

LOG-CONCAVITY IN COMBINATORICS

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TODAY

Abstract

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Acknowledgements

I would like to thank

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Chapter 1

Conventions and Notation

Chapter 2

Introduction

Chapter 3

Combinatorial Structures

3.1 Partially Ordered Sets

Our main reference for partially ordered sets is [16].

Definition 3.1.1. A **partially ordered set** is an ordered pair (P, \leq) of a set P and a binary relation \leq on P such that

(P1) $x \leq x$ for all $x \in P$.

(P2) If $x \leq y$ and $y \leq x$, then $x = y$.

(P3) If $x \leq y$ and $y \leq z$, then $x \leq z$.

Example 3.1.1.

3.1.1 Linear Extensions

Definition 3.1.2. Let (P, \leq) be a poset on n elements. A **linear extension** is any bijective map $f : P \rightarrow [n]$ satisfying $f(x) < f(y)$ for all $x, y \in P$ satisfying $x < y$.

3.1.2 Lattices

Definition 3.1.3. We say a poset \mathcal{L} is a lattice if for any $x, y \in \mathcal{L}$ there exist (unique) poset elements $x \vee y$ (called the **meet** or **greatest lower bound**) and $x \wedge y$ (called the **join** or **least upper bound**) satisfying

(L1) $x \vee y \geq x$, $x \vee y \geq y$, and for any $z \in \mathcal{L}$ satisfying $z \geq x$ and $z \geq y$, we have $z \geq x \vee y$.

(L2) $x \wedge y \leq x$, $x \wedge y \leq y$, and for any $z \in \mathcal{L}$ satisfying $z \leq x$ and $z \leq y$, we have $z \leq x \wedge y$.

The only lattice that we will be concerned with are geometric lattices.

3.2 Graph Theory

We assume a working knowledge of basic graph theory. This includes concepts such as connected components, paths, trees, etcetera. We provide the definition of a graph that we will use in this paper. Note that we allow multi-edges and but not loops. This is because in later results concerning graphic matroids, the presense of loops does not affect log-concavity. Our main reference for basic notions in graph theory come from [4]. For more the more sophistication notions that come from spectral graph theory, we refer to [3].

Definition 3.2.1. A **graph** is an ordered pair (V, E) of vertices and edges such that each edge is associated with either two distinct vertices or one vertex. If an edge is associated with two distinct vertices, then we call it a **simple edge**. If an edge is associated with one vertex, then we call it a **loop**.

3.2.1 Spectral Graph Theory

Definition 3.2.2. Given a graph G with n vertices v_1, \dots, v_n , we define its Laplacian matrix $L := L_G$ to be the $n \times n$ matrix defined element-wise as

$$L_{i,j} := \begin{cases} \deg(v_i) & \text{if } i = j \\ -E_{v_i, v_j} & \text{if } i \neq j \text{ and } v_i \text{ is adjacent to } v_j \end{cases}$$

where E_{v_i, v_j} is the number of edges between v_i and v_j .

Definition 3.2.3. Let G be a graph and let $V = \{v_1, \dots, v_n\}$ be an arbitrary ordering of the vertices. For this ordering, we can define a $|V| \times |E|$ matroid B_G called the **incidence matrix** where the entry indexed by $v \in V$ and $e = \{v_i, v_j\} \in E$ where $v_i > v_j$ is equal to

$$B_{ve} = \begin{cases} 1, & \text{if } v = v_i \\ -1, & \text{if } v = v_j \\ 0, & \text{otherwise.} \end{cases}$$

3.3 Matroids

Matroids can be thought of as combinatorial objects which abstracts and generalizes the properties of both linear independence in vectors spaces and cyclelessness in graphs. Despite the seemingly limited scope of this interpretation, matroids successfully describe many objects relevant to other areas in mathematics such as topology [7], graph theory [10], combinatorial optimization [5], algebraic geometry [6], and convex geometry [9]. In this subsection, we provide an introduction to the basic notions in matroid theory that we will need in the remainder of this thesis. We use the excellent monographs [13] and [18] as our main references for the basic theory.

3.3.1 Independent Sets

Definition 3.3.1. A **matroid** is an ordered pair $M = (E, \mathcal{I})$ consisting of a finite set E and a collection of subsets $\mathcal{I} \subseteq 2^E$ which satisfy the following three properties:

(I1) $\emptyset \in \mathcal{I}$.¹

(I2) If $X \subseteq Y$ and $Y \in \mathcal{I}$, then $X \in \mathcal{I}$.²

(I3) If $X, Y \in \mathcal{I}$ and $|X| > |Y|$, then there exists some element $e \in X \setminus Y$ such that $Y \cup \{e\} \in \mathcal{I}$.³

The set E is called the ground set of the matroid and the collection of subsets \mathcal{I} are called independent sets. This naming convention is motivated by Example 3.3.1 where the independent sets consist exactly of sets of linearly independent vectors.

Example 3.3.1 (Linear Matroids). Let V be a k -vector space and let $E = \{v_1, \dots, v_n\}$ be a finite set of vectors from V . If we let \mathcal{I} consist of all subsets of E which are linearly independent, then (E, \mathcal{I}) forms a matroid. We call any matroid which can be constructed in this way a **linear matroid**.

Example 3.3.2 (Graphic Matroids). Let $G = (V, E)$ be a graph and let \mathcal{I} be the collection of subsets of E which consist of edges containing no cycles. Then (E, \mathcal{I}) forms a matroid called the **cycle matroid** of the graph G . We call any matroid which is isomorphic to a cycle matroid of a graph a **graphic matroid**. In Proposition 3.3.1, we prove that graphic matroids are not only linear, but they can be represented by a totally unimodular matrix.

Proposition 3.3.1.

¹(I1) is the **non-emptiness** axiom

²(I2) is the **hereditary** axiom

³(I3) is the **exchange** axiom

Given a matroid $M = (E, \mathcal{I})$, we call a subset $X \subseteq E$ a **dependent set** if and only if $X \notin \mathcal{I}$. Given a matroid, we call any minimal dependent set a **circuit**. Matroids are uniquely determined by their circuits. For a proof of this fact, see Corollary 1.1.5 in [13].

3.3.2 Bases

We define a basis $B \in \mathcal{I}$ to be a maximal independent set. From property **(I3)**, we can deduce that all bases have the same number of elements. Indeed, if B_1 and B_2 are bases satisfying $|B_1| < |B_2|$ then from **(I3)** there exists some element $e \in B_2 \setminus B_1$ satisfying $B_1 \cup \{e\} \in \mathcal{I}$. But, this means that $B_1 \cup \{e\}$ is an independent set strictly larger than B_1 . This contradicts the maximality of B_1 and implies that all bases contain the same number of elements.

Definition 3.3.2. Let $M = (E, \mathcal{I})$ be a matroid and let \mathcal{B} be the collection of bases. Then, the following three properties (Lemma 1.2.2 in [13]):

(B1) \mathcal{B} is non-empty.

(B2) If B_1 and B_2 are members of \mathcal{B} and $x \in B_1 \setminus B_2$, then there is an element y of $B_2 \setminus B_1$ such that $(B_1 - x) \cup y \in \mathcal{B}$.

(B3) If B_1 and B_2 are members of \mathcal{B} and $x \in B_1 \setminus B_2$, then there is an element $y \in B_2 \setminus B_1$ such that $(B_2 - y) \cup x \in \mathcal{B}$.

Associated with the bases of a matroid, we can define the **basis generating function** of a matroid $M = (E, \mathcal{B})$ as

$$f_M(x) := \prod_{B \in \mathcal{B}} x^B \in \mathbb{R}[x_e : e \in E]$$

which is a homogeneous polynomial in the ring of polynomials with variables indexed by the ground set of our matroid.

3.3.3 Rank Functions

Definition 3.3.3. For any matroid $M = (E, \mathcal{I})$, we define its rank function $\text{rank}_M : 2^E \rightarrow \mathbb{N}$ to be equal to

$$\text{rank}_M(X) := \max\{|I| : I \in \mathcal{I}, I \subseteq X\}.$$

The rank function satisfies the following three properties (Lemma 1.3.1 in [13]):

(R1) If $X \subseteq E$, then $0 \leq r(X) \leq |X|$.

(R2) If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.

(R3) If X and Y are subsets of E , then $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.

3.3.4 Closure Operators and Lattice of Flats

Definition 3.3.4. For any matroid $M = (E, \mathcal{I})$, we define its closure operator $\text{clo}_M : 2^E \rightarrow 2^E$ to be

$$\text{clo}_M(X) := \overline{X} = \{x \in E : \text{rank}(X \cup \{x\}) = \text{rank}(X)\}.$$

The closure operator satisfies the following four properties (Lemma 1.4.3 in [13]):

(C1) If $X \subseteq E$, then $X \subseteq \text{clo}_M(X)$.

(C2) If $X \subseteq Y \subseteq E$, then $\text{clo}_M(X) \subseteq \text{clo}_M(Y)$.

(C3) If $X \subseteq E$, then $\text{clo}_M(\text{clo}_M(X)) = \text{clo}_M(X)$.

(C4) If $X \subseteq E$ and $x \in E$, and $y \in \text{clo}_M(X \cup \{x\}) \setminus \text{clo}_M(X)$, then $x \in \text{clo}_M(X \cup \{y\})$.

Example 3.3.3. Let $M := M(\{v_1, \dots, v_n\})$ be the linear matroid represented by vectors v_1, \dots, v_n in a k -vector space V . Then, the closure of a subset $X \subseteq \{v_1, \dots, v_n\}$ is precisely $\text{clo}_M(X) := \{v_1, \dots, v_n\} \cap \text{span}_k X$.

If $X \subseteq E$ satisfies $X = \text{clo}_M(X)$, then we say X is a **closed set** or **flat**. For any matroid $M = (E, \mathcal{I})$, let $\mathcal{L}(M)$ denote the partially ordered set consisting of the flats of M equipped with set inclusion.

