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SHORTER NOTES

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THE ROOTS OF A POLYNOMIAL VARY CONTINUOUSLY AS A FUNCTION OF THE COEFFICIENTS

GARY HARRIS AND CLYDE MARTIN

ABSTRACT. We present an elementary topological proof that the roots of a polynomial vary continuously as a function of the coefficients.

It is, or should be, common knowledge that the roots of a polynomial over $\mathbb{C}$ vary continuously as a function of the coefficients. Here we present an elementary topological proof of this important fact. Of course, the coefficients vary continuously as a function of the roots; indeed, the coefficients can be given as a symmetric polynomial mapping of the roots. It is natural to ask if this mapping is continuously invertible, and this is the question we answer in Theorem A.

Before becoming more precise, we should mention that proofs, probably in large numbers, exist in the literature in differing contexts. For example, for polynomials of degree 2, 3, or 4, Galois theory tells us the roots can be found by explicit formulas of the coefficients involving radicals. ([**4**] is a good source.) However, a general study of continuity can still be tricky. For polynomials without multiple roots, the Complex Implicit Function Theorem can be used to show the roots vary analytically with the coefficients [**1**]. For general polynomials our Theorem B follows from an application of Rouché's theorem [**3**].

As stated above, our purpose is to give a topological proof that the roots of a polynomial are given as a continuous function of the coefficients. We may assume all polynomials P of degree n are normalized so

$$P(z) = z^n + a_1 z^{n-1} + \cdots + a_n.$$

We will identify P with the vector $a = (a_1, \ldots, a_n)$ in $\mathbb{C}^n$. By the Fundamental Theorem of Algebra, we know P can be factored as

$$P(z) = \prod_{j=1}^{n} (z - \xi_j)$$

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for a unique sequence $\{\xi_1, \ldots, \xi_n\}$ of elements of $\mathbb{C}$. We also know for each $1 \leq j \leq n$,

$$a_j = \sigma_j(\xi_1, \ldots, \xi_n)$$

for symmetric polynomials $\sigma_1, \ldots, \sigma_n$ in n variables. Let $\sigma\colon \mathbb{C}^n \to \mathbb{C}^n$ be the continuous mapping defined by

$$\sigma(\xi) \doteq (\sigma_1(\xi), \ldots, \sigma_n(\xi)).$$

We are assuming $\mathbb{C}^n$ has the euclidean norm and are using the notation $\xi \doteq (\xi_1, \ldots, \xi_n)$. It follows from the Fundamental Theorem of Algebra that σ is surjective.

The map σ is not injective since it is independent of the arrangement of the components of the vector ξ. For $\mu \in S_n$, the group of permutations on n elements, and $\xi = (\xi_1, \ldots, \xi_n)$ in $\mathbb{C}^n$ let $\xi_\mu \doteq (\xi_{\mu(1)}, \ldots, \xi_{\mu(n)})$. We define an equivalence relation on $\mathbb{C}^n$ by

$$\xi \sim \beta \Leftrightarrow \exists \mu \in S_n \ni \xi_\mu = \beta.$$

Let $\mathbb{C}^n/\sim$ be the quotient space endowed with the quotient topology induced by the canonical projection $\pi\colon \mathbb{C}^n \to \mathbb{C}^n/\sim$, and let $\hat{\sigma}$ be the unique mapping of $\mathbb{C}^n/\sim$ onto $\mathbb{C}^n$ for which $\hat{\sigma} \circ \pi = \sigma$. It follows that $\hat{\sigma}$ is a continuous bijection. (All the facts we use about the quotient topology can be found in any good text on topology, for example [**2**, pp. 94–99].)

