Log-concavity in Combinatorics

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Abstract

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I would like to thank

Declaration

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Contents

Chapter 1

Introduction

Chapter 2

Combinatorial Structures and Convex Geometry

In this chapter, we review some basic structures from combinatorics and convex geometry. We begin in Section 2.1 by reviewing the notions of a partially ordered set and related concepts. We will not cover order theory too deeply, and will content ourselves in reviewing the basic definitions of linear extensions, lattices, and covering relations. In Section 2.2, we will go over basic notions in graph theory. The graphs that we consider contain no loops but contain multiple edges. We will over introduce some basic matrices in spectral graph theory. Arguably, Section 2.3 will be the most important section to read in terms of background. The properties of matroids and combinatorial geometries will be center stage for many of our applications in later chapters. In Section 2.4, we will cover the notion of mixed discriminants. These objects arise as the polarization form of the determinant polynomial, and were introduced by Alexandrov in [2] to study the mixed volumes of convex bodies. Finally, in Section 2.5, we outline basic notions in convex geometry and combinatorial convexity including the aforementioned mixed volumes. Our goal with this chapter is not to provide an exhaustive overview of the topics mentioned, but to cover the definitions, results, and applications needed for the rest of the thesis. For the reader who wishes to dive more deeply into these individual topics, we will refer to references within each section for further reading.

2.1 Partially Ordered Sets

In this section, we review basic notions in the theory of partially ordered sets (posets). Our treatment of posets is similar to that of [50] and [46]. Given a finite set P, we abstractly define a binary relation on P as simply a subset of $P \times P$. A **partial order** on a set is a binary relation that is reflexive, antisymmetric, and transitive. These properties are described in Definition 2.1.1.

Definition 2.1.1 (Definition 1.1.1 in [50]). A **partially ordered set** is an ordered pair $(P, \leq)$ of a set P and a binary relation $\leq$ on P which is reflexive, antisymmetric, and transitive. Explicitly, we have the following conditions.

(a) $x \leq x$ for all $x \in P$.

(b) If $x \leq y$ and $y \leq x$, then $x = y$.

(c) If $x \leq y$ and $y \leq z$ then $x \leq z$.

When $x \leq y$ and $x \neq y$, we can also write $x < y$ or $y > x$.

In this thesis, we will only consider posets where the ground set is finite. Let $(P, \leq)$ be a finite partially ordered set. We call two elements $x, y \in P$ **comparable** if and only if $x \leq y$ or $y \leq x$. We write $x \sim y$ if and only if x and y are comparable. If x and y are not comparable, we say that are incomparable. We call a subset $C \subseteq P$ a **chain** if every pair of elements in C are comparable. In a finite poset, a chain will always have the form $\{x_1 < \ldots < x_k\}$. For every elements pair of elements x, y, we can define the closed interval $[x, y] := \{z \in P : x \leq z \leq y\}$. We say that y **covers** x (or x is covered by y) if $[x, y] = \{x, y\}$. In this case, we call $x \lessdot y$ a **covering relation**, and we write $x \lessdot y$ or $y \gtrdot x$. We can diagrammatically visualize posets by drawing each element as a point and drawing the covering relations. Such a diagram is called a **Hasse diagram**. For an example of a Hasse diagram, see Figure 4.1. It is not difficult to see that the covering relations of a poset determine the poset uniquely. A poset element is called a **minimal** element if it is not greater than any other element. Similarly, we call an element a **maximal** element if it is not less than any other element. Every pair of elements $x, y \in P$ satisfying $x \leq y$ has a maximal chain C with x as the minimal element in the chain and y as the maximal element in the chain. Any maximal chain from x to y will be of the form

$$x = z_0 \lessdot z_1 \lessdot \ldots \lessdot z_{n-1} \lessdot z_n = y.$$

Posets appear all throughout mathematics. For example, the set of integers $\mathbb{Z}$ can be equipped with divisibility to give it a partially ordered set structure. The Möbius function associated with the lattice of integers is a common object in the study of analytic number theory (see [4]). In category theory, posets are examples of the most basic form of categories. In particular, they are categories where the morphisms between any two objects consists of a single element. Given a collection of sets, they can be given a poset structure with set inclusion as the partial order. Finally, in graph theory, we can equip the vertices of a directed graph with a poset structure where two vertices are comparable if and only if one can be reached from the other.

2.1.1 Morphisms of Posets and Linear Extensions

Given two posets $(P_1, \leq_1)$ and $(P_2, \leq_2)$, we define a poset morphism to be a map $f : P_1 \to P_2$ of the underlying sets satisfying the condition $f(x) \leq_2 f(y)$ whenever $x, y \in P_1$ satisfy $x \leq_1 y$. Two posets are isomorphism if and only if there is a bijection poset morphism between them. If $|P| = n$, we call any isomorphism $f : P \to [n]$ a linear extension.

Definition 2.1.2. Let $(P, \leq)$ be a poset on n elements. A **linear extension** is any bijective map $f : P \to [n]$ satisfying $f(x) < f(y)$ for all $x, y \in P$ satisfying $x < y$.

Recall that a partial order on P is called a **total order** if every pair of elements is comparable. Every partial order can be viewed a total order where some comparability information is missing. A linear extension is simply a way to extend a partial order to a total order. In later sections, we will be interested in the enumeration of linear extensions, and its connections to log-concavity.

2.1.2 Lattices

Let P be a poset and let $x, y \in P$ be two arbitrary elements. We say that $z \in P$ is a **greatest lower bound** or **meet** of x and y if $z \geq x$, $z \geq y$, and for any $w \in P$ satisfying $w \geq x, w \geq y$ we have that $w \geq z$. Similarly, we say z is a **least upper bound** or **join** if $z \leq x$, $z \leq y$, and for any $w \in P$ satisfying $w \leq x, w \leq y$, we have that $w \leq z$. If the meet or join of two elements exist, then it must be unique. In a general poset, the meet and join of two elements does not necessarily exist. When, they exist for every pair of elements, we call the poset a lattice.

Definition 2.1.3. Let $\mathcal{L}$ be a finite poset. We say $\mathcal{L}$ is a lattice if every pair of elements in $\mathcal{L}$ has a meet and a join. When $x, y \in \mathcal{L}$, we let $x \vee y$ and $x \wedge y$ denote the meet and join of x and y.

Given a lattice, it is not hard to show that the meet and join operations are commutative and associative. The partialloy ordered set of natural numbers equipped with divisibility forms a lattice under the greatest common denominator and the least common multiple. A lattice will automatically have a unique minimal element which is a global minimum and a unique maximal element which is a global maximal. We call the unique global minimum the 0 element and the unique global maximum element the 1 element, if they exist. Then, if $x \in P$ covers 0, we call x an **atom** and if x is covered by 1 we call x a **co-atom**. In later sections, we will be introduced to the lattice of faces of a polytope and the lattice of flats of a matroid. The lattice of flats of a matroid satisfy the extra conditions of a **geometric lattice**. For a thorough treatment of geometric lattices and their connections to matroids, we refer the reader to [41].

2.2 Graph Theory

In this section, we briefly review some notions in graph theory. We assume that the reader has some basic background knowledge on graph theory such as the definitions of connected components, paths, trees, etc. In Definition 2.2.1, we provide the definition of a graph that will adopt in this thesis. In particular, to each graph G we will attach some arbitrary total ordering to the vertices. For a thorough reading of graph theory, we refer the reader to [18]. For notions in spectral graph theory, we refer the reader to [17].

Definition 2.2.1. A **graph** is an ordered pair (V, E) of vertices and edges such that each edge is associated with either two distinct vertices or one vertex. If an edge is associated with two distinct vertices, then we call it a **simple edge**. If an edge is associated with one vertex, then we call it a **loop**. We also equip V with an arbitrary total ordering.

The role of the total ordering in Definition 2.2.1 will show up in Definition 2.2.3 when we define the incidence matrix. When two vertices in G contain an edge, we say that they are **adjacent**. If two vertices v and w are adjacent, we write $v \sim w$. This corresponds to comparability in the reachability poset of the graph. If an edge contains a vertex, we say that the edge is **incident** to the vertex. Given a connected graph G, we say a subgraph $T \subseteq G$ is a **spanning tree** if it is a tree that is incident to all vertices of G. In general, when G is a graph (not necessarily connected), we call T a **spanning forest** if it is a forest that is incident to all vertices of G.

2.2.1 Spectral Graph Theory

For the rest of the graph theoretic definitions, we will assume that our graph G is loopless. For every vertex $v \in V$, we define $\deg(v)$ to be the number of edges incident to v. For any pair of distinct vertices $v, w \in V$, we define $e(v, w)$ to be the number of edges between v and w. In particular, we have that

$$\deg(v) = \sum_{\substack{w \in V \\ w \neq v}} e(v, w).$$

Definition 2.2.2. Let $G = (V, E)$ be a (loopless) graph where $V = \{v_1, \ldots, v_n\}$. We define its Laplacian matrix $L := L_G$ to be the $n \times n$ matrix where the (i, j) entry is given by

$$L_{i,j} := \begin{cases} \deg(v_i) & \text{if } i = j \\ -E_{v_i,v_j} & \text{if } i \neq j \text{ and } v_i \sim v_j \end{cases}$$

where E_{v_i,v_j} is the number of edges between v_i and v_j.

Definition 2.2.3. Let G be a graph and let $V = \{v_1 < \ldots < v_n\}$ be an arbitrary ordering of the vertices. We can then define a $|V| \times |E|$ matroid $B := B_G$ called the **incidence matrix** where the entry indexed by the vertex v and the edge $e = \{v_i < v_j\} \in E$ is equal to

$$B_{ve} = \begin{cases} 1, & \text{if } v = v_i \\ -1, & \text{if } v = v_j \\ 0, & \text{otherwise.} \end{cases}$$

The matrix C_G which is obtained by removing the last row of B_G is called the **reduced incidence matrix**. The incidence matrix and reduced incidence matrix both satisfy Proposition 2.2.1. In this sense, these two matrices capture the property of being a cycle in the graph G.

Proposition 2.2.1. *For a graph $G = (V, E)$ the incidence matrix B_G and reduced incidence matrix C_G both satisfy the property that a set of columns are linearly independent if and only if the graph formed by the corresponding edges does not contain a cycle.*

Proof. See Example 5.4 of [9]. □

The incidence matrix and Laplacian of a matrix are related by Proposition 2.2.2. This relation will reappear in Section 4.3 when we prove Theorem 4.9 in the case of graphic matroids.

Proposition 2.2.2. *For a graph $G = (V, E)$, let L be its Laplacian matrix and let B be its incidence matrix. Then, we have that $L = BB^T$.*

Proof. For every $e = \{v_i, v_j\} \in E$ with $v_i < v_j$, we define $\pi_e(v_i) = -1$ and $\pi_e(v_j) = 1$. The function π_e indicators which vertex in an edge is the smaller one with respect to the total ordering on the vertices. We can compute that

$$(BB^T)_{v,w} = \sum_{e \in E: v,w \in e} \pi_e(v)\pi_e(w).$$

When $v = w$, then each summand is equal to 1 and we get exactly $\deg(v_i)$. If v and w are not adjacent, then the sum is empty and the netry is equal to zero. Otherwise, each summand is -1 and there are E_{v_i,v_j} elements in this sum. This suffices for the proof. $\square$

2.3 Matroids

Matroids can be thought of as combinatorial objects which abstract and generalize several properties in linear algebra, graph theory, and geometric concurrence. For example, it generalizes the notion of cyclelessness in graphs, the notion of linear independence in vector spaces, the poset structure of linear subspaces in a vector space, and collinearity in points configurations. Despite the seemingly limited conditions imposed on a matroid, matroids successfully describe many objects relevant to other areas in mathematics such as topology [25], graph theory [30], combinatorial optimization [19], algebraic geometry [24], and convex geometry [28]. In this section, we provide an introduction to the basic notions in matroid theory that we will need in the remainder of this thesis. We use the excellent monographs [41] and [59] as our main references for this theory. We begin by describing matroids as a set with a collection of independent sets.

2.3.1 Independent Sets

We first describe some properties of linearly independent vectors in a vector space as motivation for the definition of a matroid. Let V be a (finite-dimensional) vector space and let $S \subseteq V$ be a subset of linearly independent vectors. Note that any subset of S will also consist of linearly independent vectors. If $T \subseteq V$ is another set of linearly independent vectors with $|S| < |T|$, then there always exists a vector $v \in T \backslash S$ such that $S \cup v$ is linearly independent. To prove this fact, suppose for the sake of contradiction that there is not $v \in T \backslash S$ for which $S \cup v$ is linearly independent. This implies that $\mathrm{span}(T) \subseteq \mathrm{span}(S)$. Since our ambient vector space is finite-dimensional, by comparing dimensions we reach a contradiction. We abstract these properties in Defnition 2.3.1 and call the resulting object a **matroid**.

Definition 2.3.1. A **matroid** is an ordered pair $M = (E, \mathcal{I})$ consisting of a finite set E and a collection of subsets $\mathcal{I} \subseteq 2^E$ which satisfy the following three properties:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $X \subseteq Y$ and $Y \in \mathcal{I}$, then $X \in \mathcal{I}$.

(I3) If $X, Y \in \mathcal{I}$ and $|X| > |Y|$, then there exists some element $e \in X \backslash Y$ such that $Y \cup \{e\} \in \mathcal{I}$.

The set E is called the **ground set** of the matroid and the collection of subsets $\mathcal{I}$ are called **independent sets**. This terminology is motivated by Example 2.3.1 where the indepedent sets consist exactly of linearly independent subsets of our ground set. Condition (I1) is referred to as the non-emptiness axiom, condition (I2) is referred to as the hereditary axiom, and condition (I3) is referred to as the exchange axiom. We say two matroids M_1 and M_2 are isomorphic if there is a bijection between their ground sets which induces a one-to-one correspondence between their independent sets.

Example 2.3.1 (Linear Matroids)**.** Let V be a k-vector space and let $E = \{v_1, \ldots, v_n\}$ be a finite set of vectors from V. Let $\mathcal{I}$ consist of all subsets of E which are linearly independent. Then, the ordered pair $(E, \mathcal{I})$ forms a matroid. For a set of linearly independent vectors or equivalently a matrix A, we let $M(A)$ denote the linear matroid generated by A. We call a matroid M a **linear matroid** if there exists a matrix A such that $M \cong M(A)$. When there is a k-vector space V such that the matroid M is generated by a set of vectors in V, we say that M is **representable** over k.

Example 2.3.2 (Graphic Matroids)**.** Let $G = (V, E)$ be a graph and let $\mathcal{I}$ be the collection of subsets of E which consist of edges such that the subgraph on V with these edges is a forest (contains no cycles). The ordered pair $(E, \mathcal{I})$ forms a matroid called the **cycle matroid** of the graph G. If G is a graph, we let $M(G)$ be the cycle matroid associated to the graph G. We call any matroid isomorphic to $M(G)$ for some graph G a **graphic matroid**.

It is not difficult to directly show that the matroid generated by a graph as defined in Example 2.3.2 is a matroid. Indirectly, we can show this fact from Proposition 2.3.2. The proposition shows that graphic matroids are linear matroids.

Proposition 2.3.1 (Proposition 1.2.9 in [41])**.** *Let M be a graphic matroid. Then $M \cong M(G)$ for some connected graph G.*

Proof. Since M is graph, there exists a graph H (not necessarily connected) such that $M \cong M(H)$. Take the connected components of H and pick a one vertex from each of them. By identifying these chosen vertices, we get a connected graph G such that $M \cong M(G)$. This suffices for the proof. $\square$

Proposition 2.3.2. *Let $G = (V, E)$ be a graph and let A be its incidence matrix or reduced incidence matrix. Then $M(G) \cong M(A)$.*

Proof. This follows immediately from Proposition 2.2.1. $\square$

Example 2.3.3 (Uniform Matroids)**.** For any integers $0 \leq k \leq n$, we can define the **uniform matroid** $U_{k,n}$ which is the matroid on $[n]$ where the independent sets consist of all subsets of $[n]$ of size at most k. The matroid $U_{n,n}$ is called the **boolean matroid** or **free matroid** on n elements.

Given a matroid $M = (E, \mathcal{I})$, we call a subset $X \subseteq E$ a **dependent set** if and only if $X \notin \mathcal{I}$. Any minimal dependent set a **circuit**. An alternative way to define matroids is through circuits. Indeed, the set of circuits on a matroid satisfy some properties, and any collection of subsets which satisfy these properties will be the set of circuits to a unique matroid (see Corollary 1.1.5 in [41]). In this thesis, we will not concern ourselves with circuit properties. In the next section, we will define a dual notion of circuits called bases.

2.3.2 Bases

We call an independent set $B \in \mathcal{I}$ a **basis** if it is a maximal independent set. A maximal independent set is maximal in the sense of set inclusion. From the properties of independent sets of a matroid, we can deduce that all bases have the same number of elements. Indeed, if B_1 and B_2 are bases satisfying $|B_1| < |B_2|$, then there must exist some element $e \in B_2 \backslash B_1$ satisfying $B_1 \cup \{e\} \in \mathcal{I}$. But, this means that $B_1 \cup \{e\}$ is an independent set strictly larger than B_1. This contradicts the maximality of B_1 and implies that all bases contain the same number of elements.

Proposition 2.3.3. *Let $M = (E, \mathcal{I})$ be a matroid and let $\mathcal{B}$ be the collection of bases. Then, the following three properties:*

(B1) $\mathcal{B}$ is non-empty.

(B2) If B_1 and B_2 are members of $\mathcal{B}$ and $x \in B_1 \backslash B_2$, then there is an element y of $B_2 \backslash B_1$ such that $(B_1 - x) \cup y \in \mathcal{B}$.

(B3) If B_1 and B_2 are members of $\mathcal{B}$ and $x \in B_1 \backslash B_2$, then there is an element of $y \in B_2 \backslash B_1$ such that $(B_2 - y) \cup x \in \mathcal{B}$.

For a proof of the properties given in Proposition 2.3.3, see Lemma 1.2.2 in [41]. This result implies that any matroid will give us a set of bases which satisfies the properties listed in Proposition 2.3.3. Conversely, for any family of subsets satisfying these properties, there is a unique matroid whose set of bases is the original family of subsets. For any matroid $M = (E, \mathcal{B})$ where $\mathcal{B} = \mathcal{B}(M)$ are the bases of M, we can define the **basis generating polynomial** of a matroid $M = (E, \mathcal{B})$ by

$$f_M(x) := \prod_{B \in \mathcal{B}} x^B \in \mathbb{R}[x_e : e \in E].$$

The polynomial f_M is a homomogeneous polynomial of degree d where d is the size of a basis in M. In Section 2.3.3, we define the number d as the **rank** or **dimension** of the matroid M.

2.3.3 Rank Functions

Recall the motivating example of a matroid as a subset of vectors $S \subseteq V$ and independence being characterized through linear independence. In this scenario, there is a natural notion of dimension or rank. For any set of vectors, we can define the rank of this set to be the dimension of the vector subspace spanned by these vectors. This defines a function from the subsets of S to the non-negative integers with a few properties. We will abstract these properties in Definition 2.3.2.

Definition 2.3.2. For any matroid $M = (E, \mathcal{I})$, we define its **rank** function $\operatorname{rank}_M : 2^E \to \mathbb{N}$ to be equal to

$$\operatorname{rank}_M(X) := \max\{|I| : I \in \mathcal{I}, I \subseteq X\}.$$

The rank function satisfies the following three properties:

(R1) If $X \subseteq E$, then $0 \leq r(X) \leq |X|$.

(R2) If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.

(R3) If X and Y are subsets of E, then $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.

For a proof of (R1), (R2), and (R3) in Definition 2.3.2, we refer the reader to Lemma 1.3.1 in [41]. Many properties of rank functions which are satisfied in the case of linear matroids and the vector space picture are satisfied in general. For example, for any subset $S \subset E$ and $e \in E \backslash S$, we can prove that $\operatorname{rank}_M(S \cup e) \in \{\operatorname{rank}_M(S), \operatorname{rank}_M(S) + 1\}$.

2.3.4 Closure and Flats

In the case of a linear matroid, the rank of a set of vectors happened to be equal to the dimension of the vector subspace spanned by our vectors. We can define a subset of vectors to be closed if the vector span of these vectors contain no other vectors in our ground set. In this sense, the vector space spanned by a set of vectors is the closure of the set. We abstract the properties of this closure operation in Definition 2.3.3.

