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An analogue of the Hodge–Riemann relations for simple convex polytopes

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Introduction

A convex polytope in the space $\mathbb{R}^d$ is said to be *simple* if each of its vertices is incident to exactly d edges. Many problems in the geometry and combinatorics of simple polytopes have proved to be closely related to algebraic geometry and, in particular, to the theory of toric varieties [1]–[4].

Let Δ be an integral polytope in $\mathbb{R}^d$. By saying that Δ is *integral* we mean that all vertices of Δ belong to the integral lattice $\mathbb{Z}^d$. Associated with Δ is a complex algebraic variety $X(\Delta)$ on which the *complex torus* $(\mathbb{C}\setminus\{0\})^n$ acts. The variety $X(\Delta)$ contains exactly one dense orbit. The boundary of this orbit is a union of lower-dimensional orbits.

An algebraic variety equipped with an algebraic action of the complex torus satisfying the above conditions is called a *toric variety* (or a *toric embedding*). Thus, for each integral polytope Δ there is an associated toric variety $X(\Delta)$. In particular, the compact torus $T^n \subseteq (\mathbb{C}\setminus\{0\})$ given by the equations $|z_j| = 1$ acts on $X(\Delta)$. The quotient space of $X(\Delta)$ with respect to the compact torus action can be naturally identified with the polytope Δ itself.

For simplicity, we consider only the case in which the variety $X(\Delta)$ is non-singular. In this case, $X(\Delta)$ can be viewed as a complex-analytic subvariety of $\mathbb{CP}^N$. The condition that the toric variety be non-singular is equivalent to the following geometric condition on the polytope. A polytope Δ is said to be *integrally simple* if for each vertex v the primitive integral vectors of the edges incident to v form a basis of the integral lattice. The variety $X(\Delta)$ is non-singular if and only if the polytope Δ is integrally simple.

Suppose that Δ is integrally simple, and hence the variety $X(\Delta)$ is smooth. We denote by

$$h_k = \dim H^{2k}(X(\Delta), \mathbb{C})$$

the even-dimensional Betti numbers of $X(\Delta)$. All odd-dimensional Betti numbers are zero. It turns out that the numbers h_k are important combinatorial characteristics of Δ. Let f_i be the number of i-faces of Δ. Then

$$h_k = \sum_{i \geqslant k} f_i (-1)^{i-k} \binom{i}{k}.$$

The variety $X(\Delta)$ is compact. Consequently,

$$h_k = h_{d-k}$$

(Poincaré duality). These relations had been known long before the relationship between the geometry of polytopes and algebraic geometry was discovered. For $d = 4$ and $d = 5$ they were written out by Dehn in 1905. All relations for arbitrary d were found in 1927 by Sommerville.

Along with the Dehn–Sommerville equations, there are some inequalities between the numbers h_k. For example, by the hard Lefschetz theorem applied to the toric variety $X(\Delta)$, the numbers h_k form a 'hill', that is, they increase up to the middle and then decrease:

$$h_k \leqslant h_{k+1}, \qquad k < \frac{d+1}{2}.$$

There are also some other inequalities related to the fact that the numbers $h_{k+1}-h_k$, $k < n/2$, are the dimensions of homogeneous components of a certain commutative graded algebra generated by elements of the first degree. This algebra is obtained as the quotient of the cohomology ring of the toric variety by a certain principal ideal (see [5] for details).

In 1971, McMullen put forth a conjecture concerning necessary and sufficient conditions on the numbers h_k for a simple convex polytope. In 1980 Stanley proved that McMullen's conditions are necessary, and in 1981 Billera and Lee proved sufficiency. Stanley's proof is based on the theory of toric varieties, whereas the proof of sufficiency by Billera and Lee is purely geometric. In 1993, McMullen suggested a geometric proof of Stanley's theorem. The proof is technically complicated. One of the problems is to describe an analogue of the cohomology ring of a toric variety in geometric terms.

The problem of giving a geometric proof of Stanley's theorem is related to the problem of describing the Hodge–Riemann relations in the cohomology ring of a toric variety in geometric language. A special case of the Hodge–Riemann relations yields the well-known Aleksandrov–Fenchel inequalities for mixed volumes when expressed in geometric language.

By definition, a *convex body* in the space $\mathbb{R}^d$ is a compact convex set. Let A and B be convex bodies. The *Minkowski sum* of A and B is the set

$$C = \{a + b \mid a \in A,\ b \in B\}.$$

One can readily verify that C is also a convex body. Convex bodies can be multiplied by positive numbers. Multiplication by negative numbers is not allowed. Thus, the set of convex bodies is equipped with the structure of a convex cone: bodies can be added and multiplied by positive numbers. This cone can be extended to a vector space by the standard Grothendieck group construction. The elements of this vector space will be called *virtual convex bodies*.

It turns out that the function that takes each convex body to its volume extends to a homogeneous polynomial of degree d on the space of virtual convex bodies. However, for each homogeneous polynomial there is an associated symmetric multilinear form, the *polarization* of the polynomial. The polarization of the volume polynomial is called the *mixed volume*. Thus, the mixed volume $\operatorname{Vol}(A_1, A_2, \dots, A_d)$ is defined for each d-tuple $A_1, A_2, \dots, A_d$ of virtual convex bodies.

In 1937 Aleksandrov proved the following remarkable inequalities relating the mixed volumes of convex bodies:

$$\operatorname{Vol}(A_1, A_2, A_3, \dots, A_d)^2 \geqslant \operatorname{Vol}(A_1, A_1, A_3, \dots, A_d) \cdot \operatorname{Vol}(A_2, A_2, A_3, \dots, A_d)$$

(the Aleksandrov–Fenchel inequalities).

If we fix the convex bodies $A_3, \dots, A_d$, then the mixed volume

$$\mathrm{Vol}(A_1, A_2, A_3, \dots, A_d)$$

becomes a quadratic form with respect to the virtual convex bodies A_1 and A_2. The Aleksandrov–Fenchel inequalities say that for actual convex bodies this quadratic form satisfies the reversed Cauchy–Schwarz–Bunyakovskii inequality. Needless to say, this is because the positive index of inertia of the quadratic form is equal to 1. The negative index of inertia is infinite: the form is defined on an infinite-dimensional space and has a d-dimensional kernel. The kernel of the mixed volume form is formed by all points of $\mathbb{R}^d$, regarded as convex bodies.

The convex polytopes analogous to a given simple convex polytope form a very important finite-dimensional subspace of the infinite-dimensional space of virtual convex bodies. Let us give an inductive definition of analogous polytopes. Two polytopes are said to be *analogous* if there is a one-to-one correspondence between their facets such that corresponding facets are parallel, have the same orientation, and are analogous. The orientation of a facet is determined by the outward normal direction. Any two segments on a line are analogous by definition.

Let us take some simple convex polytope Δ. We note that the Minkowski sum of two convex polytopes analogous to Δ is also a polytope analogous to Δ. The linear span of all polytopes analogous to Δ in the space of virtual convex bodies is a finite-dimensional vector space, which will be denoted by $\mathcal{P}^*(\Delta)$. The dimension of this space is equal to the number of facets of Δ.

We can introduce a natural coordinate system in the space $\mathcal{P}^*(\Delta)$. Every polytope analogous to Δ has the same direction covectors (outward normals) $\xi_1, \dots, \xi_n$ of the facets. Let us consider a polytope P analogous to Δ. The polytope P is uniquely determined by its *support numbers*

$$H_i = \max_{p \in P} (\xi_i, p).$$

Thus, the support numbers specify a coordinate system in the space $\mathcal{P}^*(\Delta)$. We denote these coordinates by $x_1, \dots, x_n$.

The volume polynomial restricted to $\mathcal{P}^*(\Delta)$ can be expressed in the coordinates as $V(x_1, \dots, x_n)$. To each polytope $P \in \mathcal{P}^*(\Delta)$ we assign a linear differential operator with constant coefficients by setting

$$L_P = \sum_{i=1}^{n} H_i \frac{\partial}{\partial x_i},$$

where the H_i are the support numbers of P. The volume of P is expressed by the formula

$$\mathrm{Vol}(P) = \frac{1}{d!} L_P^d V.$$

Accordingly, the mixed volume of a system of polytopes $P_1, \dots, P_d$ is equal to

$$\mathrm{Vol}(P_1, \dots, P_d) = \frac{1}{d!} L_{P_1} \cdots L_{P_d} V.$$

Let us consider a linear differential operator α with constant coefficients on the space $\mathcal{P}^*(\Delta)$. We suppose that this operator is homogeneous of degree k. The operator α is said to be *primitive* with respect to the polytope Δ if

$$\alpha L_\Delta^{d-2k+1} V = 0.$$

The following analogue of the Hodge–Riemann relations is valid for primitive operators:

$$(-1)^k \alpha^2 L_\Delta^{d-2k} V \geqslant 0;$$

moreover, equality is attained only if $\alpha V = 0$. For $k = 1$ we obtain a special case of the Aleksandrov–Fenchel inequalities:

$$\operatorname{Vol}(P_1, P_2, \Delta, \dots, \Delta)^2 \geqslant \operatorname{Vol}(P_1, P_1, \Delta, \dots, \Delta) \cdot \operatorname{Vol}(P_2, P_2, \Delta, \dots, \Delta).$$

These are the classical Brunn–Minkowski inequalities.

There is a mixed analogue of the Hodge–Riemann relations. Let α be a differential operator with constant coefficients on $\mathcal{P}^*(\Delta)$ that is homogeneous of degree k. The operator α is said to be *primitive* with respect to the polytope Δ and a family of polytopes $P_1, \dots, P_{d-2k} \in \mathcal{P}^*(\Delta)$ if

$$\alpha L_\Delta L_{P_1} \cdots L_{P_{d-2k}} V = 0.$$

Primitive operators satisfy the following mixed analogue of the Hodge–Riemann relations:

$$(-1)^k \alpha^2 L_{P_1} \cdots L_{P_{d-2k}} V \geqslant 0,$$

with equality only if $\alpha V = 0$. For $k = 1$, we obtain the Aleksandrov–Fenchel inequalities.

In the present paper, we give a survey of some methods and results of the combinatorial theory of simple convex polytopes. We pay special attention to geometric structures whose introduction was motivated by similar structures in the geometry of toric varieties. In particular, we define an analogue of the cohomology ring of a toric variety. This analogue was constructed by Pukhlikov and Khovanskii [6] in connection with a multidimensional generalization of the Euler–Maclaurin formula. Its geometric definition enables one to state and prove results like the hard Lefschetz theorem and the Hodge–Riemann relations in geometric terms. The analogue of the hard Lefschetz theorem implies Stanley's theorem. The new geometric proof of Stanley's theorem given here is based on McMullen's argument [7] and the description due to Pukhlikov and Khovanskii of the polytope algebra. The proof of the analogue of the Hodge–Riemann relations is based on the homotopy method, which goes back to Aleksandrov [8], who used this method to prove the Aleksandrov–Fenchel inequalities.

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§ 1. Combinatorics of simple polytopes

In this section, we present the main definitions pertaining to the combinatorics of simple convex polytopes. We give the definitions of the f-vector and the h-vector of a simple convex polytope, we state and prove the Dehn–Sommerville equations, and we state Stanley's theorem.

1.1. Simple polytopes. Let Δ be a convex polytope in the space $\mathbb{R}^d$. The polytope Δ is said to be *simple* if in a neighbourhood of each vertex it looks like the cone over a $(d-1)$-simplex (that is, exactly d facets meet at each vertex). Obviously, a polytope whose facets are in general position is simple. In particular, a simple polytope remains simple if we move its faces slightly.

Let Δ be a simple polytope. We denote by f_i the number of i-faces (faces of dimension i) in Δ (the polytope itself is considered to be the d-face, so that $f_d = 1$).

Definition. The sequence of numbers $f_0, \dots, f_d$ is called the *f-vector* of the polytope Δ (the letter 'f' stands for 'face').

Definition. The *F-polynomial* of the polytope Δ is the generating function of the numbers of faces, that is, the polynomial

$$F(t) = \sum_{i=0}^{d} f_i t^i.$$

It is often more convenient to deal with another polynomial, from which the F-polynomial can be uniquely reconstructed:

$$H(t) = F(t-1) = \sum_{k=0}^{d} h_k t^k, \qquad h_k = \sum_{i \geqslant k} f_i (-1)^{i-k} \binom{i}{k}.$$

Definition. The numbers $h_0, \dots, h_d$ form the *h-vector* of the polytope Δ (the letter 'h' stands for 'homology').

It is clear from the above formulae that specifying the h-vector is equivalent to specifying the f-vector. Namely, the f-vector of a simple polytope is expressed via the h-vector by the formula

$$F(t) = H(t+1), \qquad f_i = \sum_{k \geqslant i} h_k \binom{k}{i}.$$

Example. Let Δ be a d-simplex. Then

$$f_i = \binom{d+1}{i+1}, \qquad i \leqslant d.$$

Indeed, each i-face of the simplex contains exactly $i+1$ vertices. On the other hand, the convex hull of any set of $i+1$ vertices of Δ is an i-face.

Thus, the F-polynomial of the simplex is equal to

$$F(t) = (d+1) + \binom{d+1}{2} t + \binom{d+1}{3} t^3 + \dots + t^d = \frac{(t+1)^{d+1} - 1}{t}.$$

We obtain the following expression for the H-polynomial of the simplex:

$$H(t) = F(t-1) = \frac{t^{d+1} - 1}{t-1} = 1 + t + \dots + t^{d-1} + t^d.$$

Hence, all components of the h-vector of the simplex are equal to 1.

Example. Let Δ_1 and Δ_2 be convex m- and $(d-m)$-polytopes, respectively. Let us express the h-vector of the direct product $\Delta = \Delta_1 \times \Delta_2$ via the h-vectors of Δ_1 and Δ_2. We note that each k-face of Δ is the direct product $\Gamma_1 \times \Gamma_2$ of some face Γ_1 of Δ_1 and some face Γ_2 of Δ_2 such that $\dim(\Gamma_1) + \dim(\Gamma_2) = k$. It follows that

$$f_k(\Delta_1 \times \Delta_2) = \sum_{i=0}^{k} f_i(\Delta_1) f_{k-i}(\Delta_2).$$

In other words, the following relation holds for the F-polynomials:

$$F_{\Delta_1 \times \Delta_2}(t) = F_{\Delta_1}(t) F_{\Delta_2}(t).$$

But then just the same relation is valid for the H-polynomials:

$$H_{\Delta_1 \times \Delta_2}(t) = H_{\Delta_1}(t) H_{\Delta_2}(t).$$

Example. Now let us suppose that Δ is a simple d-polytope with $d+2$ facets (that is, Δ has one more facet than a simplex of the same dimension).

Proposition 1.1.1. *Every simple convex d-polytope Δ with $d+2$ facets is combinatorially equivalent to the Cartesian product of two simplices.*

Proof. The polytope Δ is bounded by $d+2$ hyperplanes. We can assume that these hyperplanes are in general position. If we move them slightly, then the combinatorial type of the polytope will remain unchanged. Let us choose $d+1$ out of the $d+2$ hyperplanes. These hyperplanes bound a simplex S. The remaining hyperplane will be denoted by H.

The hyperplane H divides the set of vertices of S into two subsets; namely, $m+1$ vertices lying on one side of H, and $d-m$ on the other.

