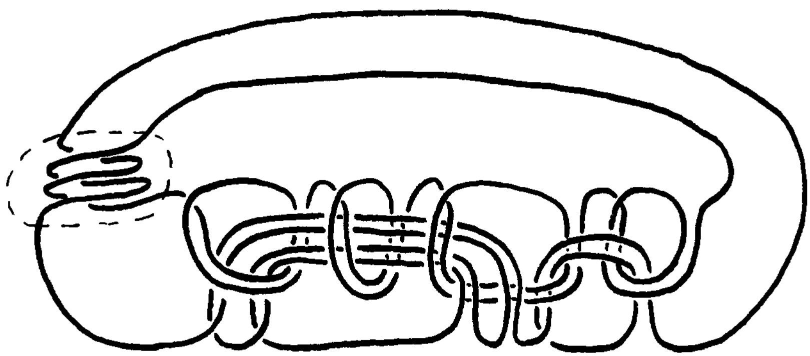


Things were contrived so that the solid tori (shaded) upstairs form a link in S^3 which is equivalent to the surgery instruction link for M^3. Each one of them doubly branch covers, by f, one of the balls downstairs. The balls upstairs are homeomorphic via f with those downstairs. As discussed in section C of this chapter, if we remove the balls downstairs and replace them with a certain twist, forming S^3 again, this corresponds to the surgery on the upstairs S^3 (removing the solid tori and replacing them differently) required to produce M^3, plus surgeries on the balls which are of no consequence. This tells us how to build a 3-fold branch covering map $M^3 \to S^3$. We can even visualize the branch set downstairs. In our example (supposing all $+1$ surgeries) it would look like:



downstairs branch set in branched covering $M^3 \to S^3$

In general the downstairs branch set will not be connected. Here's how to remedy that. Consider a ball in S^3 which intersects our branch set in exactly two arcs, in different components of the branch set, such that meridians at these two arcs are represented by the permutations (01) and (12). Then we may remove this ball from S^3, give it a 3/2 twist (as in proposition F10 above) and replace it. This may be covered in the upstairs M^3 by removing its preimage, also a ball, and replacing it with a twist. This does not change the homeomorphism type of M^3, and by performing such operations enough times we may revise the branched covering $M^3 \to S^3$ to make the downstairs branch set connected.



revised branch set which is connected

2. EXERCISE : Using the procedure outlined in the discussion above, give a formal proof of theorem 1 . (See remark 9I8)

3. REMARK : The branched covering $M^3 \to S^3$ constructed in this way has the further property that it is <u>simple</u>. This means that each meridian is represented by a permutation which is a single transposition, of the form (ij) in the permutation group. The corresponding representation of $\pi_1(S^3 - K)$ in S_3 , K = downstairs branch set, is not only transitive but surjective.

H. FIBRED KNOTS AND LINKS

A mapping $f : E \to B$ is said to be a <u>fibration</u> with <u>fibre</u> F if each point of B has a neighbourhood U and a 'trivializing' homeomorphism $h : f^{-1}(U) \to U\times F$ for which the following diagram commutes.

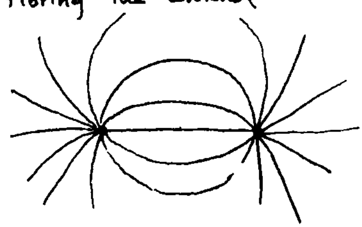
$$\begin{tikzcd} f^{-1}(U) \arrow[rr, "h"] \arrow[dr, "f"'] & & U\times F \arrow[dl, "\text{projection}"] \\ & U & \end{tikzcd}$$

E and B are known as the <u>total</u> and <u>base</u> spaces, respectively. Each set $f^{-1}(b)$ is called a fibre and is homeomorphic with F. We will be concerned with fibrations with S^1 as base space.

DEFINITION : A knot or link L^n in S^{n+2} is <u>fibred</u> if there exists a fibration map $f: S^{n+2} - L \to S^1$. We require further that the fibration be well-behaved near L. That is, each component L_i is to have a neighbourhood framed as $S^n \times D^2$, with $L_i \cong S^n \times 0$, in such a way that the restriction of f to $S^n \times (D^2 - 0)$ is the map into S^1 given by $(x,y) \to y/|y|$.

It follows that each $f^{-1}(x) \cup L$, $x \in S^1$, is an $(n+1)$-manifold with boundary L; in fact a Seifert surface for L. Exercise 5B2 shows every knot is "nearly" fibred. Nevertheless, some are and some aren't.

fibring the unknot



2. EXAMPLE: The unknot $S^n \subset S^{n+2}$ is fibred by the projection map $(S^n * S^1) - S^n \to S^1$. Fibres are $(n+1)$-disks.

Consider now the infinite cyclic cover $\tilde{X}$ of the complement of a fibred knot. If $\exp : R^1 \to S^1$ denotes the covering map $r \to e^{ir}$ we have a commutative diagram:

$$\begin{array}{ccc} \tilde{X} & \longrightarrow & R^1 \\ \downarrow & & \downarrow \exp \\ X & \longrightarrow & S^1 \end{array}$$

So $\tilde{X}$ is the total space of a fibration over R^1, which may be regarded as the "pullback" of the fibration of X. It may also be regarded as the pullback of the fibration (discrete fibre) $R^1 \to S^1$. By elementary fibration theory, any fibration with contractible base space is trivial, the total space being homeomorphic with the product of the fibre and the base, and the fibre map being merely projection onto the base. Therefore $\tilde{X} \cong R^1 \times F$. In particular, $\tilde{X}$ has the homotopy type of the fibre.

3. PROPOSITION : If $K^n \subset S^{n+2}$ is a fibred knot, then the commutator subgroup of its knot group is finitely generated. Also the higher homotopy $\pi_i(X)$, $i > 1$, of the knot complement is isomorphic with that of the fibre; in particular these groups are all finitely-generated.

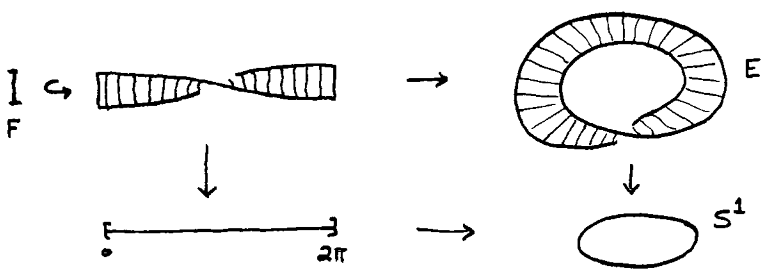
PROOF : $\pi_i(\tilde{X}) \cong \pi_i(F) \cong \pi_i(F \cup K)$, and the latter space is a compact manifold with boundary. The proposition follows from the fact that $\pi_1(\tilde{X}) \cong [\pi_1(X), \pi_1(X)]$ and $\pi_i(\tilde{X}) \cong \pi_i(X)$, $i > 1$, via the covering map.

4. REMARK : This is also true for links. Stallings [1961] has proved a converse for classical knots: if $[\pi_1 X, \pi_1 X]$ is finitely-generated, the knot is fibred. We won't, however, need to make use of this result.

5. EXAMPLE : In 7G5 we considered a knotted S^4 in S^6 whose complement has second homotopy group isomorphic with the dyadic rationals. It follows that this is not a fibred knot.

6. EXERCISE : Show that the commutator subgroup of the group of the knot 5_2 pictured at the right isn't finitely-generated, and hence this is <u>not</u> a fibred knot. [Hint: as a Λ-module it is cyclic of order $2 - 3t + 2t^2$] . This will also follow from the following analysis of the Alexander invariant of fibred knots.

Here is one way to understand a fibration $E \to S^1$. Using the exponential map $[0,2\pi] \to S^1$ we may pull back the fibration to a fibration over $[0,2\pi]$, which must be trivial (homeomorphic with $F\times[0,2\pi]$) since the base is contractible.



Thus the total space E is $F\times[0,2\pi]$ with an identification according to a homeomorphism $h : F \to F$ by which $F\times\{0\}$ is attached to $F\times\{2\pi\}$. This is sometimes called a <u>characteristic homeomorphism</u> and its induced homomorphism $h_* : H_*(F) \to H_*(F)$ is called the <u>monodromy map</u>.

7. EXERCISE : Let $K \subset S^3$ be a fibred knot with monodromy $h_*: H_1(F) \to H_1(F)$ Choose a basis for H_1F and let M be the integral matrix corresponding to h_*. Show that if I is the identity matrix of appropriate size, then $M - tI$ is a presentation matrix for the Alexander invariant $H_1(\tilde{X})$ of K, as a module over $\Lambda = Z[t,t^{-1}]$.

8. COROLLARY : For fibred knots in S^3, the Alexander polynomial is equal to the characteristic polynomial of the monodromy map.

9. COROLLARY : The Alexander polynomial of a fibred knot in S^3 is monic. That is, the first (and last) nonzero coefficients of $\Delta(t)$ are ± 1.

PROOF : The first corollary follows from the exercise by the very definition of the characteristic polynomial as $\det(M - tI)$. For the second, note that $\Delta(t) = \det(M - tI)$ gives the Alexander polynomial in "normalized" form $\Delta(t) = c_0 + c_1 t + \cdots + c_r t^r$. Then we calculate $c_0 = \Delta(0) = \det(M) = \pm 1$ since M represents the isomorphism h_*.

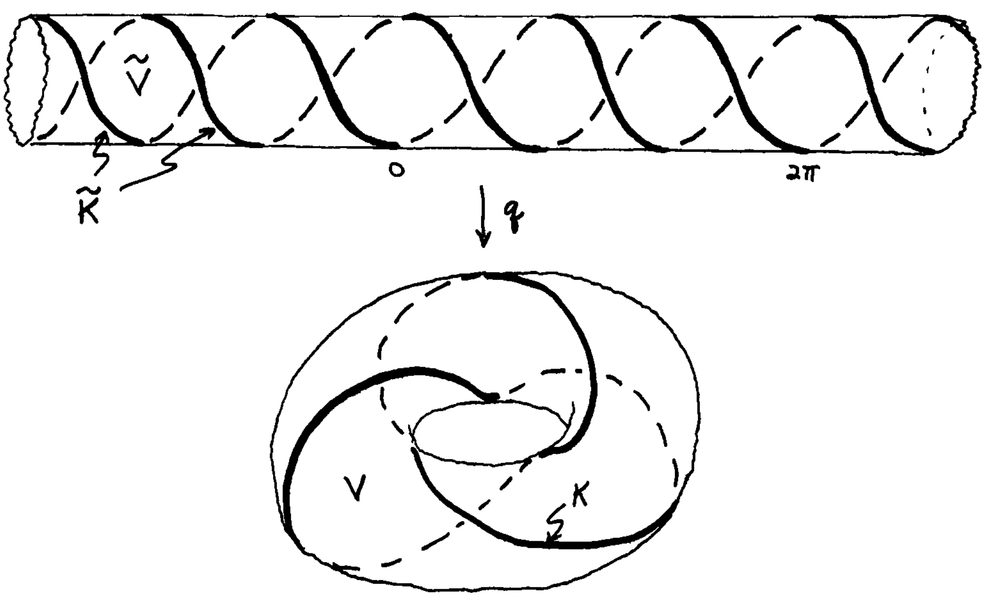
10. EXAMPLE : This is a practical detector of non-fibred knots. For example the only twist knots which can be fibred are the trefoil, figure-eight, and unknot. We'll see presently that the trefoil and figure-eight are, indeed, fibred.

11. EXERCISE : The connected sum of two fibred knots is fibred.

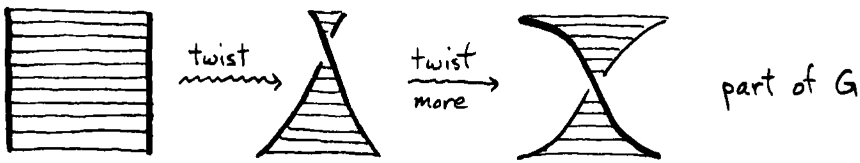
I. FIBERING THE COMPLEMENT OF A TREFOIL KNOT. (another lengthy example)

Here is an explicit construction of a fibration $p : S^3 - K \to S^1$ of the complement of the trefoil K in S^3. Each fibre F_θ is a Seifert surface of genus = 1. From the construction it will be easy to describe the characteristic map $h : F_0 \to F_0$ and its monodromy $h_* : H_1(F_0) \to H_1(F_0)$.

Regard K as lying on the boundary of a standardly embedded solid torus V in S^3. To aid visualization, consider the universal cover $q : \tilde{V} \to V$. We may realize $\tilde{V}$ as the infinite tube $\{y^2 + z^2 \leq 1\}$ in (x,y,z)-space; the covering translations may be taken to be shifts in the x-direction by multiples of 2π. Then the preimage $\tilde{K} = q^{-1}(K)$ is a double helix, as shown below (with equations $y = \pm \sin \frac{3}{2}x$, $z = \pm \cos \frac{3}{2}x$).



Now construct a surface G in $\tilde{V}$ by taking the ruled surface as indicated (in the region $0 \leq x \leq \frac{2}{3}\pi$), together with all its translates by multiples of $\frac{2}{3}\pi$ in the x-direction.



Thus G is the set in R^3 of all points of the form

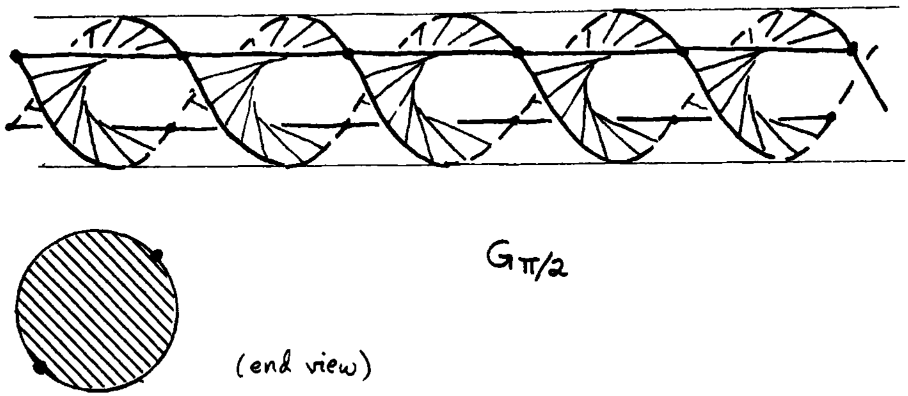
$$\begin{aligned} & r(x, \pm \sin \tfrac{3}{2}x , \pm \cos \tfrac{3}{2}x) \\ & + (1-r)(\tfrac{2\pi}{3} - x , \pm \sin \tfrac{3}{2}(\tfrac{2\pi}{3}-x) , \pm \cos \tfrac{3}{2}(\tfrac{2\pi}{3}-x)) \\ & + (\tfrac{2}{3}\pi n , 0, 0) \end{aligned}$$

$0 \leq x \leq \frac{2\pi}{3}$, $0 \leq r \leq 1$, $n = 0, \pm 1, \pm 2, \ldots$.

Let R_θ denote the screwing motion of $\tilde{V}$ which turns $\tilde{V}$ by an angle $\theta/2$ while moving it to the right so as to keep $\tilde{K}$ invariant. An explicit formula is

$$R_\theta(x,y,z) = (x + \theta/3 , y \cos \frac{\theta}{2} + z \sin \frac{\theta}{2} , -y \sin \frac{\theta}{2} + z \cos \frac{\theta}{2}).$$

Then define $G_\theta = R_\theta(G)$, $0 \leq \theta \leq 2\pi$.



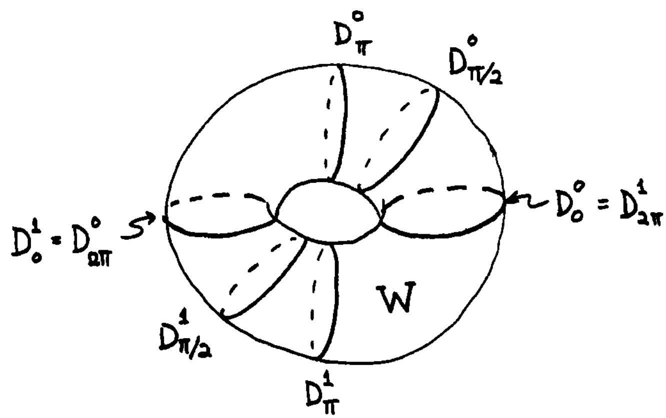
We now have a family G_θ , $0 \leq \theta \leq 2\pi$ of (pinched) 2-manifolds which fill up $\tilde{V}$ and are disjoint, except that $G_0 = G_{2\pi}$ and $\tilde{K}$ lies in every G_θ . The boundary of G_θ consists of $\tilde{K}$ plus two horizontal lines on $\partial\tilde{V}$ which are opposite each other and rotated by an angle $\theta/2$ from the vertical. Clearly every G_θ is invariant under the covering translations of $\tilde{V}$, so we can project to a family of surfaces, say

$$F_\theta' = q(G_\theta)$$

which fill up V and all contain our trefoil K . In addition F_θ' has two longitudinal circles C_θ^0 , C_θ^1 on ∂V as boundary; note that $C_{2\pi}^0 = C_0^1$ and $C_{2\pi}^1 = C_0^0$. To complete the fibration of $S^3 - K$, consider the solid torus

$$W = \overline{S^3 - V}$$

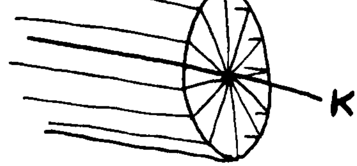
complementary to V . Here C_θ^0 and C_θ^1 are meridians and bound families of disks D_θ^0 and D_θ^1 , disjoint except that $D_{2\pi}^0 = D_0^1$ and $D_{2\pi}^1 = D_0^0$.



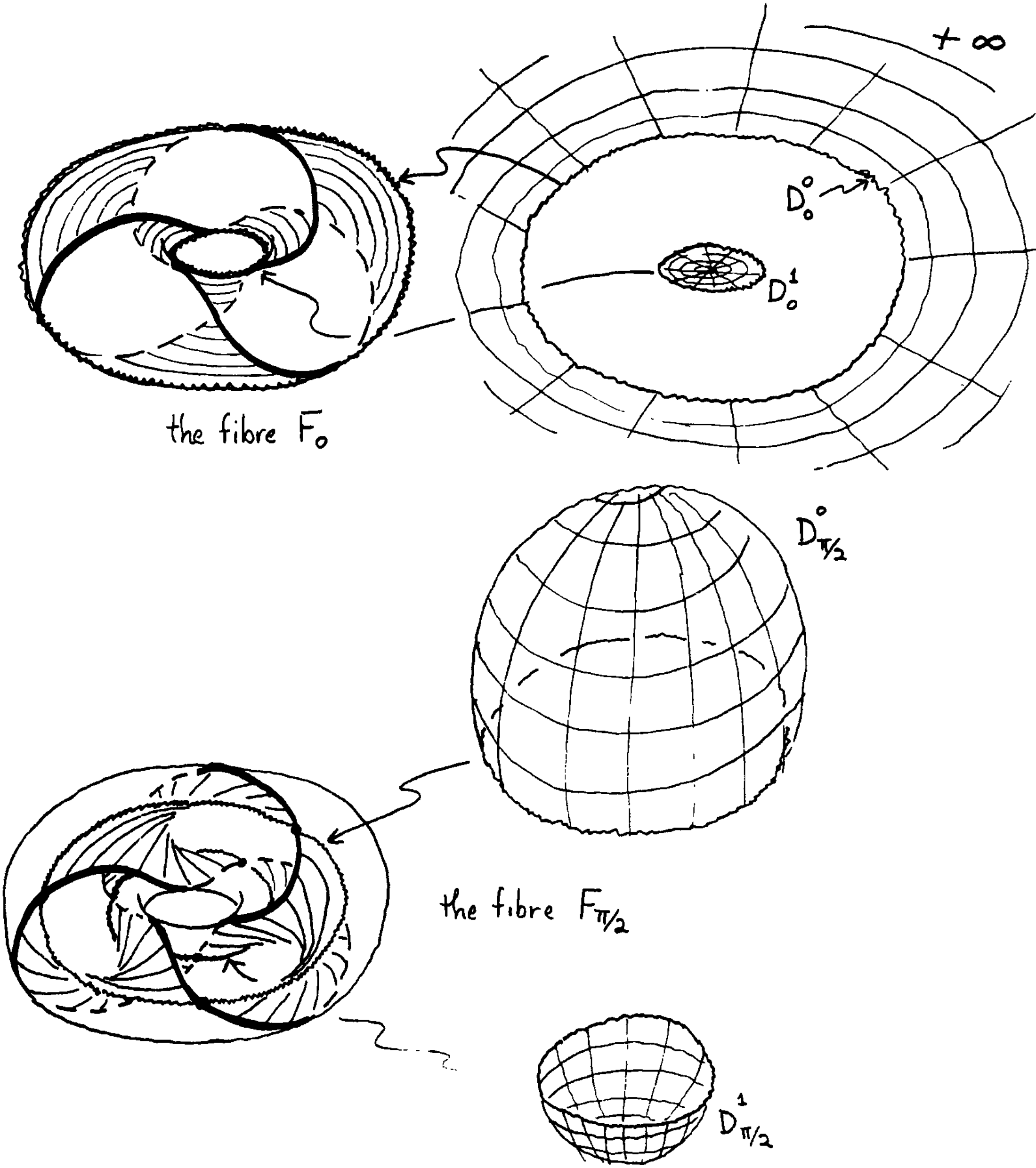
Now define $F_\theta = (F'_\theta \cup D^0_\theta \cup D^1_\theta) - K$. We can define a map $p : S^3 - K \to S^1$ by

$$p(x) = e^{i\theta} \text{ where } \theta \text{ is the unique number in } [0, 2\pi) \text{ such that } x \in F_\theta$$

It is routine to verify that p is continuous and has the local triviality property, so $p : S^3 - K \to S^1$ is a fibration with fibres F_θ. Moreover, there is clearly a tubular neighbourhood of K which intersects the fibres as indicated here:

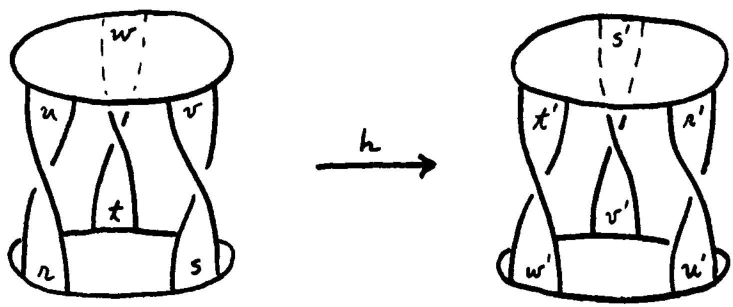


Here are two typical fibres, viewed in $R^3 = S^3 - \infty$.



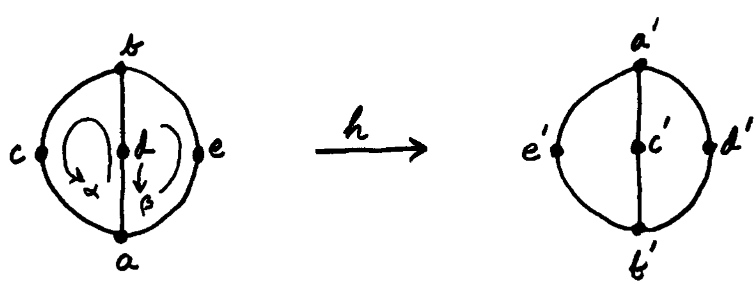
Now let's look at the characteristic map $h : F_0 \to F_0$.

Clearly it's just the composite $h = q \circ R_{2\pi} \circ q^{-1}$ (you should check that this is well-defined.) Thus h interchanges the disks D_0^0 and D_0^1 and cyclically permutes the three twisted squares which make up F_0', as well as turning them around. Here is a schematic:



Clearly h is periodic of period 6 (i.e. $h^6 = \text{id}$, $h^n \neq \text{id}$ if $1 \leq n < 6$.

To compute the monodromy $h_* : H_1(F_0) \to H_1(F_0)$, we may as well restrict attention to the 1-dimensional skeleton which is a deformation retract of F_0 :



$H_1(F_0)$ is the free abelian group with two generators, say α and β, which we may take to be represented by the oriented curves adbca and aebda. Thus we may compute:

$$h_*(\alpha) = [a'd'b'c'a'] = [beadb] = -\beta$$

$$h_*(\beta) = [a'e'b'd'a'] = [bcaeb] = \alpha + \beta .$$

With respect to the basis α and β for $H_1(F_0)$, h_* has matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

The characteristic polynomial of h_* is then

$$\det(h_*-tI) = t^2 - t + 1 ,$$

which checks with our previous calculation of the Alexander polynomial.

/ REMARK: In Milnor [1968][2] a fibration of $S^3 - K$ over S^1 is obtained by taking $K = f^{-1}(0) \cap S_\varepsilon$, where $f(z,w) = z^3 + w^2$ and S_ε is a small sphere in C^2 centered at the origin. His fibration theorem states that the mapping

$$\phi(z,w) = \frac{f(z,w)}{|f(z,w)|}$$

from $S_\varepsilon - K$ to the unit circle is the projection map of a smooth fibre bundle. Presumably this is topologically equivalent to the above.

2. EXERCISE : The k-fold (unbranched) cyclic cover $\tilde{X}_k$ of the complement of any fibred knot may be constructed from the product of the fibre and an interval by glueing the ends together by the k^{th} iterate h^k of a characteristic homeomorphism: $\tilde{X}_k \cong F \times [0,2\pi] \,/\, (y,2\pi) \sim (h^k(y),0)$.

3. EXERCISE : Show how to derive all the torsion invariants of a fibred knot, knowing the monodromy map.

Exercise 2 reveals the real geometric reason for the periodicity (period 6 in k) of the k^{th} torsion invariants of the trefoil, which has already been noted in example 8D7. Since for the characteristic map of the trefoil we have $h^k = h^{k+6}$, we may conclude the following.

4. PROPOSITION : The finite cyclic covers of the complement of the trefoil are homeomorphic as follows: $\tilde{X}_k \cong \tilde{X}_{k+6}$.

5. EXERCISE : Show also that for the trefoil $\tilde{X}_{6n+i} \cong \tilde{X}_{6n-i}$. [Hint: compare glueing the ends of $F \times [0,2\pi]$ together by a homeomorphism and by its inverse.]

