Selecta Mathematica, New Series
Vol. 1, No. 3 (1995)

1022–1824/95/030459–36\$1.50 + 0.20/0

Wonderful Models of Subspace Arrangements

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§0. Introduction

In this paper we describe, for any given finite family of subspaces of a vector space or for linear subspaces in affine or projective space, a smooth model, proper over the given space, in which the complement of these subspaces is unchanged but the family of subspaces is replaced by a divisor with normal crossings. This model can be described explicitly in a combinatorial way and also by an explicit sequence of blow ups. It is suitable for several computations as cohomology or rational homotopy and it is in some sense minimal.

The motivation stems from our attempt to understand Drinfeld's construction (cf. [Dr2]) of special solutions of the Khniznik-Zamolodchikov equation (cf. [K-Z]) with some prescribed asymptotic behavior and its consequences for some universal constructions associated to braiding: universal unipotent monodromy representations of braid groups, the construction of a universal Vassiliev invariant for knots, braided categories etc.

The K-Z connection is a special flat meromorphic connection on $\mathbb{C}^n$ with simple poles on a family of hyperplanes. It turns out that the prescription of the asymptotic behavior for such connections is controlled by the geometry of a suitable modification of $\mathbb{C}^n$ in which the union of the polar hyperplanes is replaced by a divisor with normal crossings. In the process of developing this geometry we realized that our constructions could be developed more generally for subspace arrangements and became aware of the paper of Fulton-MacPherson [F-M] in which a Hironaka model is described for the complement of the big diagonal in the power of a smooth variety X. It became clear to us that our techniques were quite similar to theirs and so we adopted their notation of *nested set* in the appropriate general form.

Although we work in a linear subspaces setting it is clear that the methods are essentially local and one can recover their results from our analysis applied to certain special configurations of subspaces. In fact the theory can be applied whenever we have a subvariety of a smooth variety which locally (in the étale topology) appears as a union of subspaces.

* Partially supported by M.U.R.S.T. 40%

We have decided to present the results relative to the holonomy computations in a separate paper "Hyperplane arrangements and holonomy" since the material in the present paper is quite independent from the problem which initially motivated our research.

This paper is entirely dedicated to the combinatorial and geometric constructions arising from subspace configurations. We describe, given a finite family $\mathcal{P}$ of linear subspaces in a vector space V or more generally in a projective space, various explicit constructions of a Hironaka model for the complement $\mathcal{A}$ of the union of the subspaces in the given family. This is a smooth irreducible variety Y with a proper map to $\mathbb{C}^n$ which is an isomorphism on the preimage of $\mathcal{A}$ and such that the complement of this preimage is a divisor with normal crossings.

The general construction consists in forming the closure of the graph of the map

$$i : \mathcal{A} \to V \times \prod_{D \in \mathcal{P}} \mathbb{P}(V/D),$$

where $\mathcal{P}$ is a family of subspaces of V, $\mathbb{P}(V/D)$ denotes the projective space of lines in V/D and the map from $\mathcal{A}$ to $\mathbb{P}(V/D)$ is the restriction of the canonical projection $V - D \to \mathbb{P}(V/D)$. We will call this variety $Y_{\mathcal{P}}$; it contains $\mathcal{A}$ as open set. For suitable classes $\mathcal{P}$ (cf. building sets) this can be obtained by a sequence of blow ups along smooth centers.

The combinatorics is quite intricate and is best described working in the dual space. In the dual let $\mathcal{C}$ be the family of subspaces dual to all the subspaces in $\mathcal{P}$ and their intersections. This is clearly closed under sum. We say that a family $\mathcal{G} \subset \mathcal{C}$ is a *building set* in $\mathcal{C}$ if every element $X \in \mathcal{C}$ is the direct sum $X = \oplus_j Y_j$ of the maximal elements Y_j of $\mathcal{G}$ contained in X. We say that $X = \oplus_j Y_j$ is the decomposition of X relative to $\mathcal{G}$. Among all building sets there is a minimal one which we call the family of irreducible elements. The elements in its complement, called reducible elements, are characterized as subspaces A expressed as a proper direct sum $A = B \oplus C$ with $B, C \in \mathcal{C}$ and such that, for every $D \subset A$ and $D \in \mathcal{C}$, we have $D \cap B, D \cap C \in \mathcal{C}$, $D = D \cap B \oplus D \cap C$.

The main theorem is that, if $\mathcal{G}$ is a building set relative to $\mathcal{C}$ the variety $Y_{\mathcal{P}}$ is smooth, the complement in $Y_{\mathcal{P}}$ of the open set $\mathcal{A}$ is a divisor with normal crossings. Moreover one can construct a sequence $\mathcal{G}_i$ of building sets (relative to different families $\mathcal{C}_i \subset \mathcal{C}$) so that $Y_{G_0} = V$ and $Y_{G_{i+1}}$ is obtained from Y_{G_i} by blowing up a smooth subvariety.

The fundamental notion to study these varieties is that of a $\mathcal{G}$ nested set, i.e., a family $A_1, \ldots, A_k$ of elements of $\mathcal{G}$ with the property that, if $B_1, \ldots, B_h$ are taken out of this family and are pairwise noncomparable elements, then they form a direct sum $X = \oplus_{i=1}^h B_i$ and this is the decomposition of X relative to $\mathcal{G}$.

For each maximal nested set we construct open charts of the model $Y_{\mathcal{G}}$ which can be given explicit coordinates corresponding to a simple local blow up. These charts cover $Y_{\mathcal{G}}$ so that one can globalize the results. In particular we deduce (§5) an explicit presentation for the cohomology of these models in the same spirit as Keel [Ke] and Fulton-MacPherson [F-M].

Using the our cohomology computations one then gives an explicit presentation of a differential graded algebra associated by Morgan to a smooth variety and a divisor with normal crossings whose minimal model is the minimal model for $\mathcal{A}$. Again following Deligne and Morgan, we can use this algebra to determine the mixed Hodge structure for $\mathcal{A}$, thus showing that it depends only on the combinatorial structure of $\mathcal{C}$ considered as a ranked poset; the same result holds for rational homotopy type by the previous remarks.

§1. Basic Definitions

1.1 Let V be a finite dimensional vector space over an infinite field K and V^* its dual. For a subset X of a vector space, we shall denote by $< X >$ the subspace spanned by X. The basic construction of our paper is the following. Let us give a finite family $\mathcal{K}$ of subspaces of V^*, and for every $A \in \mathcal{K}$ consider its orthogonal $A^\perp \subset V$. We also denote by $\mathbb{P}_A$ the projective space of lines in $V/A^\perp$ and remark that a basis of A is a system of projective coordinates in $\mathbb{P}_A$.

Let $V_{\mathcal{K}} := \cup_{A\in\mathcal{K}} A^\perp$ be the union of all the subspaces $A^\perp$ and $\mathcal{A}_{\mathcal{K}}$ be the open set of V complement of $V_{\mathcal{K}}$. By construction the rational map $\pi_A : V \to V/A^\perp \to \mathbb{P}_A$ is defined outside $A^\perp$ and thus we have a regular morphism $\mathcal{A}_{\mathcal{K}} \to \prod_{A\in\mathcal{K}} \mathbb{P}_A$. Its graph is a closed subset of $\mathcal{A}_{\mathcal{K}} \times \prod_{A\in\mathcal{K}} \mathbb{P}_A$ which embeds as open set into $V \times \prod_{A\in\mathcal{K}} \mathbb{P}_A$. Finally we have an embedding

$$\rho : \mathcal{A}_{\mathcal{K}} \to V \times \prod_{A\in\mathcal{K}} \mathbb{P}_A$$

as a locally closed subset.

DEFINITION. We let $Y_{\mathcal{K}}$ be the closure of the image of $\mathcal{A}_{\mathcal{K}}$ under ρ.

We want to study this variety under various conditions on our family $\mathcal{K}$.

From the very construction of this variety, it is clear that it does not depend on the 1-dimensional subspaces A since in this case $\mathbb{P}_A$ is a single point. Thus if we set $\tilde{V}_{\mathcal{K}} := \cup_{A\in\mathcal{K}} A^\perp$, $\dim(A) \geq 2$ and $\tilde{\mathcal{A}}_{\mathcal{K}}$ is the open set of V complement of $\tilde{V}_{\mathcal{K}}$ we still have a regular morphism $\tilde{\mathcal{A}}_{\mathcal{K}} \to \prod_{A\in\mathcal{K}} \mathbb{P}_A$; thus also $\tilde{\mathcal{A}}_{\mathcal{K}}$ is an open set in $Y_{\mathcal{K}}$.

We first make some general remarks.

REMARKS.

(1) The projection $p : V \times \prod_{A\in\mathcal{K}} \mathbb{P}_A \to V$ to the first factor, restricted to $Y_\mathcal{K}$, is a proper birational map $p_\mathcal{K}$ which is an isomorphism on $\mathcal{A}_\mathcal{K}$. We denote by $D_\mathcal{K} := p_\mathcal{K}^{-1} V_\mathcal{K}$ the complement of $\mathcal{A}_\mathcal{K}$ in $Y_\mathcal{K}$.

(2) Given two families $\mathcal{K}_1 \subset \mathcal{K}_2$ we have a canonical proper projection $p_{\mathcal{K}_1}^{\mathcal{K}_2} : Y_{\mathcal{K}_2} \to Y_{\mathcal{K}_1}$ extending the identity on $\mathcal{A}_{\mathcal{K}_2}$. If f is a linear isomorphism of V, set $f(\mathcal{K}) := \{f(A)|A \in \mathcal{K}\}$; then f extends to an isomorphism $Y_\mathcal{K} \to Y_{f(\mathcal{K})}$, in particular the group of linear symmetries of $\mathcal{K}$ acts on $Y_\mathcal{K}$.

(3) Given A minimal in $\mathcal{K}$ set $\mathcal{K}' := \mathcal{K} - \{A\}$ and define the family $\mathcal{K}(A)$ in $V^*/A = (A^\perp)^*$ by
$$\mathcal{K}(A) := \{B + A/A | B \in \mathcal{K}'\}.$$

a) By the identification we have made $(B + A/A)^\perp = B^\perp \cap A^\perp$, then $V_{\mathcal{K}'} \cap A^\perp = V_{\mathcal{K}(A)}$, $\mathcal{A}_{\mathcal{K}'} \cap A^\perp = \mathcal{A}_{\mathcal{K}(A)}$, $\mathcal{A}_\mathcal{K} = \mathcal{A}_{\mathcal{K}'} - A^\perp$.

b) The closure of $\mathcal{A}_{\mathcal{K}(A)}$ in $Y_\mathcal{K}$ is isomorphic to $Y_{\mathcal{K}(A)}$ and we have a commutative diagram:
$$\begin{tikzcd} Y_{\mathcal{K}(\mathcal{A})} \arrow[r, "i"] \arrow[d, "p_{\mathcal{K}(\mathcal{A})}"'] & Y_{\mathcal{K}'} \arrow[d, "p_{\mathcal{K}'}"'] \\ A^\perp \arrow[r] & V. \end{tikzcd} \tag{1.1.1}$$

Proof. We should only comment on the proof of 3b). If $B \in \mathcal{K}'$ then $B + A \neq A$. Set $\overline{B} := B + A/A \neq 0$. Then
$$\mathbb{P}_{\overline{B}} = \mathbb{P}_{B+A/A} = \mathbb{P}(A^\perp/B^\perp \cap A^\perp) \subset \mathbb{P}(V/B^\perp) = \mathbb{P}_B$$
and we have a canonical commutative diagram in which all maps are injective:
$$\begin{tikzcd} A_{\mathcal{K}(\mathcal{A})} \arrow[r, "i"] \arrow[d] & A_{\mathcal{K}'} \arrow[d] \\ A^\perp \times \prod_{\overline{B}\in\mathcal{K}(\mathcal{A})} \mathbb{P}_{\overline{B}} \arrow[r] & V \times \prod_{B\in\mathcal{K}'} \mathbb{P}_B. \end{tikzcd} \tag{1.1.2}$$
□

1.2 We start our analysis with a basic case given by the following:

DEFINITION. A set $\mathcal{S}$ of subspaces in V^* is nested if, given any $U_1, \ldots, U_k \in \mathcal{S}$ pairwise non comparable, they form a direct sum $U = \oplus_i U_i$ and $U \notin \mathcal{S}$.

Remark that a nested set is necessarily finite. A nested set can be recursively constructed as follows. Choose subspaces A_i which form direct sum. Then in each A_i choose a nested set $\mathcal{S}_i$ containing A_i. Set $\mathcal{S}$ to be the union of the $\mathcal{S}_i$.

Let $\mathcal{S}$ be a nested set of subspaces. For every set $A \subset V^*$ containing a non zero vector, the set of subspaces in $\mathcal{S} \cup \{V^*\}$ containing A is linearly ordered and non empty. We let $p_{\mathcal{S}}(A)$ (or $p(A)$) to be the minimum of them.

LEMMA. *The set of $x \in V^* - \{0\}$ with $p_{\mathcal{S}}(x) = U$ equals $U - \cup U_i - \{0\}$ where U_i are the maximal proper subspaces of U in $\mathcal{S}$. Given a subspace G there is an $x \in G$ such that $p(x) = p(G)$.*

Proof. The first part is clear. As for the second, set $U = p(G)$ and let the U_i's be as above. If a subspace is contained in $\cup_i U_i$ then it must be contained in one of the U_i, say U_1, hence $p(G) \subset U_1$ against the assumption $p(G) = U$. □

1.3 Let $\mathcal{S}$ be a nested set of subspaces.

DEFINITION.

(1) A basis b of V^* is adapted to $\mathcal{S}$ if, for all $A \in \mathcal{S}$, the set

$$b_A := b \cap A = \{v \in b | p(v) \subset A\}$$

is a basis of A.

(2) A marking of a basis b adapted to $\mathcal{S}$ is a choice, for all $A \in \mathcal{S}$, of an element $x_A \in b$ with $p(x_A) = A$.

LEMMA. *An adapted basis always exists and $\forall A \in \mathcal{S}$, there exists a $x \in b_A$ such that $p(x) = A$.*

Proof. The proof is by induction using the recursive construction of nested sets. Suppose we have constructed a basis adapted to the nested sets $\mathcal{S}_i$ in the maximal subspaces A_i of $\mathcal{S}$. We can then complete it to a basis of V^* which is automatically adapted to $\mathcal{S}$. The second statement is an immediate consequence of the definitions. □

We define an order in a marked basis by setting $x \leq y$ if $p(x) \subseteq p(y)$ and y is marked. Under this ordering each subspace A of the nested set has as basis the set of elements $\{v \in b | v \leq x_A\}$. Remark that, given $x \in b$, the set of $y \in b$ with $y \geq x$ consists of x and of the x_B, with $B \supseteq p(x)$. By the discussion preceding Lemma 1.2, this is a linearly ordered set.

1.4 We need now a small digression of a general nature. Given a partial order μ on a finite set E we can define a map $\rho_\mu : K^E \to K^E$ by the formula

$$a := \prod_{b \geq a} u_b \tag{1.4.1}$$

where $a \in E$ are chosen as coordinates on the target space while $u_a, a \in E$ are coordinates on the source of the map. Assume that the partial ordering is such that the elements greater than any given one a form a linearly ordered set. Then we claim that this map is a birational morphism.

To define its inverse, if a is not maximal, set $c(a)$ equal to the minimal element properly greater than a. Set formally $1 = c(a)$ if a is maximal.

$$u_a := \frac{a}{c(a)} = \frac{\prod_{b \geq a} u_b}{\prod_{b > a} u_b} \tag{1.4.2}$$

is the inverse of the map 1.4.1 (as birational map). In particular we do this for the ordering on a marked basis of a nested set.

Consider a space K^b with coordinates u_x indexed by the basis elements and set $u_A := u_y$ where y is the marked element associated to A. Define

$$v = \begin{cases} u_v \prod_{B \supset A} u_B, & \text{if } A = p(v) \text{ and } v \text{ is not marked} \\ \prod_{B \supset A} u_B, & \text{if } v = x_A. \end{cases} \tag{1.4.3}$$

This is the monomial map associated to the given ordering, and since b is a basis of V^*, we can consider it as a map $\rho_\mu : K^b \to V$.

PROPOSITION. *The map ρ_μ restricts to an isomorphism between the open set where all u_A's are different from 0 and the open set where all v_A are different from 0, $A \in \mathcal{S}$ and maps the hyperplane defined by $u_A = 0$ in the subspace $A^\perp$.*

Proof. Immediate from the explicit formulas. □

1.5 Let us now fix a nested set $\mathcal{S}$ of subspaces and consider the variety $Y_{\mathcal{S}} \subset V \times \prod_{A \in \mathcal{S}} \mathbb{P}_A$ as in 1.1.

To study $Y_{\mathcal{S}}$ choose a basis b adapted to $\mathcal{S}$ give it a marking and consider the map ρ_μ defined in 1.4.3. We notice first that the composition of ρ_μ with the rational map $\pi_A := V \to \mathbb{P}_A$, $A \in \mathcal{S}$ is given, in the projective coordinates for $\mathbb{P}_A$ coming from the basis b_A of A, by the formula 1.4.3. Thus as monomials in the u_x, these coordinates are all divisible by the monomial expressing x_A; we deduce that:

LEMMA. *The map $\pi_A \rho_\mu$ is a morphism to the affine part of $\mathbb{P}_A$ where $x_A = 1$.* □

Let us denote by $\mathbb{P}_A^0$ this affine set.

