Log-concavity in Combinatorics

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today

Abstract

(write something flowery)

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I would like to thank

Declaration

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Chapter 1

Conventions and Notation

Chapter 2

Introduction

Chapter 3

Combinatorial Structures

3.1 Partially Ordered Sets

3.2 Matroids

Our main reference for matroids are (cite)

Definition 3.2.1. A **matroid** is an ordered pair $M = (E, \mathcal{I})$ consisting of a finite set E and a collection of subsets $\mathcal{I} \subseteq 2^E$ which satisfy the following three properties:

(**I1**) $\emptyset \in \mathcal{I}$.

(**I2**) If $X \subseteq Y$ and $Y \in \mathcal{I}$, then $X \in \mathcal{I}$.

(**I3**) If $X, Y \in \mathcal{I}$ and $|X| > |Y|$, then there exists some element $e \in X \backslash Y$ such that $Y \cup \{e\} \in \mathcal{I}$.

The set E is called the ground set of the matroid and the collection of subsets $\mathcal{I}$ are called independent sets.

Given a matroid $M = (E, \mathcal{I})$, we call a subset $X \subseteq E$ a **dependent set** if and only if $X \notin \mathcal{I}$. We define a basis $B \in \mathcal{I}$ to be a maximal independent set.

Proposition 3.2.1. *Let $M = (E, \mathcal{I})$ be a matroid and B_1, B_2 be bases. Then $|B_1| = |B_2|$.*

Proof. If $|B_1| < |B_2|$, then by (I3), there exists some element $e \in B_2 \backslash B_1$ satisfying $B_1 \cup \{e\} \in \mathcal{I}$. But $|B_1 \cup \{e\}| = |B_1| + 1 > |B_1|$ which contradicts the maximality of B_1. □

(DEFINE WHAT A SIMPLE MATROID IS)

3.3 Convex Bodies

In this section, we review the notions of convexity and convex bodies. Our main reference for the theory of convex bodies is (cite).

Definition 3.3.1. A **convex body** is a compact, convex subset of $\mathbb{R}^n$.

3.3.1 Mixed Volumes

Chapter 4

Mechanisms for Log-concavity

4.1 Alexandrov's Inequality for Mixed Discriminants

4.2 Alexandrov-Fenchel Inequality

4.3 Lorentzian Polynomials

4.4 Combinatorial Atlas

Chapter 5

Log-concavity Results for Posets and Matroids

5.1 Stanley's Inequality

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Chapter 6

Hard Lefschetz and Hodge-Riemann Relations

6.1 Chow Ring of the Basis Generating Polynomial (???)

In this section, we study a cohomology ring studied in (CITE Murai). For every homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ we can associate a graded ring which is a proxy for cohomology (measuring socles??? Think about this later)

Definition 6.1.1. Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a homomgeneous polynoimal and let $S := \mathbb{R}[\partial_1, \ldots, \partial_n]$ be the polynomial ring of differentials where $\partial_i := \partial_{x_i}$. We define the ring

$$A_f^\bullet := S/\operatorname{Ann}_S(f).$$

which we call the Chow (???) ring.

We first prove that the Chow ring is a graded $\mathbb{R}$-algebra. To do this, we first prove Lemma (REFERENCE)

Lemma 6.1.1. *Let $\xi \in \mathbb{R}[\partial_1, \ldots, \partial_n]$ and $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial. We can decompose $\xi = \xi_0 + \xi_1 + \ldots$ into its homogeneous parts. If $\xi(f) = 0$, then $\xi_d(f) = 0$ for all $d \geq 0$.*

Proof. Let $d = \deg(f)$. If $\xi_i(f) \neq 0$, then $i \leq d$ and $\xi_i(f)$ is a homomgeneous polynomial of degree $d - i$. Thus,

$$\xi(f) = \xi_0(f) + \xi_1(f) + \ldots$$

will be the homomgeneous decomposition of the polynomial $\xi(f)$. Since this is equal to 0, all components of the decomposition are equal to zero. This proves the proposition. $\square$