We can now prove our

THEOREM A. *The mapping $\hat{\sigma}\colon \mathbb{C}^n/\sim \to \mathbb{C}^n$ is a homeomorphism.*

PROOF. Let d be the metric defined on $\mathbb{C}^n/\sim$ by

$$d(\pi(\xi), \pi(\beta)) \doteq \min\{|\xi' - \beta'|\colon \xi' \in \pi(\xi) \text{ and } \beta' \in \pi(\beta)\}.$$

To see that d is a metric let $\xi, \beta, \eta \in \mathbb{C}^n$. Clearly

$$d(\pi(\xi), \pi(\beta)) = \min\{|\xi - \beta'|\colon \beta' \in \pi(\beta)\}.$$

We can choose $\eta' \in \pi(\eta)$ so that $d(\pi(\xi), \pi(\eta)) = |\xi - \eta'|$. For each $\beta' \in \pi(\beta)$, $|\xi - \beta'| \leq |\xi - \eta'| + |\eta' - \beta'|$. So

$$\begin{aligned} d(\pi(\xi), \pi(\beta)) &\leq \min\{|\xi - \eta'| + |\eta' - \beta'|\colon \beta' \in \pi(\beta)\} \\ &= d(\pi(\xi), \pi(\eta)) + d(\pi(\eta), \pi(\beta)). \end{aligned}$$

It follows immediately that $\pi\colon \mathbb{C}^n \to \mathbb{C}^n/\sim$ is continuous if $\mathbb{C}^n/\sim$ is endowed with the topology induced by d. Moreover for any set A in $\mathbb{C}^n$ we have $\pi^{-1}(\pi(A))$ is open (closed) if A is open (closed). Thus π is an open and closed map. Hence the topology induced by d is the quotient topology. Let $M > 0$ be given and let $B(0, M)$ denote the open ball about 0 in $\mathbb{C}^n/\sim$ of radius M. We claim that $\hat{\sigma}|_{\overline{B(0,M)}}$ is a homeomorphism. It suffices to show that $\hat{\sigma}$ is closed. Let $C \subset \overline{B(0, M)}$ be closed, then $\pi^{-1}(C)$ is a closed and bounded subset of $\mathbb{C}^n$, hence compact. Thus $\hat{\sigma}(C) = \sigma(\pi^{-1}(C))$ is compact, and hence closed in $\mathbb{C}^n$. We can now complete the proof by showing that $\hat{\sigma}\colon \mathbb{C}^n/\sim \to \mathbb{C}^n$ is open. Let U be an open subset of $\mathbb{C}^n/\sim$ and $x \in U$. Choose $\varepsilon > 0$ and $M > 0$ so that $B(x, \varepsilon) \subset U$ and $\overline{B(x, \varepsilon)} \subset B(0, M)$. Since $\hat{\sigma}$ is open on $B(0, M)$ it follows that $\hat{\sigma}(x)$ is in the interior of $\hat{\sigma}(B(x, \varepsilon))$. Since x was arbitrary this completes the proof.

Let $\tau\colon \mathbb{C}^n \to \mathbb{C}^n/\sim$ denote the continuous inverse of $\hat{\sigma}$. Given $a \in \mathbb{C}^n$ (a identified with a polynomial P as above) and $\varepsilon > 0$, Theorem A tells us there exists $\delta > 0$ so that $\tau(B(a,\delta)) \subset B(\tau(a),\varepsilon)$. Writing out what this means in terms of the euclidean metric on $\mathbb{C}^n$ and the metric d on $\mathbb{C}^n/\sim$, we obtain

THEOREM B. *Suppose*

$$P(z) = z^n + a_1 z^{n-1} + \cdots + a_n = \prod_{j=1}^{s} (z - \xi_j)^{m_j}$$

for distinct $\xi_1, \ldots, \xi_s$. *Let* $\varepsilon > 0$ *be given so that* $i \neq j$ *implies that* $B(\xi_i,\varepsilon) \cap B(\xi_j,\varepsilon) = \varnothing$. *Then there exists* $\delta > 0$ *so that* $b \in B(a,\delta)$ *implies the polynomial*

$$Q(z) \doteq z^n + b_1 z^{n-1} + \cdots + b_n$$

has exactly m_j *roots (counting multiplicity) in* $B(\xi_j,\varepsilon)$.

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DEPARTMENT OF MATHEMATICS, TEXAS TECH UNIVERSITY, LUBBOCK, TEXAS 79409