PROPOSITION. *The map ρ_μ lifts to an open embedding into $Y_{\mathcal{S}}$.*

Proof. In $V \times_{A \in \mathcal{S}} \mathbb{P}_A$, $Y_{\mathcal{S}}$ is closed while $V \times_{A \in \mathcal{S}} \mathbb{P}_A^0$ is open. Also the map $\rho_\mu : K^b \to V$ is birational. In order to obtain our proposition it suffices to show that the map of $K^b \to V \times_{A \in \mathcal{S}} \mathbb{P}_A^0$, which is regular by the previous lemma, is a closed embedding. Its image is then the intersection of $Y_{\mathcal{S}}$ with $V \times_{A \in \mathcal{S}} \mathbb{P}_A^0$. For this it suffices to show that each function u_v on K^b is the restriction of a function on $V \times_{A \in \mathcal{S}} \mathbb{P}_A^0$. By 1.4.3, $v = u_v \prod_{B \supset A} u_B$, if $A = p(v)$ and v is not marked, while $v = \prod_{B \supset A} u_B$, if $v = x_A$. If $V^* \notin \mathcal{S}$, $p(x) = V^*$ then the function u_v is induced by the function v on V.

Assume $p(v) = A \in \mathcal{S}$. If v is not marked the function u_v is the composition of our embedding with the affine coordinate v/x_A on $\mathbb{P}_A^0$. If $v = x_A$ is marked we have two cases: if $p(v)$ is maximal then u_v comes from the function v on V and otherwise u_v is the composition with the affine coordinate $x_A/x_{c(A)}$ on $\mathbb{P}_{c(A)}^0$. □

We denote by $\mathcal{U}_{\mathcal{S}}^b$ the open set in $Y_{\mathcal{S}}$ given by the previous proposition and identify the restriction to $\mathcal{U}_{\mathcal{S}}^b$ of the projection from $Y_{\mathcal{S}}$ to V with ρ_μ.

REMARK. The open set $\mathcal{U}_{\mathcal{S}}^b$ depends only on the marked elements of the basis and not on the full basis.

Proof. We may reduce to the case in which we substitute a vector $v \in b$ not marked with another vector $v' = \alpha v + w$ where $\alpha \neq 0$ and w is a linear combination of the remaining vectors in $b_{p(v)}$. The coordinates u_x relative to the new basis coincide with the previous ones except for $u_{v'} = \alpha u_v + Q(u_x)$, $x \in b_{p(v)}$ where $Q(u)$ is a polynomial. Thus $u_{v'}$ is a regular function on $\mathcal{U}_{\mathcal{S}}^b$ and by symmetry the claim follows. □

REMARK. The maximal torus of elements diagonal in a given adapted basis stabilizes the nested set and hence acts on Y_S which is thus a torus embedding obtained by successive blow-ups of the standard torus embedding V associated to a simplex. The sequence of blow-ups is described by a sequence of subdivisions of faces.

1.6 Let $A \in \mathcal{S}$, and $\mathcal{S}'$ denote the set $\mathcal{S} - \{A\}$ (removing A). Clearly $\mathcal{S}'$ is still nested and we want to compare the two varieties $Y_\mathcal{S}$, $Y_{\mathcal{S}'}$. By construction there is a birational morphism $p : Y_\mathcal{S} \to Y_{\mathcal{S}'}$.

Let us then take a basis adapted to $\mathcal{S}$ with a marking and consider the corresponding map $\rho_\mu : \mathcal{U}^b_\mathcal{S} \to V$. The same basis is also adapted to $\mathcal{S}'$ and it still has a marking. The main difference is that the element marked v_A with respect to $\mathcal{S}$ should not be considered marked for $\mathcal{S}'$. We have a corresponding map $\rho_{\mu'} : \mathcal{U}^b_{\mathcal{S}'} \to V$, and ρ_μ equals the composition of p restricted to $\mathcal{U}^b_\mathcal{S}$ with $\rho_{\mu'}$.

In order to explicit our maps let us denote by u_v, resp. u'_v the coordinates in $\mathcal{U}^b_\mathcal{S}$ resp. $\mathcal{U}^b_{\mathcal{S}'}$. We need also to distinguish the map c on the basis b in the two cases and so will call c, c' the two maps. With these conventions one easily verifies

$$c'(v) = \begin{cases} c(v) & \text{if } c(v) \neq v_A \\ c(v_A) & \text{if } c(v) = v_A. \end{cases}$$

We can now compare the various maps using the relations:

$$u_a := \frac{a}{c(a)},\ u'_a := \frac{a}{c'(a)} = u_a \frac{c(a)}{c'(a)} \tag{1.6.1}$$

so $u_a = u'_a$ if $c(a) \neq v_A$ and $u'_a = u_a u_A$ if $c(a) = v_A$. These are exactly the explicit maps of the blow up of $\mathcal{U}^b_{\mathcal{S}'}$ along the subvariety $u'_A = 0, u'_v = 0,\ c(v) = v_A$ in the chart corresponding to the marked coordinate

$$\begin{tikzcd} \mathcal{U}^b_\mathcal{S} \arrow[r, "\rho_\mu"] \arrow[d, "p"'] & V \arrow[d, "1_V"'] \\ \mathcal{U}^b_{\mathcal{S}'} \arrow[r, "\rho_{\mu'}"] & V. \end{tikzcd} \tag{1.6.2}$$

In particular we want to apply this when A is minimal. In this case fix an adapted basis b as before and mark it for $\mathcal{S}'$. If no marked vector belongs to A, we can complete the marking to $\mathcal{S}$ in $n = \dim A$ different ways getting n marked bases b_i and n maps $\mathcal{U}^{b_i}_\mathcal{S} \to \mathcal{U}^b_{\mathcal{S}'}$. These cover the blow up of $\mathcal{U}^b_{\mathcal{S}'}$ along the subspace defined by the equations $u'_v = 0,\ v \in A$, hence the induced map $\cup_i \mathcal{U}^{b_i}_\mathcal{S} \to \mathcal{U}^b_{\mathcal{S}'}$ is a proper map from the open set $\cup_i \mathcal{U}^{b_i}_\mathcal{S} \subset Y_\mathcal{S}$. We claim that the variety we blow up in $\mathcal{U}^b_{\mathcal{S}'}$ is exactly the proper transform of the subspace $A^\perp$. In fact by the formulas 1.4.3 we have that $v = u'_v \prod_{B \supsetneq A,\, B \in \mathcal{S}'} u'_B$ for $v \in b_A$ and dividing by $\prod_{B \supsetneq A,\, B \in \mathcal{S}'} u'_B$

the claim follows. If on the other hand one of the marked vectors is in A, the open embedding $\mathcal{U}^b_{\mathcal{S}'} \to Y_{\mathcal{S}'}$ lifts to an open embedding into $Y_{\mathcal{S}}$ by the same argument as in Lemma 1.5.

Given $\mathcal{S}$ one possible way to select adapted marked bases is the following. Choose for every $B \in \mathcal{S}$ a basis b^B of B of vectors not contained in any $C \in \mathcal{S}$ properly contained in B. Choose a vector $x_B \in b^B$ for every B. One sees that these vectors are linearly independent and thus can be completed to a basis b adapted to $\mathcal{S}$ in which they are marked. As we have seen in Remark 1.5, the corresponding open set $\mathcal{U}^b_{\mathcal{S}}$ does not depend on the way in which we complete the basis. In this way we get a finite family $\mathcal{M}$ of open sets.

PROPOSITION.

(1) *The variety $Y_{\mathcal{S}}$ is covered by the open sets $\mathcal{U}^b_{\mathcal{S}}$ in the family $\mathcal{M}$.*

(2) *Given a minimal element $A \in \mathcal{S}$ and letting $\mathcal{S}' = \mathcal{S} - \{A\}$, $Y_{\mathcal{S}}$ it is the blow up of $Y_{\mathcal{S}'}$ along the proper transform Z_A of the subspace $A^\perp$ which is a smooth subvariety. Furthermore Z_A is canonically isomorphic to $Y_{\mathcal{T}}$ where $\mathcal{T} := \{B/A \in V^*/A \mid B \in \mathcal{S}'\}$.*

(3) *Consider $\mathcal{A} = V - \cup_{A\in\mathcal{S}} A^\perp$ embedded as an open set in $Y_{\mathcal{S}}$. Then $Y_{\mathcal{S}} - \mathcal{A}$ is a divisors with normal crossing with smooth irreducible components $D^{\mathcal{S}}_A$ for $A \in \mathcal{S}$ with $D^{\mathcal{S}}_A$ equal to the closure of $p^{-1}(A^\perp - \cup_{B\supsetneq A} B^\perp)$.*

(4) *With the notations of* (2), (3) *we have*

$$D^{\mathcal{S}'}_B \cap Z_A = D^{\mathcal{T}}_{B/A}.$$

(5) *All intersections of the divisors D_A are irreducible.*

Proof. (1) and (2) are proved together by induction on the cardinality of $\mathcal{S}$. Take A minimal in $\mathcal{S}$ and $\mathcal{S}'$ as above. By induction the result holds for $\mathcal{S}'$ and so it holds for $\mathcal{S}$, since a blow up is proper, and the preceding discussion. Notice that $A^\perp$ meets the complement $\mathcal{A}'$ of the union of the subspaces $B^\perp$, $B \in \mathcal{S}'$ and $A^\perp \cap \mathcal{A}' = A^\perp - \cup_{B/A\in\mathcal{T}}(B/A)^\perp$.

Under the map $\mathcal{A}' \to V \times \prod_{B\in\mathcal{S}'} \mathbb{P}_B$, $A^\perp \cap \mathcal{A}'$ maps to $A^\perp \times \prod_{B/A\in\mathcal{T}} \mathbb{P}_{B/A}$.

(3) By Proposition 1.4 we deduce that for every open set $\mathcal{U}^b_{\mathcal{S}}$, $p^{-1}(A^\perp) \cap \mathcal{U}^b_{\mathcal{S}}$ is the divisor of equation $\prod_{B\supset A} u_B$. Thus, if we set D_A equal to the closure of $p^{-1}(A^\perp - \cup_{B\supsetneq A} B^\perp)$ we obtain a smooth divisor whose intersection with $\mathcal{U}^b_{\mathcal{S}}$ is the hyperplane of equation $u_A = 0$. Furthermore $Y_{\mathcal{S}} - \mathcal{A} = \cup_{A\in\mathcal{S}} D_A$. This proves (3) and (4) except for the irreducibility of D_A. We prove this by induction on the cardinality of $\mathcal{S}$. As in (1) we assume the statement for $\mathcal{S}' = \mathcal{S} - \{A\}$ (A minimal). Considering by (2) $Y_{\mathcal{S}}$ as the blow up of $Y_{\mathcal{S}'}$ along Z_A we deduce that D_A is the exceptional divisor so that its irreducibility follows from the one of Z_A while for $B \neq A$, D_B is the proper transform of the divisor corresponding to B in $Y_{\mathcal{S}'}$ so it is irreducible by the inductive hypothesis.

(4)–(5) Follow from the previous analysis and the local description. □

§2. Some combinatorics of arrangements

2.1 Let $\mathcal{C}$ denote a finite set of non zero subspaces of V^* closed under sum.

DEFINITION. Given a subspace $U \in \mathcal{C}$ a *decomposition* of U is a a list of non zero subspaces $U_1, U_2, \ldots, U_k \in \mathcal{C}$ with $U = U_1 \oplus U_2 \oplus \ldots \oplus U_k$ and, for every subspace $A \subset U$ in $\mathcal{C}$, also $A \cap U_1,\ A \cap U_2, \ldots, A \cap U_k$ lie in $\mathcal{C}$ and $A = (A \cap U_1) \oplus (A \cap U_2) \oplus \ldots \oplus (A \cap U_k)$.

If a subspace does not admit a decomposition it is called irreducible. The set of irreducible subspaces of $\mathcal{C}$ is denoted $\mathcal{F}$ or $\mathcal{F}_{\mathcal{C}}$.

PROPOSITION. *Every subspace $U \in \mathcal{C}$ has a unique decomposition $U := \oplus_{i=1}^k U_i$ into irreducible subspaces. This is called the irreducible decomposition of U.*

If $A \subset U$ is irreducible then $A \subset U_i$ for some i.

Proof. In the first step the existence of the decomposition follows by induction on the dimension. The uniqueness clearly will follow from the second part comparing two decompositions. Let $A \subset U = \oplus_{i=1}^k U_i$. By definition of decomposition we must have $A = \oplus_{i=1}^k (A \cap U_i)$. If now $B \subset A$ again $B = \oplus_{i=1}^k (B \cap U_i) = \oplus_{i=1}^k B \cap (A \cap U_i)$. We thus have that A is reducible unless $A = A \cap U_i$ for some i. □

We notice 3 simple consequences which will be useful later and which we leave to the reader to verify.

COROLLARY. *Given a direct sum $U = \oplus_{i=1}^k U_i$ in $\mathcal{C}$ this is a decomposition if and only if, for every irreducible A, $A \subset U$ implies $A \subset U_i$ for some i.*

Given two irreducible subspaces A, B either $A + B$ is irreducible or $A \oplus B$ is an irreducible decomposition.

If a subspace M is sum of irreducible subspaces M_i, its irreducible components are also sum of subfamilies of the family M_i. □

2.2 We will also need the following.

PROPOSITION. *Let $U = \oplus_{i=1}^k U_i$ in $\mathcal{C}$ be a decomposition and let $C \subsetneq U$ be an element of $\mathcal{C}$ maximal in U. Then there is an index $j \leq k$, an element $C_j \in \mathcal{C}$ with $C_j \subsetneq U_j$ maximal in U_j, and $C = C_j \oplus (\oplus_{i \neq j} U_i)$.*

Proof. By definition of decomposition $C = \oplus_{i=1}^k (U_i \cap C)$ and for at least one j, we must have $C_j := U_j \cap C \subsetneq U_j$. Then $C \subset C_j \oplus (\oplus_{i \neq j} U_i) \subsetneq U$ by maximality of C satisfying the claim. □

2.3 Let $\mathcal{G}$ be a set of subspaces in V^*. Set $\mathcal{C}_\mathcal{G}$ equal to the set of subspaces which are sums of subspaces in $\mathcal{G}$ and $\mathcal{F}_\mathcal{G} = \mathcal{F}_{\mathcal{C}_\mathcal{G}}$ the irreducibles in $\mathcal{C}_\mathcal{G}$.

THEOREM. *The following* 3 *conditions on* $\mathcal{G}$ *are equivalent:*

(1) $\mathcal{G}$ *satisfies:*

a) $\mathcal{G} \supset \mathcal{F}_\mathcal{G}$.

b) *If* $A, B \in \mathcal{G}$, $A = \oplus_{i=1}^t F_i$ *is the irreducible decomposition of* A *in* $\mathcal{F}_\mathcal{G}$ *and* $B \supset F_i$ *for some* i, *then* $B + A \in \mathcal{G}$.

(2) *Every element* C *of* $\mathcal{C}_\mathcal{G}$ *is the direct sum* $C = G_1 \oplus G_2 \oplus \ldots \oplus G_k$ *of the set of maximal elements* $G_1, G_2, \ldots, G_k$ *of* $\mathcal{G}$ *contained in* C.

Furthermore under the previous hypotheses if $B \in \mathcal{C}_\mathcal{G}$ *and* $B \subset C$ *then* $B = \oplus_{i=1}^k (B \cap G_i)$, *i.e.* $C = G_1 \oplus G_2 \oplus \ldots \oplus G_k$ *is a decomposition.*

(3) $\mathcal{G}$ *satisfies* a) *and*

b') *If* $A, B \in \mathcal{G}$ *and* $B + A$ *is not a decomposition (in particular if* $A \cap B \neq \{0\}$*), then* $B + A \in \mathcal{G}$.

Proof. (1) implies (2). Decompose C into irreducibles, $C = A_1 \oplus A_2 \oplus \ldots \oplus A_k$. Let $G_1, \ldots, G_k$ be the maximal elements in $\mathcal{G}$ contained in C. Consider for instance G_1 decomposed as $G_1 = F_1 \oplus F_2 \oplus \ldots \oplus F_r$, each F_i must be contained in one of the A_j then by b) of (1), $G_1 + A_j \in \mathcal{G}$ and by maximality we must have $A_j \subset G_1$, hence each G_i is a direct sum of some of the A_j. Again by b) if we had two different G_i containing the same A_j their sum is in $\mathcal{G}$ contradicting again maximality. Since $\mathcal{G}$ contains all irreducibles, it contains all the A_j, hence each A_j is contained in one of the G_i's. Necessarily we have a partition $S_1, S_2, \ldots, S_k$ of the set of the indices $1, 2, \ldots, h$ and $G_i = \oplus_{j \in S_j} A_j$.