Proposition 6.1.1. *The Chow ring $A_f^\bullet$ is a graded $\mathbb{R}$-algebra where A_f^k consists of the forms of degree k.*

Proof. Let us define A_f^k as in the statement of the lemma. Let $d = \deg(f)$ be the degree of the homogeneous polynomial. Whenver $k > d$, the ring A_f^k is clearly trivial. From Lemma 6.1.1, we have the direct sum decomposition

$$A_f^\bullet = \bigoplus_{k=0}^{d} A_f^k.$$

It is also clear that multiplication induces maps $A_f^r \times A_f^s \to A_f^{r+s}$ for all $r, s \geq 0$. $\square$

The following result (Proposition (CITE)) is well-known and it follows from Theorem 2.1 in CITE(Maeno, Watanabe, Lefschetz elements of ARtinian Gorenstein algebras and Hessians of homogeneous polynomials)

Proposition 6.1.2. *Let f be a homogeneous polynoimal of degree d. Then, the ring $A_f^\bullet$ is a Poincare-Duality algebra. That is, the ring satisfies the following two properties:*

(a) $A_f^d \simeq A_f^0 \simeq \mathbb{R}$*;*

(b) The pairing induced by multiplication $A_f^{d-k} \times A_f^k \to A_f^d \simeq \mathbb{R}$ *is non-degenerate for all* $0 \leq k \leq d$*.*

We first make a small remark on non-degeneracy to make it clear what we mean when we say a pairing is non-degenerate.

Lemma 6.1.2. *Let $B : V \times W \to k$ be a bilinear pairing between two finite-dimensional k-vector spaces V and W. Then, any two of the following three conditions imply the third.*

(i) The map $B_V : V \to W^*$ defined by $v \mapsto B(v, \cdot)$ has trivial kernel.

(ii) The map $B_W : W \to V^*$ defined by $w \mapsto B(\cdot, w)$ has trivial kernel.

(iii) $\dim V = \dim W$.

Proof. Condition (i) implies $\dim V \leq \dim W$ and Condition (ii) implies $\dim W \leq \dim V$. Thus (i) and (ii) both imply (iii). Now, suppose that (i) and (iii) are true. Then B_V is an isomorphism between V and $W*$ [3.69 in Axler (CITE???)]. Let $v_1, \ldots, v_n$ be a basis for V. Then

$B_V(v_1), \ldots, B_V(v_n)$ is a basis of W^*. Let $w_1, \ldots, w_n$ be the dual basis in W with respect to this basis of W^*. Suppose that $\sum \lambda_i w_i \in \ker B_W$. Then for all $v \in V$, we have

$$\sum_{i=1}^{n} \lambda_i B_V(v)(w) = B\left(v, \sum_{i=1}^{n} \lambda_i w_i\right) = 0.$$

By letting $v = v_1, \ldots, v_n$, we get $\lambda_i = 0$ for all i. $\square$

We say a pairing between finite-dimensional vector spaces is non-degenerate whenever all three conditions in Lemma 6.1.2 hold. As a corollary, we have the following result.

Corollary 6.1.1. *Let f be a homogeneous polynomial of degree $d \geq 2$ and let k, $0 \leq k \leq d$, by a non-negative integer. Then* $\dim_{\mathbb{R}} A_f^k = \dim_{\mathbb{R}} A_f^{d-k}$.

In Proposition(CITE), we can define $\deg_f : A_f^d \to \mathbb{R}$ to be the isomorphism defined by evaluation at f. That is, for any $\xi \in A_f^d$, we can define $\deg_f(\xi) := \xi(f)$ where ξ acts on f by differentiation. FOllowing (CITE HODGE THEORY OF COMBINATORIAL GEOMETRIES), we formulate the following definition

Definition 6.1.2. Let f be a homogeneous polynomial of degree d and let $k \leq d/2$ be a non-negative integer. For an element $l \in A_f^1$, we define the following notions:

(a) The **Lefschetz operator** on A_f^k associated to l is the map $L_l^k : A_f^k \to A_f^{d-k}$ defined by $\xi \mapsto l^{d-2k} \cdot \xi$.