Thus for the trefoil the unbranched k-fold cyclic covers fall into four homeomorphism classes, according as $k \equiv 0, \pm 1, \pm 2$, or 3 (mod 6). In particular, the covers with $k = 5, 7, 11, 13, 17, 19, \ldots$ are all homeomorphic with the complement X itself, an interesting example in covering space theory. We see also that for these values of k , the branched covers Σ_k are all constructible by surgery on the trefoil itself. WARNING: the homeomorphisms among the $\tilde{X}_k$ don't necessarily extend to the

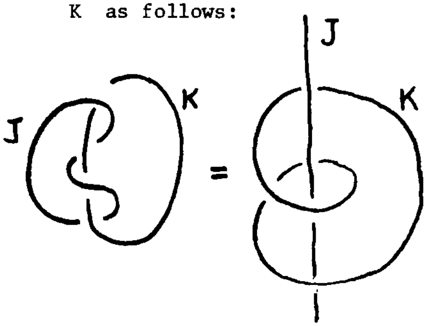
J. CONSTRUCTING FIBRATIONS .

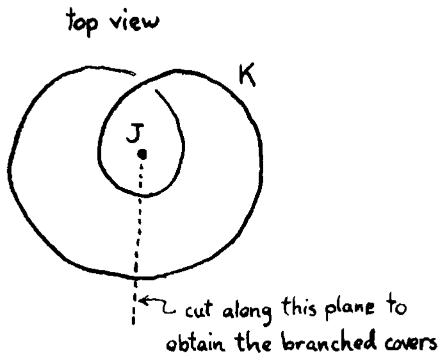
One important class of fibred knots are the so-called algebraic knots (and links) which arise in algebraic geometry. If $f(z_1, \dots, z_r)$ is a polynomial function of r complex variables and the origin is an isolated singularity (where the Jacobian vanishes), consider the intersection of a small (2r-1)-sphere centered at the origin of complex r-space with the algebraic variety V consisting of all the zeros of f. In general $V \cap S_\varepsilon$ is a smooth (2r-3)-manifold in S_ε^{2r-1} and V locally looks like the cone on this submanifold. In many important cases, $V \cap S_\varepsilon$ is a knot or link. Milnor [1968] shows that for small ε, $S_\varepsilon - V$ is the total space of a fibration over S^1, with fibre map the restriction of $f/|f|$. In classical dimensions (r=2) this shows that all torus knots and links are fibred. More generally this provides fibrations for certain "iterated cables" of the unknot.

A nice method of constructing plenty of examples of fibred knots and links in S^3 is due to Deborah Goldsmith [1975]. Let J and K be disjoint trivial knots in S^3 situated in such a way that J is braided in the open solid torus $S^3 - K \cong \mathring{B}^2 \times S^1$. In other words, J meets each fibre of the trivial fibration $S^3 - K \to S^1$ transversally in some fixed number of points. Since J is also trivial, we may form the k-fold branched cyclic cover $p : S^3 \to S^3$, branched downstairs along J. Consider the set $K' = p^{-1}(K)$ in S^3. It is either a knot or a link and Goldsmith notes that $S^3 - K'$ fibres over S^1; we leave verification as an exercise. In the following pages are a number of examples of fibred knots and links constructed in this manner.

1. EXERCISE : Construct a map $S^3 - K' \to S^1$ and verify that it is a fibration. Show that the fibre is, in a natural way, a branched cover of S^2, branched about n points, where $n = |\ell k(J,K)|$.

2. EXAMPLES: We can construct the torus knots and links $T_{2,n}$ ($T_{2,n}$ will be a knot if n is odd and a link if n is even) by situating J and K as follows:

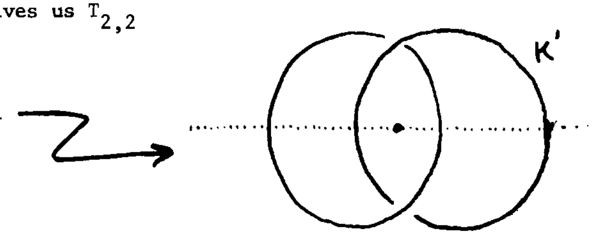




(think of $S^3 = R^3 + \infty$ and $\infty \in J$)

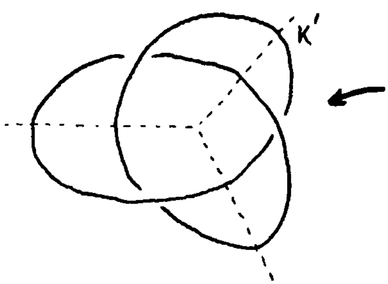
Then 2-fold branching gives us $T_{2,2}$

a fibred link



The typical fibre is a "twisted annulus".

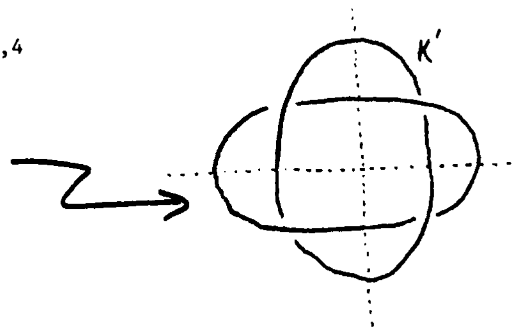
The 3-fold branched cover gives us $T_{2,3}$



a fibred knot, as we already know

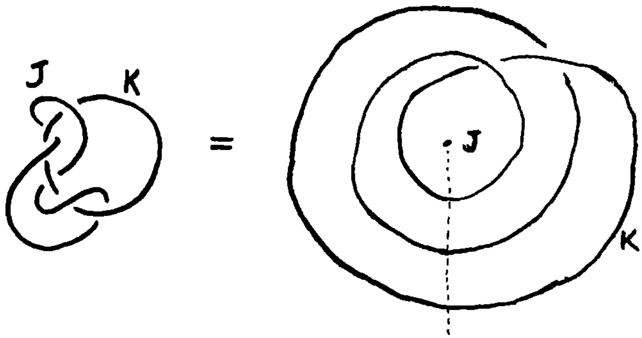
4-fold branching gives us $T_{2,4}$

a fibred link



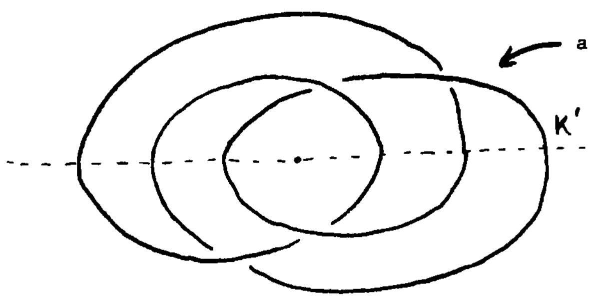
3. EXERCISE: Show how to obtain a fibration for any torus link in this way

4. EXAMPLES: If we start out with J and K situated like this



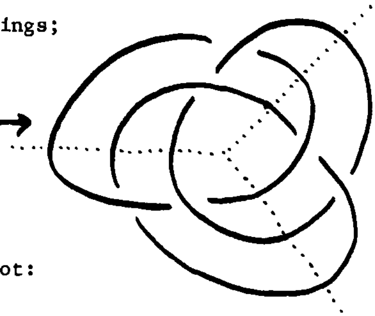
then 2-fold branching gives us the 4-knot (figure 8 knot)

a fibred knot

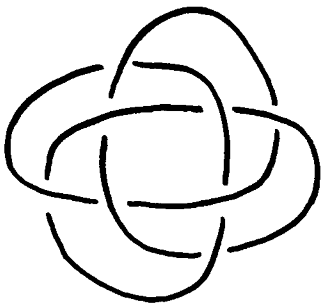


3-fold branching yields the Borromean rings;

a fibred link



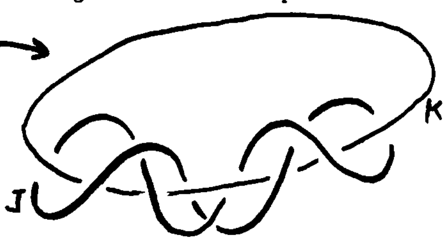
4-fold branching gives a Turk's head knot:



a fibred knot

5. EXERCISE : Show that Whitehead's link

is fibred by considering first the link pictured here



6. PUZZLE : The true lover's knot is fibred. What about the false lover's knot (reverse the two middle crossings)? The knot 8_{15} looks very similar, but is <u>not</u> fibred.

7. REMARK : Recall the construction in section 3L of twist-spun knots. Zeeman proves in [1965] the remarkable result that <u>every</u> k-twist spun knot in S^4, $k \neq 0$, is fibred. Moreover, he identifies the fibre. If K in S^3 is the knot which gets twisted and spun (actually we use the associated knotted arc in R^3_+), let Σ_k be the k-fold branched cyclic cover of S^3 branched over K. Puncture this manifold by removing a standard 3-ball and denote the resulting open 3-manifold by $\mathrm{punc}(\Sigma_k)$. Zeeman's main theorem asserts that the knot obtained by k-twist spinning K is fibred and $\mathrm{punc}(\Sigma_k)$ is the fibre. In particular if $k = 1$ we have $\mathrm{punc}(\Sigma_1) = \mathrm{punc}(S^3) \cong \mathcal{B}^3$ and the 1-twist spun knot is always unknotted, since it bounds a disk. As he points out, the 5-twist spun trefoil is a fibred knot in S^4 with fibre the punctured Poincaré dodecahedral space. Thus not only does this punctured manifold embed in S^4, but one even has a nice circle's worth of disjoint copies of them filling up all of S^4 save the twist-spun knot itself.

As might be expected, the process of k-twist spinning generalizes to higher dimensions and Zeeman proves a corresponding fibration theorem there, too.

K. OPEN BOOK DECOMPOSITIONS.

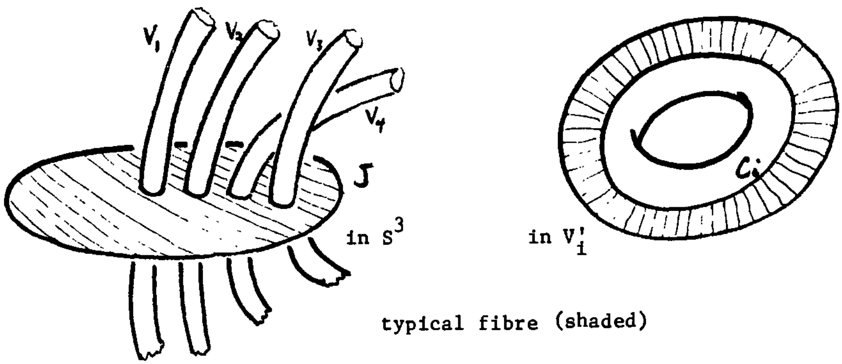
An <u>open book decomposition</u> of an n-manifold M consists of a codimension two submanifold N^{n-2}, called the <u>binding</u>, and a fibration $f : M - N \to S^1$. The fibres are called the <u>pages</u>. One may also require, as we will, the fibration to be well-behaved near N. I. e. that N have a tubular neighbourhood $N \times D^2$ so that f restricted to $N \times (D^2 -$ is the map $(x,y) \to y/|y|$. Thus a fibred link in S^n is a special case of an open book decomposition.

1. THEOREM : Every closed orientable 3-manifold M^3 has an open book decomposition.

PROOF : M may be assumed connected. By the fundamental theorem of surgery there are disjoint solid tori $V_1, \ldots, V_r$ in S^3 and $V_1', \ldots, V_r'$ in M and a homeomorphism

$$h : S^3 - (\mathring{V}_1 \cup \cdots \cup \mathring{V}_r) \to M - (\mathring{V}_1' \cup \cdots \cup \mathring{V}_r')$$

which carries meridians of V_i to longitudes of V_i'. As in remark 9I8 we may assume each V_i wraps once, in a monotone sense, around some fixed unknot J in S^3. The binding of our open book decomposition for M will consist of $h(J)$ together with the central curves C_i of the V_i'. Now the unknot J in S^3 is, of course, fibred and we may take a fibration $g : S^3 - J \to S^1$ each of whose fibres intersects each V_i in a meridinal disk. Then $gh^{-1} : M - (\mathring{V}_1' \cup \cdots \cup \mathring{V}_r' \cup h(J)) \to S^1$ is a fibration. Since the fibres intersect ∂V_i always in longitudinal curves this extends to a fibration of $M - (C_i \cup \cdots \cup C_r \cup h(J))$ of the sort reqired to give the decomposition, and we're done. The following picture of a typical fibre should make the construction clear.



typical fibre (shaded)

2. REMARK : The pages of this construction are actually planar (genus 0) 2-manifolds.

3. EXERCISE : If M^3 has an open book decomposition with more than one component in its binding, show how to construct a new decomposition with fewer components in the binding, at the expense of raising the genus of the pages. In particular, every M^3 has an open book decomposition with connected binding. (This result is due, I think, to González.)

4. EXERCISE : Show that if a simply-connected 3-manifold has an open book decomposition whose binding lies interior to a tame 3-ball, then it is S^3. (Hint: the complement of the binding has a covering space which embeds in R^3 and hence can contain no "fake" 3-balls.) (This observation of Simon, together with the previous exercise gives an alternative proof for the theorem of Bing stated in section 9E.)

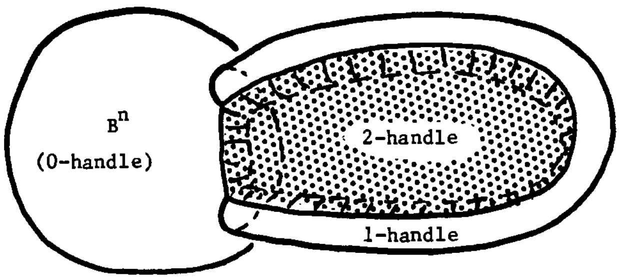
CHAPTER ELEVEN . A HIGHER-DIMENSIONAL SAMPLER

Most of our discussion so far has dealt with 3-dimensional topology and classical knot theory. Now I'd like to turn to some results in higher dimensions which are related to knots and their applications to manifolds. By no means the whole story, this rather sketchy chapter should be regarded as an invitation for further study on the reader's part.

A. TYING KNOTS BY ADDING HANDLES. An n-dimensional handle of index i, $0 \leqq i \leqq n$, is an n-ball of the form $D^i \times D^{n-i}$. The index indicates the intention to attach it to something else along the $S^{i-1} \times D^{n-i}$ part of its boundary. Although an n-dimensional object, such an " i-handle " is, in spirit, an i-dimensional disk which is thickened so that we may remain in the realm of n-manifolds (with boundary). If an i-handle is attached to the boundary of an M^n the resulting space may be denoted $M + h^{(i)}$, the dimensions and the attaching homeomorphism being understood from the context.

1. EXAMPLES : $h^{(0)}$ is just an n-ball, attached to nothing. A 3-dimensional handlebody (see 2G) may be denoted $h^{(0)} + h^{(1)} + \cdots + h^{(1)}$ We saw in 9I that any closed connected orientable 3-manifold bounds a 4-manifold of the form $h^{(0)} + h^{(2)} + \cdots + h^{(2)}$.

2. EXAMPLE : The following picture illustrates an n-ball (0-handle) to which a 1-handle is attached, turning it into an $S^1 \times D^{n-1}$. Then a 2-handle is attached in such a way that it becomes a ball again. This simple example illustrates the phenomenon of "cancellation" of handles of adjacent index (which is one of the key tricks in the proof of the celebrated h-cobordism theorem, by the way).

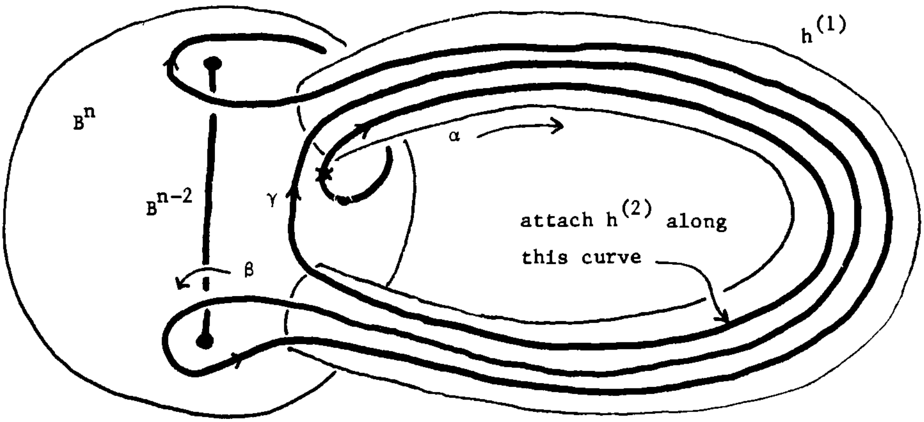


A 2-handle cancelling a 1-handle

This simple observation will form the basis for nearly everything to follow in this chapter. As a starter, consider the following trick. Let (B^n, B^{n-2}) denote the standard ball pair. Attach a 1-handle to the boundary of the B^n, along two $(n-1)$-disks which miss the B^{n-2}, to form an $S^1 \times D^{n-1}$ just as above. The fundamental group of $B^n + h^{(1)}$ is infinite cyclic, with generator α represented by a loop running through the $h^{(1)}$. On the other hand, the fundamental group of $B^n + h^{(1)} - B^{n-2}$ is free on two generators: α and a loop β which links the B^{n-2} . Construct an embedded simple closed curve in the boundary of $B^n + h^{(1)} - B^{n-2}$ which represents the element, say,

$$\gamma = \alpha^2 \beta \alpha^{-1} \beta^{-1} .$$

It might look something like this:



This particular expression was chosen because it becomes simply $\gamma = \alpha$ if we kill the element β. Now attach an $h^{(2)}$ to $B^n + h^{(1)} - B^{n-2}$ along a tubular neighbourhood of γ in the boundary. This has the effect of killing the element γ in fundamental groups (van Kampen's theorem) and we see that $B^n + h^{(1)} + h^{(2)}$ is simply-connected. What's more, if we allow it to pass through the B^{n-2}, there is an isotopy of $B^n + h^{(1)}$ moving γ over to the attaching curve of example 2 and we may conclude that $B^n + h^{(1)} + h^{(2)}$ is actually an n-ball.* B^{n-2} is still a subset of this ball, but we no longer have the standard ball pair, for van Kampen's theorem implies that

$$\pi_1(B^n + h^{(1)} + h^{(2)} - B^{n-2}) \cong (\alpha,\beta \ ; \ \alpha^2\beta\alpha^{-1}\beta^{-1} = 1)$$

So B^{n-2} is a knotted ball in the ball $\hat{B}^n = B^n + h^{(1)} + h^{(2)}$.

* for this part we need n > 3

3. EXERCISE : If $n \geqq 5$, similar calculations hold for the boundary sphere pair $(\partial\hat{B}^n, \partial B^{n-2})$. Thus in all dimensions higher than the classical, there is a codimension-two knot whose knot group is $(\, x,y \,;\, x^2yx^{-1}y^{-1} = 1)$ By its very construction, it is a slice knot.

4. EXERCISE : Use the methods of 7D to show that the Alexander polynomial of this knot is $\Delta(t) = 2t - 1$. No knot in S^3 has this group.

5. EXERCISE : Generalize the trick of this section to give an alternative proof (c. f. 7H1) that any integral polynomial satisfying $\Delta(1) = \pm 1$ is the Alexander polynomial of a higher-dimensional knot, in fact of a slice knot.

B. TRIVIAL SPHERE PAIRS CONTAIN NONTRIVIAL BALL PAIRS. Recall the non-cancellation theorem 4B7. Another way of stating this is that for any tame 3-ball B in S^3 which intersects the standard unknot S^1 in a proper arc, the ball pair $(B, B \cap S^1)$ is equivalent to the standard pair (B^3, B^1). In other words, any tame (3,1)-dimensional ball pair which lies in the trivial (3,1)-dimensional sphere pair must itself be trivial. This fails in higher dimensions.

1. PROPOSITION : If $n \geqq 4$, the standard sphere pair (S^n, S^{n-2}) contains a nontrivial (PL) ball pair of dimension $(n,n-2)$.

PROOF : This will be an embellishment of the construction of the previous section. Consider the standard (B^n, B^{n-2}) to lie in (S^n, S^{n-2}) as hemispheres (or equivalently as unit balls of (R^n, R^{n-2}), which is then compactified). With a little exercise of the imagination, the reader should be able to find a 2-disk bounded by γ which lies in S^n and whose interior misses both S^{n-2} and B^n.* So the $h^{(1)}$ and $h^{(2)}$ of the previous section may be added to B^n, within S^n itself. Then $\hat{B}^n$ is a subset of S^n and $\hat{B}^n \cap S^{n-2} = B^{n-2}$, so we have the desired example.

Here's another viewpoint of the same phenomenon. The boundary of $\hat{B}^n$ is a bicollared (n-1)-sphere in S^n. By the generalized Schönflies theorem, there is a homeomorphism of S^n which takes $\partial\hat{B}^n$ onto the standard S^{n-1} of S^n. This carries the standard S^{n-2} onto some knot K in S^n, which of course is unknotted. Its intersection with S^{n-1} is just the image of ∂B^{n-2} under the homeomorphism. We've seen that for $n \geq 5$, $(\partial\hat{B}^n, \partial B^{n-2})$ is knotted. This proves the following.

2. PROPOSITION : If $n \geq 5$, there is an unknotted (n-2)-sphere K in S^n whose intersection $K \cap S^{n-1}$ with the equator of S^n is a knotted (n-3)-sphere in S^{n-1}. Equivalently, there is an unknot in R^n which intersects the hyperplane R^{n-1} in a nontrivial knot.†

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* if $n > 4$, the disk may be constructed from a PL homotopy, using general position. An explicit construction, if you give up, may be found in Hudson and Sumners [1966], the source of this section.

† This can be improved by one dimension. A knot which is a slice of an unknot is called "doubly null-cobordant."

C. THE SMITH CONJECTURE. A homeomorphism $h : X \to X$ for which some iterate h^p is the identity map is called a <u>periodic transformation</u>. The least positive p for which h^p = identity is called its <u>period</u>. One of the most tantalizing open problems of classical knot theory is the following.

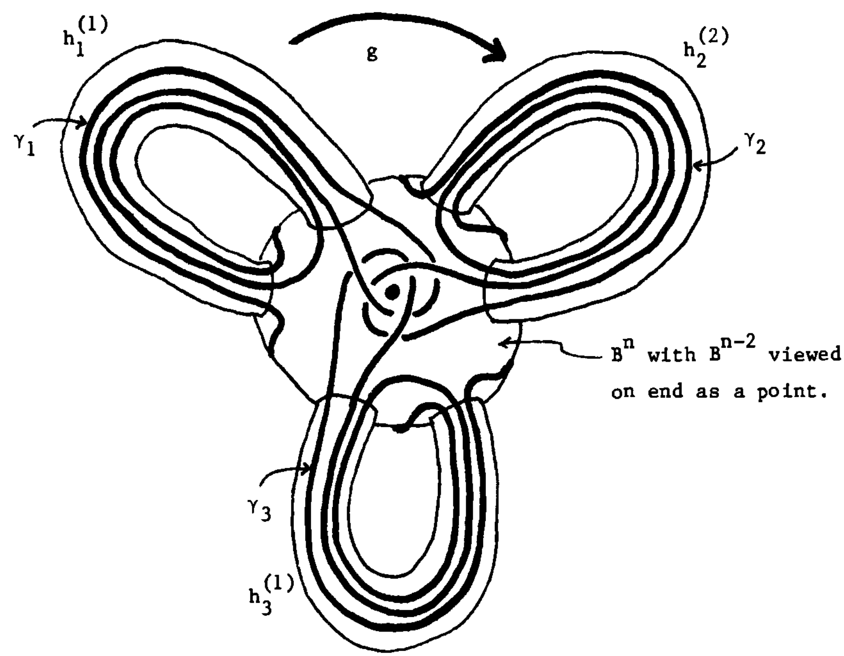
1. SMITH CONJECTURE : Suppose $h : S^3 \to S^3$ is a periodic homeomorphism whose fixed-point set $\{ x ; h(x) = x \}$ is a tame knot K in S^3. Then K is unknotted.

2. REMARK : This has been solved affirmatively for period 2, PL transformations by F. Waldhausen [1969]. For more on this conjecture and its relation to branched coverings, see Fox [1962].

In this section we'll see that the higher-dimensional version of the Smith conjecture is false. This was first discovered by Giffen [1966]. The construction given here is based on the improvement of Giffen's results by Sumners [1975].

3. THEOREM : For every $p \geqq 1$ and $m \geqq 4$, there is a periodic transformation $h : S^m \to S^m$, of period p, whose fixed-point set is a knotted $(m-2)$-sphere in S^m. (h may be taken to be smooth or PL, as desired).

PROOF : This will be another variation on the handle technique used in section A. Let $n = m+1 \geq 5$, and consider again the standard ball pair (B^n, B^{n-2}). There is a periodic transformation $g : B^n \to B^n$ whose fixed point set is the axis B^{n-2}. It may be obtained by rotating the first factor of $D^2 \times B^{n-2} \cong B^n$ by a p^{th} root of unity. Now it is possible to attach p disjoint 1-handles to the boundary of B^n, missing B^{n-2}, in such a way that they are attached equivariantly with respect to g. There is clearly an extension of g to a periodic transformation $g : B^n + h_1^{(1)} + \cdots + h_p^{(1)} \to B^n + h_1^{(1)} + \cdots + h_p^{(1)}$, which cyclicly permutes the new handles. Again, the fixed point set of g is exactly B^{n-2}. The fundamental group of $B^n + h_1^{(1)} + \cdots + h_p^{(1)}$ is free on generators $\alpha_1, \ldots, \alpha_p$ corresponding to the handles, and the group of the complement of B^{n-2} has one extra generator β. Now find p disjoint simple closed curves in the boundary which miss B^{n-2}, are cyclically permuted by g, and represent the words $\gamma_i = \alpha_i^2 \beta \alpha_i^{-1} \beta^{-1}$. This picture suggests how they might look for the case $p = 3$.



Then attach p disjoint 2-handles along these curves to form

$$\bar{B}^n = B^n + h_1^{(1)} + \cdots + h_p^{(1)} + h_1^{(2)} + \cdots + h_p^{(2)} .$$

Since n is large enough, we can argue just as before that $\bar{B}^n$ is an n-dimensional ball; the handles cancel. Moreover we can extend g to a transformation $g : \bar{B}^n \to \bar{B}^n$ which still has period p and still has B^{n-2} as fixed-point set. The fundamental group of $\bar{B}^n - B^{n-2}$, as well as that of $\partial\bar{B}^n - \partial B^{n-2}$, work out to have presentations $(\alpha_1, \ldots , \alpha_p, \beta ;\ \alpha_1^2\beta\alpha_1^{-1}\beta^{-1} = \cdots = \alpha_p^2\beta\alpha_p^{-1}\beta^{-1} = 1)$, which is easily checked to be a nontrivial knot group. Let $f : \partial\bar{B}^n \to S^m$ be a homeomorphism. Then the map $h = fgf^{-1} : S^m \to S^m$ is a period p transformation with fixed point set $f(\partial B^{n-2})$, a knotted $(m-2)$-sphere in S^m and the theorem is proved.