Definition 2.3.3. For any matroid $M = (E, \mathcal{I})$, we define its closure operator $\text{clo}_M : 2^E \to 2^E$ to be

$$\text{clo}_M(X) := \overline{X} = \{x \in E : \text{rank}(X \cup \{x\}) = \text{rank}(X)\}.$$

The closure operator satisfies the following four properties:

(C1) If $X \subseteq E$, then $X \subseteq \text{clo}_M(X)$.

(C2) If $X \subseteq Y \subseteq E$, then $\text{clo}_M(X) \subseteq \text{clo}_M(Y)$.

(C3) If $X \subseteq E$, then $\text{clo}_M(\text{clo}_M(X)) = \text{clo}_M(X)$.

(C4) If $X \subseteq E$ and $x \in E$, and $y \in \text{clo}_M(X \cup \{x\}) \setminus \text{clo}_M(X)$, then $x \in \text{clo}_M(X \cup \{y\})$.

For a proof of (C1), (C2), (C3), and (C4) in Definition 2.3.3, we refer the reader to the proof of Lemma 1.4.3 in [41]. From this definition, it is not difficult to check that the closure operation in a linear matroid returns the collection of all vectors contained in a given subspace. If $X \subseteq E$ satisfies $X = \text{clo}_M(X)$, then we say X is a **closed set** or **flat**. For any matroid $M = (E, \mathcal{I})$, let $\mathcal{L}(M)$ denote the partially ordered set consisting of the flats of M equipped with set inclusion. From Theorem 1.7.5 in [41], we have that $\mathcal{L}(M)$ is a geometric lattice with

$$\begin{aligned} X \wedge Y &:= X \cap Y \\ X \vee Y &:= \text{clo}_M(X \cup Y) \\ \text{rank}_{\mathcal{L}}(X) &:= \text{rank}_M(X) \end{aligned}$$

where $\text{rank}_{\mathcal{L}}$ is the rank function associated to $\mathcal{L}$ as a graded poset. In fact, from Theorem 1.7.5 in [41], a lattice is geometric if and only if it is the lattice of flats of a matroid. A geometric lattice determines a matroid up to simplification. We define the notion of simplification in Section 2.3.5. Although this motivates a geometric lattice-theoretic approach to the study of matroids, we will rarely use this perspective in the present thesis.

2.3.5 Loops, Parallelism, and Simplification

Let $M = (E, \mathcal{I})$ be a matroid. We say an element of the ground set $e \in E$ is a **loop** in M if $\{e\}$ is a dependent set. Equivalently, $e \in E$ is a loop if $\text{rank}(\{e\}) = 0$. In a graphic matroid, this corresponds to a loop in the underlying graph. We define E_0 as the set of loops. An element $e \in E$ in the ground set is called a **coloop** if it is contained in every basis. We say that two elements $x_1, x_2 \in E \backslash E_0$ are **parallel** if and only if $\{x_1, x_2\} \notin \mathcal{I}$ or $x_1 = x_2$. When this happens, we write $x_1 \sim_M x_2$. Equivalently, $x_1 \sim_M x_2$ if and only if $\operatorname{rank}(\{x_1, x_2\}) = 1$. For a thorough treatment of loops and parallelism, we refer the reader to Section 1.4 of [59].

Proposition 2.3.4. *Let $M = (E, \mathcal{I})$ be a matroid. Then $\sim_M$ is an equivalence relation on $E \backslash E_0$.*

Proof. For any $x \in E \backslash E_0$, we have $\operatorname{rank}(\{x, x\}) = \operatorname{rank}(\{x\}) = 1$ since $x \notin E_0$. Thus $x \sim_M x$. For $x, y \in E \backslash E_0$, we have $\operatorname{rank}(\{x, y\}) = \operatorname{rank}\{y, x\}$. This proves that $\sim_M$ is reflexive. To prove transitivity, suppose that we have elements $x, y, z \in E$ that satisfy $x \sim_M y$ and $y \sim_M z$. If $x = y$ or $y = z$, then we automatically get $x \sim_M z$. Suppose that they are all distinct. Then, we have $\{x, y\}, \{y, z\} \notin \mathcal{I}$. For the sake of contradiction, suppose that $\{x, z\} \in \mathcal{I}$. From (I3) applied to $\{x, z\}$ and $\{y\}$, we have that either $\{x, y\} \in \mathcal{I}$ or $\{y, z\} \in \mathcal{I}$. This is a contradiction. This proves that $\sim_M$ is transitive and is an equivalence relation. □

From Proposition 2.3.4, for any non-loop $e \in E \backslash E_0$, we can consider its equivalence class $[e] := [e]_M$ under the equivalence relation $\sim_M$. We call the equivalence class $[e]$ the **parallel class** of e. Then, we can partition the ground set E into $E = E_1 \sqcup E_2 \sqcup \ldots \sqcup E_s \sqcup E_0$ where $E_1, \ldots, E_s$ are the distinct parallel classes and E_0 are the loops. In Proposition 2.3.5, we give a characterization of the atoms of the lattice of flats in terms of the parallel classes and loops.

Proposition 2.3.5. *Let $M = (E, \mathcal{I})$ be a matroid and let $e \in E$ be an element of the matroid that is not a loop. Then, $\overline{e} = [e] \cup E_0$.*

Proof. Since any independent set contains no loops, we know that $E_0 \subseteq \overline{e}$. For any $f \in [e]$, we have that $\operatorname{rank}(\{e, f\}) = 1 = \operatorname{rank}(\{e\})$ by definition of $\sim_M$. This proves that $[e] \subseteq \overline{e}$. To prove the opposite inclusion, let $f \in \overline{e}$. Then $\operatorname{rank}(\{e, f\}) = 1$. If f is a loop, then $f \in E_0$. Otherwise, $f \sim_M e$ and $f \in [e]$. This suffices for the proof of the proposition. □

We call a matroid **simple** if it contains no loops and no parallel elements. To every matroid M, we can associate a simple matroid $\widetilde{M}$ called the **simplification** of the matroid M. For any matroid $M = (E, \mathcal{I})$, let $E(\widetilde{M})$ be the set of rank 1 flats of M. In particular, if $x_1, \ldots, x_s$ are the representatives for the parallel classes in M, then we can define the ground set of $\widetilde{M}$ as

$$E\left(\widetilde{M}\right) := \{[x_1], \ldots, [x_s]\}.$$

We can define a map $\pi_M : E\backslash E_0 \to E(\widetilde{M})$ to be the map which sends $e \in E\backslash E_0$ to the rank one flat $\overline{e}$. For any $\alpha \in E(\widetilde{M})$, we define $\operatorname{fiber}(\alpha) := \pi_e^{-1}(\alpha)$. In other words, this is the elements of E in the parallel class α. We can define the following collection of subsets of $E(\widetilde{M})$:

$$\mathcal{I}\left(\widetilde{M}\right) := \left\{\{[x_{i_1}], \ldots, [x_{i_l}]\} : \{x_{i_1}, \ldots, x_{i_l}\} \in \mathcal{I}(M)\right\}.$$

We would like $\mathcal{I}(\widetilde{M})$ to be a collection of independent sets for $\widetilde{M}$. To prove that this is the case, it suffices to prove Proposition 2.3.6.

Proposition 2.3.6. *Let $M = (E, \mathcal{I})$ be a matroid and let $e \sim f$ be two parallel elements. If $I \in \mathcal{I}$ is an independent set which contains e, then $(I\backslash e) \cup f \in \mathcal{I}$. In other words, in an independent set we can freely replace elements by parallel ones without breaking the independence structure.*

Proof. Since e and f are parallel elements, by definition they are not loops. In particular, f is independent. By applying the exchange axiom for independent sets repeatedly for f and I, we must have that $(I\backslash e) \cup f$ is independent. This is because we can never exchange e from I to the independent set containing f due to the fact that $\{e, f\}$ is dependent. □

As a consequence of Proposition 2.3.6, we know that $\mathcal{I}(\widetilde{M})$ provides a well-defined collection of indepedent sets on the ground set $E(\widetilde{M})$. Given any matroid $M = (E, \mathcal{I})$, we define the matroid $\widetilde{M} = (E(\widetilde{M}), \mathcal{I}(\widetilde{M}))$ to be the **simplification** of M.

Remark 1. According to James Oxley in [41], the notation $\widetilde{M}$ for the simplification of a matroid is rarely used in the literature anymore. Currently, the convention is to use $\operatorname{si}(M)$ for the simplification.

2.3.6 Restriction, Deletion, and Contraction

Given a graph, there are well-defined notions of edge deletion and edge contraction. In this section, we generalize these graph operations to general matroids. When we apply this generalization of deletion and contraction to graphic matroids, we recover the graph-theoretic model of deletion and contraction.

Definition 2.3.4 (Restriction, Deletion, and Contraction)**.** Let $M = (E, \mathcal{I})$ be a matroid and let $T \subseteq E$ be a subset. We call $M|_T$ the **restriction** of M on T. This is defined as the matroid on T with independent sets given by

$$\mathcal{I}(M|_T) := \{I \in \mathcal{I} : I \subseteq T\}.$$

We call the matroid $M\backslash T$ the deletion of T from M. This is defined to be $M|_{E\backslash T}$, the restriction of M on $E\backslash T$. Let B_T be a basis of $M|_T$. We call M/T the contraction of M by T. This is the matroid on $E\backslash T$ with independent sets given by

$$\mathcal{I}(M/T) := \{I \subset E\backslash T : I \cup B_T \in \mathcal{I}(M)\}.$$

In the definition of matroid contraction, there is the question of whether or not different bases B_T will result in different matroids. It turns out that this is not an issue as the definition of the contraction will be independent from our choice of basis. In fact, we can define contraction without choosing a basis with the matroid dual: the contraction M/T is defined to be $(M^*\backslash T)^*$. We will not concern ourselves with the dual matroid and refer the interested reader to Section 3.1 in [41]. In Lemma 2.3.1, we describe how contraction and deletion effect the basis generating polynomial.

Lemma 2.3.1. *Let $M = (E, \mathcal{B})$ be a matroid with basis generating polynomial f_M.*

(a) If $e \in E$ is a loop, then $\partial_e f_M = 0$.

(b) If $e \in E$ is not a loop, then $\partial_e f_M = f_{M/e}$.

(c) If $e \in E$ is not a coloop, then $f_M = x_e f_{M/e} + f_{M\backslash e}$

Proof. Since every basis contains no loops, we have $\partial_e f_M = 0$ whenever a is a loop. The second claim follows from the fact that the remaining monomials in $\partial_e f_M$ will correspond to sets of the form $B\backslash e$ where $e \in B$ and B is a basis of M. This is exactly the set of bases of M/e. For the third claim, this follows from the fact that the monomials in f_M which do not contain x_e will correspond to bases of M which do not contain e. That is the exactly the bases of $M\backslash e$. This implies that we can write $f_M = x_e p + f_{M\backslash e}$ where $p \in \mathbb{R}[x_e : e \in E]$ and p contains no monomial with x_e. Taking the partial derivative with respect to x_e, we get $p = f_{M/e}$ from (a). This suffices for the proof. ☐

2.3.7 Matroid Sum and Truncation

Given two matroids, there is a notion of adding these two matroids to produce another. There is also a notion of truncating a matroid so that the rank lowers by 1. The first operation is called the matroid sum of two matroids while the second operation will be called the truncation of a matroid.

To describe the matroid sum, let $M = (E, \mathcal{I}_M)$ and $N = (F, \mathcal{I}_N)$ be matroids. We define the **matroid sum** of M and N to be the matroid $M \oplus N$ on the set $E \sqcup F$ such that the independent sets of $M \oplus N$ are subsets of $E \sqcup F$ of the form $I \cup J$ where $I \in \mathcal{I}_M$ and $J \in \mathcal{I}_N$. In other words, we define

$$\mathcal{I}(M+N) := \{I \cup J : I \in \mathcal{I}(M), J \in \mathcal{I}(N)\}.$$

It is not difficult to see that this produces a well-defined collection of independent sets on the set $E \sqcup F$. Now, we describe the truncation of a matroid. For a matroid $M = (E, \mathcal{I})$, we define TM to be the **truncation** of M. This is the matroid on the same ground set E with independent sets given by

$$\mathcal{I}(TM) := \{I \in \mathcal{I} : \mathrm{rank}_M(I) \leq \mathrm{rank}_M(M) - 1\}.$$

This collection of sets inherits the properties of independent sets from $\mathcal{I}(M)$. Thus, our definition TM gives a well-defined matroid. Note that we can repeatedly apply our truncation operation to get a matroid $T^k(M)$ of rank $\mathrm{rank}_M(M) - k$ for any $0 \leq k \leq \mathrm{rank}_M(M)$.

2.3.8 Regular Matroids

In this section, we will study the subclass of regular matroids. As a preliminary definition, these are the matroids which are representable over any field. The content of Definition 2.3.5 illustrates the many different equivalent definitions of regular matroids.

Definition 2.3.5. We say that a matroid M is **regular** if it satisfies any of the following equivalent conditions:

(a) M is representable over any field.

(b) M is $\mathbb{F}_2$ and $\mathbb{F}_3$ representable.

(c) M is representable over $\mathbb{F}_2$ and k where k is any field of characteristic other than 2.

(d) M is representable over $\mathbb{R}$ by a totally unimodular matrix.

For a proof of the equivalences of the conditions in Definition 2.3.5, we refer the reader to Theorem 5.16 in [41]. In the definition, we define a **totally unimodular matrix** to be a real matrix for which every square submatrix has determinant in the set $\{0, \pm 1\}$. In this thesis, we use the terminology totally unimodular and unimodular interchangeably even though they have different meanings in the literature. The main property that we will use for regular matroids is (d) in Definition 2.3.5. Note that when we represent M by a unimodular matrix over $\mathbb{R}$, property (d) gives not indication on how small the dimension of the ambient space can be made. Ideally, we want the column vectors of the matrix to lie in a d-dimensional real vector space where d is the rank of the matroid. This is the minimu possible dimension of an ambient vector space for which a matroid can be embedded. Fortunately, Theorem 2.1 implies that we can achieve this minimum dimension.

Theorem 2.1 (Lemma 2.2.21 in [41]). *Let $\{e_1, \ldots, e_r\}$ be a basis of a matroid M of non-zero rank. Then M is regular if and only if there is a totally unimodular matrix $[I_r|D]$ representing M over $\mathbb{R}$ whose first r columns are labelled, in order, $e_1, e_2, \ldots, e_r$.*

Thus, for any regular matroid M of rank n, there is an injection $v : E(M) \to \mathbb{R}^n$ where the columns $\{v(e) : e \in M\}$ form a totally unimodular matrix. We call such a map a **unimodular coordinatization** of M. Note that it not only does it assign unimodular coordinates to our matroid, but it does so in a way which minimizes the dimension of the ambient space. From Theorem 2.1, we know that such a coordinatization exists for all regular matroids.

Proposition 2.3.7. *Let $G = (V, E)$ be a graph and let $M = M(G)$ be the graphic matroid associated with G. Then, M is also the linear matroid generated by the incidence matrix and reduced incidence matrix of the graph. As a consequence, graphic matroids are regular.*

Proof. From Proposition 2.2.1, it suffices to prove that the incidence matrix is totally unimodular. But this follows from Lemma 4.4 in [23]. □

In the proof of Proposition 2.3.7, we have used the fact that the incidence and reduced incidence matrix of a graph are totally unimodular. For any graphic matroid M, Proposition 2.3.1 gives us a connected graph G so that $M \cong M(G)$. In this case, the rank of M is $|V(G)| - 1$. Hence, the reduced incidence matrix of G is a unimodular coordinatization of M.

2.4 Mixed Discriminants

In this section, we discuss a symmetric multilinear form called the mixed discriminant which arises as the polarization form of the determinant. The notion of mixed discriminants was introduced by Alexandrov in his paper [2] where he proved studied the Alexandrov-Fenchel inequality. Similarly to mixed volumes, mixed discriminants also satisfy a similar inequality called the Alexandrov inequality for mixed discriminants. This inequality is the subject of Theorem 3.3. Our treatment of mixed discriminants is inspired by the exposition in [9]. Using Cholesky factorization, we will show how to compute mixed discriminants for rank 1 matrices. This will allow us to extend the computation to all positive semi-definite matrices.

Definition 2.4.1. let $n \geq 1$ be a positive integer. Suppose that for each $k \in [n]$, we are given real matrix $A_k := \left(a_{ij}^{(k)}\right)_{i,j=1}^{n} \in \mathbb{R}^{n \times n}$. Then, we define the **mixed discriminant** of the collection $(A_1, \ldots, A_n)$ to be

$$\mathsf{D}(A^1, \ldots, A^n) := \frac{1}{n!} \sum_{\sigma \in S_n} \det \begin{bmatrix} a_{11}^{\sigma(1)} & \ldots & a_{1n}^{\sigma(n)} \\ \vdots & \ddots & \vdots \\ a_{n1}^{\sigma(1)} & \ldots & a_{nn}^{\sigma(n)} \end{bmatrix} = \frac{1}{n!} \sum_{\sigma \in S_n} \mathrm{Det}\left(v_1^{\sigma(1)}, \ldots, v_n^{\sigma(n)}\right). \tag{2.1}$$

In Equation 2.1, the group S_n is the symmetric group on n letters and Det refers to the determinant as a multilinear n-form on $(\mathbb{R}^n)^n$.

From Definition 2.4.1, the mixed discriminant is multilinear and symmetric in its entries. Moreover, for any symmetric matrix A, we have $\mathsf{D}(A, \ldots, A) = \det(A)$.

2.4.1 Polarization Form of a Homogeneous Polynomial

Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be an arbitrary homogeneous polynomial of degree d. For any choice of vectors $v_1, \ldots, v_d \in \mathbb{R}^n$ we can study the coefficients of $f(\lambda_1 v_1 + \ldots + \lambda_d v_d)$ as a polynomial in $\lambda_1, \ldots, \lambda_d$. In Definition 2.4.2, we define the polarization form associated to a homogeneous polynomial. In the literature, this form is also called the complete homogeneous form. These objects are briefly mentioned in Section 3.2 of [43], Section 5.5 of [49], and Section 4.1 of [12]. In this section, we hope to provide an accessible and self-contained exposition of the polarization form.

Definition 2.4.2. Let k be a field of characteristic 0. Let $f \in k[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree d. We define the **polarization form** or **complete homogeneous form** of f to be the function $F_f : (k^n)^d \to k$ defined by

$$F_f(v_1, \ldots, v_d) := \frac{1}{d!} \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_d} f(x_1 v_1 + \ldots + x_d v_d).$$

From the definition, we can see that the form is k-multilinear, symmetric, and $F_f(v, \ldots, v_n) = f(v)$ for all $v \in k^n$. Let $f \in k[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree d. Then, we can write this polynomial in the form

$$f(x_1, \ldots, x_n) = \sum_{\alpha_1, \ldots, \alpha_d = 1}^{n} c_{\alpha_1, \ldots, \alpha_d} \cdot x_{\alpha_1} \ldots x_{\alpha_d}.$$

where $c_{\alpha_1, \ldots, \alpha_d}$ is symmetric in $\alpha_1, \ldots, \alpha_d$. Let $v_1, \ldots, v_d \in k^n$ be vectors where $v_i = v_1^{(i)} e_1 + \ldots +$

$v_n^{(i)} e_n$ for all $1 \le i \le d$ and $e_1, \ldots, e_n$ is the standard basis in k^n. We can define $y_i := \sum_{j=1}^d x_j v_i^{(j)}$ for all $1 \le i \le d$. Then, we have $f(x_1 v_1 + \ldots + x_d v_d) = f(y_1, \ldots, y_d)$. This allows us to compute

$$\begin{aligned}
F_f(v_1, \ldots, v_d) &= \frac{1}{d!}[x_1 \ldots x_d] f(y_1, \ldots, y_d) \\
&= \frac{1}{d!}[x_1 \ldots x_d] \sum_{\alpha_1, \ldots, \alpha_d = 1}^{n} c_{\alpha_1, \ldots, \alpha_d} y_{\alpha_1} \cdots y_{\alpha_d} \\
&= \frac{1}{d!} \sum_{\sigma \in S_n} \sum_{\alpha_1, \ldots, \alpha_d = 1}^{n} c_{\alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(d)}} v_{\alpha_{\sigma(1)}}^{(1)} \cdots v_{\alpha_{\sigma(d)}}^{(d)} \\
&= \sum_{\alpha_1, \ldots, \alpha_d = 1}^{n} c_{\alpha_1, \ldots, \alpha_d} v_{\alpha_1}^{(1)} \cdots v_{\alpha_d}^{(d)}.
\end{aligned}$$

This gives us a formula for the polarization form explicitly in the vectors $v_1, \ldots, v_d$ and the coefficients of f. Using this explicit formula, we prove the well-known polarization formula given in Theorem 2.2.