Let H^+ be the half-space bounded by H and containing Δ. We shall assume that $m+1$ vertices of S lie in H^+. Then the convex hull F of these vertices is a face of Δ. The convex hull of the remaining vertices of S will be denoted by G.

The polytope Δ is fibred into $(d-m)$-simplices of the form

$$\operatorname{conv}(\{x\} \cup G) \cap H^+, \qquad x \in F,$$

($\operatorname{conv}(X)$ denotes the convex hull of a set X). These simplices are parametrized by the point $x \in F$. This fibring specifies the structure of a Cartesian product on Δ. Thus, Δ can be identified with the product of the m-simplex F by some $(d-m)$-simplex. This identification obviously specifies a combinatorial equivalence.

Let S_m be the m-simplex. We have just proved that each simple d-polytope Δ with $d+2$ facets is combinatorially equivalent to some product

$$S_m \times S_{d-m}.$$

Thus, the H-polynomial of Δ is given by the formula

$$H_\Delta(t) = H_{S_m}(t) H_{S_{d-m}}(t) = \frac{(t^{m+1}-1)(t^{d-m+1}-1)}{(1-t)^2}.$$

It follows that, in particular, distinct products of simplices are not combinatorially equivalent (we do not distinguish between products that differ only in the order of the factors).

1.2. The Dehn–Sommerville equations. The following natural question arises: what sequences of integers are f-vectors of simple convex polytopes? We can readily write out some necessary conditions. For example, Euler's theorem states that

$$\sum_{i=0}^{d}(-1)^i f_i = 1.$$

This is true for arbitrary convex polytopes, not necessarily simple. For arbitrary convex polytopes, this theorem exhausts all linear relations on the f-vector. However, for simple polytopes there are also other linear relations. For example, one can readily obtain a relation between the number of edges and the number of vertices: $f_1 = \dfrac{d \cdot f_0}{2}$. Both relations are special cases of the *Dehn–Sommerville equations*, which can be written most conveniently in terms of the h-vectors:

$$h_k = h_{n-k}, \qquad h_0 = h_d = 1.$$

In other words, the h-vector is symmetric about the middle.

We give a proof of the Dehn–Sommerville equations following the paper [9], where it is also explained why the argument is apparently similar to the standard type of argument in Morse theory. The point is that the Dehn–Sommerville equations correspond to Poincaré duality for the associated toric variety. The construction used in Morse theory for the proof of the Poincaré duality theorem is carried over to the polytope with the help of the moment map.

Let Δ be a simple convex polytope in $\mathbb{R}^d$. We consider an arbitrary linear function l on $\mathbb{R}^d$ that is not constant on any face of Δ of positive dimension. Such functions will be called *generic linear functions* on the polytope. A vertex of index m (with respect to a generic linear function l) is a vertex from which exactly m edges descend (that is, the function l restricted to these edges is decreasing). The other edges ascend.

We denote by $B(l, m)$ the number of vertices of index m in the polytope Δ with respect to a generic linear function l.

Proposition 1.2.1. *The number $B(l, m)$ is independent of the specific choice of a generic linear function l and can be explicitly expressed via the components of the f-vector of the polytope*:

$$B(l, m) = h_m.$$

Proof. To prove this assertion, we note that a vertex of index m is the point of maximum of our linear function restricted to some face of dimension $\leqslant m$. Let us now calculate f_i as follows. To each i-face of Δ we assign the vertex at which the restriction of l to this face attains the maximum value. This is a vertex of index $m \geqslant i$. However, exactly $\binom{m}{i}$ i-faces descend from each such vertex. Consequently, to calculate the number of i-faces in Δ, we must count all vertices of index $\geqslant i$, and moreover, each vertex of index m should be counted $\binom{m}{i}$ times. Thus,

$$f_i = \sum_{m \geqslant i} B(l, m)\binom{m}{i}.$$

Thus, $B(l,m) = h_m$ by the definition of the h-vector. It follows that the number $B(l,m)$ is independent of the choice of the linear function l.

Let us change the sign of the linear function l. On the one hand, by the above proposition, the number $B(l,m)$ remains the same. On the other hand, all points of index m become points of index $d-m$. Hence

$$h_m = B(l,m) = B(-l,d-m) = h_{d-m}.$$

The Dehn–Sommerville equations are thereby proved.

1.3. Stanley's theorem. The Dehn–Sommerville equations exhaust all linear relations between the components of the h-vector of a simple convex polytope. However, there are also some inequalities. Necessary and sufficient conditions on the h-vector of a simple polytope were stated by McMullen and proved by Stanley [5] (necessity) and by Billera and Lee [10] (sufficiency).

Prior to stating McMullen's conditions, let us give the following definition. Suppose that a and r are positive integers. A *canonical r-representation* of a is a decomposition of the form

$$a = \binom{a_r}{r} + \binom{a_{r-1}}{r-1} + \dots + \binom{a_i}{i}, \qquad a_r > a_{r-1} > \dots > a_i \geqslant i \geqslant 1.$$

One can show that for given a and r the decomposition exists and is unique. Let s be another positive integer. Then the *partial degree* is defined as follows:

$$a^{\langle s|r\rangle} = \binom{a_r + s - r}{s} + \binom{a_{r-1} + s - r}{s-1} + \dots + \binom{a_i + s - r}{i + s - r}.$$

One usually sets $0^{\langle s|r\rangle} = 0$.

Definition. A sequence of integers $(g_0, g_1, \dots)$ is called an *M-sequence* if the following relations hold:

$$g_0 = 1, \qquad 0 \leqslant g_{r+1} \leqslant g_r^{\langle r+1|r\rangle} \quad (\forall\, r \geqslant 0).$$

It is known that the dimensions of the homogeneous components of a commutative graded algebra generated by elements of the first degree form an M-sequence. Now we are in a position to state McMullen's conditions.

- The h-vector satisfies the Dehn–Sommerville equations.
- The h-vector satisfies the 'hill' theorem, that is, the components of the h-vector increase up to the middle and then decrease:

$$h_i \leqslant h_{i+1}, \qquad i < d/2.$$

- The numbers

$$g_0 = 1,\ g_1 = h_1 - h_0,\ g_2 = h_2 - h_1,\ \dots,\ g_r = h_r - h_{r-1},\ \dots$$

form an M-sequence.

Stanley's proof reduces the 'hill' theorem to the Lefschetz decomposition in the cohomology ring of the corresponding toric variety. In the following we outline the main ideas of this proof.

In 1993, McMullen published a proof of the 'hill' theorem [7] without resorting to algebraic geometry. For a given polytope, he constructed a certain commutative algebra and proved an analogue of the Lefschetz decomposition and the Hodge–Riemann relations for this algebra.

In the present paper we give a proof of the Hodge–Riemann relations for simple convex polytopes on the basis of a more transparent description of the polytope algebra (this description was suggested by Pukhlikov and Khovanskii in [6]). The proof given here was obtained from McMullen's argument by replacing one algebra by another. However, this replacement enables us to avoid some technical difficulties as well as to make the proof more geometric.

The remaining subsections of the present section serve only to motivate the main results and are not necessary for understanding the subsequent material.

1.4. Kähler manifolds. The classical Hodge–Riemann relations are defined in the cohomology ring of a Kähler manifold. In what follows we give the main definitions from the theory of Kähler manifolds needed to state the Hodge–Riemann relations.

Let M be a compact complex manifold of dimension d. Then the space Λ of all smooth differential forms on M can be decomposed into the direct sum

$$\Lambda = \bigoplus_{p,q \leqslant d} \Lambda^{p,q}.$$

The bihomogeneous component $\Lambda^{p,q}$ is generated by products of p holomorphic and q antiholomorphic differentials. Suppose that a *closed* 2-form

$$\omega \in \Lambda^{1,1}$$

is given that is positive at each point, which means that the inequality

$$\omega(\xi, i\xi) > 0 \tag{1.4.1}$$

holds for each non-zero tangent vector ξ. In other words, the restriction of ω to each complex plane is equal to the volume form of the plane with a positive coefficient. Inequality (1.4.1) is equivalent to the assertion that the form ω is the imaginary part of some positive-definite Hermitian metric. The condition $\omega \in \Lambda^{1,1}$ means that the form is invariant with respect to multiplication of the arguments by the imaginary unit:

$$\omega(i\xi, i\eta) = \omega(\xi, \eta).$$

A manifold M equipped with a form ω with such properties is called a *Kähler manifold.*

1.5. The Hodge–Riemann form on a Kähler manifold

Definition. A form $\alpha \in \Lambda^{p,q}$ on a Kähler manifold is said to be *primitive* if

$$[\alpha \wedge \omega^{d-p-q+1}] = 0$$

(the square brackets are used to indicate the cohomology class).

This notion depends only on the cohomology class of α, and hence it makes sense to speak of primitive cohomology classes. The cohomology classes have the same decomposition into bihomogeneous components as the differential forms (the Hodge decomposition):

$$H^*(M, \mathbb{C}) = \bigoplus_{p,q \leqslant d} H^{p,q}(M, \mathbb{C}).$$

Definition. The *Hodge–Riemann form* on the bihomogeneous component $H^{p,q}$ is defined by

$$q(\alpha) = i^{p-q}(-1)^{\frac{(d-p-q)(d-p-q-1)}{2}} \int_M \alpha \wedge \overline{\alpha} \wedge \omega^{d-p-q}.$$

Theorem 1.5.1 (the Hodge–Riemann relations). *The Hodge–Riemann form is positive definite on primitive cohomology classes.*

The proof of the Hodge–Riemann relations can be found, for example, in [11]. In any case, the proof splits into two parts. The first part is a local (linear) version of the Hodge–Riemann relations, which deals with constant differential forms on $\mathbb{C}^n$. The second part deals with globalization. Here one uses the theory of harmonic forms. The main idea is that in the space of all differential forms on a compact Kähler manifold one considers the finite-dimensional subspace of *harmonic forms*, which satisfy the following conditions:

- in each cohomology class there is a unique harmonic representative, which is the form of minimum length with respect to some Euclidean metric on the space of differential forms;
- if a harmonic form represents the zero cohomology class, then it is identically zero;
- any exterior product of harmonic forms and all powers of ω are harmonic forms.

Using the above properties of harmonic forms, one can readily reduce the general Hodge–Riemann relations to their local version. The point is that if α is a harmonic form whose cohomology class is primitive, that is, $[\alpha \wedge \omega^{d-p-q+1}] = 0$, then the equality $\alpha \wedge \omega^{d-p-q+1} = 0$ holds at each point.

1.6. The Lefschetz decomposition. Let M be a compact Kähler manifold. We denote by L the operator of multiplication by the class of the form ω in the cohomology ring. In other words, the operator L takes each class $[\alpha] \in H^{p,q}$ (where α is a closed (p,q)-form) to the class $[\omega \wedge \alpha] \in H^{p+1,q+1}$. This is obviously well defined. We can now state the hard Lefschetz theorem.

Theorem 1.6.1 (Lefschetz)**.** *The operator L^{d-p-q} establishes an isomorphism between the spaces $H^{p,q}$ and $H^{d-q,d-p}$.*

The modern proof of this theorem uses the fact that there is a natural representation of the algebra sl_2 on the cohomology ring of a Kähler manifold, and moreover, the matrix

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

is represented by the operator L. See [12] for details.

Corollary 1.6.2. *For $p+q \leqslant d$, the restriction of the Hodge–Riemann form to the component $H^{p,q}$ of the cohomology ring of a compact Kähler manifold is non-degenerate.*

Proof. It suffices to do the following: for each differential form $\alpha \in \Lambda^{p,q}$ specifying a non-zero cohomology class, we must find a form $\beta \in \Lambda^{p,q}$ such that

$$\int_M \alpha \wedge \beta \wedge \omega^{d-p-q} \neq 0.$$

Indeed, by the hard Lefschetz theorem, the form $\alpha \wedge \omega^{d-p-q}$ specifies a non-zero cohomology class. We can assume without loss of generality that α is harmonic. Then the form

$$\beta = *(\alpha \wedge \omega^{d-p-q})$$

is also harmonic (since the Hodge star operator takes harmonic forms to harmonic forms) and hence closed. Now we use the well-known equation

$$\alpha_1 \wedge *\alpha_2 = (\alpha_1, \alpha_2) \cdot \mathrm{vol}\,.$$

Here α and β are bihomogeneous forms of the same bidegree, and vol is the volume form. By virtue of this equation,

$$\pm \int_M \alpha \wedge \beta \wedge \omega^{d-p-q} > 0$$

in our case, as desired.

Corollary 1.6.3 (the Lefschetz decomposition)**.** *The following decomposition into an orthogonal direct sum holds*:

$$H^{p,q} = \mathrm{Im}(L) \oplus \mathrm{Ker}\big(L^{d-p-q+1}\big) \qquad (p+q>0).$$

Here orthogonality is understood in the sense of the Hodge–Riemann form. In particular, each bihomogeneous cohomology class can be uniquely represented as the sum of a multiple of $[\omega]$ and a primitive class. Moreover, the summands are orthogonal with respect to the Hodge–Riemann form.

Proof. First, let us prove that the direct sum decomposition is valid. To this end it suffices to note that the restriction of the operator $L^{d-p-q+1}$ to the subspace $\mathrm{Im}(L)$

is an isomorphism. This follows from the hard Lefschetz theorem applied to the operator

$$L^{d-p-q+2}\colon H^{p-1,q-1} \to H^{d-q+1,d-p+1}.$$

It remains to verify that the direct sum is orthogonal. Indeed, let

$$L\alpha \in \mathrm{Im}(L), \qquad \beta \in \mathrm{Ker}\bigl(L^{d-p-q+1}\bigr).$$

The value of the Hodge–Riemann form on these elements is equal to

$$\pm \int_M \bigl(\omega \wedge \alpha\bigr) \wedge \beta\omega^{d-p-q} = \pm \int_M \alpha \wedge \bigl(\omega^{d-p-q+1} \wedge \beta\bigr) = 0,$$

so that these elements are indeed orthogonal.

Corollary 1.6.4. *The dimension of the primitive subspace of the homogeneous component $H^{p,q}$ of the cohomology ring of a compact Kähler manifold is equal to*

$$\dim \mathrm{Ker}\bigl(L^{d-p-q+1}\bigr) = \dim H^{p,q} - \dim H^{p-1,q-1}.$$

Corollary 1.6.4 readily follows from Corollary 1.6.3 and the hard Lefschetz theorem.

Corollary 1.6.5. *The restriction of the Hodge–Riemann form to each primitive subspace* $\mathrm{Ker}\bigl(L^{d-p-q+1}\bigr)$, $p+q \leqslant d$, *is non-degenerate.*

This readily follows from Corollaries 1.6.2 and 1.6.3. Corollary 1.6.5 implies, in particular, that the Hodge numbers $h^{p,p} = \dim H^{p,p}$ form a 'hill' (they increase up to the middle term of the sequence and then decrease).

1.7. Integrally simple polytopes. A polytope in $\mathbb{R}^n$ is said to be *integral* if all its vertices have integer coordinates, that is, lie in $\mathbb{Z}^n$. Associated with each integral polytope Δ is a so-called toric variety $X(\Delta)$. This is an algebraic variety whose topological properties are related to the combinatorial properties of Δ. A polytope is said to be *integrally simple* if for each of its vertices the integral direction vectors of the edges incident to this vertex form a basis of the integer lattice. (Here it is assumed that one takes integral direction vectors of minimum length.)