D. KERVAIRE'S CHARACTERIZATION OF KNOT GROUPS. Which groups arise as the fundamental group of the complement of a tame knot in S^3 ? The problem of characterizing, in a purely algebraic manner, the classical knot groups is still unsolved. For an abstract group G to be a classical knot group, we've seen that the following conditions are necessary:

(1) G is finitely presentable.

(2) The abelianization of G is infinite cyclic.

(3) The normal closure of some single element is all of G.*
(In other words, the weight of G equals 1).

(4') There is a presentation with one fewer relation than there are generators. (G has defect 1).

But they aren't sufficient. For example $G = (x,y \; ; \; x^2yx^{-1}y^{-1} = 1)$ satisfies these conditions, but is not a classical knot group. (See A4).

What about higher dimensions? It is clear from the spinning construction that any group of a smooth codimension-two knot in S^n is also the group of a knot in S^{n+1} ; as dimensions rise the classes of knot groups are related by inclusion. But notice that already for $n = 4$ the condition (4') is no longer necessary. For (4') implies that the knot has an Alexander polynomial, and example 7D7 shows that this may not be the case in S^4 . By considering a weaker condition than (4'), Kervaire [1965] was able to completely solve the characterization problem for $n \geq 5$. A remarkable consequence is that in this range the answer is independent of n . He used the C^∞ category, but we'll do it PL .

---

* for such an element, take a meridian.

1. KERVAIRE'S CHARACTERIZATION THEOREM : Let G be a group. Then there exists a PL knot K^{n-2} in S^n , $n \geq 5$, with $G \cong \pi_1(S^n - K)$ if and only if G satisfies conditions (1), (2), (3) and

(4) $H_2(G) = 0$.

This last condition -- that the second homology of the group be trivial -- requires some explanation. Although homology of groups can be defined in a strictly algebraic fashion, without reference to topology, I find the following definition most convenient. If G is a group, one may construct a space $K(G,1)$ whose fundamental group equals G and whose higher homotopy groups are all trivial. One then defines the homology of G by

$$H_i(G) = H_i(\ K(G,1)\ )$$

using, say, singular homology with integer coefficients. That this is well-defined is a consequence of the fact that any two spaces with the $K(G,1)$ property are homotopy-equivalent. Notice that condition (2) could be restated as $H_1(G) \cong Z$. To clarify this concept, let's prove the following.

2. PROPOSITION : If a group G satisfies (1), (2), (3) and (4'), then it satisfies (1), (2), (3) and (4) .

PROOF : We'll actually show that (2) ... ume, then that G abelianizes to Z and has a deficiency one presentation

$$G \cong (x_1, \ldots , x_p ; r_1, \ldots , r_{p-1}) .$$

A $K(G,1)$ space may be constructed as follows. Let X_1 be a wedge of p circles, one for each generator. Attach $p-1$ disjoint 2-disks to X_1 along their boundaries, which are to represent the relations, and call the result X_2. Then $\pi_1(X_2) = G$. Then add 3-disks to kill any 2-dimensional homotopy. Add 4-disks, 5-disks and so on to construct $K(G,1)$ as a possibly infinite union of disks. It is easy to see, however, that $H_2(K(G,1))$ is isomorphic with $H_2(X_2)/\partial_3(H_3(X_2,X_1))$. The exact sequence

$$\begin{array}{ccccccccccccc} \to & H_2(X_1) & \to & H_2(X_2) & \to & H_2(X_2,X_1) & \to & H_1(X_1) & \to & H_1(X_2) & \to & H_1(X_2,X_1) & \to \cdots \\ & \| & & & & \| & & \| & & \| & & \| & \\ & 0 & & & & Z^{p-1} & & Z^p & & Z & & 0 & \end{array}$$

shows by a rank argument that $H_2(X_2) = 0$, proving the proposition.

3. EXERCISE : Show that knot groups of any dimension satisfy (1), (2) and (3) assuming the knot is PL and codimension two.

4. LEMMA : If K^{n-2} is a PL knot in S^n, $n \geq 2$, and G is its knot group, then $H_2(G) = 0$. That is, condition (4) is necessary.

PROOF : By Alexander duality, $H_2(S^n - K) = 0$. We may construct a $K(G,1)$ from $S^n - K$ (which may be regarded as a finite complex) by adjoining cells of dimension 3 or higher. This introduces no new homology in dimension 2, so $H_2(G) = H_2(K(G,1)) \cong H_2(S^n - K) \cong 0$.

5. PROOF OF KERVAIRE'S THEOREM : We've just established that any PL knot group satisfies (1) - (4) . For the converse, assuming $n \geq 5$, consider a group $G \cong (x_1, \dots , x_p ; r_1, \dots , r_q)$ satisfying the four conditions. Now (3) implies that some word in this presentation would kill the group if adjoined as a relation. Introduce a new generator x_0 and a relator r_0 which says that x_0 equals this killer word. Then

$$G \cong (x_0, \dots , x_p ; r_0, \dots , r_q) \quad \text{and} \quad G/\{x_0\} \cong 1 .$$

Now we'll use the method of handles to produce a knot with group G . Consider the standard ball pair (B^{n+1}, B^{n-1}). Attach p 1-handles to the boundary of B^{n+1}, missing B^{n-1}, in an orientable way. The fundamental group of $B^{n+1} + h_1^{(1)} + \cdots + h_p^{(1)}$ is then free on p generators, which we identify with $x_1, \dots , x_p$. The complement of B^{n-1} in this handlebody has group free with one extra generator, linking the B^{n-1}, which we identify with x_0 . Now construct $q+1$ embedded curves in the boundary of the handlebody, missing B^{n-1} and representing the words $r_0, \dots , r_q$. Attach 2-handles along these curves to form the $(n+1)$-manifold

$$M^{n+1} = B^{n+1} + h_1^{(1)} + \cdots + h_p^{(1)} + h_0^{(2)} + \cdots + h_q^{(2)} .$$

The construction was arranged so that:

$$\pi_1(M^{n+1} - B^{n-1}) \cong \pi_1(\partial M^{n+1} - \partial B^{n-1}) \cong G \quad \text{and}$$

$$\pi_1(M^{n+1}) \cong \pi_1(\partial M^{n+1}) \cong G/\{x_0\} = 1$$

The assumption that $H_2(G) = 0$ implies [EXERCISE] that $H_2(M) \cong H_2(\partial M) = 0$ in the case the group has deficiency zero.

If the deficiency is not zero, we have $H_2(M) \cong H_2(\partial M) \cong Z^{q+1-p}$. To complete the construction, proceed as follows: let $X = \partial M \backslash \partial B^{n-1}$. Then

(1) show $H_2(X) \xrightarrow{\cong} H_2(\partial M) \xrightarrow{\cong} H_2(M)$.

(2) $\pi_1(X) = G$ and $H_2(G) = 0$ imply that $\pi_2(X)$ maps onto $H_2(X) \cong H_2(M) \cong Z^{q+1-p}$.

(3) By taking care in the construction, we may assume M is parallelizable.

(4) By (2) and (3) we may represent a basis for $H_2(M)$ by disjoint embedded 2-spheres in X having trivial normal bundles.

(5) Kill $H_2(M)$ by attaching 3-handles to M along those 2-spheres in X; let $\bar{M}$ denote the resulting $(n+1)$-manifold.

Then $\pi_1(\bar{M} \backslash B^{n-1}) \cong \pi_1(\partial\bar{M} \backslash \partial B^{n-1}) \cong G$ and $\pi_1(\bar{M}) \cong \pi_1(\partial\bar{M}) \cong G/\{x_0\} \cong 1$. Moreover $H_j(\bar{M}) \cong 0$ for $j \geq 1$ and $H_j(\partial\bar{M}) \cong H_j(S^n)$ for all j.

Conclude $\partial\bar{M}$ is a PL n-sphere, using the generalized Poincaré conjecture (see for example Rourke and Sanderson [1972]) and ∂B^{n-1} is the desired knot in $\partial\bar{M}$

6. REMARKS : In the case of zero deficiency the argument works also in the dimension n = 4. However, the PL Poincaré conjecture is still open for 4-manifolds, though the topological version has been proved by M. Freedman. Therefore, we obtain only the weaker result that a group satisfying (1), (2), (3), and (4') is the group of a knot in a PL homotopy sphere, or a topological 4-sphere.

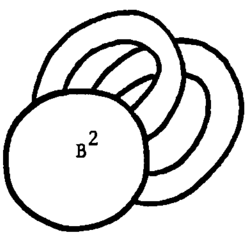
As a bonus in this construction, the knot K constructed is locally flat. What's more one can deduce that M is a PL ball and hence that K is a slice knot. As a result, in dimensions ≥ 5 one cannot use the knot group alone to distinguish local knotting or to detect non-slice knots (the latter <u>is</u> possible in dimension 3).

E. CONTRACTIBLE 4-MANIFOLDS . The classical Poincaré conjecture can be restated as follows: Every compact contractible 3-manifold (with boundary) is homeomorphic with the ball B^3 *. In this section we'll see that the analogue of this statement is false in dimension 4 (although the 4-dimensional Poincaré conjecture is still an open question). The following was discovered at more-or-less the same time by Mazur [1961], Poenaru [1960] and de Rham [published 1965].

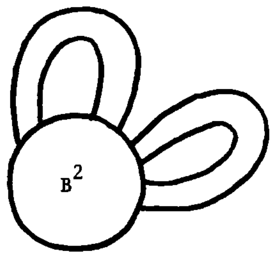
1. THEOREM : There exists a compact contractible 4-manifold whose boundary is not homeomorphic with S^3 .

2. REMARK : The boundary is necessarily a homology 3-sphere. It is easy to check that the 4-dimensional Poincaré conjecture is equivalent to the assertion that a contractible compact 4-manifold bounded by an S^3 is necessarily homeomorphic with a 4-dimensional ball.

We'll prove the theorem by giving Mazur's construction. Again the device of adding handles appears. But this time they don't cancel, but rather they almost cancel. As Zeeman has pointed out, the following examples illustrate the principle.



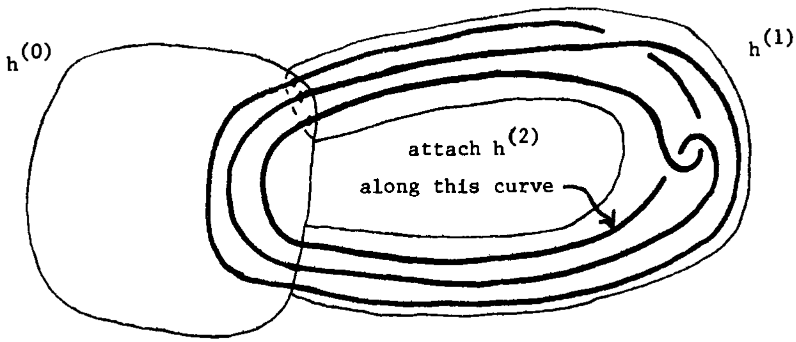
Two ways to add a pair of 1-handles to a disk.



* but recall the example of Whitehead (see 3I) showing that there are unusual noncompact contractible 3-manifolds.

Each is a disk with two 1-handles attached. But they aren't homeomorphic, in fact one has connected boundary while the other doesn't. If we take the cartesian product of each with the unit interval I, however, they become homeomorphic. This is basically because the extra dimension gives enough room for the 1-handles (which are still 1-handles after multiplication by I) to slide around.

Here is Mazur's example. Start with a 4-ball (0-handle). Add a 1-handle. Then add a 2-handle along the curve in the boundary indicated in the following picture:



The curve is drawn in a $D^2 \times S^1$, which is half of the boundary of $h^{(0)} + h^{(1)} = D^3 \times S^1$. Call the resulting 4-manifold

$$W^4 = h^{(0)} + h^{(1)} + h^{(2)}$$

3. EXERCISE : Verify Mazur's calculation that the boundary of W^4 has fundamental group $(x, y ; y^5 = x^7, y^4 = x^2yx^2)$. This is nontrivial, so the boundary of W^4 is not S^3.

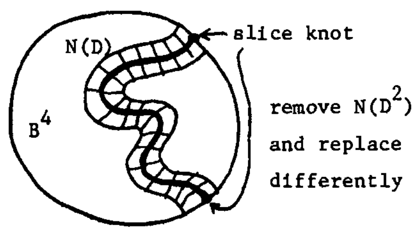
4. EXERCISE : $W^4 \times I$ is homeomorphic with the 5-ball. [Hint: with the extra dimension, one may move the 2-handle to make it cancel the $h^{(1)}$.]

It follows, of course, that W^4 is contractible and is the sort of example required by the theorem. Here are some related items. The <u>double</u> $2M$ of a manifold M with boundary is the union of two copies of M pasted together by the identity map on their boundaries.

5. EXERCISE : $2W^4$ is homeomorphic with S^4 .

6. EXERCISE : There is an involution (period 2 transformation) of S^4 whose fixed-point set is a non-simply-connected 3-manifold. It follows that this is non-standard in the sense of not being conjugate to a linear involution.

7. REMARK : Another method of constructing contractible 4-manifolds involves slice knots. If $K \subset S^3$ bounds a disk D^2 in B^4 which has a tubular neighbourhood $N(D)$, we may remove $N(D)$ and replace it differently, in a way, say, corresponding to a ± 1 surgery along K in S^3 . The result is a contractible 4-manifold whose boundary is the homology sphere obtained by ± 1 surgery on S^3 along K, and thus very likely not S^3 . One may also put together two different $B^4 - N(D^2)$'s to obtain such 4-manifolds. See Gordon [1975] for further details.



APPENDIX A

COVERING SPACES AND SOME ALGEBRA IN A NUTSHELL

COVERING SPACES. This is just a summary. The ambitious reader should be able to supply his own proofs. An excellent reference which goes into much more detail is Massey [1967]. A <u>covering space</u> is a triple $(\tilde{X}, p, X)$ consisting of two path-connected spaces and a map $p : \tilde{X} \to X$ such that X may be covered by open sets U satisfying: each component C of $p^{-1}(U)$ is open in $\tilde{X}$ and $p|C : C \to U$ is a homeomorphism. We call X the <u>base</u>, $\tilde{X}$ the <u>total space</u> or <u>cover</u>, and $p^{-1}(x)$ a <u>fibre</u>, $x \in X$.

LIFTING PROPERTIES. Let $p : \tilde{X} \to X$ denote a fixed covering space. A map $f : Y \to X$ is said to <u>lift</u> to a map $F : Y \to \tilde{X}$ if $f = p \circ F$.

UNIQUENESS LEMMA. If Y is a connected space and a fixed map $f : Y \to X$ lifts to two maps $F, F' : Y \to \tilde{X}$, then $\{y \in Y; F(y) = F'(y)\}$ is either empty or all of Y.

PATH LIFTING THEOREM. Given any path $\omega : I \to X$ and any specified $\tilde{x} \in p^{-1}\omega(0) \subset \tilde{X}$, there is a lifting $\tilde{\omega}$ of ω satisfying $\tilde{\omega}(0) = \tilde{x}$. By the uniqueness lemma, $\tilde{\omega}$ is unique.

COROLLARY. Any two fibres $p^{-1}(x)$, $p^{-1}(x')$, $x, x' \in X$ have the same cardinality.

HOMOTOPY LIFTING THEOREM. If $f_t : Y \to X$ is a homotopy and $F_0 : Y \to \tilde{X}$ is a lifting of f_0, then there is a (unique) homotopy $F_t : Y \to \tilde{X}$ which lifts f_t and extends F_0.

COROLLARY. If $\tilde{x} \in p^{-1}(x)$, then $p_* : \pi_k(\tilde{X},\tilde{x}) \to \pi_k(X,x)$ is an isomorphism when $k \geq 2$ and a monomorphism when $k = 1$.

Thus $\pi_1(\tilde{X},\tilde{x})$ is isomorphic to the subgroup of $\pi_1(X,x)$ corresponding to those loops in X (based at x) which lift to loops (rather than just paths) in $\tilde{X}$ based at $\tilde{x}$.

MAP LIFTING THEOREM. If Y is connected and locally path connected, then a map $f : (Y,y) \to (X,x)$ lifts to a map $\tilde{f} : (Y,y) \to (\tilde{X},\tilde{x})$, $\tilde{x} \in p^{-1}(x)$ specified, if and only if $f_*\pi_1(Y,y) \subset p_*\pi_1(\tilde{X},\tilde{x})$.

In other words, the commutative triangle of maps:

$$\begin{tikzcd} & \tilde{X} \arrow[d] \\ Y \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & X \end{tikzcd}$$

is possible if and only it is possible on the level of fundamental groups.

COVERINGS AND SUBGROUPS OF π_1. In the following, X denotes a connected, locally-path-connected topological space. A covering $\tilde{X} \to X$ is <u>universal</u> if $\tilde{X}$ is simply-connected.

EXISTENCE THEOREM: If X has a cover $\{U_\alpha\}$ by open path-connected subsets such that each inclusion homomorphism $\pi_1(U_\alpha) \to \pi_1(X)$ is the zero map, then X has a universal covering space.

THEOREM: Suppose X has a universal covering. Let $G < \pi_1(X,x)$ be any specified subgroup. Then there exists a covering space $p : \tilde{X} \to X$ and basepoint $\tilde{x}_0 \in p^{-1}(x) \subset \tilde{X}$ such that $p_*\pi_1(\tilde{X},\tilde{x}_0) = G$. (Recall that p_* is injective). If $\tilde{x}_1$ is any other point of $p^{-1}(x)$, then $p_*\pi_1(\tilde{X},\tilde{x}_1)$ is a subgroup of $\pi_1(X,x)$ which is conjugate to G. Moreover all subgroups conjugate to G arise by choosing different basepoints of $\tilde{X}$ in the fibre over x.

UNIQUENESS THEOREM: Suppose $p_i : (\tilde{X}_i,\tilde{x}_i) \to (X,x)$, $i = 1,2$, are coverings (with preferred basepoints) which correspond to the same subgroup of $\pi_1(X,x)$, i.e. $p_{1*}\pi_1(\tilde{X},\tilde{x}_1) = p_{2*}\pi_1(\tilde{X}_2,\tilde{x}_2)$. Then they are equivalent in the sense that there is a homeomorphism $h : (\tilde{X}_1,\tilde{x}_1) \to (\tilde{X}_2,\tilde{x}_2)$ such that the following diagram commutes:

$$\begin{tikzcd} X_1 \arrow[rr, "h", "\cong"'] \arrow[dr, "p_1"'] & & X_2 \arrow[dl, "p_2"] \\ & X & \end{tikzcd}$$

The hypothesis of the existence theorem is satisfied by most reasonable spaces, e.g. manifolds, simplicial complexes, and open subsets thereof (assuming connectedness). The content of the above is that for these spaces, there is a one-to-one correspondence:

{Coverings of X(with specified basepoint)} $\leftrightarrow$ {subgroups of $\pi_1(X,x)$}

If one wishes to neglect basepoints, the correspondence becomes:

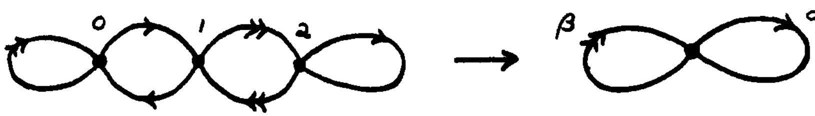
{Coverings of X} $\leftrightarrow$ {conjugacy classes of subgroups of $\pi_1(x)$}

COVERING AUTOMORPHISMS AND REGULAR COVERINGS. If $p : \tilde{X} \to X$ is a covering space, then a homoemorphism $\tau : \tilde{X} \to \tilde{X}$ is called a <u>covering automorphism</u> (or <u>translation</u>) if the following commutes:

$$\begin{tikzcd} \tilde{X} \arrow[rr, "\tau"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

By the uniqueness lemma, τ is determined by its action on any one point of $\tilde{X}$, in particular by its action on a fibre $p^{-1}(x)$. Unless τ = identity, it has no fixed points.

EXAMPLE: The following covering space of $S^1 \vee S^1$ has no covering automorphisms other than the identity.



A covering space $\tilde{X} \to X$ is said to be <u>regular</u> if for some $x \in X$ and $\tilde{x} \in p^{-1}(x)$ the subgroup $p_*\pi_1(\tilde{X},\tilde{x})$ is <u>normal</u> in $\pi_1(X,x)$. The definition is independent of the choices of basepoints. The covering of the example above is not regular (why?).

PROPOSITION: A covering space $\tilde{X} \to X$ is regular if and only if, for any two points $\tilde{x}_1, \tilde{x}_2$ in the fibre $p^{-1}(x)$ there is a covering automorphism $\tau : \tilde{X} \to \tilde{X}$ such that $\tau(\tilde{x}_1) = \tilde{x}_2$.

Consider a regular covering $p : \tilde{X} \to X$ and choose basepoints $x_0 \in X$ and $\tilde{x}_0 \in \tilde{X}$. For a loop $\alpha : [0, 1] \to X$ with $\alpha(0) = \alpha(1) = x_0$, let $\tilde{\alpha} : [0, 1] \to \tilde{X}$ denote the lift of α with $\tilde{\alpha}(0) = \tilde{x}_0$. By the proposition above there is a (unique) covering automorphism, call it τ_α, which carries $\tilde{\alpha}(0)$ to $\tilde{\alpha}(1)$. Clearly τ_α depends only on the class

$[\alpha] \in \pi_1(X, x_0)$. If β denotes another loop in X based at x_0 , it is an interesting exercise to verify that the composite $\tau_\alpha \circ \tau_\beta$ coincides with the covering automorphism $\tau_{\alpha\beta}$ corresponding to the product of the two loops (Hint: consider their effects on x_0 .) That is the main ingredient to the following proposition.

PROPOSITION: If $p : \tilde{X} \to X$ is regular, then the correspondence above is a homomorphism of $\pi_1(X,x_0)$ onto the group of covering automorphisms $\mathrm{Aut}(\tilde{X},p)$. Its kernel is exactly the subgroup $p_*\pi_1(\tilde{X})$, basepoints being inmaterial. Thus for regular coverings,

$$\mathrm{Aut}(\tilde{X},p) \cong \pi_1(X)/p_*\pi_1(\tilde{X}) .$$

In particular, the automorphism group of the universal cover of X is isomorphic with the fundamental group of X. The situation for non-regular coverings is more complicated, see Massey [1967]. Examples of regular coverings are the universal cover and the trivial cover $X \to X$, corresponding to the extreme subgroups of $\pi_1(X)$. Another important example is the <u>universal abelian</u> covering of X, which is the cover corresponding to the commutator subgroup $[\pi_1(X), \pi_1(X)]$ of $\pi_1(X)$. Its group of covering automorphisms is just the abelianization of $\pi_1(X)$. In the case that X is a knot complement this has automorphism group isomorphic with Z, and is the only covering with this property, so it is also called the _infinite cyclic_ covering. One more class of regular coverings we'll use quite a bit (Chapter 6) are the _finite cyclic_ coverings of a knot complement X. The k-fold finite cyclic covering is defined to be the one which corresponds to the kernel of the composite homomorphism:

$$\pi_1(X) \xrightarrow{\text{abelianization}} Z \xrightarrow[\text{projection}]{\text{natural}} Z/k.$$

The homology groups of the cyclic coverings of a knot complement are sometimes called the _abelian invariants_ of the knot.

GROUP PRESENTATIONS. It is assumed that the reader is familiar with the concept of a free group. See, for example, Massey [1967] or Crowell and Fox [1963], which also discuss group presentations. Let $r_1, r_2, \ldots$ be words (i.e. finite strings) in the symbols $x_1, x_2, \ldots$ Let G be a group. We write

$$G \cong (x_1, x_2, \ldots; r_1, r_2, \ldots)$$

if G is isomorphic with the quotient of the free group on the symbols $x_1, x_2, \ldots,$ modulo the smallest normal subgroup which contains $r_1, r_2, \ldots$ The x_i are called <u>generators</u> and the r_i <u>relators</u> of this <u>presentation</u> of G. A more informal notation is to write the equation $v = w$, called a <u>relation</u>, instead of the relator vw^{-1}, where v and w are words in the generators. A relator r becomes a relation by writing $r = 1$.

Every group has a presentation. G is said to be <u>finitely generated</u> (resp. <u>finitely presentable</u>) if it has a presentation with finitely many generators (resp. generators and relations).

If G is presented as above, and H is another group, then a function $f : \{x_1, x_2, \ldots\} \to H$ defines (uniquely) a homomorphism $G \to H$, provided $f(r_i) = 1 \in H$

for each relator r_i. In other words, each relation becomes "true", under f, in H.

Any element of the normal closure of the r_i is called a <u>consequence</u> of relators. In terms of relations, an equation is then a consequence of the relations if it can be derived from them by the usual formal rules of algebra. In practice, one often replaces the relations $r_1, \ldots$ by another set of relations $r_1', \ldots$ in the same generators. The presentations $(x_1, \ldots; r_1, \ldots)$ and $(x_1, \ldots; r_1', \ldots)$ give isomorphic groups if each r_i' is a consequence of $\{r_i\}$ and vice-versa. More generally one may "change variables" as follows. Let f be a function which assigns to each x_i a word in the symbols $x_1', x_2', \ldots,$ and let g be another function which assigns to each x_i' a word in $x_1, x_2, \ldots$. Then the presentations $(x_1, \ldots; r_1, \ldots)$ and $(x_1', \ldots; r_1', \ldots)$ describe isomorphic groups provided:

(1) each $f(r_i)$ is a consequence of $r_1', \ldots$ and each $g(r_i')$ is a consequence of $r_1, \ldots$.

(2) $gf(x_i) = x_i$ is a consequence of $r_1, \ldots$ and $fg(x_i') = x_i'$ is a consequence of $r_1', \ldots$.

In general it is difficult to decide when two presentations define the same group (up to isomorphism). Or even if a given presentation describes the trivial group.

The <u>free product</u> of two groups can be obtained from presentations of them simply by taking the generators of both (assumed distinct) and all the relations from both presentations.

PRESENTATIONS OF MODULES AND ABELIAN GROUPS. Let A denote a commutative ring with identity 1. An abelian group M is called a (left unitary) A-module provide there is defined a multiplication $A \times M \to M$ satisfying the axioms

$$\begin{aligned} a(m + m') &= am + am' \\ (a + a')m &= am + a'm \\ (aa')m &= a(a'm) \\ 1m &= m. \end{aligned}$$

A <u>submodule</u> is a subgroup which is closed under multiplication by elements of A. The quotient of a module by a submodule is the quotient group with the ring multiplication induced in the usual manner.

The free A-module on the symbols $x_1, x_2, \ldots$ is the set of formal linear combinations

$$a_1 x_1 + a_2 x_2 + \ldots$$

in which all but finitely many terms are zero. The group addition and module multiplication are the obvious ones.