Assume (2). Take $C \in \mathcal{C}_\mathcal{G}$. By (2) we can write $C = G_1 \oplus G_2 \oplus \ldots \oplus G_k$, where $G_1, G_2, \ldots, G_k$ are the maximal elements in $\mathcal{G}$ contained in C. Let us prove that we get a decomposition. Take $C \supset B \in \mathcal{C}_\mathcal{G}$. Write also $B = A_1 \oplus A_2 \oplus \ldots \oplus A_s$, where $A_1, A_2, \ldots, A_s$ are the maximal elements in $\mathcal{G}$ contained in B. By the maximality of $G_1, G_2, \ldots, G_k$ each A_i must be contained in one G_j and $B = \oplus_{i=1}^k (B \cap G_i)$ while each term $B \cap G_i$ is a direct sum of A_j's.

(2) implies (3). Let us prove a). Take F irreducible in $\mathcal{C}_\mathcal{G}$. Then F has a decomposition in $\mathcal{G}$, but since F is irreducible $F \in \mathcal{G}$. Now we prove b'). Suppose $A, B \in \mathcal{G}$ and $A + B$ is not a decomposition. Decompose $A + B = \oplus_i G_i$ in $\mathcal{G}$. Each A, B must be contained in one of the G_i's, since $A + B$ is not a decomposition they must be contained in the same, say G_1, hence $G_1 = A + B$.

(3) implies (1). Suppose $\mathcal{G}$ satisfies properties a) and b'). Let $A, B \in \mathcal{G}$ be such that $B \supset F_i$ for some i. Then, since $B + A$ is not a decomposition, $B + A \in \mathcal{G}$. □

DEFINITION. A set $\mathcal{G}$ satisfying the 3 equivalent conditions of the previous theorem will be called a *building set.*

We shall refer to the decomposition $C = G_1 \oplus G_2 \oplus \ldots \oplus G_k$ as the decomposition of C in $\mathcal{G}$ and the elements of $\mathcal{G}$ are thought of as *$\mathcal{G}$-indecomposables.*

The decomposition of C in $\mathcal{G}$ is clearly unique (by the maximality of the G_j) and characterized by the property that, if $A \in \mathcal{G}$, $A \subset C$, then there is an i with $A \subset G_i$.

We may remark that we can apply in particular the notion of decomposition to the sum of all the subspaces in $\mathcal{C}_\mathcal{G}$. Thus we have maximal elements in $\mathcal{G}$ forming direct sum and the set $\mathcal{G}$ is the disjoint union of the subfamilies contained in each of these maximal ones.

The following proposition gives us some examples of building sets

PROPOSITION. *Let $\mathcal{C}$ be a set of non zero subspaces closed under sum. Then*

(1) *$\mathcal{C}$ is a building set.*

(2) *$\mathcal{F}_\mathcal{C}$ is a building set.*

(3) *Let $\mathcal{G} \subset \mathcal{F}_\mathcal{C}$ be a subset with the property that, if $X \in \mathcal{C}_\mathcal{G}$, then all its irreducible components in $\mathcal{F}_\mathcal{C}$ lie in $\mathcal{G}$. Then $\mathcal{G}$ is a building set.*

(4) *Let $A \in \mathcal{F}_\mathcal{C}$ be a minimal element. Denote by $\phi : V^* \to V^*/A$ the quotient morphism, and consider the set $\mathcal{G} = \{\phi(B) | B \in \mathcal{F}_\mathcal{C} - \{A\}\}$. We also use the notation $\phi(G) = \overline{G} = G + A/A$:*

   a) *The set $\mathcal{H} := \mathcal{F}_\mathcal{C} - \{A\}$ is a building set.*

   b) *If $\overline{G} \in \mathcal{F}_\mathcal{G}$ then either G is irreducible in $\mathcal{C}$ or $G \supset A$ and there is an irreducible $G_0 \subset G$, such that $G = G_0 \oplus A$ is the irreducible decomposition of G.*

   c) *$\mathcal{G}$ is a building set in V^*/A.*

Proof. (1) is obvious.

(2) follows from Corollary 2.1.

(3) The fact that $\mathcal{G} \supset \mathcal{F}_\mathcal{G}$ is clear from the definitions. Let us check property (1) b) in Theorem 2.3. Take $A, B \in \mathcal{G}$. Let $A = \oplus_{i=1}^t F_i$ be the irreducible decomposition of A in $\mathcal{F}_\mathcal{G}$ and suppose $B \supset F_i$ for some i. Then by Corollary 2.1 $B + A \in \mathcal{F}_\mathcal{C}$. By the definition of $\mathcal{G}$ we deduce that $B + A \in \mathcal{G}$ as desired.

(4) a) We verify property (2) in Theorem 2.3. Take a sum $\sum_j F_j$, $F_j \in \mathcal{F}_\mathcal{C} - \{A\}$, and decompose it in irreducibles. Each of these irreducibles must be sum of some of the F_j, hence A cannot appear in the decomposition by its minimality.

b)–c) Assume $A \subset G$ and $G = H_1 \oplus H_2 \ldots \oplus H_k$ is the irreducible decomposition. We may assume, by Proposition 2.1, that $A \subset H_1$. We claim that $G/A = H_1/A \oplus \overline{H}_2 \ldots \oplus \overline{H}_k$ is a decomposition. In fact let $\overline{B} = B/A$ be a subspace in $\mathcal{C}_{\mathcal{G}}$ contained in G/A. We must have $B = (B \cap H_1) \oplus (B \cap H_2) \ldots \oplus (B \cap H_k)$ and $A \subset B \cap H_1$, hence $\overline{B} = (\overline{B} \cap \overline{H}_1) \oplus (\overline{B} \cap \overline{H}_2) \ldots \oplus (\overline{B} \cap \overline{H}_k)$.

To prove b) assume that $\overline{G}$ is irreducible. We must have, either $\overline{G} = \overline{H}_1$ and $k = 1$ or $\overline{H}_1 = 0$ and $k = 2$ as desired.

To prove c) we verify property (2) in Theorem 2.3. Consider an element $\phi(C)$ in $\mathcal{G}$ contained in G/A, $C \subset G$ irreducible. Then $C \subset H_i$ for some i, hence the spaces $\overline{H}_i$ are all the maximal elements in $\mathcal{G}$ contained in $\overline{G}$. □

2.4 DEFINITION. Let $\mathcal{G}$ be a building set. A subset $\mathcal{S} \subset \mathcal{G}$ will be called *nested relative to $\mathcal{G}$* or $\mathcal{G}$-nested if

(1) $\mathcal{S}$ is nested.

(2) Given a subset $\{A_1, \ldots, A_h\}$ of pairwise incomparable elements in $\mathcal{S}$, then $C = \oplus_{i=1}^{h} A_h$ is the decomposition of C in $\mathcal{G}$.

REMARK. Given a nested set $\mathcal{S}$, the minimal set of subspaces $\mathcal{C}$ closed under sum and containing $\mathcal{S}$ is formed by the direct sums of the families of noncomparable elements of $\mathcal{S}$. Furthermore $\mathcal{S}$ coincides with the set $\mathcal{F}$ of irreducible elements of $\mathcal{C}$.

LEMMA.

(1) *A subset $\mathcal{S} \subset \mathcal{G}$ is nested relative to $\mathcal{G}$ if and only if for all sets of pairwise noncomparable elements $A_i \in \mathcal{S}$, $i = 1, \ldots, h$, $h > 1$, we have $\sum_i A_i \notin \mathcal{G}$.*

(2) *A $\mathcal{G}$ nested set $\mathcal{S}$ is obtained recursively, choosing elements $A_1, \ldots, A_k \in \mathcal{G}$ such that $\oplus_{i=1}^{k} A_i$ is a decomposition, a $\mathcal{G}$-nested set $\mathcal{S}_i$ inside A_i and containing A_i for each i and setting $\mathcal{S} = \cup_i \mathcal{S}_i$.*

(3) *We can give a rank to the elements of a nested set, setting $rk_{\mathcal{S}}(A_i) = 1$ for the maximal elements and, if $B \in \mathcal{S}_i$, $rk_{\mathcal{S}}(B) := rk_{\mathcal{S}_i}(B) + 1$. We have then that the elements greater or equal of a given $B \in \mathcal{S}$ are a linearly ordered set with $rk(B)$ elements.*

Proof. (1) If $\mathcal{S}$ is nested it satisfies the condition; conversely take a set of pairwise non comparable elements $A_i \in \mathcal{S}$, $i = 1, \ldots, h$, $h > 1$ we have $C := \sum_i A_i \notin \mathcal{G}$. Decompose $C = G_1 \oplus \ldots \oplus G_m$ in $\mathcal{G}$. Each G_j is the sum of the A_j contained in it. Again by the same condition this sum must be reduced to a single term, we have $m = h$ and we may assume $G_i = A_i$ for all i.

(2) and (3) are immediate. □

Let $\mathbb{F} := C_1 \supset C_2 \supset \ldots \supset C_k$ be a flag in $\mathcal{C}_{\mathcal{G}}$. If $C_i = \oplus_j A_i^{(j)}$ is the decomposition into $\mathcal{G}$-indecomposables the family $\mathcal{D}_{\mathcal{G}}(\mathbb{F}) := \{A_i^{(j)}; i, j\}$ will be called the decomposition of $\mathbb{F}$ (in $\mathcal{G}$).

Let $\mathcal{M} = \mathcal{M}_0$ be a set of subspaces, we set $\mathcal{M}_1$ to be the set of nonmaximal elements of $\mathcal{M}$ and recursively $\mathcal{M}_{k+1} := (\mathcal{M}_k)_1$. Denote by $< \mathcal{M} > := \sum_{A \in \mathcal{M}} A$ their common span. We clearly have $< \mathcal{M}_{k+1} > \subseteq < \mathcal{M}_k >$. We can define a (descending) flag associated to $\mathcal{M}$ setting $\mathbb{F}(\mathcal{M}) := \{\mathbb{F}_k(\mathcal{M}) := < \mathcal{M}_k >\}$.

THEOREM. *The map $\mathbb{F} \to \mathcal{D}_{\mathcal{G}}(\mathbb{F})$ is surjective from flags to $\mathcal{G}$-nested sets with left inverse $\mathcal{S} \to \mathbb{F}(\mathcal{S})$.*

Proof. The inclusion $\oplus_j A_i^{(j)} \subset \oplus_k A_{i-1}^{(k)}$ implies, from the properties of decompositions, that the $A_i^{(j)}$ are subdivided into groups contained in the various $A_{i-1}^{(k)}$. This clearly implies the nestedness.

Conversely proceed by induction on the cardinality of $\mathcal{S}$. Let $A_1, \ldots, A_h$ be the maximal elements in $\mathcal{S}$ and as before $\mathcal{S}_1 := \mathcal{S} - \{A_1, \ldots, A_h\}$. Then $\mathbb{F}_1(\mathcal{S}) := A_1 \oplus \ldots \oplus A_h$ is a decomposition. By induction $\mathcal{S}_1$ is obtained decomposing the flag $\mathbb{F}_2(\mathcal{S}) \supset \ldots \supset \mathbb{F}_k(\mathcal{S})$. □

We should remark that the previous correspondence is compatible with the obvious inclusion relations. In particular, to maximal flags correspond surjectively maximal $\mathcal{G}$-nested sets. For each maximal flag the largest element of the flag must necessarily be the sum of all the spaces in $\mathcal{C}_{\mathcal{G}}$, hence every maximal nested set contains all the maximal elements in $\mathcal{G}$.

We should notice a feature of the previous construction. Fix a class of subspaces $\mathcal{C}$ closed under sum; in general $\mathcal{C} = \mathcal{C}_{\mathcal{G}}$ for many different classes $\mathcal{G}$, of which the minimal is $\mathcal{F}$ (the irreducibles) the maximal is $\mathcal{C}$ itself.

The set of maximal flags in $\mathcal{C}$ does not depend on the choice of $\mathcal{G}$ while the nested sets depend on it, since they depend on the decomposition of the spaces in the flag. In particular two different maximal flags may give the same nested set (with some difference in the indexing of the components $A_i^{(j)}$). Notice that for the maximal family $\mathcal{C}$ nested sets are necessarily flags.

We collect two properties of $\mathcal{G}$-nested sets which will be used later.

PROPOSITION.

(1) *If $\mathcal{M}$ is a set of subspaces in $\mathcal{G}$ which is not $\mathcal{G}$-nested, there is a subset of elements $A_1, \ldots, A_k \in \mathcal{M}$ such that $B := \sum_{i=1}^k A_i \in \mathcal{G}$ and $A_i \subsetneq B$ for all i.*

(2) *Let $\mathcal{T}$ be a maximal nested set relative to $\mathcal{G}$ and $D \in \mathcal{G}$. Let $A = p_{\mathcal{T}}(D)$ (cf. 1.2) and $A_1, \ldots, A_n$ be the maximal elements in $\mathcal{T}$ contained in A. Then $D + \oplus A_i = A \in \mathcal{T}$.*

Proof. (1) Let $\mathbb{F}(\mathcal{M})$ be the flag of $\mathcal{M}$. Since $\mathcal{M}$ is not $\mathcal{G}$-nested there is an $A \in \mathcal{M}$ which does not appear in the decomposition of the flag. Let h be a maximal index such that $A \subset \mathbb{F}_h(\mathcal{M})$ and let $\oplus_j G_j$ be the decomposition of $\mathbb{F}_h(\mathcal{M})$. Thus $A \subset G_j$ for some j. By construction G_j is a sum of subspaces in $\mathcal{M}$. Now A is maximal among the subspaces of $\mathcal{M}$ contained in G_j otherwise we would have $A \subset \mathbb{F}_{h+1}(\mathcal{M})$ and by assumption $A \neq G_j$, thus $B := G_j \notin \mathcal{M}$. Since B is clearly a sum of elements in $\mathcal{M}$, B is the required space.

(2) Let $\mathbb{F}_i$ be a maximal flag associated to the maximal nested set $\mathcal{T}$. Since $D + \mathbb{F}_1 \in \mathcal{C}_\mathcal{G}$, by maximality we must have that $D \subset \mathbb{F}_1$. Let then h be maximal such that $D \subset \mathbb{F}_h$. We have $\mathbb{F}_h \supset D + \mathbb{F}_{h+1} \supset \mathbb{F}_{h+1}$. By maximality of the flag and the hypothesis $D \not\subset \mathbb{F}_{h+1}$ we must have $D + \mathbb{F}_{h+1} = \mathbb{F}_h$. Let G be the component of $\mathbb{F}_h$ containing D. We must have $G = A = p(D)$ and moreover A must be the sum of D and the components of $\mathbb{F}_{h+1}$ contained in A. By construction of the flag these are the maximal elements in $\mathcal{T}$ contained in A. □

2.5 The following proposition tells us how to obtain the nested sets for a building set $\mathcal{G}$ from those relative to $\mathcal{F}_{\mathcal{C}_\mathcal{G}}$ in an inductive way.

PROPOSITION.

(1) *Let $\mathcal{G}$ be a building set. Let G be a minimal element in $\mathcal{G} - \mathcal{F}_{\mathcal{C}_\mathcal{G}}$. Then $\mathcal{G}' = \mathcal{G} - \{G\}$ is also a building set.*

(2) *Let G be as in (1) and $G = \oplus_{i=1}^t F_i$ be its irreducible decomposition in $\mathcal{F}_{\mathcal{C}_\mathcal{G}}$. Then a maximal nested set relative to $\mathcal{G}$ is either a maximal nested set relative to $\mathcal{G}'$ not containing $\{F_1, \dots, F_t\}$ or is obtained from a maximal nested set relative to $\mathcal{G}'$ and containing $\{F_1, \dots, F_t\}$ substituting one of the F_i's with G and conversely this operation produces a maximal nested set relative to $\mathcal{G}$.*

(3) *If $\mathcal{S}$ is a maximal nested set in $\mathcal{G}'$ which does not contain $\{F_1, \dots, F_t\}$, there exists $i = 1, \dots, t$ such that $p_\mathcal{S}(F_i) = p_\mathcal{S}(G)$.*

(4) *If $\mathcal{S}$ is a maximal nested set in $\mathcal{G}'$ which contains $\{F_1, \dots, F_t\}$, then $c(F_i) = c(F_j) = p_\mathcal{S}(G)$ for all $i, j = 1, \dots, t$.*

Proof. (1) We verify property (1) of Theorem 2.2. Notice that $\mathcal{C}_\mathcal{G} = \mathcal{C}_{\mathcal{G}'}$ and the fact that $\mathcal{G}' \supset \mathcal{F}_{\mathcal{C}_\mathcal{G}}$ is obvious. Suppose now that $A, B \in \mathcal{G}'$ and B contains an irreducible component of A in $\mathcal{F}_{\mathcal{C}_\mathcal{G}}$. If A itself is in $\mathcal{F}_{\mathcal{C}_\mathcal{G}}$ then $B \supset A$ so $A + B = B \in \mathcal{G}'$. Otherwise $A + B \in \mathcal{G}$ and since $A + B \supset A$ we cannot have $A + B = G$. It follows that $A + B \in \mathcal{G}'$.