Proposition 3.3.2 (Theorem 1.7.5 in [13]). *Let $M = (E, \mathcal{I})$ be a matroid. Then, the poset $\mathcal{L}(M)$ is a geometric lattice with join and meet operations defined as $X \vee Y := \text{clo}_M(X \cup Y)$, $X \wedge Y := X \cap Y$, and rank function $\text{rank}_{\mathcal{L}}(X) := \text{rank}_M(X)$.*

3.3.5 Loops and Parallel Elements

We say $e \in E$ is a **loop** in M if $\{e\}$ is a dependent set. Equivalently, $e \in E$ is a loop if $\text{rank}(\{e\}) = 0$. We define E_0 as the set of loops. An element $e \in E$ in the ground set is called a **coloop** if it is contained in every basis. We say $x_1, x_2 \in E \setminus E_0$ satisfy $x_1 \sim_M x_2$ if and only if $\{x_1, x_2\} \notin \mathcal{I}$ or $x_1 = x_2$. Equivalently, $x_1 \sim_M x_2$ if and only if $\text{rank}(\{x_1, x_2\}) = 1$. When $x_1 \sim_M x_2$, we say that x_1 and x_2 are **parallel**. For a more in-depth treatment of loops and parallelism, I refer the reader to section 1.4 of [18].

Proposition 3.3.3. *Let $M = (E, \mathcal{I})$ be a matroid. Then \sim_M is an equivalence relation on $E \setminus E_0$.*

Proof. For any $x \in E \setminus E_0$, we have $\text{rank}(\{x, x\}) = \text{rank}(\{x\}) = 1$ since $x \notin E_0$. Thus $x \sim_M x$. For $x, y \in E \setminus E_0$, we have $\text{rank}(\{x, y\}) = \text{rank}\{y, x\}$. This proves that \sim_M is reflexive. To prove transitivity, suppose that we have $x, y, z \in E$ that satisfy $x \sim_M y$ and $y \sim_M z$. If $x = y$ or $y = z$, then we automatically get $x \sim_M z$. Suppose that they are all distinct. Then, we have $\{x, y\}, \{y, z\} \notin \mathcal{I}$. For the sake of contradiction, suppose that $\{x, z\} \in \mathcal{I}$. From **(I3)** applied to $\{x, z\}$ and $\{y\}$, we have that either $\{x, y\} \in \mathcal{I}$ or $\{y, z\} \in \mathcal{I}$. This is a contradiction. This proves that \sim_M is transitive and an equivalence relation. \square

From Proposition 3.3.3, for any non-loop $e \in E \setminus E_0$, we can consider its equivalence class $[e] := [e]_M$ under the equivalence relation \sim_M . We call the equivalence class $[e]$ the **parallel class** of e . Then, we can partition the ground set E into $E = E_1 \sqcup E_2 \sqcup \dots \sqcup E_s \sqcup E_0$ where E_1, \dots, E_s are the distinct parallel classes and E_0 are the loops. In Proposition 3.3.4, we give a characterization of the atoms of the lattice of flats in terms of the parallel classes and loops.

Proposition 3.3.4. *Let $M = (E, \mathcal{I})$ be a matroid and let $e \in E$ be an element of the matroid that is not a loop. Then, $\bar{e} = [e] \cup E_0$.*

Proof. Since any independent set contains no loops, we know that $E_0 \subseteq \bar{e}$. For any $f \in [e]$, we have that $\text{rank}(\{e, f\}) = 1 = \text{rank}(\{e\})$ by definition of \sim_M . This proves that $[e] \subseteq \bar{e}$. To prove the opposite inclusion, let $f \in \bar{e}$. Then $\text{rank}(\{e, f\}) = 1$. If f is a loop, then $f \in E_0$. Otherwise, $f \sim_M e$ and $f \in [e]$. This suffices for the proof of the proposition. \square

We call a matroid **simple** if it contains no loops and no parallel elements. To every matroid M , we can associate a simple matroid \widetilde{M} called the **simplification** of the matroid M . For any matroid $M = (E, \mathcal{I})$, let $\pi_M : E \rightarrow 2^E$ be the map $\pi_M(e) = \bar{e}$ which sends a matroid element to its closure. When e is not a loop, then \bar{e} is a one-dimensional flat. If e is a loop, then $\bar{e} = E_0$. From Proposition 3.3.4, we can think of $\pi_M(E)$ as the set of one-dimensional flats and the set of loops.

3.3.6 Contraction and Deletion

In graph theory, there are notions of edge contraction and edge deletion. In Definition, we generalize contraction and deletion from graphs to general matroids. When applying contraction and deletion in the matroid sense to graphic matroids, we recover edge contraction and deletion in graph theory.

Definition 3.3.5 (Restriction, Contraction, and Deletion). Let $M = (E, \mathcal{I})$ be a matroid and let $T \subseteq E$ be a subset. We define $M|_T$ to be the restriction of M on T as the matroid on T with independent sets

$$\mathcal{I}(M|_T) := \{I \in \mathcal{I} : I \subseteq T\}.$$

The deletion $M \setminus T$ of T from M is then defined to be the restriction $M|_{E \setminus T}$. Let B_T be a basis of $M|_T$. The contraction M/T of the matroid M by T is the matroid on $E \setminus T$ defined by

$$\mathcal{I}(M/T) := \{I \subset E \setminus T : I \cup B_T \in \mathcal{I}(M)\}.$$

For a proof of why the definition of $\mathcal{I}(M/T)$ is independent of our choice of B_T we refer the reader to Proposition 3.1.7 of [13].

Lemma 3.3.1. *Let $M = (E, \mathcal{B})$ be a matroid with basis generating polynomial f_M .*

- (a) *If $e \in E$ is a loop, then $\partial_e f_M = 0$.*
- (b) *If $e \in E$ is not a loop, then $\partial_e f_M = f_{M/e}$.*
- (c) *If $e \in E$ is not a coloop, then $f_M = x_e f_{M/e} + f_{M \setminus e}$*

Proof. The first claim follows from the fact that any basis will contain no loops. Hence, for a loop $e \in E$ the variable x_e will not appear in f_M . The second claim follows from the fact that the remaining monomials in $\partial_e f_M$ will correspond to sets of the form $B \setminus e$ where $e \in B$ and B is a basis of M . This is exactly the set of bases of M/e . For the third claim, this follows because the monomials in f_M which do not contain x_e will correspond to bases of M which do not contain e . That is the exactly the bases of $M \setminus e$. This implies that we can write $f_M = x_e p + f_{M \setminus e}$ where $p \in \mathbb{R}[x_e : e \in E]$ and p contains no monomial with x_e . Taking the partial derivative with respect to x_e , we get $p = f_{M/e}$ from (a). This suffices for the proof. \square

3.3.7 Regular Matroids

Definition 3.3.6. A **regular matroid** is a matroid which is isomorphic to a linear matroid with respect to a real matrix which is totally unimodular. A totally unimodular matrix is a matrix for which every square submatrix has determinant in $\{0, -1, 1\}$.

3.4 Mixed Discriminants

In this section, we discuss a symmetric multilinear form called the mixed discriminant which arises as the polarization of the determinant. This object will be useful in future sections as it satisfies the log-concavity inequality in Theorem (CITE). Our main reference for the theory of mixed discriminants will come from [1].

Definition 3.4.1. let $n \geq 1$ be a positive integer. Suppose that for each $k \in [n]$, we are given an $n \times n$ matrix $A^k := (a_{ij}^k)_{i,j=1}^n$. Then, we define the **mixed discriminant** of (A^1, \dots, A^n) as the expression

$$D(A^1, \dots, A^n) := \frac{1}{n!} \sum_{\sigma \in S_n} \det \begin{bmatrix} a_{11}^{\sigma(1)} & \dots & a_{1n}^{\sigma(n)} \\ \vdots & \ddots & \vdots \\ a_{n1}^{\sigma(1)} & \dots & a_{nn}^{\sigma(n)} \end{bmatrix}.$$

Here, S_n is the symmetric group on n letters.

3.4.1 Polarization Form

Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be an arbitrary homogeneous polynomial of degree d . Then, for fixed $v_1, \dots, v_d \in \mathbb{R}^n$, we can consider the coefficients of $f(\lambda_1 v_1 + \dots + \lambda_d v_d)$ as a homogeneous polynomial in $\lambda_1, \dots, \lambda_d$ of degree d . From section 3.2 of [14] and section 5.5 of [15], there is form $\tilde{f}(v_1, \dots, v_d)$ which is linear in each vector and symmetric in its arguments such that $\tilde{f}(v, \dots, v) = f(v)$. Indeed, since f is homogeneous of degree d , we can always write

$$f(x) = \sum_{i_1, \dots, i_d=1}^n a_{i_1, \dots, i_d} \cdot x_{i_1} \dots x_{i_d}$$

where $x = (x_1, \dots, x_n)$ and the coefficients a_{i_1, \dots, i_d} are symmetric. Now, if we define the vectors $v_i = (c_1^{(i)}, c_2^{(i)}, \dots, c_n^{(i)})$ for all $1 \leq i \leq d$, then, we can define the multilinear form

$$\tilde{f}(v_1, \dots, v_n) := \sum_{i_1, \dots, i_d=1}^n a_{i_1, \dots, i_d} \cdot c_{i_1}^{(1)} \dots c_{i_d}^{(d)}$$

which is symmetric in its arguments. This form satisfies the following “polarization identity”

$$f(\lambda_1 v_1 + \dots + \lambda_d v_d) = \sum_{1 \leq i_1, \dots, i_d \leq d} \lambda_{i_1} \dots \lambda_{i_d} \tilde{f}(v_{i_1}, \dots, v_{i_d}).$$

By taking partial derivatives of this identity and exploiting the symmetric of the form, we get the following formula in terms of the partial derivatives of f :

$$\tilde{f}(v_1, \dots, v_n) := \frac{1}{d!} \cdot \frac{\partial}{\partial \lambda_1} \dots \frac{\partial}{\partial \lambda_d} f(\lambda_1 v_1 + \dots + \lambda_d v_d) |_{\lambda=0}. \quad (3.1)$$

The mixed discriminant of matrices manifests as the polarization form of the determinant function. Indeed, Theorem 3.4.1 is well-known and can be found in [19] or [15].