By analogy with group presentations, let $r_1, r_2, \ldots$ be elements of the free A-module on the symbols $x_1, x_2, \ldots$ Then an A-module M is said to have the <u>A-module</u> presentation

$$M \cong (x_1, x_2, \ldots; r_1, r_2, \ldots)$$

if M is isomorphic with the quotient of the free module on $x_1, x_2, \ldots,$ module the submodule generated by $r_1, r_2, \ldots.$ (There is no need to worry about normal closure here, since M is abelian).

One may manipulate module presentations in much the same manner as group presentations.

An abelian group by itself is the same thing as a Z-module, where Z is the ring of integers. Here ng has the usual meaning of the sum of $|n|$ copies of g (or $-g$ if $n < 0$). A presentation of G as a Z-module is called an <u>abelian group presentation</u>. This is quite different from a group presentation, although the same notation is used. If we write $G \cong (x_1, \ldots; r_1, \ldots)$ we'll mean this to be a group presentation (in the sense of the previous section) unless we specifically say it's a presentation <u>as an abelian group</u>. For example $(x, y;\ x^2 = y^2 = 1)$

presents the free product of $Z/2$ with itself, an infinite group. As an abelian group presentation it gives the direct sum $Z/2 \oplus Z/2$ which has order four.

Suppose we have a presentation $G \cong (x_1,\ldots;r_1,\ldots)$ as an abelian group. Then it is true, though not completely trivial, that G has the following presentation as a group:

$$G \cong (x_1,\ldots;r_1,\ldots, x_1x_2 = x_2x_1,\ldots).$$

where the additional relations say that the generators commute (see Crowell and Fox [1963]).

VAN KAMPEN'S THEOREM AND THE MAYER-VIETORIS SEQUENCE. These are the basic tools for computing the fundamental group and the homology groups of a space which is built of pieces whose groups are known. In this section, X denotes a topological space with open subsets X_0, X_1, X_2 such that:

$$X = X_1 \cup X_2 \qquad \text{and} \qquad X_0 = X_1 \cap X_2$$

Denote the inclusion maps as follows:

$$\begin{tikzcd} & X_0 \arrow[dl, "i_1"'] \arrow[dr, "i_2"] & \\ X_1 \arrow[dr, "j_1"'] & & X_2 \arrow[dl, "j_2"] \\ & X & \end{tikzcd}$$

The homomorphisms they induce on either π_1 or homology will be denoted by a subscript "*".

The first theorem (also attributed to Seifert) describes $\pi_1(X)$ in terms of a universal mapping property The basepoint for the fundamental group is assumed to lie in X_0.

VAN KAMPEN'S THEOREM: Suppose each X_i, $i = 0, 1, 2,$ is nonempty and path-connected. Further suppose there are given any group G and homomorphisms $h_i : \pi_1(X_i) \to G$, $i = 1, 2,$ which form a commutative diagram:

$$\begin{tikzcd} & \pi_1(X_0) \arrow[dl, "i_{1*}"'] \arrow[dr, "i_{2*}"] & \\ \pi_1(X_1) \arrow[dr, "h_1"'] & & \pi_1(X_2) \arrow[dl, "h_2"] \\ & G & \end{tikzcd}$$

Then there is a unique homomorphism $g : \pi_1(X) \to G$ such that the following diagram commutes:

$$\begin{tikzcd}
& \pi_1(X_0) \arrow[dl, "i_{1*}"'] \arrow[dr, "i_{2*}"] & \\
\pi_1(X_1) \arrow[r, "j_{1*}"] \arrow[dr, "h_1"'] & \pi_1(X) \arrow[d, dashed, "g"] & \pi_1(X_2) \arrow[l, "j_{2*}"'] \arrow[dl, "h_2"] \\
& G &
\end{tikzcd}$$

An alternative form of this theorem, in terms of generators and relations, is often more useful in practice.

VAN KAMPEN'S THEOREM: Suppose X_i, $i = 0, 1, 2$, is nonempty and path-connected and suppose there are presentations:

$$\pi_1(X_1) \cong (x_1, \ldots; r_1, \ldots)$$
$$\pi_1(X_2) \cong (y_1, \ldots; s_1, \ldots)$$
$$\pi_1(X_0) \cong (z_1, \ldots; t_1, \ldots)$$

Then the fundamental group of X has the presentation

$$\pi_1(X) \cong (x_1, \ldots, y_1, \ldots; r_1, \ldots, s_1, \ldots, i_{1*}(z_1) = i_{2*}(z_1), .$$

In other words, one throws together the generators and relations from $\pi_1(X_1)$ and $\pi_1(X_2)$, plus one relation for each generator z_i of $\pi_1(X_0)$ which says that its images in $\pi_1(X_1)$ and $\pi_1(X_2)$ are equal. Notice that the z_i do not appear as generators of $\pi_1(X)$ and that the relations $t_1, \ldots$ play no role in the calculation.

In the special case that $\pi_1(X_0) \cong 1$, $\pi_1(X)$ is the free product $\pi_1(X_1) * \pi_1(X_2)$.

Somewhat more generally, suppose the inclusion homomorphisms i_{1*} and i_{2*} are injective. Then one may deduce that j_{1*} and j_{2*} are also injective, so that all three of $\pi_1(X_i)$, $i=1,2,3$, may be regarded as subgroups of $\pi_1(X)$. In this case $\pi_1(X)$ is said to be the free product of $\pi_1(X_1)$ and $\pi_1(X_2)$ with amalgamated subgroup $\pi_1(X_0)$.

Now we consider the integral homology groups $H_n(X)$, although other coefficient groups may be used. A sequence of homomorphisms of groups.

$$\cdots \to G_{n+1} \xrightarrow{h_{n+1}} G_n \xrightarrow{h_n} G_{n-1} \to \cdots$$

is said to be <u>exact</u> if the image of each h_{n+1} is equal to the kernel of h_n, both being subgroups of G_n.

MAYER-VIETORIS THEOREM: There is an infinite exact sequence of homomorphisms (terminating on the right with $H_0(X)$):

$$\cdots \to H_{n+1}(X) \xrightarrow{\Delta} H_n(X_0) \xrightarrow{g} H_n(X_1) \oplus H_n(X_2) \xrightarrow{h} H_n(X) \xrightarrow{\Delta} H_{n-1}(X_0) \to$$

Although stated for open subsets, this theorem is true for more general classes of subsets of X. One may also use reduced homology groups, provided all the spaces are nonempty.

Two of these homomorphisms are easy to write down:

$$g(\alpha_0) = (i_{1*}(\alpha_0), \pm i_{2*}(\alpha_0))$$
$$h(\alpha_1, \alpha_2) = j_{1*}(\alpha_1) \mp j_{2*}(\alpha_2)$$

where $\alpha_r \in H_n(X_r)$, $r = 0,1,2$. For definition of Δ and further explanations, see e.g. Greenberg [1967].

APPENDIX B. DEHN'S LEMMA AND THE LOOP THEOREM

(guest lectures by David Gillman)

Dehn's Lemma and the Loop Theorem. It is a standard theorem of general topology that pathwise connectedness implies arcwise connectedness. We now will prove an even weaker result, namely, that this result holds in the special case of piecewise linear paths in 2-manifolds. The rationale for this simple exercise is to motivate the methods to be used in Dehn's Lemma, an analogous result in one higher dimension.

Theorem 1. If $f : [0,1] \to M^2$ is a piecewise linear path in the 2-manifold M^2, with $f(0) \neq f(1)$, then there exists an embedding $g : [0,1] \to M^2$ such that $g(0) = f(0)$ and $g(1) = f(1)$.

Proof. We assume that f is a general position map. At any point p of self-intersection of the path $f([0,1])$, a neighborhood N of p in M^2 can be found which looks like R^2, with the path represented by both x-axis and y-axis. See Figure 1.

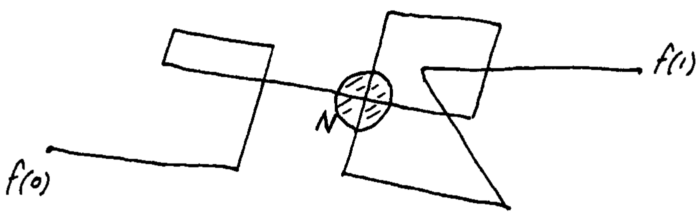
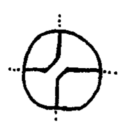


FIGURE I

We alter the path within N by removal of the self-intersection. Our new path in N looks like this:



or like this:

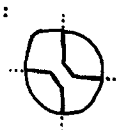
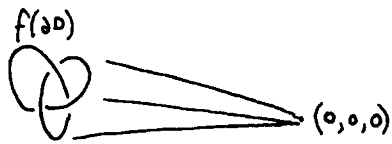


FIGURE II

After making such an alteration at every point of self-intersection, we have formed a set consisting of an arc from $f(0)$ to $f(1)$, plus several simple closed curves which are unnecessary and we simply ignore. The arc is the embedding g that we were seeking. QED

How can Theorem 1 be generalized to one higher dimension? Here is an obvious <u>conjecture</u>: If f is a piecewise linear map of a 2-dimensional disk D into a 3-manifold M^3 such that $f/\partial D$ is an embedding, then there exists an embedding $g : D \to M^3$ such that $g(\partial D) = f(\partial D)$. But this is false, unfortunately, as can be seen by defining f to carry ∂D to a trefoil knot in R^3, f carries the center point of D to $(0,0,0) \in R^3$, and f is extended linearly on each radius of D:



<u>Exercise</u>: Verify that f is a counterexample to the Conjecture.
A strengthening of the hypothesis is thus necessary. We essentially require that $f(\text{Int } D) \cdot f(\partial D)$ is also empty. Formally, let the <u>singular set of f</u>, $S(f)$, be $\{x \in D \mid \exists y \in D : y \neq x \text{ and } f(x) = f(y)\}$.

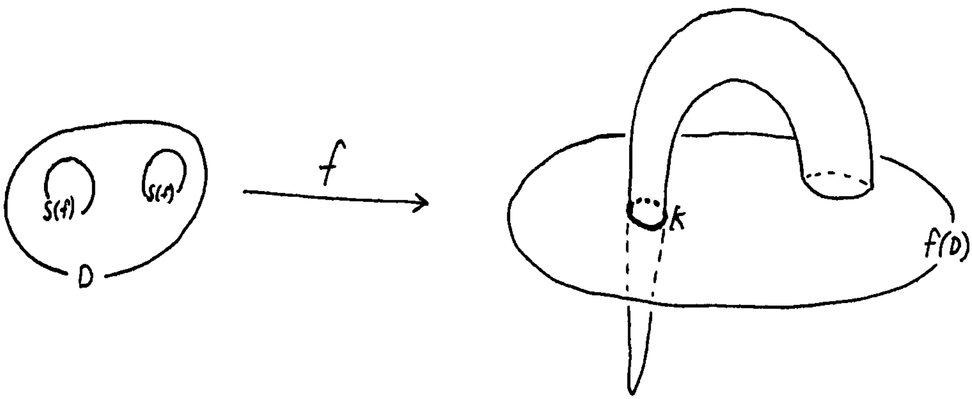
<u>Dehn's Lemma.</u> If $f : D \to M^3$ is a piecewise linear map with $\overline{S(f)} \cdot \partial D = \phi$, then there exists an embedding $g : D \to M^3$ with $g(\partial D) = f(\partial D)$. (In this section, $\cdot$ means intersection)

<u>Note</u>: The requirement $\overline{S(f)} \cdot \partial D = \phi$ guarantees us an annulus containing $f(\partial D)$ without singularities. Thus, all difficulties are a finite

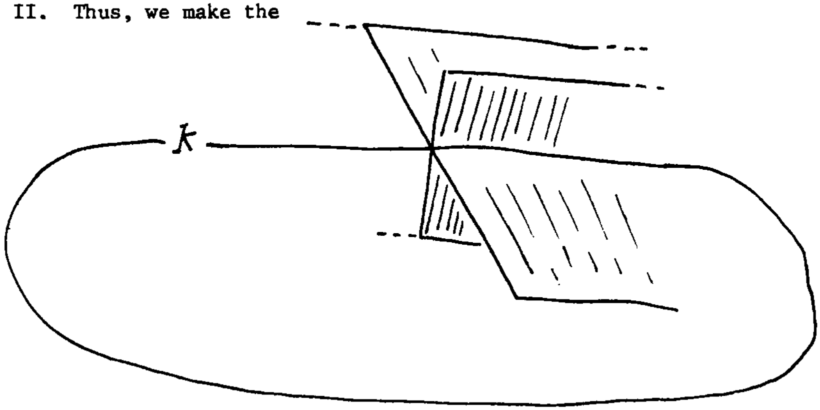
distance from the boundary of our singular disk. If we only required $S(f) \cdot \partial D = \phi$, the result is still true but more complicated. If we further allow $f(\partial D)$ to be a wild embedding, then the problem is unsolved.

<u>Proof of A SPECIAL CASE of Dehn's Lemma</u>:

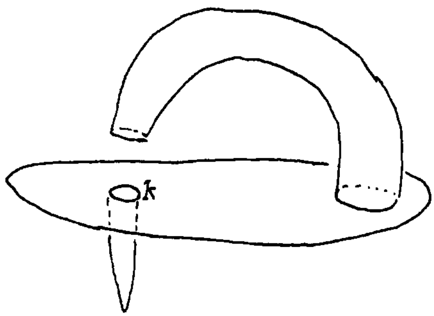
Let us deal only with the special case pictured here. A flat disk in R^3 has a feeler stretched upwards from its right side, then bent and pushed down through itself on the left side. The singular sct $S(f)$ is two curves in D . Their common image in R^3 is the darkened simple closed curve k



To form a nonsingular disk, we alter $f(D)$ near the k only! Pictorially, a cross-sectional slice of this curve looks just like Figure II. Thus, we make the

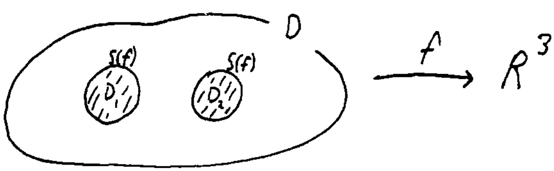


analogous alteration to that of Figure II, only this time all the way around I , forming the desired non-singular disk. An exaggerated picture of our new disk:



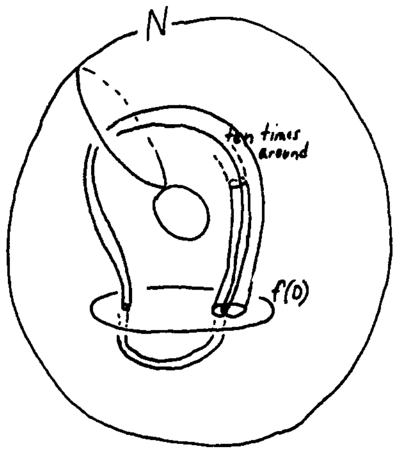
The old feeler is cut off and capped off before intersecting the surface A at K . A new "feeler" now exists starting at k and stretching downwards.

In terms of the set S(f) , we may view this alteration as switching the two disks D_1 and D_2 ;

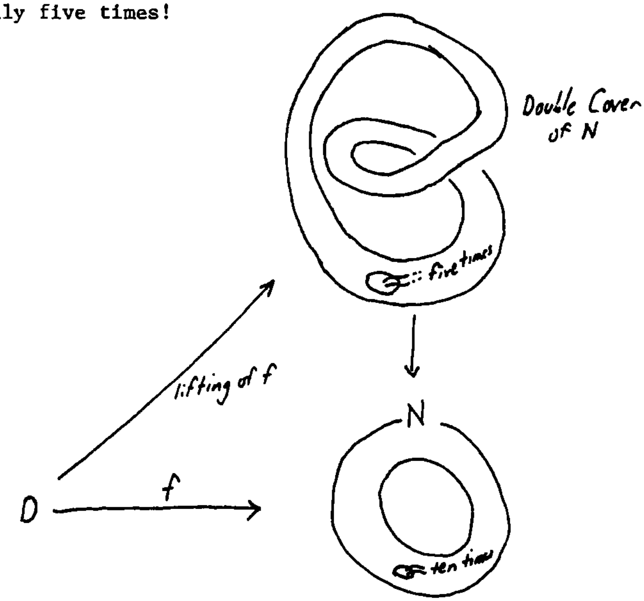


This completes the Special Case.

<u>Proof of a slightly worse case</u>: This time let us stretch a feeler around and around through itself ten times.



The set S(f) is too complicated now to draw. We no longer have just <u>double</u> points, but triple points (quadruple points?) The trick is to select a solid torus neighborhood N of f(D) , then form the double cover of N . It is itself a solid torus, but lifting the map f to the double cover gives a map which stretches the feeler around and around only five times!



So we have made progress. Take a double cover of this double cover and it goes around only $2\frac{1}{2}$ times. Another double cover and yet another and we finally have an embedding. We call this construction a *tower* of double covers. The embedding at the top of the tower is pushed down the tower, one level at a time, by the method of the SPECIAL CASE. The special case handled only *double* points of singularities, but a double cover will give us only double points, so the problem of triple points and other horrible self intersections has been entirely avoided by pushing down the tower only one double cover at a time. We are now ready for the general proof:

Formal Proof of Dehn's Lemma: We rename $f_o = f$, and let R_o be a regular neighborhood of $f_o(D)$ in $M^3 = M_o$. This is the 0th storey of our tower.

Form the double cover M_1 of R_o, and call f_1 the lifting of f_o

$$\begin{tikzcd} & M_1 \arrow[d, "\rho_1"] \\ D_o \arrow[ur, dashed, "f_1"] \arrow[r, "f_o"] & R_o \end{tikzcd}$$

A regular neighborhood R_1 of $f_1(D)$ in M_1 completes our construction of the 1st storey, and a double cover M_2 of R_1 begins the 2nd storey.

Lemma 1. This process terminates after a finite number of stories. That is, there exists an n for which no double cover of R_n exists.

Proof. For the above commutative diagram, it follows that $S(f_1) \subseteq S(f_o)$; that is, the singular set gets smaller (or stays the same) as we proceed up the tower. This is the idea of the proof: show that the singular set must in fact gradually disappear. Unfortunately, we need the following statement in our argument: If $S(f_1) = S(f_o)$, then $f_1(D)$ and $f_o(D)$ are homeomorphic under ρ_1. But is this so? Because

<u>Exercise</u>: Construct topological spaces X and Y and maps $\alpha : X \to Y$ a $\beta : X \to Y$ with $S(\alpha) = S(\beta)$, yet $\alpha(X)$ is not homeomorphic with $\beta(X)$.

Thus, we are forced to alter our definition of $S(f)$ a bit: Given $f : X \to Y$, let $\tilde{S}(f)$ be the subset of $X \times X$ consisting of those pairs (x_1, x_2) such that $f(x_1) = f(x_2)$. Now, things work perfectly:

<u>Exercise:</u> If $\tilde{S}(f_1) = \tilde{S}(f_o)$, then $f_1(D)$ and $f_o(D)$ are homeomorphic.

The following diagram is commutative, where i and j are natur injections:

$$\begin{tikzcd} f_1(D) \arrow[r, "i"] \arrow[d, "\rho_1 \wr"] & M_1 \arrow[d, "\rho_1"] \\ f_o(D) \arrow[r, "j"'] & R_o \end{tikzcd}$$

Thus

$$\begin{tikzcd} \pi_1(f_1(D)) \arrow[r, "4"] \arrow[d, "1"] & \pi_1(M_1) \arrow[d, "3"] \\ \pi_1(f_o(D)) \arrow[r, "2"] & \pi_1(R_o) \end{tikzcd}$$

commutes.

Homomorphism (1) is an isomorphism, since it comes from a homeomorphism. Homomorphism(2) is an isomorphism, since R_o is a regular neighborhood of $f_o(D)$. Thus (3) must be an epimorphism for the diagram to commute. But this homomorphism of the double cover maps onto a subgroup of index 2, yielding the desired contradiction.

Thus $\tilde{S}(f_i) \subsetneq \tilde{S}(f_{i-1})$, and since these sets are piecewise linear, we must discard at least one simplex for each storey. This proves Lemma 1.

It would be ideal if in the top storey the map $f_n : D \to M_n$ were an embedding, as occurred in the special cases. Unfortunately, this may not be so, but we can prove something almost as good: ∂R_n consists of a finite number of 2-spheres.

Lemma 2. If a compact, connected 3-manifold-with-boundary, M, has no double cover, then ∂M is a finite number of 2-spheres.

Proof. We must break the proof into seven steps.

1. The manifold M is orientable, as otherwise a double cover could be constructed.
2. Let ∂M^n have components $B_1, B_2, \ldots B_K$. Then each B_i is orientable. This can be shown by orienting each 3-simplex of M by step 1, thus inducing an orientation on B_i.
3. The group $H_1(M)$ is finite. If not, then $H_1(M)$ would have a Z factor. Thus, we could construct an epimorphism of $H_1(M)$ to Z_2. Thus, we may carry $\pi_1(M)$ to $H_1(M)$ by abelianizing, then onto Z_2 by the above epimorphism. But the composition $\pi_1(M) \to Z_2$ gives us a double cover,

contrary to assumption.

Exercise. Give an example of a 3-manifold-with-boundary M for which $\pi_1(M)$ is finite and nontrivial.

4. We denote the Euler Characteristic of M by $\chi(M)$. We form the "double" of M, a 3-manifold without boundary obtained by sewing two copies of M together along their boundaries; the "double" has Euler Characteristic zero, by duality, but also has Euler Characteristic equal to $2\chi(M) - \chi(\partial M)$. Thus,

$$2\chi(M) = \chi(\partial M) .$$

5. $\chi(M) = r_o - r_1 + r_2 - r_3$ where $r_i = \text{rank } (H_i(M))$. We know that $r_o = 1$, as M is connected; that $r_1 = 0$, as $H_1(M)$ is finite; that $r_2 \geq K - 1$, as $B_1, B_2, B_3, \ldots B_{K-1}$ are a set of linearly independent generators of $H_2(M)$; that $r_3 = 0$ as M has a boundary, so admits no fundamental 3-cycle. Combining and simplifying gives

$$\chi(M) \geq K$$

6. Combining steps 4 and 5 yields $\chi(\partial M) \geq 2K$, so

$$\chi(B_1) + \chi(B_2) + \cdots + \chi(B_K) \geq 2K .$$

7. Recall the Euler characteristics of orientable 2-manifolds:

$$\chi(\text{2-sphere}) = 2$$

$$\chi(\text{torus}) = 0$$

$$\chi(\text{double torus}) = -2$$

$$\chi(\text{triple torus}) = -4$$

etc.

This fact together with step 6 shows that B_i must be a 2-sphere for all

i , thus proving Lemma 2 .

Lemma 3. The conclusion of Dehn's Lemma holds in the top storey M_n of the tower.

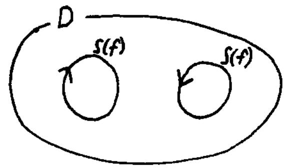
Proof. Since ∂R^n is a collection of 2-spheres, we need only to adjust $f_n(\partial D)$ so that it lies in one of them to establish Lemma 3. Indeed, it seems that $f_n(\partial D)$ lies close to ∂R^n and a little "push" is all that is necessary to make this adjustment. Perhaps this is so, but how to rigorously describe this "push"? To avoid this problem, we go back to the tower construction and change the definition of the regular neighborhood R_i : Let R_i be all simplices in the 2nd barycentric subdivision which intersect $f(\text{Int } D)$. Thus R_i is now defined in a fashion which yields $f(\partial D) \subset \partial R_i$. A glance back at Lemma 1 and Lemma 2 is now necessary to check that this redefinition does not upset any of the arguments there.

Lemma 4. If the conclusion of Dehn's Lemma holds on the $(j+1)$th storey, then it also holds on the jth storey.

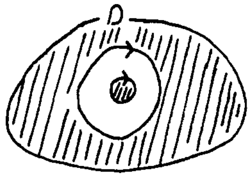
Proof. Let g_{j+1} be our embedding of D in M_{j+1} . Our first approximation to the desired map is the composition $g = \rho_{j+1} \circ g_{j+1}$, carrying D into M_j , but with much self-intersection. At least the singular set of g avoids ∂D , as usual. Why are we better off in this situation than we were at the very beginning of Dehn's Lemma? Because we have two properties now that we lacked previously:

1. g is locally an embedding of D into M_i . (Exercise: Define this property rigorously and verify that g satisfies it)

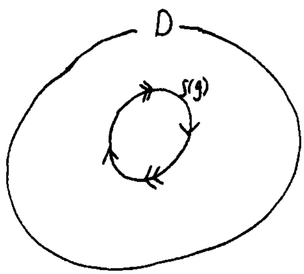
2. S(g) is composed only of <u>double</u> points. Triple points, quadruple points, etc. are gone, since ρ_{j+1} is a double cover. Thus, if g is a general position map, then S(g) will be a 1-manifold. Suppose S(g) looks like two circles paired as indicated by the arrows:



This is exactly the situation discussed in the special case and we can easily remove it. If the curves were concentric



then we could remove them either by removing the annulus between them and sewing it back turned inside out, or by removing this annulus altogether and sewing the two shaded regions together directly. Think about this process! We similarly may remove all pairs of curves in S(g) . The only remaining difficulty is a curve of S(g) in which opposite points are identified under g :



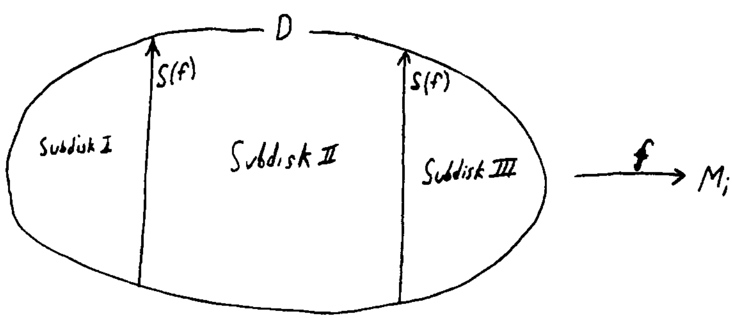
This singularity is removed by removing the subdisk of D bounded by $S(g)$, turning it 180°, then sewing it back in. This is hard to visualise. Can we make such an alteration? Can you describe what a neighbourhood of $S(g)$ looks like?

This completes the proof of Dehn's Lemma. Our next goal is the Loop Theorem, as its proof is basically the same as that of Dehn's Lemma. This theorem tells us that self-intersections may be removed from loops as well as disks.

Loop Theorem. If M is a 3-manifold-with-boundary and $\pi_1(\partial M) \xrightarrow{i_*} \pi_1(M)$ has non-trivial kernel, then there exists an embedding of a disk D in M such that ∂D lies in ∂M, and represents a nontrivial element of $\pi_1(\partial M)$.

