On A. D. Aleksandrov's Inequalities for Mixed Discriminants

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Introduction. Let μ be a real symmetric matrix of order n and define $P_k(\mu)$ by means of the equation

$$\det(\delta + x\mu) = \sum_{k=0}^{n} \binom{n}{k} P_k(\mu) x^k,$$

where δ denotes the unit matrix and x is a parameter. $P_k(\mu)$ is thus a homogeneous polynomial of degree k in the elements of the matrix μ. If $\mu_1, \cdots, \mu_k$ are positive definite symmetric matrices, and if $P_k(\mu_1, \cdots, \mu_k)$ denotes the completely polarized form of the polynomial $P_k(\mu)$, then the following inequality holds:

$$P_k^k(\mu_1, \cdots, \mu_k) \geqq P_k(\mu_1) \cdots P_k(\mu_k). \tag{1}$$

There is strict inequality unless $\mu_1, \cdots, \mu_k$ are pairwise proportional.

This algebraic inequality has been used by Chern [3] in proving uniqueness theorems of differential geometry. In [3], the inequality (1) is deduced from a more general one for hyperbolic polynomials by Gårding [4], and it is asked there for an elementary proof of this particular case. Now, (1) can be derived from A. D. Aleksandrov's [1] inequalities for mixed discriminants (see also Busemann [2], pp. 51–56), and for these inequalities we shall give here a new proof, which seems to be a little more simple than Aleksandrov's original one, and which leads to a slightly more general result.

We mention that also Aleksandrov's general inequalities have proved useful in geometry; so Aleksandrov [1] himself used them in his second proof of Fenchel's inequalities for the mixed volumes of convex bodies. Another application is given by Petty [7].

The inequalities for mixed discriminants. Let $\mu_1, \cdots, \mu_m$ be real symmetric matrices of order n. By means of the equation

$$\det(x_1\mu_1 + \cdots + x_m\mu_m) = \sum x_{i_1} \cdots x_{i_n} D(\mu_{i_1}, \cdots', \mu_{i_n}), \tag{2}$$

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(with real parameters $x_1, \cdots, x_m$), where the indices $i_1, \cdots, i_n$ run independently from 1 to m, and where the coefficient of $x_{i_1} \cdots x_{i_n}$ is required to be symmetric in $i_1, \cdots, i_n$, the numbers $D(\mu_{i_1}, \cdots, \mu_{i_n})$ are defined. $D(\mu_{i_1}, \cdots, \mu_{i_n})$ (which depends only on $\mu_{i_1}, \cdots, \mu_{i_n}$) is called the mixed discriminant of the quadratic forms of which $\mu_{i_1}, \cdots, \mu_{i_n}$ are the coefficient matrices.

Theorem. *Let $\mu_1, \cdots, \mu_{n-r}, \mu$ $(1 \leqq r \leqq n-1)$ be real symmetric matrices of order n, the first $n-r$ of which are positive definite. Then*

$$D^2(\mu_1, \cdots, \mu_{n-r}, \underbrace{\mu, \cdots, \mu}_{r})$$

$$\geqq D(\mu_1, \cdots, \mu_{n-r}, \mu_{n-r}, \underbrace{\mu, \cdots, \mu}_{r-1}) D(\mu_1, \cdots, \mu_{n-r-1}, \underbrace{\mu, \cdots, \mu}_{r+1}),$$

where equality holds only if $\mu = c\mu_{n-r}$ with a real number c.

For $r = 1$, this is Aleksandrov's inequality.

Proof of the theorem. If there are given m different matrices $\mu_1, \cdots, \mu_m$, we write

$$D(\underbrace{\mu_1, \cdots, \mu_1}_{r_1}, \underbrace{\mu_2, \cdots, \mu_2}_{r_2}, \cdots, \underbrace{\mu_m, \cdots, \mu_m}_{r_m}) = D^{(m)}_{r_1 r_2 \cdots r_m}.$$

Then (2) can be written in the form

$$\det(x_1\mu_1 + \cdots + x_m\mu_m) = \sum \frac{n!}{r_1! \cdots r_m!} D^{(m)}_{r_1 \cdots r_m} x_1^{r_1} \cdots x_m^{r_m}, \tag{3}$$

where $0 \leqq r_i \leqq n$, $i = 1, \cdots, m$, and $r_1 + \cdots + r_m = n$.