Theorem 2.2 (Polarization Identity)**.** *Let k be a field of characteristic 0. Let $v_1, \ldots, v_m \in k^n$ be arbitrary vectors. Then, we have the identity*

$$f(x_1 v_1 + \ldots + x_m v_m) = \sum_{i_1, \ldots, i_d = 1}^{m} F_f(v_{i_1}, \ldots, v_{i_d}) \cdot x_{i_1} \ldots x_{i_d}.$$

Proof. For all $1 \le i \le n$, we have $v_i = v_i^{(1)} e_1 + \ldots + v_i^{(n)} e_n$ for some constants $v_i^{(j)}$. Let $y_i = \sum_{j=1}^d x_j v_i^{(j)}$ for all $1 \le i \le n$. Then, we have that

$$\begin{aligned}
f(x_1 v_1 + \ldots + v_m v_m) &= f(y_1, \ldots, y_m) \\
&= \sum_{\alpha_1, \ldots, \alpha_d = 1}^{n} c_{\alpha_1, \ldots, \alpha_d} \sum_{i_1, \ldots, i_d = 1}^{m} x_{i_1} \ldots x_{i_d} \cdot v_{\alpha_1}^{(i_1)} \ldots v_{\alpha_d}^{(i_d)} \\
&= \sum_{i_1, \ldots, i_d = 1}^{m} \left(\sum_{\alpha_1, \ldots, \alpha_d = 1}^{n} c_{\alpha_1, \ldots, \alpha_d} v_{\alpha_1}^{(i_1)} \ldots v_{\alpha_d}^{(i_d)} \right) x_{i_1} \ldots x_{i_d} \\
&= \sum_{i_1, \ldots, i_d = 1}^{m} F_f(v_{i_1}, \ldots, v_{i_d}) \cdot x_{i_1} \ldots x_{i_d}.
\end{aligned}$$

This suffices for the proof. □

2.4.2 Polarization for Mixed Discriminants

We can view the mixed discriminant as the polarization form of the determinant function. This fact is the content of Theorem 2.3. For proofs of this result in other sources, we refer the reader to [60] or [49].

Theorem 2.3. *For $n \times n$ matrices $A_1, \ldots, A_m$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$, the determinant of the linear combination $\lambda_1 A_1 + \ldots + \lambda_m A_m$ is a homogeneous polynomial of degree n in the λ_i and is given by*

$$\det(\lambda_1 A_1 + \ldots + \lambda_m A_m) = \sum_{1 \le i_1, \ldots, i_n \le m} \mathsf{D}(A_{i_1}, \ldots, A_{i_n}) \lambda_{i_1} \ldots \lambda_{i_n}.$$

Proof. From the definition of the determinant, there exists a multilinear function $\mathrm{Det} : (\mathbb{R}^n)^n \to \mathbb{R}$ defined by

$$\mathrm{Det}(v_1, \ldots, v_n) := \det \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix}$$

where we take the determinant of the matrix with columns $v_1, \ldots, v_n$. If $v_1^{(i)}, \ldots$ and $v_n^{(i)}$ be the columns of A_i for $1 \le i \le m$, we have that

$$\begin{aligned} \det\left(\sum_{i=1}^m \lambda_i A_i\right) &= \mathrm{Det}\left(\sum_{i=1}^m \lambda_i v_1^{(i)}, \ldots, \sum_{i=1}^m \lambda_i v_n^{(i)}\right) \\ &= \sum_{i_1, \ldots, i_n = 1}^m \mathrm{Det}(\lambda_{i_1} v_1^{(i_1)}, \ldots, \lambda_{i_n} v_n^{(i_n)}) \\ &= \sum_{i_1, \ldots, i_n = 1}^m \lambda_{i_1} \ldots \lambda_{i_n} \cdot \mathrm{Det}(v_1^{(i_1)}, \ldots, v_n^{(i_n)}). \end{aligned}$$

Looking at the coefficient in front of $\lambda_1^{r_1} \ldots \lambda_m^{r_m}$ where $r_1 + \ldots + r_m = n$, it is equal to

$$\begin{aligned} [\lambda_1^{r_1} \ldots \lambda_M^{r_m}] \det\left(\sum_{i=1}^m \lambda_i A_i\right) &= \frac{1}{(r_1)! \ldots (r_m)!} \sum_{\sigma \in S_n} \mathrm{Det}(v_1^{i_{\sigma(1)}}, \ldots, v_n^{i_{\sigma(n)}}) \\ &= \binom{n}{r_1, \ldots, r_m} \mathsf{D}(A_1[r_1], \ldots, A_m[r_m]) \end{aligned}$$

where the multiset $\{i_1, \ldots, i_m\}$ is equal to $\{1[r_1], \ldots, m[r_m]\}$. This coincides with the right hand side in Theorem 2.3. □

As an application of the polarization identity, we will compute the mixed discriminants of rank 1 matrices. Since positive semi-definite matrices can be written as the sum of rank 1 matrices, this computation will extend to the mixed discriminants of positive semi-definite matrices.

Example 2.4.1 (Mixed discriminants of rank 1 matrices)**.** Let $x_1, \ldots, x_n \in \mathbb{R}^n$ be real vectors. For any $\lambda_1, \ldots, \lambda_n > 0$, define the vectors $y_i = \sqrt{\lambda_i} x_i$. Let X be the matrix with the x_i as column vectors and let Y be the matrices with the y_i as column vectors. Then, we have that

$$\det\left(\sum_{i=1}^{n} \lambda_i x_i x_i^T\right) = \det\left(\sum_{i=1}^{n} y_i y_i^T\right) = \det(YY^T) = (\det(Y))^2 = \lambda_1 \ldots \lambda_n (\det(X))^2.$$

From Theorem 2.3, we get the identity:

$$\mathsf{D}(x_1 x_1^T, \ldots, x_n x_n^T) = \frac{1}{n!}\left[\operatorname{Det}(x_1, \ldots, x_n)\right]^2. \tag{2.2}$$

From Equation 2.2, we see that the mixed discriminant of rank 1 matrices is exactly a determinant of vectors generating the rank 1 matrices. In particular, it can serve as an indicator for when a collection of vectors forms a basis.

2.4.3 Positivity

To extend the calculation in Example 2.4.1 to positive semi-definite matrices, we first recall the following linear algebra factorization result.

Theorem 2.4 (Cholesky Factorization, Theorem 4.2.5 in [26]). *If $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, then there exists a unique lower triangular $L \in \mathbb{R}^{n\times n}$ with positive diagonal entries such that $A = LL^T$. When A is positive semi-definite, then there exists a (not necessarily unique) lower triangular $L \in \mathbb{R}^{n\times n}$ with $A = LL^T$.*

Let A be a positive semi-definite matrix. Theorem 2.4 implies that there exists some matrix $X \in \mathbb{R}^{n\times n}$ satisfying $A = XX^T$. We can decompose X into $X = X_1 + \ldots + X_n$ where X_i is the matrix with x_i in the ith column and 0 everywhere else. These matrices satisfy the properties that $X_i X_i^T = x_i x_i^T$ and $X_i X_j^T = 0$ when $i \neq j$. Thus, we have that

$$A = XX^T = \left(\sum_{i=1}^{n} X_i\right)\left(\sum_{i=1}^{n} X_i^T\right) = \sum_{i=1}^{n} x_i x_i^T.$$

We have just proved that all positive semi-definite matrices can be written as the sum of rank 1 matrices of the form xx^T. In Lemma 2.4.1, we will use this fact to give an explicit formula for the mixed discriminant on positive semi-definite matrices.

Lemma 2.4.1 (Lemma 5.2.1 in [9]). *Let $A_1, \ldots, A_n$ be positive semi-definite $n \times n$ matrices, and* *suppose that* $A_k = X_k X_k^T$ *for each* k*. Then*

$$\mathsf{D}(A_1, \dots, A_n) = \frac{1}{n!} \sum_{\substack{x_j \in X_j \\ 1 \le j \le n}} [\mathrm{Det}(x_1, \dots, x_n)]^2$$

where the sum is over all choices of x_k *in the columns of* X_k*.*

Proof. From the multi-linearity of the mixed discriminant, we have that

$$\mathsf{D}(A_1, \dots, A_n) = \mathsf{D}\left(\sum_{x_1 \in \mathsf{col}(X_1)} x_1 x_1^T, \dots, \sum_{x_n \in \mathsf{col}(X_n)} x_n x_n^T\right) = \sum_{\substack{x_j \in X_j \\ 1 \le j \le n}} \mathsf{D}(x_1 x_1^T, \dots, x_n x_n^T).$$

The lemma follows from the computation in Example 2.4.1. □

Corollary 2.4.1. *Let* $A_1, \dots, A_n$ *be positive semi-definite* $n \times n$ *real symmetric matrices. Then*

$$\mathsf{D}(A_1, \dots, A_n) \ge 0.$$

If $A_1, \dots, A_n$ *are positive definite, then* $\mathsf{D}(A_1, \dots, A_n) > 0$*.*

Proof. This follows from Lemma 2.4.1 and Theorem 2.4. □

2.5 Convex Bodies

In this section, we review the notions of convexity and convex bodies. We use [49] as our main reference for the theory of convex bodies and Brunn-Minkowski theory. Recall that a subset $C \subseteq \mathbb{R}^n$ is convex if for every $x, y \in C$, the line segment $[x, y]$ is contained in C. Even though convexity is quite a rigid condition to impose, there still exist topologically wild convex sets. For example, let $S \subseteq \mathbb{S}^{n-1}$ be an arbitrary subset of the unit sphere. Then, the set $S \cup \{x : \|x\|_2 < 1\}$ is always a convex set. Measure theoretically, this same example gives cases where convex sets might not be Borel measurable. In this thesis, we will only consider a subclass of convex sets called convex bodies. A **convex body** is a non-empty, compact, convex subset of $\mathbb{R}^n$. Let K^n be the space of convex bodies in $\mathbb{R}^n$. We can define an associative, commutative binary operation on the set of convex bodies called the Minkowski sum.

Definition 2.5.1. For any convex bodies $K, L \subseteq \mathbb{R}^n$, we define the **Minkowski sum** of the two bodies to be the convex body

$$K + L := \{x + y \in \mathbb{R}^n : x \in K, y \in L\}.$$

For any $x_0, x_1 \in K$ and $y_0, y_1 \in K$ we have that $\lambda x_0 + \tau x_1 \in K$ and $\lambda y_0 + \tau y_1 \in L$ for any $\lambda, \tau \geq 0$ satisfying $\lambda + \tau = 1$. Thus, we have that

$$\lambda(x_0 + y_0) + \tau(x_1 + y_1) = (\lambda x_0 + \tau x_1) + (\lambda y_0 + \tau y_1) \in K + L.$$

This proves that $K + L$ is a well-defined convex body. In fact, a subset $K \subseteq \mathbb{R}^n$ is a convex body if and only if $\alpha K + \beta K = (\alpha + \beta)K$ for all $\alpha, \beta \geq 0$. For a thorough treatment of convex sets and convex bodies, we recommend the excellent by Schneider [49]. We will only cite the results from the book that we need for this thesis.

2.5.1 Support and Facial Structure

For a convex body $K \subseteq \mathbb{R}^n$, we define its dimension as $\dim K := \dim \operatorname{aff} K$. The boundary structure of convex bodies are somewhat well understood. The boundary can be characterized in terms of supporting hyperplanes and faces. We call H a **supporting hyperplane** of K if K lies on one side of the hyperplane and $K \cap H \neq \emptyset$. A **supporting hyperspace** will then by the half-space corresponding to a supporting hyperplane which contains K. Any convex body will be equal to the intersection of all of its supporting hyperspaces. To explore the boundary structure in greater detail, we define the following the notion of a face and an exposed face.

Definition 2.5.2. Let $K \subseteq \mathbb{R}^n$ be a convex body and let $F \subseteq K$ be a subset.

(a) If F is a convex subset such that each segment $[x, y] \subseteq K$ with $F \cap \operatorname{relint}[x, y] \neq \emptyset$ is contained in F, then we call F a **face**. If $\dim F = i$, then we call F an i-face.

(b) If there is a supporting hyperplane H such that $K \cap H = F$, we call F an **exposed face**. The exposed faces of codimension 1 are called **facets** and the exposed faces of dimension 0 are called **vertices**.

For $i \geq -1$, we define $\mathcal{F}_i(K)$ be the set of i-faces of K. By convention, we let $\mathcal{F}_{-1}(K) = \{\emptyset\}$ and consider $\emptyset$ to be the unique face of dimension -1. The set of all faces $\mathcal{F}(K) := \bigcup_i \mathcal{F}_i(K)$ equipped with set inclusion forms a poset.

Remark 2. In general, the notions of faces and exposed faces are not the same. It is not hard to check that an exposed face is a face. On the other hand, a face is not necessarily an exposed face. For example, in Figure 2 the top semi-circle is a face which is not exposed. But, in the special case where our convex body is a polytope, the notions of faces and exposed faces are the same. We will talk about these types of convex bodies subsequently.

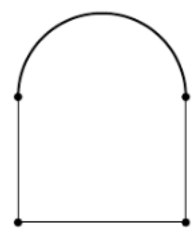

Figure 2.1: Convex body with a face which is not exposed.

For any convex body, there exists an exposed face in any direction. Indeed, for any convex body $K \in \mathsf{K}^n$ and $u \in \mathbb{R}^n \backslash \{0\}$, we can define the **support function** $h_K(u)$ of K in the direction u and the **exposed face** $F_K(u)$ of K in the direction u as

$$h_K(u) := \sup_{x \in K} \langle u, x \rangle$$
$$F_K(u) := K \cap \{x \in \mathbb{R}^n : \langle u, x \rangle = h_K(u)\}.$$

With respect to Minkowski sum, it is not difficult to prove that the support function $h_\bullet$ and exposed face function $F_\bullet$ satisfy

$$h_{K+L} = h_K + h_L, \quad \text{and} \quad F_{K+L} = F_K + F_L.$$

Moreover, for any positive scalar $\lambda > 0$, we have $h_{\lambda K} = \lambda h_K$ and $F_{\lambda K} = \lambda F_K$. Geometrically, for a unit vector u, the support value $h_K(u)$ is the (signed) distance of the furthest point on K in the direction u. The exposed face $F_K(u)$ consists of the subset of K which achieves this maximum distance in the direction u. The support function h_K of a convex body K completely determines the convex body because of the equation

$$K = \bigcap_{u \in \mathbb{S}^{n-1}} H^-_{u,h_K(u)} = \bigcap_{u \in \mathbb{S}^{n-1}} \{x \in \mathbb{R}^n : \langle x, u \rangle \leq h_K(u)\}.$$

Thus, from the properties of support functions, we have the following cancellation law in Proposition 2.5.1.

Proposition 2.5.1 (Cancellation Law for Minkowski Addition). *Let $K, L, M \subseteq \mathbb{R}^n$ be convex bodies such that $K + M = L + M$. Then $K = L$.*

Proof. Since $K + M = L + M$, we have that $h_K + h_M = h_{K+M} = h_{L+M} = h_L + h_M$. So, we have that $h_K = h_L$ and from our remark that the support function determines the convex body we have $K = L$. □

Proposition 2.5.1 gives the set of convex bodies the structure of a commutative, associative monoid with a cancellation law.

2.5.2 Polytopes and SSI Polytopes

In this section, we define a subclass of convex bodies which are combinatorial in nature and allow us to approximate general convex bodies. We call a convex body $P \subseteq \mathbb{R}^n$ a **polytope** if it can be written as the convex hull of a finite number of points.

Proposition 2.5.2 (Properties of Polytopes). *Let $P \subseteq \mathbb{R}^n$ be an arbitrary polytope. Then, P satisfies the following properties:*

(a) The exposed faces of P are exactly the faces of P.

(b) P has a finite number of faces.

(c) Let $F_1, \ldots, F_k$ be the facets of P with normal vectors $u_1, \ldots, u_k$. Then

$$P = \bigcap_{i=1}^{k} H^-_{u_i, h_P(u_i)}$$

In particular, the numbers $h_P(u_1), \ldots, h_P(u_k)$ determine P uniquely.

*(d) The face poset $\mathcal{F}(P)$ of P is a graded lattice where the poset rank function is the dimension of the corresponding face. The face lattice satisfies the **Jordan-Dedekind chain condition**. In other words, if we have a face $F^j \in \mathcal{F}_j(P)$ and a face $F^k \in \mathcal{F}_k(P)$ satisfying $F^j \subset F^k$, then there are faces $F^i \in \mathcal{F}_i(P)$ for $j + 1 \leq i \leq k - 1$ such that*

$$F^j \subset F^{j+1} \subset \ldots \subset F^{k-1} \subset F^k.$$

Proof. These properties follow from Corollary 2.4.2, Theorem 2.4.3, Corollary 2.4.4, and Corollary 2.4.8 in [49]. □

We now give a few examples of special polytopes and their combinatorially properties. The calculations performed in these examples will reappear in future chapters.

Example 2.5.1 (Order Polytope)**.** Let $(P, \leq)$ be a finite poset. Let $\mathbb{R}^P$ be Euclidean space of dimension $|P|$ where the coordinates are indexed by the elements of P. Every vector $v \in \mathbb{R}^P$ can be written in the form $v = \sum_{\omega \in P} v_\omega \cdot e_\omega$ where $\{e_\omega\}_{\omega \in P}$ is a chosen orthonormal basis for $\mathbb{R}^P$. Then, we can define the **order polytope** of P to be the polytope given by

$$\mathcal{O}_P := \{x \in [0,1]^P : x_i \geq x_j \text{ whenever } i \geq j \text{ in } P\}.$$

Given any linear extension $\sigma : P \to [n]$, we can define the **linear extension simplex**

$$\Delta_l := \{x \in [0,1]^n : 0 \leq x_{l^{-1}(1)} \leq \ldots \leq x_{l^{-1}}(n) \leq 1\}.$$

Let $e(P)$ be the set of linear extensions of P. Then, we can triangulate the order polytope with linear extension simplicies as

$$\mathcal{O}_P := \bigsqcup_{l \in e(P)} \Delta_l.$$

In the union, the simplices are disjoint except possibly in a set of measure zero. This implies that $\text{Vol}_n(\mathcal{O}_P) = e(P)/(n!)$. When P is the matroid on n elements with an empty partial order, we recover the familiar fact that the $\text{Vol}_n(\Delta) = 1/n!$ for a simplex Δ whose vertices lie in $\{0,1\}^n$ and no two vertices lie in the same affine hyperplane orthogonal to $(1, \ldots, 1)$.

Example 2.5.2 (Zonotope)**.** For any vectors $v_1, \ldots, v_l \in \mathbb{R}^n$, we can define the **zonotope** generated by these vectors by

$$Z(v_1, \ldots, v_l) := [0, v_1] + \ldots + [0, v_l] = \left\{ \sum_{i=1}^{l} \lambda_i v_i : \lambda_i \in [0,1] \right\}.$$

We call a polytope $P \subseteq \mathbb{R}^n$ **simple** if $\text{int}(P) \neq \emptyset$ and each of its vertices is contained in exactly n facets. We say two polytopes P_1, P_2 are called **strongly isomorphic** if $\dim F_{P_1}(u) = \dim F_{P_2}(u)$ for all $u \in \mathbb{R}^n \backslash \{0\}$. Strong isomorphism is an equivalence relation between polytopes which implies that two polytopes have isomorphic face lattices with corresponding faces being parallel to each other. To illustrate the strength of this equivalence relation, it can be shown that given two strongly isomorphic polytopes, the corresponding faces are also strongly isomorphic. This follows from the fact that the faces of a polytope come from the intersections of facets.

Lemma 2.5.1. *If P_1, P_2 are strongly isomorphic polytopes, then for each $u \in \mathbb{R}^n \backslash \{0\}$, the faces $F_{P_1}(u)$ and $F_{P_2}(u)$ are strongly isomorphic.*

Proof. See Lemma 2.4.10 in [49]. $\square$

Given two polytopes, it is not difficult to construct an infinite family of polytopes which are in the same strong isomorphism class. Indeed, any pair of positively-weighted Minkowski sums of two polytopes are strongly isomorphic.