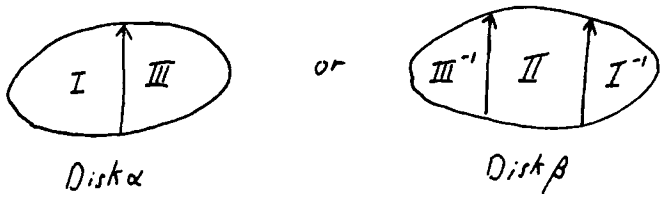
Outline of Proof.

The tower is built as before, and Lemmas 1,2, and 3 proceed identically. In Lemma 4, in addition to singularities we already encountered, we have a new type: pairs of arcs ending on ∂D.

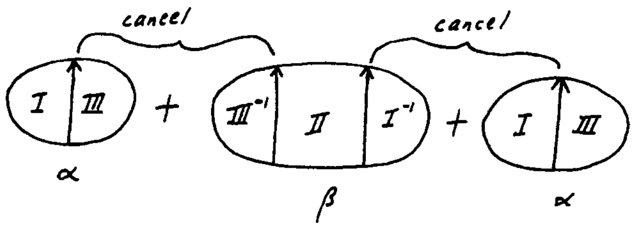


But we have a choice here. We may form a new Disk α by combining

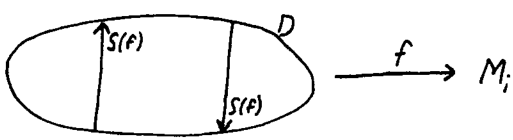
Subdisk I and III, or Disk β by reversing the positions of subdisks I and III , then resewing them to Subdisk II .



If the projection of g all the way down the tower and into ∂M yielded nontrivial element of $\pi_1(\partial M)$, then one of these two alterations will do the same. Because sewing together α and β in the proper way yields our original disk and map g .



<u>Exercise</u>: If the two arcs in S(f) had opposite orientations,



then a similar argument removes the similarities.

This completes our discussion of the Loop Theorem. For more details, see Stallings: On the Loop Theorem, Annals of Mathematics, Vol. 72 pp. 12-19.

APPENDIX C

TABLE OF KNOTS AND LINKS

Compiled by: James Bailey Drawn by: Ali Roth

The following table lists all the prime knots of 10 crossings or less and links of 9 crossings or less.* The basic source is the enumeration by Conway [1970] which lists knots to 11, and links to 10 crossings, but without pictures. (A few errors were corrected and hopefully no new ones were introduced). The ordering is the same as that of the table of Alexander and Briggs [1926] as far as that goes (knots to 9 crossings) and their notation has been extended to provide easy reference.

Conway's notation (which actually describes the knot or link) and the Alexander polynomial are also listed, immediately to the right of the picture. For example:

Extended A & B notation for the 16th link of 2 components and 9 crossings. $\longrightarrow$ 9^2_{16} 311, 2, 2 $\longleftarrow$ Conway's notation

$$\begin{bmatrix} 2 & -5 & 4 & 0 \\ 0 & 4 & -5 & 2 \end{bmatrix} \longleftarrow$$ Alexander polynomial

For convenience of display, the Alexander polynomial has been abbreviated as follows:

* Trivial and splittable examples are omitted. The completeness of the table and the primeness of the knots have not been established rigorously, so far as I know.

For knots: $[a+b+c+\ldots$ stands for $a+b(x+x^{-1}) + c(x^2+x^{-2})+$ As an example, for the knot 10_{20}, $[11-9+3$ means that the Alexander polynomial is $\Delta(x) = 3x^{-2} - 9x^{-1} + 11 - 9x + 3x^2$, or equivalently $3 - 9x + 11x^2 - 9x^3 + 3x^4$.

For 2-component links: A matrix is given so that if one numbers the columns from left to right and rows from bottom to top, starting with zero, the entry in the i^{th} column, j^{th} row is the coefficient of $x^i y^j$ in the Alexander polynomial $\Delta(x,y)$. Thus the link 9^2_{26} has polynomial

$$\begin{bmatrix} -2 & 6 & -4 & 1 \\ 1 & -4 & 6 & -2 \end{bmatrix} \qquad \Delta(x,y) = \begin{matrix} -2y + 6xy - 4x^2y + x^3y \\ +1 - 4x + 6x^2 - 2x^3 \end{matrix}$$

For 3-component links: In the range of this table it turns out that all the Alexander polynomials are of the form $\Delta(x,y,z) = P(x,y) - \bar{P}(x,y)z$, where P and $\bar{P}$ are related polynomials. The matrix displayed gives P by the same convention as above. To find $\bar{P}$, number the columns from right to left and rows from top to bottom, starting with zero, and the entry in the i^{th} column, j^{th} row gives the coefficient of $x^i y^j$ in $\bar{P}(x,y)$. As an example, for the link 9^3_1 the matrix

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & -2 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

signifies that the Alexander polynomial is

$$\Delta(x,y,z) = \begin{Bmatrix} & + & xy^2 & - & x^2y^2 \\ +y & - & 2xy & + & 2x^2y \\ -1 & + & 2x & - & x^2 \end{Bmatrix} - \begin{Bmatrix} & + & xz & - & z \\ +x^2yz & - & 2xyz & + & 2yz \\ -x^2y^2z & + & 2xy^2z & - & y^2z \end{Bmatrix}$$

<u>For 4-component links</u>: There are only four of these with 9 or fewer crossings, and their Alexander polynomials happen to have the special form

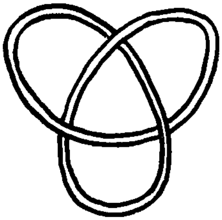
$$\Delta(x,y,z,w) = P(x,y) + Q(x,y)z + \bar{Q}(x,y)w + \bar{P}(x,y)zw$$

Two matrices, A and B, are given in the table. One computes P, $\bar{P}$ from A, and Q, $\bar{Q}$ from B in the same manner as above. Example: The link 8^4_1 has matrices $\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, and so

$$\Delta(x,y,z,w) = \begin{Bmatrix} -y + xy \\ -x \end{Bmatrix} + \begin{Bmatrix} 2y - xy \\ -1 + x \end{Bmatrix} z + \begin{Bmatrix} 2x - 1 \\ -xy + y \end{Bmatrix} w + \begin{Bmatrix} -x + 1 \\ -y \end{Bmatrix} zw$$

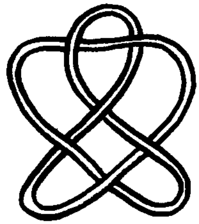
$$= -x - y - z - w + xy + xz + 2xw + 2yz + yw + zw - xyz - xyw - xzw - yzw$$

The determinant $|\Delta(-1)|$ of a knot can easily be computed from $[a + b + c + \ldots,$ being simply $a - 2b + 2c - \ldots$ Other invariants, such as signature, Minkowski units, Conway's potential function, and symmetries can be found in Conway [1969]. One can find the second and third torsion numbers for knots (up to 9 crossings) in Reidemeister [1948].



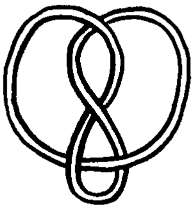
3_1 3

[1-1



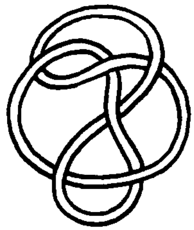
6_2 312

[3-3+1



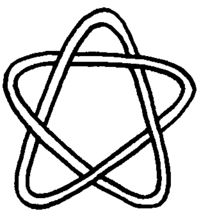
4_1 22

[3-1



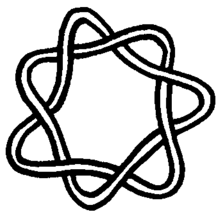
6_3 2112

[5-3+1



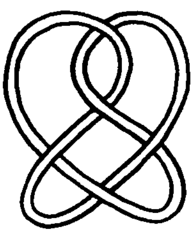
5_1 5

[1-1+1



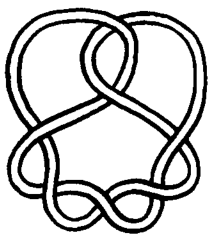
7_1 7

[1-1+1-1



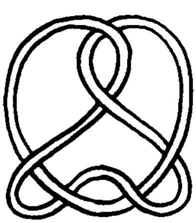
5_2 32

[3-2



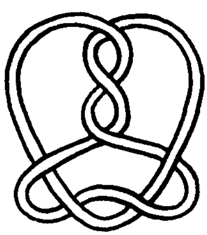
7_2 52

[5-3



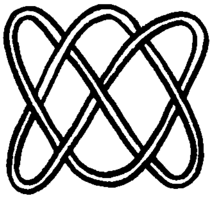
6_1 42

[5-2



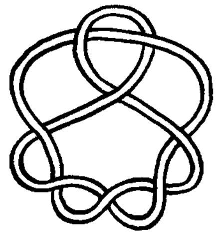
7_3 43

[3-3+2



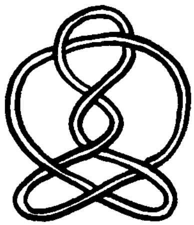
7_4 313

[7-4



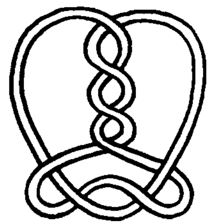
8_2 512

[3-3+3-1



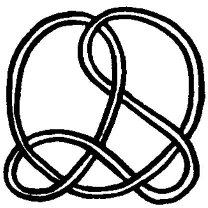
7_5 322

[5-4+2



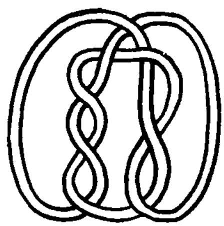
8_3 44

[9-4



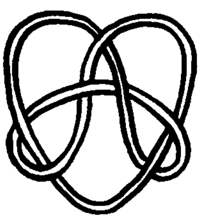
7_6 2212

[7-5+1



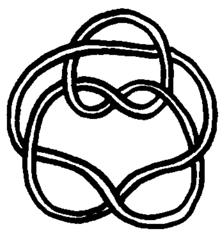
8_4 413

[5-5+2



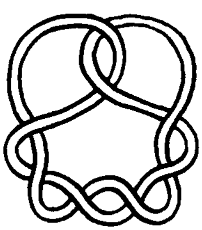
7_7 21112

[9-5+1



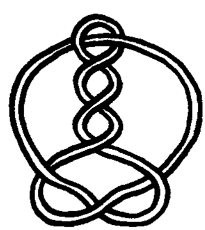
8_5 3,3,2

[5-4+3-1



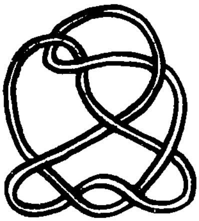
8_1 62

[7-3



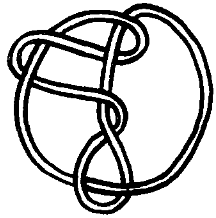
8_6 332

[7-6+2



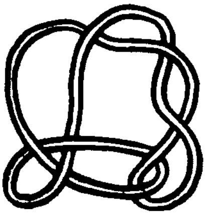
8_7 4112

[5-5+3-1



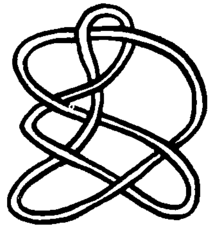
8_{12} 2222

[13-7+1



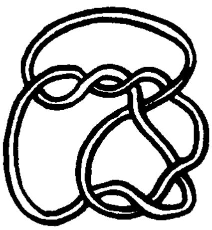
8_8 2312

[9-6+2



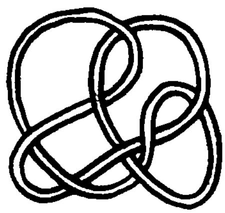
8_{13} 31112

[11-7+2



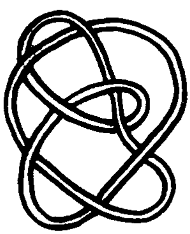
8_9 3113

[7-5+3-1



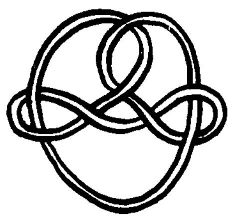
8_{14} 22112

[11-8+2



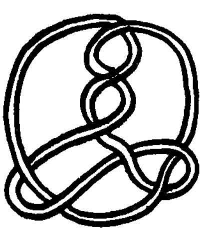
8_{10} 3,21,2

[7-6+3-1



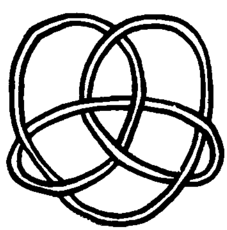
8_{15} 21,21,2

[11-8+3



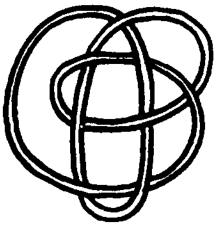
8_{11} 3212

[9-7+2



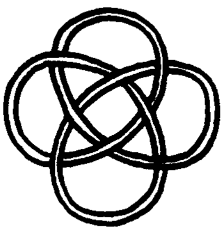
8_{16} .2.20

[9-8+4-1



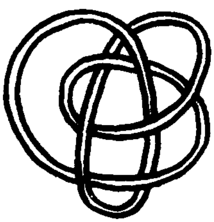
8_{17} .2.2

[11-8+4-1



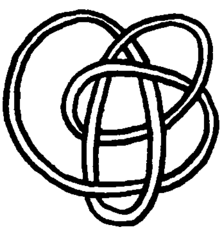
8_{18} 8^{*}.