Theorem 3.4.1. *For $n \times n$ matrices A_1, \dots, A_m and $\lambda_1, \dots, \lambda_m \in \mathbb{R}$, the determinant of the linear combination $\lambda_1 A_1 + \dots + \lambda_m A_m$ is a homogeneous polynomial of degree n in the λ_i and is given by*

$$\det(\lambda_1 A_1 + \dots + \lambda_m A_m) = \sum_{1 \leq i_1, \dots, i_n \leq m} D(A_{i_1}, \dots, A_{i_n}) \lambda_{i_1} \dots \lambda_{i_n}.$$

Example 3.4.1 (Mixed discriminants of rank 1 matrices). Let $x_1, \dots, x_n \in \mathbb{R}^n$ be real matrices. Then, for any $\lambda_1, \dots, \lambda_n > 0$, we can define $y_i = \sqrt{\lambda_i} x_i$. Then, we have that

$$\begin{aligned} \det \left(\sum_{i=1}^n \lambda_i x_i x_i^T \right) &= \det \left(\sum_{i=1}^n y_i y_i^T \right) \\ &= \det \left[\begin{pmatrix} y_1 & \dots & y_n \end{pmatrix} \cdot \begin{pmatrix} y_1^T \\ \vdots \\ y_n^T \end{pmatrix} \right] \\ &= \left[\det \begin{pmatrix} y_1 & \dots & y_n \end{pmatrix} \right]^2 \\ &= \lambda_1 \dots \lambda_n \cdot \left[\det \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} \right]^2. \end{aligned}$$

From Equation 3.1, we get the formula

$$D(x_1 x_1^T, \dots, x_n x_n^T) = \frac{1}{n!} \left[\det \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} \right]^2.$$

3.4.2 Positivity

In this section, we discuss when the mixed discriminant is positive. First, recall the following factorization result in linear algebra for positive (semi-)definite matrices.

Theorem 3.4.2 (Cholesky Factorization, Theorem 4.2.5 in [8]). *If $A \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix, then there exists a unique lower triangular $L \in \mathbb{R}^{n \times n}$ with positive diagonal entries*

such that $A = LL^T$. When A is positive semi-definite, then there exists a (not necessarily unique) lower triangular $L \in \mathbb{R}^{n \times n}$ with $A = LL^T$.

Lemma 3.4.1 (Lemma 5.2.1 in [1]). *Let A_1, \dots, A_n be positive semi-definite $n \times n$ matrices, and suppose that $A_k = X_k X_k^T$ for each k . Then*

$$D(A_1, \dots, A_n) = \frac{1}{n!} \sum_{\substack{x_j \in \text{col}(X_j) \\ 1 \leq j \leq n}} \left[\det \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix} \right]^2$$

where the sum is over all choices of x_k in the columns of X_k .

Proof. From the multi-linearity of the mixed discriminant, we have that

$$D(A_1, \dots, A_n) = D \left(\sum_{x_1 \in \text{col}(X_1)} x_1 x_1^T, \dots, \sum_{x_n \in \text{col}(X_n)} x_n x_n^T \right) = \sum_{\substack{x_j \in \text{col}(X_j) \\ 1 \leq j \leq n}} D(x_1 x_1^T, \dots, x_n x_n^T).$$

The Lemma follows from the computation in Example 3.4.1. □

Corollary 3.4.1. *Let A_1, \dots, A_n be positive semi-definite $n \times n$ real symmetric matrices. Then*

$$D(A_1, \dots, A_n) \geq 0.$$

Proof. This follows from Lemma 3.4.1 and Theorem 3.4.2. □

3.5 Convex Bodies

In this section, we review the notions of convexity and convex bodies. We use [15] as our main reference for the theory of convex bodies and Brunn-Minkowski inequalities.

Definition 3.5.1. A **convex body** is a compact, convex subset of \mathbb{R}^n .

3.5.1 Mixed Volumes

Chapter 4

Mechanisms for Log-concavity

4.1 Log-Concavity and Ultra Log-Concavity

Definition 4.1.1.

(Many sequences in combinatorics are proven to be log-concave ... give examples: Read's conjecture, Mason's Conjecture, etc.)

4.2 Alexandrov's Inequality for Mixed Discriminants

In this section we introduce a fundamental inequality given in Theorem 4.2.1 in the theory of mixed discriminants. This inequality is similar to the inequality for mixed volumes of convex bodies given in Theorem 4.3.1. We first give a

Proposition 4.2.1.

Proof.

□

Theorem 4.2.1 (Alexandrov's Inequality for Mixed Discriminants). *Let $X, Y, A_1, \dots, A_{n-2}$ be real symmetric positive definite $n \times n$ matrices. Then*

$$D(X, Y, A_1, \dots, A_{n-2})^2 \geq D(X, X, A_1, \dots, A_{n-2}) \cdot D(Y, Y, A_1, \dots, A_{n-2})$$

where equality holds if and only if $B = \lambda A$ for a real number λ .

Corollary 4.2.1. *Let A, B be $n \times n$ positive definite symmetric real matrices. For $0 \leq k \leq n$, define the mixed discriminant*

$$D_k := D(\underbrace{A, \dots, A}_{k \text{ times}}, \underbrace{B, \dots, B}_{n-k \text{ times}}).$$

Then the sequence D_0, D_1, \dots, D_n is log-concave.

Corollary 4.2.2. *Let A, B, D_k for $1 \leq k \leq n$ be as in Corollary 4.2.1. If $D_k^2 = D_{k-1}D_{k+1}$ for some $1 \leq k \leq n-1$, then $D_k^2 = D_{k-1}D_{k+1}$ holds for all $k = 1, \dots, n-1$.*

Proof. This follows from the equality case in Theorem 4.2.1. □

4.3 Alexandrov-Fenchel Inequality

Theorem 4.3.1 (Alexandrov-Fenchel Inequality). *Let $K_1, K_2, \dots, K_{d-2} \subseteq \mathbb{R}^d$ be convex bodies. For any convex bodies $X, Y \subseteq \mathbb{R}^d$, we have the inequality*

$$V_d(X, Y, K_1, \dots, K_{d-2})^2 \geq V_d(X, X, K_1, \dots, K_{d-2}) \cdot V_d(Y, Y, K_1, \dots, K_{d-2}).$$

4.4 Lorentzian Polynomials

4.5 Combinatorial Atlas

Chapter 5

Log-concavity Results for Posets and Matroids

5.1 Stanley's Inequality

5.2 Kahn-Saks Inequality

5.3 Stanley's Matroid Inequality

Chapter 6

Hard Lefschetz Property and Hodge-Riemann Relations

6.1 Gorenstein ring associated to a polynomial

In this section, we study a ring introduced in by Toshiaki Maeno and Yasuhide Numata in [11]. The Gorenstein Ring associated to the basis generating polynomial of a matroid is intimately related to graded Mobius(") algebra associated to a matroid and the Chow ring associated to a matroid. The Gorenstein stein ring associated to a polynomial is defined based on the action of differential forms on a fixed homogeneous polynomial.

Definition 6.1.1. Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be a homomogeneous polynoimal and let $S := \mathbb{R}[\partial_1, \dots, \partial_n]$ be the polynomial ring of differentials where $\partial_i := \partial_{x_i}$. We define the ring

$$A_f^\bullet := S / \text{Ann}_S(f).$$

We call A_f^\bullet the Gorenstein ring associated to the polynomial f . Alternatively, we view the polynomial ring $\mathbb{R}[x_1, \dots, x_n]$ as a module over the polynomial ring $\mathbb{R}[X_1, \dots, X_n]$ where the module structure is defined by the relation

$$p(X_1, \dots, X_n) \cdot q(x_1, \dots, x_n) := p(\partial_1, \dots, \partial_n)q(x_1, \dots, x_n).$$

Then the Gorenstein ring can be equivalently defined as $\mathbb{R}[X_1, \dots, X_n] / \text{Ann}(f)$. Note that this

is trivially equivalent to the first definition. We note this alternative perspective because both conventions are used in the literature [12, 11].

The Gorenstein ring associated to a polynomial has a natural grading with respect to the degree of the differential form. Before we prove that this actually gives A_f^\bullet a graded ring structure, we first prove Lemma 6.1.1.

Lemma 6.1.1. *Let $\xi \in \mathbb{R}[\partial_1, \dots, \partial_n]$ and $f \in \mathbb{R}[x_1, \dots, x_n]$ be a homogeneous polynomial. We can decompose $\xi = \xi_0 + \xi_1 + \dots$ into its homogeneous parts. If $\xi(f) = 0$, then $\xi_d(f) = 0$ for all $d \geq 0$.*

Proof. Let $d = \deg(f)$. If $\xi_i(f) \neq 0$, then $i \leq d$ and $\xi_i(f)$ is a homogeneous polynomial of degree $d - i$. Thus,

$$\xi(f) = \xi_0(f) + \xi_1(f) + \dots$$

will be the homogeneous decomposition of the polynomial $\xi(f)$. Since this is equal to 0, all components of the decomposition are equal to zero. This proves the proposition. \square

Proposition 6.1.1. *The Chow ring A_f^\bullet is a graded \mathbb{R} -algebra where A_f^k consists of the forms of degree k .*

Proof. Let us define A_f^k as in the statement of the lemma. Let $d = \deg(f)$ be the degree of the homogeneous polynomial. Whenever $k > d$, the ring A_f^k is clearly trivial. From Lemma 6.1.1, we have the direct sum decomposition

$$A_f^\bullet = \bigoplus_{k=0}^d A_f^k.$$

It is also clear that multiplication induces maps $A_f^r \times A_f^s \rightarrow A_f^{r+s}$ for all $r, s \geq 0$. \square

The following result (Proposition (CITE)) is well-known and it follows from Theorem 2.1 in CITE(Maeno, Watanabe, Lefschetz elements of ARTinian Gorenstein algebras and Hessians of homogeneous polynomials)

Proposition 6.1.2. *Let f be a homogeneous polynomial of degree d . Then, the ring A_f^\bullet is a Poincare-Duality algebra. That is, the ring satisfies the following two properties:*

- (a) $A_f^d \simeq A_f^0 \simeq \mathbb{R}$;
- (b) *The pairing induced by multiplication $A_f^{d-k} \times A_f^k \rightarrow A_f^d \simeq \mathbb{R}$ is non-degenerate for all $0 \leq k \leq d$.*

We first make a small remark on non-degeneracy to make it clear what we mean when we say a pairing is non-degenerate.