(2) A maximal nested set is given by the decomposition of a maximal flag $C_1 \supset C_2 \dots \supset C_h$ in $\mathcal{C}_\mathcal{G} = \mathcal{C}_{\mathcal{G}'}$. If the decomposition of the C_i in $\mathcal{G}$ and $\mathcal{G}'$ coincide, we have the same nested set. The other possibility is that G appears in the $\mathcal{G}$ decomposition of C_h for some h, in which case for every such h the summand G of the $\mathcal{G}'$ decomposition is replaced by the summands $\{F_1, \dots, F_t\}$. Take a maximal

h with G a summand of C_h so that $\{F_1, \dots, F_t\}$ appear in the decomposition of C_h in $\mathcal{G}'$ but not in the one of C_{h+1}, by maximality of h. We can now apply Proposition 2.2, getting that, in the decomposition of C_{h+1} which is maximal in C_h, all the F_i except exactly one appear. The claim follows.

(3) Is clearly equivalent to see that there exists an i such that $p_{\mathcal{S}}(F_i) \supset p_{\mathcal{S}}(F_j)$ for all $j = 1, \dots, t$. Suppose the contrary. Since $\{F_1, \dots, F_t\}$ is not contained in $\mathcal{S}$ there is an i such that $p_{\mathcal{S}}(F_i) \supsetneq F_i$. Up to reordering we can assume that $C = p_{\mathcal{S}}(F_1)$ is maximal among the elements with the previous property and thus that $C \supsetneq F_i$ for $i = 2, \dots, r$ while $p_{\mathcal{S}}(F_j)$ is not comparable with C for $j > r$. It follows that $C \supsetneq \oplus_{i=1}^r F_i$ (since $\oplus_{i=1}^r F_i$ is not in $\mathcal{G}'$) and the set $\{C, F_{r+1}, \dots, F_t\}$ is nested relative to $\mathcal{G}'$ and consists of non comparable elements. On the other hand $C \oplus \oplus_{j=r+1}^t F_j = C + G \supsetneq G$. We deduce from Theorem 2.3 (3) that $C \oplus \oplus_{j=r+1}^t F_j \in \mathcal{G}'$ getting a contradiction.

(4) is clear from (2). $\square$

§3. The geometry of $Y_{\mathcal{G}}$

3.1 Let us now fix a building set $\mathcal{G}$ and let $\mathcal{A}_{\mathcal{G}} = \mathcal{A}$ for simplicity be the complement of the union of $G^\perp$, $G \in \mathcal{G}$. As in 1.1 we take the variety $Y_{\mathcal{G}}$ as the closure of the embedding of $\mathcal{A}$ in $V \times \prod_{G \in \mathcal{G}} \mathbb{P}_G$.

Take a $\mathcal{G}$-nested set $\mathcal{S}$ and a marked basis b adapted to it. Let $p(X) = p_{\mathcal{S}}(X)$ be defined as in 1.2 for any subset $X \subset V$ not coinciding with $\{0\}$. Let $\rho_m : K^b \to V$ be the morphism associated to $\mathcal{S}$ and B in 1.4.3.

LEMMA.

(1) *Given any $x \in V^* - \{0\}$, suppose $A = p(x) \in \mathcal{S}$. Then $x = x_A P_x(u_v)$, where P_x is a polynomial depending only on the variables u_v, $v < x_A$.*

(2) *If $\mathcal{S}$ is maximal nested and $G \in \mathcal{G}$ then $p(G) = A \in \mathcal{S}$ and there is an $x \in G$ with $A = p(x)$ such that, writing $x = x_A P_x(u_v)$, P_x does not vanish in 0.*

Proof. Since $b_A = \{v \in b | v \leq x_A\}$ is a basis of A we have an expression $x = \sum_{v \leq x_A} a_v v = x_A(a_A + \sum_{v < x_A} a_v \frac{v}{x_A})$. Substitute for the v's their expression in the u's. The claim follows since $\frac{v}{x_A} = \prod_{v \leq y < x_A} u_y$.

For the second part the fact that $p(G) \in \mathcal{S}$ follows from the maximality of $\mathcal{S}$.

Let $\{A_i\}_{i=1,\dots,n}$ be the set of maximal elements in $\mathcal{S}$ properly contained in A. From Proposition 2.4.2, $G + \oplus A_i = A$ hence there must be $x \in G$ written as $x = \sum_{v \leq x_A} a_v v = x_A(a_A + \sum_{v < x_A} a_v \frac{v}{x_A})$ with $a_A \neq 0$. $\square$

From the previous lemma, for a $G \in \mathcal{G}$ we shall define polynomials $P_x^G(u)$, $x \in G$ (functions on K^b) by the formula $x = x_A P_x^G(u)$. In K^b the subvariety defined by the vanishing of these polynomials will be denoted by Z_G. It is defined in such a way that the map $K^b \to V \to V/G^\perp$ given by the coordinate functions $x \in G$ can be composed in $K^b - Z_G$ with the rational map $V/G^\perp \to \mathbb{P}(V/G^\perp) = \mathbb{P}_G$ giving a regular morphism.

DEFINITION. Given a $\mathcal{G}$ nested set $\mathcal{S}$ define the open set $\mathcal{U}_\mathcal{S}^b$ or $\mathcal{U}_\mathcal{S}^b(\mathcal{G})$ as the complement in K^b of the union of all the varieties Z_G, $G \in \mathcal{G}$.

On the set $\mathcal{U}_\mathcal{S}^b$ all the rational morphisms to $\mathbb{P}_G$ are defined and so we get an embedding $j_\mathcal{S}^b$ of $\mathcal{U}_\mathcal{S}^b$ in $Y_\mathcal{G}$.

REMARK. Given a $\mathcal{G}$ nested set $\mathcal{S}$ if this is also a $\mathcal{H}$ nested set for another set $\mathcal{H} \supset \mathcal{G}$ then, by its very definition, we obtain that $\mathcal{U}_\mathcal{S}^b(\mathcal{H})$ is an open subset of $\mathcal{U}_\mathcal{S}^b(\mathcal{G})$.

From the construction of the open set $\mathcal{U}_\mathcal{S}^b$ and the formula $x_A = \prod_{B \supset A} u_B$ it is clear that the complement in it of the divisors of equations $u_A = 0$, $A \in \mathcal{S}$ maps to the open set $\mathcal{A}$ injectively. The divisor $u_A = 0$ maps to $A^\perp$.

We can now prove

THEOREM.

(1) *The map $j_\mathcal{S}^b$ is an open embedding.*

(2) $Y_\mathcal{G} = \cup_\mathcal{S} j_\mathcal{S}^b(\mathcal{U}_\mathcal{S}^b)$. *In particular $Y_\mathcal{G}$ is smooth.*

(3) *Set $D_\mathcal{S}^b$ equal to the divisor in $\mathcal{U}_\mathcal{S}^b$ defined by $\prod_{A \in \mathcal{S}} u_A = 0$. Set $D = \cup_\mathcal{S} j_\mathcal{S}^b(D_\mathcal{S}^b)$. Then $\mathcal{A} = Y_\mathcal{G} - D$ and D is a divisor with normal crossings.*

Proof. Consider the open embedding $i_\mathcal{S}^b : K^b \to Y_\mathcal{S}$ defined in Proposition 1.5. Then (1) is clear by the diagram

$$\begin{tikzcd} \mathcal{U}_\mathcal{S}^b \arrow[r, "j_\mathcal{S}^b"] \arrow[d, "i"'] & Y_\mathcal{G} \arrow[d, "\pi"'] & \\ K^b \arrow[r, "i_\mathcal{S}^b"] & Y_\mathcal{S} \arrow[r] & V \times_{A \in \mathcal{S}} \mathbb{P}_A \end{tikzcd} \tag{3.1.1}$$

since π is birational and $i_\mathcal{S}^b$, i are open embeddings. From now on we shall identify $\mathcal{U}_\mathcal{S}^b$ with its image via $j_\mathcal{S}^b$.

To see the second part, we are now going to show, using the valuative criterion for properness, that the projection map $\delta : Y_\mathcal{G} \to V$ is proper when restricted to $\cup_\mathcal{S} \mathcal{U}_\mathcal{S}^b$.

For this take a curve $f(t)$ in V defined in a neighborhood of 0 and with values in $\mathcal{A}$ in a punctured neighborhood of 0. For every $A \in \mathcal{C}_{\mathcal{G}}$ let n_A be the order in 0 of tangency to $A^{\perp}$ which by definition is the minimum of the orders of the functions $< x|f(t) >$ in 0 as $x \in A$. Let $\mathcal{C}_{\mathcal{G},m} := \{A \in \mathcal{C}_{\mathcal{G}} | n_A \geq m\}$. This set is clearly closed under sum and let C_m be the maximum element in it. The subspaces C_m form a flag, its decomposition $C_m := \oplus_i A_i^{(m)}$ in $\mathcal{G}$ is a nested set which we can complete to a maximal one $\mathcal{S}$ of $\mathcal{G}$.

If $G \in \mathcal{G}$ and $m = n_G$ then necessarily $G \subset A_i^{(m)}$ for some i and the generic element x of G gives a function $< x|f(t) >$ of order m in 0 while $n_{p(G)} = m$. Since the spaces $A_i^{(m)}$ are the components of C_m it is clear that we can choose an adapted basis b in a generic way, such that if $U \in \mathcal{S}$ is contained in $\mathcal{C}_{\mathcal{G},m-1}$ and not in $\mathcal{C}_{\mathcal{G},m}$ then for the vectors $v \in U$ in the adapted basis not in $A_i^{(m)}$, the order of the function $< v|f(t) >$ in 0 is $m-1$. We consider the open set $\mathcal{U}_{\mathcal{S}}^b$ with its coordinates, compose with the curve obtaining functions $u_v(t) = \frac{<v|f(t)>}{<c(v)|f(t)>}$ where the order of the denominator is always less or equal than the order of the numerator, or $u_A = x_A$ if A is maximal. We claim then that $f(t)$ has a limit in $\mathcal{U}_{\mathcal{S}}^b$ as t tends to zero. By the definition of the set $\mathcal{S}$ it follows that the functions u_A are regular in 0, thus $f(t)$ has a limit in K^b for t tending to zero. We still have to show that the limit in 0 does not lie in any of the subvarieties Z_G that we have removed. For this let $G \in \mathcal{G}$ we know that for a generic $x \in G$ we have $p(x) = A$ and $n_x = n_A$ set $x = x_A P_x$. Since we have that $n_x = n_{x_A}$ the order of $P_x(t)$ in 0 is zero as desired.

(3) The fact that D is a divisor with normal crossings is clear. As for the rest it follows immediately, since from Proposition 1.4 we deduce that the image of $D_{\mathcal{S}}^b$ under the projection to V is contained in $V - \mathcal{A}$ while $\mathcal{U}_{\mathcal{S}}^b - D_{\mathcal{S}}^b \subset \mathcal{A}$. □

3.2 We want to prove now by induction a more precise statement which will be useful in cohomology computations. We shall use the notations introduced above.

THEOREM.

(1) *The complement D of $\mathcal{A}$ in $Y_{\mathcal{G}}$ is the union of smooth irreducible divisors D_G indexed by the elements $G \in \mathcal{G}$, where D_G is the unique irreducible component in D such that $\delta(D_G) = G^{\perp}$.*

(2) *The divisors $D_{A_1}, \dots, D_{A_n}$ have nonempty intersection if and only if the set $\{A_1, \dots, A_n\}$ is nested in $\mathcal{G}$. In this case the intersection is transversal and irreducible.*

Consider the following two cases:

a) *G is a minimal element in $\mathcal{G} - \mathcal{F}_{\mathcal{G}}$ and $\mathcal{G}' = \mathcal{G} - \{G\}$.*

b) *G is a minimal element in $\mathcal{G} = \mathcal{F}_{\mathcal{G}}$ and $\mathcal{G}' = \mathcal{G} - \{G\}$.*

Then,

(3) *in case* a) *let* $G = \oplus_{i=1}^{t} F_i$ *be the irreducible decomposition of* G. $Y_{\mathcal{G}}$ *is obtained from* $Y_{\mathcal{G}'}$ *by blowing up the transversal intersection* T_G *of the divisors* D_{F_i} *corresponding to the elements* F_i *decomposing* G.

(4) *In case* b) $Y_{\mathcal{G}}$ *is obtained from* $Y_{\mathcal{G}'}$ *by blowing up the proper transform* T_G *of* $G^{\perp}$ *and* T_G *isomorphic to the variety* $Y_{\overline{\mathcal{G}}}$ *where* $\overline{\mathcal{G}}$ *is induced in* V^*/G *by* $\mathcal{F}_{\mathcal{G}}$ *as in Prop.* 2.3 (4).

Proof. We proceed by induction on the cardinality of $\mathcal{G}$.

We first assume $\mathcal{G} \supsetneq \mathcal{F}_{\mathcal{G}}$ and take G minimal in $\mathcal{G} - \mathcal{F}_{\mathcal{G}}$ as in case a) above. So, by induction we assume that (1) and (2) hold for $\mathcal{G}' = \mathcal{G} - \{G\}$ and show (3).

Consider the projection $\pi : Y_{\mathcal{G}} \to Y_{\mathcal{G}'}$. Take a maximal nested set $\mathcal{T}$ relative to $\mathcal{G}'$, a marked basis adapted to $\mathcal{T}$ and take the open set $\mathcal{U}_{\mathcal{T}}^{b} \subset Y_{\mathcal{G}'}$. We distinguish two cases. If $\mathcal{T}$ does not contain $\{F_1, \ldots, F_t\}$ then by Proposition 2.5 (2) $\mathcal{T}$ is also a maximal nested set relative to $\mathcal{G}$ and $\mathcal{U}_{\mathcal{T}}^{b}$ does not intersect $\cap_{i=1}^{t} D_{F_i}$. We claim that the embedding of $\mathcal{U}_{\mathcal{T}}^{b}$ into $Y_{\mathcal{G}'}$ lifts to an embedding into $Y_{\mathcal{G}}$. To see this we have to show that the rational morphism $\mathcal{U}_{\mathcal{T}}^{b} \to \mathbb{P}_G$ is a well defined regular morphism.

Set $A = P_{\mathcal{S}}(G)$. Take $x \in G$ and write it as in 3.1 as $x = x_A P_x(u)$. To see that the above morphism is regular we need to see that the polynomials $P_x(u)_{|x \in G}$ have no common zero in $\mathcal{U}_{\mathcal{T}}^{b}$. But using Proposition 2.5.3 we can take F_i with $p_{\mathcal{S}}(F_i) = A$ and write $x = x_A P_x(u)$ for $x \in F_i$. By the definition of $\mathcal{U}_{\mathcal{T}}^{b}$ the polynomials $P_x(u)_{|x \in F_i}$ have no common zero in $\mathcal{U}_{\mathcal{T}}^{b}$ and this obviously implies the claim.

Suppose now that $\mathcal{T}$ does contain $\{F_1, \ldots, F_t\}$. It follows from Proposition 1.4 that the intersection of D_{F_i} with $\mathcal{U}_{\mathcal{T}}^{b}$ is the divisor of equation u_{F_i}. Thus we need to show that the subscheme where the rational morphism $\mathcal{U}_{\mathcal{T}}^{b} \to \mathbb{P}_G$ is not defined has equations $u_{F_i} = 0$ for $i = 1, \ldots, t$. Using Proposition 2.5.4 we have that $c(F_i) = A$ does not depend on $i = 1, \ldots, t$, so that, as basis for G, we can take $b_G := \cup_{i=1}^{t} b_{F_i}$. We deduce that our subscheme has equations given by

$$\begin{aligned} u_v u_{F_i} &= 0 \quad \text{if } v \in b_{F_i} \text{ and } v \text{ is not marked,} \\ u_{F_i} &= 0 \quad \text{otherwise.} \end{aligned}$$

This immediately implies the claim.

We now assume $\mathcal{G} = \mathcal{F}_{\mathcal{G}}$ and take G minimal in $\mathcal{F}_{\mathcal{G}}$ as in case b) above. By induction we assume that (1) and (2) hold for $\mathcal{G}' = \mathcal{G} - \{G\}$ and show (4). The fact that the proper transform T_G of G in $Y_{\mathcal{G}'}$ equals $Y_{\overline{\mathcal{G}}}$ is immediate once we notice that for any $H \in \mathcal{G}'$ the restriction of the rational morphism $V \to \mathbb{P}_H$ to $G^{\perp}$ factors through the rational morphisms

$$G^{\perp} \to \mathbb{P}(G^{\perp}/G^{\perp} \cap H^{\perp}) = \mathbb{P}(G^{\perp}/(G+H)^{\perp}) = \mathbb{P}_{\overline{H}}.$$

It remains to see that the subscheme where the rational morphism $Y_{\mathcal{G}'} \to \mathbb{P}_G$ is not defined coincides with T_G. Take a maximal nested set $\mathcal{T}$ relative to $\mathcal{G}'$ a marked basis adapted to $\mathcal{T}$ and take the open set $\mathcal{U}^b_{\mathcal{T}}(\mathcal{G}') \subset Y_{\mathcal{G}'}$. There are two possibilities: either $\mathcal{T}$ is also maximal in $\mathcal{G}$, in this case the projection map $Y_{\mathcal{G}} \to Y_{\mathcal{G}'}$ induces an open embedding $\mathcal{U}^b_{\mathcal{T}}(\mathcal{G}) \subset \mathcal{U}^b_{\mathcal{T}}(\mathcal{G}')$ (Remark 3.1), or $\mathcal{T} \cup \{A\}$ is maximal nested in $\mathcal{G}$. We can embed $\mathcal{U}^b_{\mathcal{T}}$ into $Y_{\mathcal{T}}$. Then using Proposition 1.6.1 we deduce that on the open set $\mathcal{U}^b_{\mathcal{T}}(\mathcal{G}')$ the map $Y_{\mathcal{G}} \to Y_{\mathcal{G}'}$ is a blow up along the intersection with T_G.

