Ultra Log-concavity of Basis Partition Sequence

Alan Yan
Senior Thesis Advisor: Professor June Huh

July 9, 2022

Abstract

Let $M = (E, \mathcal{I})$ be a matroid of rank r and let $E = A \sqcup B$ be a fixed partition of the ground set. For $0 \leq k \leq r$, let N_k be the number of bases whose intersection with A has size k. We consider whether or not the sequence $\{N_k\}$ is ultra-log-concave.

Contents

1 Introduction

The question is motivated by the results in [2]. In particular, Stanley proves that

Theorem 1.1. *If M is regular, then N_k is r-ultra-log-concave.*

Stanley's proof relates the sequence N_k to a sequence of mixed volume. The ultra-log-concave follows from the Alexandrov-Fenchel inequality. The relationship relies on the unimodular coordinazation that exists for regular matroids. It is unclear whether or not the result holds true for general matroids. According to [1], we say that a non-negative sequence a_i is ultra-logconcave of order m denoted $ULC(m)$ if $a_i = 0$ for $i > m$ and the sequence $a_i/\binom{m}{i}$ is logconcave. Note that $ULC(m) \implies ULC(m+1)$.

Example 1.1. *Let $M = U_{n,r}$ be the uniform matroid of rank r on n elements. Consider the partition $[n] = \{1, \ldots, m\} \cup \{m+1, \ldots, n\}$ for some $1 \leq m \leq n-1$. Then, we can explicitly compute for $k \in [m]$ the value*

$$N_k = \binom{m}{k} \cdot \binom{n-m}{r-k}.$$

We want to prove under the assumption that $k \leq r-1$ and $N_{k+1}, N_{k-1} \neq 0$ that

$$\frac{N_k^2}{N_{k+1}N_{k-1}} \cdot \frac{\binom{r}{k-1}\binom{r}{k+1}}{\binom{r}{k}^2} \geq 1.$$

It seems like the calculation I did with June Huh on the board was wrong ... I'll check again tomorrow ... This example shows that ultra-log-concavity does not hold in general (at least not in the rank). However, N_k is still log-concave.

2 Combinatorial Objects

References

[1] Thomas M Liggett. Ultra logconcave sequences and negative dependence. *Journal of Combinatorial Theory, Series A*, 79(2):315–325, 1997.

[2] Richard P Stanley. Two combinatorial applications of the aleksandrov-fenchel inequalities. *Journal of Combinatorial Theory, Series A*, 31(1):56–65, 1981.