We have to prove that for positive definite $\mu_1, \cdots, \mu_{m-1}$ $(m \geqq 2)$ and for $r_{m-1} \geqq 1$, $r_m \geqq 1$,

$$(D^{(m)}_{r_1 \cdots r_m})^2 \geqq D^{(m)}_{r_1 \cdots r_{m-2}, r_{m-1}+1, r_m-1} D^{(m)}_{r_1 \cdots r_{m-2}, r_{m-1}-1, r_m+1}.$$

Together with this inequality, we prove the following assertion:

(4) If μ_m is positive definite, too, all the numbers $D^{(m)}_{r_1 \cdots r_m}$ are positive.

We proceed to the proof by induction with respect to m, the number of given matrices. First let $m = 2$. The roots of the polynomial

$$f(x) = \det(x\mu_1 + \mu_2) = \sum_{i=0}^{n} \binom{n}{i} D^{(2)}_{i,n-i} x^i \tag{5}$$

are the negative eigenvalues of the symmetric matrix μ_2 relative to the symmetric positive definite matrix μ_1, hence all of them are real. From this it follows that

$$(D^{(2)}_{i,n-i})^2 \geqq D^{(2)}_{i+1,n-i-1} D^{(2)}_{i-1,n-i+1}, \qquad 1 \leqq i \leqq n-1,$$

with equality for any i only if all the roots of $f(x)$ are equal (see Hardy–Littlewood–Polya [5], p. 104). In the case of equality, the matrix μ_2 has an n-fold eigenvalue relative to μ_1, therefore μ_1 and μ_2 are proportional. Thus the theorem is true for $m = 2$.

In order to prove (4) for $m = 2$, we observe that all the eigenvalues of μ_2 relative to μ_1 are positive if μ_2 is positive definite; hence in this case the numbers $D^{(2)}_{i,n-i}$, which are, according to (5), the elementary symmetric functions of these eigenvalues, are positive.

Now assume that the theorem and the assertion (4) hold for an $m \geqq 2$. Let $\mu_1, \cdots, \mu_{m+1}$ be given real symmetric matrices of order n, the first m of them being positive definite. If we write $\bar{\mu} = y\mu_m + \mu_{m+1}$, where y is a real parameter, we have

$$(6) \qquad \det(x_1\mu_1 + \cdots + x_{m-1}\mu_{m-1} + x_m\bar{\mu}) = \sum \frac{n!}{r_1! \cdots r_m!} D^{(m)}_{r_1 \cdots r_m}(y) x_1^{r_1} \cdots x_m^{r_m},$$

where the coefficients

$$D^{(m)}_{r_1 \cdots r_m}(y) = D(\underbrace{\mu_1, \cdots, \mu_1}_{r_1}, \cdots, \underbrace{\mu_{m-1}, \cdots, \mu_{m-1}}_{r_{m-1}}, \underbrace{\bar{\mu}, \cdots, \bar{\mu}}_{r_m})$$

depend on y. The left-hand side of (6) can also be written in the form

$$\det(x_1\mu_1 + \cdots + x_{m-1}\mu_{m-1} + x_m y\mu_m + x_m\mu_{m+1}) = \sum \frac{n!}{r_1! \cdots r_{m-1}!\, i!\, (r_m - i)!} D^{(m+1)}_{r_1 \cdots r_{m-1}, i, r_m - i} y^i x_1^{r_1} \cdots x_m^{r_m}.$$

Comparing the coefficients of $x_1^{r_1} \cdots x_m^{r_m}$ and writing $r_m = k$, we get

$$D^{(m)}_{r_1 \cdots r_{m-1}, k}(y) = \sum_{i=0}^{k} \binom{k}{i} D^{(m+1)}_{r_1 \cdots r_{m-1}, i, k-i} y^i.$$