Proposition 2.5.3. *If $P_1, P_2 \subseteq \mathbb{R}^n$ are polytopes which are strongly isomorphic, then $\lambda_1 P_1 + \lambda_2 P_2$ with $\lambda_1, \lambda_2 > 0$ are strongly isomorphic. If P_1, P_2 were strongly isomorphic to begin with, then the polytopes $\lambda_1 P_1 + \lambda_2 P_2$ with $\lambda_1, \lambda_2 \geq 0$ and $\lambda_1 + \lambda_2 > 0$ are strongly isomorphic.*

Proof. See Corollary 2.4.12 in [49]. $\square$

Any polytope is determined by its facets. The facets of a polytope are determined by their normal vectors and the polytope's support values in the direction of these normals. Let α be a strong isomorphism class of polytopes. All of the polytopes in this isomorphism class will share the same facet normals $\mathcal{U}$. These, all of the polytopes in α are *uniquely* determined by their support values at each of the facet normals. In other words, there is injective map $h : \alpha \to \mathbb{R}^{\mathcal{U}}$ from the strong isomorphism class to the finite-dimensional vector space $\mathbb{R}^{\mathcal{U}}$ defined by

$$h_P := h(P) = \left(H^-_{u, h_P(u)}\right)_{u \in \mathcal{U}}$$

For a polytope $P \in \alpha$, we call $h_P := h(P)$ the **support vector** of P. Clearly, the support vector map is not surjective. For example, if we pick a vector in $\mathbb{R}^{\mathcal{U}}$ in which all of the coordinates of $x \in \mathbb{R}^{\mathcal{U}}$ sufficiently negative, there is no corresponding polytope with those support values. In Lemma 2.5.2 we prove that simple strong isomorphic polytopes are robust under small perturbations. That is, after perturbing a polytope in α by a sufficiently small amount, it will remain in α. As a corollary, we prove that any vector in $\mathbb{R}^{\mathcal{U}}$ can be written as a scalar multiple of the difference of two support vectors.

Lemma 2.5.2. *Let P be a simple n-polytope with facet normals $\mathcal{U}$. Then, there is a number $\beta > 0$*

such that every polytope of the form

$$P' := \bigcap_{u \in \mathcal{U}} H^-_{u, h_P(u) + \alpha_u}$$

with $|\alpha_u| \leq \beta$ *is simple and strongly isomorphic to* P*.*

Corollary 2.5.1 (Lemma 5.1 in [51])**.** *Let* α *be the strong isomorphic class of a simple polytope* P *with facet normals* $\mathcal{U}$*. For any* $x \in \mathbb{R}^{\mathcal{U}}$ *there are* $a > 0$ *and* $Q \in \alpha$ *such that* $x = a(h_Q - h_P)$*.*

Proof. From Lemma 2.5.2 there exists $a^{-1} > 0$ sufficiently small and $Q \in \alpha$ such that $h_Q = a^{-1}(x + h_P)$. By rearranging the equation, we get $x = a(h_Q - h_P)$. □

Given a collection of convex bodies, we will be interested in approximating all of these polytopes simultaneously by polytopes which are simpe and in the same strong isomorphism class. To make sense of approximation of polytopes, we must equip the convex bodies with a metric structure. In Section 2.5.3, we will define a metric on the space of convex bodies called the Hausdorff metric.

2.5.3 Hausdorff Metric on Convex Bodies

We define a metric δ called the **Hausdorff metric** on K^n such that for any pair of elements $K, L \in \mathsf{K}^n$, we have

$$\begin{aligned} \delta(K, L) &= \max\left\{ \sup_{x \in K} \inf_{y \in L} |x - y|, \sup_{y \in L} \inf_{x \in K} |x - y| \right\} \\ &= \inf\{\varepsilon \geq 0 : K \subseteq L + \varepsilon B^n, L \subseteq K + \varepsilon B^n\} \\ &= \|h_K - h_L\|_\infty . \end{aligned}$$

For a proof of the equivalence of these three descriptions, we refer the reader to the proof of Theorem 3.2 in [29]. In Proposition 2.5.4, we prove that δ is a metric on the space of convex bodies. Theorem 2.5 implies that any set of convex bodies can be approximated by simple polytopes such that the polytopes in the approximation are strongly isomorphic.

Proposition 2.5.4. *The ordered pair* (K^n, δ) *is a metric space.*

Proof. For $K, L, M \in \mathsf{K}^n$, we have

$$\delta(K, M) = \|h_K - h_M\|_\infty \leq \|h_K - h_L\|_\infty + \|h_L - h_M\|_\infty = \delta(K, L) + \delta(L, M).$$

It is clear that $\delta(K, L) = \delta(L, K)$. Finally, we have $\delta(K, L) = 0$ if and only if $\|h_K - h_L\|_\infty = 0$. Since h_K and h_L are continuous function, this is true if and only if $h_K = h_L$. Since K and L are the intersections of their support hyperplanes, this implies that $K = L$. This suffices for the proof. $\square$

Theorem 2.5. *Let $K_1, \ldots, K_m \in \mathsf{K}^n$ be convex bodies. To every $\varepsilon > 0$, there are simple strongly isomorphic polytopes $P_1, \ldots, P_m$ such that $\delta(K_i, P_i) < \varepsilon$ for $i = 1, \ldots, m$.*

Proof. See Theorem 2.4.15 in [49]. $\square$

We define a few continuous functions with respect to the Hausdorff distance on K^n. This will allow us to compute the function values of general convex bodies through approximation by simple strongly isomorphic polytopes. Since convex bodies are compact, they are Lebesgue measurable. This implies that there is a well-defined function $\mathrm{Vol}_n : \mathsf{K}^n \to \mathbb{R}_{\geq 0}$ defined by

$$\mathrm{Vol}_n(K) := \int_{x \in \mathbb{R}^n} \mathbb{1}_K(x) \, d\lambda(x)$$

where λ is the Lebesgue measure on $\mathbb{R}^n$. From Theorem 1.8.20 in [49], the volume functional $\mathrm{Vol}_n(\cdot)$ is a continuous function on (K^n, δ). For another example of a continuous map, consider the projection map $p : \mathsf{K}^n \times \mathbb{R}^n \to \mathbb{R}^n$ which maps $(K, x) \mapsto p(K, x)$ where $p(K, x)$ is the projection of x onto K. For the proof that this map is continuous, see Section 1.8 in [49]. Finally, the map induced by the Minkowski sum $\mathsf{K}^n \times \mathsf{K}^n \to \mathsf{K}^n$ is continuous.

2.5.4 Mixed Volumes

Recall that in the setting of mixed discriminants, for $n \times n$ matrices $A_1, \ldots, A_m$ we had the identity

$$\det(\lambda_1 A_1 + \ldots + \lambda_m A_m) = \sum_{i_1, \ldots, i_n = 1}^{m} \lambda_{i_1} \ldots \lambda_{i_n} \cdot \mathsf{D}(A_{i_1}, \ldots, A_{i_n}) \tag{2.3}$$

In this section, we study the mixed volume analog of the left hand side of Equation 2.3. Specifically, for fixed convex bodies $K_1, \ldots, K_m$, we consider the function

$$(\lambda_1, \ldots, \lambda_m) \longmapsto \mathrm{Vol}_n\left(\lambda_1 K_1 + \ldots + \lambda_m K_m\right) \tag{2.4}$$

Similar to the situation in mixed volumes, the function described in Equation 2.4 ends up being a homogeneous polynomial of degree n. The coefficients of this polynomial become convex body analogs of the mixed discriminants called mixed volumes.

Example 2.5.3. In this example, we commute Equation 2.4 in the dimension $n = 2$. Let $K \subseteq \mathbb{R}^2$ be a polygon and $L \subseteq \mathbb{R}^2$ be the unit disc. Then by drawing $\lambda K + \mu L$ on the plane, we can compute

$$\mathrm{Vol}_2(\lambda K + \mu L) = \lambda^2 \,\mathrm{Vol}_2(K) + \lambda\mu \cdot \mathrm{perimeter}(K) + \mu^2 \,\mathrm{Vol}_2(L).$$

Note that the coefficients of the polynomial encode geometric information about K and L. These coefficients will be the mixed volumes of the convex bodies in the mixed Minkowski sum.

We first define mixed volumes for polytopes in Definition 2.5.3. Then we prove that the expression for mixed volumes for polytopes admits a continuous extension to all convex bodies. This continuous extension will be the definition for mixed volumes for arbitrary convex bodies.

Definition 2.5.3. For $n \geq 2$, let $P_1, \ldots, P_n \subseteq \mathbb{R}^n$ be polytopes and let $\mathcal{U}$ be the set of unit facet normals of $P_1 + \ldots + P_{n-1}$. We define their mixed volume $\mathsf{V}_n(P_1, \ldots, P_n)$ inductively by

$$\mathsf{V}_n(P_1, \ldots, P_n) = \frac{1}{n} \sum_{u \in \mathcal{U}} h_{P_n}(u) \cdot \mathsf{V}_{n-1}(F_{P_1}(u), \ldots, F_{P_{n-1}}(u)).$$

On the right hand side of the equation though $F_{P_1}(u), \ldots, F_{P_{n-1}}(u)$ are in $\mathbb{R}^n$, since they are in parallel hyperplanes we can consider them as subsets of $\mathbb{R}^{n-1}$ by projecting them orthogonally on the same hyperplane isomorphic to $\mathbb{R}^{n-1}$. Thus, the mixed volume $\mathsf{V}_{n-1}(F_{P_1}(u), \ldots, F_{P_{n-1}}(u))$ is well-defined. For $n = 1$, we define $V_1([a, b]) = b - a$.

Theorem 2.6 (Theorem 3.7 in [29])**.** *Let $P_1, \ldots, P_m \subseteq \mathbb{R}^n$ be polytopes. Then, we have*

$$\mathrm{Vol}_n(\lambda_1 P_1 + \ldots + \lambda_m P_m) = \sum_{i_1, \ldots, i_n = 1}^{m} \lambda_{i_1} \ldots \lambda_{i_n} \cdot \mathsf{V}_n(P_{i_1}, \ldots, P_{i_n}).$$

Proof. It suffices to prove the equality when $\lambda_1, \ldots, \lambda_m > 0$. The general expansion will follow from the continuity of polynomials, the volume functional, and Minkowski sums. Since the λ_i are all strictly positive, Proposition 2.5.3 implies that $\lambda_1 P_1 + \ldots + \lambda_m P_m$ and $P_1 + \ldots + P_m$ are strongly isomorphic. In particular, the set of facet normals $\mathcal{U}$ are the same for both of these polytopes. We can compute

$$\begin{aligned}
\mathrm{Vol}_n(\lambda_1 P_1 + \ldots + \lambda_m P_m) &= \frac{1}{n}\sum_{u\in\mathcal{U}} h_{\lambda_1 P_1+\ldots+\lambda_m P_m}(u)\cdot \mathrm{Vol}_{n-1}\left(F_{\lambda_1 P_1+\ldots+\lambda_m P_m}(u)\right)\\
&= \frac{1}{n}\sum_{u\in\mathcal{U}}\sum_{i=1}^{n}\lambda_i h_{P_i}(u)\cdot \mathrm{Vol}_{n-1}\left(\sum_{i=1}^{m}\lambda_i F_{P_i}(u)\right)\\
&= \frac{1}{n}\sum_{u\in\mathcal{U}}\sum_{i_1,\ldots,i_n=1}^{m}\lambda_{i_1}\ldots\lambda_{i_n}\cdot h_{P_{i_n}}(u)\cdot \mathsf{V}_{n-1}\left(F_{P_{i_1}}(u),\ldots,F_{P_{i_{n-1}}}(u)\right)\\
&= \sum_{i_1,\ldots,i_n=1}^{m}\lambda_{i_1}\ldots\lambda_{i_n}\sum_{u\in\mathcal{U}}\frac{1}{n}h_{P_{i_n}}(u)\cdot \mathsf{V}_{n-1}\left(F_{P_{i_1}}(u),\ldots,F_{P_{i_{n-1}}}(u)\right)\\
&= \sum_{i_1,\ldots,i_n=1}^{m}\lambda_{i_1}\ldots\lambda_{i_n}\cdot \mathsf{V}_n\left(F_{P_{i_1}}(u),\ldots,F_{P_{i_n}}(u)\right).
\end{aligned}$$

This suffices for the proof. □

From Definition 2.5.3, it is not clear that the mixed volume is symmetric. The symmetry for mixed volumes will follow from the inversion formula given in Theorem 2.7. This result was prove [49] as Lemma 5.1.4. This result will not only prove that the mixed volume is symmetric, but it gives a continuous extension of the mixed volume to all convex bodies.

Theorem 2.7 (Inversion Formula, Lemma 5.1.4 in [49])**.** *For polytopes $P_1,\ldots,P_n \subseteq \mathbb{R}^n$, we have*

$$\begin{aligned}
\mathsf{V}_n(P_1,\ldots,P_n) &= \frac{1}{n!}\sum_{k=1}^{n}(-1)^{n+k}\sum_{1\le r_1<\ldots<r_k\le n}\mathrm{Vol}_n(P_{r_1}+\ldots+P_{r_k})\\
&= \frac{1}{n!}\sum_{I\subseteq[n]}(-1)^{n-|I|}\,\mathrm{Vol}_n\left(\sum_{i\in I}P_i\right).
\end{aligned}$$

Proof. We present the proof in [49] for the sake of completeness. We can define the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(\lambda_1,\ldots,\lambda_n) := \frac{1}{n!}\sum_{k=1}^{n}(-1)^{n+k}\sum_{1\le r_1<\ldots<r_k\le n}\mathrm{Vol}_n(\lambda_{r_1}P_{r_1}+\ldots+\lambda_{r_k}P_{r_k}).$$

From Theorem 2.6, we get that f is either 0 or a homogeneous polynomial of degree n. If we let $\lambda_1 = 0$, then we have

$$n!(-1)^n f(0,\lambda_2,\ldots,\lambda_n) := S_1 + \sum_{k=2}^{n} S_k$$

where we define $S_1 = -\sum_{2 \leq r_1 \leq n} \mathrm{Vol}_n(\lambda_{r_1} P_{r_1})$ and for $k \geq 2$ we define

$$S_k = \sum_{k=1}^{n} (-1)^k \left(\sum_{2 \leq r_2 < \ldots < r_k \leq n} \mathrm{Vol}_n(\lambda_{r_2} P_{r_2} + \ldots + \lambda_{r_k} P_{r_k}) + \sum_{2 \leq r_1 < \ldots < r_k \leq n} \mathrm{Vol}_n(\lambda_{r_1} P_{r_1} + \ldots + \lambda_{r_k} P_{r_k}) \right).$$

Note that this is a telescoping sum and $f(0, \lambda_2, \ldots, \lambda_n) = 0$. By symmetry, we know that f is a scalar multiple of $\lambda_1 \ldots \lambda_n$. The only term in f which contributes $\lambda_1 \ldots \lambda_n$ is in $\mathrm{Vol}_n(\sum_{i=1}^n \lambda_i P_i)$. Thus, we have that

$$f(\lambda_1, \ldots, \lambda_n) = \lambda_1 \ldots \lambda_n \cdot \mathsf{V}_n(P_1, \ldots, P_n).$$

By substituting $\lambda_1 = \ldots = \lambda_n = 1$, this completes the proof. □

As an application of the inversion formula, we will compute the mixed volume of a collection of line segments. This calculation will be instrumental for the calculation of the volume of a general zonotope.

Example 2.5.4. Let $v_1, \ldots, v_n \in \mathbb{R}^n$ be vectors. From Theorem 2.7, we have that

$$\mathsf{V}_n([0, v_1], \ldots, [0, v_n]) = \frac{1}{n!} \mathrm{Vol}_n([0, v_1] + \ldots + [0, v_n])$$

where the other summands are zero since they are contained in lower dimensional affine spaces. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear map which sends $Te_i = v_i$ for $1 \leq i \leq n$. Then, $[0, v_1] + \ldots + [0, v_n] = T([0, 1]^n)$. This gives us the identity

$$\mathsf{V}_n([0, v_1], \ldots, [0, v_n]) = \frac{1}{n!} \mathrm{Vol}_n(T([0, 1]^n)) = \frac{1}{n!} |\mathrm{Det}(v_1, \ldots, v_n)|.$$

Since the volume functional and Minkowski sum is continuous on the space of convex bodies, the function on the right hand side of Theorem 2.7 is also a continuous function on the space of convex bodies. This allows us to extend the definition of mixed volumes to all convex bodies.

Definition 2.5.4. Let $K_1, \ldots, K_n \subseteq \mathbb{R}^n$ be convex bodies. Then, we define the **mixed volume** of $K_1, \ldots, K_n$ as

$$\mathsf{V}_n(K_1, \ldots, K_n) := \frac{1}{n!} \sum_{I \subseteq [n]} (-1)^{n-|I|} \mathrm{Vol}_n \left(\sum_{i \in I} P_i \right).$$

The function defined in Definition 2.7 is continuous. Let $K_1, \ldots, K_n \subseteq \mathbb{R}^n$ be a collection of convex bodies. Given a sequence $P_k^{(i)}$ of polytopes for all $1 \leq i \leq n$ satisfying $P_i^{(k)} \longrightarrow K_i$ as $k \to \infty$

for all $1 \leq i \leq n$, we have

$$\mathsf{V}_n(K_1, \ldots, K_n) = \lim_{k\to\infty} \mathsf{V}_n(P_1^{(k)}, \ldots, P_n^{(k)})$$

from the continuity of the mixed volume. This allows us to extend Theorem 2.6 to general convex bodies.

Theorem 2.8. *Let $K_1, \ldots, K_m \subseteq \mathbb{R}^n$ be convex bodies. Then, we have*

$$\mathrm{Vol}_n(\lambda_1 K_1 + \ldots + \lambda_m K_m) = \sum_{i_1,\ldots,i_n=1}^{m} \lambda_{i_1} \ldots \lambda_{i_n} \cdot \mathsf{V}_n(K_{i_1}, \ldots, K_{i_n}).$$

Proof. For $1 \leq i \leq n$, let $P_i^{(k)}$ for $k \geq 1$ be a sequence of polytopes converging to K_i in Hausdorff distance. Then, we have

$$\begin{aligned}
\mathrm{Vol}_n\left(\sum_{i=1}^{n} \lambda_i K_i\right) &= \lim_{k\to\infty} \mathrm{Vol}_n\left(\sum_{i=1}^{n} \lambda_i P_i^{(k)}\right) \\
&= \lim_{k\to\infty} \sum_{i_1,\ldots,i_n=1}^{m} \lambda_{i_1} \ldots \lambda_{i_n} \cdot \mathsf{V}_n(P_{i_1}^{(k)}, \ldots, P_{i_n}^{(k)}) \\
&= \sum_{i_1,\ldots,i_n=1}^{m} \lambda_{i_1} \ldots \lambda_{i_n} \cdot \lim_{k\to\infty} \mathsf{V}_n(P_{i_1}^{(k)}, \ldots, P_{i_n}^{(k)}) \\
&= \sum_{i_1,\ldots,i_n=1}^{m} \lambda_{i_1} \ldots \lambda_{i_n} \cdot \mathsf{V}_n(K_{i_1}, \ldots, K_{i_n}).
\end{aligned}$$

This suffices for the proof. $\square$

Example 2.5.5 (Volume of a Zonotope)**.** From Theorem 2.8, we can compute the volume of a zonotope. Let $v_1, \ldots, v_l \in \mathbb{R}^n$ be vectors. Then, we have that

$$\begin{aligned}
\mathrm{Vol}_n(Z(v_1, \ldots, v_l)) &= \mathrm{Vol}_n\left(\sum_{i=1}^{l} [0, v_i]\right) \\
&= \sum_{i_1,\ldots,i_n=1}^{l} \mathsf{V}_n([0, v_{i_1}], \ldots, [0, v_{i_n}]) \\
&= \sum_{i_1,\ldots,i_n=1}^{l} \frac{1}{n!} |\mathrm{Det}(v_{i_1}, \ldots, v_{i_n})| \\
&= \sum_{1\leq i_1<\ldots<i_n\leq l} |\mathrm{Det}(v_{i_1}, \ldots, v_{i_n})|
\end{aligned}$$

where the equality in the third line follows from the computation in Example 2.5.4.