For an integrally simple polytope Δ, the variety $X(\Delta)$ is a smooth algebraic variety embedded in projective space and hence equipped with the induced Kähler metric. It turns out that the numbers $h^{p,p}(X(\Delta))$ coincide with the Dehn–Sommerville numbers $h_p(\Delta)$. Thus, the 'hill' theorem for the polytope Δ follows from the Lefschetz decomposition on the corresponding toric variety $X(\Delta)$.

The variety corresponding to an integral polytope that is not integrally simple is a singular toric variety with elementary singularities. In the Macpherson cohomology of such varieties there is also a Lefschetz decomposition. Using this fact, we can prove the hill theorem for an arbitrary simple polytope. The point is that each simple polytope is combinatorially equivalent to an integral polytope. The simplest way to see this is to consider the dual simplicial polytope. By slightly moving the vertices of a simplicial polytope, we can make them rational without changing the combinatorial type. The vertices of a simple polytope dual to a simplicial polytope with rational vertices are also rational. A homothety with an appropriate integer ratio makes the polytope integral. Thus, the hill theorem for a simple polytope reduces to the Lefschetz decomposition in the cohomology of a toric variety. This is just Stanley's proof (see [5]).

§ 2. The volume polynomial and the polytope algebra

In the present section we describe the algebra of a simple convex polytope in a convenient form. This algebra is obtained from the algebra of differential operators with constant coefficients by passing to the quotient by the ideal of operators annihilating the volume polynomial. We write out the generators of this ideal in explicit form and prove that the dimensions of homogeneous components of the polytope algebra coincide with the components of the h-vector of the polytope.

2.1. The volume polynomial.

Definition. Let Δ be a convex polytope in $\mathbb{R}^d$, and let Γ be a facet of Δ. A *direction covector* (or an *outward normal*) of the facet Γ is a linear functional ξ on the space $\mathbb{R}^d$ such that the restriction of ξ to Δ attains its maximum value at each point of Γ. The direction covector of a facet is defined uniquely up to multiplication by a positive constant.

Informally speaking, two polytopes are *analogous* if their faces have the same combinatorial structure and the same orientation in space. Two polytopes are *combinatorially equivalent* if their faces have the same combinatorial structure (in particular, their f-vectors coincide). Now let us give the precise definitions by induction.

Definition. Two polytopes are said to be *analogous* if there is a one-to-one correspondence between their facets such that corresponding facets have the same outward normal direction and are analogous. The coincidence of the outward normals to corresponding facets simply means that these facets are parallel and have the same orientation. Two polytopes are said to be *combinatorially equivalent* if there is a one-to-one correspondence between their facets such that corresponding facets are combinatorially equivalent. Any two segments on a line are combinatorially equivalent and analogous by definition.

One can readily verify that combinatorially equivalent polytopes have the same f-vectors. Needless to say, two analogous polytopes are combinatorially equivalent.

Let us consider a simple convex polytope P in the space $\mathbb{R}^d$. We denote by $\mathcal{P}(P)$ the set all polytopes analogous to P. Let $P', P'' \in \mathcal{P}(P)$. Then the Minkowski sum

$$P' + P'' = \{p' + p'' \mid p' \in P',\ p'' \in P''\}$$

is also a simple polytope analogous to P. Next, if P' is a polytope analogous to P, then the polytope

$$\lambda P' = \{\lambda p' \mid p' \in P'\}, \qquad \lambda > 0,$$

is also analogous to P. Thus, on the set $\mathcal{P}(P)$ there are naturally defined operations of addition and multiplication by positive numbers. In other words, the set $\mathcal{P}(P)$ possesses the structure of a convex cone.

We can extend the cone $\mathcal{P}(P)$ to a vector space by considering formal differences. We denote this vector space by $\mathcal{P}^*(P)$ and call it the *space of virtual polytopes analogous to* P. The elements of $\mathcal{P}^*(P)$ that are not actual polytopes will be called *virtual* polytopes. A geometric definition of virtual polytopes can be found in [13].

Theorem 2.1.1. *The function that takes each polytope in $\mathcal{P}(P)$ to its volume extends to a homogeneous polynomial of degree d on the space $\mathcal{P}^*(P)$.*

This polynomial will be called the *volume polynomial* and denoted by V.

Proof of Theorem 2.1.1. We proceed by induction on the dimension d. For $d = 1$ the theorem is obvious. Suppose that the theorem has already been proved for all $(d-1)$-polytopes. Now let us consider a simple d-polytope P. We can assume without loss of generality that the origin O lies inside P.

Let $P_1, \dots, P_n$ be the facets of P. Then P is the union of the cones

$$C_1 = \operatorname{conv}(\{O\} \cup P_1), \ \dots, \ C_n = \operatorname{conv}(\{O\} \cup P_n).$$

We note that the volumes of these cones can be calculated by the formula

$$\operatorname{Vol}(C_i) = \frac{1}{d} \operatorname{Vol}(P_i),$$

whose right-hand side contains the $(d-1)$-dimensional volume of P_i. Now we have the following formula for the volume of P:

$$\operatorname{Vol}(P) = \frac{1}{d} \sum H_i \operatorname{Vol}(P_i). \tag{2.1.1}$$

One can readily verify that the support numbers of the polytope P_i considered inside its affine hull can be linearly expressed via the support numbers of P. Theorem 2.1.1 follows from this remark, formulae (2.1.1), and the inductive hypothesis.

We denote the facets of P by $P_1, \dots, P_n$. Let $\xi_1, \dots, \xi_n$ be their direction covectors. Let us consider some polytope $P' \in \mathcal{P}(P)$ analogous to P. The polytope P' is characterized by the *support numbers* (or *support parameters*)

$$H_i(P') = \max_{p \in P'} (\xi_i, p).$$

The linear functional ξ_i restricted to P' attains its maximum value $H_i(P')$ on the facet P'_i. The facet P'_i corresponds to the facet P_i, since P' and P are analogous.

We can introduce a natural coordinate system in the vector space $\mathcal{P}^*(P)$: the function that takes each polytope P' analogous to P to the support number in the direction of a given covector ξ_i extends to a linear functional on the space $\mathcal{P}^*(P)$.

The linear functionals thus obtained form a coordinate system in $\mathcal{P}^*(P)$. The coordinates of a virtual polytope will sometimes be referred to as its support numbers. One can readily verify that each n-tuple $H_1, \dots, H_n$ of real numbers can be obtained as the set of support parameters of some (possibly virtual) polytope.

The coordinates introduced above on the space $\mathcal{P}^*(P)$ will be denoted by $x_1, \dots, x_n$. Although these coordinates have the meaning of support parameters, to avoid misunderstanding we use different letters to denote the coordinates themselves and the support numbers of specific polytopes (that is, the values of these coordinates).

The volume polynomial on the space $\mathcal{P}^*(P)$ can be rewritten in the coordinates as $V(x_1, \dots, x_n)$. Then

$$V\bigl(H_1(P'), \dots, H_n(P')\bigr) = \operatorname{Vol}(P')$$

for each polytope P' analogous to P.

Example (the volume polynomial of a simplex)**.** Let us consider a d-simplex S in the space $\mathbb{R}^d$ and the corresponding space $\mathcal{P}^*(S)$ of virtual simplices analogous to S. A virtual simplex S' analogous to S can be interpreted as a simplex analogous (that is, homothetic) to the simplex $-S$ and taken with the minus sign. The space $\mathcal{P}^*(S)$ is equipped with the coordinates $x_1, \dots, x_{d+1}$.

Proposition 2.1.2. *The volume polynomial of the simplex S is equal to*

$$V = (a_1 x_1 + \dots + a_{d+1} x_{d+1})^d,$$

where all coefficients $a_1, \dots, a_{d+1}$ are positive.

Proof. For each simplex $S' \in \mathcal{P}(S)$, consider the length of the altitude to the facet of some fixed direction. The function thus defined extends to a linear functional ψ on the vector space $\mathcal{P}^*(S)$.

The functional ψ is invariant under translations: $\psi(S' + a) = \psi(S')$ for each vector $a \in \mathbb{R}^d$ and each (possibly virtual) simplex $S' \in \mathcal{P}^*(S)$. Since ψ depends on $d+1$ variables and the group of translations is d-dimensional, we see that ψ actually depends on a single parameter. In other words, any two translation-invariant linear functionals on $\mathcal{P}^*(S)$ are proportional.

Let us consider V_S/ψ^d. This ratio can be viewed as a translation- and homothety-invariant function of a virtual simplex analogous to S. However, each virtual simplex analogous to S can be obtained from S by a translation and a homothety (possibly, with negative ratio). Consequently, $V_S/\psi^d = \text{const}$.

The constant is positive (if the altitude is positive, then so is the volume). Thus, by multiplying the functional ψ by an appropriate positive coefficient, we can ensure that the constant is equal to 1.

It remains only to prove that ψ is a linear combination with positive coefficients of the support parameters $x_1, \dots, x_{d+1}$. However, this follows from the fact that if all support parameters are positive, then the functional ψ also takes a positive value.

2.2. The polarization of a homogeneous polynomial. As we have seen in the preceding subsection, the space $\mathcal{P}^*(P)$ is equipped with a natural coordinate system $x_1, \dots, x_n$. The operator $\partial/\partial x_i$ of differentiation with respect to x_i will be denoted for brevity by ∂_i.

Lemma 2.2.1. *Let V be a homogeneous polynomial of degree d in the variables $x_1, \dots, x_n$. We denote by L_a the operator of differentiation in the direction $a = (a_1, \dots, a_n)$:*

$$L_a = \sum_{i=1}^{n} a_i \partial_i.$$

Then

$$V(a) = \frac{1}{d!} L_a^d V.$$

Proof. Let us consider the polynomial $\varphi(t) = V(at)$. This is a homogeneous polynomial of degree d. Hence, by Maclaurin's formula,

$$V(a) = \varphi(1) = \frac{1}{d!} \varphi^{(d)}(0).$$

Now it suffices to note that, by the chain rule, $\varphi^{(k)}(t) = L_a^k V(at)$. In particular,

$$\varphi^{(d)}(t) \equiv \varphi^{(d)}(0) = L_a^d V,$$

as desired.

Lemma 2.2.2. *Under the assumptions of the preceding lemma, there is a unique symmetric n-linear functional $V(a_1, \dots, a_n)$ such that*

$$V(a, \dots, a) = V(a)$$

for each $a \in \mathbb{R}^n$.

Proof. Let us prove the uniqueness. For this it suffices to verify that if $V(a_1, \dots, a_n)$ is a symmetric multilinear functional such that

$$V(a, \dots, a) = 0 \tag{2.2.1}$$

for each $a \in \mathbb{R}^n$, then the functional $V(a_1, \dots, a_n)$ is identically zero. Indeed, let us choose vectors $a_1, \dots, a_n$ and consider the homogeneous polynomial

$$V(\lambda_1 a_1 + \dots + \lambda_n a_n, \dots, \lambda_1 a_1 + \dots + \lambda_n a_n)$$

in the variables $\lambda_1, \dots, \lambda_n$. By condition (2.2.1), this polynomial is identically zero. Consequently, all its coefficients are zero. On the other hand, one can readily note that the coefficient of the monomial $\lambda_1 \cdots \lambda_n$ in this polynomial is equal to $V(a_1, \dots, a_n)$. Thus $V(a_1, \dots, a_n) = 0$, as desired.

It remains to prove existence. To this end, we set

$$V(a_1, \dots, a_n) = \frac{1}{d!} L_{a_1} \cdots L_{a_n}(V).$$

This form is obviously multilinear and symmetric. Finally, $V(a, \dots, a) = V(a)$ by Lemma 2.2.1.

Definition. The form $V(a_1, \dots, a_n)$ is called the *polarization* of the homogeneous polynomial V.

2.3. A differential operator connected with a polytope. For each simple polytope P, we consider the following differential operator, whose coefficients are the support numbers of P:

$$L_P = \sum_{i=1}^{n} H_i \partial_i.$$

Let us now give an invariant definition. We regard the polytope P as a vector in the space $\mathcal{P}^*(P)$. Then the operator L_P is just the operator of differentiation in the corresponding direction.

Now we can apply the results of the preceding subsection to the volume polynomial.

Corollary 2.3.1. *The volume of a polytope Q analogous to P is given by the formula*

$$\mathrm{Vol}(Q) = \frac{1}{d!} L_Q^d V_P.$$

Definition. The polarization of the volume polynomial on the space $\mathcal{P}^*(P)$ is called the *mixed volume form*. The mixed volume of polytopes $P_1, \dots, P_d$ is denoted by $\mathrm{Vol}(P_1, \dots, P_d)$. The definition of the polarization readily implies the following properties of the mixed volume:

- the mixed volume is a symmetric multilinear function of its arguments;
- the restriction of the mixed volume form to the diagonal is the usual volume polynomial:

$$\mathrm{Vol}(P, \dots, P) = \mathrm{Vol}(P).$$

Corollary 2.3.2. *The mixed volume of polytopes $P_1, \dots, P_d$ analogous to P is given by the formula*

$$\mathrm{Vol}(P_1, \dots, P_d) = \frac{1}{d!} L_{P_1} \cdots L_{P_d} V_P.$$

2.4. The volume polynomial of a face. Let us state an elementary fact from the combinatorics of simple convex polytopes.

Lemma 2.4.1. *Let $P_{i_1}, \dots, P_{i_k}$ be distinct facets of a simple polytope P. If*

$$F = P_{i_1} \cap \cdots \cap P_{i_k} \neq \varnothing, \tag{$*$}$$

then F is a face of codimension k in P (if P were not simple, then the F might have smaller codimension). Conversely, each face of codimension k in P can be represented in the form $(*)$.

Suppose that F is a face of codimension k in a polytope P. Let us consider a polytope P' analogous to P. We denote by F' the face of P' corresponding to F. Then by an appropriate translation we can move F' into the plane of F. Thus, we obtain a map $\mathcal{P}(P) \to \mathcal{P}(F)$, which extends to an epimorphism $Y_F \colon \mathcal{P}^*(P) \to \mathcal{P}^*(F)$. The volume polynomial V_F defined on the space $\mathcal{P}^*(F)$ can be carried over with the help of this epimorphism to the space $\mathcal{P}^*(P)$. The resulting polynomial will be denoted by $\widetilde{V}_F$.

Let $H_1, \dots, H_n$ be the support numbers of the polytope P. The volume of the facet P_i is then given by the obvious formula

$$\mathrm{Vol}(P_i) = \frac{\partial V_P}{\partial x_i}(H_1, \dots, H_n).$$

It readily follows that

$$\widetilde{V}_{P_i} = \partial_i V.$$

Proposition 2.4.2. *The volume of a simple convex polytope P is equal to*

$$\operatorname{Vol}(P) = \frac{1}{d} L_P V_P(H_1, \dots, H_n) = \frac{1}{d} \sum_{i=1}^{n} H_i \operatorname{Vol}(P_i).$$

This proposition readily follows from Euler's theorem on homogeneous functions.