Lemma 6.1.2. *Let $B : V \times W \rightarrow k$ be a bilinear pairing between two finite-dimensional k -vector spaces V and W . Then, any two of the following three conditions imply the third.*

(i) *The map $B_V : V \rightarrow W^*$ defined by $v \mapsto B(v, \cdot)$ has trivial kernel.*

(ii) *The map $B_W : W \rightarrow V^*$ defined by $w \mapsto B(\cdot, w)$ has trivial kernel.*

(iii) $\dim V = \dim W$.

Proof. Condition (i) implies $\dim V \leq \dim W$ and Condition (ii) implies $\dim W \leq \dim V$. Thus (i) and (ii) both imply (iii). Now, suppose that (i) and (iii) are true. Then B_V is an isomorphism between V and W^* [3.69 in Axler (CITE???)]. Let v_1, \dots, v_n be a basis for V . Then $B_V(v_1), \dots, B_V(v_n)$ is a basis of W^* . Let w_1, \dots, w_n be the dual basis in W with respect to this basis of W^* . Suppose that $\sum \lambda_i w_i \in \ker B_W$. Then for all $v \in V$, we have

$$\sum_{i=1}^n \lambda_i B_V(v)(w_i) = B\left(v, \sum_{i=1}^n \lambda_i w_i\right) = 0.$$

By letting $v = v_1, \dots, v_n$, we get $\lambda_i = 0$ for all i . □

We say a pairing between finite-dimensional vector spaces is non-degenerate whenever all three conditions in Lemma 6.1.2 hold. As a corollary, we have the following result.

Corollary 6.1.1. *Let f be a homogeneous polynomial of degree $d \geq 2$ and let k , $0 \leq k \leq d$, be a non-negative integer. Then $\dim_{\mathbb{R}} A_f^k = \dim_{\mathbb{R}} A_f^{d-k}$.*

In Proposition(CITE), we can define $\deg_f : A_f^d \rightarrow \mathbb{R}$ to be the isomorphism defined by evaluation at f . That is, for any $\xi \in A_f^d$, we can define $\deg_f(\xi) := \xi(f)$ where ξ acts on f by differentiation. Following (CITE HODGE THEORY OF COMBINATORIAL GEOMETRIES), we formulate the following definition

Definition 6.1.2. Let f be a homogeneous polynomial of degree d and let $k \leq d/2$ be a non-negative integer. For an element $l \in A_f^1$, we define the following notions:

- (a) The **Lefschetz operator** on A_f^k associated to l is the map $L_l^k : A_f^k \rightarrow A_f^{d-k}$ defined by $\xi \mapsto l^{d-2k} \cdot \xi$.
- (b) The **Hodge-Riemann form** on A_f^k associated to l is the bilinear form $Q_l^k : A_f^k \times A_f^k \rightarrow \mathbb{R}$ defined by $Q_l^k(\xi_1, \xi_2) = (-1)^k \deg(\xi_1 \xi_2 l^{d-2k})$.

(c) The **primitive subspace** of A_f^k associated to l is the subspace

$$P_l^k := \{\xi \in A_f^k : l^{d-2k+1} \cdot \xi = 0\} \subseteq A_f^k.$$

Definition 6.1.3. Let f be a homogeneous polynomial of degree d , let $k \leq d/2$ be a non-negative integer, and let $l \in A_f^1$ be a linear differential form. We define the following notions:

- (a) (Hard Lefschetz Property) We say A_f satisfies HL_k with respect to l if the Lefschetz operator L_l^k is an isomorphism.
- (b) (Hodge-Riemann Relations) We say A_f satisfies HRR_k with respect to l if the Hodge-Riemann form Q_l^k is positive definite on the primitive subspace P_l^k .

Sometimes, instead of saying A_f satisfies Hodge-Riemann Relations or the Hard Lefschetz Property, we will say that f satisfies Hodge-Riemann Relations or the Hard Lefschetz property. For any $a \in \mathbb{R}^n$, we can define the linear differential form $l_a := a_1 \partial_1 + \dots + a_n \partial_n$. We say that f satisfies HL or HRR with respect to a if and only if it satisfies HL or HRR with respect to l_a . Most applications of the Hodge-Riemann Relations only end up using the relations up to degree $k \leq 1$. (GIVE EXAMPLE OF AN APPLICATION WHICH USES HIGHER DIMENSION,, maybe Chris Eur???)

Proposition 6.1.3 (Lemma 3.4 in [12]). *Let $f \in \mathbb{R}[x_1, \dots, x_n]$ be a homogeneous polynomial of degree $d \geq 2$ and $a \in \mathbb{R}^n$. Assume that $f(a) > 0$. Then,*

- (a) A_f has HL_1 with respect to l_a if and only if $Q_{l_a}^1$ is non-degenerate.
- (b) Suppose that A_f satisfies HL_1 . Then A_f has HRR_1 with respect to l_a if and only if $-Q_{l_a}^1$ has signature $(+, -, \dots, -)$.

Proof. We include a proof for completeness. We first prove the statement in (a). Suppose that A_f has HL_1 with respect to l_a . We have the following commutative diagram:

$$\begin{array}{ccc} A_f^1 \times A_f^1 & \xrightarrow{\text{id} \times L_{l_a}^1} & A_f^1 \times A_f^{d-1} \\ & \searrow -Q_{l_a}^1 & \swarrow \\ & \mathbb{R} & \end{array}$$

where the missing mapping is multiplication. If A_f has HL_1 with respect to l_a , then the top map between $A_f^1 \times A_f^1 \rightarrow A_f^1 \times A_f^{d-1}$ is an bijection. Thus the non-degeneracy of $Q_{l_a}^1$ follows from the non-degeneracy of the multiplication pairing as stated in Proposition 6.1.2. Now, suppose that

$Q_{l_a}^1$ is non-degenerate. Then, the map $B : A_f^1 \rightarrow (A_f^1)^*$ defined by $\xi \mapsto -Q_{l_a}^1(\xi, \cdot)$ is given by $m(L_{l_a}^1 \xi, \cdot)$ where $m : A_f^1 \rightarrow A_f^{d-1} \rightarrow \mathbb{R}$ is the multiplication map. This is the composition of $A_f^1 \rightarrow A_f^{d-1} \rightarrow (A_f^1)_*$ where the first map is $L_{l_a}^1$ and the second map is injective from the non-degeneracy of the multiplication map. This proves that $L_{l_a}^1$ is injective. From Corollary 6.1.1, the map $L_{l_a}^1$ is an isomorphism. This suffices for the proof of (a).

To prove (b), consider the commutative diagram

$$\begin{array}{ccccccc} \mathbb{R} & \xrightarrow{\cong} & A_f^0 & \xrightarrow{\times l_a} & A_f^1 & \xrightarrow{L_{l_a}^1} & A_f^{d-1} & \xrightarrow{\times l_a} & A_f^d & \xrightarrow{\cong} & \mathbb{R} \\ & & & & & \searrow L_{l_a}^0 & & \nearrow & & & \end{array}$$

Note that $L_{l_a}^0$ is an isomorphism because

$$\deg L_{l_a}^0(1) = l_a^d(f) = d!f(a) \neq 0.$$

Thus, we have $A_f^1 = \mathbb{R}l_a \oplus P_l^1$ where the direct sum is orthogonal over the Hodge-Riemann form by definition of the primitive subspace. Now, note that

$$-Q_{l_a}^1(l_a, l_a) = l_a^d(f) = -d!f(a) > 0.$$

Thus, the signature of $-Q_{l_a}^1$ is $(+, -, \dots, -)$ if and only if $Q_{l_a}^1$ is positive definite over the primitive subspace if and only if A_f satisfies HRR_1 with respect to l_a . \square

From Sylvester's Law of Inertia and the fact that Hessians of Lorentzian polynomials always have at most one positive eigenvalue on the positive orthant. (cite in previous part of thesis maybe)

Lemma 6.1.3 (Lemma 3.5 in [12]). *If $f \in \mathbb{R}[x_1, \dots, x_n]$ is Lorentzian, then for any $a \in \mathbb{R}_{\geq 0}^n$ with $f(a) > 0$, A_f^1 has HL_1 with respect to l_a if and only if f has the HRR_1 with respect to l_a .*

6.2 Local Hodge-Riemann Relations

Suppose that we are trying to prove the Hodge-Riemann Relations property for a Lorentzian polynomial. To do so, it is actually enough to show that it satisfies a “local” version of the Hodge-Riemann relations. This will imply that it satisfies the Hard Lefschetz Property, which by Lemma 6.1.3 is enough to prove that it satisfies Hodge-Riemann relations. We define the local HRR as in [12].

Definition 6.2.1. A homogeneous polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ of degree $d \geq 2k + 1$ satisfies the **local HRR_k** with respect to a form $l \in A_f^1$ if for all $i \in [n]$, either $\partial_i f = 0$ or $\partial_i f$ satisfies **HRR_k** with respect to l .