This equation holds for each $(m - 1)$-tuple $r_1, \cdots, r_{m-1}$ with $r_1 + \cdots + r_{m-1} \leqq n$, $k = n - r_1 - \cdots - r_{m-1}$, and for all real y. We want to show that, if μ_{m+1} and μ_m are not proportional, each polynomial $D^{(m)}_{r_1 \cdots r_{m-1}, k}(y)$ has all its roots real and simple. This is trivial if $r_1 + \cdots + r_{m-2} = n$ or $n - 1$, so we assume $r_1 + \cdots + r_{m-2} \leqq n - 2$. By R we denote the (in the following fixed) $(m - 2)$-tuple $r_1, \cdots, r_{m-2}$, and by s the difference $s = n - (r_1 + \cdots + r_{m-2})$. Furthermore, we write $D^{(m)}_{R, s-k, k}(y) = Q_k(y)$.

Since the theorem is assumed true for m, we have

$$(7) \qquad (D^{(m)}_{R, s-k, k})^2 \geqq D^{(m)}_{R, s-k+1, k-1} D^{(m)}_{R, s-k-1, k+1}$$

or

$$Q_k^2(y) \geqq Q_{k-1}(y) Q_{k+1}(y)$$

for $1 \leqq k \leqq s - 1$ and all real y.

Now we suppose that the matrices μ_{m+1} and μ_m are not proportional. Then it is easy to see that there is strict inequality in (7) if y is a zero of Q_k. Indeed, if we have

$$0 = Q_k(y_0) = Q_{k-1}(y_0)Q_{k+1}(y_0),$$

then by the inductive assumption we have

$$\bar{\mu} = y_0\mu_m + \mu_{m+1} = c\mu_{m-1} .$$

Here $c \neq 0$, since μ_m and μ_{m+1} are assumed to be not proportional. Since μ_{m-1} is positive definite, the matrix $\bar{\mu}$ is definite, too. If $c > 0$, $\bar{\mu}$ is positive definite, hence all the numbers $D^{(m)}_{R,s-k,k}(y_0)$ must be positive, since the assertion (4) is assumed true for m. If $c < 0$, the matrix $-\bar{\mu}$ is positive definite, thus (6) and (4) yield that $(-1)^k D^{(m)}_{R,s-k,k}(y_0)$ must be positive. But both cases contradict

$$D^{(m)}_{R,s-k,k}(y_0) = Q_k(y_0) = 0.$$

Next we show that in each of the polynomials $Q_k(y)$, the highest coefficient $D^{(m+1)}_{R,s-k,k,0}$ is positive: Developing det $(x_1\mu_1 + \cdots + x_m\mu_m + x_{m+1}\mu_{m+1})$ and putting $x_{m+1} = 0$, we get

$$D^{(m+1)}_{R,s-k,k,0} = D^{(m)}_{R,s-k,k} ,$$

which is positive since the assertion (4) holds for the m positive definite matrices $\mu_1 , \cdots , \mu_m$.

Now we can prove:

Lemma. *The polynomial $Q_k(y)$ has k different real roots* $(1 \leqq k \leqq s)$. *The roots of $Q_{k-1}(y)$ separate the roots of $Q_k(y)$* $(2 \leqq k \leqq s)$.

Proof by induction: $Q_1(y)$ is linear and not a constant, because the highest coefficient is positive. Let $q^{(1)}$ denote the zero of $Q_1(y)$. The inequality (7) together with the subsequent remark on the impossibility of the equality sign gives

$$0 = Q_1^2(q^{(1)}) > Q_0(q^{(1)})Q_2(q^{(1)}).$$

Q_0 is a positive constant, hence $Q_2(q^{(1)}) < 0$. Since the highest coefficient of $Q_2(y)$ is positive, $Q_2(y)$ has therefore two different real roots $q_1^{(2)} < q_2^{(2)}$, which are separated by the root of $Q_1(y)$.