We now define the notion of mixed area measures, a measure-theoretic interpretation of mixed volumes. This notion will become essential when considering the equality cases of the Alexandrov-Fenchel inequality. Our treatment of mixed area measures is inspired by Chapter 4 in [29]. Recall that for polytopes $P_1, \ldots, P_n$, we define the mixed volume as

$$\mathsf{V}_n(P_1, \ldots, P_n) = \frac{1}{n} \sum_{u \in \mathbb{S}^{n-1}} h_{P_n}(u) \cdot \mathsf{V}_{n-1}(F_{P_1}(u), \ldots, F_{P_{n-1}}(u)).$$

The sum is well-defined because the number of facet normals for polytopes is finite. Outside of these facet normals, the mixed volume inside the sum vanishes. For any convex body $K \subseteq \mathbb{R}^n$ which is not necessarily a polytope, we can take a sequence of polytopes convering to K to get the identity

$$\mathsf{V}_n(P_1, \ldots, P_{n-1}, K) = \frac{1}{n} \sum_{u \in \mathbb{S}^{n-1}} h_K(u) \cdot \mathsf{V}_{n-1}(F_{P_1}(u), \ldots, F_{P_{n-1}}(u)).$$

Measure-theoretically, we can define a measure on the unit sphere defined by

$$S_{P_1, \ldots, P_{n-1}} := \sum_{u \in \mathbb{S}^{n-1}} \mathsf{V}_{n-1}(F_{P_1}(u), \ldots, F_{P_{n-1}}(u)) \cdot \delta_u$$

where δ_u is the dirac delta measure at u. Then, we have

$$\mathsf{V}_n(P_1, \ldots, P_{n-1}, K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K(u) \, S_{P_1, \ldots, P_{n-1}}(du).$$

We have defined this measure in the case where the first $n-1$ convex bodies in the mixed volume are polytopes. From the Riesz representation theorem, it can be shown that such a measure exists for general convex bodies (see Theorem 4.1 in [29] and Theorem 2.14 in [47]). The existence of this measure will follow from Theorem 2.9.

Theorem 2.9. *For convex bodies $K_1, \ldots, K_{n-1} \subseteq \mathbb{R}^n$, there exists a uniquely determined finite Borel measure $S_{K_1, \ldots, K_{n-1}}$ on $\mathbb{S}^{n-1}$ such that*

$$\mathsf{V}(K_1, \ldots, K_{n-1}, K) = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K(u) \, S_{K_1, \ldots, K_{n-1}}(du)$$

for all convex bodies $K \subseteq \mathbb{R}^n$.

Proof. For a proof of this result, see Theorem 4.1 in [29]. □

Definition 2.5.5. For convex bodies $K_1, \ldots, K_{n-1} \subseteq \mathbb{R}^n$, we call the measure $S_{K_1, \ldots, K_{n-1}}$ the

mixed area measure associated with $(K_1, \dots, K_{n-1})$.

Just as the mixed volume is a generalization of volume, the mixed area measure is a generalization of surface area. For example, in the case our convex bodies satisfy $K_1 = \dots = K_{n-1} = \mathbb{B}^n$ where $\mathbb{B}^n$ is the closed unit ball, we have that $S_{\mathbb{B}^n,\dots,\mathbb{B}^n}$ is exactly the spherical Lebesgue measure on $\mathbb{S}^{n-1}$. Thus, for any convex body $K \subseteq \mathbb{R}^n$ have

$$\mathsf{V}_n(K, \mathbb{B}^n, \dots, \mathbb{B}^n) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_K(u)\, S_{\mathbb{B}^n,\dots,\mathbb{B}^n}(du) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_K(u)\, \sigma(du).$$

This implies that the mixed volume $\mathsf{V}_n(K, \mathbb{B}^n, \dots, \mathbb{B}^n)$ is exactly the surface area of K. This calculation explains the appearance of the perimeter in Example 2.5.3. In the next section, we discuss positivty of mixed volumes and the support of mixed area measures.

2.5.5 Positivity and Normal Directions

We begin this section with necessary and sufficient conditions for a mixed volume to be strictly positive. The conditions are collected in Lemma 2.5.3 for reference. For a proof of this result, we refer the reader to Lemma 2.2 in [53].

Lemma 2.5.3 (Lemma 2.2 in [53]). *For conex bodies $C_1, \dots, C_n \subseteq \mathbb{R}^n$, the following two conditions are equivalent.*

(a) $\mathsf{V}_n(C_1, \dots, C_n) > 0$.

(b) There are segments $I_i \subseteq C_i$, $i \in [n]$ with linearly independent directions.

(c) $\dim(C_{i_1} + \dots + C_{i_k}) \geq k$ *for all* $k \in [n]$, $1 \leq i_1 < \dots < i_k \leq n$.

Next, we provide necessary and sufficient conditions for a vector to be in the support of the mixed area measure $S_{B,P_1,\dots,P_{n-2}}$ where $P_1, \dots, P_{n-2}$ are polytopes. The terminology used in Definition 2.5.6 was introduced in Lemma 2.3 of [53].

Definition 2.5.6 (Lemma 2.3 in [53]). For $P_1, \dots, P_{n-2} \subseteq \mathbb{R}^n$ convex polytopes and $u \in \mathbb{S}^{n-1}$. We call the vector u a $(B, P_1, \dots, P_{n-2})$-extreme **normal direction** if and only if at least one of the following three equivalent conditions hold:

(a) $u \in \operatorname{supp} S_{B,P_1,\dots,P_{n-2}}$.

(b) There are segments $I_i \subseteq F(P_i, u)$, $i \in [n-2]$ with linearly independent directions.

(c) $\dim(F(P_{i_1}, u) + \ldots + F(P_{i_k}, u)) \geq k$ for all $k \geq [n-2]$, $1 \leq i_1 < \ldots < i_k \leq n-2$.

The fact that (a)-(c) are equivalent in Definition 2.5.6 is exactly the content of Lemma 2.3 in [53]. Both Lemma 2.5.3 and Definition 2.5.6 will come to the fore in later chapters when we write about Kahn-Saks inequality.

Chapter 3

Mechanisms for Log-concavity

In this chapter, we study a handful of tools which have been used to prove log-concavity conjectures in combinatorics. We begin the chapter with an introduction on log-concavity, ultra-log-concavity, and some of the log-concavity phenomenon present in combinatorics. Then, we prove Cauchy's interlacing theorem. This theorem has been used frequently for example in the theory of Lorentzian polynomials in [12] and Hodge Riemann relations in [37]. In the next two sections, we introduce Alexandrov's inequality for mixed discriminants as well as the Alexandrov-Fenchel inequality for mixed volumes. We will also consider the equality cases of these inequalities. We will apply the results about equality in later chapters. We end the chapter on a section on Lorentzian polynomials and its analog to the Alexandrov-Fenchel inequality. According to private communications with Ramon van Handel, there exist avenues to characterize the equality cases of the Lorentzian analogs to the Alexandrov-Fenchel inequality. In this chapter, we were only able to cover an ε percentage of the tools used to prove log-concavity conjectures. For a wider survey of the techniques available, we refer the reader to [56]. We also chose not to discuss the recent technology of the combinatorial atlas [14, 13] due to space and time constraints.

3.1 Log-concavity and Ultra-log-concavity

In this thesis we are interested in the log-concavity and ultra-log-concavity of sequences which come from combinatorial structures. Let $a_1, \ldots, a_n$ be a sequence of non-negative numbers. We say that the sequence is log-concave if we have $a_i^2 \geq a_{i-1}a_{i+1}$ for all $1 \leq i \leq n$ where $a_{-1} = a_{n+1} = 0$. We say that the sequence is ultra-log-concave if the sequence $a_i/\binom{n}{i}$ is log-concave. Perhaps the most prototypical example of an ultra-log-concave sequence which arises from combinatorics is the sequence of binomial coefficients $\binom{n}{k}$. Ultra-log-concavity is a stronger condition than log-concave. In the literature, there are many instances in which a conjecture was made about a sequence being unimodal, and then the conjecture was solved by proving that it is log-concave. Indeed, a log-concave sequence with no internal zeroes will automatically be unimodal. log-concavity conjectures started out as unimodality conjectures. This is another occurence of the common phenomenon in mathematics where it is easer to prove a stronger result than what was asked. We now give some examples of solved log-concavity conjectures in combinatorics.

Example 3.1.1 (Read's Conjecture and Heron-Rota-Welsh Conjecture)**.** Let G be a finite graph and let $\chi_G(x)$ be the chromatic polynomial of G. In his 1968 paper [45], Ronald Read conjectured that the absolute values of the coefficients of the chromatic polynomial of a graph G are unimodal. The conjecture was then strengthed by Heron, Rota, and Welsh to the log-concavity of the coefficients of the characteristic polynomial of an arbitrary matroid. Read's conjecture was first proved by June Huh in his paper [30] as a result of proving the Heron-Rota-Welsh conjecture for representable matroids. The Heron-Rota-Welsh conjecture then proved in full generality by Karim Adiprasito, June Huh, and Eric Katz in their paper [1] by proving that the Chow ring satisfies the hard Lefschetz theorem and Hodge-Riemann relations.

Example 3.1.2 (The Mason Conjectures)**.** Let M be a matroid of rank r. For $0 \leq i \leq r$, let I_i denote the number of independent sets of M with i elements. Then there are three conjectures of increasing strength related to the log-concavity of this sequence.

1. (Mason Conjecture) $I_k^2 \geq I_{k-1}I_{k+1}$.
2. (Strong Mason Conjecture) $I_k^2 \geq \left(1 + \frac{1}{k}\right) I_{k-1}I_{k+1}$.
3. (Ultra-Strong Mason Conjecture) $I_k^2 \geq \left(1 + \frac{1}{k}\right)\left(1 + \frac{1}{n-k}\right) I_{k-1}I_{k+1}$.

The Mason conjecture says that the sequence I_i is log-concave while the ultra-strong Mason conjecture states that the sequence I_i is ultra-log-concave. All three of these conjectures have been proven. In [1], Karim Adiprasito, June Huh, and Eric Katz proved the Mason conjecture using a hodge theory for matroids. Later, the strong Mason conjecture was proven by June Huh, Benjamin Schro"ter, and Botong Wang in [32] and Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant in [3] independently.

3.2 Cauchy's Interlacing Theorem

Given an $n \times n$ matrix A, we call a submatrix B a **principal submatrix** if it is obtained from A by deleting some rows and the corresponding columns. When A is a Hermitian matrix and B is a principal submatrix of size $(n-1) \times (n-1)$, Theorem 3.1 gives a mechanism for controlling the eigenvalues of B based on A. We present a short proof from [22] of this result. For different proofs using the intermediate value theorem, Sylvester's law of inertia, and the Courant-Fischer minimax theorem, we refer the reader to [33], [42], and [26], respectively.

Theorem 3.1 (Cauchy Interlace Theorem)**.** *Let A be a Hermitian matrix of order n, and let B be a principal submatrix of A of order $n-1$. If $\lambda_n \leq \lambda_{n-1} \leq \ldots \leq \lambda_2 \leq \lambda_1$ are the eigenvalues of A and $\mu_n \leq \mu_{n-1} \leq \ldots \leq \mu_3 \leq \mu_2$ are the eigenvalues of B. Then, we have*

$$\lambda_n \leq \mu_n \leq \lambda_{n-1} \leq \mu_{n-1} \leq \ldots \leq \lambda_2 \leq \mu_2 \leq \lambda_1.$$

Before proving this result, we need Theorem 3.2. This is a theorem about when roots of polynomials interlace. Suppose that $f, g \in \mathbb{R}[x]$ are real polynomials with only real roots. We say that the polynomials f and g **interlace** if their roots $r_1 \leq \ldots \leq r_n$ and $s_1 \leq \ldots \leq s_{n-1}$ satisfy

$$r_1 \leq s_1 \leq \ldots \leq r_{n-1} \leq s_{n-1} \leq r_n.$$

Theorem 3.2. *Let $f, g \in \mathbb{R}[x]$ be polynomials with only only real roots. Suppose that $\deg(f) = n$ and $\deg(g) = n-1$. Then, f and g interlace if and only if the linear combinations $f + \alpha g$ have all real roots for all $\alpha \in \mathbb{R}$.*

Proof. See Theorem 6.3.8 in [44]. $\square$

Proof of Theorem 3.1. We follow the proof in [22]. Without loss of generality, we can decompose

$$A = \begin{bmatrix} B & v \\ v^T & c \end{bmatrix}$$

where $v \in \mathbb{R}^{(n-1)\times 1}$ and $c \in \mathbb{R}$. Consider the polynomials $f(x) := \det(A - xI)$ and $g(x) := \det(B - xI)$. Note that the roots of f and g are all real and are exactly the eigenvalues of A and B, respectively. For any $\alpha \in \mathbb{R}$, we have

$$\begin{aligned} f(x) + \alpha g(x) &= \det \begin{bmatrix} B - xI & v \\ v^T & d - x \end{bmatrix} + \alpha \det \begin{bmatrix} B - xI & v \\ 0 & 1 \end{bmatrix} \\ &= \det \begin{bmatrix} B - xI & v \\ v^T & d - x \end{bmatrix} + \det \begin{bmatrix} B - xI & v \\ 0 & \alpha \end{bmatrix} \\ &= \det \begin{bmatrix} B - xI & v \\ v^T & d + \alpha - x \end{bmatrix}. \end{aligned}$$

Thus $f + \alpha g$ is the characteristic polynomial of a Hermitian matrix and has real roots. From Theorem 3.2, the proof is complete. □

Corollary 3.2.1. *Let M be a $n \times n$ matrix with exactly one positive eigenvalue. If A is a principal matrix A of size 2×2 such that there exists $v \in \mathbb{R}^2$ with $v^T A v > 0$, then* $\det(A) \leq 0$.

Proof. From Theorem 3.1, the matrix A has at most one positive eigenvalue. Since $v^T A v > 0$, we know that A has exactly one positive eigenvalue. The other eigenvalue must be at most 0. Hence, we have $\det(A) \leq 0$. This suffices for the proof. □

3.3 Alexandrov's Inequality for Mixed Discriminants

In this section we prove a log-concavity inequality involving mixed discriminants. The inequality, call the Alexandrov inequality for mixed discriminants, is given in the statement of Theorem 3.3. This inequality is similar to Alexandrov-Fenchel inequality in Section 3.4, which is the analog for mixed volumes of convex bodies.

Theorem 3.3 (Alexandrov's Inequality for Mixed Discriminants)**.** *Let $A_1, \ldots, A_{n-2}$ be real symmetric positive definite $n \times n$ matrices. Let X be a real symmetric positive definite $n \times n$ square matrix and let Y be a real symmetric positive semidefinite $n \times n$ square matrix. Then*

$$\mathsf{D}(X, Y, A_1, \ldots, A_{n-2})^2 \geq \mathsf{D}(X, X, A_1, \ldots, A_{n-2}) \cdot \mathsf{D}(Y, Y, A_1, \ldots, A_{n-2})$$

where equality holds if and only if $B = \lambda A$ for a real number λ.

We will provide a self-contained proof of Theorem 3.3 given in [9] and [48]. Before we begin the proof, we first need the following result from [27].

Lemma 3.3.1 ([27] and Lemma 5.3.2 in [9]). *Let $p(x) = \sum_{k=0}^{n} \binom{n}{k} a_k x^k$ be a real-rooted univariate polynomial. Then, we have $a_k^2 \geq a_{k-1}a_{k+1}$ for all $1 \leq i \leq n-1$. If $a_0 \neq 0$ then equality occurs for any i if and only if all the roots of $f(x)$ are equal.*

Proof. From Rolle's Theorem, if a polynomial is real-rooted, then so is all of its derivatives. It is also clear that when we reverse the coefficients of a real-rooted polynomial, then it remains real-rooted. We call the operation for reversing the coefficients **reciprocation** of the polynomial. This implies through a sequence of derivatives and reciprocations, we get the polynomial $\alpha_{i-1} + 2\alpha_i x + \alpha_{i+1}x^2$. Since we arrived at this polynomial throuhg a sequence of derivatives and reciprocations, we know that this polynomial is real-rooted. Hence, the discriminant is non-negative and $\alpha_i^2 \geq \alpha_{i-1}\alpha_{i+1}$. The equality case follows from the fact that Rolle's Theorem implies that the roots of the derivative of a real-rooted polynomial interlaces the roots of the original polynomial. $\square$

Proof of Theorem 3.3. Let $A_1, \ldots, A_m$ be positive semi-definite $n \times n$ matrices. For $(r_1, \ldots, r_m) \in \Delta_m^n$ we define the term

$$D(r_1, \ldots, r_m) := \mathsf{D}_n \left(\underbrace{A_1, \ldots, A_1}_{r_1 \text{ times}}, \ldots, \underbrace{A_m, \ldots, A_m}_{r_m \text{ times}} \right).$$

It suffices to prove that

$$D(r_1, \ldots, r_{m-2}, r_{m-1}, r_m)^2 \geq D(r_1, \ldots, r_{m-2}, r_{m-1}+1, r_m - 1) \cdot D(r_1 \ldots, r_{m-2}, r_{m-1}-1, r_m+1).$$

To prove this, we induct on m. Suppose that $m = 2$. Consider the polynomial

$$\det(xA_1 + A_2) = \sum_{i=0}^{n} \binom{n}{i} D(i, r-i)x^i.$$

Since A_1 is positive definite, there is some positive definite Σ_1 such that $\Sigma_1^2 = A_1$. Thus, we have

$$\det(xA_1 + A_2) = \det(x\Sigma_1^2 + A_2) = \det(\Sigma_1)^2 \cdot \det(x + \Sigma_1^{-1}A_2\Sigma_1^{-1}).$$

Thus, the polynomial $\det(xA_1 + A_2)$ is real-rooted. From Lemma 3.3.1, this implies that $D(i, r-i)^2 \geq D(i-1, r-i+1) \cdot D(i+1, r-i-1)$ where equality holds whenever all of the roots are the same. In this case, we have $\Sigma_1^{-1}A_2\Sigma_1^{-1} = \lambda I \implies A_2 = \lambda A_1$ for some $\lambda \in \mathbb{R}$. This proves the base case.

Now suppose that the claim is true for m. Consider a $m+1$-tuple $(r_1, \ldots, r_{m+1}) \in \Delta_{m+1}^n$

satisfying $r_1 + \ldots + r_{m+1} = n$. We want to prove that

$$D(r_1, \ldots, r_{m-1}, r_m, r_{m+1})^2 \geq D(r_1, \ldots, r_{m-1}, r_m - 1, r_{m+1} + 1) \cdot D(r_1, \ldots, r_{m-1}, r_m + 1, r_{m+1} - 1).$$

We can define $B_y := yA_m + A_{m+1}$ where y is a real parameter. Then, we have that

$$\det\left(\sum_{i=1}^{m-1} x_i A_i + x_m B_y\right) = \sum_{r_1+\ldots+r_m=n} \binom{n}{r_1, \ldots, r_m} D(r_1, \ldots, r_m)(y) \cdot x_1^{r_1} \ldots x_m^{r_m}$$

where $D(r_1, \ldots, r_m)(y)$ is a polynomial in y of degree r_m. We can also write

$$\det\left(\sum_{i=1}^{m-1} x_i A_i + x_m B_y\right) = \det\left(\sum_{i=1}^{m-1} x_i A_i + x_m y A_m + x_m A_{m+1}\right)$$

$$= \sum_{r_1+\ldots+r_{m+1}=n} \binom{n}{r_1, \ldots, r_{m+1}} D_{m+1}(r_1, \ldots, r_{m+1}) \cdot (x_1^{r_1} \ldots x_{m-1}^{r_{m-1}}) \cdot x_m^{r_m+r_{m+1}} \cdot y^{r_m}$$

$$= \sum_{r_1+\ldots+r_{m-1}+r_m=n} \left\{\sum_{i=0}^{r_m} \binom{n}{r_1, \ldots, r_{m-1}, i, r_m - i} D_{m+1}(r_1, \ldots, r_{m-1}, i, r_m - i) \cdot y^i\right\} \cdot x_1^{r_1} \cdot x_m^{r_m}.$$

Comparing coefficients, we have that

$$\begin{aligned} D(r_1, \ldots, r_m)(y) &= \sum_{i=0}^{r_m} \binom{n}{r_1, \ldots, r_{m-1}, i, r_m - i} D_{m+1}(r_1, \ldots, r_{m-1}, i, r_m - i) y^i \\ &= \binom{n}{r_1, \ldots, r_m} \sum_{i=0}^{r_m} \binom{r_m}{i} D_{m+1}(r_1, \ldots, r_{m-1}, i, r_m - i) y^i. \end{aligned}$$

Now we show that if A_m and A_{m+1} are not proportional, then $D(r_1, \ldots, r_m)(y)$ is real-rooted with simple roots. If this is shown, then we are done from Lemma 3.3.1. Following the notation in [48], for fixed $r_1 + \ldots + r_{m-2} \neq n$, we define the polynomial

$$\begin{aligned} Q_k(y) &:= D_m(r_1, \ldots, r_{m-2}, s - k, k)(y) \\ &= \mathsf{D}_m(\underbrace{A_1, \ldots, A_1}_{r_1}, \ldots, \underbrace{A_{m-2}, \ldots, A_{m-2}}_{r_{m-2}}, \underbrace{A_{m-1}, \ldots, A_{m-1}}_{s-k}, \underbrace{B_y, \ldots, B_y}_{k}). \end{aligned}$$

where $s = n - r_1 - \ldots - r_{m-2} = r_{m-1} + r_m$. From the inductive hypothesis, we have

$$Q_k(y)^2 \geq Q_{k-1}(y) \cdot Q_{k+1}(y)$$

where equality holds if and only if $B_y = \lambda_y A_{m-1}$ for some $\lambda_y \in \mathbb{R}$. Let y_0 be a root of Q_k. If equality held at y_0, we would have $0 = Q_k^2(y_0) = Q_{k-1}(y_0)Q_{k+1}(y_0)$. We have $B_{y_0} = \lambda A_{m-1}$ for some $\lambda \in \mathbb{R}$ from the inductive hypothesis. We have

$$y_0 A_m + A_{m+1} = \lambda A_{m-1}.$$

Note that $\lambda \neq 0$ because then A_m and A_{m+1} would be proportional to each other. If $\lambda > 0$, then B_{y_0} would be positive definite. But Corollary 2.4.1 implies that $Q_k(y_0) > 0$, which cannot happen. If $\lambda < 0$, then B_{y_0} is negative definite. This implies that the sign of $Q_k(y_0)$ is $(-1)^k$ and is not zero. This also cannot happen. It must be the case that at all roots of $Q_k(y)$ there is strict inequality.