One existing inductive mechanism for proving Hodge-Riemann relations is Lemma 6.2.1.

Lemma 6.2.1 (Lemma 3.7 in [12]). *Let $f \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ be a homogeneous polynomial of degree d and k a positive integer with $d \geq 2k + 1$, and $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. Suppose that f has the local **HRR_k** with respect to l_a .*

- (i) *If $a \in \mathbb{R}_{> 0}^n$, then A_f has the **HL_k** with respect to l_a .*
- (ii) *If $a_1 = 0, a_2, \dots, a_n > 0$ and $\{\xi \in A_f^k : \partial_i \xi = 0 \text{ for } i = 2, \dots, n\} = \{0\}$, then A_f has the **HL_k** with respect to l_a .*

This is effective in proving that all Lorentzian polynomials satisfy **HRR₁** with respect to l_a for any $a \in \mathbb{R}_{> 0}^n$. Indeed, this is easy to prove for Lorentzian polynomials of small degree. For an arbitrary Lorentzian polynomial, all of its partial derivatives are also Lorentzian (CITE THIS RESULT SOMEWHERE IN THE THESIS). Thus, by the inductive hypothesis all of these polynomials satisfy **HRR₁**. But this implies from Lemma 6.2.1 that the original polynomial satisfies **HL₁**, hence **HRR₁** since it is Lorentzian. This proves Theorem 6.2.1.

Theorem 6.2.1 (Theorem 3.8 in [12]). *If $f \in \mathbb{R}[x_1, \dots, x_n]$ is Lorentzian, then f has **HRR₁** with respect to l_a for any $a \in \mathbb{R}_{> 0}^n$.*

If the value of a is not restricted to the positive orthant and additionally we know that $\partial_1, \dots, \partial_n$, then we have the following computational result.

Corollary 6.2.1. *Let f be a homogeneous polynomial of degree $d \geq 2$. If $\partial_1 f, \dots, \partial_n f$ are linearly independent in $\mathbb{R}[x_1, \dots, x_n]$ and $f(a) > 0$ for some $a \in \mathbb{R}^n$, then A_f satisfies **HRR₁** with respect to l_a if and only if $\text{Hess}_f|_{x=a}$ has signature $(+, -, \dots, -)$.*

Proof. Since $\partial_1 f, \dots, \partial_n f$ are linearly independent, the partials ∂_i form a basis for A_f^1 . Thus, the signature of $Q_{l_a}^1$ is actually the signature of matrix of $Q_{l_a}^1$ with respect to the set $\{\partial_1, \dots, \partial_n\}$. We have

$$-Q_{l_a}^1(\partial_i, \partial_j) = \partial_i \partial_j l_a^{d-2} f = l_a^{d-2} \partial_i \partial_j f = (d-2)! \partial_i \partial_j f(a).$$

This proves that the signature of $-Q_{l_a}^1$ is the same as the signature of $\text{Hess}_f|_{x=a}$. We are done from Lemma 6.1.3(b). □

6.3 The Gorenstein ring associated to the basis generating polynomial of a matroid

Definition 6.3.1. Let M be a matroid. We define $A^\bullet(M) := A_{f_M}^\bullet$ to be the **Gorenstein ring associated to the basis generating polynomial** of M . In this case, we define

$$S_M := \mathbb{R}[\partial_e : e \in E]$$

to be the polynomial ring in variables indexed by the ground set. We also define

$$\text{Ann}_M := \text{Ann}_{S_M}(f_M)$$

which is the annihilator of f_M with respect to S_M . Then $A(M) = S_M / \text{Ann}_M$.

Remark. For the sake of brevity, we will refer to the ring $A(M)$ as simply the **Gorenstein ring** of the matroid M . As a warning, note that this is not standard in the literature since there are **many** Gorenstein rings associated to matroids. Immediate examples are the Chow ring and augmented Chow ring of a matroid [2].

We will now prove that the Gorenstein ring associated to the basis generating polynomial of matroid only depends on its simplification. Indeed, if $a, b \in E(M)$ are parallel elements, then $\partial_a - \partial_b$ gets annihilated and if $e \in E(M)$ is a loop then ∂_e gets annihilated. We will describe this isomorphism in Theorem 6.3.1. Before proving this result, we first describe some common elements in the ideal Ann_M .

Proposition 6.3.1 (Proposition 3.1 in [11]). *The annihilator Ann_M contains the elements*

$$\Lambda_M := \{x_e^2 : e \in E(M)\} \cup \{x^S : S \notin \mathcal{I}(M)\} \cup \{x^A - x^{A'} : A, A' \in \mathcal{I}(M), \overline{A} = \overline{A'}\}.$$

Example 6.3.1. The elements Λ_M unfortunately do not generate Ann_M in general. In [11], they give an example of matroid which is the matroid described by the linear independence of the following

five vectors.

$$\begin{aligned}
e_1 &= (1, 0, 0) \\
e_2 &= (0, 1, 0) \\
e_3 &= (0, 0, 1) \\
e_4 &= (1, 1, 0) \\
e_5 &= (0, 1, 1).
\end{aligned}$$

The bases are 123, 125, 134, 135, and 145. From Example 3.5 in [11], it happens that

$$\text{Ann}_M = \langle \Lambda_M \rangle + (\partial_1 \partial_3 + \partial_4 \partial_5 - \partial_1 \partial_5 - \partial_3 \partial_4).$$

Let $M = (E, \mathcal{I})$ be a matroid and let \widetilde{M} be its simplification. Recall that \widetilde{M} is a matroid on the ground set of rank-1 flats $E(\widetilde{M}) = \{\bar{x} : x \in E(M) \setminus E_0(M)\}$ whose independent sets consist of the subsets of $E(\widetilde{M})$ where by taking one representative from each rank-1 flat, we get an independent set of M . We can define $\phi : S_M \rightarrow S_{\widetilde{M}}$ and $\psi : S_{\widetilde{M}} \rightarrow S_M$ by

$$\phi(\partial_{x_e}) := \partial_{\bar{x}_e}, \quad \text{and} \quad \psi(\partial_{\bar{x}}) := \frac{1}{|\bar{x}|} \sum_{e \in \bar{x}} \partial_{x_e}.$$

The entire maps ϕ and ψ are then defined by using the universal property of polynomial rings to extend over the whole space.

Theorem 6.3.1. *The maps $\phi : S_M \rightarrow S_{\widetilde{M}}$ and $\psi : S_{\widetilde{M}} \rightarrow S_M$ induce isomorphisms between $A(M)$ and $A(\widetilde{M})$.*

Proof. We first prove that ϕ and ψ induce homomorphisms between the Chow rings. To show that ϕ induces a homomorphism, consider the diagram in Equation 6.1.

$$\begin{array}{ccccc}
S_M & \xrightarrow{\phi} & S_{\widetilde{M}} & \xrightarrow{\pi_{\widetilde{M}}} & A(\widetilde{M}) \\
& \searrow \pi_M & & \nearrow \exists! \Phi & \\
& & A(M) & &
\end{array} \tag{6.1}$$

Let $\xi \in S_M$ be an element satisfying $\xi(f_M) = 0$. We will prove that $\phi(\xi)(f_{\widetilde{M}}) = 0$. In other words, we want to prove that $\text{Ann}_M \subseteq \ker \pi_{\widetilde{M}} \circ \phi$. From Proposition 6.1.1, it suffices to consider

the case where ξ is homogeneous. Let e_1, \dots, e_s be representatives of all parallel classes. Then, we have that $M = \overline{e_1} \sqcup \dots \sqcup \overline{e_s} \cup E_0$ where E_0 denotes the loops in M . In terms of the basis generating polynomials, we have

$$f_{\widetilde{M}}(x_{\overline{e_1}}, \dots, x_{\overline{e_s}}) = \sum_{\substack{1 \leq i_1 < \dots < i_r \leq s \\ \{\overline{e_{i_1}}, \dots, \overline{e_{i_r}}\} \in \mathcal{B}(\widetilde{M})}} x_{\overline{e_{i_1}}} \dots x_{\overline{e_{i_r}}}$$

$$f_M(x_1, \dots, x_n) = f_{\widetilde{M}}(y_1, \dots, y_s)$$

where for $1 \leq i \leq s$, we define $y_i := \sum_{e \in \overline{e_i}} x_e$. Since ξ is homogeneous of degree k , we can write it in the form $\xi = \sum_{\substack{\alpha \subseteq [n] \\ |\alpha| = k}} c_\alpha \partial^\alpha$. Then, we have

$$\xi(f_M) = \sum_{\beta \in \mathcal{B}} \xi(x^\beta) = \sum_{\beta \in \mathcal{B}(M)} \sum_{\substack{\alpha \subseteq [n] \\ |\alpha| = k}} c_\alpha \partial^\alpha x^\beta = \sum_{\gamma \in \mathcal{I}_{r-k}(M)} \left(\sum_{\substack{\alpha \in \mathcal{I}_k \\ \alpha \cup \gamma \in \mathcal{I}_r(M)}} c_\alpha \right) x^\gamma. \quad (6.2)$$

Since $\xi(f_M) = 0$, we know that all of the coefficients on the right hand side of Equation 6.2 are equal to 0. Thus, for any $\gamma \in \mathcal{I}_{r-k}(M)$, we have

$$\sum_{\substack{\alpha \in \mathcal{I}_k \\ \alpha \cup \gamma \in \mathcal{I}_r(M)}} c_\alpha = 0.$$

On the other hand, we have

$$\phi(\xi) = \sum_{\substack{\alpha \subseteq [n] \\ |\alpha| = k}} c_\alpha \prod_{e \in \alpha} \partial_{\overline{e}} = \sum_{\beta \in \mathcal{I}_k(\widetilde{M})} \left(\sum_{\alpha \in \text{fiber}(\beta)} c_\alpha \right) \partial^\beta.$$

We can let this differential act on $f_{\widetilde{M}}$ to get the expression

$$\phi(\xi)(f_{\widetilde{M}}) = \sum_{\gamma \in \mathcal{I}_{r-k}(\widetilde{M})} \left(\sum_{\substack{\beta \in \mathcal{I}_k(\widetilde{M}) \\ \beta \cup \gamma \in \mathcal{B}(\widetilde{M})}} \sum_{\alpha \in \text{fiber}(\beta)} c_\alpha \right) x^\gamma = \sum_{\gamma \in \mathcal{I}_{r-k}(\widetilde{M})} \left(\sum_{\substack{\alpha \in \mathcal{I}_k(M) \\ \alpha \cup \gamma_0 \in \mathcal{B}(M)}} c_\alpha \right) x^\gamma = 0. \quad (6.3)$$

In Equation 6.3, the independent set $\gamma_0 \in \text{fiber}(\gamma)$ is an arbitrary element in the fiber of γ . This proves that $\text{Ann}_M \subseteq \ker \pi_{\widetilde{M}} \circ \phi$. Thus, there is a unique ring homomorphism $\Phi : A(M) \rightarrow A(\widetilde{M})$ which makes Equation 6.1 commute.