Theorem 2.4.3. *Let $P_{i_1}, \dots, P_{i_k}$ be distinct facets of a simple polytope P. Then the following assertions hold*:

- *if $P_{i_1} \cap \dots \cap P_{i_k} = \varnothing$, then $\partial_{i_1} \cdots \partial_{i_k} V = 0$;*
- *if $F = P_{i_1} \cap \dots \cap P_{i_k}$ is a face of codimension k in P, then $\partial_{i_1} \cdots \partial_{i_k} V = C\widetilde{V}_F$, where C is a positive coefficient independent of the variables x_i.*

Proof. We proceed by induction on k. For $k = 1$ the theorem has already been proved. Suppose that the theorem is known to be valid for some k. Let us consider $k+1$ facets $P_{i_1}, \dots, P_{i_k}, P_{i_{k+1}}$. If $P_{i_1} \cap \dots \cap P_{i_k} = \varnothing$, then certainly $P_{i_1} \cap \dots \cap P_{i_k} \cap P_{i_{k+1}} = \varnothing$. Next, by the inductive hypothesis, $\partial_{i_1} \cdots \partial_{i_k} V = 0$. But then, certainly $\partial_{i_1} \cdots \partial_{i_k} \partial_{i_{k+1}} V = 0$. Thus, we can restrict ourselves to the case $P_{i_1} \cap \dots \cap P_{i_k} = F \neq \varnothing$. Suppose first that $P_{i_1} \cap \dots \cap P_{i_k} \cap P_{i_{k+1}} = \varnothing$. This means that the facet $P_{i_{k+1}}$ has no points common with F. Consequently, the volume of F does not change under small displacements of $P_{i_{k+1}}$. Thus, $\partial_{i_{k+1}} V_F = 0$. It follows from this and from the inductive hypothesis that $\partial_{i_1} \cdots \partial_{i_k} \partial_{i_{k+1}} V = 0$.

Now we suppose that $P_{i_1} \cap \dots \cap P_{i_k} \cap P_{i_{k+1}} = F_{i_{k+1}} \neq \varnothing$. As the facet $P_{i_{k+1}}$ moves in the positive direction, the facet $F_{i_{k+1}}$ of the polytope F moves in the plane of F also in the positive direction (but possibly at a different speed). The other facets of F remain in place. Thus, the derivative $\partial_{i_{k+1}} \widetilde{V}_F$ is proportional with a positive coefficient to the polynomial $\widetilde{V}_{F_{i_{k+1}}}$. To complete the proof of the theorem it remains to use the inductive hypothesis.

2.5. Construction of an algebra for a given polynomial. Let V be an arbitrary homogeneous polynomial of degree d on the finite-dimensional vector space $\mathbb{R}^n$. We denote by Diff the algebra of differential operators with constant coefficients on $\mathbb{R}^n$. This algebra has a natural grading:

$$\operatorname{Diff} = \bigoplus_{k \in \mathbb{N}} \operatorname{Diff}_k.$$

Each homogeneous component Diff_k consists of differential operators of degree k. Let us consider the ideal

$$I(V) = \operatorname{Ann}(V) = \{\alpha \in \operatorname{Diff} \mid \alpha V = 0\}.$$

Definition. The *algebra corresponding to the polynomial* V is the quotient algebra

$$A(V) = \operatorname{Diff} / I(V).$$

This algebra inherits a grading from the algebra of differential operators:

$$A(V) = \bigoplus_{k=0}^{d} A_k(V).$$

Let $\alpha \in \operatorname{Diff}$ be a differential operator. We denote by $[\alpha]$ the class of α in the quotient algebra $A(V)$.

Proposition 2.5.1. *The formula* $([\alpha],[\beta])=\alpha\beta V$ *specifies a non-degenerate pairing*

$$A_k(V)\otimes A_{d-k}(V)\to\mathbb{R}.$$

Proof. The map is obviously well defined. Let us prove non-degeneracy: for each non-zero $\beta\in A_{d-k}$, there is an element $\alpha\in A_k$ such that $(\alpha,\beta)\neq 0$. We note that βV is a non-zero homogeneous polynomial of degree k. Hence there is an operator α of degree k that does not annihilate the polynomial βV, as desired.

Corollary 2.5.2. *The following duality relation holds*: $\dim A_k(V)=\dim A_{d-k}(V)$.

Under certain conditions, we identify algebras corresponding to various polynomials on various spaces. Let us describe these conditions in detail. Suppose that we are given a homogeneous polynomial V on the space $\mathbb{R}^s$. Let us consider an arbitrary linear epimorphism $Y\colon\mathbb{R}^n\to\mathbb{R}^s$. Using this linear map, we can define the polynomial $\widetilde{V}=Y^*V$ on the space $\mathbb{R}^n$. In other words, $\widetilde{V}(x)=V(Y(x))$.

Proposition 2.5.3. *Let us consider an element* $\alpha\in A(V)$. *There is a unique element* $\beta\in A(\widetilde{V})$ *such that*

$$\beta\widetilde{V}=Y^*(\alpha V).$$

The map $\alpha\mapsto\beta$ *is an isomorphism between the algebras* $A(V)$ *and* $A(\widetilde{V})$. *The element* $L_{Y(a)}\in A(V)$ *is taken by this isomorphism to the element* $L_a\in A(\widetilde{V})$.

Proof. This assertion is almost obvious. Roughly speaking, it means that the algebra $A(V)$ is not changed if we append inessential variables to the polynomial V. Nevertheless, let us give a careful proof.

We choose compatible coordinate systems in the spaces $\mathbb{R}^n$ and $\mathbb{R}^s$. Suppose that $y=Y(x)$, x_i are the coordinates of a vector $x\in\mathbb{R}^n$, and y_i are the coordinates of a vector $y\in\mathbb{R}^s$. By saying that the coordinate systems are compatible, we mean that $y_i=x_i$, $i=1,\dots,s$. In other words, the map Y can be viewed as the standard projection of a coordinate space on a coordinate subspace. The polynomial $\widetilde{V}$ is obtained from V by appending 'dummy' variables. Now the proposition becomes obvious: the operator β is obtained from the operator α by replacing $\partial/\partial y_i$ by $\partial/\partial x_i$ for all $i=1,\dots,s$.

The algebras $A(V)$ and $A(\widetilde{V})$ can be identified using this isomorphism.

2.6. The polytope algebra. Let P be a simple polytope and let $A=A(V_P)$ be the algebra corresponding to the volume polynomial of this polytope. The algebra A will be called the *polytope algebra of* P. One can readily verify that if P is integrally simple, then A is isomorphic to the cohomology algebra of the toric variety $X(P)$.

Theorem 2.6.1. *The dimensions of homogeneous components of the algebra of a simple convex polytope are equal to* $\dim A_k=h_k$.

We shall prove this theorem later.

The polytope algebra of P is obtained from the algebra of differential operators on the space $\mathcal{P}^*(P)$ by passing to the quotient algebra by the ideal $I=I(V_P)$. The ideal I will be called the *ideal corresponding to the simple polytope* P.

We can explicitly write down a system of multiplicative generators of I.

Let a be a vector (a point) in $\mathbb{R}^d$. Then the element $(P+a)-P$ of the group $\mathcal{P}^*(P)$ can be identified with a. Adding a point to a polytope results in a translation of the polytope. In particular, the volume of the polytope remains unchanged. Hence, the derivative of the volume polynomial in the direction of this point (viewed as an element of $\mathcal{P}^*(P)$) is zero. This fact can be expressed in coordinates as follows. Let H_i be the system of support numbers of a (with respect to the direction covectors of P), that is, $H_i = (a, \xi_i)$. Then the operator

$$L = \sum_{i=1}^{n} H_i \partial_i$$

is just the derivative in the direction of a in $\mathcal{P}^*(P)$. Hence $LV = 0$, that is, $L \in I$.

Definition. Relations of the form $LV_P = 0$ will be called *translation invariance relations*. They express the fact that the volume polynomial is preserved under translations of the polytope.

Next, by Theorem 2.2, the ideal I contains also the operators

$$L = \partial_{i_1} \cdots \partial_{i_r},$$

where $P_{i_1} \cap \cdots \cap P_{i_r} = \varnothing$.

Definition. Relations of the form $LV_P = 0$ will be called *incidence relations*.

Theorem 2.6.2. *The ideal corresponding to a simple convex polytope is multiplicatively generated by the incidence and translation invariance relations.*

The proofs of Theorems 2.6.1 and 2.6.2 are based on 'Morse theory for polytopes'.

We note that each face F of a polytope P can be represented in the form $F = P_{i_1} \cap \cdots \cap P_{i_k}$, where $P_{i_1}, \cdots, P_{i_k}$ are facets. To the face F we assign the differential monomial

$$\partial_F = \partial_{i_1} \cdots \partial_{i_k}.$$

Lemma 2.6.3. *Every differential operator with constant coefficients can be expressed modulo the ideal I via the differential monomials corresponding to the faces of P. Moreover, each operator with constant coefficients can be reduced to a linear combination of differential monomials of the form ∂_F by using incidence and translation invariance relations alone.*

Proof. It suffices to verify this assertion for a differential monomial. Suppose that the monomial does not contain multiple differentiations with respect to any variable. Then either it is zero in the algebra A by virtue of incidence relations, or it corresponds to some face of P. Suppose that the monomial contains multiple differentiations:

$$\alpha = \partial_{i_1}^{k_1} \cdots \partial_{i_s}^{k_s}, \qquad k_1, \dots, k_s > 0.$$

The number $(k_1 - 1) + \cdots + (k_s - 1)$ will be called the *multiplicity* of the monomial α. If the multiplicity is greater than zero (that is, the monomial contains multiple differentiations), then we can reduce the monomial to a linear combination of monomials of lower multiplicity by using translation invariance relations alone.

Indeed, suppose, for example, that $k_1 > 1$. If $\alpha \neq 0$ in the algebra A, then $P_{i_2} \cap \cdots \cap P_{i_s}$ is a face of positive dimension. Thus, there are two distinct vertices $O, X \in P_{i_2} \cap \cdots \cap P_{i_s}$ such that $(\xi_{i_1}, \overrightarrow{OX}) \neq 0$.

We use the point O as the origin. Let $H_i = (\xi_i, \overrightarrow{OX})$ be the support numbers of X. Since the segment OX belongs to the facets $P_{i_2}, \ldots, P_{i_s}$, it follows that $H_{i_2} = \cdots = H_{i_s} = 0$.

Let us consider the differential operator $L = \sum_{i=1}^n H_i \partial_i$ corresponding to the point X. By the relation

$$L \equiv 0 \pmod{I}$$

we can write

$$\partial_{i_1} \equiv \frac{1}{H_{i_1}} \sum_{i \neq i_1} (-H_i)\partial_i = L_i \pmod{I(P)}.$$

We note that the operators $\partial_{i_1}, \ldots, \partial_{i_s}$ occur on the right-hand side with zero coefficients. Hence, the operator

$$L_i \partial_{i_1}^{k_1-1} \partial_{i_2}^{k_2} \cdots \partial_{i_s}^{k_s}$$

is equal to the operator α modulo the ideal I but is represented by a linear combination of operators of lower multiplicity. Thus, each operator coincides modulo I with some linear combination of differential monomials corresponding to faces of P. This coincidence can be seen with the help of incidence and translation invariance relations alone.

Definition. Let us choose a generic linear function l on a polytope P. A k-face of P is called a *separatrix* face if the function l restricted to this face attains its maximum at a vertex of index k. In other words, a separatrix face is a face 'spanned' by all edges descending from a vertex of index k. The differential monomial corresponding to a separatrix face will be called a *separatrix monomial*.

Lemma 2.6.4. *The component A_k is generated by separatrix monomials. Moreover, each differential operator in* Diff *can be reduced to a linear combination of separatrix monomials using incidence and translation invariance relations alone.*

Proof. We define an ordering on the set of faces of P by saying that *one face is less than another* if the maximum value of l on the first face is less than the maximum value of l on the second.

Let us prove that if F is not a separatrix face, then the corresponding operator ∂_F can be expressed via monomials corresponding to smaller faces using translation invariance relations.

We denote by O the maximum vertex of F. Let $P_{i_1}, \ldots, P_{i_d}$ be the facets containing O, and let G be the lower separatrix emanating from O. Then F is a face of G. At least one of the facets $P_{i_1}, \ldots, P_{i_d}$ contains F but not G. Without loss of generality, we can assume that this facet is P_{i_1}. In this case,

$$\partial_F = \partial_G \partial \partial_{i_1},$$

where ∂ is some differential monomial.

Let us consider the edge $OX = P_{i_2} \cap \cdots \cap P_{i_d}$. We use the point O as the origin. Let $H_i = (\xi_i, \overrightarrow{OX})$ be the support numbers of the point X. Since the segment OX belongs to the facets $P_{i_2}, \dots, P_{i_d}$, it follows that $H_{i_2} = \cdots = H_{i_d} = 0$.

Let us consider the differential operator $L = \sum_{i=1}^{n} H_i \partial_i$ corresponding to X. By the relation

$$L = 0 \pmod{I}$$

we can write

$$\partial_{i_1} \equiv \frac{1}{H_{i_1}} \sum_{i \neq i_1} (-H_i)\partial_i = L_i \pmod{I}.$$

We note that the operators $\partial_{i_1}, \dots, \partial_{i_d}$ occur on the right-hand side with zero coefficients. The operator

$$\partial_G \partial L_i$$

is equal to ∂_F modulo the ideal I. On the other hand, its decomposition contains only differential monomials corresponding to faces of G that do not contain the vertex O. But all such faces are less than F. Thus, we have expressed the operator ∂_F in terms of smaller faces modulo the ideal I.

Lemma 2.6.5. *The separatrix monomials are linearly independent in the polytope algebra.*

Proof. Let us consider a linear combination (with non-zero coefficients) of separatrix monomials,

$$\alpha = \lambda_1 \partial_{F_1} + \cdots + \lambda_s \partial_{F_s}.$$

We can assume that all monomials have the same degree. Thus, the separatrices $F_1, \dots, F_s$ have the same dimension. Suppose that $\alpha = 0$ in the polytope algebra. Let us consider the maximum separatrix occurring in the linear combination α. We can assume that this is F_1. Let G_1 be the 'upper separatrix' corresponding to F_1. The face G_1 is spanned by all edges ascending from the maximum vertex of F_1.

Let us multiply the linear combination α by the differential monomial ∂_{G_1}. By Theorem 2.4.3, all products of the form $\partial_{G_1}\partial_{F_2}, \dots, \partial_{G_1}\partial_{F_s}$ are zero in the polytope algebra. The point is that in the linear combination α all monomials except the first correspond to faces of P lying entirely below G_1. Thus, $G_1 \cap F_2 = \cdots = G_1 \cap F_s = \varnothing$. But the product $\partial_{G_1}\partial_{F_1}$ is non-zero. Consequently, it follows from the relation $\partial_{G_1}\alpha = 0$ that $\lambda_1 = 0$. This contradicts the assumption that all coefficients are non-zero.

Let us now note that the above lemmas imply Theorems 2.6.1 and 2.6.2. By Lemmas 2.6.4 and 2.6.5, the separatrix monomials form a basis in the vector space A_k. But the number of separatrix monomials of dimension k coincides with the number h_k of vertices of index k. Theorem 2.6.1 follows. On the other hand, the incidence and translation invariance relations are sufficient for reducing each operator to a linear combination of separatrix monomials. Hence, all relations in the algebra A are exhausted by the incidence and translation invariance relations and their consequences. But the relations in A are just elements of the ideal I. Thus, I is generated by the incidence and translation invariance relations. This is precisely what Theorem 2.6.2 claims.