[13-10+5-1



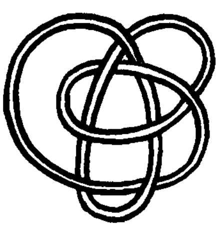
8_{19} 3,3,2-

[1+0-1+1



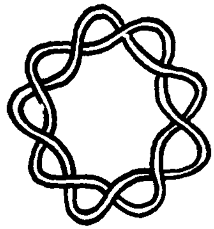
8_{20} 3,21,2-

[3-2+1



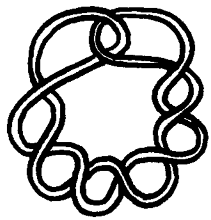
8_{21} 21,21,2-

[5-4+1



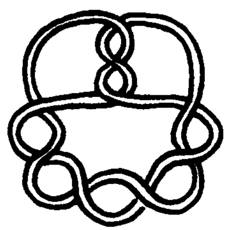
9_{1} 9

[1-1+1-1+1



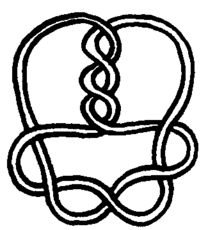
9_{2} 72

[7-4



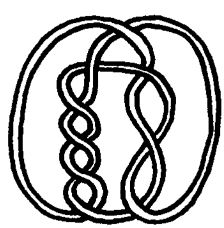
9_{3} 63

[3-3+3-2



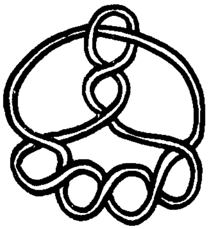
9_{4} 54

[5-5+3



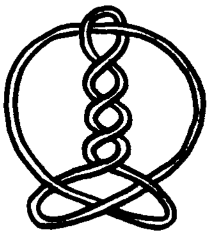
9_{5} 513

[11-6



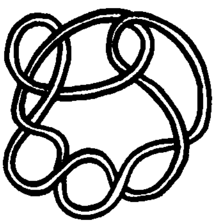
9_6 522

[5-5+4-2



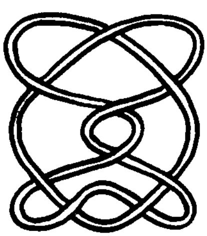
9_7 342

[9-7+3



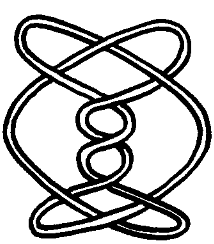
9_8 2412

[11-8+2



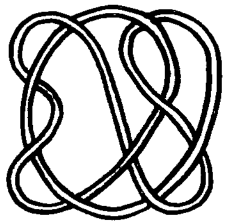
9_9 423

[7-6+4-2



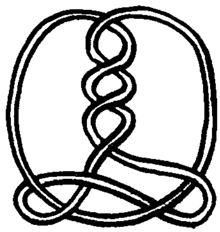
9_{10} 333

[9-8+4



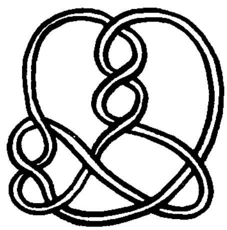
9_{11} 4122

[7-7+5-1



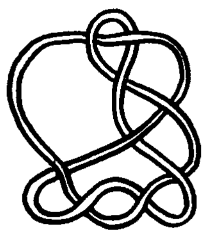
9_{12} 4212

[13-9+2



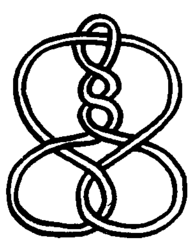
9_{13} 3213

[11-9+4



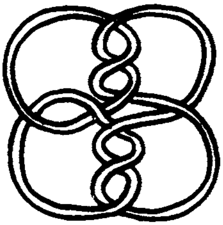
9_{14} 41112

[15-9+2



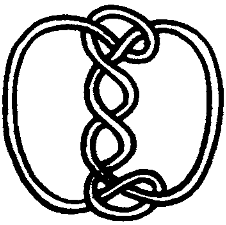
9_{15} 2322

[15-10+2



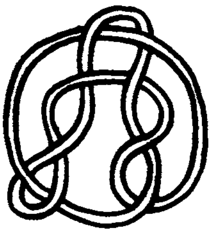
9_{16} 3,3,2+

[9-8+5-2



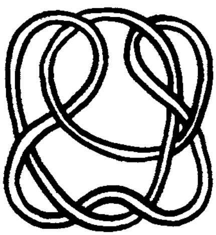
9_{17} 21312

[9-9+5-1



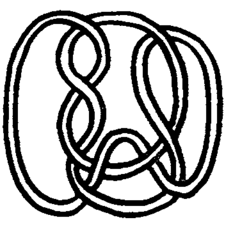
9_{18} 3222

[13-10+4



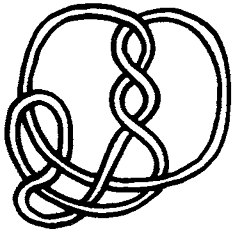
9_{19} 23112

[17-10+2



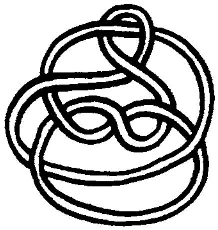
9_{20} 31212

[11-9+5-1



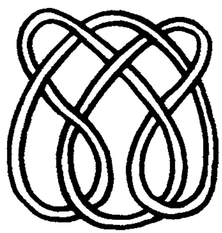
9_{21} 31122

[17-11+2



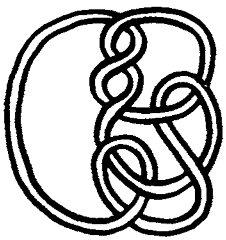
9_{22} 211,3,2

[11-10+5-1



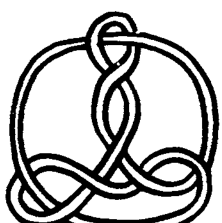
9_{23} 22122

[15-11+4



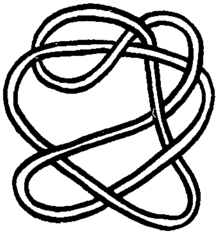
9_{24} 3,21,2+

[13-10+5-1



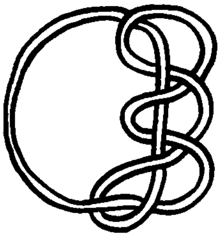
9_{25} 22,21,2

[17-12+3



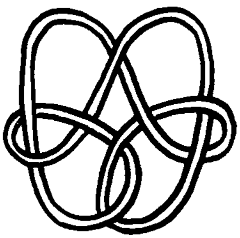
9_{26} 311112

[13-11+5-1



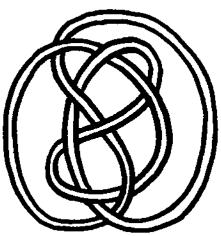
9_{27} 212112

[15-11+5-1



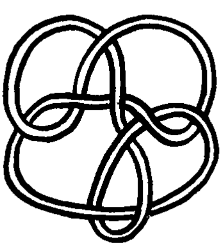
9_{28} 21,21,2+

[15-12+5-1



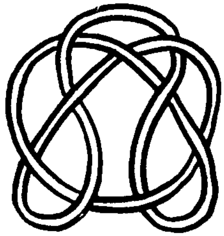
9_{29} .2.20.2

[15-12+5-1



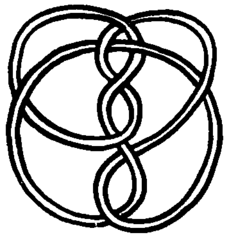
9_{30} 211,21,2

[17-12+5-1



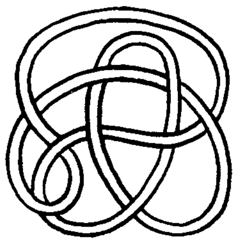
9_{31} 2111112

[17-13+5-1



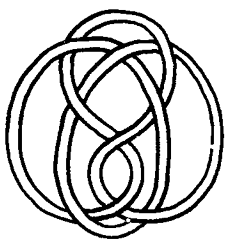
9_{32} .21.20

[17-14+6-1



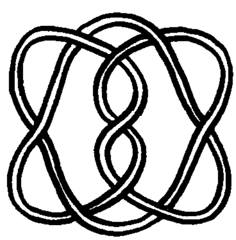
9_{33} .21.2

[19-14+6-1



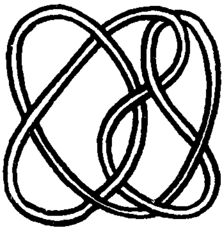
9_{34} 8^*20

[23-16+6-1



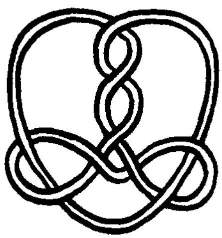
9_{35} 3,3,3

[13-7



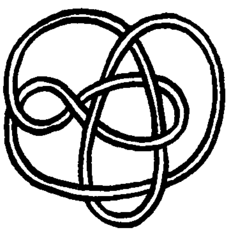
9_{36} 22,3,2

[9-8+5-1



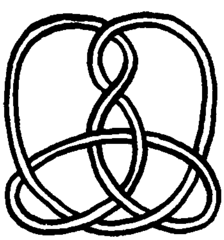
9_{37} 3,21,21

[19-11+2



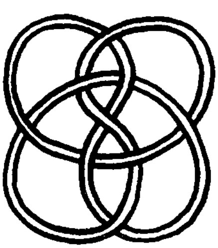
9_{38} .2.2.2

[19-14+5



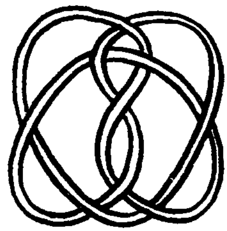
9_{39} 2:2:20

[21-14+3



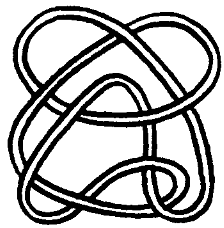
9_{40} 9^*

[23-18+7-1



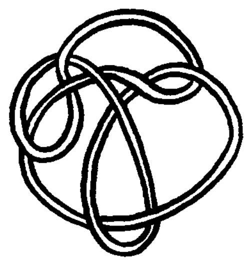
9_{41} 20:20:20

[19-12+3



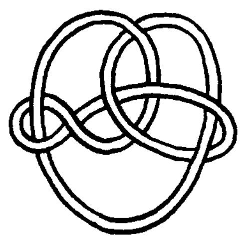
9_{42} 22,3,2-

[1-2+1



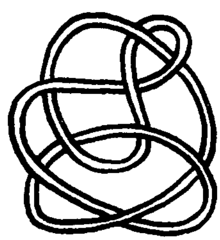
9_{43} 211,3,2-

[1-2+3-1



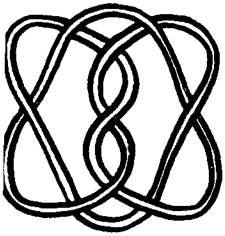
9_{44} 22,21,2-

[7-4+1



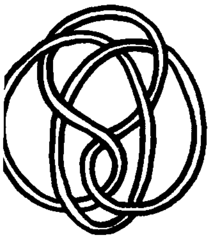
9_{45} 211,21,2-

[9-6+1



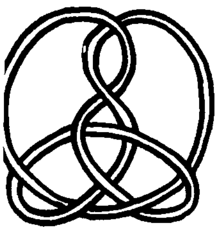
9_{46} 3,3,21-

[5-2



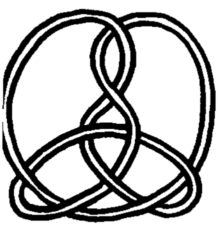
9_{47} 8^{*}-20

[5-6+4-1



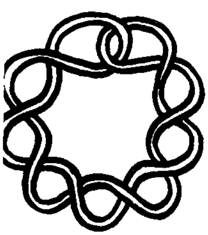
9_{48} 21,21,21-

[11-7+1



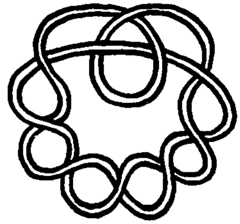
9_{49} -20:-20:-20

[7-6+3



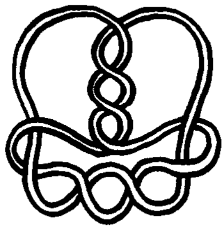
10_{1} 82

[9-4



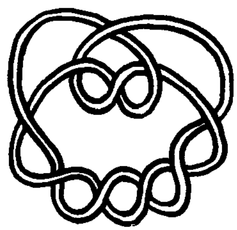
10_{2} 712

[3-3+3-3+1



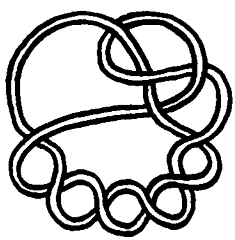
10_{3} 64

[13-6



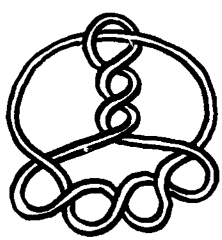
10_{4} 613

[7-7+3



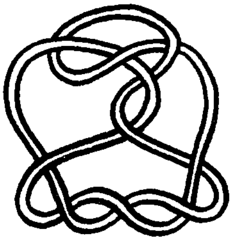
10_{5} 6112

[5-5+5-3+1



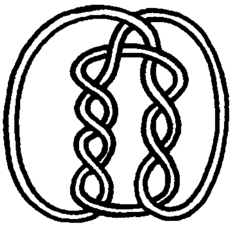
10_{6} 532

[7-7+6-2



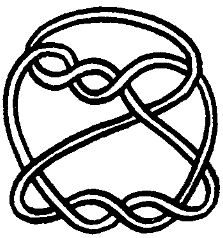
10_7 5212

[15-11+3



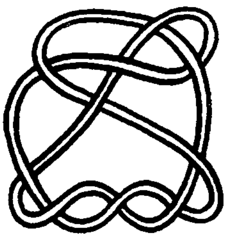
10_8 514

[5-5+5-2



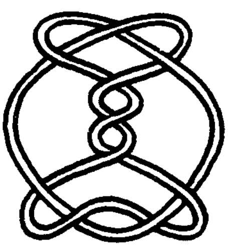
10_9 5113

[7-7+5-3+1



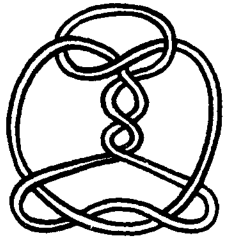
10_{10} 51112

[17-11+3



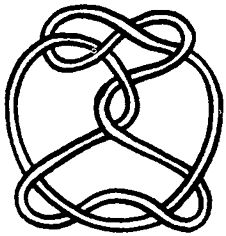
10_{11} 433

[13-11+4



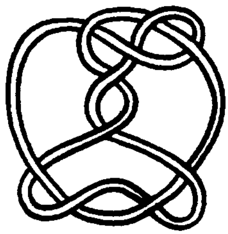
10_{12} 4312

[11-10+6-2



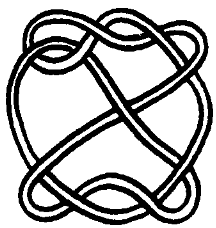
10_{13} 4222

[23-13+2



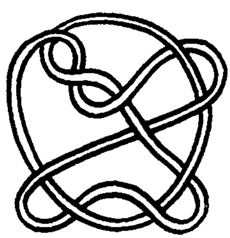
10_{14} 42112

[13-12+8-2



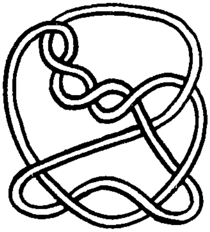
10_{15} 4132

[9-9+6-2



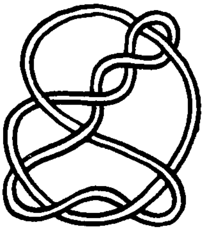
10_{16} 4123

[15-12+4



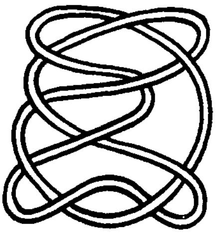
10_{17} 4114

[9-7+5-3+1



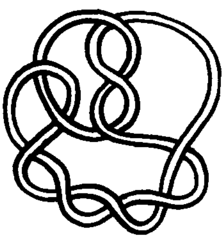
10_{18} 41122

[19-14+4



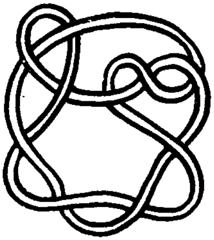
10_{19} 41113

[11-11+7-2



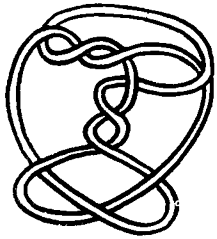
10_{20} 352

[11-9+3



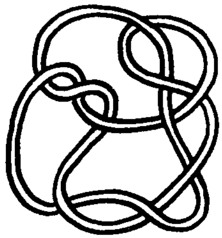
10_{21} 3412

[9-9+7-2



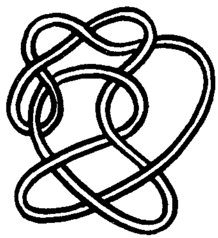
10_{22} 3313

[13-10+6-2



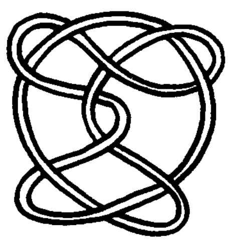
10_{23} 33112

[15-13+7-2



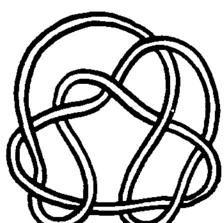
10_{24} 3232

[19-14+4



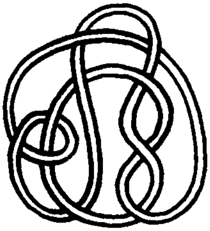
10_{25} 32212

[17-14+8-2



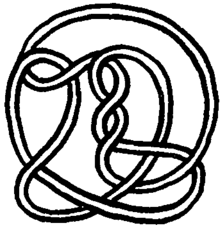
10_{26} 32113

[17-13+7-2



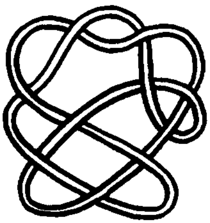
10_{27} 321112

[19-16+8-2



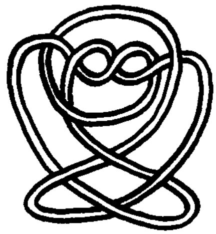
10_{32} 311122

[19-15+8-2



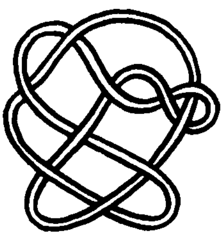
10_{28} 31312

[19-13+4



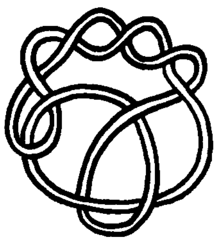
10_{33} 311113

[25-16+4



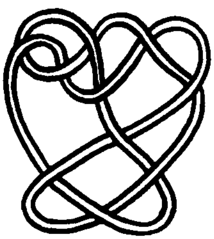
10_{29} 31222

[17-15+7-1



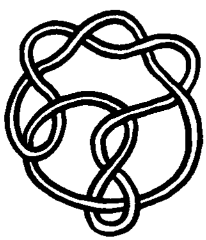
10_{34} 2512

[13-9+3



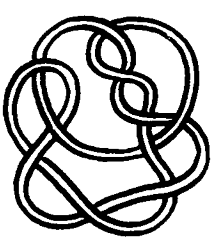
10_{30} 312112

[25-17+4



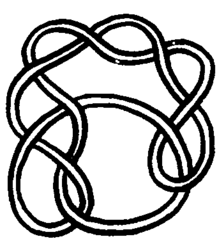
10_{35} 2422

[21-12+2



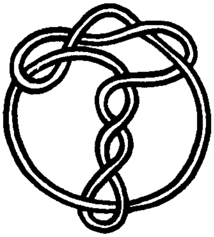
10_{31} 31132

[21-14+4



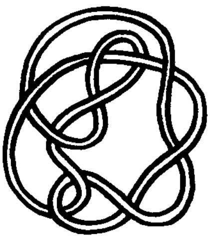
10_{36} 24112

[19-13+3



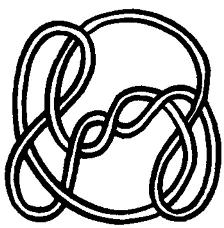
10_{37} 2332

[19-13+4



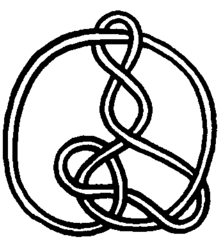
10_{38} 23122

[21-15+4



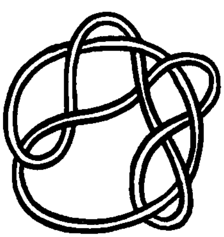
10_{39} 22312

[15-13+8-2



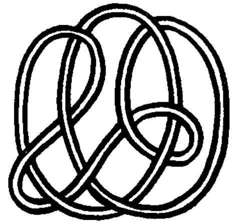
10_{40} 222112

[21-17+8-2



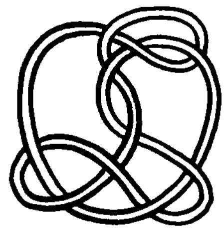
10_{41} 221212

[21-17+7-1



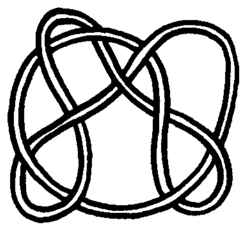
10_{42} 2211112

[27-19+7-1



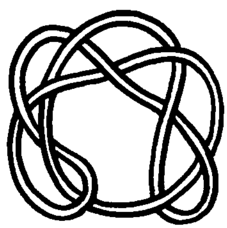
10_{43} 212212

[23-17+7-1



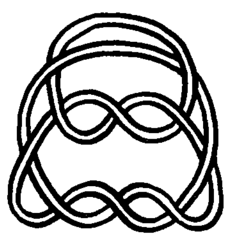
10_{44} 2121112

[25-19+7-1



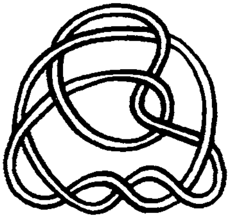
10_{45} 21111112

[31-21+7-1



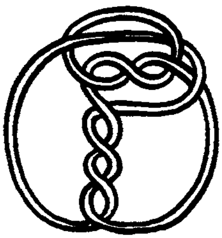
10_{46} 5,3,2

[5-5+4-3+1



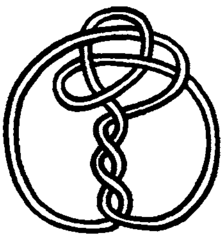
10_{47} 5,21,2

[7-7+6-3+1



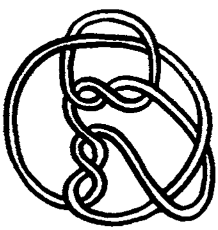
10_{48} 41,3,2

[11-9+6-3+1



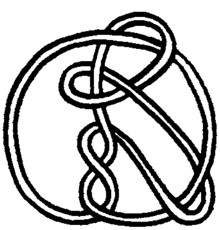
10_{49} 41,21,2

[13-12+8-3



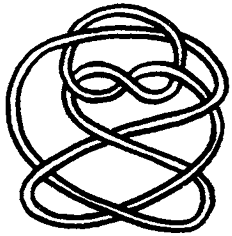
10_{50} 32,3,2

[13-11+7-2



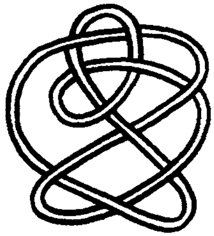
10_{51} 32,21,2

[19-15+7-2



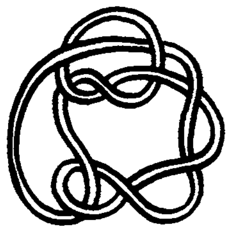
10_{52} 311,3,2

[15-13+7-2



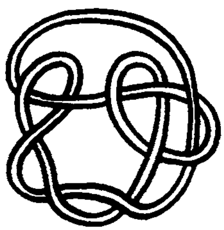
10_{53} 311,21,2

[25-18+6



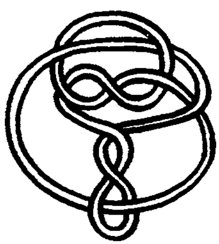
10_{54} 23,3,2

[11-10+6-2



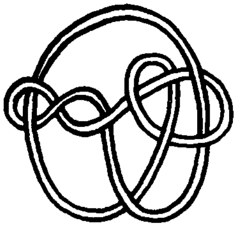
10_{55} 23,21,2

[21-15+5



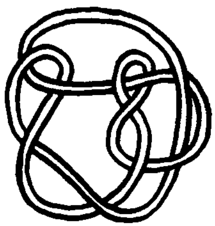
10_{56} 221,3,2

[17-14+8-2



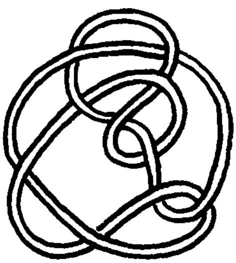
10_{57} 221,21,2

[23-18+8-2



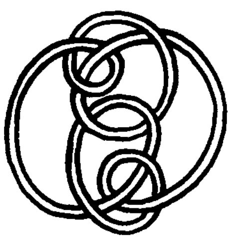
10_{58} 22,22,2

[27-16+3



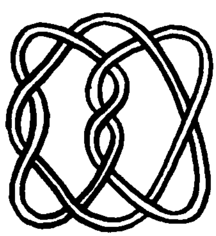
10_{59} 22,211,2

[23-18+7-1



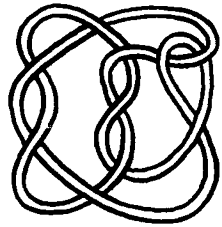
10_{60} 211,211,2

[29-20+7-1



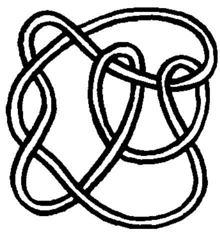
10_{61} 4,3,3

[7-6+5-2



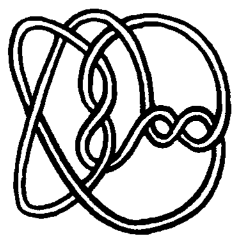
10_{62} 4,3,21

[9-8+6-3+1



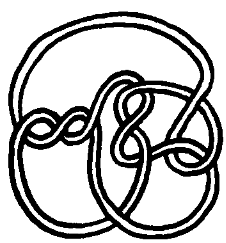
10_{63} 4,21,21

[19-14+5



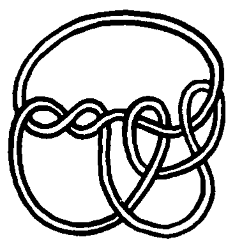
10_{64} 31,3,3

[11-10+6-3+1



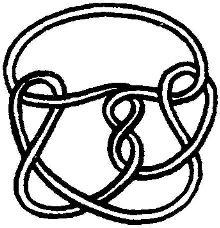
10_{65} 31,3,21

[17-14+7-2



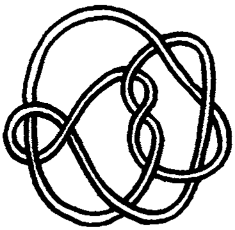
10_{66} 31,21,21

[19-16+9-3



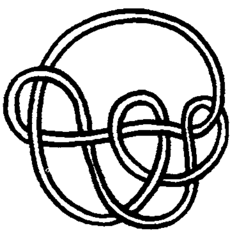
10_{67} 22,3,21

[23-16+4



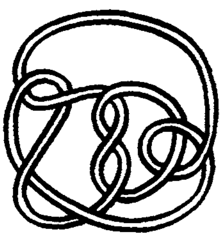
10_{68} 211,3,3

[21-14+4



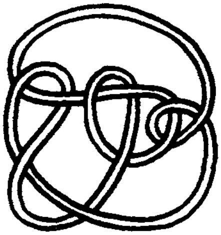
10_{69} 211,21,21

[29-21+7-1



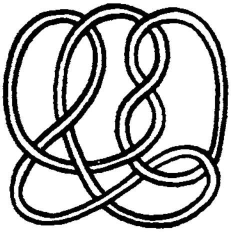
10_{70} 22,3,2+

[19-16+7-1



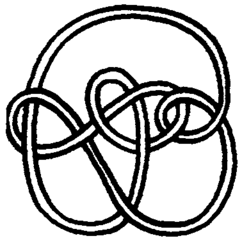
10_{71} 22,21,2+

[25-18+7-1



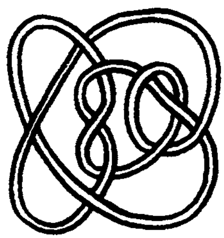
10_{72} 211,3,2+

[19-16+9-2



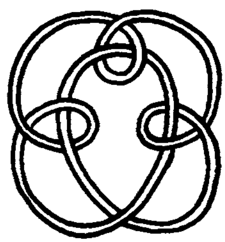
10_{73} 211,21,2+

[27-20+7-1



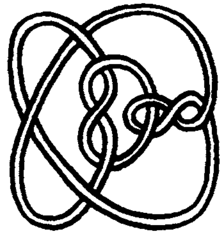
10_{74} 3,3,21+

[23-16+4



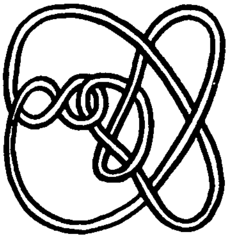
10_{75} 21,21,21+

[27-19+7-1



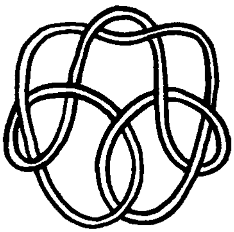
10_{76} 3,3,2++

[15-12+7-2



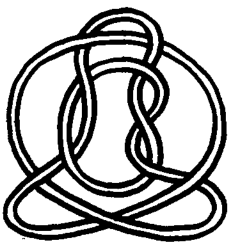
10_{77} 3,21,2++

[17-14+7-2



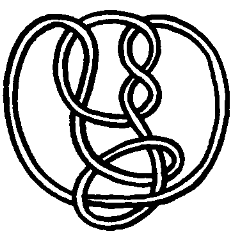
10_{78} 21,21,2++

[21-16+7-1



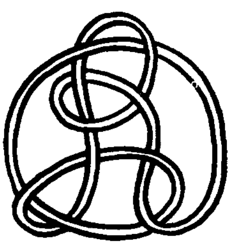
10_{79} (3,2)(3,2)

[15-12+7-3+1



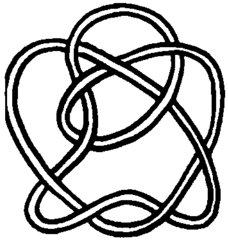
10_{80} (3,2)(21,2)

[17-15+9-3



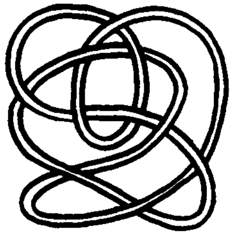
10_{81} (21,2)(21,2)