Having proved (3) and (4) under our inductive assumption we deduce (1) and (2).

(1) is immediate setting D_G equal to the exceptional divisor of the blow up $Y_{\mathcal{G}} \to Y_{\mathcal{G}'}$ and for $H \in \mathcal{G}'$ equal to the proper transform of the corresponding divisor D'_H in $Y_{\mathcal{G}'}$. We only have to remark that the intersection of D'_H with T_G in case b) equals the smooth irreducible divisor $D_{\overline{H}}$.

As we have already pointed out, given one of the open sets $\mathcal{U}^b_{\mathcal{T}}$, the divisor D_A meets it if and only if $A \in \mathcal{T}$. Furthermore by Remark 1.4, if $A \in \mathcal{T}$ then $D_A \cap \mathcal{U}^b_{\mathcal{T}}$ is the divisor of equation $u_A = 0$. This fact immediately implies all the statements in (2) except for the irreducibility of $Z = D_{A_1} \cap \ldots \cap D_{A_n}$ in the case in which the set $\{A_1, \ldots, A_n\}$ is nested relative to $\mathcal{G}$. To see this notice that if $\{A_1, \ldots, A_n\}$ does not contain G, then by the above analysis Z is just the proper transform of the corresponding variety in $Y_{\mathcal{G}'}$.

If $G \in \{A_1, \ldots, A_n\}$, say $G = A_1$ then in case a) above Z is the preimage in $Y_{\mathcal{G}}$ of the variety $T_G \cap D'_{A_1} \cap \ldots \cap D'_{A_n}$ which is irreducible by the inductive hypothesis.

In case b) Z is the preimage of the subvariety $D_{\overline{A}_1} \cap \ldots \cap D_{\overline{A}_n}$ in T_G which is again irreducible by the inductive hypothesis. □

DEFINITION. Given a nested set $\mathcal{S} \subset \mathcal{G}$, we shall set $D_{\mathcal{S}}$ equal to the smooth irreducible subvariety of $Y_{\mathcal{G}}$ which is the transversal intersection of the divisors D_A with $A \in \mathcal{S}$.

REMARK. By the explicit induction we can see also directly the validity of the first theorem 3.1.

Let us observe a consequence of the previous construction. From point (3) and (4) we deduce that the map $p : Y_{\mathcal{G}} \to Y_{\mathcal{G}'}$ is an isomorphism over the complement of T_G and the preimage of T_G, the exceptional divisor D_G, is the projectivized normal bundle of T_G in $Y_{\mathcal{G}}$.

Consider the commutative diagram:

$$\begin{tikzcd} D_G \arrow[r, "i"] \arrow[d, "p"] & Y_{\mathcal{G}} \arrow[r, "j"] \arrow[d] & Y_{\mathcal{G}'} \times \mathbb{P}_G \arrow[d] \\ T_G \arrow[r, "i'"] & Y_{\mathcal{G}'} \arrow[r, "1"] & Y_{\mathcal{G}'}. \end{tikzcd} \tag{3.2.1}$$

We deduce a commutative diagram

$$\begin{tikzcd} D_G \arrow[r, "i"] \arrow[d, "p"'] & T_G \times \mathbb{P}_G \arrow[d] \\ T_G \arrow[r, "1"] & T_G \end{tikzcd} \tag{3.2.2}$$

which by the previous discussion implies that $i : D_G \to Z_G \times \mathbb{P}_G$ is an isomorphism.

§4. Projective arrangements and the construction of Fulton-MacPherson

4.1 Associated to a nonempty family $\mathcal{K}$ of non zero subspaces in V^* we can also consider the configuration of linear subspaces $\mathbb{P}(A^\perp)$ in $\mathbb{P}(V)$. As in 1.1 we set $\overline{V}_\mathcal{K} := \cup_{A\in\mathcal{K}}\mathbb{P}(A^\perp)$ to be the union of all the subspaces $\mathbb{P}(A^\perp)$ and $\overline{\mathcal{A}}_\mathcal{K}$ the open set of $\mathbb{P}(V)$ complement of $\overline{V}_\mathcal{K}$.

With the same notations as in 1.1, the multiplicative group K^* acts on $\mathcal{A}_\mathcal{K}$ and $\overline{\mathcal{A}}_\mathcal{K} = (\mathcal{A}_\mathcal{K})/K^*$. The regular morphism $\mathcal{A}_\mathcal{K} \to \prod_{A\in\mathcal{K}} \mathbb{P}_A$, is constant on K^* orbits and we get a morphism $\overline{\mathcal{A}}_\mathcal{K} \to \prod_{A\in\mathcal{K}} \mathbb{P}_A$, its graph is a closed subset of $\overline{\mathcal{A}}_\mathcal{K} \times \prod_{A\in\mathcal{K}} \mathbb{P}_A$ which embeds as open set into $\mathbb{P}(V) \times \prod_{A\in\mathcal{K}} \mathbb{P}_A$. Finally we have an embedding

$$\rho : \overline{\mathcal{A}}_\mathcal{K} \to \mathbb{P}(V) \times \prod_{A\in\mathcal{K}} \mathbb{P}_A$$

as a locally closed subset.

DEFINITION. We let $\overline{Y}_\mathcal{K}$ to be the closure of the image of $\overline{\mathcal{A}}_\mathcal{K}$ under ρ.

The regular morphism $\mathcal{A}_\mathcal{K} \to V \times \prod_{A\in\mathcal{K}} \mathbb{P}_A$ is K^*-equivariant where K^* acts on $\mathcal{A}_\mathcal{K}$, V by multiplication and trivially on $\prod_{A\in\mathcal{F}} \mathbb{P}_A$. Thus K^* acts naturally on $Y_\mathcal{K}$ and the projection $p : Y_\mathcal{K} \to V$ is equivariant.

$Y^0_\mathcal{K} = Y_\mathcal{K} - p^{-1}(0)$ is open in $Y_\mathcal{K}$, K^* stable and it is clearly the closure of $\mathcal{A}_\mathcal{K}$ in the embedding $\mathcal{A}_\mathcal{K} \to V - \{0\} \times \prod_{A\in\mathcal{K}} \mathbb{P}_A$. Furthermore, we have the following commutative diagram in which all the vertical maps are quotients by the free action of K^*, $i, \overline{i}$ are open embeddings, $j, \overline{j}$ closed embeddings:

$$\begin{tikzcd} \mathcal{A}_\mathcal{K} \arrow[r, "i"] \arrow[d] & Y^0_\mathcal{K} \arrow[r, "j"] \arrow[d] & V - \{0\} \times \prod_{A\in\mathcal{K}} \mathbb{P}_A \arrow[d] \\ \overline{\mathcal{A}}_\mathcal{K} \arrow[r, "\overline{i}"] & \overline{Y}_\mathcal{K} \arrow[r, "\overline{j}"] & \mathbb{P}(V) \times \prod_{A\in\mathcal{F}} \mathbb{P}_A \end{tikzcd} \tag{4.1.1}$$

since $V - \{0\} \times \prod_{A\in\mathcal{K}} \mathbb{P}_A/K^* = \mathbb{P}(V) \times \prod_{A\in\mathcal{K}} \mathbb{P}_A$ we clearly have that $\overline{Y}_\mathcal{K} = \overline{Y}^0_\mathcal{K}/K^*$.

REMARK. Since the canonical rational map of V to $\mathbb{P}(V) = \mathbb{P}_{V^*}$ is defined on $Y^0_{\mathcal{K}}$, there is no restriction to assume that V^* is an element of $\mathcal{K}$.

Add V^* to $\mathcal{K}$ getting a new family $\mathcal{K}'$ and consider the total space $\mathbb{E}_V$ of the tautological bundle of $\mathbb{P}(V)$. It is the closure of the natural map $V - \{0\} \to V \times \mathbb{P}(V)$.

The diagram 4.1.1 can thus be modified to

$$\begin{tikzcd} \mathcal{A}_{\mathcal{K}} \arrow[r, "i"] \arrow[d] & Y^0_{\mathcal{K}} \arrow[r, "j"] \arrow[d] & \mathbb{E}_V \times \prod_{A \in \mathcal{K}} \mathbb{P}_A \arrow[d] \\ \overline{\mathcal{A}}_{\mathcal{K}} \arrow[r, "\bar{i}"] & \overline{Y}_{\mathcal{K}} \arrow[r, "\bar{j}"] & \mathbb{P}(V) \times \prod_{A \in \mathcal{F}} \mathbb{P}_A. \end{tikzcd} \tag{4.1.2}$$

Thus we obtain a fiber product diagram:

$$\begin{tikzcd} Y_{\mathcal{K}} \arrow[r, "j"] \arrow[d] & \mathbb{E}_V \arrow[d] \\ \overline{Y}_{\mathcal{K}} \arrow[r, "\bar{j}"] & \mathbb{P}(V). \end{tikzcd} \tag{4.1.3}$$

It is then clear that $Y^0_{\mathcal{K}} = Y^0_{\mathcal{K}'}$ and that $Y_{\mathcal{K}}$ is the pullback under the canonical map $\overline{Y}_{\mathcal{K}} \to \mathbb{P}(V)$ of the tautological line bundle.

We apply all this to the case of a building set $\mathcal{G}$ containing V^* and deduce immediately

THEOREM.

(1) *$\overline{Y}_{\mathcal{G}}$ is a smooth variety.*

(2) *$Y_{\mathcal{G}}$ is the total space of a line bundle on $p^{-1}(0) = D_{V^*}$ and $\overline{Y}_{\mathcal{G}}$ is isomorphic to $p^{-1}(0) = D_{V^*}$.*

Proof. From the previous diagram it follows that $Y_{\mathcal{G}}$ is the total space of a line bundle on $\overline{Y}_{\mathcal{G}}$ hence $\overline{Y}_{\mathcal{G}}$ is isomorphic to the 0 section of this line bundle which is smooth by Theorem 3.1 (2). □

An explicit description in local coordinates is the following.

Take $\mathcal{S}$ nested (containing V^*) and consider a marked basis b and the associated open set $\mathcal{U}^b_{\mathcal{S}}$. By the explicit description of the coordinates u_v, we immediately see that $\mathcal{U}^b_{\mathcal{S}}$ is stable under K^*, the coordinates u_v for v not maximal are fixed, while the maximal coordinate x_{V^*} is multiplied by the scalars in K^*.

If the nested set is maximal contained in a building set $\mathcal{G}$ we can consider, as in 3.1, the open set $\mathcal{U}^b_{\mathcal{S}}$ defined as the complement of the union of the varieties Z_G

of equations P_x, $x \in G$. From the same description the polynomials P_x depend only on the invariant coordinates. Thus $\mathcal{U}_{\mathcal{S}}^b$ is a trivial line bundle on the fixed points of K^* with action on the fiber by scalar multiplication. The fixed points are defined by $x_{V^*} = 0$ and are the intersection of $\mathcal{U}_{\mathcal{S}}^b/K^*$ with the 0 fiber $p^{-1}(0)$. The divisors D_A are K^* stable, and the divisor $D_{V^*} = p^{-1}(0)$.

4.2 We have seen in Theorem 4.1 that the variety $\overline{Y}_{\mathcal{G}}$ equals the divisor D_{V^*} in the corresponding variety $Y_{\mathcal{G}}$. Thus all the relevant geometric properties of $\overline{Y}_{\mathcal{G}}$ are implicit in Theorem 3.1, we make them explicit again, denoting by $\mathcal{G}^0$ the set $\mathcal{G}$ minus the element V^*, and omitting the proof which is a special case of 3.1:

THEOREM.

(1) *The variety $\overline{Y}_{\mathcal{G}}$ is smooth projective and irreducible.*

(2) *The morphism $p : \overline{Y}_{\mathcal{G}} \to \mathbb{P}(V)$ is surjective and restricts to an isomorphism on $\overline{\mathcal{A}}_{\mathcal{G}}$.*

(3) *$\overline{D} = \overline{Y}_{\mathcal{G}} - \overline{\mathcal{A}}_{\mathcal{G}}$ is a divisor with normal crossings. The irreducible components of $\overline{D}$ are smooth and in $1-1$ correspondence with the elements $F \in \mathcal{G}^0$. If we denote by $\overline{D}_F$ the divisor corresponding to F, we have $\pi^{-1}(\mathbb{P}(F^{\perp})) = \cup_{G \supseteq F} \overline{D}_G$.*

(4) *Given a subset $\mathcal{S} \subset \mathcal{G}^0$ the corresponding divisors have nonempty intersection if and only if $\mathcal{S}$ is nested (in $\mathcal{G}$). Furthermore under the identification of $\overline{Y}_{\mathcal{G}}$ with the divisor D_{V^*}, the variety $\cap_{A \in \mathcal{S}} \overline{D}_A$ is identified with $\cap_{A \in \mathcal{S} \cup \{V^*\}} D_A$ and in particular it is irreducible.*

REMARK. We end this section by remarking that if $\mathcal{G}$ is a building set in V^*, then the set $\mathcal{G}'$ of subspaces in $V^* \oplus K$ formed by the subspaces $H \oplus \{0\}$, $H \in \mathcal{G}$, and the line $L = \{0\} \oplus K$ is again a building set and the complement of the union of the corresponding linear spaces in $\mathbb{P}(V \oplus K)$ equals the open set $\mathcal{A}_{\mathcal{G}}$. Thus applying the above results, we obtain a *compactification* of $\mathcal{A}_{\mathcal{G}}$ in which the complement of $\mathcal{A}_{\mathcal{G}}$ is a divisor with normal crossings whose irreducible components, if we identify $\mathcal{G}'$ with $\mathcal{G} \cup \{L\}$ in the obvious way, are in bijection with the subsets $\mathcal{S} \subset \mathcal{G}'$ which are either $\mathcal{G}$-nested subsets or are such that $\mathcal{S} - \{L\}$ is $\mathcal{G}$-nested.

4.3 We are interested here to describe the structure of the subvarieties $D_{\mathcal{S}}$, with $\mathcal{S}$ a nested set, in the variety $\overline{Y}_{\mathcal{G}}$ in the projective case. So we shall assume that $V^* \in \mathcal{G}$ and that $\mathcal{S}$ does contains $\{V^*\}$.

Given $A \in \mathcal{S}$ we consider the space $W_A = A / \sum_{B \in \mathcal{S}\ B \subsetneq A} B$. We set π equal to the canonical projection from A to W_A. Notice that all W_A's are non zero. In W_A we consider the family $\mathcal{C}_A^{\mathcal{S}} = \{D \subset W_A |$ there exists $B \in \mathcal{C}, B \subset A$ with $\pi(B) = D\}$.

It is clear that $\mathcal{C}_A^{\mathcal{S}}$ is closed under sum. We now consider the family $\mathcal{G}_A^{\mathcal{S}} \subset \mathcal{C}_A^{\mathcal{S}}$ consisting of those elements $D \in \mathcal{C}_A^{\mathcal{S}}$ for which there exists a $B \in \mathcal{G}$ with $D = \pi(B)$.

LEMMA.

(1) *$\mathcal{G}_A^{\mathcal{S}}$ is a building family in $\mathcal{C}_A^{\mathcal{S}}$.*

(2) *Let A be a minimal element in $\mathcal{S}$. Consider the projection $\delta : V^* \to V^*/A$. Then the family $\delta(\mathcal{G}) = \{D \subset V^*/A|$ there exits $\tilde{D} \in \mathcal{G}$ with $\delta(\tilde{D}) = D\}$ is a building family and the set $\delta(\mathcal{S} - \{A\})$ is $\delta(\mathcal{G})$-nested.*

Proof. (1) Let $B_1, \dots, B_s$ be the maximal elements in $\mathcal{S}$ which are properly contained in A so that $\sum_{B\in\mathcal{S}\ B\subset A} B = B_1 \oplus \cdots \oplus B_s$.

Let $D \in \mathcal{C}_A^{\mathcal{S}}$. Set $\tilde{D} = \pi^{-1}(D)$. Clearly $\tilde{D} \in \mathcal{C}$, so that if $\tilde{H}_1, \dots, \tilde{H}_t$ are the maximal elements in $\mathcal{G}$ contained in $\tilde{D}$ we have $\tilde{D} = \tilde{H}_1 \oplus \cdots \oplus \tilde{H}_t$. Set $H_i = \pi(\tilde{H}_i)$. By the maximality of the $\tilde{H}_i$'s it follows that for each index $i = 1, \dots, s$ there exists a unique $j = 1, \dots, t$ such that $\tilde{H}_j \supseteq B_i$. It follows from this that the non zero spaces among the H_j's form a direct sum equal to D and lie in $\mathcal{G}_A^{\mathcal{S}}$.