(b) The **Hodge-Riemann form** on A_f^k associated to l is the bilinear form $Q_l^k : A_f^k \times A_f^k \to \mathbb{R}$ defined by $Q_l^k(\xi_1, \xi_2) = (-1)^k \deg(\xi_1 \xi_2 l^{d-2k})$.

(c) The **primitive subspace** of A_f^k associated to l is the subspace

$$P_l^k := \{\xi \in A_f^k : l^{d-2k+1} \cdot \xi = 0\} \subseteq A_f^k.$$

Definition 6.1.3. Let f be a homogeneous polynomial of degree d, let $k \leq d/2$ be a non-negative integer, and let $l \in A_f^1$ be a linear differential form. We define the following notions:

(a) (Hard Lefschetz Property) We say A_f satisfies HL_k with respect to l if the Lefschetz operator L_l^k is an isomorphism.

(b) (Hodge-Riemann Relations) We say A_f satisfies HRR_k with respect to l if the Hodge-Riemann form Q_l^k is positive definite on the primitive subspace P_l^k.

Sometimes, instead of saying A_f satisfies Hodge-Riemann Relations or the Hard Lefschetz Property, we will say that f satisfies Hodge-Riemann Relations or the Hard Lefschetz property. For any $a \in \mathbb{R}^n$, we can define the linear differential form $l_a := a_1\partial_1 + \ldots + a_n\partial_n$. We say that f satisfies HL or HRR with respect to a if and only if it satisfies HL or HRR with respect to l_a. Most applications of the Hodge-Riemann Relations only end up using the relations up to degree $k \leq 1$. (GIVE EXAMPLE OF AN APPLICATION WHICH USES HIGHER DIMENSION,, maybe Chris Eur???)

Proposition 6.1.3 (Lemma 3.4 in (CITE MURAI))**.** *Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d \geq 2$ and $a \in \mathbb{R}^n$. Assume that $f(a) > 0$. Then,*

(a) A_f has HL_1 with respect to l_a if and only $Q^1_{l_a}$ is non-degenerate.

(b) A_f has HRR_1 with respect to l_a if and only if $-Q^1_{l_a}$ has signature $(+, -, \ldots, -)$.

Proof. We include a proof for the sake of completeness. We first prove the statement in (a). Suppose that A_f has HL_1 with respect to l_a. We have the following commutative diagram:

$$\begin{tikzcd} A^1_f \times A^1_f \arrow[rr, "\mathrm{id}\times L^1_{l_a}"] \arrow[dr, "-Q^1_{l_a}"'] & & A^1_f \times A^{d-1}_f \arrow[dl] \\ & \mathbb{R} & \end{tikzcd}$$

where the missing mapping is multiplication. If A_f has HL_1 with respect to l_a, then the top map between $A^1_f \times A^1_f \to A^1_f \times A^{d-1}_f$ is an bijection. Thus the non-degeneracy of $Q^1_{l_a}$ follows from the non-degeneracy of the multiplication pairing as stated in Proposition 6.1.2. Now, suppose that $Q^1_{l_a}$ is non-degenerate. Then, the map $B : A^1_f \to (A^1_f)^*$ defined by $\xi \mapsto -Q^1_{l_a}(\xi, \cdot)$ is given by $m(L^1_{l_a}\xi, \cdot)$ where $m : A^1_f \to A^{d-1}_f \to \mathbb{R}$ is the multiplication map. This is the composition of $A^1_f \to A^{d-1}_f \to (A^1_f)_*$ where the first map is $L^1_{l_a}$ and the second map is injective from the non-degeneracy of the multiplication map. This proves that $L^1_{l_a}$ is injective. From Corollary 6.1.1, the map $L^1_{l_a}$ is an isomorphism. This suffices for the proof of (a).