Now we prove that $Q_k(y)$ has k simple real roots where the roots of $Q_{k-1}(y)$ and $Q_k(y)$ are interlaced for $2 \leq k \leq s$. To prove that $Q_1(y)$ satisfies the claim, note that

$$\begin{aligned}
Q_1(y) &= D_m(r_1, \dots, r_{m-2}, s-1, 1)(y) \\
&= \mathsf{D}_m(\underbrace{A_1, \dots, A_1}_{r_1}, \dots, \underbrace{A_{m-2}, \dots, A_{m-2}}_{r_{m-2}}, \underbrace{A_{m-1}, \dots, A_{m-1}}_{s-1}, B_y) \\
&= \mathsf{D}_m(A_1[r_1], \dots, A_{m-2}[r_{m-2}], A_{m-1}[s-1], A_m)y + \mathsf{D}_m(A_1[r_1], \dots, A_{m-2}[r_{m-2}], A_{m-1}[s-1], A_{m+1}).
\end{aligned}$$

The coefficient in front of y is positive from Corollary 2.4.1. Thus the polynomial $Q_1(y)$ has 1 simple real zero. Let $q_1 := q_1^{(1)}$ be the unique root of this polynomial. Then, from our earlier discussion, we have the strict inequality $Q_1(q_1)^2 > Q_0(q_1) \cdot Q_2(q_1)$. This implies that $Q_2(q_1) < 0$ since $Q_0(q_1) > 0$ from Corollary 2.4.1. Since $Q_2(y)$ is a quadratic with a positive leading coefficient and $Q_2(q_1) < 0$, the intermediate value theorem implies that there exist $q_2^{(1)}$ and $q_2^{(2)}$ which are roots of Q_2 and $q_2^{(1)} < q_1^{(1)} < q_2^{(2)}$. Now, suppose that the roots of Q_{k-1} are $q_{k-1}^{(1)}, \dots, q_{k-1}^{(k-1)}$ and the roots of Q_k are $q_k^{(1)}, \dots, q_k^{(k)}$ satisfy

$$q_k^{(1)} < q_{k-1}^{(1)} < q_k^{(2)} < \ldots < q_k^{(k-1)} < q_{k-1}^{(k-1)} < q_k^{(k)}.$$

Note that we have

$$Q_k(q_k^{(i)})^2 > Q_{k-1}(q_k^{(i)}) \cdot Q_{k+1}(q_k^{(i)}).$$

From the inductive hypothesis, $Q_{k-1}(q_k^{(i)})$ alternates in sign. Thus $Q_{k+1}(q_k^{(i)})$ also alternates in sign. Thus the roots of Q_{k+1} are simple, real, and interlace Q_k. This proves the auxiliary claim that Q_k has k simple real roots.

This claim proves that $D_m(r_1, \ldots, r_{m-1}, k)(y)$ has k simple real roots. This proves the desired inequality and also guarentees that it is strict in the case where A_m and A_{m+1} are not proportional. This completes the induction and suffices for the proof. $\square$

Corollary 3.3.1. *Let A, B be $n \times n$ positive definite symmetric real matrices. For $0 \leq k \leq n$, define the mixed discriminant*

$$D_k := \mathsf{D}(\underbrace{A, \ldots, A}_{k \text{ times}}, \underbrace{B \ldots, B}_{n-k \text{ times}}).$$

Then the sequence $D_0, D_1, \ldots, D_n$ is log-concave.

Proof. This is an immediate consequence of Theorem 3.3. $\square$

Corollary 3.3.2. *Let A, B, D_k for $1 \leq k \leq n$ be as in Corollary 3.3.1. If $D_k^2 = D_{k-1}D_{k+1}$ for some $1 \leq k \leq n-1$, then $D_k^2 = D_{k-1}D_{k+1}$ holds for all $k = 1, \ldots, n-1$.*

Proof. This follows from the equality case in Theorem 3.3. $\square$

3.4 Alexandrov-Fenchel Inequality

As mentioned the previous section, Theorem 3.4 is a mixed volume analog of Theorem 3.3. One difference between the two inequalities is that Alexandrov's inequality on mixed discriminants have relatively reasonable equality cases. However, in the case of Theorem 3.4, the equality cases are not so clear. We will discuss more about equality cases in Section 3.4.1.

Theorem 3.4 (Alexandrov-Fenchel Inequality)**.** *Let $K_1, K_2, \ldots, K_{d-2} \subseteq \mathbb{R}^d$ be convex bodies. For any convex bodies $X, Y \subseteq \mathbb{R}^d$, we have the inequality*

$$\mathsf{V}_d(X, Y, K_1, \ldots, K_{d-2})^2 \geq \mathsf{V}_d(X, X, K_1, \ldots, K_{d-2}) \cdot \mathsf{V}_d(Y, Y, K_1, \ldots, K_{d-2}).$$

In this section, we will present the proof of the inequality given in [51]. From Theorem 2.5, we can approximate our convex bodies by simple strongly isomorphic polytopes. The general result will then follow from taking a suitable limit in the space of convex bodies. When restricted to an isomorphism class α of a simple polytope with facet normals $\mathcal{U}$, the mixed volumes $\mathsf{V}_d(X, Y, P_1, \ldots, P_{d-2})$ can be viewed as a bilinear form applied to the support vectors of X and Y. Indeed, for any support vectors h_P, h_Q, we can define

$$\mathsf{V}_d(h_P, h_Q, P_1, \ldots, P_{d-2}) := \mathsf{V}_d(P, Q, P_1, \ldots, P_{d-2}).$$

We can extend this definition to all of $\mathbb{R}^{\mathcal{U}}$ using Corollary 2.5.2. Indeed, for any $x, y \in \mathbb{R}^{\mathcal{U}}$ there are polytopes $A_1, A_2, B_1, B_2 \in \alpha$ such that $x = h_{A_2} - h_{A_1}$ and $y = h_{B_2} - h_{B_1}$. We define

$$\mathsf{V}_d(x, y, P_1, \ldots, P_{d-2}) = \sum_{i,j=1}^{2} (-1)^{i+j} \mathsf{V}_d(h_{A_i}, h_{B_j}, P_1, \ldots, P_{d-2}).$$

To prove that this extension is well-defined, it suffices to prove that if $x = h_K - h_L = h_P - h_Q$ for $K, L, P, Q, P_0 \in \alpha$, then

$$\mathsf{V}_d(h_K, \mathcal{P}) - \mathsf{V}_d(h_L, \mathcal{P}) = \mathsf{V}_d(h_P, \mathcal{P}) - \mathsf{V}_d(h_Q, \mathcal{P})$$

where $\mathcal{P} = (P_0, P_1, \ldots, P_{d-2})$. Note that $x = h_K - h_L = h_P - h_Q$ implies that $h_K + h_Q = h_L + h_P$. Thus $h_{K+Q} = h_{L+P}$ where $K + Q, L + P \in \alpha$ from Proposition 2.5.3 and so $K + Q = L + P$. Thus

$$\begin{aligned}
\mathsf{V}_d(h_K, \mathcal{P}) + \mathsf{V}_d(h_Q, \mathcal{P}) &= \mathsf{V}_d(K, \mathcal{P}) + \mathsf{V}_d(Q, \mathcal{P}) \\
&= \mathsf{V}_d(K + Q, \mathcal{P}) \\
&= \mathsf{V}_d(L + P, \mathcal{P}) \\
&= \mathsf{V}_d(L, \mathcal{P}) + \mathsf{V}_d(P, \mathcal{P}) \\
&= \mathsf{V}_d(h_L, \mathcal{P}) + \mathsf{V}_d(h_P, \mathcal{P}).
\end{aligned}$$

This proves that the extension of $\mathsf{V}_d(\cdot, \cdot, P_1, \ldots, P_{d-2})$ to $\mathbb{R}^{\mathcal{U}}$ is well-defined. Thus, it suffices to prove the inequality in Theorem 3.5. In the same way, for all $u \in \mathcal{U}$ and $x \in \mathbb{R}^{\mathcal{U}}$, there is a well-defined extension

$$\mathsf{V}_{d-1}(F(x, u), F(\mathcal{P}, u)) := \mathsf{V}_{d-1}(F(Q, u), F(\mathcal{P}, u)) - \mathsf{V}_{d-1}(F(Q', u), F(\mathcal{P}, u))$$

where $F(\mathcal{P}, u) := (F(P_3, u), \ldots, F(P_d, u))$ and $x = h_Q - h_{Q'}$. We can define the matrix $\widetilde{A} : \mathbb{R}^{\mathcal{U}} \to \mathbb{R}^{\mathcal{U}}$ defined by

$$\widetilde{A}x := \frac{1}{d} \sum_{u \in \mathcal{U}} \mathsf{V}_{d-1}(F(x, u), F(P_3, u), \ldots, F(P_d, u)) \cdot e_u.$$

This matrix satisfies the property that

$$\begin{aligned}
\langle h_Q, \widetilde{A} h_P \rangle &= \frac{1}{d} \sum_{u \in \mathcal{U}} h_Q(u) \cdot \mathsf{V}_{d-1}(F(P, u), F(P_3, u), \ldots, F(P_d, u)) \\
&= \mathsf{V}_d(P, Q, P_3, \ldots, P_d).
\end{aligned}$$

By linearity, we have the equality $\langle x, \widetilde{A}y\rangle = \mathsf{V}_d(x, y, P_3, \ldots, P_d)$ and $\widetilde{A}$ is a symmetric matrix. We now prove that $\widetilde{A}$ is irreducible by proving that the non-zero entries correspond exactly to the graph on facets where two facets are adjacent if and only if their intersection is a facet of size $d-2$.

Lemma 3.4.1. *Let $d \geq 3$. Then the matrix $\widetilde{A}$ is a symmetric irreducible matrix with non-negative off-diagonal entries.*

Proof. Let $\mathcal{U} = \{u_1, \ldots, u_m\}$ be the facet normals of the strong isomorphism class of our polytopes where the facet corresponding to u_i is F_i. For $i, j \in [m]$ we write $i \sim j$ if and only if $F_i \cap F_j$ is a face of dimension $d-2$. We write $F_{ij} = F_i \cap F_j$ when we consider $F_i \cap F_j$ as a facet of F_i. Let u_{ij} be the facet norm of F_{ij} in F_i. For $i \sim j$, let θ_{ij} be the angle satisfying $\langle u_i, u_j\rangle = \cos\theta_{ij}$. Note that $\operatorname{codim} F_i = \operatorname{codim} F_j = 1$, $\operatorname{codim} F_{ij} = 2$. Since no two of u_i, u_j, u_{ij} are linearly independent, there are coefficients $a_i, a_{ij} \in [-1, 1]$ such that $u_j = a_{ij}u_{ij} + a_i u_i$ where $a_i^2 + a_j^2 = 1$. By taking inner products with u_i, we get that $a_i = \cos\theta_{ij}$. This implies that $a_{ij} = \pm\cos\theta_{ij}$. By negating θ_{ij} is necessary, we have

$$u_j = (\cos\theta_{ij})u_i + (\sin\theta_{ij})u_{ij} \implies u_{ij} = (\csc\theta_{ij})u_j - (\cot\theta_{ij})u_i.$$

We can then compute the support values of F_{ij}

$$\begin{aligned} h_{F(P,u_i)}(u_{ij}) &= \sup_{x \in F(P,u_i)} \langle u_{ij}, x\rangle \\ &= \sup_{x \in F_i(P)} \langle (\csc\theta_{ij})u_j - (\cot\theta_{ij})u_i, x\rangle \\ &= (\csc\theta_{ij}) \sup_{x \in F(P,u_i)} \langle u_j, x\rangle - (\cot\theta_{ij})h_P(u_i) \\ &= (\csc\theta_{ij})h_P(u_j) - (\cot\theta_{ij})h_P(u_i). \end{aligned}$$

For $i \sim j$, we can define the constants

$$A_{ij} := \frac{\mathsf{V}_{d-2}(F(F(P_3, u_i), u_{ij}), \ldots, F(F(P_d, u_i), u_{ij}))}{d(d-1)}.$$

Then, for $x = h_P = \sum_{i=1}^m x_i e_i$ where $x_i = h_P(u_i)$, we have that

$$\begin{aligned}
\widetilde{A}x &= \frac{1}{d} \sum_{i \in [m]} \mathsf{V}_{d-1}(F(P, u_i), F(P_3, u_i), \ldots, F(P_d, u_i)) \cdot e_i \\
&= \sum_{i \in [m]} \left(\sum_{j \sim i} A_{ij} h_{F(P,u_i)}(u_{ij}) \right) \cdot e_i \\
&= \sum_{i \in [m]} \left(\sum_{j \sim i} A_{ij}(\csc \theta_{ij}) x_j - A_{ij}(\cot \theta_{ij}) x_i \right) \cdot e_i \\
&= \sum_{i \in [m]} \left(\sum_{j \sim i} A_{ij}(\csc \theta_{ij}) x_j \right) e_i - \sum_{i \in [m]} \left(\sum_{j \sim i} A_{ij} \cot \theta_{ij} \right) x_i \cdot e_i.
\end{aligned}$$

For $i \in [m]$, we have that

$$(\widetilde{A})_{ii} = \langle e_i, \widetilde{A} e_i \rangle = -\sum_{j \sim i} A_{ij} \cot \theta_{ij}.$$

For $i, j \in [m]$ distinct, we have

$$(\widetilde{A})_{ij} = \langle e_i, \widetilde{A} e_j \rangle = \mathbb{1}_{i \sim j} \cdot (A_{ij} \csc \theta_{ij}).$$

When $i \sim j$, then $(\widetilde{A})_{ij} > 0$. This implies that the non-zero entries of $\widetilde{A}$ except the diagonals have the same non-zero positions as the non-zero entries in the adjacency matrix of the graph on facets where two facets are adjacent if and only if $i \sim j$. This graph is clearly strongly-connected, which prove that it is irreducible. $\square$

Theorem 3.5 (Alexandrov-Fenchel Inequality for Simple Strongly Isomorphic Polytopes)**.** *Let α be a strong isomorphism class with facet normals $\mathcal{U} = \{u_1, \ldots, u_m\}$ of simple strongly isomorphic polytopes $P_2, \ldots, P_d$. Then, for all $x, y \in \mathbb{R}^m$ we have the inequality*

$$\mathsf{V}_d(x, P_2, \mathcal{P})^2 \geq \mathsf{V}_d(x, x, \mathcal{P}) \cdot \mathsf{V}_d(P_2, P_2, \mathcal{P})$$

where $\mathcal{P} := (P_3, \ldots, P_d)$.

The inequality of Theorem 3.5 with respect to a bilinear form implies that the bilinear form $\mathsf{V}_d(x, y, \mathcal{P})$ has similar properties to a bilinear form with respect to a **hyperbolic** matrix. We call a symmetric matrix $M \in \mathbb{R}^{d \times d}$ **hyperbolic** if for all $v, w \in \mathbb{R}^d$ satisfying $\langle w, Mw \rangle \geq 0$, we have

$$\langle v, Mw \rangle^2 \geq \langle v, Mv \rangle \langle w, Mw \rangle.$$

From Lemma 1.4 in [51], we find necessary and sufficient conditions for a matrix to be hyperbolic.

Lemma 3.4.2 (Lemma 1.4 in [51]). *Let M be a symmetric matrix. Then, the following conditions are equivalent:*

(a) M is hyperbolic.

(b) The positive eigenspace of M has dimension at most one.

Proof of Theorem 3.5. This proof follows that of [51]. We induct on the dimension d. For the base case $d = 2$, see Lemma 6.1.1 in the appendix. Now suppose that the claim is true for dimensions less than d. Currently, it is not clear that the matrix $\widetilde{A}$ is hyperbolic. We will alter it to become a matrix which is hyperbolic. For $u \in \mathcal{U}$, we can define the matrix $A \in \mathbb{R}^{\mathcal{U}\times\mathcal{U}}$ and diagonal matrix $P = \text{Diag}(p_u : u \in \mathcal{U}) \in \mathbb{R}^{\mathcal{U}\times\mathcal{U}}$ such that

$$\begin{aligned} Ax &:= \sum_{u\in\mathcal{U}} \frac{h_{P_3}(u)\mathsf{V}_{d-1}(F(x,u), F(P_3,u), \ldots, F(P_n,u))}{\mathsf{V}_{d-1}(F(P_3,u), F(P_3,u), \ldots, F(P_n,u))} \cdot e_u \\ p_u &:= \frac{1}{d}\frac{\mathsf{V}_{d-1}(F(P_3,u), F(P_3,u), \ldots, F(P_d,u))}{h_{P_3}(u)}. \end{aligned}$$

We can always translate our polytopes so that $0 \in \text{int}(P_3)$ or equivalently $h_{P_3} > 0$. If we define the inner product $\langle x, y\rangle_P := \langle x, Py\rangle$, then we have that

$$\langle x, Ay\rangle_P = \langle x, \widetilde{A}y\rangle = \mathsf{V}_d(x, y, P_3, \ldots, P_d)$$

since $\widetilde{A} = PA$. In particular, the matrix A is self-adjoint with respect to the inner product $\langle \cdot, \cdot\rangle_P$. From Lemma 3.4.1, we know that A is irreducible with non-negative entries in the off-diagonal entries. Moreover, we have that

$$A(h_{P_3}) = \sum_{u\in\mathcal{U}} \frac{h_{P_3}(u)\mathsf{V}_{d-1}(F(P_3,u), F(P_3,u), \ldots, F(P_n,u))}{\mathsf{V}_{d-1}(F(P_3,u), F(P_3,u), \ldots, F(P_n,u))} \cdot e_u = \sum_{u\in\mathcal{U}} h_{P_3}(u) \cdot e_u = h_{P_3}.$$

Thus A has an eigenvector of eigenvalue 1. Let $c > 0$ be sufficiently large so that $A + cI$ has non-negative entries. From the Perron-Frobenius Theorem (see Theorem 1.4.4 in [9]), the vector h_{P_3} is an eigenvector of eigenvalue $1 + c$ of $A + cI$ and since it has strictly positive entries this is the largest eigenvector and happens to be simple. Thus, the largest eigenvalue of A is 1 and this eigenvalue has multiplicity 1. From the inductive hypothesis, we have that

$$\begin{aligned}
\langle Ax, Ax\rangle_P &= \sum_{u\in\mathcal{U}} (Ax)_u^2 p_u \\
&= \sum_{u\in\mathcal{U}} \frac{1}{d}\cdot \frac{h_{P_3}(u)\mathsf{V}_{d-1}(F(x,u), F(P_3,u),\dots,F(P_d,u))^2}{\mathsf{V}_{d-1}(F(P_3,u),F(P_3,u),\dots,F(P_d,u))} \\
&\geq \sum_{u\in\mathcal{U}} \frac{1}{d} h_{P_3}(u)\cdot \mathsf{V}_{d-1}(F(x,u),F(x,u),F(P_4,u),\dots,F(P_d,u))^2 \\
&= \mathsf{V}_d(x,x,P_3,\dots,P_d) \\
&= \langle x, Ax\rangle_P.
\end{aligned}$$

Let λ be an arbitrary eigenvalue of A. For the corresponding eigenvector v, we have that

$$\langle Av, Av\rangle_P \geq \langle v, Av\rangle_P \implies \lambda^2 \geq \lambda.$$

Thus, the eigenvalues satisfy $\lambda \geq 1$ or $\lambda \leq 0$. From Lemma 3.4.2, we know that A is hyperbolic. Since $\langle h_{P_2}, Ah_{P_2}\rangle_P = \mathsf{V}_d(P_2,P_2,\mathcal{P}) > 0$, we know from hyperbolicity that

$$\begin{aligned}
\mathsf{V}_d(x,P_2,\mathcal{P})^2 &= \langle x, Ah_{P_2}\rangle_P^2 \\
&\geq \langle x, Ax\rangle_P \cdot \langle h_{P_2}, Ah_{P_2}\rangle_P \\
&= \mathsf{V}_d(x,x,\mathcal{P})\cdot \mathsf{V}_d(P_2,P_2,\mathcal{P}).
\end{aligned}$$

This completes the induction and suffices for the proof. ☐

Corollary 3.4.1. *Let $K, L \subseteq \mathbb{R}^n$ be convex bodies. For $0 \leq k \leq n$, we can define the mixed volumes*

$$V_k := \mathsf{V}_n(\underbrace{K,\dots,K}_{k \text{ times}}, \underbrace{L,\dots,L}_{n-k \text{ times}}).$$

Then, the sequence $V_0, V_1, \dots, V_n$ is log-concave.