Suppose now that $H \in \mathcal{G}_A^{\mathcal{S}}$ and $H \subset D$. Take a space $\tilde{H} \in \mathcal{G}$ with $\pi(\tilde{H}) = H$. Then clearly $\tilde{H} \subset \tilde{D}$ and by maximality there exists a unique $j = 1, \dots, s$ with $\tilde{H} \subset \tilde{H}_j$. Thus $H \subset H_j$ and our claim follows.

The fact that $\delta(\mathcal{G})$ is a building family is a special case of (1) taking $\mathcal{G}_{V^*}^{\{A,V^*\}}$. We leave the rest to the reader. □

Thus we have associated to each element A in $\mathcal{S}$ a vector space W_A together with a building family $\mathcal{G}_A^{\mathcal{S}}$.

Thus by our previous construction we obtain a projective variety $\overline{Y}_{\mathcal{G}_A^{\mathcal{S}}}$.

We now have

THEOREM. *Let $\mathcal{S}$ be a nested set in $\mathcal{G} - \{V^*\}$. Let $D_{\mathcal{S}} \subset \overline{Y}_{\mathcal{G}}$ be the corresponding subvariety. Then we have a natural isomorphism of $D_{\mathcal{S}}$ with $\prod_{A\in\mathcal{S}} \overline{Y}_{\mathcal{G}_A^{\mathcal{S}}}$.*

Proof. Notice that since $\sum_{A\in\mathcal{S}} \dim W_A = \dim V$, we have that $\dim D_{\mathcal{S}} = \dim \prod_{A\in\mathcal{S}} \overline{Y}_{\mathcal{G}_A^{\mathcal{S}}}$, so in order to prove our claim it suffices to give an embedding of $D_{\mathcal{S}}$ into $\prod_{A\in\mathcal{S}} \overline{Y}_{\mathcal{G}_A^{\mathcal{S}}}$.

We start with the case in which $\mathcal{S} = \{A, V^*\}$.

We divide $\mathcal{G}$ in $\mathcal{G}_1 = \mathcal{G}_A^{\mathcal{S}} = \{B \in \mathcal{G} | B \subseteq A\}$ and $\mathcal{G}_2 = \mathcal{G} - \mathcal{G}_1$.

Recall that $\overline{Y}_{\mathcal{G}}$ is the closure in $\prod_{B\in\mathcal{G}} \mathbb{P}(V/B^\perp) = \prod_{B\in\mathcal{G}_1} \mathbb{P}(V/B^\perp) \times \prod_{C\in\mathcal{G}_2} \mathbb{P}(V/C^\perp)$ of the set $\mathcal{A} = \mathbb{P}(V) - \cup_{B\in\mathcal{C}} \mathbb{P}(B^\perp)$.

Set p_1 (resp. p_2) equal to the projection of $\prod_{B\in\mathcal{G}} \mathbb{P}(V/B^\perp)$ onto $\prod_{B\in\mathcal{G}_1} \mathbb{P}(V/B^\perp)$ (resp. $\prod_{B\in\mathcal{G}_2} \mathbb{P}(V/B^\perp)$) and let $\overline{Y}_1 = p_1(\overline{Y}_{\mathcal{G}})$, $\overline{Y}_2 = p_2(\overline{Y}_{\mathcal{G}})$. Notice that, by definition, these are just the closure of the image of $\mathcal{A}$ under the two projection maps.

Notice that, if we set $\mathcal{A}'$ equal to the the image of $\mathcal{A}$ in $\mathbb{P}(V/A^{\perp})$, clearly the restriction of p_1 to $\mathcal{A}$ factors through $\mathcal{A}'$ and $\mathcal{A}'$ embeds in $\prod_{B\in\mathcal{G}_1}\mathbb{P}(V/B^{\perp})$. By the very definitions its closure is exactly $\overline{Y}_{\mathcal{G}_A^{\mathcal{S}}}$.

Thus we get a map $\overline{Y}_{\mathcal{G}} \to \overline{Y}_{\mathcal{G}_A^{\mathcal{S}}}$ which we can restrict to D_A.

Let us examine now the second projection. Consider the open set in $\mathcal{A}_A \subset \mathbb{P}(A^{\perp})$ defined by $\mathcal{A}_A = \mathbb{P}(A^{\perp}) - \cup_{B\in\mathcal{C}\ B\not\subseteq A}\mathbb{P}((A+B)^{\perp})$. One knows that, under the projection map $p : \overline{Y}_{\mathcal{G}} \to \mathbb{P}(V)$, D_A equals the closure of $p^{-1}(\mathcal{A}_A)$. On the other hand the embedding $\mathcal{A} \to \prod_{B\in\mathcal{G}_2}\mathbb{P}(V/B^{\perp})$ clearly extends to and embedding of the larger open set $\tilde{\mathcal{A}} = \mathbb{P}(V) - \cup_{B\in\mathcal{C}\ B\not\subseteq A}\mathbb{P}(B^{\perp})$ which contains $\mathcal{A}_A$. Furthermore again by definition, the closure of $\mathcal{A}_A$ equals $\overline{Y}_{\mathcal{G}_{V^*}^{\mathcal{S}}}$ and by the above remark we deduce that $p_2(D_A) = \overline{Y}_{\mathcal{G}_{V^*}^{\mathcal{S}}}$.

From the above consideration we deduce the desired embedding $D_A \to \overline{Y}_{\mathcal{G}_A^{\mathcal{S}}} \times \overline{Y}_{\mathcal{G}_{V^*}^{\mathcal{S}}}$ hence our claim.

We now pass to the general case. We proceed by induction on the cardinality of $\mathcal{S}$ and assume that $\mathcal{S}$ has at least three elements. Let A be a minimal element in $\mathcal{S}$. Set $\mathcal{T} = \{A, V^*\}$. Remark that $\overline{Y}_{\mathcal{G}_A^{\mathcal{T}}} = \overline{Y}_{\mathcal{G}_A^{\mathcal{S}}}$, so that the fact that $D_A = \overline{Y}_{\mathcal{G}_A^{\mathcal{T}}} \times \overline{Y}_{\mathcal{G}_{V^*}^{\mathcal{T}}}$ gives us an inclusion $D_{\mathcal{S}} \to \overline{Y}_{\mathcal{G}_A^{\mathcal{S}}} \times \overline{Y}_{\mathcal{G}_{V^*}^{\mathcal{T}}}$. On the other hand one easily sees that $\overline{Y}_{\mathcal{G}_{V^*}^{\mathcal{T}}} = \overline{Y}_{\delta(\mathcal{G})}$ and, under the projection $p_2 : D_A \to \overline{Y}_{\mathcal{G}_{V^*}^{\mathcal{T}}}$, the subvariety $D_{\mathcal{S}}$ maps onto $D_{\delta(\mathcal{S})}$. By induction $D_{\delta(\mathcal{S})} = \prod_{B\in\delta(\mathcal{S})\cup\{V^*/A\}}\overline{Y}_{\delta(\mathcal{G})_B^{\delta(\mathcal{S})}} = \prod_{B\in\mathcal{S}\ B\neq A}\overline{Y}_{\mathcal{G}_B^{\mathcal{S}}}$ and our claim follows. □

REMARKS.

(1) One can treat in the same way the case of the varieties $D_{\mathcal{S}}$ in $Y_{\mathcal{G}}$. The result one obtains is the following. For simplicity we assume that $V^* \in \mathcal{G}$ (the general case follows easily from this). Then we have two cases. If $V^* \in \mathcal{S}$ we are in the projective case which has already been treated. Otherwise set $\mathcal{T} := \mathcal{S} \cup \{V*\}$. Then $D_{\mathcal{S}} = \prod_{A\in\mathcal{S}}\overline{Y}_{\mathcal{G}_A^{\mathcal{T}}} \times Y_{\mathcal{G}_{V^*}^{\mathcal{T}}}$.

(2) Notice that, if $\mathcal{S}$ is a maximal nested set, then for every $A \in \mathcal{S}$ one has that $\overline{Y}_{\mathcal{G}_A^{\mathcal{S}}}$ is a projective space. This follows immediately from maximality.

(3) If one applies our results in the case of the hyperplane configuration of type A_n one recovers the geometry of the moduli space of stable curves of genus zero with $n+2$ labeled pairwise distinct points (see [Ke]).

4.4 Let X be a smooth variety, let Δ be the big diagonal in the product X^n and A its complement. In [F-M] Fulton and MacPherson construct a model Y of i.e. a smooth variety Y with a proper birational morphism $Y \to X$ which is an isomorphism outside Δ and replaces Δ by a divisor with normal crossings. In fact the construction is essentially local and, if X is a vector space Δ is a configuration of subspaces and their model coincides with the canonical model constructed in this paper. We want only remark that there is a somewhat general construction available by the same methods.

Suppose we fix a family of smooth irreducible subvarieties $\mathcal{C}$, closed under intersection, in a variety V. Define in V the function $\phi : V \to \mathcal{C}$ by $\phi(v)$ is the minimal subvariety in $\mathcal{C}$ containing v.

DEFINITION. Given a smooth variety X and a function $\psi : X \to \mathcal{C}$, we say that the pair X, ψ is a $\mathcal{C}$-model if, for every point $p \in X$ there is an analytic isomorphism i between an open neighborhood U_1 of p in X and an open set V_1 of V making the diagram

$$\begin{tikzcd} U_1 \arrow[r, "i"] \arrow[d, "\psi"] & V_1 \arrow[d, "\phi"] \\ \mathcal{C} \arrow[r, "1_{\mathcal{C}}"] & \mathcal{C} \end{tikzcd}$$

commutative.

We consider $\mathcal{C}$ as partially ordered by inclusion. Given any $a \in \mathcal{C}$ set

$$X_a := \{p \in X | \psi(p) \leq a\}.$$

Clearly X_a is either empty or a smooth subvariety of dimension equal to the one of a.

As basic example we take a vector space W of dimension n, $V = W^m$, $\mathcal{C}$ the family of subspaces giving the big diagonal. It is indexed by partitions of $[1, \ldots, m]$. Given a smooth variety M of dimension n let $X := M^m$, define ψ as the function from $X \to \mathcal{C}$ which associates to a point $x_1, \ldots, x_m$ the partition induced by coincidence of coordinates. Then clearly we get a model.

Start now from $\mathcal{C}$ a family of subspaces in a vector space V. According to §3 one can construct a sequence $\mathcal{G}_i$ of building sets and of varieties $Y_i := Y_{\mathcal{G}_i}$ so that, starting from $V = Y_0$, one can obtain Y_{i+1} blowing up Y_i along a smooth locus $a_i \in \mathcal{C}_i$. Let $\mathcal{C}_i$ be the family of subvarieties of Y_i closed under intersection and generated by the divisors D_A, $A \in \mathcal{G}_i$ and by the proper transforms of the subspaces in $\mathcal{C}$. We claim that under these hypotheses there is a sequence of varieties X_i and maps $\psi_i : X_i \to \mathcal{C}_i$ such that, for all i the pair X_i, ψ_i is a $\mathcal{C}_i$ model, and X_{i+i} is obtained from X_i blowing up the subvariety $(X_i)_{a_i}$.

Once we arrive at a Hironaka model for $\mathcal{C}$, we also arrive to such a model for the complement of the varieties X_a in X.

§5. The cohomology

5.1 As in the preceding sections, let $\mathcal{C}$ be a family of subspaces in V^* closed under sum, and let $\mathcal{F} \subset \mathcal{C}$ be the subset of irreducible elements. Let $\mathcal{G} \supset \mathcal{F}$ be a building set. In section 3 we have considered the variety $Y_{\mathcal{G}}$ which is the closure of $\mathcal{A} = V - \cup_{X \in \mathcal{C}} X^{\perp}$ in $V \times (\times_{A \in \mathcal{G}} \mathbb{P}(V/A^{\perp}))$. We have shown that the complement of $\mathcal{A}$ in $Y_{\mathcal{G}}$ is a divisor with normal crossing which is the union of smooth irreducible components D_A indexed by the elements $A \in \mathcal{G}$. Furthermore for any subset $\mathcal{S} = \{A_1, \dots, A_h\} \subset \mathcal{G}$ the transversal intersection $D_{\mathcal{S}} = \cap_{i=1}^{h} D_{A_i}$ is irreducible, and nonempty if and only if $\mathcal{S}$ is nested relative to $\mathcal{G}$. In order to have a uniform notation we shall set $D_{\emptyset} := Y_{\mathcal{G}}$. Our computation of cohomology will be based on the basic cohomology classes $[D_B]$ associated to the divisors $D_B,\ B \in \mathcal{G}$.

For these divisors we want to make the following remark.

Given a maximal G-nested set $\mathcal{S}$ and a choice of a marked adapted basis b, we have defined an open set $\mathcal{U}_{\mathcal{S}}^{b}$. Let $B \in \mathcal{G}$, we define the function $u_B^{\mathcal{S}}$ on $\mathcal{U}_{\mathcal{S}}^{b}$ to be 1 if $B \notin \mathcal{S}$, and u_B (with the notations of 1.4) otherwise. In either case the divisor of $u_B^{\mathcal{S}}$ is $D_B \cap \mathcal{U}_{\mathcal{S}}^{b}$.

By the very definitions the functions $\frac{u_B^{\mathcal{S}}}{u_B^{\mathcal{T}}}$ on $\mathcal{U}_{\mathcal{S}}^{b} \cap \mathcal{U}_{\mathcal{T}}^{b'}$ can be considered as the transition functions of the line bundle $\mathcal{L}_B$ associated to the divisor D_B, and the functions $u_B^{\mathcal{S}}$ define a section s_B of this line bundle with divisor D_B. Let us now use the notations of 1.6. Assume that $A \in \mathcal{F}$ is a minimal element and let $\mathcal{G} := \mathcal{F} - \{A\}$. Fix a basis $v_1, \dots, v_k$ for A. For every maximal nested set $\mathcal{S}$ relative to $\mathcal{G}$ containing A let $\mathcal{S}'$ be the nested set $\mathcal{S}' := \mathcal{S} - \{A\}$ and let b be a a marked basis adapted to $\mathcal{S}'$ extending the given basis for A.

As $\mathcal{S}$ and b vary, the open sets $\mathcal{U}_{\mathcal{S}'}^{b}$ cover a neighborhood $\mathcal{N}$ of the variety Z_A which is the proper transform of $A^{\perp}$ in $Y_{\mathcal{G}}$. Local equations for Z_A in $\mathcal{U}_{\mathcal{S}'}^{b}$ are $u'_v = 0,\ v \in \{v_1, \dots, v_k\}$. Thus the equations $u'_{v_i} = 0$ define divisors D_i in $\mathcal{N}$ and Z_A is the transversal intersection of these divisors. By the formulas $v = u'_v \prod_{B \supsetneq A,\ B \in \mathcal{S}'} u'_B$ we get that the sum of the divisors $\sum_{B \supsetneq A} D_B + D_i$ is the divisor of the function v_i. It follows that:

PROPOSITION. *The normal bundle of the variety Z_A in $Y_{\mathcal{G}}$ is isomorphic to the direct sum of k copies of the line bundles $\otimes_{B \supsetneq A} \mathcal{L}_B^*$.* □

This implies that:

COROLLARY. *The Chern polynomial of the normal bundle of Z_A is*

$$\left(t - \sum_{B \supsetneq A} [D_B] \right)^{\dim(A)}$$

□

We should also remark that, if we consider the composition

$$Y_{\mathcal{G}} \to V \times (\times_{A \in \mathcal{G}} \mathbb{P}(V/A^{\perp})) \to \mathbb{P}_A$$

the pullback of the hyperplane class is given by $\sum_{B \supset A}[D_B]$, as follows from the geometric analysis preceding Theorem 3.1.

Suppose now we have fixed a $\mathcal{G}$-nested set $\mathcal{S} \subset \mathcal{G}$. Take a subset $\mathcal{H} \subset \mathcal{G}$ and an element $B \in \mathcal{G}$ such that $B \supsetneq A$ for all $A \in \mathcal{H}$. Set $\mathcal{S}_B = \{A \in \mathcal{S} | A \subsetneq B\}$ and define

$$d^{\mathcal{S}}_{\mathcal{H},B} = \dim B - \dim \left(\sum_{A \in \mathcal{H} \cup \mathcal{S}_B} A \right).$$

Notice that $d^{\mathcal{S}}_{\mathcal{H},B} \geq 0$ and, if $\mathcal{S}' \supset \mathcal{S}$ are two $\mathcal{G}$-nested sets $d^{\mathcal{S}}_{\mathcal{H},B} \geq d^{\mathcal{S}'}_{\mathcal{H},B}$.

To these data we associate the polynomial in $\mathbb{Z}[c_A]_{A \in \mathcal{G}}$ given by

$$P^{\mathcal{S}}_{\mathcal{H},B} = \prod_{A \in \mathcal{H}} c_A \left(\sum_{C \supset B} c_C \right)^{d^{\mathcal{S}}_{\mathcal{H},B}}. \tag{5.1.1}$$

We let $I_{\mathcal{S}}$ be the ideal in $\mathbb{Z}[c_A]$ generated by these polynomials, for fixed $\mathcal{S}$ as $\mathcal{H}, B$ vary. Notice again that, if $\mathcal{S}' \supset \mathcal{S}$ are two $\mathcal{G}$-nested sets, the polynomial $P^{\mathcal{S}'}_{\mathcal{H},B}$ divides $P^{\mathcal{S}}_{\mathcal{H},B}$ so $I_{\mathcal{S}'} \supset I_{\mathcal{S}}$ and also that we can assume that $\mathcal{H} \cap \mathcal{S}_B = \emptyset$.