To prove (b)

$\square$

6.1.1 Chow Ring of the Basis Generating Polynomial

Definition 6.1.4. Let M be a matroid. We define $A^\bullet(M) := A^\bullet_{f_M}$ to be the Chow ring of the basis generating polynomial of M.

For simple matroids, the vector space structure of $A^1(M)$ is simple.

Lemma 6.1.3 (Theorem 2.5 in (CITE MURAI))**.** *If $M = ([n], \mathcal{I})$ is simple, then $\partial_1, \ldots, \partial_n$ is a basis of $A^1(M)$.*

We are interested on the domain in which $A(M)$ satisfies HRR_k. From cite (MURAI), the following is known.

Theorem 6.1.1 (Theorem 3.8 in (MURAI))**.** *If $f \in \mathbb{R}[x_1, \ldots, x_n]$ is Lorentzian, then f has HRR_1 with respect to l_a for any $a \in \mathbb{R}^n_{>0}$.*

In particular, since f_M is Lorentzian (CITE PREVIOUS PART OF THESIS), the Chow ring $A(M)$ satisfies HRR_1 on the positive orthant. We can be more precise if A^1_f has $\partial_1, \ldots, \partial_n$ as a basis.

Proposition 6.1.4. *Let f be a homogeneous polynomial of degree $d \geq 2$. If $\partial_1 f, \ldots, \partial_n f$ are linearly independent in $\mathbb{R}[x_1, \ldots, x_n]$ and $f(a) > 0$ for some $a \in \mathbb{R}^n$, then A_f satisfies HRR_1 with respect to l_a if and only if $\mathrm{Hess}_f|_{x=a}$ has signature $(+, -, \ldots, -)$.*

Let $M = (E, \mathcal{I})$ be a matroid and let $\widetilde{M}$ be its simplification. Recall that $\widetilde{M}$ is a matroid on the ground set of rank-1 flats $E(\widetilde{M}) = \{\bar{x} : x \in E(M) \backslash E_0(M)\}$ whose independent sets consist of the subsets of $E(\widetilde{M})$ where by taking one representative from each rank-1 flat, we get an independent set of M. We can define $\phi : S_M \to S_{\widetilde{M}}$ and $\psi : S_{\widetilde{M}} \to S_M$ defined by

$$\begin{aligned} \phi(\partial_{x_e}) &:= \partial_{\overline{x_e}} \\ \psi(\partial_{\overline{x}}) &:= \frac{1}{|\overline{x}|} \sum_{e \in \overline{x}} \partial_{x_e} \end{aligned}$$

and extending linearly to the whole space.

Theorem 6.1.2. *The maps $\phi : S_M \to S_{\widetilde{M}}$ and $\psi : S_{\widetilde{M}} \to S_M$ induce isomorphisms between $A(M)$ and $A(\widetilde{M})$.*

Proof. We first prove that ϕ and ψ induce homomorphisms between the Chow rings. To show that ϕ induces a homomorphism, consider the diagram

$$\begin{tikzcd} S_M \arrow[r, "\phi"] \arrow[dr, "\pi_M"'] & S_{\widetilde{M}} \arrow[r, "\pi_{\widetilde{M}}"] & A(\widetilde{M}) \\ & A(M) \arrow[ur, dashed, "\exists ! \Phi"'] & \end{tikzcd}$$