To prove that ψ induces a map between the Chow rings, consider the diagram in Equation 6.4.

$$\begin{array}{ccccc}
S_{\widetilde{M}} & \xrightarrow{\psi} & S_M & \xrightarrow{\pi_M} & A(M) \\
& \searrow \pi_{\widetilde{M}} & & \nearrow \exists! \Psi & \\
& & A(\widetilde{M}) & &
\end{array} \tag{6.4}$$

Consider a differential $\xi \in S_{\widetilde{M}}$ satisfying $\xi(f_{\widetilde{M}}) = 0$. We can write this as $\xi = \sum_{\alpha \in \mathcal{I}_k(\widetilde{M})} c_\alpha \partial^\alpha$ for some real constants c_α . Then, its image under ψ is equal to

$$\psi(\xi) = \sum_{\alpha \in \mathcal{I}_k(\widetilde{M})} \frac{c_\alpha}{\prod_{e \in \alpha} |e|} \sum_{\beta \in \text{fiber}(\alpha)} \partial^\beta.$$

Fix a $\alpha \in \mathcal{I}_k(\widetilde{M})$ and a $\beta \in \text{fiber}(\alpha)$. Since $\partial y_i / \partial x_e = \mathbb{1}_{e \in \overline{e_i}}$, we have

$$\partial^\beta f_M(x_1, \dots, x_n) = \partial^\beta f_{\widetilde{M}}(y_1, \dots, y_s) = \partial^\alpha f_{\widetilde{M}}(x_1, \dots, x_s)|_{x_1=y_1, \dots, x_s=y_s} = 0.$$

Thus, we have that $\psi(\xi)(f_M) = 0$ and $\text{Ann}_{\widetilde{M}} \subseteq \ker \pi_M \circ \psi$. This proves that there is a unique ring homomorphism $\Psi : A(\widetilde{M}) \rightarrow A(M)$ which causes the diagram in Equation 6.4 to commute. Since the maps Ψ and Φ are inverses of each other, they are both isomorphisms. This suffices for the proof. \square

From Theorem 6.3.1, we get Corollary 6.3.1 and Corollary 6.3.2 immediately.

Corollary 6.3.1. *Let $M = (E, \mathcal{I})$ be a matroid. For any $a \in \mathbb{R}^E$, we can define the linear form $l_a := \sum_{e \in E} a_e \cdot \partial_{x_e} \in A^1(M)$. Let $\widetilde{l}_a := \Phi(l_a) = \sum_{e \in E} a_e \cdot \partial_{x_{\overline{e}}} \in A^1(\widetilde{M})$. Then, the following diagram commutes:*

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{\times l_a} & A^{i-1}(M) & \xrightarrow{\times l_a} & A^i(M) & \xrightarrow{\times l_a} & A^{i+1}(M) \xrightarrow{\times l_a} \cdots \\
& & \Phi \downarrow & & \Phi \downarrow & & \Phi \downarrow \\
\cdots & \xrightarrow{\times \widetilde{l}_a} & A^{i-1}(\widetilde{M}) & \xrightarrow{\times \widetilde{l}_a} & A^i(\widetilde{M}) & \xrightarrow{\times \widetilde{l}_a} & A^{i+1}(\widetilde{M}) \xrightarrow{\times \widetilde{l}_a} \cdots
\end{array}$$

Corollary 6.3.2. *Let $M = (E, \mathcal{I})$ be a matroid. Then $A(M)$ satisfies HRR_k with respect to l if and only if $A(\widetilde{M})$ satisfies HRR_k with respect to $\Phi(l)$.*

From Theorem 6.3.1, Corollary 6.3.1, and Corollary 6.3.2, we are led to study $A(M)$ in the case where \widetilde{M} is simple. In [12], Satoshi Murai, Takahiro Nagaoka, and Akiko Yazawa prove that if M is a simple matroid on $[n]$, then $\dim A^1(M) = n$ with basis $\partial_1, \dots, \partial_n$.

Lemma 6.3.1 (Theorem 2.5 in [12]). *If $M = ([n], \mathcal{I})$ is simple, then $\partial_1, \dots, \partial_n$ is a basis of $A^1(M)$.*

From Corollary 6.2.1, this implies that when M is simple, the Hessian of the basis generating polynomial f_M has signature $(+, -, \dots, -)$ at every point in the positive orthant. Thus, the basis generating polynomial for a simple matroid is **strictly** log-concave on the positive orthant. We are interested in log-concavity on the facets of the positive orthant. In other words, we want to extend the region in which we know $A(M)$ satisfies HRR_1 . From Lemma 6.2.1, we already know that $A(M)$ satisfies HRR_1 on $\mathbb{R}_{\geq 0}^n$. For a matroid $M = (E, \mathcal{I})$, we can write

$$\text{bd } \mathbb{R}^E = \bigcup_{e \in E} H_e$$

where $H_e := \{x \in \mathbb{R}_{\geq 0}^E : x_e = 0\}$. In the next section, we prove necessary and sufficient conditions for the basis generating polynomial to satisfy HRR_1 on the relative interior of one of the facets.

Before moving on to the study of HRR_1 on the facets of the positive orthant, our current results allow us to find necessary and sufficient conditions to tell us when $A(M)$ satisfies HRR_1 with respect to $a \in \mathbb{R}^n$ in the case $f_M(a) > 0$ and M is simple.

Corollary 6.3.3. *Let $M = (E, \mathcal{I})$ be a simple matroid of rank $r \geq 2$. If $a \in \mathbb{R}_{\geq 0}^E$ satisfies $f_M(a) > 0$, then $A(M)$ satisfies HRR_1 with respect to l_a if and only if $\text{Hess}_{f_M}|_{x=a}$ is non-singular.*

Proof. This follows immediately from Lemma 6.1.3, Corollary 6.2.1, and Lemma 6.3.1. □

Theorem 6.3.2. *Let $M = (E, \mathcal{I})$ be a simple matroid of rank $r \geq 2$. If $a \in \mathbb{R}_{\geq 0}^E$ satisfies $f_M(a) > 0$ and $a_e = 0$ for some $e \in E$ which is not a co-loop, then $A(M)$ satisfies HRR_1 with respect to l_a if and only if*

$$\det \left(\nabla f_{M/e}^\top \cdot \text{Hess}_{f_{M \setminus e}}^{-1} \cdot \nabla f_{M/e} \right) |_{x=a} \neq 0.$$

Proof. Without loss of generality, we can assume that $E(M) = [n]$ and $e = n$. In particular, this means that $a = (a_1, \dots, a_{n-1}, 0) \in \mathbb{R}^n$ with $a_i \geq 0$ for all i . From Corollary 6.3.3, HRR_1 is satisfied if and only if the Hessian is non-singular. To compute the Hessian at $x = a$, note that because n is a co-loop, we can rewrite the basis generating polynomial as

$$f_M = x_n f_{M/n} + f_{M \setminus n}.$$

The Hessian when computed at $(a_1, \dots, a_{n-1}, 0)$ will be equal to

$$\text{Hess}_{f_M} = \begin{bmatrix} \text{Hess}_{f_{M \setminus n}} & \nabla f_{M/n} \\ (\nabla f_{M/n})^\top & 0 \end{bmatrix}.$$

We know that $M \setminus n$ is still simple. Thus $\text{Hess}_{f_{M \setminus n}}$ is invertible since it has the same signature as the Hodge-Riemann form. Recall the following linear algebraic lemma.

Lemma 6.3.2. *When D is invertible and A is a square matrix, we have*

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A - BD^{-1}C) \det(D).$$

From the lemma, we have

$$\det \text{Hess}_{f_M} = \det \left(\nabla f_{M/n}^\top \cdot \text{Hess}_{f_{M \setminus n}}^{-1} \cdot \nabla f_{M/n} \right).$$

This suffices for the proof. □

6.4 Hodge-Riemann relations on the facets of the positive orthant

Lemma 6.4.1. *Let $M = (E, \mathcal{I})$ be a matroid satisfying $\text{rank}(M) \geq 2$. If $e \in M$ is not a co-loop of M , then e will not be a co-loop of M/i for any $i \in E \setminus e$.*

Proof. We assume that e is not a loop since otherwise it would not be a co-loop of any non-trivial matroid. We can also assume that i is not a loop otherwise the statement is trivial. Suppose for the sake of contradiction that e is a co-loop of M/i and i is not a loop. In the language of the original matroid, this means that any basis containing i also contains e . Since e is not a co-loop, the deletion M/e will have $\text{rank}(M/e) = \text{rank } M$. Since i is not a loop, it is contained in a basis of M/e . This would be a basis in the original matroid M not containing e . This is a contradiction. □

(what was the socle interpretation?)