[27-20+8-1



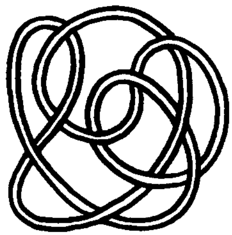
10_{82} .4.2

[13-12+8-4+1



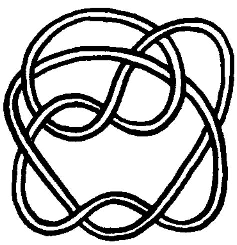
10_{83} .31.20

[23-19+9-2



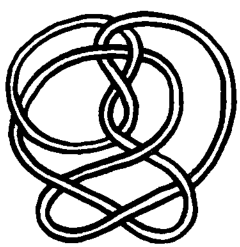
10_{84} .22.2

[25-20+9-2



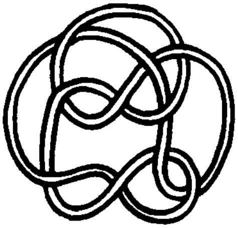
10_{85} .4.20

[11-10+8-4+1



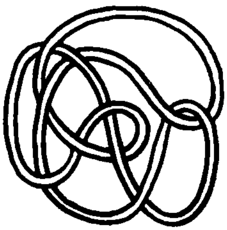
10_{86} .31.2

[25-19+9-2



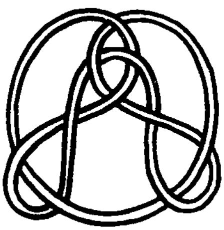
10_{87} .22.20

[23-18+9-2



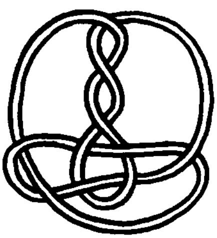
10_{88} .21.21

[35-24+8-1



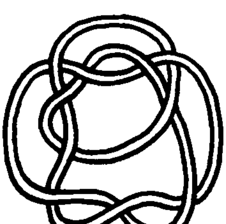
10_{89} .21.210

[33-24+8-1



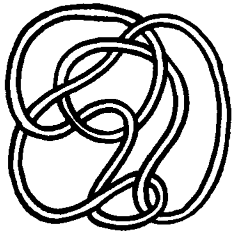
10_{90} .3.2.2

[23-17+8-2



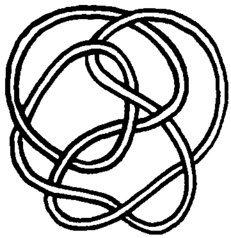
10_{91} .3.2.20

[17-14+9-4+1



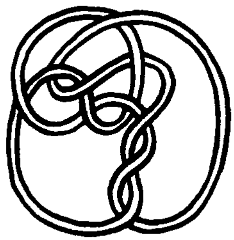
10_{92} .21.2.2

[25-20+10-2



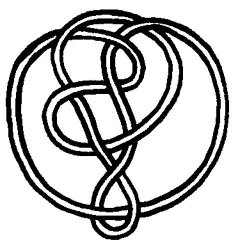
10_{93} .3.20.2

[17-15+8-2



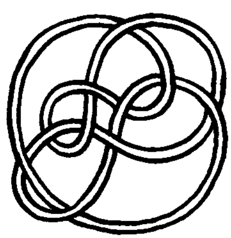
10_{94} .30.2.2

[15-14+9-4+1



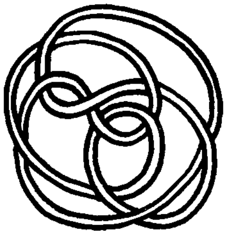
10_{95} .210.2.2

[27-21+9-2



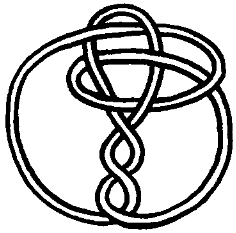
10_{96} .2.21.2

[33-22+7-1



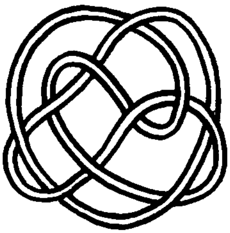
10_{97} .2.210.2

[33-22+5



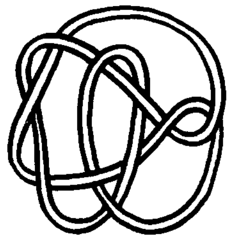
10_{102} 3:2:20

[21-16+8-2



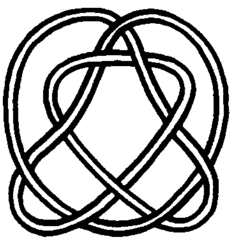
10_{98} .2.2.2.20

[23-18+9-2



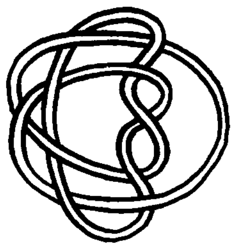
10_{103} 30:2:2

[21-17+8-2



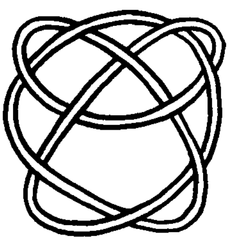
10_{99} .2.2.20.20

[19-16+10-4+1



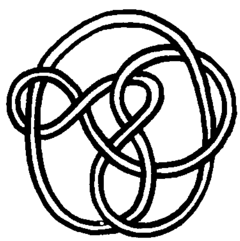
10_{104} 3:20:20

[19-15+9-4+1



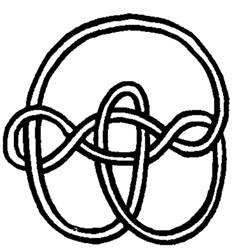
10_{100} 3:2:2

[13-12+9-4+1



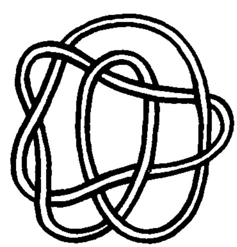
10_{105} 21:20:20

[29-22+8-1



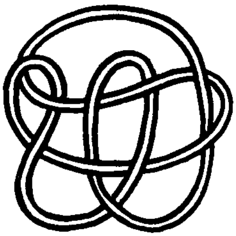
10_{101} 21:2:2

[29-21+7



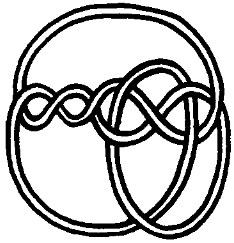
10_{106} 30:2:20

[17-15+9-4+1



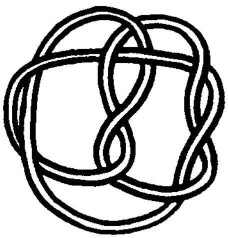
10_{107} 210:2:20

[31-22+8-1



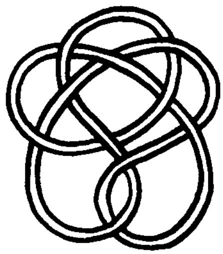
10_{112} $8^{*}3$

[19-17+11-5+1



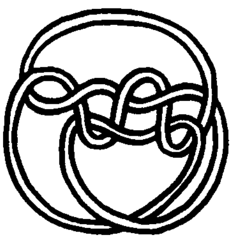
10_{108} 30:20:20

[15-14+8-2



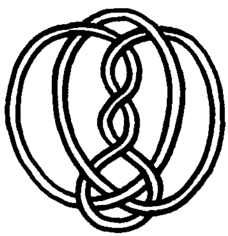
10_{113} $8^{*}21$

[33-26+11-2



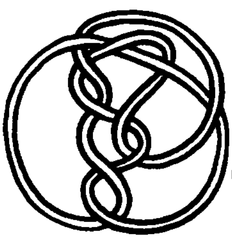
10_{109} 2.2.2.2

[21-17+10-4+1



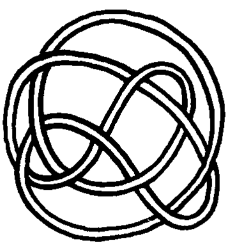
10_{114} $8^{*}30$

[27-21+10-2



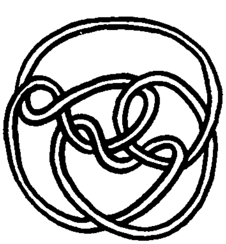
10_{110} 2.2.2.20

[25-20+8-1



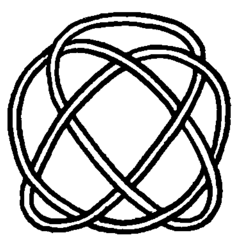
10_{115} $8^{*}20.2$

[37-26+9-1



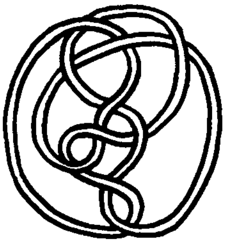
10_{111} 2.2.20.2

[21-17+9-2



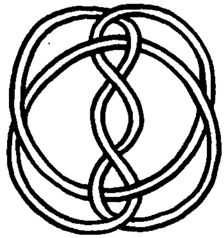
10_{116} 8*2:2

[21-19+12-5+1



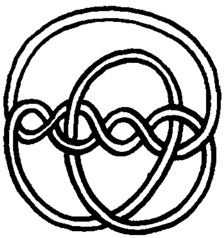
10_{117} $8^*2:20$

[31-24+10-2



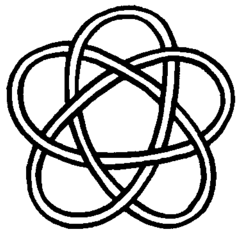
10_{122} 9^*20

[31-24+11-2



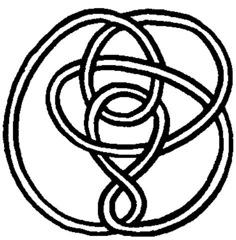
10_{118} $8^*2:.2$

[23-19+12-5+1



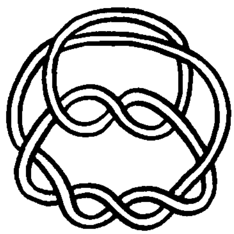
10_{123} 10^*

[29-24+15-6+1



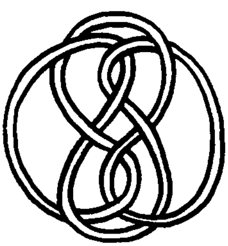
10_{119} $8^*2:.20$

[31-23+10-2



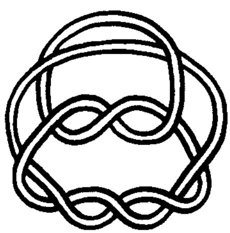
10_{124} 5,3,2-

[1-1+0+1-1



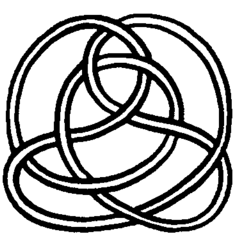
10_{120} $8^*20::20$

[37-26+8



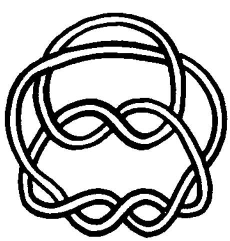
10_{125} 5,21,2-

[1-2+2-1



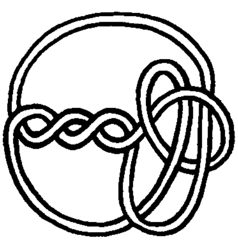
10_{121} 9^*20

[35-27+11-2



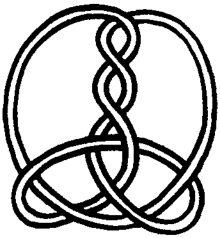
10_{126} 41,3,2-

[5-4+2-1



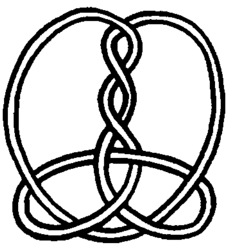
10_{127} 41,21,2-

[7-6+4-1



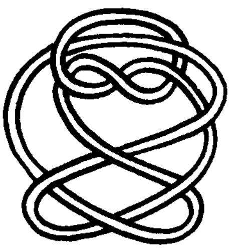
10_{128} 32,3,2-

[1+1-3+2



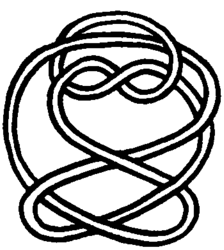
10_{129} 32,21,2-

[9-6+2



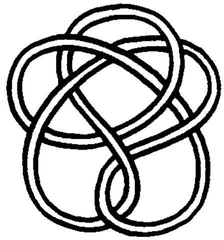
10_{130} 311,3,2-

[5-4+2



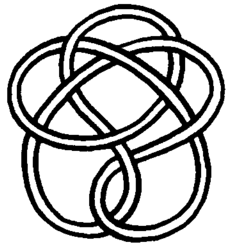
10_{131} 311,21,2-

[11-8+2



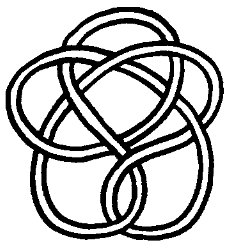
10_{132} 23,3,2-

[1-1+1



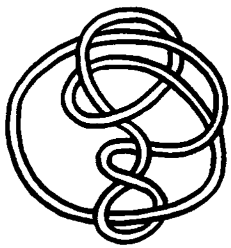
10_{133} 23,21,2-

[7-5+1



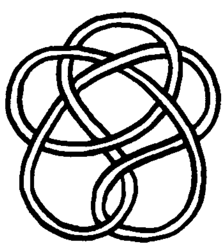
10_{134} 221,3,2-

[3-4+4-2



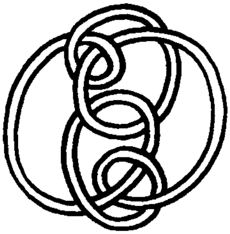
10_{135} 221,21,2-

[13-9+3



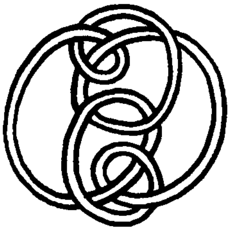
10_{136} 22,22,2-

[5-4+1



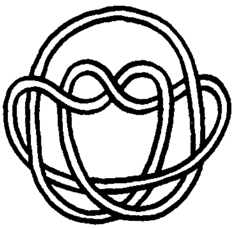
10_{137} 22,211,2-

[11-6+1



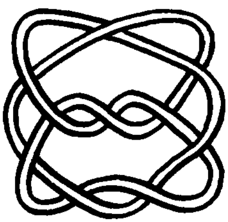
10_{138} 211,211,2-

[7-8+5-1



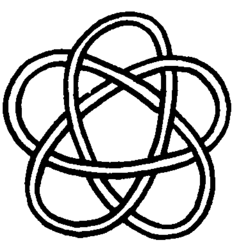
10_{139} 4,3,3-

[3-2+0+1-1



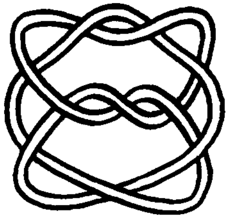
10_{140} 4,3,21-

[3-2+1



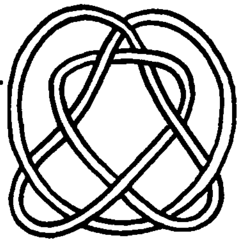
10_{141} 4,21,21-

[5-4+3-1



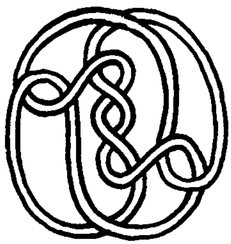
10_{142} 31,3,3-

[1-2+3-2



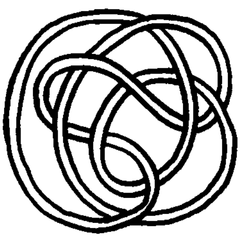
10_{143} 31,3,21-

[7-6+3-1



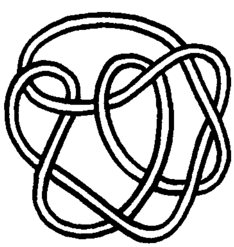
10_{144} 31,21,21-

[13-10+3



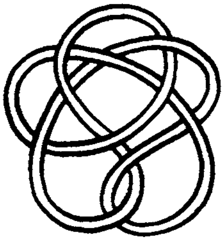
10_{145} 22,3,3-

[3-1-1



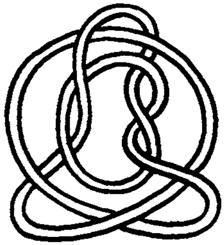
10_{146} 22,21,21-

[13-8+2



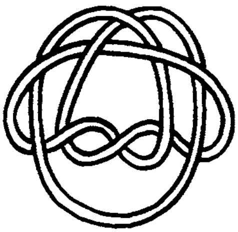
10_{147} 211,3,21-

[9-7+2



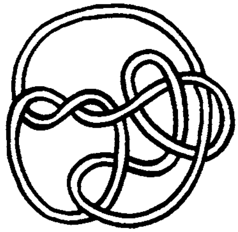
10_{152} (3,2)-(3,2)

[5-4+1+1-1



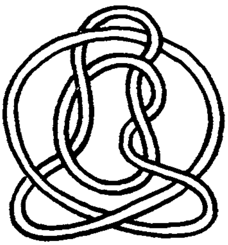
10_{148} (3,2)(3,2-)

[9-7+3-1



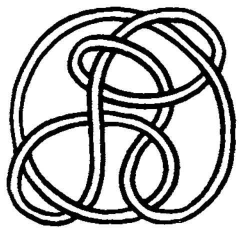
10_{153} (3,2)-(21,2)

[3-1-1+1



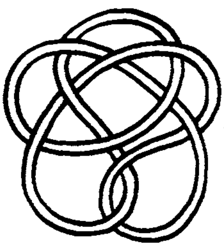
10_{149} (3,2)(21,2-)

[11-9+5-1



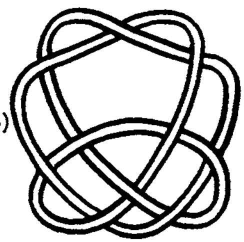
10_{154} (21,2)-(21,2)

[7-4+0+1



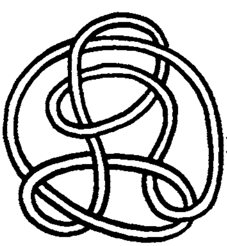
10_{150} (21,2)(3,2-)

[7-6+4-1



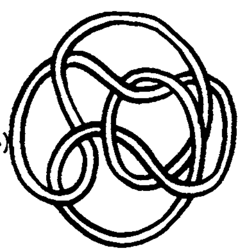
10_{155} -3:2:2

[7-5+3-1



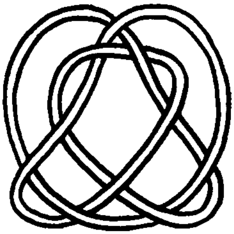
10_{151} (21,2)(21,2-)

[13-10+4-1



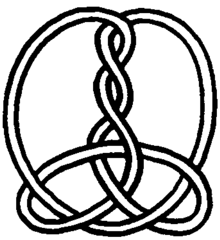
10_{156} -3:2:20

[9-8+4-1



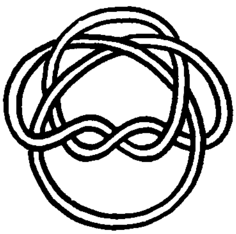
10_{157} -3:20:20

[13-11+6-1



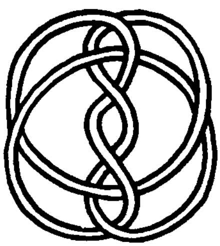
10_{158} -30:2:2

[15-10+4-1



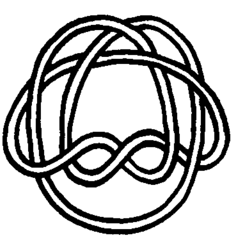
10_{159} -30:2:20

[11-9+4-1



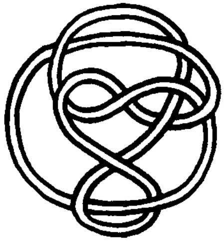
10_{160} -30:20:20

[3-4+4-1



10_{161} 3:-20:-20

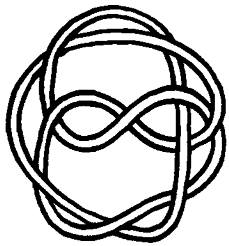
[3-2+0+1



10_{162} 21:-20:-20

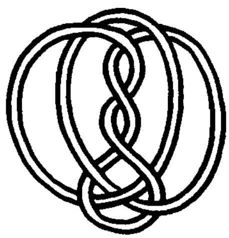
[3-2+0+1

(K. Perko has pointed out that 10_{162} is actually equivalent to 10_{161}.)



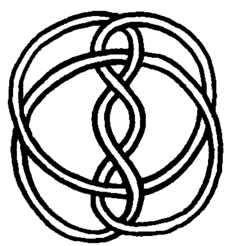
10_{163} -30:-20:-20

[11-9+3



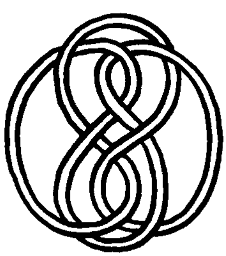
10_{164} 8^*-30

[15-12+5-1



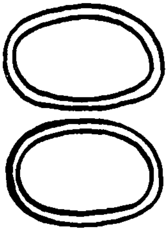
10_{165} 8^*2:-20

[17-11+3



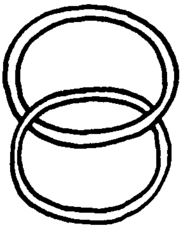
10_{166} 8^*2:.-20

[15-10+2



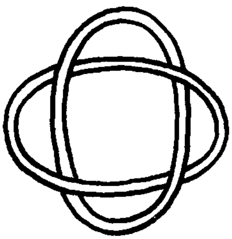
0^2_1 0

$$\begin{bmatrix} 0 \end{bmatrix}$$



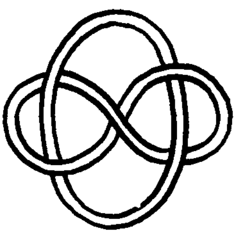
2^2_1 2

$$\begin{bmatrix} 1 \end{bmatrix}$$



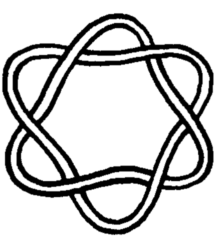
4^2_1 4

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$



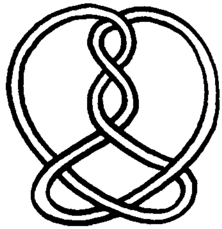
5^2_1 212

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$



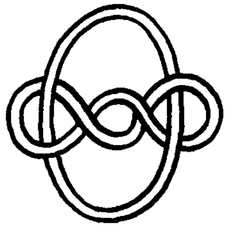
6^2_1 6

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$



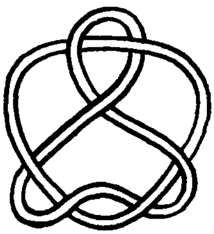
6^2_2 33

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$



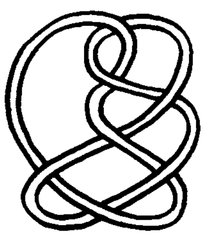
6^2_3 222

$$\begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$



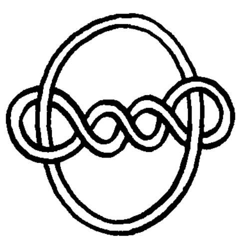
7^2_1 412

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$



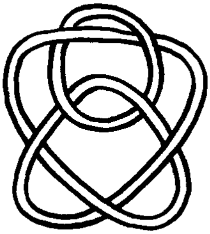
7^2_2 3112

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$



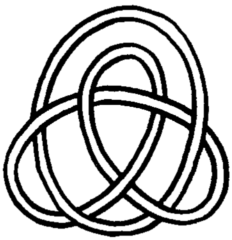
7^2_3 232

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$



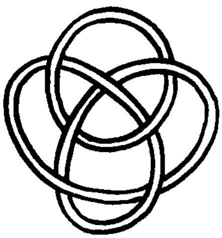
7^2_4 3,2,2

$$\begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$



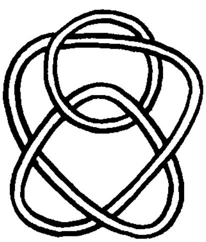
7^2_5 21,2,2

$$\begin{bmatrix} 0 & 2 & -2 & 1 \\ 1 & -2 & 2 & 0 \end{bmatrix}$$



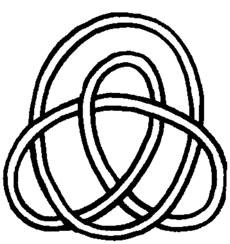
7^2_6 .2

$$\begin{bmatrix} -1 & 2 & -2 & 1 \\ 1 & -2 & 2 & -1 \end{bmatrix}$$



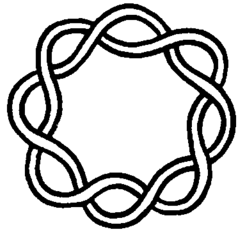
7^2_7 3,2,2-

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



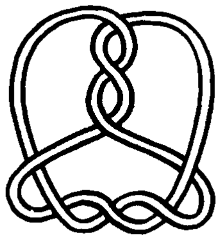
7^2_8 21,2,2-

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$



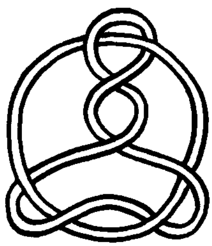
8^2_1 8

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$



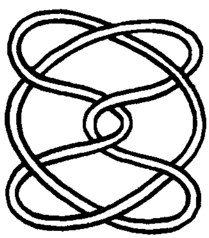
8^2_2 53

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



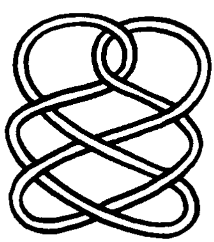
8^2_3 422

$$\begin{bmatrix} 0 & -1 & 2 \\ -1 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$



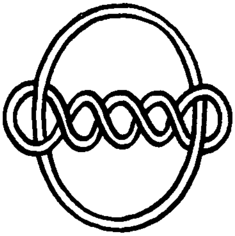
8^2_4 323

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & -2 & 1 \\ 1 & -2 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$



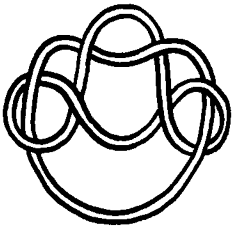
8^2_5 3122

$$\begin{bmatrix} 0 & -2 & 1 \\ -2 & 3 & -2 \\ 1 & -2 & 0 \end{bmatrix}$$



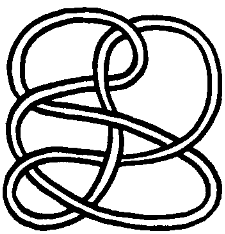
8^2_6 242

$$\begin{bmatrix} -2 & 3 \\ 3 & -2 \end{bmatrix}$$



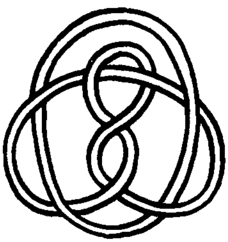
8^2_7 21212

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$



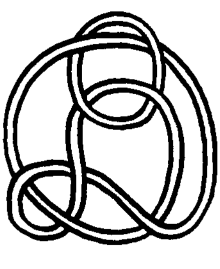
8^2_8 211112

$$\begin{bmatrix} 1 & -2 & 1 \\ -2 & 5 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$



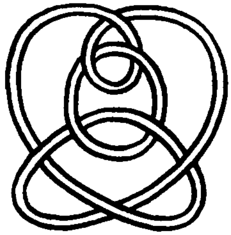
8^2_9 22,2,2

$$\begin{bmatrix} -1 & 4 & -2 & 0 \\ 0 & -2 & 4 & -1 \end{bmatrix}$$



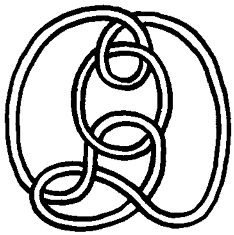
8^2_{10} 211,2,2

$$\begin{bmatrix} -1 & 3 & -3 & 1 \\ 1 & -3 & 3 & -1 \end{bmatrix}$$



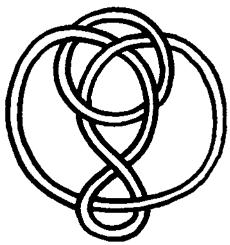
8^2_{11} 3,2,2+

$$\begin{bmatrix} 2 & -2 & 2 & -1 \\ -1 & 2 & -2 & 2 \end{bmatrix}$$



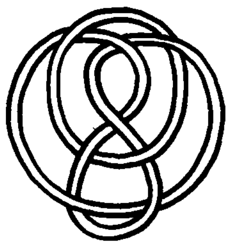
8^2_{12} 21,2,2+

$$\begin{bmatrix} -1 & 3 & -3 & 1 \\ 1 & -3 & 3 & -1 \end{bmatrix}$$



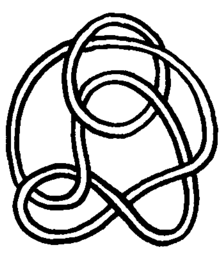
8^2_{13} .21

$$\begin{bmatrix} -1 & 4 & -4 & 1 \\ 1 & -4 & 4 & -1 \end{bmatrix}$$



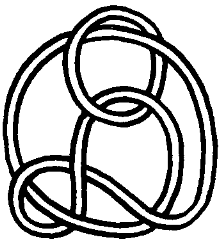
8^2_{14} .2:2

$$\begin{bmatrix} 0 & 4 & -4 & 1 \\ 1 & -4 & 4 & 0 \end{bmatrix}$$



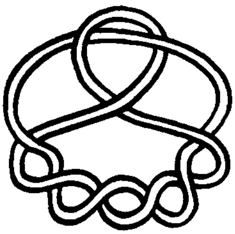
8^2_{15} 22,2,2-

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$



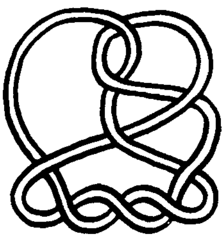
8^2_{16} 211,2,2-

$$\begin{bmatrix} -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$



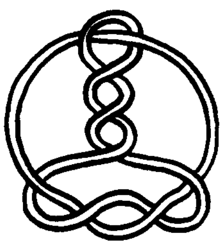
9^2_1 612

$$\begin{bmatrix} 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & -1 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$



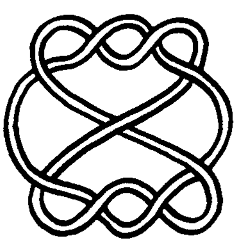
9^2_2 5112

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$



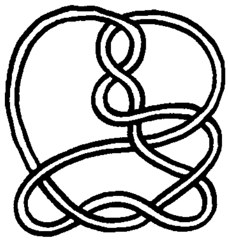
9^2_3 432

$$\begin{bmatrix} 0 & -2 & 2 \\ -2 & 3 & -2 \\ 2 & -2 & 0 \end{bmatrix}$$



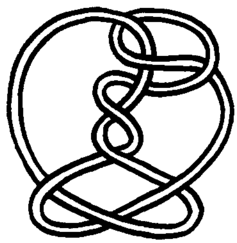
9^2_4 414

$$\begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$



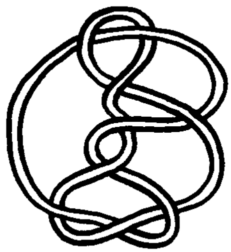
9^2_5 4113

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$



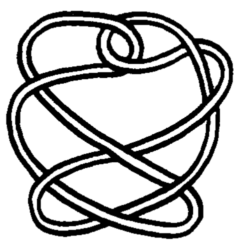
9^2_6 3312

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & -2 & 3 & -1 \\ -1 & 3 & -2 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$



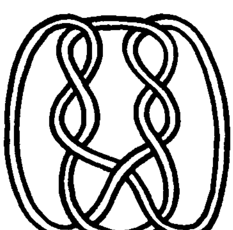
9^2_7 32112

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & -3 & 4 & -1 \\ -1 & 4 & -3 & 1 \\ 0 & -1 & 1 & 0 \end{bmatrix}$$



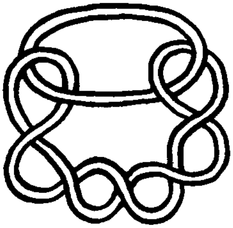
9^2_8 3132

$$\begin{bmatrix} -2 & 2 & 0 \\ 2 & -5 & 2 \\ 0 & 2 & -2 \end{bmatrix}$$



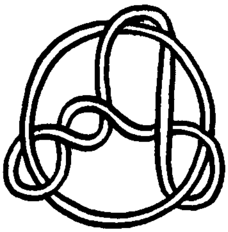
9^2_9 31113

$$\begin{bmatrix} 0 & -1 & 1 & 0 \\ -1 & 3 & -3 & 1 \\ 1 & -3 & 3 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$



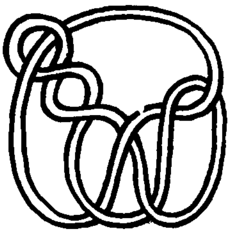
9^2_{10} 252

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$$



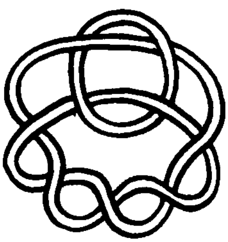
9^2_{11} 22212

$$\begin{bmatrix} 1 & -3 & 2 \\ -3 & 5 & -3 \\ 2 & -3 & 1 \end{bmatrix}$$



9^2_{12} 221112

$$\begin{bmatrix} 2 & -3 & 1 \\ -3 & 7 & -3 \\ 1 & -3 & 2 \end{bmatrix}$$

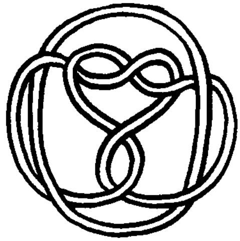


9^2_{13} 5,2,2

$$\begin{bmatrix} -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

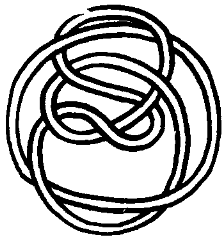
9^2_{14} 41,2,2

$$\begin{bmatrix} 1 & -2 & 2 & -2 & 2 & 0 \\ 0 & 2 & -2 & 2 & -2 & 1 \end{bmatrix}$$



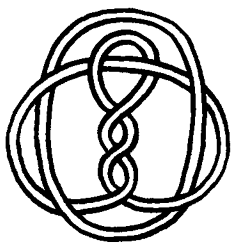
9^2_{15} 32,2,2

$$\begin{bmatrix} -2 & 3 & -3 & 2 \\ 2 & -3 & 3 & -2 \end{bmatrix}$$



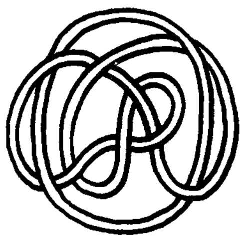
9^2_{16} 311,2,2

$$\begin{bmatrix} 2 & -5 & 4 & 0 \\ 0 & 4 & -5 & 2 \end{bmatrix}$$



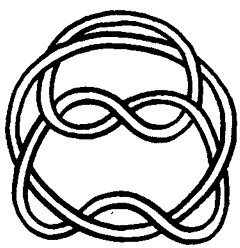
9^2_{17} 23,2,2

$$\begin{bmatrix} 2 & -4 & 3 & 0 \\ 0 & 3 & -4 & 2 \end{bmatrix}$$



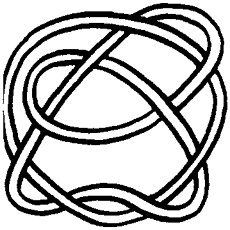
9^2_{18} 221,2,2

$$\begin{bmatrix} -2 & 4 & -4 & 2 \\ 2 & -4 & 4 & -2 \end{bmatrix}$$



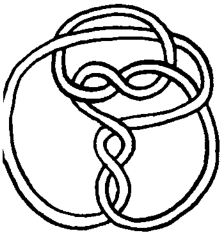
9^2_{19} 4,3,2

$$\begin{bmatrix} -1 & 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & -1 \end{bmatrix}$$