Proof. From Theorem 3.4, we immediately get $V_k^2 \geq V_{k-1}V_{k+1}$. This suffices for the proof. ☐

3.4.1 Equality Cases

Unlike Alexandrov's Inequality for Mixed Discriminants, the equality cases of the Alexandrov-Fenchel inequality are rich and full of flavor. In this thesis, we do not provide any deep explanations for the mechanisms involved in the equality of the Alexandrov-Fenchel inequality as this is not our main focus. For the reader interested in these questions, we direct them to read [52] or [53] where many of the extremals of the Alexandrov-Fenchel inequality are resolved. Instead, we will provide a rather surface-level overview of the equality cases of the Alexandrov-Fenchel inequality that we will need in later sections of this thesis when we cover log-concavity of combinatorial sequences.

The Alexandrov-Fenchel inequality can be viewed as a generalization of Minkowski's Inequality and various isoperimetric inequalities. Recall, the Brunn-Minkowski inequality and its equality cases given in Theorem 3.6.

Theorem 3.6 (Brunn-Minkowski Inequality, Theorem 3.13 in [29]). *Let $K, L \subseteq \mathbb{R}^n$ be convex bodies and $\alpha \in (0,1)$. Then*

$$\sqrt[n]{V(\alpha K + (1-\alpha)L)} \geq \alpha \sqrt[n]{V(K)} + (1-\alpha)\sqrt[n]{V(L)}$$

with equality if and only if K and L lie in parallel hyperplanes or K and L are homothetic.

The inequality in Theorem 3.6 will explain the equality cases of Minkowski's Inequality (Theorem 3.7) which follows from the Alexandrov-Fenchel inequality.

Theorem 3.7 (Minkowski's Inequality, Theorem 3.14 in [29]). *Let $K, L \subseteq \mathbb{R}^n$ be convex bodies. Then,*

$$\mathsf{V}_n(\underbrace{K, \dots, K}_{n-1 \text{ times}}, L)^n \geq \mathrm{Vol}_n(K)^{n-1} \cdot \mathrm{Vol}_n(L).$$

Equality occurs if and only if $\dim K \leq n-2$ or K and L lie in parallel hyperplanes or K and L are homothetic. When $\mathsf{V}_n(K[k], L[n-k]) > 0$ for all k, then equality occurs if and only if

$$\mathsf{V}_n(\underbrace{K, \dots, K}_{k \text{ times}}, \underbrace{L, \dots, L}_{n-k \text{ times}})^2 = \mathsf{V}_n(\underbrace{K, \dots, K}_{k-1 \text{ times}}, \underbrace{L, \dots, L}_{n-k+1 \text{ times}}) \cdot \mathsf{V}_n(\underbrace{K, \dots, K}_{k+1 \text{ times}}, \underbrace{L, \dots, L}_{n-k-1 \text{ times}})$$

for all k.

Proof. For k, $0 \leq k \leq n$, define the mixed volume $V_k = \mathsf{V}_n(K[k], L[n-k])$. Then, we want to prove that $V_{n-1}^n \geq V_n^{n-1} V_0$. Assuming that $V_k > 0$ for all k, from Theorem 3.4, we have that

$$\frac{V_{n-1}}{V_n} \geq \frac{V_{n-2}}{V_{n-1}} \geq \dots \geq \frac{V_0}{V_1}.$$

Thus, we have that

$$\left(\frac{V_{n-1}}{V_n}\right)^n \geq \prod_{k=1}^{n-1} \frac{V_k}{V_{k-1}} = \frac{V_0}{V_n}.$$

This proves that $V_{n-1}^n \geq V_n^{n-1} \cdot V_0$. When $V_k > 0$, equality occurs if and only if all of the ratios V_k/V_{k-1} are equal, which is equivalent to the Alexandrov-Fenchel inequality holding at each k: $V_k^2 = V_{k-1}V_{k+1}$. Now, suppose we remove the assumption that $V_k > 0$ for all k. Then from the proof of Theorem 3.14 in [29], using the concavity of $f(t) := \mathrm{Vol}_n(K + tL)^{1/n}$ we can prove that

$$\mathrm{Vol}_n(K)^{\frac{1}{n}-1}\mathsf{V}_n(K[n-1], L) \geq \mathrm{Vol}_n(K+L)^{\frac{1}{n}} - \mathrm{Vol}_n(K)^{\frac{1}{n}} \geq \mathrm{Vol}_n(L)^{\frac{1}{n}}$$

where the second inequality is exactly Theorem 3.6. The equality conditions in Theorem 3.6 then give us the equality conditions in the theorem. ☐

In general, the equality cases of the Alexandrov-Fenchel inequality are complicated. Indeed, the previous pattern of having only degenerate equality cases or homothetic equality cases ends when one considers even the simplest examples of Minkowski's Quadratic inequality. Minkowski's quadratic ienquality is version of the Alexandrov-Fenchel inequality where $n = 3$ and we have convex bodies $K, L, M \subseteq \mathbb{R}^3$ satisfying

$$\mathsf{V}_3(K, L, M)^2 \geq \mathsf{V}_3(K, K, M)\mathsf{V}_3(L, L, M). \tag{3.1}$$

In [52], Yair Shenfeld and Ramon van Handel completely characterize the extremals of Equation 3.1. Even in the case where $M = K, L = B$ the equality cases are non-trivial. The inequality becomes

$$\mathsf{V}_3(B, K, K)^2 \geq \mathsf{V}_3(B, B, K)\mathsf{V}_3(K, K, K). \tag{3.2}$$

Each of these mixed volumes has a geometric interpretation: $\mathsf{V}_3(B, K, K)$ is the surface area of K, $\mathsf{V}_3(B, B, K)$ is the mean width of K, and $\mathsf{V}_3(K, K, K)$ is the volume of K. The problem of finding equality cases to Equation 3.2 can be rephrased as an isoperimetric inequality: given a fixed mean width and fixed volume, what convex body K will achieve the minimum surface area? This is a generalization of the classical isoperimetric inequality:

$$\mathsf{V}_2(B, K)^2 \geq \mathsf{V}_2(B, B)\mathsf{V}_2(K, K)$$

which the equality cases is trivial. Unlike the classical case, the equality cases of Equation 3.2 involves new objects called cap bodies which are the convex hull of the ball with some collection of points satisfying some disjointness conditions. From [10], the cap bodies end up being the only equality cases of the quadratic isoperimetric inequality.

3.4.2 Equality Cases for Polytopes

In [52], Yair Shenfeld and Ramon van Handel completely characterize the equality cases of the Alexandrov-Fenchel inequality for polytopes. We record some of their results in this section for future use in this thesis. In [53], Shenfeld and van Handel define the notion of a supercritical collection of convex bodies.

Definition 3.4.1 (Definition 2.14 in [53]). A collection of convex bodies $\mathcal{C} = (C_1, \ldots, C_{n-2})$ is **supercritical** if $\dim(C_{i_1} + \ldots + C_{i_k}) \geq k + 2$ for all $k \in [n-2]$, $1 \leq i_1 < \ldots < i_k \leq n-2$.

The paper [53] gives a characterization of the equality cases of

$$\mathsf{V}_n(K, L, P_1, \ldots, P_{n-2})^2 \geq \mathsf{V}_n(K, K, P_1, \ldots, P_{n-2})\mathsf{V}_n(L, L, P_1, \ldots, P_{n-2})$$

in both the case where $(P_1, \ldots, P_{n-2})$ is critical and the case where $(P_1, \ldots, P_{n-2})$ is supercritical. In the case where the collection is supercritical, the characterization simplifies. We will only be concerned with the supercritical case since all of our applications will happen to only involve supercritical collections. For an application of the critical case, see the paper [36] which studies the equality cases of the general Stanley's poset inequality. In this general case, the corresponding collection of polytopes is critical.

Theorem 3.8 (Corollary 2.16 in [53]). *Let $\mathcal{P} = (P_1, \ldots, P_{n-2})$ be a supercritical collection of polytopes in $\mathbb{R}^n$, and let K, L be convex bodies such that $\mathsf{V}_n(K, L, P_1, \ldots, P_{n-2}) > 0$. Then*

$$\mathsf{V}_n(K, L, P_1, \ldots, P_{n-2})^2 = \mathsf{V}_n(K, K, P_1, \ldots, P_{n-2})\mathsf{V}_n(L, L, P_1, \ldots, P_{n-2})$$

if and only if there exist $a > 0$ and $v \in \mathbb{R}^n$ so that K and $aL+v$ have the same supporting hyperplanes in all $(B, \mathcal{P})$-extreme normal directions.

In Theorem 3.8, the constant $a > 0$ determines the ratio between the mixed volumes. To see why this is the case, define $\mathsf{V}(A, B) := \mathsf{V}_n(A, B, P_1, \ldots, P_{n-2})$ for any convex bodies $A, B \subseteq \mathbb{R}^n$. From the mixed area measure interpretation of mixed volumes, the value of $\mathsf{V}(A, B)$ only depends on the support function values of A and B in the directions of the support of the mixed area measure. From the result in Theorem 3.8, the polytopes K and $aL + v$ have the same support values in all relevant directions. Thus, we can interchange the two polytopes when calculating the mixed volume. This implies that

$$\mathsf{V}(K,K) = \mathsf{V}(K, aL+v) = a\mathsf{V}(K,L)$$
$$\mathsf{V}(K,L) = \mathsf{V}(aL+v, L) = a\mathsf{V}(L,L).$$

Hence, the constant $a > 0$ indicates the ratio between the mixed volumes.

3.5 Basic Theory of Lorentzian polynomials

In this section, we give a brief overview of the theory of Lorentzian polynomials. Lorentzian polynomials are objects which offer natural explanations as to why many sequences in combinatorics end up being log-concave. According to [8], the notion of Lorentzian polynomials was independently discovered by Brändén-Huh [12] and Anari-Oveis-Garan-Vinzant [3]. For our main reference on the basic theory of Lorentzian polynomials, we primarily use the paper [12] by Brändén-Huh. Following their notation, we let H_n^d be the degree d homogeneous subring defined by

$$H_n^d := \{f \in \mathbb{R}[x_1,\ldots,x_n] : f \text{ is homogeneous}, \deg(f) = d\}.$$

We can turn H_n^d into a topological space by equipping it with the Euclidean topology on its coefficients. There are many different equivalent definitions of Lorentzian polynomials. These different definitions make it easier to prove that certain polynomials are Lorentzian. We first define the notion of a strictly Lorentzian polynomial in Definition 3.5.1.

Definition 3.5.1. Let $\underline{\mathrm{L}}_n^2 \subseteq H_n^2$ be the set of quadratic forms in $\mathbb{R}[x_1,\ldots,x_n]$ with positive coefficients such that the Hessian has signature $(+,-,\ldots,-)$. For $d \geq 3$, we can define $\underline{\mathrm{L}}_n^d$ inducively as

$$\underline{\mathrm{L}}_n^d := \{f \in H_n^d : \partial_i f \in \underline{\mathrm{L}}_n^{d-1} \text{ for all } i\}.$$

We call polynomials in $\underline{\mathrm{L}}_n^d$ **strictly Lorentzian polynomials**.

The space of strictly Lorentzian polynomials $\underline{\mathrm{L}}_n^d$ forms an open subset of H_n^d. One defintion of Lorentzian polynomials is that it is any polynomial in the closure of $\underline{\mathrm{L}}_n^d$ as a subset in the topological space H_n^d. The next definition of Lorentzian polynomials combines analytic and combinatorial properties of the polynomial. Specifically, the properties of the polynomial as a function on $\mathbb{C}^n$ and the support structure of its monomials.

Definition 3.5.2. A polynomial $f \in \mathbb{R}[w_1, \dots, w_n]$ is **stable** if f is non-vanishing on $\mathbb{H}^n$ where $\mathbb{H}$ is the open upper half plane in $\mathbb{C}$. We denote by S_n^d the set of degree d homogeneous stable polynomials in n variables.

In this thesis, we will not concern ourselves with the notion of stable polynomials. We have only written Definition 3.5.2 because it is a part of an alternative definition of Lorentzian polynomials. For background on stable polynomials, we refer the reader to [58].

Definition 3.5.3. We define $J \subseteq \mathbb{N}^n$ to be M**-convex** if for any $\alpha, \beta \in J$ and any index i satisfying $\alpha_i > \beta_i$, there is an index j satisfying $\alpha_j < \beta_j$ and $\alpha - e_i + e_j \in J$. Equivalently, we have that for any $\alpha, \beta \in J$ and any index i satisfying $\alpha_i > \beta_i$, there is an index j satisfying $\alpha_j < \beta_j$ and $\alpha - e_i + e_j \in J$ and $\beta - e_j + e_i \in J$.

We will also not concern ourselves too much with M-convex sets. In this thesis, the only fact that we use related to M-convex sets is that the set of bases of a matroid form a M-convex set. Indeed, any M-convex set restricted to the hypercube is a matroid. For more information about M-convex sets, we refer the reader to [40]. We will consider the M-convexity of the support of a homogeneous polynomial. For a polynomial $f = \sum_{\alpha \in \Delta_n^d} c_\alpha x^\alpha$ in H_n^d, we define the support of f to be the exponents for which the corresponding coefficient is non-zero. Specifically, we have

$$\operatorname{supp}(f) := \{\alpha \in \Delta_n^d : c_\alpha \neq 0\}.$$

For example, the newton polygon of any polynomial will depend only on its support. We let M_n^d be the set of degree d homogeneous polynomials in $\mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ whose support are M-convex. With these notions on hands, we can now define Lorentzian polynomials equivalently as the homogeneous polynomials with non-negative coefficients and M-convex support such that all codimension 2 partial derivatives are stable polynomials. We write the full description of this set in Definition 3.5.4.

Definition 3.5.4 (Definition 2.6 in [12])**.** We set $L_n^0 = S_n^0$, $L_n^1 = S_n^1$, and $L_n^2 = S_n^2$. For $d \geq 3$, we define

$$\begin{aligned} L_n^d &= \{f \in M_n^d : \partial_i f \in L_n^{d-1} \text{ for all } i \in [n]\} \\ &= \{f \in M_n^d : \partial^\alpha f \in S_n^2 \text{ for all } \alpha \in \Delta_n^{d-2}\}. \end{aligned}$$

Definition 3.5.5. We call a degree d homogeneous polynomial $f \in \mathbb{R}_{\geq 0}[x_1, \ldots, x_n]$ a **Lorentzian polynomial** if it satisfies any of the following equivalent conditions:

(a) $f \in L_n^d$.

(b) There exists a sequence of polynomials $f_k \in \underline{\mathrm{L}}_n^d$ such that $f_k \to f$ as $k \to \infty$.

(c) For any $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq d-2$, the polynomial $\partial^\alpha f$ is identically zero or log-concave at any $a \in \mathbb{R}^n_{>0}$.

Polynomials which satisfy condition (c) in Definition 3.5.5 are also known in the literature as **strongly log-concave polynomials**. These polynomials were studies in [3]. The fact that the conditions in Definition 3.5.5 are equivalent follows from the work in [12]. As an application of this definition, we will now prove that the basis generating polynomial of a matroid is Lorentzian.

Theorem 3.9. *Let M be a matroid. Then the basis generating polynomial f_M is Lorentzian.*

Proof. We prove the result by induction on the size of our matroid. On matroids of size 2, the only possible basis generating polynomials are $x_1x_2, x_1 + x_2, x_1, x_2$, and 0. All of these polynomials are Lorentzian because they are stable. Now, suppose that the claim holds for all matroids of smaller size. Since the support of f_M is the collection of bases of a matroid, the support is M-convex. Hence, it suffices to prove that all of its partial derivatives are Lorentzian. For all $i \in E$, we know that $\partial_i f_M = f_{M/i}$ for all $i \in E$. From the inductive hypothesis, the partial derivative is Lorentzian because it the basis generating polynomial of a smaller matroid. This completes the induction and suffices for the proof. $\square$

In Proposition 3.5.1, we have properties of Lorentzian polynomials which are of the most interest to us. These properties will allow us to prove that certain polynomials are Lorentzian by relating them to known Lorentzian polynomials.

Proposition 3.5.1. *Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a Lorentzian polynomial of degree d. Then, f satisfies the following properties:*

(a) Let A be any $n \times m$ matrix with nonnegative entries. Then $f : \mathbb{R}^m \to \mathbb{R}$ defined by $f(y) := f(Ay)$ is Lorentzian.

(b) For any $a_1, \ldots, a_n \geq 0$, the polynomial $\sum_{i=1}^n a_i \partial_i f$ is Lorentzian.

(c) If $f \neq 0$, then $\mathrm{Hess}_f(a)$ has exactly one positive eigenvalue for all $a \in \mathbb{R}^n_{>0}$.

Proof. For a proof of these properties, see Theorem 2.10, Corollary 2.11, and Proposition 2.14 in [12]. ☐

Lorentzian polynomials satisfy log-concavity inequalities similar to that of Alexandrov's inequality on mixed discriminants and the Alexandrov-Fenchel inequality. We present the proofs of two analogs of these log-concavity inequalities found in [12].

Proposition 3.5.2 (Proposition 4.4 in [12])**.** *If $f = \sum_{\alpha \in \Delta_n^d} \frac{c_\alpha}{\alpha!} x^\alpha$ is a Lorentzian polynomial, then $c_\alpha^2 \geq c_{\alpha+e_i-e_j} c_{\alpha-e_i+e_j}$ for any $i, j \in [n]$ and any $\alpha \in \Delta_n^d$.*

Proof. Consider the differential operator $\partial^{\alpha-e_i-e_j}$. Applying D to the polynomial f, we have

$$\begin{aligned} Df &= \sum_{\beta \in \Delta_n^d} \frac{c_\beta}{\beta!} \partial^{\alpha-e_i-e_j} x^\beta \\ &= \frac{c_{\alpha+e_i-e_j}}{(\alpha+e_i-e_j)!} \cdot \frac{(\alpha+e_i-e_j)!}{2} x_i^2 + \frac{c_{\alpha-e_i+e_j}}{(\alpha-e_i+e_j)!} \cdot \frac{(\alpha-e_i+e_j)!}{2} x_j^2 + \frac{c_\alpha}{\alpha!} \cdot \alpha! \cdot x_i x_j \\ &= \frac{1}{2}\left(c_{\alpha+e_i-e_j} x_i^2 + 2 c_\alpha x_i x_j + c_{\alpha-e_i+e_j} x_j^2\right). \end{aligned}$$

From Proposition 3.5.1, the polynomial Df is Lorentzian. This suffices for the proof. ☐

Recall that the mixed discriminant $\mathsf{D}_n(A_1, \ldots, A_n)$ can be defined as the polarization form of the polynomial $\det(X)$ where $X \in \mathbb{R}^{n\times n}$. It happens to be the case that the polarization of this polynomial satisfies a log-concavity inequality given by the Alexandrov inequality for mixed discriminants. An analog of this phenomenon occurs for general Lorentzian polynomials.