§ 3. Flips

In the present section, we give auxiliary information needed to understand what follows. All results here can be found in [7]. We deal with flips of simple polytopes. If we begin to move the facets of a simple polytope generically, then at some instant the combinatorial type of the polytope changes. This is a flip. In all there are $d+2$ types of flip (including flips in which a simplex appears or disappears). Each simple polytope can be obtained from a simplex by a continuous deformation through finitely many flips.

3.1. Flips. Let P be a simple polytope in $\mathbb{R}^d$. Then there is a simple polytope Δ in $\mathbb{R}^{d+1}$ and a generic linear function l such that some section of Δ by a horizontal hyperplane $l = \text{const}$ coincides with P. Let us raise this hyperplane. As long as it does not pass through the vertices of Δ, the sections are analogous polytopes. Suppose that at some instant the hyperplane passes through a vertex. (Without loss of generality we can assume that no two vertices lie on the same horizontal hyperplane; otherwise, it suffices to perturb the linear function l slightly.) We say that the polytope P has undergone a *flip of order* m if the vertex is of index m. Thus, d-polytopes admit $d+2$ types of flip.

Example 1 (flips of the simplest types). A flip of order 0 is just the creation of a simplex, and a flip of order $d+1$ is the destruction of a simplex. A flip of order 1 is the creation of a new facet. Under a flip of this type, some vertex is cut off, that is, is replaced by a simplicial facet. A flip of order d is the collapse of a simplicial facet to a point.

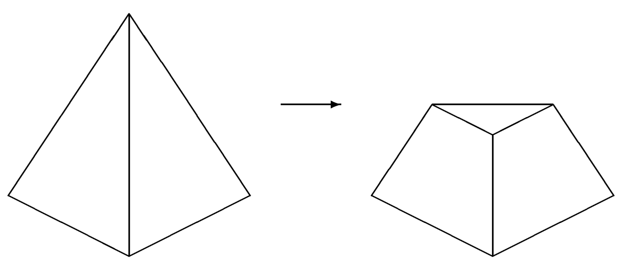

Figure 1. A flip of order 1 of a 3-polytope

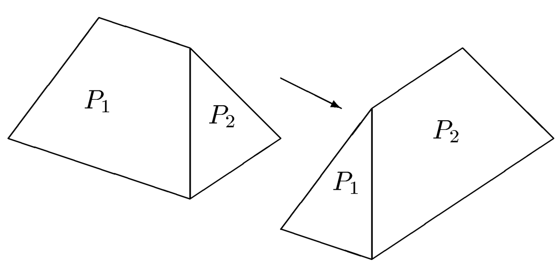

Figure 2. A flip of order 2 of a 3-polytope

Example 2 (flips of 3-polytopes)**.** For 3-polytopes there is a unique non-trivial flip of order 2 in addition to the flips of orders 0, 1, 3, and 4 already described. This flip is shown in Fig. 2.

The faces (front and back) that had non-empty intersection before the flip become disjoint after the flip. Conversely, the lateral faces were disjoint before the flip and have non-empty intersection after the flip. We note that this flip preserves the number of faces in each dimension and hence preserves the h-vector.

Theorem 3.1.1. *Every simple polytope can be obtained from a simplex by a continuous deformation through finitely many flips. More precisely, there is a continuous deformation that takes a given simple polytope to a simplex and can be described as follows*:

- *first the polytope moves slightly without changing the combinatorial type*;
- *then all facets move in a parallel way, and all changes in the combinatorial type* (*and even in the class of analogous polytopes*) *occur by flips.*

Proof. Let P be an arbitrary simple d-polytope. We place this polytope in the space $\mathbb{R}^{d+1}$.

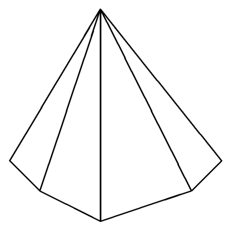

Figure 3. Resolution of a simple polytope

We can think of the polytope P as lying in a horizontal hyperplane. Let us take a point outside this hyperplane and construct a cone over P with vertex at that point (see Fig. 3). Now let us slightly move the facets of the cone. We obtain a simple polytope whose lower facet is obtained by a small perturbation of P and, in particular, is combinatorially equivalent to P.

3.2. A combinatorial description of flips. Let us give a direct combinatorial description of a flip. Suppose that a polytope P is obtained from a polytope Q by a flip of order m, $0 < m < d+1$. Then we can consider a simple polytope $\Delta \in \mathbb{R}^{d+1}$ and a generic linear function l such that

- the lower facet of Δ coincides with P (more precisely, there is an embedding of the plane of P in $\mathbb{R}^{d+1}$ such that the image of P under this embedding coincides with the lower facet of Δ with respect to l);
- the upper facet of Δ coincides with Q;
- Δ has exactly one vertex that lies neither on P nor on Q (this vertex is of index m; it will be denoted by O).

This construction is shown in Fig. 4. Since Δ is simple, it follows that exactly $d+1$ facets meet at the vertex O. We denote these facets by $\Gamma_1, \dots, \Gamma_{d+1}$. Suppose that we have chosen a numbering such that the facets $\Gamma_1, \dots, \Gamma_m$ contain the

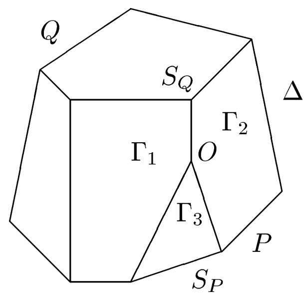

Figure 4. A flip of order $m = 2$ of a 2-polytope

'*upper separatrix*' (that is, contain all edges ascending from the vertex O), and the facets $\Gamma_{m+1}, \dots, \Gamma_{d+1}$ contain the '*lower separatrix*' (that is, contain all edges descending from O). It is a fact that each of the facets $\Gamma_1, \dots, \Gamma_{d+1}$ contains either the upper or the lower separatrix. This follows from the fact that each of these facets contains all but one of the edges incident to O.

We introduce the following notation:

$$S_P = P \cap \Gamma_{m+1} \cap \dots \cap \Gamma_{d+1},$$
$$S_Q = Q \cap \Gamma_1 \cap \dots \cap \Gamma_m.$$

The face S_P of the polytope P is an $(m-1)$-simplex. The facets of this simplex are the faces $S_P \cap \Gamma_1, \dots, S_P \cap \Gamma_m$. Likewise, the face S_Q of the polytope Q is the $(d-m)$-simplex with faces $S_Q \cap \Gamma_{m+1}, \dots, S_Q \cap \Gamma_{d+1}$. The flip in question destroys the simplex S_P and creates the simplex S_Q 'in its place'.

We note the following obvious fact.

Proposition 3.2.1. *Suppose that a polytope Q can be obtained from a polytope P by a flip of order m. Then P can be obtained from Q by a flip of order $d+1-m$.*

We note that flips of all orders except for 0, 1, d, and $d+1$ neither destroy nor create facets.

3.3. The transformation of the h-vector under a flip.

Theorem 3.3.1. *Suppose that a polytope Q is obtained from a polytope P by a flip of order $m \leqslant \frac{d+1}{2}$. Then*

$$h_k(Q) = \begin{cases} h_k(P) & \text{if } k < m, \\ h_k(P) + 1 & \text{if } m \leqslant k \leqslant d - m, \\ h_k(P) & \text{if } k > d - m. \end{cases}$$

Proof. First, let us find out what happens to the f-vector under a flip of this type. As noted above, the flip destroys the simplex S_P and creates the simplex S_Q. Thus, if we count all faces before and after the flip, then after the flip the faces of S_P are missing, but we count the faces of the newly created polytope S_Q. Hence, the F-polynomials of P and Q are related by the equation

$$F_Q = F_P - F_{S_P} + F_{S_Q}.$$

But then the same equation is valid for the H-polynomials, that is,

$$H_Q = H_P - H_{S_P} + H_{S_Q}.$$

In § 1.1 we established that the H-polynomial of a k-simplex has the form

$$t^k + t^{k-1} + \dots + 1 = \frac{t^{k+1} - 1}{t - 1}.$$

Consequently,

$$H_Q(t) = H_P(t) + t^m + \dots + t^{d-m}.$$

By equating coefficients of like powers of t, we obtain the assertion of the theorem.

Remark. The flips of higher orders $m > \frac{d+1}{2}$ are the inverses of the flips of lower orders $m \leqslant \frac{d+1}{2}$. Hence the following formula holds for $m > \frac{d+1}{2}$:

$$h_k(Q) = \begin{cases} h_k(P) & \text{if } k < d - m + 1, \\ h_k(P) - 1 & \text{if } d - m + 1 \leqslant k \leqslant m - 1, \\ h_k(P) & \text{if } k > m - 1. \end{cases}$$

3.4. The action of a flip on faces. We use the notation of § 3.2. Let us describe the action of a flip on facets.

Theorem 3.4.1. *Suppose that a polytope P undergoes a flip of order m. Then*

- *the facets $P \cap \Gamma_1, \dots, P \cap \Gamma_m$ undergo flips of order $m - 1$;*
- *the facets $P \cap \Gamma_{m+1}, \dots, P \cap \Gamma_{d+1}$ undergo flips of order m;*
- *the rest of the facets remain analogous to themselves.*

Proof. We note that the vertex O is of index $m - 1$ with respect to the facets $\Gamma_1, \dots, \Gamma_m$. Indeed, these facets do not contain the lower separatrix. Hence, each of them contains all descending edges but one. Next, the facets $\Gamma_{m+1}, \dots, \Gamma_{d+1}$ contain the lower separatrix.

Thus, the vertex O is of index m with respect to these facets. All other facets of Δ do not contain O. Hence nothing happens to their hyperplane section as this vertex is passed through.

3.5. Transition polytopes. Suppose that a simple polytope Q is obtained from a simple polytope P by a flip of order $0 < m < d + 1$. Then we can assume that P and Q are horizontal sections of some $(d + 1)$-polytope Δ, and moreover, there is exactly one vertex of Δ between these sections, and this vertex is of index m. Let us consider the section T of Δ by the horizontal hyperplane passing through this vertex. The polytope T is called the *transition polytope*. For $1 < m < d$ this polytope is not simple: $d + 1$ facets meet at one of the vertices. The exceptions are flips of orders $m = 1$ and $m = d$. In these cases the transition polytope is simple.

A flip of order $1 < m < d$ goes as follows. First, the polytope P varies within its class of analogous polytopes. Then, at some instant, it degenerates into a transition polytope T, and then T instantaneously changes into a polytope analogous to Q. For $m = 1$ no degeneration occurs at the transition instant, but then some vertex is destroyed instantaneously, and a $(d-1)$-simplex appears in its place. For $m = d$ the opposite sequence of events occurs. Some $(d-1)$-simplex collapses into a vertex at the transition instant.

It turns out that the transition polytope T 'remembers' the flip in which it appeared (up to a transition to the opposite flip, of course, since flips of order m and $d+1-m$ can yield the same transition polytope). The information is stored in the combinatorics of the arrangement of the facets meeting at the singular vertex.

Proposition 3.5.1. *Suppose that a transition polytope T is obtained by a flip of order $m \leqslant \frac{d+1}{2}$. Then the number m can be recovered from T (more precisely, from the mutual position of the facets meeting at the singular vertex).*

Proof. If T is simple, then $m = 1$. Now we suppose that T has a singular vertex with $d+1$ incident facets. Then in a neighbourhood of the singular vertex the polytope T looks like a cone over some simple $(d-1)$-polytope U. We note that U has exactly $d+1$ facets. By Proposition 1.1.1, U is combinatorially equivalent to the product $S_{m-1} \times S_{d-m}$ of two simplices. Since distinct products of simplices are not combinatorially equivalent (see the end of § 1.1), we see that the number m is determined uniquely.

We note that each facet of U corresponds to a facet of the transition polytope. But the facets of U fall into two categories: the first consists of facets of the form

$$S_{m-1} \times (\text{a facet of } S_{d-m}),$$

and the second consists of facets of the form

$$(\text{a facet of } S_{m-1}) \times S_{d-m}.$$

If $m < \frac{d+1}{2}$, then facets belonging to distinct categories have distinct combinatorial types, and hence the categories are characterized by intrinsic criteria.

For $m = \frac{d+1}{2}$ we can also divide the facets into two categories, but these categories turn out to be equivalent. Thus, we consider the polytope

$$U = S_{m-1} \times S_{m-1}.$$

Let us define an equivalence relation on the set of simplicial $(m-1)$-faces of U as follows. Two faces are said to be equivalent if they are disjoint. One can readily verify that this is indeed an equivalence relation, which induces a partition of the set of simplicial $(m-1)$-faces into two equivalence classes with the same number of elements. Elements belonging to distinct equivalence classes necessarily intersect in a point, and their direct product is the entire polytope U. Now we can divide the facets of U into two categories as follows:

- the first category consists of facets of the form (an element of the first equivalence class) $\times$ (a facet of an element of the second equivalence class);
- the second category consists of facets of the form (a facet of an element of the first equivalence class) $\times$ (an element of the second equivalence class).

Let $T_1, \dots, T_m$ be the facets of the transition polytope corresponding to facets of U of the first category, and let $T_{m+1}, \dots, T_{d+1}$ be the facets of the transition polytope corresponding to facets of U of the second category. To reconstruct the 'polytope before the flip' from T, we must slightly lower the facets $T_{m+1}, \dots, T_{d+1}$ (that is, move them towards the interior of the polytope). In fact, it suffices to move only one of these faces slightly.

The resulting simple polytope will be called a *regularization* of the polytope T and denoted by $\operatorname{reg} T$. We note that for $m < \frac{d+1}{2}$ any two regularizations of the same transition polytope are analogous. However, this assertion is not valid in the exceptional case $m = \frac{d+1}{2}$. Indeed, in this case

$$m - 1 = d - m = \frac{d-1}{2},$$

and it is not clear which of the two categories of facets is the first and which is the second. In this case the transition polytope possesses two non-analogous regularizations. A flip of the type $m = \frac{d+1}{2}$ is remarkable in yet another respect: it does not change the h-vector at all (but, generally speaking, does change the combinatorial type of the polytope).

§ 4. Flips and the volume polynomial

In the present section, we describe how the volume polynomial and the polytope algebra change under a flip. A flip of order m adds a power of an affine functional with a sign depending on m to the volume polynomial. We explicitly describe how the annihilator ideal of the volume polynomial varies under a flip. The results proved in this section are needed in the proof of the analogue of the Hodge–Riemann relations, but they are possibly of interest in themselves.

4.1. Variation of the volume polynomial. Suppose that a simple polytope Q is obtained from a simple polytope P by a flip of order m. We retain the notation of § 3.2. The faces $\Gamma_1, \dots, \Gamma_{d+1}$ of the polytope Δ cut out some simplex S in the plane of P. This simplex is shown in Fig. 5 in the affine hull of P.