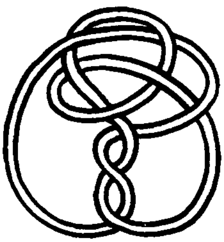
9^2_{20} 4,21,2

$$\left[\begin{matrix}1 & -2 & 2 & 0 & 0\\ 0 & 2 & -3 & 2 & 0\\ 0 & 0 & 2 & -2 & 1\end{matrix}\right.$$



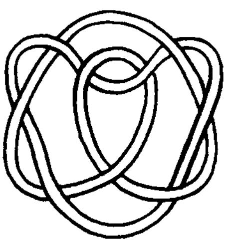
9^2_{21} 31,3,2

$$\left[\begin{matrix}0 & -1 & 1 & -1 & 1\\ -1 & 3 & -3 & 3 & -1\\ 1 & -1 & 1 & -1 & 0\end{matrix}\right.$$



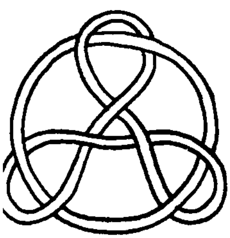
9^2_{22} 31,21,2

$$\left[\begin{matrix}0 & 1 & -2 & 2 & 0\\ 1 & -3 & 5 & -3 & 1\\ 0 & 2 & -2 & 1 & 0\end{matrix}\right.$$



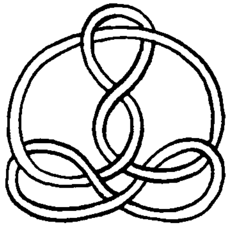
9^2_{23} 3,3,21

$$\left[\begin{matrix}-1 & 1 & 0 & 0\\ 1 & -4 & 2 & 0\\ 0 & 2 & -4 & 1\\ 0 & 0 & 1 & -1\end{matrix}\right.$$



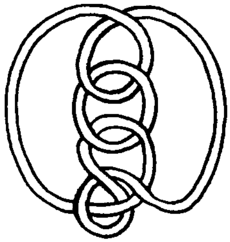
9^2_{24} 21,21,21

$$\left[\begin{matrix}1 & -3 & 3\\ -3 & 7 & -3\\ 3 & -3 & 1\end{matrix}\right.$$



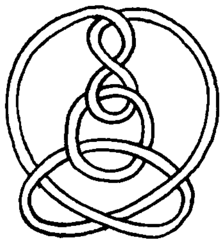
9^2_{25} 22,2,2+

$$\left[\begin{matrix}-1 & 5 & -5 & 1\\ 1 & -5 & 5 & -1\end{matrix}\right.$$



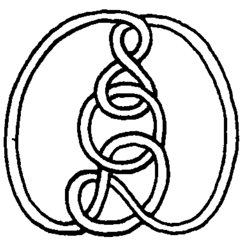
9^2_{26} 211,2,2+

$$\left[\begin{matrix}-2 & 6 & -4 & 1\\ 1 & -4 & 6 & -2\end{matrix}\right.$$



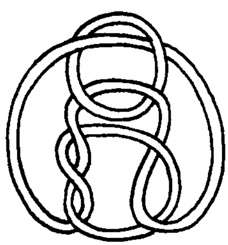
9^2_{27} 3,2,2++

$$\left[\begin{matrix}-2 & 3 & -3 & 2\\ 2 & -3 & 3 & -2\end{matrix}\right.$$



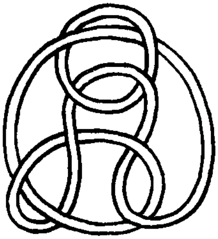
9^2_{28} 21,2,2++

$$\left[\begin{matrix}-2 & 4 & -4 & 1\\ 1 & -4 & 4 & -2\end{matrix}\right.$$



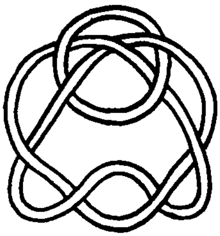
9^2_{29} (3,2)(2,2)

$$\left[\begin{matrix}0 & 2 & -3 & 3 & -2\\ 1 & -2 & 3 & -3 & 2\end{matrix}\right.$$



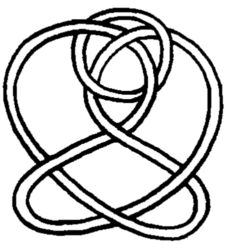
9^2_{30} (21,2)(2,2)

$$\left[\begin{array}{rrrr} 3 & -5 & 4 & -1 \\ -1 & 4 & -5 & 3 \end{array}\right.$$



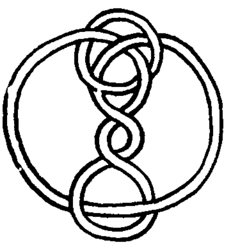
9^2_{31} .4

$$\left[\begin{array}{rrrrrr} -1 & 2 & -2 & 2 & -2 & 1 \\ 1 & -2 & 2 & -2 & 2 & -1 \end{array}\right.$$



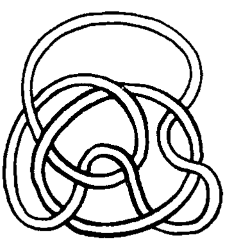
9^2_{32} .31

$$\left[\begin{array}{rrrr} -2 & 5 & -5 & 2 \\ 2 & -5 & 5 & -2 \end{array}\right.$$



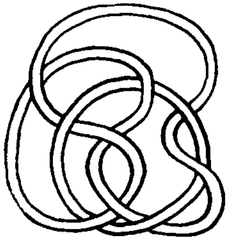
9^2_{33} .22

$$\left[\begin{array}{rrrr} -2 & 5 & -5 & 2 \\ 2 & -5 & 5 & -2 \end{array}\right.$$



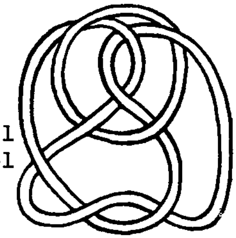
9^2_{34} .3.2

$$\left[\begin{array}{rrrrr} 0 & -1 & 2 & -2 & 1 \\ -1 & 3 & -5 & 3 & -1 \\ 1 & -2 & 2 & -1 & 0 \end{array}\right.$$



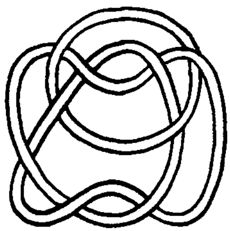
9^2_{35} .3.20

$$\left[\begin{array}{rrrrr} 0 & -1 & 2 & -2 & 1 \\ -1 & 3 & -3 & 3 & -1 \\ 1 & -2 & 2 & -1 & 0 \end{array}\right.$$



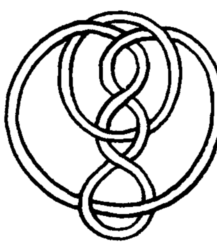
9^2_{36} .3:2

$$\left[\begin{array}{rrrr} -2 & 4 & -4 & 2 \\ 2 & -4 & 4 & -2 \end{array}\right.$$



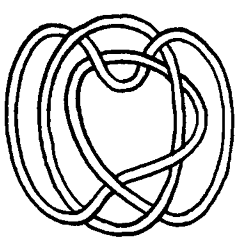
9^2_{37} .3:20

$$\left[\begin{array}{rrrrrr} -1 & 2 & -3 & 3 & -2 & 1 \\ 1 & -2 & 3 & -3 & 2 & -1 \end{array}\right.$$



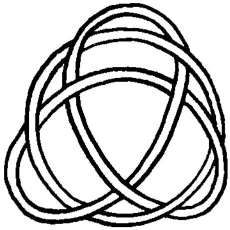
9^2_{38} .21:20

$$\left[\begin{array}{rrrr} -2 & 6 & -5 & 2 \\ 2 & -5 & 6 & -2 \end{array}\right.$$



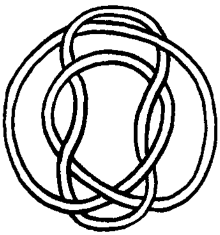
9^2_{39} .2.2∘20

$$\left[\begin{array}{rrrrr} 0 & -1 & 2 & -2 & 1 \\ -1 & 4 & -5 & 4 & -1 \\ 1 & -2 & 2 & -1 & 0 \end{array}\right.$$



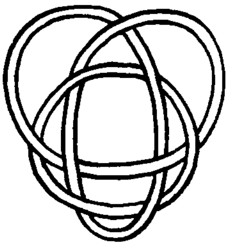
9^2_{40} 2:2:2

$$\begin{bmatrix} 0 & 0 & 3 & -3 & 1 \\ 0 & 3 & -5 & 3 & 0 \\ 1 & -3 & 3 & 0 & 0 \end{bmatrix}$$



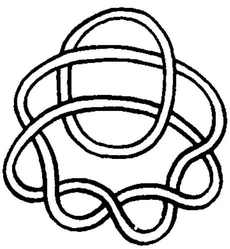
9^2_{41} 2:20:20

$$\begin{bmatrix} 0 & 1 & -2 & 1 \\ 0 & -4 & 5 & -1 \\ -1 & 5 & -4 & 0 \\ 1 & -2 & 1 & 0 \end{bmatrix}$$



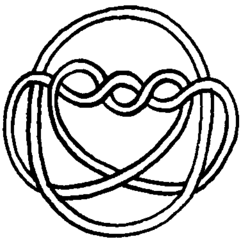
9^2_{42} 8^*2

$$\begin{bmatrix} 0 & -1 & 3 & -3 & 1 \\ -1 & 4 & -7 & 4 & -1 \\ 1 & -3 & 3 & -1 & 0 \end{bmatrix}$$



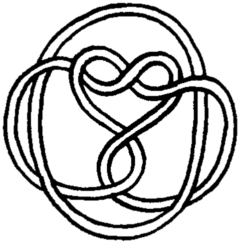
9^2_{43} 5,2,2-

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



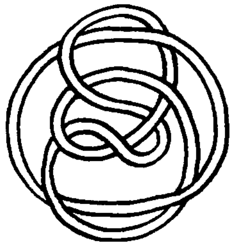
9^2_{44} 41,2,2-

$$\begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$



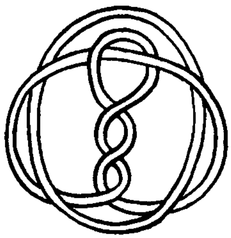
9^2_{45} 32,2,2-

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$



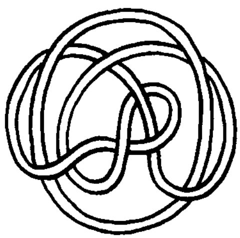
9^2_{46} 311,2,2-

$$\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}$$



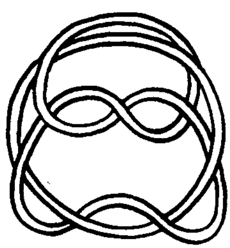
9^2_{47} 23,2,2-

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$



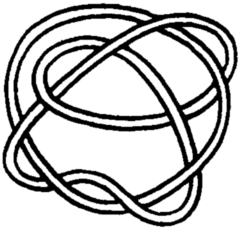
9^2_{48} 221,2,2-

$$\begin{bmatrix} 2 & -2 & 1 & 0 \\ 0 & 1 & -2 & 2 \end{bmatrix}$$



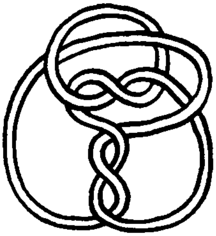
9^2_{49} 4,3,2-

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$



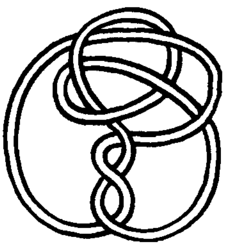
9^2_{50} 4,21,2-

$$\left[\begin{matrix}1 & 0 & 0\\ -1 & 1 & -1\\ 0 & 0 & 1\end{matrix}\right.$$



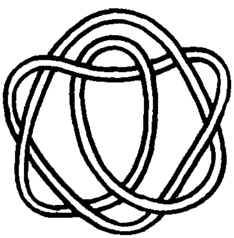
9^2_{51} 31,3,2-

$$\left[\begin{matrix}0 & 1 & 0 & 0 & 0\\ 1 & -1 & 1 & -1 & 1\\ 0 & 0 & 0 & 1 & 0\end{matrix}\right.$$



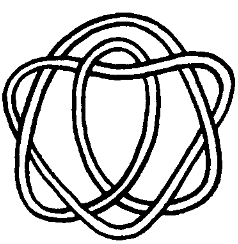
9^2_{52} 31,21,2-

$$\left[\begin{matrix}0 & -2 & 1\\ -1 & 3 & -1\\ 1 & -2 & 0\end{matrix}\right.$$



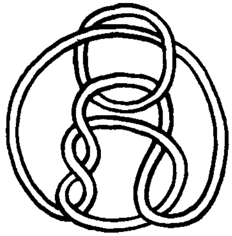
9^2_{53} 3,3,3-

$$\left[\begin{matrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{matrix}\right.$$



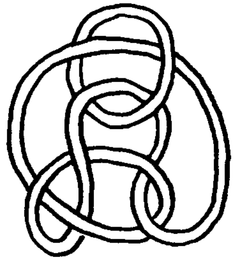
9^2_{54} 3,21,21-

$$\left[\begin{matrix}-1 & 1 & 0\\ 1 & -3 & 1\\ 0 & 1 & -1\end{matrix}\right.$$



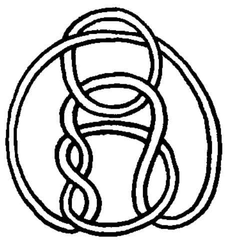
9^2_{55} (3,2)(2,2-)

$$\left[\begin{matrix}-1 & 2 & -2 & 1\\ 1 & -2 & 2 & -1\end{matrix}\right.$$



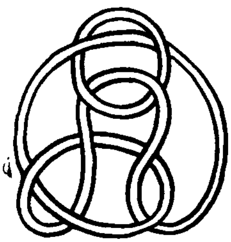
9^2_{56} (21,2)(2,2-)

$$\left[\begin{matrix}-1 & 2 & -2 & 1\\ 1 & -2 & 2 & -1\end{matrix}\right.$$



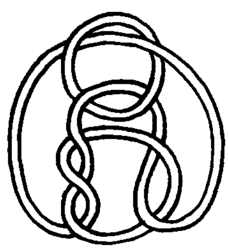
9^2_{57} (3,2-)(2,2)

$$\left[\begin{matrix}-1 & 2 & -1 & 1\\ 1 & -1 & 2 & -1\end{matrix}\right.$$



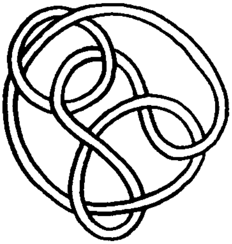
9^2_{58} (21,2-)(2,2)

$$\left[\begin{matrix}-1 & 3 & -2 & 1\\ 1 & -2 & 3 & -1\end{matrix}\right.$$



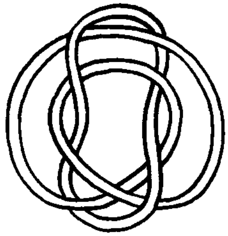
9^2_{59} (3,2)-(2,2)

$$\left[\begin{matrix}0 & 0 & 1 & -1 & 0 & 1\\ 1 & 0 & -1 & 1 & 0 & 0\end{matrix}\right.$$



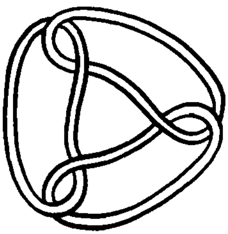
9^2_{60} (21,2)-(2,2)

$$\begin{bmatrix} 1 & 0 & -1 & 1 \\ 1 & -1 & 0 & 1 \end{bmatrix}$$



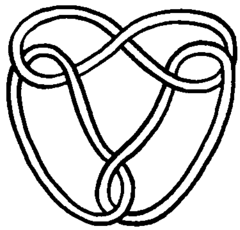
9^2_{61} 2:-20:-20

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$



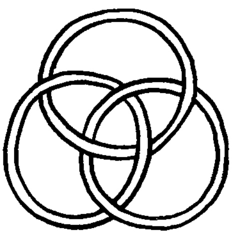
6^3_1 2,2,2

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$



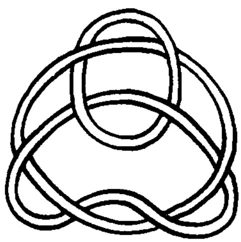
7^3_1 2,2,2+

$$\begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$$



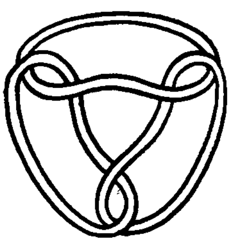
6^3_2 .1

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$



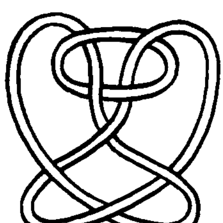
8^3_1 4,2,2

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$



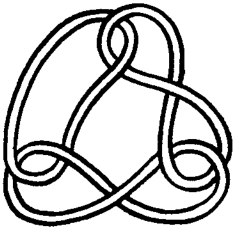
6^3_3 2,2,2-

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$



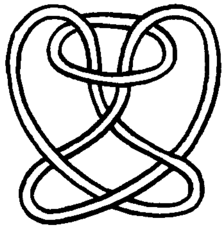
8^3_2 31,2,2

$$\begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$



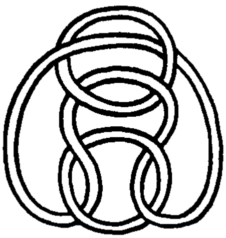
8^3_3 2,2,2++

$$\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$$



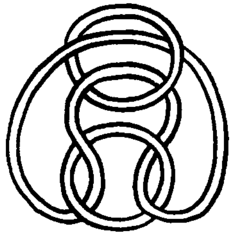
8^3_8 31,2,2-

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$



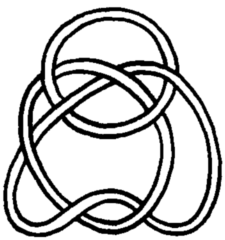
8^3_4 (2,2)(2,2)

$$\begin{bmatrix} 0 & 1 & -2 & 1 \\ 0 & -2 & 2 & 0 \end{bmatrix}$$



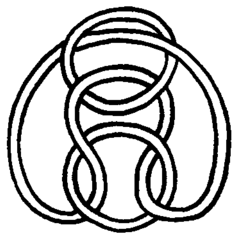
8^3_9 (2,2)(2,2-)

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$



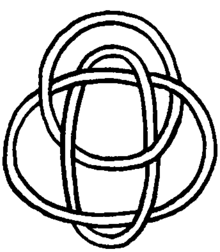
8^3_5 .3

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$



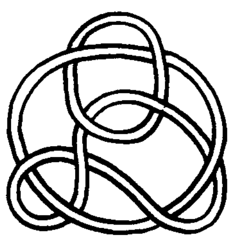
8^3_{10} (2,2)-(2,2)

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{bmatrix}$$



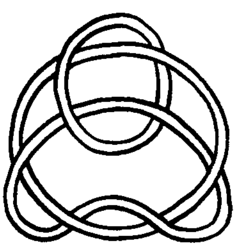
8^3_6 .2:20

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & -3 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$



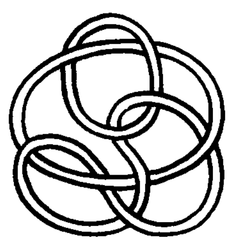
9^3_1 212,2,2

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & -2 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$



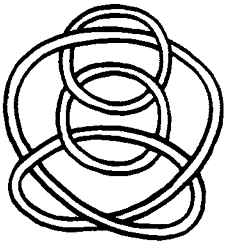
8^3_7 4,2,2-

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$



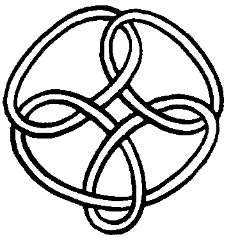
9^3_2 2111,2,2

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$



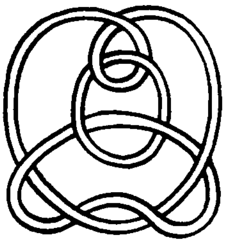
9^3_3 3,2,2,2

$$\begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & -2 & 2 & -2 \end{bmatrix}$$



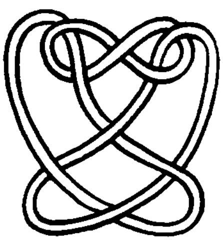
9^3_4 21,2,2,2

$$\begin{bmatrix} 1 & 3 & -3 & 1 \\ 0 & -2 & 2 & -1 \end{bmatrix}$$



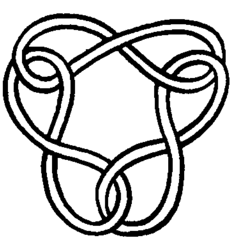
9^3_5 4,2,2+

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$



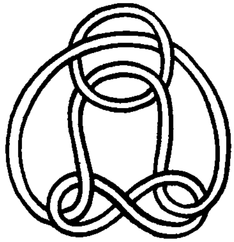
9^3_6 31,2,2+

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & 0 \end{bmatrix}$$



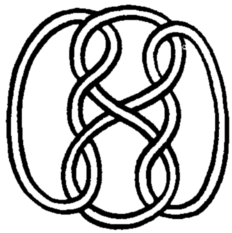
9^3_7 2,2,2+++

$$\begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix}$$



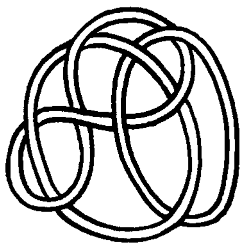
9^3_8 (2,2+)(2,2)

$$\begin{bmatrix} 0 & 2 & -3 & 1 \\ 0 & -2 & 3 & -1 \end{bmatrix}$$



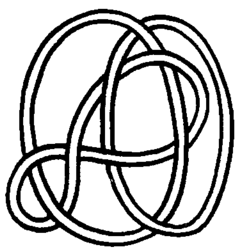
9^3_9 (2,2)1(2,2)

$$\begin{bmatrix} 1 & -2 & 2 & -1 \\ -1 & 2 & -2 & 1 \end{bmatrix}$$



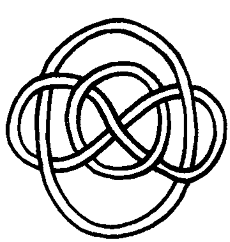
9^3_{10} .211

$$\begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$



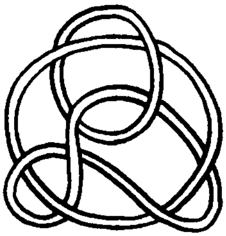
9^3_{11} .21:2

$$\begin{bmatrix} -1 & 2 & -1 \\ 2 & -4 & 2 \\ -1 & 2 & 0 \end{bmatrix}$$



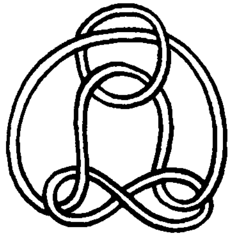
9^3_{12} .(2,2)

$$\begin{bmatrix} 1 & -3 & 3 & -1 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$



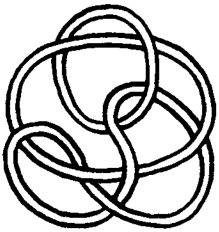
9^3_{13} 212,2,2-

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$



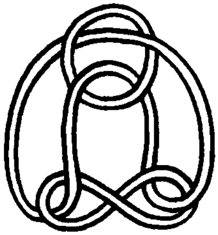
9^3_{18} (2,2+)(2,2-)

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$



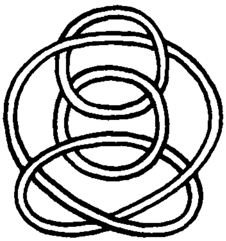
9^3_{14} 2111,2,2-

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 0 \\ -1 & 1 & 0 \end{bmatrix}$$



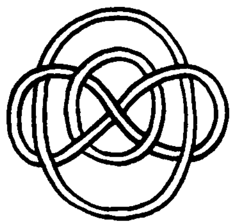
9^3_{1} (2,2+)-(2,2)

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$



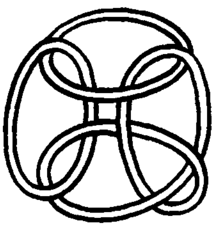
9^3_{15} 3,2,2,2-

$$\begin{bmatrix} -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$



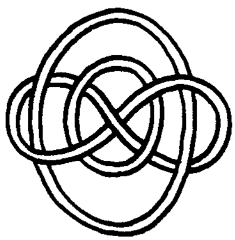
9^3_{20} .(2,2-)

$$\begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$



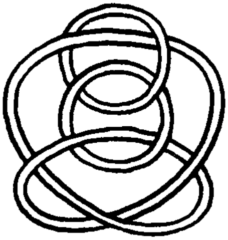
9^3_{16} 21,2,2,2-

$$\begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 2 & -2 & 1 \end{bmatrix}$$



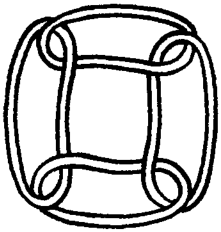
9^3_{21} .-(2,2)

[0



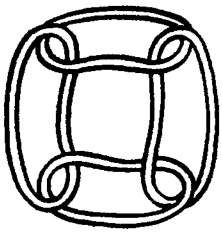
9^3_{17} 3,2,2,2--

$$\begin{bmatrix} 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$



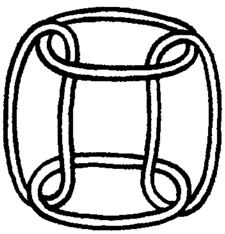
8^4_1 2,2,2,2

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$



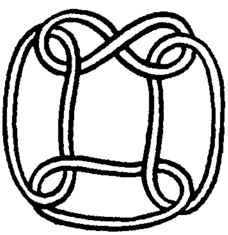
8^4_2 2,2,2,2-

$$\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$



8^4_3 2,2,2,2--

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$$



9^4_1 2,2,2,2+

$$\begin{bmatrix} -2 & 2 \\ 1 & -2 \end{bmatrix} \quad \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

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