LEMMA. *Let $\mathcal{H} \subset \mathcal{G}$ be such that $\mathcal{H} \cup \mathcal{S}$ is not $\mathcal{G}$-nested, then $\prod_{A \in \mathcal{H}} c_A \in I_{\mathcal{S}}$.*

Proof. We apply Proposition 2.4 and find a $B \in \mathcal{G}$ which is a sum of a family of subspaces $\mathcal{H}' \cup \mathcal{S}'$ ($\mathcal{H}' \subset \mathcal{H}$, $\mathcal{S}' \subset \mathcal{S}$) and properly containing all these summands. Since $\mathcal{S}$ is $\mathcal{G}$-nested necessarily $\mathcal{H}'$ is not empty and $\mathcal{S}_B \supset \mathcal{S}'$. Then $d^{\mathcal{S}}_{\mathcal{H}',B} = 0$ and $P^{\mathcal{S}}_{\mathcal{H}',B} = \prod_{A \in \mathcal{H}'} c_A$ divides the product $\prod_{A \in \mathcal{H}} c_A$ which therefore lies in $I_{\mathcal{S}}$ as claimed. □

Notice that, if $\overline{\mathcal{H}}$ denotes the set of maximal elements of $\mathcal{H}$ we have that $P^{\mathcal{S}}_{\overline{\mathcal{H}},B}$ divides $P^{\mathcal{S}}_{\mathcal{H},B}$. Thus $I_{\mathcal{S}}$ is generated by the polynomials $\prod_{A \in \mathcal{H}} c_A$ for $\mathcal{H} \cup \mathcal{S}$ not $\mathcal{G}$-nested, and $P^{\mathcal{S}}_{\mathcal{H},B}$ for $\mathcal{H} \cup \mathcal{S}$, $\mathcal{G}$-nested; moreover the spaces in $\overline{\mathcal{H}}$ form the decomposition of their span $< \mathcal{H} >$ (since we can assume that they are also non comparable).

5.2 Let us now compute the cohomology of the varieties $D_{\mathcal{S}}$. We have

THEOREM. *The natural map*

$$\phi_{\mathcal{S}} : \mathbb{Z}[c_A] \to H^*(D_{\mathcal{S}}, \mathbb{Z}) \tag{5.2.1}$$

defined by sending c_A to the cohomology class $[D_A]$ (restricted to $D_{\mathcal{S}}$) induces an isomorphism between $H^*(D_{\mathcal{S}}, \mathbb{Z})$ and $\mathbb{Z}[c_A]/I_{\mathcal{S}}$. In particular if $\mathcal{S}' \supset \mathcal{S}$ the restriction map $H^*(D_{\mathcal{S}}, \mathbb{Z}) \to H^*(D_{\mathcal{S}'}, \mathbb{Z})$ is surjective.

Proof. The proof will be given in various steps. In each step we are going to use the following lemma [Ke].

LEMMA.

(1) *Let Z be a smooth variety $W \subset Z$ a smooth subvariety. Assume that the restriction map $H^*(Z, \mathbb{Z}) \to H^*(W, \mathbb{Z})$ is surjective with kernel J. Let Z' denote the blow up of Z along W. Denote by E both the exceptional divisor and its class in $H^2(Z', \mathbb{Z})$. Then*

$$H^*(Z', \mathbb{Z}) \equiv H^*(Z, \mathbb{Z})[E]/(J \cdot E, P_{Z/W}(-E)),$$

where $P_{Z/W}(x) \in H^*(Z, \mathbb{Z})[x]$ is any polynomial whose restriction to $H^*(W, \mathbb{Z})$ $[x]$ gives the Chern polynomial of the normal bundle of W in Z.

(2) *$H^*(E, \mathbb{Z}) \equiv H^*(W, \mathbb{Z})[E]/(P_{Z/W}(-E))$, where $P_{Z/W}(x) \in H^*(W, \mathbb{Z})[x]$ denotes the Chern polynomial of the normal bundle of W in Z.* □

Let us now start with the first step in the proof.

Step 1. Assume that our theorem holds for $\mathcal{F}$. Then it holds for any building set $\mathcal{G}$ with $\mathcal{F} \subset \mathcal{G} \subset \mathcal{C}_{\mathcal{F}}$.

We proceed by induction with the notations and assumptions of 2.5 and assume the theorem proved for $\mathcal{G}' = \mathcal{G} - \{G\}$. For a $\mathcal{G}'$-nested subset $\mathcal{S} \subset \mathcal{G}'$ set $D'_{\mathcal{S}}$ equal to the corresponding subvariety in $Y_{\mathcal{G}'}$ and let

$$\pi : Y_{\mathcal{G}} \to Y_{\mathcal{G}'}$$

denote the natural projection. Recall that $Y_{\mathcal{G}}$ is obtained by blowing up $Y_{\mathcal{G}'}$ along the subvariety $D'_{\{F_1,\dots,F_t\}}$, where $\{F_1, \dots, F_t\}$ is the irreducible decomposition of G in $\mathcal{G}'$. In particular this implies that

$$\pi^*([D'_A]) = \begin{cases} [D_A] & \text{if } A \neq F_i \\ [D_A] + [D_G] & \text{if } A = F_i. \end{cases} \tag{5.2.2}$$

Let us now take a $\mathcal{G}$-nested set $\mathcal{S}$. We distinguish two cases.

Case 1. $G \notin \mathcal{S}$. In this case $\mathcal{S}$ is also $\mathcal{G}'$-nested and $D_{\mathcal{S}}$ is obtained from $D'_{\mathcal{S}}$ blowing up its intersection with $D'_{\{F_1,\dots,F_t\}}$. From the above considerations we have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[c'_A]_{A \in \mathcal{G}'} & \xrightarrow{\phi'_{\mathcal{S}}} & H^*(D'_{\mathcal{S}}, \mathbb{Z}) \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\pi^*} \\ \mathbb{Z}[c_A]_{A \in \mathcal{G}} & \xrightarrow{\phi_{\mathcal{S}}} & H^*(D_{\mathcal{S}}, \mathbb{Z}) \end{array} \tag{5.2.3}$$

where

$$\gamma(c'_A) = \begin{cases} c_A & \text{if } A \neq F_i \\ c_A + c_G & \text{if } A = F_i. \end{cases} \tag{5.2.4}$$

We have two subcases:

Subcase a) $\mathcal{S} \cup \{F_1, \ldots, F_t\}$ is not nested relative to $\mathcal{G}'$.

The intersection $D'_{\mathcal{S}} \cap D'_{\{F_1,\ldots,F_t\}}$ is empty and π induces an isomorphism between $D_{\mathcal{S}}$ and $D'_{\mathcal{S}}$. In this case the kernel of $\phi_{\mathcal{S}}$ is the ideal generated by $\gamma(I'_{\mathcal{S}})$ and by c_G.

Let us show that $\ker \phi_{\mathcal{S}} \subset I_{\mathcal{S}}$. The fact that $\mathcal{S} \cup \{F_1, \ldots, F_t\}$ is not $\mathcal{G}'$-nested clearly implies, by our description of $\mathcal{G}$-nested sets (Prop. 2.5), that $\mathcal{S} \cup \{G\}$ is not nested relative to $\mathcal{G}$ so that $c_G \in I_{\mathcal{S}}$ by Lemma 5.1. On the other hand by the definition of γ it follows immediately that $\gamma(I'_{\mathcal{S}}) \subset I_{\mathcal{S}} + (c_G) \subset I_{\mathcal{S}}$.

Let us now prove that $I_{\mathcal{S}} \subset \ker \phi_{\mathcal{S}}$. Take one of the generators $P^{\mathcal{S}}_{\mathcal{H},B}$ of $I_{\mathcal{S}}$. If $G \in \mathcal{H}$ there is nothing to prove otherwise one immediately verifies, from the definition of γ, that $P^{\mathcal{S}}_{\mathcal{H},B} = \gamma(P^{\mathcal{S}}_{\mathcal{H},B})$ modulo (c_G) as desired.

Subcase b) Assume now that $\mathcal{S} \cup \{F_1, \ldots, F_t\}$ is $\mathcal{G}'$-nested. Then $D_{\mathcal{S}}$ is obtained blowing up $D'_{\mathcal{S}}$ along $D'_{\mathcal{S}\cup\{F_1,\ldots,F_t\}}$. We can assume up to reordering that $\{F_1,\ldots, F_h\}$ equals the subset of elements in $\{F_1, \ldots, F_t\}$ which are not in $\mathcal{S}$. Thus $D_{\mathcal{S}}$ is obtained blowing up $D'_{\mathcal{S}}$ along the transversal intersection of the divisors $D'_{\mathcal{S}} \cap D'_{F_i}$ for $i = 1, \ldots, h$.

We can thus apply in this situation Lemma 5.2 and obtain that $\ker \phi_{\mathcal{S}}$ is the ideal generated by $\gamma(I'_{\mathcal{S}})$, $c_G\gamma(I'_{\mathcal{S}\cup\{F_1,\ldots,F_t\}})$ and $\prod_{i=1}^h c_{F_i}$.

Let us show that $\ker \phi_{\mathcal{S}} \subset I_{\mathcal{S}}$. The fact that $\prod_{i=1}^h c_{F_i} \in I_{\mathcal{S}}$ follows since $\mathcal{S} \cup \{F_1, \ldots, F_h\}$ is not nested relative to $\mathcal{G}$. Take now a generator $P^{\mathcal{S}\cup\{F_1,\ldots,F_t\}}_{\mathcal{H},B}(c'_A)$ of $I'_{\mathcal{S}\cup\{F_1,\ldots,F_t\}}$; by definition

$$\gamma\left(P^{\mathcal{S}\cup\{F_1,\ldots,F_t\}}_{\mathcal{H},B}(c'_A)\right) = P^{\mathcal{S}\cup\{F_1,\ldots,F_t\}}_{\mathcal{H},B}(c_A).$$

Remark that either $\{B, G\}$ is $\mathcal{G}$-nested and B and G are not comparable or $B + G \in \mathcal{G}'$. In the first situation we get that $P^{\mathcal{S}\cup\{F_1,\ldots,F_t\}}_{\mathcal{H},B}(c_A) \in I_{\mathcal{S}}$.

In the second notice that, if $A \supset B$ but $A \not\supseteq G$, then $\{A, G\}$ is not $\mathcal{G}$-nested, otherwise, since $B \subset A$,$\{B, G\}$ would also be $\mathcal{G}$-nested.

From this we deduce that $\overline{d} := d^{\mathcal{S}\cup\{F_1,\ldots,F_t\}}_{\mathcal{H},B} - d^{\mathcal{S}}_{\mathcal{H}\cup\{G\},B+G} \geq 0$ and so

$$c_G P^{\mathcal{S}\cup\{F_1,\ldots,F_t\}}_{\mathcal{H},B} = P^{\mathcal{S}}_{\mathcal{H}\cup\{G\},B+G}\left(\sum_{A\supseteq B+G} A\right)^{\overline{d}} \mod I_{\mathcal{S}},$$

and $c_G\gamma(I'_{\mathcal{S}\cup\{F_1,\dots,F_t\}}) \subset I_{\mathcal{S}}$.

Take now a generator $P^{\mathcal{S}}_{\mathcal{H},B}(c'_A)$ of $I'_{\mathcal{S}}$. Set $\mathcal{H}' = \mathcal{H} - \{F_1, \dots, F_t\}$ and remark that the polynomial $\prod_{A\in\mathcal{H}'} c_A(\sum_{C\supset B} c_C)^{d^{\mathcal{S}}_{\mathcal{H},B}}$ lies in $I'_{\mathcal{S}\cup\{F_1,\dots,F_t\}}$. But then

$$\gamma(P^{\mathcal{S}}_{\mathcal{H},B}(c'_A)) = P^{\mathcal{S}}_{\mathcal{H},B}(c_A) + c_G Q \prod_{A\in\mathcal{H}'} c_A \left(\sum_{C\supset B} c_C\right)^{d^{\mathcal{S}}_{\mathcal{H},B}} \in I_{\mathcal{S}} \tag{5.2.5}$$

with $Q \in \mathbb{Z}[c_A]$, as desired.

Let us now show that $I_{\mathcal{S}} \subset \ker \phi_{\mathcal{S}}$.

Formula 5.2.5 shows that if $P^{\mathcal{S}}_{\mathcal{H},B}(c_A)$ is a generator of $I_{\mathcal{S}}$ with $G \notin \mathcal{H}$, then $P^{\mathcal{S}}_{\mathcal{H},B}(c_A) \in \ker \phi_{\mathcal{S}}$. Suppose now $G \in \mathcal{H}$, we can clearly assume that none of the F_i's lies in $\mathcal{H}$. Then the polynomial $\frac{P^{\mathcal{S}}_{\mathcal{H},B}(c_A)}{c_G} = \gamma(P^{\mathcal{S}\cup\{F_1,\dots,F_t\}}_{\mathcal{H},B}(c'_A))$. We immediately deduce that $P^{\mathcal{S}}_{\mathcal{H},B}(c_A) \in \ker \phi_{\mathcal{S}}$ as desired.

Case 2. Assume $G \in \mathcal{S}$. Set $\mathcal{S}' = \mathcal{S} - \{G\}$. It follows that $D_{\mathcal{S}} = \pi^{-1}(D_{\mathcal{S}'\cup\{F_1,\dots,F_t\}})$ and the result follows in a completely analogous way to that of Case 1 using the second part of Lemma 5.2. We leave the details to the reader.

Step 2. We assume now that $Y_{\mathcal{F}}$ is obtained from $Y_{\mathcal{G}}$, where $\mathcal{G} := \mathcal{F} - \{A\}$, A a minimal irreducible, by blowing up the proper transform Z_A of $A^{\perp}$. We take a $\mathcal{G}$-nested set $\mathcal{S}$ in $\mathcal{F}$ and want to compute the cohomology of $D_{\mathcal{S}}$. Let us denote by $\overline{\mathcal{G}}$ the family of subspaces $\overline{B} = B + A/A$ with $B \in \mathcal{G}$ defining the variety $Z_A = Y_{\overline{\mathcal{G}}}$. In general, if $\mathcal{H} = \{H_1, \dots, H_r\} \subset \mathcal{G}$, we shall set $\overline{\mathcal{H}} = \{\overline{H_1}, \dots, \overline{H_r}\} \subset \overline{\mathcal{G}}$.

We need to distinguish several cases, if $\mathcal{S} \subset \mathcal{G}$ we wish to distinguish the variety $D_{\mathcal{S}} \subset Y_{\mathcal{S}}$ defined by $\mathcal{S}$ in $\mathcal{F}$ by the variety defined by the same set in $\mathcal{G}$ which we will denote by $D'_{\mathcal{S}}$, similarly for $I_{\mathcal{S}}$ and $I'_{\mathcal{S}}$, $P^{\mathcal{S}}_{\mathcal{H},B}, {P'}^{\mathcal{S}}_{\mathcal{H},B}$.

As in Step 1 we present the cohomology of $Y_{\mathcal{G}}$ and its subvarieties under consideration as quotients of the polynomial ring $\mathbb{Z}[c'_B]$ for $B \in \mathcal{G}$, where the generators c'_B correspond to the cohomology classes of the divisors D'_B.

Remark that under the blowing up map

$$\pi^*([D'_B]) = [D_B]. \tag{5.2.6}$$

From the above considerations we have a commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[c'_B]_{B\in\mathcal{G}'} & \xrightarrow{\phi'} & H^*(Y_{\mathcal{G}}, \mathbb{Z}) \\ \gamma \big\downarrow & & \big\downarrow \pi^* \\ \mathbb{Z}[c_B]_{B\in\mathcal{G}} & \xrightarrow{\phi} & H^*(Y_{\mathcal{F}}, \mathbb{Z}) \end{array} \tag{5.2.7}$$

where

$$\gamma(c'_B) = c_B. \tag{5.2.8}$$

Moreover for the inclusion $i : Z_A = Y_{\overline{\mathcal{G}}} \to Y_{\mathcal{G}}$ we have a commutative diagram

$$\begin{tikzcd} \mathbb{Z}[c'_B]_{B\in\mathcal{G}'} \arrow[r, "\phi'"] \arrow[d, "\overline{\gamma}"'] & H^*(Y_{\mathcal{G}}, \mathbb{Z}) \arrow[d, "i^*"] \\ \mathbb{Z}[c_{\overline{B}}]_{B\in\mathcal{G}} \arrow[r, "\phi"] & H^*(Y_{\overline{\mathcal{G}}}, \mathbb{Z}) \end{tikzcd} \tag{5.2.9}$$

where

$$\overline{\gamma}(c'_B) = \begin{cases} c_{\overline{B}} & \text{if } \{B, A\} \text{ is } \mathcal{G}\text{-nested} \\ 0 & \text{otherwise.} \end{cases} \tag{5.2.10}$$

Notice that in particular, since A is minimal, $\mathcal{H} \cup \mathcal{S} \subset \mathcal{G}, B \in \mathcal{G}$,

$$P^{\mathcal{S}}_{\mathcal{H},B} = \gamma {P'}^{\mathcal{S}}_{\mathcal{H},B}. \tag{5.2.11}$$

Case 1. $\mathcal{S} \cup \{A\}$ is not $\mathcal{F}$-nested.