Theorem 6.4.1. *Let $M = (E, \mathcal{I})$ be a matroid satisfying $\text{rank}(M) \geq 3$. Let $S \subseteq E(M)$ be a subset*

with $\text{rank}(S) \leq \text{rank}(M) - 2$. Then

$$\{\xi \in A_f^1 : \xi(\partial_i f) = 0 \text{ for } i \in E \setminus S\} = \{0\}.$$

Proof. Let $r = \text{rank}(M) \geq 3$. We want to prove that if a linear form $\xi = \sum_{e \in E} c_e \cdot \partial_e$ satisfies $\xi(\partial_e f_M) = 0$ for all $e \in E \setminus S$, then we have $\xi(f_M) = 0$. For $i \in E \setminus S$, we have

$$0 = \xi(\partial_i f_M) = \sum_{e \in E} c_e \partial_e \partial_i f_M = \sum_{e \in E} c_e \sum_{\substack{\alpha \in \mathcal{I}_{r-2}(M) \\ \alpha \cup \{e, i\} \in \mathcal{I}_r(M)}} x^\alpha = \sum_{\alpha \in \mathcal{I}_{r-2}(M)} \left(\sum_{\substack{e \in E \\ \alpha \cup \{i, e\} \in \mathcal{I}_r(M)}} c_e \right) x^\alpha. \quad (6.5)$$

By setting all of the coefficients of the right hand side of Equation 6.5, we have that

$$\sum_{\substack{e \in E \\ \alpha \cup \{i, e\} \in \mathcal{I}_r(M)}} c_e = 0$$

for all $\alpha \in \mathcal{I}_{r-2}(M)$ and $i \in E \setminus S$. For any $\beta \in \mathcal{I}_{r-1}(M)$, we know that $\beta \not\subseteq S$ since $\text{rank}(\beta) = r - 1 > \text{rank}(S)$. There exists some $i \in \beta \setminus S$. Thus, we can write $\beta = \alpha \cup \{i\}$ where $\alpha \in \mathcal{I}_{r-2}(M)$ and $i \in E \setminus S$. This proves that for any $\beta \in \mathcal{I}_{r-1}(M)$, we have

$$\sum_{\substack{e \in E \\ \beta \cup \{e\} \in \mathcal{I}_r}} c_e = \sum_{\substack{e \in E \\ \alpha \cup \{i, e\} \in \mathcal{I}_r}} c_e = 0.$$

Finally, we have that

$$\xi(f_M) = \sum_{e \in E} c_e \partial_e f_M = \sum_{e \in E} c_e \sum_{\substack{\beta \in \mathcal{I}_{r-1}(M) \\ \beta \cup \{e\} \in \mathcal{I}_r}} x^\beta = \sum_{\beta \in \mathcal{I}_{r-1}(M)} \left(\sum_{\substack{e \in E \\ \beta \cup \{e\} \in \mathcal{I}_r}} c_e \right) x^\beta = 0.$$

This suffices for the proof of the Theorem. \square

Theorem 6.4.2 (Higher Degree Socles). *Let $M = (E, \mathcal{I})$ be a matroid. Let $S \subseteq E$ be a subset such that $\text{rank}(S) \leq \text{rank}(M) - k - 1$. Then,*

$$\{\xi \in A^k(M) : \xi(\partial_e f_M) = 0 \text{ for all } e \in E \setminus S\} = \{0\}.$$

Proof. Any $\xi \in A^k(M)$ can be written as $\xi = \sum_{\alpha \in \mathcal{I}_k} c_\alpha \partial^\alpha$. For any $e \in E \setminus S$, we have

$$0 = \xi \partial_e f_M = \sum_{\alpha \in \mathcal{I}_k} c_\alpha \partial_e \partial^\alpha f_M = \sum_{\alpha \in \mathcal{I}_k} c_\alpha \sum_{\substack{\gamma \in \mathcal{I}_{r-k-1} \\ \gamma \cup \alpha \cup \{e\} \in \mathcal{I}_r}} x^\gamma = \sum_{\gamma \in \mathcal{I}_{r-k-1}} \left(\sum_{\substack{\alpha \in \mathcal{I}_k \\ \gamma \cup \alpha \cup \{e\} \in \mathcal{I}_r}} c_\alpha \right) x^\gamma.$$

This implies that for all $\gamma \in \mathcal{I}_{r-k-1}$ and $e \in E \setminus S$, we have

$$\sum_{\substack{\alpha \in \mathcal{I}_k \\ \gamma \cup \alpha \cup \{e\} \in \mathcal{I}_r}} c_\alpha = 0$$

for all $\gamma \in \mathcal{I}_{r-k-1}$ and $e \in E \setminus S$. Let $\beta \in \mathcal{I}_{r-k}$ be an arbitrary independent set. Note that $\text{rank}(\beta) = r - k > \text{rank}(S)$. Thus, we cannot have $\beta \subseteq S$. This implies that we can find $e \in \beta \setminus S$ such that $\beta = \alpha \cup \{e\}$ for $e \in E \setminus S$. Thus,

$$\sum_{\substack{\alpha \in \mathcal{I}_k \\ \beta \cup \alpha \in \mathcal{I}_r}} c_\alpha = 0$$

for all $\beta \in \mathcal{I}_{r-k}$. Then, we have

$$\xi(f_M) = \sum_{\alpha \in \mathcal{I}_k} c_\alpha \partial^\alpha f_M = \sum_{\alpha \in \mathcal{I}_k} c_\alpha \sum_{\substack{\beta \in \mathcal{I}_{r-k} \\ \beta \cup \alpha \in \mathcal{I}_r}} x^\beta = \sum_{\beta \in \mathcal{I}_{r-k}} \left(\sum_{\substack{\alpha \in \mathcal{I}_k \\ \beta \cup \alpha \in \mathcal{I}_r}} c_\alpha \right) x^\beta = 0.$$

This suffices for the proof. \square

Theorem 6.4.3. *Let $M = (E, \mathcal{I})$ be a matroid which satisfies $\text{rank}(M) \geq 2$. For any $e \in E(M)$, the basis generating polynomial f_M satisfies HRR_1 on $\text{relint}(H_e)$ if and only if e is not a co-loop.*

Proof. Without loss of generality, let M be a matroid on the set $[n]$ and let $e = 1$. Then, we want to prove that whenever $a_2, \dots, a_n > 0$, the ring $A(M)$ satisfies HRR_1 on $a = (0, a_2, \dots, a_n)$ if and only if e is not a co-loop. We first prove that if 1 is not a co-loop, then $A(M)$ satisfies HRR_1 . To prove this, we induct on the rank of M . For the base case $\text{rank}(M) = 2$, Corollary 6.3.2 implies that it suffices to prove that $A(\widetilde{M})$ satisfies HRR_1 on $\Phi(l_a)$. The only simple matroid of rank 2 is the uniform matroid of rank 2. From Corollary 6.3.3, it suffices to check that the signature of the Hessian is $(+, -, \dots, -)$. But the Hessian of a rank 2 simple matroid at every point is the same

matrix. If the matroid is on n elements, the Hessian matrix is

$$A(K_n) = \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 0 \end{bmatrix}.$$

This happens to be the adjacency matrix of a complete graph. From Proposition 1.5 in [17], this matrix has an eigenvalue of -1 with multiplicity $n - 1$ and an eigenvalue of $n - 1$ with multiplicity 1. Hence, its signature is $(+, -, \dots, -)$ which proves the base case. Now suppose that the claim is true for all matroids of rank less than r . Let M be a matroid of rank r . We want to prove that the claim is true for M . By the same reasoning, it suffices to prove the result when M is loopless. For all $i \in E \setminus \{1\}$, we know that e is not a co-loop of M/i from Lemma 6.4.1. Since M is simple, we know that $\text{rank}(M/i) = \text{rank}(M) - 1 < \text{rank}(M)$. Hence, from the inductive hypothesis, we know that $A(M/i) = A_{\partial_i f_M}^\bullet$ satisfies HRR_1 for l_a .

Now, we have enough information to directly prove that $A(M)$ satisfies HRR_1 with respect to l_a . From Lemma 6.1.3, it suffices to prove that $A(M)$ satisfies HL_1 with respect to l_a . Since $\dim_{\mathbb{R}} A^1(M) = \dim_{\mathbb{R}} A^{r-1}(M)$ from being a Poincare Duality algebra, it suffices to prove that the Lefschetz operator $L_{l_a}^1 : A^1(M) \rightarrow A^{r-1}(M)$ is injective. Let $\Xi \in A^1(M)$ be the kernel of $L_{l_a}^1$. This is the same as saying $\Xi l_a^{r-2} = 0$ in $A(M)$. We want to prove that $\Xi = 0$ in $A^1(M)$. Since $\Xi l_a^{r-2} = 0$ in $A(M)$, we have

$$0 = -Q_{l_a}^1(\Xi, \Xi) = \deg_M(\Xi^2 \cdot l_a^{r-2}) = \sum_{i=2}^n a_i \deg_M(\Xi^2 \cdot l_a^{r-3} \cdot \partial_i).$$

Note that

$$\deg_M(\Xi^2 \cdot l_a^{r-3} \cdot \partial_i) = (\Xi^2 \cdot l_a^{r-3})(\partial_i f_M) = \deg_{M/i}(\Xi^2 \cdot l_a^{r-3}) = -Q_{M/i}(\Xi, \Xi).$$

where $Q_{M/i}$ is the Hodge-Riemann form of degree 1 with respect to l associated with $A(M/i)$. Thus,

$$\sum_{i=2}^n a_i Q_{M/i}(\Xi, \Xi) = 0.$$

In $A(M/i)$, the linear form Ξ is in the primitive subspace. Hence, since we know that $A(M/i)$ satisfies HRR_1 with respect to l , we know that the Hodge-Riemann form $Q_{M/i}$ is negative-definite on $\mathbb{R}\Xi$. Since $a_i > 0$ for $i = 2, \dots, n$, this implies that $Q_{M/i}(\Xi, \Xi) = 0$ for all such i . Hence $\Xi = 0$ in $A(M/i)$ for all $i \in [2, n]$. In terms of polynomials, this means that $\Xi(\partial_i f_M) = 0$ for all $i \in [2, n]$. From Theorem 6.4.1, this implies that $\Xi(f_M) = 0$. Thus, $A(M)$ satisfies HRR_1 with respect to l .