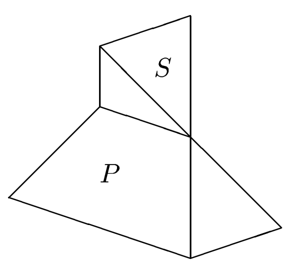

Figure 5. The polytope P and the simplex S

Lemma 4.1.1. *Let $\xi_1, \dots, \xi_{d+1}$ be the direction covectors of the facets $P \cap \Gamma_1, \dots, P \cap \Gamma_{d+1}$ of the polytope P. Then the direction covectors of the facets of the simplex S are as follows:*

$$\xi_1' = \xi_1, \ \dots, \ \xi_m' = \xi_m, \ \xi_{m+1}' = -\xi_{m+1}, \ \dots, \ \xi_{d+1}' = -\xi_{d+1}.$$

Accordingly, the support numbers $H'_1, \dots, H'_{d+1}$ *of* S *are related to the support numbers of* P *as follows*:

$$H'_1 = H_1,\ \dots,\ H'_m = H_m,\ H'_{m+1} = -H_{m+1},\ \dots,\ H'_{d+1} = -H_{d+1}.$$

Proof. We use the notation of § 3.2. Let $e_1, \dots, e_m$ be the direction vectors of the edges of Δ descending from O, and let $e_{m+1}, \dots, e_{d+1}$ be the direction vectors of the edges ascending from O. Then by choosing the lengths of the vectors $e_1, \dots, e_{d+1}$ appropriately (that is, by multiplying these vectors by appropriate non-zero coefficients), we can ensure that the points

$$O + e_1,\ \dots,\ O + e_m,\ O - e_{m+1},\ \dots,\ O - e_{d+1}$$

are the vertices of S (and hence lie in the plane of P). In the space $\mathbb{R}^{d+1}$ we introduce a Euclidean metric such that $(e_1, \dots, e_{d+1})$ is an oriented basis with respect to this metric. Then the simplex S consists of the vectors v that lie in the hyperplane of P and satisfy the inequalities

$$(v, e_1) \geqslant 0,\ \dots,\ (v, e_m) \geqslant 0,\ (v, e_{m+1}) \leqslant 0,\ \dots,\ (v, e_{d+1}) \leqslant 0.$$

For comparison we note that the vectors $w \in P$ satisfy the inequalities

$$(w, e_1) \geqslant 0,\ \dots,\ (w, e_m) \geqslant 0,\ (w, e_{m+1}) \geqslant 0,\ \dots,\ (w, e_{d+1}) \geqslant 0.$$

Now it remains only to note that the direction covectors $\xi_1, \dots, \xi_{d+1}$ of the facets of P are the orthogonal projections of the vectors $-e_1, \dots, -e_{d+1}$ on the plane of P (of course, we identify vectors with covectors using the metric).

Let P' be a polytope analogous to P. Then the simplex S' corresponding to S is well defined. S' is bounded by the facets of P' corresponding to the facets of P that bound S. Thus, associated with each polytope $P' \in \mathcal{P}(P)$ is the simplex $S' = S(P')$, and moreover, the map $P' \mapsto S(P')$ is linear:

$$S(P' + P'') = S(P') + S(P''), \qquad S(\lambda P') = \lambda S(P').$$

To every polytope $P' \in \mathcal{P}(P)$ we assign the volume of the corresponding simplex S'. On the cone of analogous polytopes we obtain a function which obviously has a unique extension to a polynomial on the space $\mathcal{P}^*(P)$. Let us denote this polynomial by V_S. By definition,

$$V_S(H_1, \dots, H_n) = \mathrm{Vol}(S),$$

where $H_1, \dots, H_n$ are the support numbers of P.

Now we note that if $1 < m < d$, then the direction covectors of the facets of P and Q coincide. Hence, we can identify elements of the spaces $\mathcal{P}^*(P)$ and $\mathcal{P}^*(Q)$ with the same support parameters (that is, the same coordinates). Thus we have identified the space $\mathcal{P}^*(P)$ with the space $\mathcal{P}^*(Q)$. However, this identification always takes an actual polytope to a virtual polytope.

If $m = 1$, then Q has one more facet than P. In this case, there is a natural epimorphism of $\mathcal{P}^*(Q)$ onto $\mathcal{P}^*(P)$, namely, the map 'forgetting' the support parameter corresponding to the extra facet of Q. This epimorphism enables one to transfer polynomials from the space $\mathcal{P}^*(P)$ to the space $\mathcal{P}^*(Q)$. In view of Proposition 2.5.3, by transferring the volume polynomial from $\mathcal{P}^*(P)$ to $\mathcal{P}^*(Q)$ we obtain an isomorphism of the corresponding algebras.

If $m = 1$, then, conversely, we can construct an epimorphism of $\mathcal{P}^*(P)$ onto $\mathcal{P}^*(Q)$, which enables one to transfer polynomials defined on the space $\mathcal{P}^*(Q)$ to the space $\mathcal{P}^*(P)$.

The above reasoning shows that if a polytope Q is obtained from a polytope P by a flip of order $0 < m < d + 1$, then the polynomials V_P, V_Q, and V_S can be viewed as polynomials defined on the same space.

Theorem 4.1.2. *Suppose that a simple polytope Q is obtained from a simple polytope P by a flip of order m. Then the volume polynomials of P and Q are related by the formula*

$$V_Q = V_P + (-1)^{d-m} V_S. \tag{4.1.1}$$

Remark. In the statement of the theorem we have used the above identification of the spaces $\mathcal{P}^*(P)$ and $\mathcal{P}^*(Q)$ (or the epimorphism of one of the spaces onto the other). We note that if the variables x are the support numbers of some *actual* polytope P' analogous to P, then the left-hand side of the above equation contains the volume of a *virtual* polytope in $\mathcal{P}^*(Q)$. Conversely, if the variables x are equal to the support numbers of an actual polytope analogous to Q, then the right-hand side of the equation contains the volumes of a virtual polytope P' analogous to P and of the virtual simplex $S' = S(P)$.

Proof of Theorem 4.1.2. We proceed by induction on $d - m$. If $d = m$, then Q is obtained from P by cutting off the simplex S. Naturally, the volume of Q is equal to the volume of P minus the volume of S. Now suppose that $d > m$. In this case we can reduce the dimension of d, thereby carrying out the inductive step.

We denote by S_i the face of S lying in the affine hull of the face $P \cap \Gamma_i$. By Theorem 3.4.1 and the inductive hypothesis, the following assertions hold:

$$\partial_i V_Q = \partial_i V_P + (-1)^{d-m} V_{S_i} \quad \text{for } i = 1, \dots, m; \tag{4.1.2}$$

$$\partial_i V_Q = \partial_i V_P + (-1)^{d-1-m} V_{S_i} \quad \text{for } i = m+1, \dots, d+1; \tag{4.1.3}$$

$$\partial_i V_Q = \partial_i V_P \quad \text{for the remaining } i. \tag{4.1.4}$$

(We assume that the argument x_i of the volume polynomial corresponds to the facet $P \cap \Gamma_i$.) Note that for $i = 1, \dots, m$ larger values of the coordinate x_i correspond to a larger simplex S, while for $i = m+1, \dots, d+1$ larger values of x_i correspond to smaller S.

By Lemma 4.1.1, the following assertions hold:

$$V_{S_i} = \partial_i V_S \quad \text{for } i = 1, \dots, m; \tag{4.1.5}$$

$$V_{S_i} = -\partial_i V_S \quad \text{for } i = m+1, \dots, d+1. \tag{4.1.6}$$

By substituting (4.1.5) and (4.1.6) into (4.1.2) and (4.1.3), respectively, we find that the derivative of the left-hand side of (4.1.1) with respect to any of the variables is equal to the corresponding derivative of the right-hand side of (4.1.1). But a homogeneous polynomial is uniquely determined by its first derivatives (Euler's theorem on homogeneous functions). Consequently, (4.1.1) holds.

4.2. The volume polynomial of a simplex. Now let us describe the polynomial V_S.

Proposition 4.2.1. *The polynomial V_S is equal to φ^d, where φ is a linear functional of the form*

$$\varphi(x) = a_1x_1 + \dots + a_mx_m - a_{m+1}x_{m+1} - \dots - a_{d+1}x_{d+1}.$$

The coefficients $a_1, \dots, a_{d+1}$ are positive. The variables $x_1, \dots, x_{d+1}$ are the coordinates in $\mathcal{P}^*(P)$ equal to the support parameters corresponding to the faces $P \cap \Gamma_1, \dots, P \cap \Gamma_{d+1}$ of the polytope P.

Proof. The polynomial V_S depends only on the variables $x_1, \dots, x_{d+1}$. We note that these variables coincide (up to a sign) with the support parameters on the space $\mathcal{P}^*(S)$. Hence, V_S can be viewed as the volume polynomial on the space of virtual simplices analogous to S. The only subtle point is that this volume polynomial is expressed via variables that are not the support parameters of a simplex but only coincide with these support parameters up to a sign.

We denote by $x'_1, \dots, x'_{d+1}$ the coordinates in $\mathcal{P}^*(S)$ having the meaning of the support parameters of the simplex S. Then by Lemma 4.1.1,

$$x_1 = x'_1,\ \dots,\ x_m = x'_m,\ x_{m+1} = -x'_{m+1},\ \dots,\ x_{d+1} = -x'_{d+1}.$$

Here it suffices to use Proposition 2.1.2.

Theorem 4.2.1 can now be rewritten as follows:

$$V_Q = V_P + (-1)^{d-m}\varphi^d. \tag{4.2.1}$$

This equation plays a key role in the subsequent argument.

Equation (4.2.1) is not new. For example, it can be found in [14].

The transition polytope T can be viewed as a virtual polytope in the space $\mathcal{P}^*(P)$ (as well as in $\mathcal{P}^*(Q)$). If $1 < m < d$, then the direction covectors of the facets of P and Q coincide. Thus, the spaces $\mathcal{P}^*(P)$ and $\mathcal{P}^*(Q)$ can be identified with the help of coordinates. The polytopes P, T, and Q can then be viewed as elements of the same vector space. Hence, on the same space we have the operators L_P, L_Q, and L_T of differentiation in the directions of the polytopes P, Q, and T, respectively.

Proposition 4.2.2. *The following relations hold*:

$$L_P\varphi > 0, \qquad L_T\varphi = 0, \qquad L_Q\varphi < 0.$$

Proof. This is almost obvious. The inequality $L_P\varphi > 0$ shows that the simplex S is actual. The number $L_P\varphi$ is a characteristic linear size of S. The equality $L_T\varphi = 0$

corresponds to the fact that at the transition instant the simplex collapses to a point. Finally, the inequality $L_Q\varphi < 0$ can be obtained from linearity considerations. Namely, we can assume that the polytope T lies on the segment joining P and Q in the space $\mathcal{P}^*(P) = \mathcal{P}^*(Q)$. The function φ is linear, assumes a positive value at one of the endpoints of this segment, and is zero at some interior point. Consequently, φ is negative at the other endpoint.

One can assign a meaning to Proposition 4.2.2 even for $m = 1$ or $m = d$.

For $m = 1$, the operator L_P is not uniquely determined. We consider volume polynomials on the space $\mathcal{P}^*(Q)$, which is epimorphically mapped onto the space $\mathcal{P}^*(P)$. Although the polytope P has many pre-images under this epimorphism, we must take one for which $L_P\varphi > 0$. Likewise, for $m = d$ the operator L_Q is determined by the condition $L_Q\varphi < 0$.

To prove Proposition 4.2.2 for $m = 1$ and $m = d$, it suffices to use linearity considerations.

Lemma 4.2.3. *Suppose that a linear functional φ and a linear differential operator L satisfy $L\varphi = 0$. Then*

$$L^r\varphi^s = 0$$

for any integers $r, s > 0$.

Proof. For $r > s$ the assertion is obvious. If $r \leqslant s$, then the differentiation formula for a power yields

$$L^r\varphi^s = \frac{s!}{(s-r)!}\,(L\varphi)^r\varphi^{s-r} = 0.$$

Corollary 4.2.4. *Let φ be the linear functional defined in Proposition* 4.2.1. *Then $L_T^r\varphi^s = 0$ if $r, s > 0$.*

4.3. The change of the polytope algebra under a flip. Let us now find out what happens to each component A_k of the polytope algebra under a flip. To this end it suffices to keep track of the variation of the component I_k of the ideal I with the help of Theorem 2.6.2. We use the notation from the preceding subsections.

Proposition 4.3.1. *If $k < m$, then $I_k(P) = I_k(Q)$.*

Proof. By Theorem 2.6.2, it suffices to verify that

$$P \cap \Gamma_{i_1} \cap \cdots \cap \Gamma_{i_k} = \varnothing \text{ if and only if } Q \cap \Gamma_{i_1} \cap \cdots \cap \Gamma_{i_k} = \varnothing.$$

Indeed, suppose, for example, that

$$P \cap \Gamma_{i_1} \cap \cdots \cap \Gamma_{i_k} = \varnothing. \tag{4.3.1}$$

Then certainly

$$P \cap \Gamma_{i_1} \cap \cdots \cap \Gamma_{i_k} \cap \Gamma_{m+1} \cap \cdots \cap \Gamma_{d+1} = \varnothing.$$

By regrouping the terms in the last equation, we see that the left-hand side is the intersection of several facets of the simplex S_P. However, an intersection of facets of a simplex can be empty only if *all* facets occur in the intersection. Thus for $k < m$ the equality (4.3.1) never holds. For the same reason, the equality

$$Q \cap \Gamma_{i_1} \cap \cdots \cap \Gamma_{i_k} = \varnothing \tag{4.3.2}$$

is also impossible. (Instead of the simplex S_P, here we must consider the simplex S_Q, which has larger dimension.)

For $k = m$ the equality (4.3.1) can hold. For example,

$$P \cap \Gamma_1 \cap \cdots \cap \Gamma_m = \varnothing.$$

This corresponds to the fact that the intersection of all the facets of the simplex S_P is empty. Thus, the differential operator

$$\sigma_k = \frac{\partial^m}{\partial x_1 \cdots \partial x_m} D_{k-m}$$

(where D_{k-m} is an arbitrary differential operator of degree $k - m$ with constant coefficients) lies in the component $I_k(P)$ of the ideal $I(P)$. We impose only one condition on the operator D_{k-m}:

$$D_{k-m}\varphi^{k-m} \neq 0.$$

Now we can describe what happens to the higher components $I_k(P)$.

Proposition 4.3.2. *If $m \leqslant k \leqslant d/2$, then $I_k(P) = I_k(Q) \oplus \langle \sigma_k \rangle$.*

Proof. By the above condition on D_{k-m} and by (4.2.1), the operator σ_k does not lie in $I_k(Q)$ but belongs to $I_k(P)$. Moreover, $I_k(Q)$ is a subspace of $I_k(P)$, since (4.3.2) cannot be realized for any $k < d-m$ (there cannot be extra generators in the space $I_k(Q)$). Thus,

$$I_k(P) \supset I_k(Q) \oplus \langle \sigma_k \rangle .$$

However, dimension considerations (see § 3.3) show that in fact equality holds.

§ 5. An analogue of the Hodge–Riemann form

In this section we prove analogues of the Hodge–Riemann relations and the Lefschetz decomposition in the algebra of a simple convex polytope (for the actual Hodge–Riemann relations and the Lefschetz decomposition in the cohomology of a compact Kähler manifold, see §§ 1.5 and 1.6).