Assume now that the lemma is true for some k and $k - 1$, where $2 \leqq k \leqq s - 1$ (we assume $s > 2$; if $s = 2$, the proof is already complete). Let $q_i^{(k-1)}$ and $q_i^{(k)}$ denote the roots of $Q_{k-1}(y)$ and $Q_k(y)$, respectively, so that by the inductive hypothesis we have

$$q_1^{(k)} < q_1^{(k-1)} < q_2^{(k)} < q_2^{(k-1)} \cdots < q_{k-1}^{(k-1)} < q_k^{(k)} .$$

Write

$$q_0^{(k-1)} = -\infty, \qquad q_k^{(k-1)} = \infty .$$

Since the highest coefficient of $Q_{k-1}(y)$ is positive, we see that

(8) $$\operatorname{sgn} Q_{k-1}(y) = (-1)^{r+k+1} \quad \text{for} \quad y \in (q_r^{(k-1)}, q_{r+1}^{(k-1)})$$

for $0 \leqq r \leqq k-1$. Applying (7), we get

$$Q_{k-1}(q_i^{(k)})Q_{k+1}(q_i^{(k)}) < Q_k^2(q_i^{(k)}) = 0, \qquad 1 \leqq i \leqq k.$$

Because of

$$q_i^{(k)} \in (q_{i-1}^{(k-1)}, q_i^{(k-1)}),$$

(8) leads to

$$\operatorname{sgn} Q_{k-1}(q_i^{(k)}) = (-1)^{i+k},$$

hence

$$\operatorname{sgn} Q_{k+1}(q_i^{(k)}) = (-1)^{i+k+1}.$$

Furthermore, for sufficiently small y, we have

$$\operatorname{sgn} Q_{k+1}(y) = (-1)^{k+1},$$

and for sufficiently large y,

$$\operatorname{sgn} Q_{k+1}(y) = 1.$$

From these facts we conclude that in $(-\infty, \infty)$ the polynomial $Q_{k+1}(y)$ changes sign $k+1$ times, and hence has $k+1$ different real roots, which are separated by the roots of $Q_k(y)$. Thus the lemma is proved.

Now the fact that the polynomial

$$Q_k(y) = D_{R,s-k,k}^{(m)}(y) = \sum_{i=0}^{k} \binom{k}{i} D_{R,s-k,i,k-i}^{(m+1)} y^i$$

has all its roots real and simple, implies

(9) $$(D_{R,s-k,i,k-i}^{(m+1)})^2 > D_{R,s-k,i+1,k-i-1}^{(m+1)} D_{R,s-k,i-1,k-i+1}^{(m+1)}$$

$(1 \leqq i \leqq k-1)$. This inequality has been proved under the assumption that the matrices μ_{m+1} and μ_m are not proportional. If, however, $\mu_{m+1} = c\mu_m$, it is easy to see that (9) holds with equality instead of strict inequality. Thus the assertion of the theorem is true for $m+1$.

It remains to prove (4) for $m+1$: If μ_{m+1} is positive definite, too, then, for $y \geqq 0$, the matrix $\bar{\mu} = y\mu_m + \mu_{m+1}$ is positive definite. By (6) and the inductive assumption, $D_{R,s-k,k}^{(m)}(y)$ is positive. Hence each polynomial $Q_k(y)$ has all its k different real roots negative. Using a rule of elementary algebra (see, *e.g.* Haupt [6], p. 409), we see that in the sequence of the coefficients of the polynomial $Q_k(-y)$, i.e. in the sequence

$$D_{R,s-k,0,k}^{(m+1)}, \quad -D_{R,s-k,1,k-1}^{(m+1)}, \quad D_{R,s-k,2,k-2}^{(m+1)}, \cdots, (-1)^k D_{R,s-k,k,0}^{(m+1)},$$

there must be k changes of sign. Since the highest coefficient $D_{R,s-k,k,0}^{(m+1)}$ is positive, all the numbers $D_{R,s-k,i,k-i}^{(m+1)}$ must be positive. Thus the theorem is proved.

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