In this case the variety Z_A does not meet the variety $D_{\mathcal{S}'}$ and we have $D_{\mathcal{S}'} = D_{\mathcal{S}}$. Also by Lemma 5.1 $c_A \in I_{\mathcal{S}}$. By 5.2.11, we immediately deduce that $I_{\mathcal{S}}$ is generated by $\gamma(I'_{\mathcal{S}})$ and c_A as desired.

Case 2. $\mathcal{S}$ does not contain A and $\mathcal{S} \cup A$ is $\mathcal{F}$-nested.

In this case the variety $D'_{\mathcal{S}}$ intersects transversally Z_A and $D_{\mathcal{S}}$ is the blow up of $D'_{\mathcal{S}}$ along $Z^{\mathcal{S}}_A := Z_A \cap D'_{\mathcal{S}}$.

The divisor D'_C, for $\{C, A\}$ $\mathcal{G}$-nested, intersects Z_A in the divisor $D_{\overline{C}}$. We thus have that $Z^{\mathcal{S}}_A$ is the subvariety $D_{\overline{\mathcal{S}}}$ in $Y_{\overline{\mathcal{G}}}$ and, by the inductive assumptions, the cohomology of $Z^{\mathcal{S}}_A$ is generated by the image of the cohomology of $D'_{\mathcal{S}}$ and we can apply Lemma 5.2. As far as the normal bundle of $Z^{\mathcal{S}}_A$ in $D'_{\mathcal{S}}$ we have, by transversality, that it is the restriction of the normal bundle of Z_A in $Y_{\overline{\mathcal{G}}}$ to $Z^{\mathcal{S}}_A$. With the notations of Lemma 5.2 in this case $P_{Z/W}(-E) = (-\sum_{B\supseteq A}[D_B])^{\dim(A)}$.

Consider the induced map on cohomology, and let $I^{\mathcal{S}}_A$ denote the kernel of map

$$\mathbb{Z}[c'_B] \xrightarrow{\phi_{\mathcal{S}}} H^*(D_{\mathcal{S}}, \mathbb{Z}) \xrightarrow{j^*} H^*(Z^{\mathcal{S}}_A, \mathbb{Z})$$

where j is the inclusion. Using the diagram 5.2.9. and the inductive assumptions we immediately deduce that $j^* \circ \phi_{\mathcal{S}}$ is surjective and that $I^{\mathcal{S}}_A$ coincides with the kernel of $\phi_{\overline{\mathcal{S}}} \circ \overline{\gamma}$.

We must thus show by Lemma 5.2 that

$$I_{\mathcal{S}} = \left(\left(\sum_{B\supseteq A} c_B \right)^{\dim(A)}, c_A\gamma(I^{\mathcal{S}}_A), \gamma(I'_{\mathcal{S}}) \right) := J_{\mathcal{S}}.$$

Let us first show that $I_{\mathcal{S}} \supset J_{\mathcal{S}}$. By 5.2.11 we get that $\gamma(I'_{\mathcal{S}}) \subset I_{\mathcal{S}}$, but we also have $(\sum_{B\supseteq A} c_B)^{\dim(A)} = P^{\mathcal{S}}_{\emptyset,A} \in I_{\mathcal{S}}$.

Finally we need to show that $c_A\gamma(I^{\mathcal{S}}_A) \subset I_{\mathcal{S}}$. Recall that $I^{\mathcal{S}}_A = \overline{\gamma}^{-1}(I^{\overline{\mathcal{S}}})$ and that $I^{\overline{\mathcal{S}}}$ is generated by the polynomials $P^{\overline{\mathcal{S}}}_{\overline{\mathcal{H}},\overline{B}}$. Notice that since $\ker\overline{\gamma}$ is generated by the c'_B such that $\{A, B\}$ is not $\mathcal{F}$-nested clearly $c_A\gamma(\ker\overline{\gamma}) \subset I_{\mathcal{S}}$. So we are reduced to showing that for any polynomial $P^{\overline{\mathcal{S}}}_{\overline{\mathcal{H}},\overline{B}}$ there is a representative $P' \in \mathbb{Z}[c'_B]$ such that $c_A\gamma(P') \in I_{\mathcal{S}}$.

Write $\overline{\mathcal{G}} = \overline{\mathcal{G}}_1 \cup \overline{\mathcal{G}}_2$ where $\overline{\mathcal{G}}_1$ is the set of elements in $\overline{\mathcal{G}}$ whose preimage under the quotient homomorphism $q : V^* \to V^*/A$ is irreducible and $\overline{\mathcal{G}}_2 = \overline{\mathcal{G}} - \overline{\mathcal{G}}_1$. Remark that, using Proposition 2.3, one sees that if $\overline{C} \subset \overline{B}$ and $\overline{C} \in \overline{\mathcal{G}}_1$ also $\overline{B} \in \overline{\mathcal{G}}_1$ and that if $\overline{B} \in \overline{\mathcal{G}}_2$ then $q^{-1}\overline{B}$ has irreducible decomposition of the form $B \oplus A$. This allows us to define a unique lifting of $\overline{\mathcal{G}}$ to $\mathcal{G}$ as follows. If $\overline{B} \in \overline{\mathcal{G}}_1$ we lift it to $q^{-1}(B)$, otherwise we lift it to the unique irreducible component of $q^{-1}(B)$ different from A.

Now take a polynomial $P^{\overline{\mathcal{S}}}_{\overline{\mathcal{H}},\overline{B}}$. Using our lifting to lift $\overline{\mathcal{H}}$ to $\mathcal{H}$ and $\overline{B}$ to B we can consider the polynomial $P'^{\mathcal{S}}_{\mathcal{H},B}$. We easily see that unless $\overline{\mathcal{H}} \cup \overline{\mathcal{S}}_{\overline{B}} \subset \overline{\mathcal{G}}_2$ and $\overline{B} \in \overline{\mathcal{G}}_1$ we have $\overline{\gamma}(P'^{\mathcal{S}}_{\mathcal{H},B}) = P^{\overline{\mathcal{S}}}_{\overline{\mathcal{H}},\overline{B}}$. If on the other hand $\overline{\mathcal{H}} \cup \overline{\mathcal{S}}_{\overline{B}} \subset \overline{\mathcal{G}}_2$ and $\overline{B} \in \overline{\mathcal{G}}_1$ then $P'^{\mathcal{S}}_{\mathcal{H},B}$ is divisible by $(\sum_{C\supseteq B} c_C)^{\dim A}$ and if we set $Q'^{\mathcal{S}}_{\mathcal{H},B} = P'^{\mathcal{S}}_{\mathcal{H},B}/(\sum_{C\supseteq B} c_C)^{\dim A}$, we obtain that $\overline{\gamma}(Q'^{\mathcal{S}}_{\mathcal{H},B}) = P^{\overline{\mathcal{S}}}_{\overline{\mathcal{H}},\overline{B}}$.

Now in the first case $P'^{\mathcal{S}}_{\mathcal{H},B} \in I'_{\mathcal{S}}$ so our claim is obvious. In the second $c_A\gamma(Q'^{\mathcal{S}}_{\mathcal{H},B}) = P^{\mathcal{S}}_{\mathcal{H}\cup\{A\},B} \in I_{\mathcal{S}}$.

Now we must show $I_{\mathcal{S}} \subset J_{\mathcal{S}}$. From the previous analysis $\gamma(P'^{\mathcal{S}}_{\mathcal{H},B}) = P^{\mathcal{S}}_{\mathcal{H},B}$ so we only have to consider the case in which A appears in $\mathcal{H}$ or $A = B$. Now $A = B$ can occur only when $\mathcal{H} = \emptyset$ and we have already treated this case. Let now $A \in \mathcal{H}$. We immediately deduce from our previous analysis that $P^{\mathcal{S}}_{\mathcal{H},B} = c_A\gamma(P')$ with $P' \in I^{\mathcal{S}}_A$.

Finally we have to consider the case $\mathcal{S} = \mathcal{S}' \cup \{A\}$. In this case $D_{\mathcal{S}}$ is the exceptional divisor in $D_{\mathcal{S}'}$ or equivalently the preimage in $Y_{\mathcal{F}}$ of $Z^{\mathcal{S}}_A$. Our claim then follows by completely similar argument to the one above using Lemma 5.2.2. We leave the details to the reader. □

REMARK. Using the results of 4.2 we get a completely analogous result in the projective case (which is in fact a special case of the above).

5.3 In [M] Morgan associates to a variety M together with a divisor $D = D_1 \cup D_2 \cup \cdots \cup D_n$ in M with normal crossings and smooth irreducible components D_i a differential graded algebra over $\mathbb{Q}$. If M is complete, the cohomology of this algebra is the rational cohomology algebra of $M - D$ and its minimal model is the minimal model of $M - D$.

The definition is as follows. For any sequence $\underline{i} = \{i_1, \ldots, i_h\}$ with $1 \leq i_1 < \ldots < i_h \leq n$ we set $D_{\underline{i}} = D_{i_1} \cap \cdots \cap D_{i_h}$ and $M_{\underline{i}} = H^*(D_{\underline{i}}, \mathbb{Q})$. We then set $M = \oplus_{\underline{i}} M_{\underline{i}}$ with grading obtained by putting $H^t(D_{\underline{i}}, \mathbb{Q})$ in degree $t + h$.

Suppose now that $\underline{i}$ and $\underline{j}$ are two sequence as above. If $\underline{i}$ is obtained from $\underline{j}$ by removing k of its elements, then we have a natural restriction map $r_{\underline{i}}^{\underline{j}} : M_{\underline{i}} \to M_{\underline{j}}$ which increases the degree by k. On the other hand, if $\underline{i}$ is obtained from $\underline{j}$ by adding one index not in $\underline{j}$, we have a Gysin morphism $G_{\underline{i}}^{\underline{j}} : M_{\underline{i}} \to M_{\underline{j}}$ which increases the degree by one.

We use these two maps to define a graded multiplication and a graded differential. Suppose that $\underline{i} \cap \underline{j} = \emptyset$. Then there is a permutation σ of the indices in $\underline{i} \cup \underline{j}$ making this sequence into an increasing sequence. We then define our multiplication on M by setting

$$ab = \begin{cases} (-1)^{sg\sigma} r_{\underline{i}}^{\underline{i}\cup\underline{j}}(a) r_{\underline{j}}^{\underline{i}\cup\underline{j}}(b) & \text{if } a \in M_{\underline{i}},\ b \in M_{\underline{j}} \text{ and } \underline{i} \cap \underline{j} = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

and our differential by setting

$$d(a) = \sum_{s=1}^{h} (-1)^s G_{\underline{i}}^{\underline{i}-\{i_s\}}(a) \quad \text{if} \quad a \in M_{\underline{i}}.$$

We wish to compute this algebra in our case. For this take a $\mathcal{G}$-nested set $\mathcal{S}$ in our building set $\mathcal{G}$. Let $\Sigma = \{A \in \mathcal{G} |\ \mathcal{S} \cup \{A\}$ is $\mathcal{G}$-nested $\}$. For every $A \in \Sigma - \mathcal{S}$, we have a smooth irreducible divisor $D_{\mathcal{S} \cup \{A\}}$ in $D_{\mathcal{S}}$ and the divisor $D = \cup_{A \in \Sigma - \mathcal{S}} D_{\mathcal{S} \cup \{A\}}$ has normal crossings. We can thus consider the Morgan algebra $M_{\mathcal{S}}^*$ associated to the pair $(D_{\mathcal{S}},\ D)$ (and a given total ordering on $\Sigma - \mathcal{S}$).

To determine $M_{\mathcal{S}}^*$ consider the polynomial algebra $\mathbb{Q}[c_A]$ with $A \in \Sigma$ and the exterior algebra over $\mathbb{Q}$, $\Lambda[e_A]$, with $A \in \Sigma - \mathcal{S}$. Take $\mathbb{Q}[c_A] \otimes \Lambda[e_A]$ and give it a grading by setting deg $c_A = 2$, deg $e_A = 1$ and a degree one differential by setting $d[e_A] = c_A$. This makes $\mathbb{Q}[c_A] \otimes \Lambda[e_A]$ into a differential graded algebra.

In a similar way to what we have done in 5.1 take two subsets $\mathcal{H}_1, \mathcal{H}_2 \subset \Sigma - \mathcal{S}$ and an element $B \in \Sigma$ such that $B \supsetneq A$ for all $A \in \mathcal{H} := \mathcal{H}_1 \cup \mathcal{H}_2$. Set $\mathcal{S}_B = \{A \in \mathcal{S} | A \subsetneq B\}$ and define

$$d_{\mathcal{H},B}^{\mathcal{S}} = \dim B - \dim \left(\sum_{A \in \mathcal{H} \cup \mathcal{S}_B} A \right).$$

To these data we associate the element in $\mathbb{Q}[c_A] \otimes \Lambda[e_A]$ given by

$$P_{\mathcal{H}_1,\mathcal{H}_2,B}^{\mathcal{S}} = \prod_{A \in \mathcal{H}_1} e_A \prod_{A \in \mathcal{H}_2} c_A \left(\sum_{C \supset B} c_C \right)^{d_{\mathcal{H},B}^{\mathcal{S}}}. \tag{5.3.1}$$

Define now $J_{\mathcal{S}}$ equal to the ideal generated by the elements 5.3.1. It is clear that $J_{\mathcal{S}}$ is a graded ideal preserved by the differential d. Define now a graded homomorphism

$$\psi_{\mathcal{S}} : \mathbb{Q}[c_A] \otimes \Lambda[e_A] \to M^*_{\mathcal{S}} \tag{5.3.2}$$

by setting $\psi_{\mathcal{S}}(c_A) = \phi_{\mathcal{S}}(c_A) \in H^2(D_{\mathcal{S}}, \mathbb{Q}) \subset M^2_{\mathcal{S}}$ and $\psi_{\mathcal{S}}[e_A]$ equal to the identity element in $H^0(D_{\mathcal{S}\cup\{A\}}, \mathbb{Q}) \subset M^1_{\mathcal{S}}$.

From Theorem 5.2 and the definition of $M_{\mathcal{S}}$, we deduce

THEOREM. *The homomorphism $\psi_{\mathcal{S}}$ is a surjective homomorphism of graded differential algebras whose kernel equals the ideal $J_{\mathcal{S}}$.*

Proof. Let $\mathcal{H} \subset \Sigma - \mathcal{S}$ be such that $\mathcal{S}' = \mathcal{H} \cup \mathcal{S}$ is nested. The element $\prod_{A\in\mathcal{H}} e_A$ is well defined up to sign, and the sign can be chosen using the total ordering on $\Sigma - \mathcal{S}$.

Remark that under the map $\psi_{\mathcal{S}}$, $\mathbb{Q}[c_A] \otimes \prod_{A\in\mathcal{H}} e_A$ maps to $H^*(D_{\mathcal{H}\cup\mathcal{S}}, \mathbb{Q})$ and is such that $\psi_{\mathcal{S}}(y \prod_{A\in\mathcal{H}} e_A) = \phi_{\mathcal{H}\cup\mathcal{S}}(y)$ for all $y \in \mathbb{Q}[c_A]$.

By Theorem 5.2 each of these components of $\psi_{\mathcal{S}}$ is surjective showing the first part. Once the surjectivity has been shown, the fact that $J_{\mathcal{S}} = \ker \psi_{\mathcal{S}}$ and that $\psi_{\mathcal{S}}$ is a homomorphism of differential algebras follow immediately from Theorem 5.2 and the definition of the Gysin morphism. We leave the verifications to the reader. □

REMARKS.

(1) As mentioned above, if $D_{\mathcal{S}}$ is complete, i.e. if $V^* \in \mathcal{S}$, the cohomology of $M_{\mathcal{S}}$ is the cohomology of $D_{\mathcal{S}} - D$ and $M_{\mathcal{S}}$ determines the rational homotopy type of $D_{\mathcal{S}} - D$.

   In particular we can apply this in the case $\mathcal{S} = \{V^*\}$ and by Theorem 4.2, determine the rational cohomology algebra and the rational homotopy type of the complement in $\mathbb{P}^n(\mathbb{C})$ of the union of any finite configuration of linear subspaces.

(2) Again if $D_{\mathcal{S}}$ is complete, the fact that all the varieties $D_{\mathcal{T}}$ have cohomology only of type (i,i), implies that the same holds for $D_{\mathcal{S}} - D$. Indeed one easily checks that the only non vanishing mixed Hodge numbers for $D_{\mathcal{S}} - D$ in degree ℓ are the $h^{i,i}$ with $\frac{\ell}{2} \leq i \leq \ell$.

We have thus shown that the mixed Hodge structure and the rational homotopy type of $\mathcal{A}$ depend only on the combinatorial structure of the rank poset associated to the subspace configuration.

One should compare these results with [G-M] and [O-S].

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