Let $\xi \in S_M$ be an element satisfying $\xi(f_M) = 0$. From Proposition 6.1.1, it suffices to consider the case where ξ is homomgeneous. Let $e_1, \ldots, e_s$ be representatives of all parallel classes. Then, we have that

$$f_{\widetilde{M}}(x_{\overline{e_1}}, \ldots, x_{\overline{e_s}}) = \sum_{\substack{1 \le i_1 < \ldots < i_r \le s \\ \{\overline{e_{i_1}}, \ldots, \overline{e_{i_r}}\} \in \mathcal{B}(\widetilde{M})}} x_{\overline{e_{i_1}}} \ldots x_{\overline{e_{i_r}}}$$

$$f_M(x_1, \ldots, x_n) = f_{\widetilde{M}}(y_1, \ldots, y_s)$$

where for $1 \le i \le s$, we define

$$y_i := \sum_{e \in \overline{e_i}} x_e.$$

Suppose that ξ is homomgeneous of degree k. Then, we can write

$$\xi = \sum_{\substack{\alpha \subseteq [n] \\ |\alpha| = k}} c_\alpha \partial^\alpha$$

Then, we have

$$\begin{aligned} \xi(f_M) &= \sum_{\beta \in \mathcal{B}} \xi(x^\beta) \\ &= \sum_{\beta \in \mathcal{B}(M)} \sum_{\substack{\alpha \subseteq [n] \\ |\alpha| = k}} c_\alpha \partial^\alpha x^\beta \\ &= \sum_{\gamma \in \mathcal{I}_{r-k}(M)} \left(\sum_{\substack{\alpha \in \mathcal{I}_k \\ \alpha \cup \gamma \in \mathcal{I}_r(M)}} c_\alpha \right) x^\gamma. \end{aligned}$$

Thus, for any $\gamma \in \mathcal{I}_{r-k}(M)$, we have

$$\sum_{\substack{\alpha \in I_k \\ \alpha \cup \gamma \in \mathcal{I}_r(M)}} c_\alpha = 0.$$

On the other hand, we have

$$\phi(\xi) = \sum_{\substack{\alpha \subseteq [n] \\ |\alpha| = k}} c_\alpha \prod_{e \in \alpha} \partial_{\overline{e}} = \sum_{\beta \in \mathcal{I}_k(\widetilde{M})} \left(\sum_{\alpha \in \text{fiber}(\beta)} c_\alpha \right) \partial^\beta.$$

We can compute

$$\phi(\xi)(f_{\widetilde{M}}) = \sum_{\gamma \in \mathcal{I}_{r-k}(\widetilde{M})} \left(\sum_{\substack{\beta \in \mathcal{I}_k(\widetilde{M}) \\ \beta \cup \gamma \in \mathcal{B}(\widetilde{M})}} \sum_{\alpha \in \mathrm{fiber}(\beta)} c_\alpha \right) x^\gamma = \sum_{\gamma \in \mathcal{I}_{r-k}(\widetilde{M})} \left(\sum_{\substack{\alpha \in \mathcal{I}_k(M) \\ \alpha \cup \gamma_0 \in \mathcal{B}(M)}} c_\alpha \right) x^\gamma = 0.$$

where $\gamma_0 \in \mathrm{fiber}(\gamma)$ is an arbitrary element in the fiber of γ. This proves that $\mathrm{Ann}_M \subseteq \ker \pi_{\widetilde{M}} \circ \phi$. Thus, there is a unique ring homomorphism $\Phi : A(M) \to A(\widetilde{M})$ which makes diagram (REFERENCE) commute.

To prove that ψ induces a map between the Chow rings, consider the diagram

$$\begin{tikzcd} S_{\widetilde{M}} \arrow[r, "\psi"] \arrow[dr, "\pi_{\widetilde{M}}"'] & S_M \arrow[r, "\pi_M"'] & A(M) \\ & A(\widetilde{M}) \arrow[ur, dotted, "\exists! \Psi"'] & \end{tikzcd}$$

Consider a differential $\xi \in S_{\widetilde{M}}$ satisfying $\xi(f_{\widetilde{M}}) = 0$. We can write

$$\xi = \sum_{\alpha \in \mathcal{I}_k(\widetilde{M})} c_\alpha \partial^\alpha.$$