For the other direction, we will prove that if e is a co-loop of M , then f_M does not satisfy HRR_1 on $\text{relint}(H_e)$. Without loss of generality, we can suppose $E(M) = [n]$ and $e = 1$. The linear differential can be written as l_a where $a = (0, a_2, \dots, a_n)$ for $a_2, \dots, a_n \geq 0$. It suffices to consider the case when M is simple because the coefficient of ∂_1 in the simplification of l_a will remain 0. This is because co-loops have no parallel elements. In the simple case, the bottom $(n-1) \times (n-1)$ sub-matrix of Hess_{f_M} will be entirely 0. This means that the Hodge-Riemann form will singular and f_M cannot satisfy HRR_1 on $\text{relint}(H_e)$. This suffices for the proof. \square

With a similar type of proof, you can prove the following result.

Theorem 6.4.4. *Let $M = (E, \mathcal{I})$ be a matroid which satisfies $\text{rank}(M) \geq 2$. For any $S \subseteq E$, if $\text{rank}(S) \leq \text{rank}(M) - 2$ and S contains no co-loops, then f_M satisfies HRR_1 on $\text{relint}(H_S)$.*

Theorem 6.4.5. *Let $M = (E, \mathcal{I})$ be a matroid and let $S \subseteq E$ be a subset such that $\text{rank}(S) \leq \text{rank}(M) - k - 1$. Let $l := \sum_{i \in E \setminus S} a_i \cdot e_i$ where $a_i > 0$. If $\partial_e f_M$ is 0 or satisfies HRR_k and with respect to l for all $e \in E \setminus S$, then $A(M)$ satisfies HL_k with respect to l .*

Proof. Note that $\partial_e f = 0$ if and only if e is a loop. In this case, the variable x_e doesn't appear in the basis generating polynomial. Hence, without loss of generality, we can suppose that M is loopless and $\partial_e f \neq 0$ for all $e \in E$. Since $A(M)$ is a Poincare duality algebra, it suffices to prove for $\Xi \in A^k(M)$ that if $\Xi l^{d-2k} = 0$ in $A^{r-k}(M)$, then $\Xi = 0$ in $A^k(M)$. We know that Ξ is in the primitive subspaces of $\partial_e f$ for all $e \in E$ and we can compute the formula

$$0 = Q^k(\Xi, \Xi) = \sum_{i \in E \setminus S} a_i Q_{\partial_i f}(\Xi, \Xi).$$

From the positive definiteness on the primitive subspaces, we have $\Xi = 0$ in $A_{\partial_i f}^k$ for all $i \in E \setminus S$. From Theorem 6.4.2, we get $\Xi = 0$ in $A^k(M)$. This suffices for the proof. \square

Conjecture 6.4.1. *For any matroid M , $A(M)$ has HL_k for all k with respect to some linear form l .*

Definition 6.4.1. Let $B : V \times V \rightarrow k$ be a symmetric bilinear form on a finite dimensional vector space V . Suppose that the signature of B has n_+ positive eigenvalues and n_- negative eigenvalues. Then, we define the **net signature** to be $\sigma(B) = n_+ - n_-$.

When our bilinear form $B : V \times V \rightarrow k$ is non-degenerate, then the net signature $\sigma(B)$ determines the exact signature of the form. Indeed, in the non-degenerate case, we have $n_+ - n_- = \sigma(B)$ and $n_+ + n_- = \dim V$.

Lemma 6.4.2. Let $A(M)$ satisfies HRR_i and HL_i with respect to l for $1 \leq i \leq k$, then, we have

$$\sigma((-1)^k Q_l^k) = \sum_{i=0}^k (-1)^i (\dim A^i(M) - \dim A^{i-1}(M)).$$

Proof. We induct on k . For the base case, we have $k = 1$ and the claim follows from Proposition 6.1.3. Suppose that the claim holds for $k - 1$. Consider the composition of maps given by the following commutative diagram:

$$\begin{array}{ccccccc} A^{k-1}(M) & \xrightarrow{\times l_a} & A^k(M) & \xrightarrow{\times l_a^{d-2k}} & A^{d-k}(M) & \xrightarrow{\times l_a} & A^{d-k+1}(M) \\ & & & \searrow \psi_a & & & \nearrow \end{array} \quad (6.6)$$

This diagram exhibits an isomorphism between $A^{k-1}(M)$ and $A^{d-k+1}(M)$ from HL_{k-1} . This implies that we can decompose $A^k(M)$ into

$$A^k(M) = l \cdot A^{k-1}(M) \oplus \ker \psi_a$$

where the two summands in the direct sum are orthogonal with respect to the Hodge-Riemann form Q_l^k . Let $u_1, \dots, u_m \in A^{k-1}(M)$ be a basis for $A^{k-1}(M)$. Then, $l \cdot u_1, \dots, l \cdot u_m$ is a basis for $A^k(M)$ and

$$(-1)^k Q_l^k(l \cdot u_i, l \cdot u_j) = (-1)^{k-1} Q_l^{k-1}(u_i, u_j).$$

Thus, the signature of Q_l^k on $l \cdot A^{k-1}(M)$ should be the negative of the signature of Q_l^{k-1} on $A^{k-1}(M)$. Since $A(M)$ satisfies HRR_k , we know that the signature of Q_l^k on $\ker \psi$ is $\dim \ker \psi$. This

gives us the formula

$$\begin{aligned}\sigma((-1)^k Q_{l_a}^k) &= \sigma((-1)^{k-1} Q_{l_a}^{k-1}) + (-1)^k (\dim A^k(M) - \dim A^{k-1}(M)) \\ &= \sum_{i=0}^k (-1)^i (\dim A^i(M) - \dim A^{i-1}(M))\end{aligned}$$

from the inductive hypothesis. This suffices for the proof. \square

Lemma 6.4.3. *Let $k \geq 2$ and $A(M)$ satisfies HRR_i and HL_i with respect to $l = l_a$ for some $a \in \mathbb{R}^n$ for all $1 \leq i \leq k-1$. Then*

(a) *$A(M)$ satisfies HL_k with respect to l if and only if Q_l^k is non-degenerate on $A^k(M)$.*

(b) *Suppose that HL_k is satisfied. Then $A(M)$ satisfies HRR_k with respect to l if and only if*

$$\sigma((-1)^k Q_l^k) = \sum_{i=0}^k (-1)^i (\dim A^i(M) - \dim A^{i-1}(M)).$$

Proof. The argument for (a) is exactly the same as the argument for (a) in Proposition 6.1.3. For part (b), consider the composition in Equation 6.6. Recall from the proof of Lemma 6.4.2, we have that

$$\sigma((-1)^k Q_{l_a}^k) = \sigma((-1)^{k-1} Q_{l_a}^{k-1}) + \sigma((-1)^k Q_{l_a}^k|_{\ker \psi_a}).$$

Since $A(M)$ satisfies HRR_k if and only if $Q_{l_a}^k$ is positive definition on $\ker \psi_a$, we know that $A(M)$ satisfies HRR_k if and only if $\sigma(Q_{l_a}^k) = \dim \ker \psi_a = \dim A^k(M) - \dim A^{k-1}(M)$. Then Lemma 6.4.2 completes the proof. \square

Lemma 6.4.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a subset such that for all $x \in \Omega$, $A(M)$ satisfies HRR_i for $i \leq k-1$ and HL_i for $i \leq k$. Suppose that there is a continuous path $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ with image $\varphi([0, 1]) \subseteq \Omega$, such that $A(M)$ satisfies HRR_k with respect to $l_{\gamma(0)}$. Then $A(M)$ satisfies HRR_k with respect to $l_{\gamma(1)}$.*

Proof. Let $\lambda_i(a)$ be the i th largest eigenvalue of $Q_{l_a}^k$ as a function in a for $1 \leq i \leq \dim A^k(M)$. Then for all $1 \leq i \leq \dim A^k(M)$, the i th eigenvalue $\lambda_i(a)$ is a continuous function in a . Along the path γ , we know that $Q_{l_a}^k$ is non-degenerate. Hence, none of the functions $\lambda_i(a)$ cross zero. This implies that on the path, the signature $\sigma((-1)^k Q_{l_a}^k)$ remains constant. From Lemma 6.4.3(b), this completes the proof. \square

In the case when $a = (1, \dots, 1)$, we have

$$Q^k(\xi, \xi) = \sum_{\substack{\alpha_1, \alpha_2 \in \mathcal{I} \\ \alpha_1 \cup \alpha_2 \in \mathcal{L}_{2k}}} \text{Ext}(\alpha_1 \cup \alpha_2) \cdot c_{\alpha_1} c_{\alpha_2}.$$

6.4.1 Graded Mobius Algebra

Definition 6.4.2 (taken word for word in [2]). For any non-negative integer k , we define a vector space

$$H^k(M) := \bigoplus_{F \in \mathcal{L}^k(M)} \mathbb{Q} y_F,$$

where the direct sum is over the set $\mathcal{L}^k(M)$ of rank k flats of M . The **graded Mobius algebra** of M is the graded vector space

$$H(M) := \bigoplus_{k \geq 0} H^k(M).$$

The multiplication in $H(M)$ is defined by the rule

$$y_{F_1} y_{F_2} = \begin{cases} y_{F_1 \vee F_2} & \text{if } \text{rank}_M(F_1) + \text{rank}_M(F_2) = \text{rank}_M(F_1 \vee F_2) \\ 0 & \text{if } \text{rank}_M(F_1) + \text{rank}_M(F_2) > \text{rank}_M(F_1 \vee F_2). \end{cases}$$

Chapter 7

Appendix

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