5.1. An analogue of the Hodge–Riemann form. In this subsection we state the main results of the paper.

Definition. Let P be a simple convex d-polytope in $\mathbb{R}^d$. Let us consider a homogeneous differential operator α of degree k with constant coefficients. Let V_P be the volume polynomial on the space $\mathcal{P}^*(P)$, and let L_P be the operator of differentiation in the direction of P (see the definition in § 2.3). The operator α is said to be *primitive* (with respect to P) if

$$\alpha L_P^{d-2k+1} V_P = 0.$$

This condition means that some homogeneous polynomial of degree $k - 1$ is zero. Every element of the ideal $I(P)$ of P is primitive. Hence we can speak of primitive elements of the polytope algebra $A(P)$.

Definition. The *Hodge–Riemann form* on the homogeneous component $A_k(P)$, $k \leqslant d/2$, of the polytope algebra is the quadratic form

$$q(\alpha) = (-1)^k \alpha^2 L_P^{d-2k} V_P, \qquad \alpha \in A_k(P).$$

On the right-hand side, a homogeneous differential operator of degree d is applied to a homogeneous polynomial of degree d. The result is a number.

Theorem 5.1.1 (the Hodge–Riemann relations)**.** *The Hodge–Riemann form is positive definite on the primitive elements of the polytope algebra.*

The Hodge–Riemann relations can also be stated without using the polytope algebra.

Let α be a homogeneous differential operator of degree k with constant coefficients. If α is primitive with respect to P, then

$$q(\alpha) = (-1)^k \alpha^2 L_P^{d-2k} V_P \geqslant 0,$$

with equality only if $\alpha V_P = 0$.

Now we shall state a certain analogue of the Hodge–Riemann relations for transition polytopes. The volume polynomial (algebra) of a transition polytope is defined as the volume polynomial (algebra) of a regularization of it. If a transition polytope admits two non-analogous regularizations, then we shall associate with it two volume polynomials and two algebras.

Definition. Let T be a transition polytope and P a regularization of it. Let us consider a homogeneous differential operator α of degree k with constant coefficients. α is said to be *primitive* (with respect to T and P) if

$$\alpha L_T^{d-2k+1} V_P = 0.$$

We recall that the operator L_T is defined as follows:

$$L_T = \sum_{i=1}^{n} H_i(T) \frac{\partial}{\partial x_i},$$

where the $H_i(T) = \max(\xi_i, T)$ are the support numbers of T in the directions of the facets of P (the ξ_i are the direction covectors of these facets).

Definition. The *Hodge–Riemann form* on the homogeneous component $A_k(T) = A_k(P)$ of the algebra corresponding to a transition polytope T is the quadratic form

$$q(\alpha) = (-1)^k \alpha^2 L_T^{d-2k} V_P, \qquad \alpha \in A_k(P).$$

Theorem 5.1.2 (the Hodge–Riemann relations)**.** *The Hodge–Riemann form for a simple polytope or a transition polytope is positive definite on the primitive elements of the polytope algebra.*

If a transition polytope has two non-analogous regularizations, then the theorem holds for both of them.

5.2. Outline of the proof. We shall prove the Hodge–Riemann relations by induction on the dimension of the polytope. The inductive step has two stages. At the first stage, we use the Hodge–Riemann relations in dimension $d-1$ to derive the Lefschetz decomposition in dimension d. Specifically, the inductive hypothesis is used in the proof of the following crucial theorem (see § 5.5).

Hard Lefschetz Theorem. *Let P be a simple polytope or a transition polytope. Then the operator of multiplication by the element L_P^{d-2k} establishes an isomorphism between the spaces $A_k(P)$ and $A_{d-k}(P)$.*

The Lefschetz decomposition will be derived in § 5.3 from the hard Lefschetz theorem by standard, purely algebraic reasoning (indicated already in § 1.6).

Lefschetz Decomposition. *The following orthogonal direct sum decomposition holds*:

$$A_k(P) = \mathrm{Im}(L_P) \oplus \mathrm{Ker}(L_P^{d-2k+1}) \qquad (k > 0).$$

The orthogonality is understood in the sense of the Hodge–Riemann form.

Once the signature of the Hodge–Riemann form on each homogeneous component of the polytope algebra is known, the Lefschetz decomposition enables one to find the signature on each primitive subspace. Hence at the second stage it suffices to calculate the signature of the Hodge–Riemann form on each homogeneous component.

The simplex algebra does not contain any non-trivial primitive subspaces. All homogeneous components of the simplex algebra are one-dimensional. The Hodge–Riemann form on the homogeneous component of degree k has the sign $(-1)^k$.

Now we use Theorem 3.1.1. By this theorem, each simple polytope can be obtained from a simplex by a continuous deformation through finitely many flips.

Suppose first that the simple polytope P is somehow deformed within its class of analogous polytopes. In this case, the algebra $A(P)$ does not change at all, and the operator L_P varies continuously. Hence the primitive subspaces also vary continuously (this is due to the fact that the primitive subspaces corresponding to distinct operators L_P have the same dimension, which follows from the Lefschetz decomposition; see § 5.3). The Hodge–Riemann form also varies continuously and never degenerates (this follows from the hard Lefschetz theorem; see § 5.3). Hence, the signature of the Hodge–Riemann form is preserved under this deformation.

Next, the signature cannot change even if the polytope P does not stay within a single class of analogous polytopes but is deformed continuously so that its combinatorial type is preserved (that is, no flips occur). This case is also covered by Theorem 3.1.1. In our case the algebra $A_k(P)$ varies continuously in a certain sense. More precisely, for each k the component $I_k(P)$ varies continuously (as a finite-dimensional subspace of the finite-dimensional space Diff_k). We can readily assign a meaning to the assertion that each primitive component, together with the Hodge–Riemann form defined on it, also varies continuously. At least, it follows that the signature of the Hodge–Riemann form does not change for this deformation of the polytope as well.

Thus, we see that the signature of the Hodge–Riemann form can change only through a flip, and it remains to find out how the signature changes under a flip of order m. This is done in § 5.6. Here (4.2.1) is the key relation, which contains all information on the signature of the Hodge–Riemann form.

5.3. An analogue of the Lefschetz decomposition

Theorem 5.3.1 (the hard Lefschetz theorem). *Let P be a simple polytope or a transition polytope. Then the operator of multiplication by the element L_P^{d-2k} is an isomorphism between $A_k(P)$ and $A_{d-k}(P)$.*

We shall prove this theorem later, but now we derive some corollaries. The proof is based on the hard Lefschetz theorem in the same dimension as is indicated in the corollaries. Thus, the corollaries proved in the present subsection can be used in the inductive argument (in particular, they will be used in the proof of the hard Lefschetz theorem itself).

Corollary 5.3.2. *The restriction of the Hodge–Riemann form to each component $A_k(P)$ is non-degenerate.*

Proof. For each non-zero element $\alpha \in A_k(P)$, it suffices to find an element $\beta \in A_k(P)$ such that

$$\alpha\beta L_P^{d-2k} V_P \neq 0. \tag{5.3.1}$$

The element $\alpha \in A_k(P)$ can be viewed as a differential operator that does not annihilate the volume polynomial. Then $\alpha V_P(x)$ is a non-zero homogeneous polynomial of degree $d-k$. There is a differential operator γ of degree $d-k$ such that $\gamma\alpha V_P \neq 0$. By the hard Lefschetz theorem,

$$\gamma = L_P^{d-2k}\beta \pmod{I(P)},$$

where β is a differential operator of degree k obviously satisfying (5.3.1). The proof of Corollary 5.3.2 is complete.

Corollary 5.3.3 (the Lefschetz decomposition). *The following orthogonal direct sum decomposition holds*:

$$A_k(P) = \operatorname{Im}(L_P) \oplus \operatorname{Ker}(L_P^{d-2k+1}) \qquad (k>0).$$

The orthogonality is understood in the sense of the Hodge–Riemann form.

Proof. First let us prove that there is a direct sum decomposition. To this end, it suffices to note that the restriction of the operator L_P^{d-2k+1} to the subspace $\operatorname{Im}(L_P)$ is an isomorphism. This follows from the hard Lefschetz theorem applied to the operator

$$L_P^{d-2k+2}\colon A_{k-1}(P) \to A_{d-k+1}(P).$$

It remains to verify that the direct sum is orthogonal. Indeed, let

$$L_P\alpha \in \operatorname{Im}(L_P), \qquad \beta \in \operatorname{Ker}(L_P^{d-2k+1}).$$

The value of the Hodge–Riemann form on the chosen elements is equal to

$$(L_P\alpha)\beta L_P^{d-2k} V_P = \alpha(L_P^{d-2k+1}\beta) V_P = 0,$$

so that these elements are indeed orthogonal.

Corollary 5.3.4. *The dimension of the primitive subspace of the space $A_k(P)$ is equal to*

$$\dim \mathrm{Ker}(L_P^{d-2k+1}) = h_k - h_{k-1}.$$

In particular, this dimension is independent of the specific choice of P within its class of analogous polytopes. Corollary 5.3.4 readily follows from Corollary 5.3.3 and the hard Lefschetz theorem.

Corollary 5.3.5. *The restriction of the Hodge–Riemann form to each primitive subspace* $\mathrm{Ker}(L_P^{d-2k+1})$ *is non-degenerate.*

This readily follows from Corollaries 5.3.2 and 5.3.3.

5.4. Some corollaries. Now we shall derive some corollaries of the Hodge–Riemann relations. The proof is based on the Hodge–Riemann relations in the same dimension as is indicated in the corollaries (see the similar remark in the preceding subsection).

Let P and Q be simple convex d-polytopes in $\mathbb{R}^d$. Suppose that Q is obtained from P by a flip of order m. We denote by T the corresponding transition polytope. Let us consider a homogeneous differential operator α of degree k with constant coefficients.

Lemma 5.4.1. *If $k \leqslant d/2$, then $\alpha L_T^{d-2k+1} V_P = \alpha L_T^{d-2k+1} V_Q$.*

The proof follows from Lemma 4.2.3 and (4.2.1).

The following proposition is a version of the Hodge–Riemann relations for transition polytopes.

Proposition 5.4.2. *Suppose that a simple polytope Q is obtained from a simple polytope P by a flip of order $m \leqslant \frac{d+1}{2}$. We denote by T the corresponding transition polytope. Then*

$$(-1)^k \alpha^2 L_T^{d-2k} V_Q \geqslant 0$$

for each homogeneous differential operator α of degree $k < d/2$ with constant coefficients such that $\alpha L_T^{d-2k+1} V_Q = 0$.

Proof. Indeed, suppose that a differential operator α of degree k satisfies the condition

$$\alpha L_T^{d-2k+1} V_Q = 0.$$

Then by Lemma 5.4.1,

$$\alpha L_T^{d-2k+1} V_P = 0$$

(that is, the operator α is primitive with respect to the polytope T). Now we can use the standard Hodge–Riemann relations

$$(-1)^k \alpha^2 L_T^{d-2k} V_P \geqslant 0$$

(Theorem 5.1.1). However, we can prove the following equation in the same way as Lemma 5.4.1:

$$(-1)^k \alpha^2 L_T^{d-2k} V_P = (-1)^k \alpha^2 L_T^{d-2k} V_Q$$

(here it is essential that $k < d/2$). Thus, the theorem follows from the standard Hodge–Riemann relations.

Let us prove another version of the Hodge–Riemann relations for transition polytopes.

Proposition 5.4.3. *Suppose that a simple polytope Q is obtained from a simple polytope P by a flip of order m. We denote by T the corresponding transition polytope. Then*

$$(-1)^k\alpha^2 L_T^{d-2k} V_Q \geqslant 0$$

for each homogeneous differential operator α of degree $k = m$ such that $\alpha L_T^{d-2k+1} = 0$.

Proof. Let us consider the family of polytopes

$$Q_\lambda = (1-\lambda)Q + \lambda T.$$

If $\lambda < 1$, then the polytope Q_λ is analogous to the polytope Q. It is only at the very last instant $\lambda = 1$ that the polytope Q_λ degenerates and coincides with the transition polytope T. For this family of polytopes, we consider the corresponding family of Hodge–Riemann forms

$$q_\lambda(\alpha) = (-1)^k \alpha^2 L_{Q_\lambda}^{d-2k} V_{Q_\lambda}.$$

Up to the very last instant $\lambda = 1$, all these forms are non-degenerate and positive definite. However, each form q_λ is defined on its own primitive subspace. If we can prove that all primitive subspaces (including the one corresponding to the very last instant) have the same dimension, then it follows (using continuity arguments) that the limit form $q_1(\alpha) = (-1)^k\alpha^2 L_T^{d-2k} V_Q$ is positive semidefinite on the last primitive space

$$\Lambda_k^Q = \bigl\{\alpha \in A_k(Q) \mid \alpha L_T^{d-2k+1} V_Q = 0\bigr\}.$$

We can find the dimension of the primitive space

$$\Lambda_k^P = \bigl\{\alpha \in A_k(P) \mid \alpha L_T^{d-2k+1} V_P = 0\bigr\}$$

using Corollary 5.3.4:

$$\dim(\Lambda_k^P) = h_k(P) - h_{k-1}(P).$$

Let us consider the space

$$\Lambda_k = \bigl\{\alpha \in \mathrm{Diff}_k \mid \alpha L_T^{d-2k+1} V_Q = \alpha L_T^{d-2k+1} V_P = 0\bigr\}.$$

We note that the following isomorphisms hold:

$$\Lambda_k^P = \frac{\Lambda_k}{I_k(P)}, \qquad \Lambda_k^Q = \frac{\Lambda_k}{I_k(Q)}.$$

Thus,

$$\begin{aligned}\dim(\Lambda_k^Q) &= \dim(\Lambda_k^P) + \dim(I_k(P)) - \dim(I_k(Q)) \\ &= h_k(P) - h_{k-1}(P) + h_k(Q) - h_k(P) = h_k(Q) - h_{k-1}(P).\end{aligned}$$

Now we use the condition $k = m$. This implies that $h_{k-1}(P) = h_{k-1}(Q)$. Hence, $\dim(\Lambda_k^Q) = h_k(Q) - h_{k-1}(Q)$.

Corollary 5.4.4. *Suppose that a polytope Q is obtained from a polytope P by a flip of order $m = d/2$. Then*

$$(-1)^k \alpha^2 L_T^{d-2k} V_Q \geqslant 0$$

for each homogeneous differential operator α of degree k with constant coefficients such that $\alpha L_T^{d-2k+1} V_Q = 0$.

Proof. This follows from Proposition 5.4.2 for $k < d/2$ and from Proposition 5.4.3 for $k = d/2$.

5.5. The hard Lefschetz theorem. In this subsection we prove the hard Lefschetz Theorem 5.3.1. First we shall give a proof for the case of a simple polytope. In the case of a transition polytope, some additional considerations will be needed.

The spaces $A_k(P)$ and $A_{d-k}(P)$ have the same dimension. Consequently, it suffices to prove that the map

$$L_P^{d-2k} \colon A_k(P) \to A_{d-k}(P)$$

is injective.

Let α be a homogeneous differential operator of degree k with constant coefficients such that

$$\alpha L_P^{d-2k} V_P = 0. \tag{5.5.1}$$

We must prove that in this case $\alpha V_P = 0$. Let us apply the operator ∂_i to both sides of (5.5.1):

$$\alpha L_P^{d-2k} \widetilde{V}_{P_i} = 0. \tag{5.5.2}$$

Let V_{P_i} be the volume polynomial of the facet P_i considered in its convex hull. By Proposition 2.5.3, we can identify the algebras $A(\widetilde{V}_{P_i})$ and $A(V_{P_i})$. Then the operator L_P corresponds to the operator L_{P_i}.