Then

$$\psi(\xi) = \sum_{\alpha \in \mathcal{I}_k(\widetilde{M})} \frac{c_\alpha}{\prod_{e \in \alpha} |e|} \sum_{\beta \in \mathrm{fiber}(\alpha)} \partial^\beta.$$

Fix a $\alpha \in \mathcal{I}_k(\widetilde{M})$ and a $\beta \in \mathrm{fiber}(\alpha)$. Since $\partial y_i / \partial x_e = \mathbb{1}_{e \in \overline{e_i}}$, we have

$$\begin{aligned} \partial^\beta f_M(x_1, \ldots, x_n) &= \partial^\beta f_{\widetilde{M}}(y_1, \ldots, y_s) \\ &= \partial^\alpha f_{\widetilde{M}}(x_1, \ldots, x_s)|_{x_1 = y_1, \ldots, x_s = y_s} \\ &= 0. \end{aligned}$$

Thus, we have that $\psi(\xi)(f_M) = 0$. Hence, $\mathrm{Ann}_{\widetilde{M}} \subseteq \ker \pi_M \circ \psi$ and there is a unique ring homomorphism $\Psi : A(\widetilde{M}) \to A(M)$ which causes the diagram (REFERENCE) to commute. It is easy to check that Ψ and Φ are inverses of each other. Thus, they are both isomorphisms between the rings $A(M)$ and $A(\widetilde{M})$. $\square$

From Theorem (CITE), we get Corollary (CITE) and (CITE) immediately.

Corollary 6.1.2. *Let $M = (E, \mathcal{I})$ be a matroid. For any $a \in \mathbb{R}^E$, we can define the linear form $l_a := \sum_{e \in E} a_e \cdot \partial_{x_e} \in A^1(M)$. Let $\widetilde{l_a} := \Phi(l_a) = \sum_{e \in E} a_e \cdot \partial_{x_{\overline{e}}} \in A^1(\widetilde{M})$. Then, the following diagram is an isomorphism of chain complexes:*

$$\begin{array}{ccccccccc} \dots & \xrightarrow{\times l_a} & A^{i-1}(M) & \xrightarrow{\times l_a} & A^i(M) & \xrightarrow{\times l_a} & A^{i+1}(M) & \xrightarrow{\times l_a} & \dots \\ & & \Phi\big\downarrow & & \Phi\big\downarrow & & \Phi\big\downarrow & & \\ \dots & \xrightarrow{\times \widetilde{l_a}} & A^{i-1}(\widetilde{M}) & \xrightarrow{\times \widetilde{l_a}} & A^i(\widetilde{M}) & \xrightarrow{\times \widetilde{l_a}} & A^{i+1}(\widetilde{M}) & \xrightarrow{\times \widetilde{l_a}} & \dots \end{array}$$

Corollary 6.1.3. *Let $M = (E, \mathcal{I})$ be a matroid. Then $A(M)$ satisfies HRR_k with respect to l if and only if $A(\widetilde{M})$ satisfies HRR_k with respect to $\Phi(l)$.*

6.2 Hodge Riemann Relations on the Boundary

Theorem 6.2.1. *Let $M = (E, \mathcal{I})$ be a matroid which satisfies $\operatorname{rank}(M) \geq 2$. For any $e \in E(M)$, the basis generating polynomial f_M satisfies HRR_1 on $\operatorname{relint}(H_e)$ if and only if e is not a co-loop.*

Proof. Without loss of generality, let M be a matroid on the set $[n]$ and let $e = 1$. Then, we want to prove that whenever $a_2, \dots, a_n > 0$, the ring $A(M)$ satisfies HRR_1 on $(0, a_2, \dots, a_n)$ if and only if e is not a co-loop. We first prove that if 1 is not a co-loop, then $A(M)$ satisfies HRR_1. To prove this, we induct on the rank of M. For the base case $\operatorname{rank}(M) = 2$, $\square$

Chapter 7

Appendix

Chapter 8

References