The dimension of P_i is one less than that of P. Hence, (5.5.2) implies that the operator α is primitive with respect to the polytope P_i. But then, by the inductive hypothesis, we have the Hodge–Riemann relation

$$(-1)^k \alpha^2 L_{P_i}^{d-1-2k} \widetilde{V}_{P_i} \geqslant 0. \tag{5.5.3}$$

Now we note that

$$L_P^{d-2k} V_P = \sum_{i=1}^{n} H_i(P) L_P^{d-1-2k} \frac{\partial}{\partial x_i} V_P = \sum_{i=1}^{n} H_i(P) L_P^{d-1-2k} \widetilde{V}_{P_i}.$$

It follows that the Hodge–Riemann form for the polytope P can be obtained by summing the left-hand sides of (5.5.3) with the coefficients $H_i(P)$. However, we can always suppose that the origin lies inside the polytope P, so that all the $H_i(P)$ are positive. Thus, by summing the inequalities (5.5.3) with positive coefficients, we obtain the inequality

$$(-1)^k \alpha^2 L_P^{d-2k} V_P \geqslant 0.$$

However, this inequality is actually an equality (by virtue of (5.5.1)). Thus, equality holds in all the inequalities (5.5.3). But then, by the inductive hypothesis,

$$\alpha\widetilde{V}_{P_i} = \alpha\frac{\partial}{\partial x_i}V_P = 0 \qquad (1 \leqslant i \leqslant n).$$

The polynomial $\alpha V_P(x)$ is homogeneous of degree $d-k$. Hence, by Euler's theorem on homogeneous functions,

$$\alpha V_P = \frac{1}{d-k}\sum_{i=1}^{n}\frac{\partial}{\partial x_i}\alpha V_P = \frac{1}{d-k}\sum_{i=1}^{n}\alpha\frac{\partial}{\partial x_i}V_P = 0,$$

as desired.

We now turn to the proof of the hard Lefschetz theorem for a transition polytope T. First, suppose that the index of T satisfies the inequality $m < \frac{d+1}{2}$. Then the above proof remains valid word for word. We need only bear in mind that each facet of the regularization $\operatorname{reg} T$ is a regularization of the corresponding facet of T.

Now let us consider the exceptional case $m = \frac{d+1}{2}$. In this case, T has two volume polynomials V_P and V_Q corresponding to distinct regularizations P and Q. Accordingly, the polytope T has two distinct algebras $A(P)$ and $A(Q)$. Let us prove the hard Lefschetz theorem for $A(P)$. Suppose that

$$\alpha L_P^{d-2k}V_P = 0.$$

The case $k = d/2$ is obvious (in this case we are dealing with the identity map). Hence we restrict our considerations to the case $k < d/2$. Let us apply the operator αL_T^{d-2k} to both sides of (4.2.1) and use Lemma 4.2.3. We obtain

$$\alpha L_T^{d-2k}(V_Q) = \alpha L_T^{d-2k}(V_P) = 0.$$

By Theorem 5.1.1 and Corollary 5.4.4, the following non-strict inequalities hold:

$$(-1)^k\alpha^2 L_T^{d-1-2k}\widetilde{V}_{P_i} \geqslant 0, \tag{5.5.4}$$

$$(-1)^k\alpha^2 L_T^{d-1-2k}\widetilde{V}_{Q_i} \geqslant 0. \tag{5.5.5}$$

These correspond to the inequalities (5.5.3) for the case of a simple polytope. Needless to say, we cannot claim that the quadratic forms on the left-hand sides in (5.5.4) and (5.5.5) are non-degenerate. In complete analogy with the proof for a simple polytope, we conclude that both inequalities (5.5.4) and (5.5.5) are in fact equalities:

$$(-1)^k\alpha^2 L_T^{d-1-2k}\widetilde{V}_{P_i} = 0, \tag{5.5.6}$$

$$(-1)^k\alpha^2 L_T^{d-1-2k}\widetilde{V}_{Q_i} = 0. \tag{5.5.7}$$

Equations (5.5.6) and (5.5.7) reflect the fact that the inequality (5.5.3) turns into an equality. We note that the regularization of the facet T_i of the polytope T is either P_i or Q_i. Consequently,

$$\text{either}\quad \alpha\widetilde{V}_{P_i} = 0, \tag{5.5.8}$$

$$\text{or}\quad \alpha\widetilde{V}_{Q_i} = 0 \tag{5.5.9}$$

for each i. Suppose that $k<\frac{d-1}{2}$. Then $A_k(P_i)$ is one-dimensional, and hence (5.5.8) and (5.5.9) are equivalent. But then they are both valid (since we know that at least one of them holds). Next, by analogy with the proof for a simple polytope, we conclude that $\alpha V_P = 0$ and $\alpha V_Q = 0$, that is, the hard Lefschetz theorem holds. It remains to consider the case $k = \frac{d-1}{2}$. In this case both quadratic forms on the left-hand sides in (5.5.6) and (5.5.7) are non-degenerate. The point is that there is no operator L in these forms. But then (5.5.6) implies (5.5.8), and (5.5.7) implies (5.5.9), since (5.5.8) and (5.5.9) are again valid simultaneously. This completes the proof of the hard Lefschetz theorem.

5.6. The signature of the Hodge–Riemann form

Definition. Let q be a quadratic form on a finite-dimensional vector space. Suppose that, in some basis,

$$q(\alpha) = \alpha_1^2 + \cdots + \alpha_a^2 - \alpha_{a+1}^2 - \cdots - \alpha_{a+b}^2,$$

where $\alpha_1, \dots, \alpha_{a+b}, \dots, \alpha_N$ are the coordinates with respect to that basis. Then the difference $a - b$ is called the signature of the quadratic form q.

Let us find out how the signature of the Hodge–Riemann form changes under a flip.

Proposition 5.6.1. *A flip of order m preserves the signature of the Hodge–Riemann form on the homogeneous components $A_k(P)$, $k < m$, of the algebra of a simple convex polytope.*

Proof. Let us apply the differential operator $(-1)^k \alpha^2 L_T^{d-2k}$, $\alpha \in A_k(P)$, to both sides of (4.2.1):

$$(-1)^k \alpha^2 L_T^{d-2k} V_Q = (-1)^k \alpha^2 L_T^{d-2k} V_P. \tag{5.6.1}$$

(Since $k < m \leqslant (d+1)/2$, we see that $k \leqslant m-1 \leqslant (d-1)/2$ and hence $2k < d$. It follows from Corollary 4.2.4 that $L_T^{d-2k}\varphi^d = 0$.)

We note that the form on the left-hand side has the same signature as the Hodge–Riemann form for the polytope P. To verify this, we consider the deformation of P into T. Then the Hodge–Riemann form of P is deformed into the Hodge–Riemann form of T, that is, the form on the right-hand side in (5.6.1). In the course of the deformation, the component A_k does not vary at all, while the operator L varies continuously. Thus, we obtain a one-parameter family of Hodge–Riemann forms depending continuously on the parameter. By Corollary 5.3.2, all these forms are non-degenerate. Hence, their signatures all coincide. Since the form on the right-hand side of (5.6.1) is non-degenerate, it follows that so is the form on the left-hand side. But then its signature (by the above considerations) coincides with the signature of the Hodge–Riemann form for Q. Thus, the Hodge–Riemann forms for P and Q have the same signature, as desired.

Corollary 5.6.2. *A flip of order $m > d/2$ preserves the signature of the Hodge–Riemann form on the component $A_k(P)$, $k > d - m$.*

Proposition 5.6.3. *A flip of order m adds the number $(-1)^{k-m}$ to the signature of the Hodge–Riemann form on the component $A_k(P)$, $m \leqslant k \leqslant d/2$.*

In the proof of this assertion, we use the following simple general fact from linear algebra.

Lemma 5.6.4. *Suppose that three quadratic forms q, q_1, and q_2 are defined on a finite-dimensional linear space, and moreover,*

$$q = q_1 + q_2.$$

Suppose that the rank of q_2 coincides with the dimension of the kernel of q_1 and the form q_2 itself is non-degenerate on this kernel. Let a, a_1, and a_2 be the signatures of the forms q, q_1, and q_2, respectively. Then

$$a = a_1 + a_2.$$

Proof of Proposition 5.6.3. Let us now apply the operator $(-1)^k\alpha^2L_P^{d-2k}$ to both sides of (4.2.1). We obtain

$$q^P(\alpha) = q_1^P(\alpha) + q_2^P(\alpha), \tag{5.6.2}$$

where

$$\begin{aligned} q^P(\alpha) &= (-1)^k\alpha^2L_P^{d-2k}V_Q, \\ q_1^P(\alpha) &= (-1)^k\alpha^2L_P^{d-2k}V_P, \\ q_2^P(\alpha) &= (-1)^k\alpha^2L_P^{d-2k}((-1)^{d-m}\varphi^d). \end{aligned}$$

We can assume that $\alpha \in A_k(Q)$. Indeed, by Proposition 4.3.2 the space $A_k(P)$ is a quotient space of $A_k(Q)$.

We note that the form q_1^P, restricted to the space $A_k(Q)$, has a one-dimensional kernel generated by the element σ_k. Indeed, let us take the quotient of $A_k(Q)$ by this element. We obtain the space $A_k(P)$, on which the form q_1^P is non-degenerate (by Corollary 5.3.2).

The form q_2^P is of rank 1 and is non-degenerate on the line generated by σ_k. By Lemma 5.6.4, the signature of the form q^P is equal to the sum of the signatures of the forms q_1^P and q_2^P.

Let us deform P into Q. After the deformation, (5.6.2) becomes

$$q^Q(\alpha) = q_1^Q(\alpha) + q_2^Q(\alpha), \tag{5.6.3}$$

where

$$\begin{aligned} q^Q(\alpha) &= (-1)^k\alpha^2L_Q^{d-2k}V_Q, \\ q_1^Q(\alpha) &= (-1)^k\alpha^2L_Q^{d-2k}V_P, \\ q_2^Q(\alpha) &= (-1)^k\alpha^2L_Q^{d-2k}((-1)^{d-m}\varphi^d). \end{aligned}$$

In the course of deformation, the kernel of q_1 remains unchanged. The form q_2 becomes degenerate at the transition instant and changes sign in general. However, after the flip it is again non-degenerate. Thus, the signature of q^Q is obtained from the signature of q_1^Q by adding the sign of the number $q_2^Q(\sigma_k)$.

It remains only to find this sign. We have

$$q_2^Q(\sigma_k) = (-1)^k \sigma_k^2 L_Q^{d-2k}((-1)^{d-m}\varphi^d) = (-1)^{k-m} \frac{d!}{(d-2k)!} (-L_Q\varphi)^{d-2k} \sigma_k^2(\varphi^{2k}).$$

This sign of the latter expression is $(-1)^{k-m}$. Indeed, $-L_Q\varphi > 0$ by Proposition 4.2.2, and $\sigma_k^2(\varphi^{2k}) = \binom{2k}{k}(\sigma_k\varphi^k)^2$.

Corollary 5.6.5. *A flip of order $d+1-m$, $m \leqslant k \leqslant d/2$, decreases the signature of the Hodge–Riemann form on the homogeneous component A_k of the polytope algebra by $(-1)^{k-m}$.*

Let

$$g_k = h_k - h_{k-1}.$$

These numbers are just the dimensions of the primitive subspaces. Using the above propositions, we can readily find the signature of the Hodge–Riemann form.

Theorem 5.6.6. *The signature of the Hodge–Riemann form on the homogeneous component $A_k(P)$ of the polytope algebra corresponding to a simple or transition polytope P is equal to*

$$s_k = g_k - g_{k-1} + g_{k-2} - g_{k-3} + \cdots.$$

Now we can calculate the signature of the Hodge–Riemann form restricted to the primitive subspace $\operatorname{Ker}(L_P^{d-2k+1})$ of the homogeneous component $A_k(P)$ of the polytope algebra of P. Let q_k be the Hodge–Riemann form on the homogeneous component $A_k(P)$, and let α be an arbitrary element of $A_k(P)$. By Corollary 5.3.3, we have the orthogonal Lefschetz decomposition $\alpha = \beta + L_P\gamma$, where the element β belongs to the primitive subspace. Let us multiply both sides of this equation by $(-1)^k L_P^{d-2k}$ and apply them to the volume polynomial. As a result, we find that $q_k(\alpha) = q_k(\beta) - q_{k-1}(\gamma)$.

Hence, the form $\beta \mapsto q_k(\beta)$ (which is just the restriction of the form q_k to the primitive subspace of the homogeneous component $A_k(P)$ of the polytope algebra) has signature

$$s_k + s_{k-1} = g_k.$$

Thus, the Hodge–Riemann form is positive definite on the primitive subspace in each homogeneous component, as required.

5.7. The mixed Hodge–Riemann relations

Definition. Let P be a simple convex polytope, and let $P_1, \dots, P_{d-2k}$ be simple polytopes analogous to P. Let us consider a homogeneous differential operator α of degree k with constant coefficients. The operator α is said to be *primitive* (with respect to the system of polytopes $P, P_1, \dots, P_{d-2k}$) if

$$\alpha L_P \Omega(V_P) = 0, \quad \text{where} \quad \Omega = L_{P_1} \cdots L_{P_{d-2k}}.$$

Each element of the ideal $I(P)$ is primitive. Hence we can speak of primitive elements of the algebra $A(P)$.

Definition. The mixed Hodge–Riemann form on the homogeneous component $A_k(P)$, $k \leqslant d/2$, of the polytope algebra is defined to be the quadratic form

$$q(\alpha) = (-1)^k \alpha^2 \Omega(V_P), \qquad \alpha \in A_k(P).$$

Now we can state a mixed analogue of the Hodge–Riemann bilinear relations for simple polytopes.

Theorem 5.7.1. *The mixed Hodge–Riemann form is positive definite on the primitive elements of the polytope algebra.*

The mixed relations can be proved as follows: first, one establishes the mixed version of the hard Lefschetz theorem (in exactly the same way as the usual version), then formal corollaries of this theorem (quite similar to the corollaries in § 5.3) are proved, and the mixed Hodge–Riemann form can be deformed to the usual Hodge–Riemann form. Moreover, by an analogue of Corollary 5.3.2, the form remains non-degenerate in the course of the deformation. Consequently, the mixed Hodge–Riemann form has the same signature as the usual form. Now we can find the signs using the mixed Lefschetz decomposition and hence establish the mixed Hodge–Riemann relations. We point out that these relations are a generalization of the classical Aleksandrov–Fenchel inequalities for polytopes.

Corollary 5.7.2. *Let $P_1, \dots, P_d$ be simple analogous polytopes. Then the following inequality holds for the mixed volumes*:

$$\mathrm{Vol}(P_1, P_2, \dots, P_d)^2 \geqslant \mathrm{Vol}(P_1, P_1, \dots, P_d) \cdot \mathrm{Vol}(P_2, P_2, \dots, P_d).$$

The Aleksandrov–Fenchel inequalities follow from the fact that the positive inertia index of the mixed Hodge–Riemann form on the component A_1 is equal to 1. For such quadratic forms, the reverse Cauchy–Schwarz–Bunyakovskii inequality holds in the interior of the isotropic cone. The reverse Cauchy–Schwarz–Bunyakovskii inequality for the mixed Hodge–Riemann form coincides with the Aleksandrov–Fenchel inequalities.

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