Proposition 3.5.3. *Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a Lorentzian polynomial of degree d. Then, for any $v_1, v_2, \ldots, v_d \in \mathbb{R}^n_{\geq 0}$, we have that*

$$F_f(v_1, v_2, v_3, \ldots, v_d)^2 \geq F_f(v_1, v_1, v_3, \ldots, v_d) \cdot F_f(v_2, v_2, v_3, \ldots, v_d).$$

Proof. The polarization identity in Theorem 2.2 gives us the equation

$$f(x_1 v_1 + \ldots + x_m v_m) = d! \sum_{\alpha \in \Delta_m^d} \frac{F_f(v_1[\alpha_1], \ldots, v_m[\alpha_m])}{\alpha!} x^\alpha \tag{3.3}$$

The result follows from Proposition 3.5.2. ☐

Remark 3. In Proposition 4.5 of [12], Bräden-Huh prove a stronger version of Proposition 3.5.3 where we allow the first vector v_1 to be any vector in $\mathbb{R}^n$. The proof of this stronger fact uses the Cauchy interlacing theorem from Section 3.2. Through private discussions with Ramon van Handel, there are ideas similar to those in [53] that may be successful in characterizing the equality cases in the Lorentzian analog of Alexandrov-Fenchel inequality. We plan to pursue questions in this direction for future work. In the expansion of Equation 3.3 for the case $m = n$, by substituting $v_i = e_i$ for all $1 \leq i \leq n$ we get the equation

$$f(x_1, \ldots, x_n) = d! \sum_{\alpha \in \Delta_n^d} \frac{F_f(e_1[\alpha_1], \ldots, e_n[\alpha_n])}{\alpha!} x^\alpha.$$

Thus a characterization for the Lorentzian analog of a characterization for the extremals of the Alexandrov-Fenchel inequality (as given in [53]) would be useful in characterizing equality in the inequalities given in Proposition 3.5.2. The characterization would involve checking when the vectors $e_1, \ldots, e_n$ satisfy the equaliy conditions.

Chapter 4

Log-concavity Results for Posets and Matroids

In this chapter, we discuss proofs of several log-concavity results from the perspective of mixed discriminants, mixed volumes, and Lorentzian polynomials. In the first section, we write about a poset inequality proved by Stnaley regarding the log-concavity of a linear extension counting sequence. We also detail a combinatorial characterization of the equality cases given in [53]. we discuss two log-concavity inequalities related to the linear extensions of a partially ordered set. Both of these results follows immediately from Theorem 3.4 for particular convex bodies. In these sections, we also discuss how the results from [53] can be used to find combinatorial characterizations of the equality cases. In the subsequent section, we discuss a poset inequality proved by Jeff Kahn and Michael Saks in [34]. We present joint work with Ramon van Handel, Xinmeng Zeng, and the author which completely characterizes the equality cases of this inequality. In the final section, we study another inequality of Stanley regarding the log-concavity of a basis counting sequence of regular matroids. By combining perspectives from both mixed discriminants and mixed volumes, we characterize the equality cases in a simplified version of Stanley's inequality. We then generalize Stanley's inequality to all matroids using the technology of Lorentzian polynomials.

4.1 Stanley's Poset Inequality

Let $(P, \leq)$ be a finite poset on n elements and let $x \in P$ be a distinguished element in the poset. For every $k \in [n]$, let N_k be the number of linear extensions of P which send x to the index k.

Then we can consider properties of the sequence $N_1, \ldots, N_n$. Intuitively, one would expect that as a function in $[n]$, the sequence should be roughly unimodal. The fact that the sequence is unimodal is originally a conjecture by Ronald Rivest in his paper [16]. Rivest proves his conjecture in the case where P can be covered by two linear orders. In [55], Richard Stanley proves the full conjecture by proving the stronger claim that the sequence is log-concave.

Theorem 4.1 (Stanley's Poset Inequality, Theorem 3.1 in [55])**.** *Let $x_1 < \ldots < x_k$ be a fixed chain in P. If $1 \leq i_1 < \ldots < i_k \leq n$, then define $N(i_1, \ldots, i_k)$ to be the number of linear extensions $\sigma : P \to [n]$ satisfying $\sigma(x_j) = i_j$ for $1 \leq j \leq k$. Suppose that $1 \leq j \leq k$ and $i_{j-1}+1 < i_j < i_{j+1}-1$, where $i_0 = 0$ and $i_{k+1} = n+1$. Then*

$$N(i_1, \ldots, i_k)^2 \geq N(i_1, \ldots, i_{j-1}, i_j - 1, i_{j+1}, \ldots, i_k) N(i_1, \ldots, i_{j-1}, i_j + 1, i_{j+1}, \ldots, i_k).$$

Rivest's conjecture follows from Theorem 4.1 in the case $k = 1$. We call the next inequality when $k = 1$ by the name Stanley's simple poset inequality. This simplified statement is given in Corollary 4.1.1.

Corollary 4.1.1 (Stanley's Simple Poset Inequality)**.** *The sequence $N_1, \ldots, N_n$ is log-concave. That is, we have $N_k^2 \geq N_{k-1}N_{k+1}$ for all $k \in \{2, \ldots, n-1\}$.*

We will only present Stanley's proof of Corollary 4.1.1 since the argument generalizes easily to his stronger result. Then, we present a sketch of the extremals of Corollary 4.1.1 due to Ramon van Handel's and Yair Shenfeld's breakthrough on the equality cases of the Alexandrov-Fenchel inequality in [53].

Proof of Corollary 4.1.1. Define the polytopes $K, L \subseteq \mathbb{R}^n$ by

$$K := \{t \in \mathcal{O}_P : t_\omega = 1 \text{ if } \omega \geq x\}$$
$$L := \{t \in \mathcal{O}_P : t_\omega = 0 \text{ if } \omega \leq x\}.$$

For any $\lambda \in [0, 1]$, we have that

$$\mathrm{Vol}_{n-1}\left((1-\lambda)K + \lambda L\right) = \sum_{l \in e(P)} \mathrm{Vol}_{n-1}\left(\Delta_l \cap ((1-\lambda)K + \lambda L)\right)$$

where we note that $\dim(1-\lambda)K + \lambda L = n - 1$. Since K and L lie in parallel hyperplanes, the expression on the left hand side is well-defined. Now suppose that l satisfies $l(x) = k$. Then, we have

$$\Delta_l \cap ((1-\lambda)K+\lambda L) = \{t \in [0,1]^n : 0 \le t_{\pi^{-1}(1)} \le \ldots \le t_{\pi^{-1}(k-1)} \le t_{\pi^{-1}(k)} = 1-\lambda \le t_{\pi^{-1}(k+1)} \le \ldots \le t_{\pi^{-1}(n)} \le 1\}$$

which has volume

$$\mathrm{Vol}_{n-1}(\Delta_l \cap ((1-\lambda)K+\lambda L)) = \frac{(1-\lambda)^{k-1}\lambda^{n-k}}{(k-1)!(n-k)!}.$$

Thus, we have the equation

$$\mathrm{Vol}_{n-1}\left((1-\lambda)K+\lambda L\right) = \sum_{k=1}^{n} \binom{n-1}{k-1} \frac{N_k}{(n-1)!}(1-\lambda)^{k-1}\lambda^{n-k}.$$

From Theorem 2.8, we have for $1 \le i \le n-1$ the equality

$$\mathsf{V}_{n-1}(K[i-1], L[n-i]) = \frac{N_i}{(n-1)!} \implies N_i = (n-1)! \cdot \mathsf{V}_{n-1}(K[i-1], L[n-i]).$$

The log-concavity of the sequence $N_1, N_2, \ldots, N_n$ follows from Theorem 3.4.1. □

The proof of Corollary 4.1.1 illustrates one strategy for proving log-concavity results. To prove that a sequence is log-concave, it is enough to associate each element in the sequence to some mixed volume. Log-concavity then follows immediately from the Alexandrov-Fenchel inequality. In the sequel, we will discuss how to use the results about the extremals of the Alexandrov Fenchel inequality from [53] to give a combinatorial characterization of the equality cases of $N_k^2 = N_{k-1}N_{k+1}$ in Stanley's simple poset inequality.

4.1.1 Equality Case for the Simple Stanley Poset Inequality

The equality cases for the Stanley poset inequality was computed in [53] using their results on the equality cases of the Alexandrov-Fenchel inequality. From the proof of Corollary 4.1.1, we have the equalities $N_k = (n-1)!\mathsf{V}_{n-1}(K[k], L[n-k-1])$ where the polytopes K and L were defined in the proof. Thus, to find the equality cases of $N_k^2 = N_{k-1}N_{k+1}$, it is enough to find the equality cases of

$$\mathsf{V}_{n-1}(K[k-1], L[n-k])^2 \ge \mathsf{V}_{n-1}(K[k], L[n-k-1]) \cdot \mathsf{V}_{n-1}(K[k-2], L[n-k+1]) \tag{4.1}$$

Our goal is to give combinatorial conditions to the poset P so that the corresponding polytopes K and L satisfy equality in Equation 4.1. We sketch the proof given in Section 15 in [53] to give an idea on how the argument works. The same type of argument will be used to charaterize the extremals of the Kahn-Saks inequality. The combinatorial characterization given in [53] is compiled in Theorem 4.2.

Theorem 4.2 (Theorem 15.3 in [53])**.** *Let $i \in \{2, \ldots, n-1\}$ be such that $N_i > 0$. Then the following are equivalent:*

(a) $N_i^2 = N_{i-1}N_{i+1}$.

(b) $N_i = N_{i-1} = N_{i+1}$.

(c) Every linear extension $\sigma : P \to [n]$ with $\sigma(x) = i$ assigns ranks $i-1$ and $i+1$ to elements of P that are incomparable to x.

(d) $|P_{<y}| > i$ *for all* $y \in P_{>x}$*, and* $|P_{>y}| > n-i+1$ *for all* $y \in P_{<x}$.

In Theorem 4.2, it is not difficult to prove that $(d) \implies (c) \implies (b) \implies (a)$. The main difficulty lies in proving that equality and positivity in $N_i^2 = N_{i-1}N_{i+1}$ implies the combinatorial conditions in (d). From here, let us assume that $N_i^2 = N_{i-1}N_{i+1}$ and $N_i > 0$. From Lemma 2.5.3, we get necessary and sufficient conditions to guarantee $N_i > 0$.

Lemma 4.1.1 (Lemma 15.2 in [53])**.** *For any $i \in [n]$, we have that $N_i = 0$ if and only if $|P_{<x}| > i-1$ or $|P_{>x}| > n-i$.*

Proof. Recall that $N_i = \mathsf{V}_{n-1}(\underbrace{K, \ldots, K}_{i-1 \text{ times}}, \underbrace{L, \ldots, L}_{n-i \text{ times}})$. From Lemma 2.5.3, we know that $N_i > 0$ if and only if $\dim K \geq i-1$, $\dim L \geq n-i$, and $\dim(K+L) \geq n-1$. From the definition of K and L, we can compute

$$\dim K = n - 1 - |P_{>x}|$$
$$\dim L = n - 1 - |P_{<x}|$$
$$\dim(K + L) = n - 1.$$

To see the details behind this computation, see Lemma 15.7 in [53]. Thus, $N_i = 0$ if and only if $|P_{>x}| > n-i$ and $|P_{<x}| > i-1$. $\square$

We get inequalities on the dimensions of $\dim K$ and $\dim L$ from the assumption $N_i > 0$. However, since $N_i^2 = N_{i-1}N_{i+1}$, we also have $N_{i-1}, N_{i+1} > 0$. This gives slightly stronger inequalities from Lemma 4.1.1 applied to N_{i-1} and N_{i+1}. Specifically, we get the inequalities

$$|P_{<x}| \le i-2, \quad \text{and} \quad |P_{>x}| \le n-i-1.$$

With these inequalities and Theorem 3.8 we get a specialized version of Corollary 2.16 in [53].

Lemma 4.1.2 (Lemma 15.8 in [53]). *Let $i \in \{2, \ldots, n-1\}$ be such that $N_i > 0$ and $N_i^2 = N_{i+1}N_{i-1}$. Then $|P_{<x}| \le i-2$, $|P_{>x}| \le n-i-1$, and there exist $a > 0$ and $v \in \mathbb{R}^{n-1}$ so that*

$$h_K(x) = h_{aL+v}(x) \text{ for all } x \in \operatorname{supp} S_{B,K[i-2],L[n-i-1]}.$$

In the beginning of our analysis, the convex bodies K and L were contained in parallel hyperplanes of $\mathbb{R}^{n-1}$. When studying equality via Lemma 4.1.2, we project these bodies to the same copy of $\mathbb{R}^{n-1}$. The vector $v \in \mathbb{R}^{n-1}$ will also lie in the same hyperplane as our convex bodies K and L. Lemma 4.1.2 will be the main tool to extract combinatorial information about P from the equality condition in the Alexandrov-Fenchel inequality. Recall in Definition 2.5.6, we are giving a characterization for vectors in the support of the mixed area measure. By specializing this result in the Stanley case, we get the result in Lemma 4.1.3.

Lemma 4.1.3. *Let $u \in \mathbb{R}^{n-1}$ be a unit vector. Then the following are equivalent.*

(a) $u \in \operatorname{supp} S_{B,K[i-2],L[n-i-1]}$.

(b) $\dim F_K(u) \ge i-2$, $\dim F_L(u) \ge n-i-1$, *and* $\dim F_{K+L}(u) \ge n-3$.

From here, the argument will involve finding suitable vectors $u \in \mathbb{S}^{n-2} \subseteq \mathbb{R}^{n-1}$ and applying Lemma 4.1.2. The suitability of the vector $u \in \mathbb{S}^{n-2}$ will depend on the corresponding inequalities in Lemma 4.1.3. When we find a "suitable" vector $u \in \mathbb{S}^{n-2}$, one of two things generally occur. In the first case, the vector u will unequivocally satisfy the dimension inequalities in Lemma 4.1.3 and from Lemma 4.1.2 we get the relation $h_K(u) = ah_L(u) + \langle v, u\rangle$. In this case, the identity we get usually tells us something about v or a. In the second case, it is unclear whether or not the vector u will satisfy the dimension inequalities in Lemma 4.1.3. The dimension inequalities will usually involve some statistics about the combinatorial structure of our object. For example, it might involve the number of elements greater than another, or the number of elements between two elements in a poset. In this case, we know that if the dimension inequalities in Lemma 4.1.3 are satisfied then we get the identity in Lemma 4.1.2. If we are lucky, the resulting identity we get will contradict some information that we know is true from the vectors in the first case. This will imply that at least one of the dimension inequalities are incorrect, giving a combinatorial condition that must be satisfied for equality.

In the Stanley case, vectors in the first case are vectors of the form $-e_\omega$ where ω is a minimal element, e_ω where ω is a maximal element, and some vectors of the form $e_{\omega_1,\omega_2} := \frac{e_{\omega_1}-e_{\omega_2}}{\sqrt{2}}$ where $\omega_1, \omega_2 \in P$ are elements in the poset such that $\omega_1 \lessdot \omega_2$. From the first two vectors, we get that $v_\omega = 1-a$ for any maximal element ω, the second vector gives $v_\omega = 0$ for any minimal element ω. The third type of vector tells us that in some situations where $\omega_1 \lessdot \omega_2$ we have $v_{\omega_1} = v_{\omega_2}$. From the proof of Corollary 15.11 of [53], it is possible to create a chain from a minimal element to a maximal element such as the adjacent relations are covering relations where the corresponding vector e_{ω_1,ω_2} is in the support of the mixed area measure. This implies that all of the coordinates of v with respect to this chain are equal to each other. In particular, the coordinate at the minimal element will be equal to the coordinate at the maximal element (that is, $a = 1$). Currently, this result is interesting in its own right. We have just proved that if $N_i^2 = N_{i-1}N_{i+1}$ and $N_i > 0$, then $N_{i+1} = N_i = N_{i-1}$.

Vectors in the second case consist of vectors of the form $-e_\omega$ where ω is a minimal element in $P_{>x}$ and also vectors of the form e_ω where ω is a maximal element in $P_{<x}$. From the information that $a = 1$, it is not difficult to show that the equality that we would get as a result of these vectors being in the support of the mixed area measure give contradictions. Thus, some of the dimension inequalities corresponding to these vectors must be incorrect. This is enough to give the combinatorial characterization in Theorem 4.2. During the present discussion, we did not show any of the computations for the sake of time and space. To see the computations in full detail and rigour, we refer the reader to the original source [53].

4.2 Extremals of the Kahn-Saks Inequality

This section covers joint work by Ramon van Handel, Xinmeng Zeng, and the author. We consider a slight generalization of the simple version of the Stanley poset inequality called the **Kahn-Saks Inequality**. We also cover aspects of the proof of our characterization of the equality cases. In the paper [34], Jeff Kahn and Michael Saks prove that any finite poset contains a pair of elements x and y such that the proportion of linear extensions of P in which x lies below y is between $\frac{3}{11}$ and $\frac{8}{11}$. This fact has consequences in theoretic computer science and sorting algorithms. The interested reader should consult the original source [34] for these applications as they are tangential to our thesis. One ingredient in their proof is a log-concavity type inequality. Let P be a finite poset and fix elements $x, y \in P$ with $x \leq y$. Let N_k be the number of linear extension f satisfying $f(y) - f(x) = k$ for all $1 \leq k \leq n-1$. The Kahn-Saks inequality written in Theorem 4.3 states that the sequence $N_1, \ldots, N_{n-1}$ is log-concave.

Theorem 4.3 (Kahn-Saks Inequality, Theorem 2.5 in [34]). *For all $k \in \{2, \ldots, n-2\}$, we have $N_k^2 \geq N_{k-1}N_{k+1}$.*

In the case where x is an initial object in P, Theorem 4.3 reduces to Corollary 4.1.1 since x will be forced to be the lowest element in any linear extension. The proof of Theorem 4.3 is similar to the proof of Stanley's inqualty. To prove log-concavity, we will associate the sequence N_k to a sequence of mixed volumes of suitable polyopes. The polytopes that we use will be cross-sections of the order polytope $\mathcal{O}_P$. For all $\lambda \in \mathbb{R}$, we define the polytope K_λ as

$$K_\lambda := \{t \in \mathcal{O}_P : t_y - t_x = \lambda\}.$$

Each of these cross-sections can be written as a convex combination of K_0 and K_1. We defer the proof of $(1-\lambda)K_0 + \lambda K_1 = K_\lambda$ to Lemma 6.2.1. From [34], we know that $N_k = (n-1)!\mathsf{V}_{n-1}(K_0[n-k], K_1[k-1])$. The mixed volume is well-defined since K_0 and K_1 lie in parallel hyperplanes. We give the full computation of this fact in Lemma 6.2.2 in the appendix. The Kahn-Saks inequality then follows immediately from the Alexandrov-Fenchel inequality. In the next section, we begin our combinatorial characterization of the extremals of the Kahn-Saks inequality. Our main results will be the content Theorem 4.4, Theorem 4.5, and Theorem 4.6.

4.2.1 Simplifications and Special Regions

Let $(P, \leq)$ be our poset with distinguished elements $x, y \in P$ satisfying $x \not\geq y$. We can always add a 0 element and 1 element to our poset. In any linear extension, these two elements will be forced to be placed in the beginning and end of the linear extension. Hence, adding these two elements to the poset will not change the values of the sequence $\{N_k\}$. If $x || y$, Lemma 6.2.4 and Lemma 6.2.5 in the appendix prove that in the poset generated by our original relations and the new relation $x \leq y$, our sequence N_k still does not change. Thus, we will assume that our poset P has a 0 element, has a 1 elements, and satisfies $x \leq y$. In our poset, we will define the several special regions which each need to be handled separately in our analysis:

$$\mathsf{END}_x := \{\omega \in P : \omega < x\} = P_{<x}$$

$$\mathsf{END}_y := \{\omega \in P : \omega > y\} = P_{>y}$$

$$\mathsf{MID} := \{\omega \in P : x < \omega < y\} = P_{x<\cdot<y}$$

$$\mathsf{MID}_x := \{\omega \in P : x < \omega \text{ and } \omega||y\} = P_{\cdot>x,\cdot||y}$$

$$\mathsf{MID}_y := \{\omega \in P : \omega < y \text{ and } \omega||x\} = P_{\cdot<y,\cdot||x}$$

These regions are disjoint, but they do not account for all elements in P. The elements of P that are still unaccounted are the elements which are incomparable to both x and y.

4.2.2 Combinatorial characterization of the extremals

The combinatorial conditions that we found were conjectured by Swee Hong Chan, Igor Pak, and Greta Panova in their paper [15]. In this paper, they define the notions of the midway and dual-midway properties recorded in Definition 4.2.1.

Definition 4.2.1. We say that (x, y) satisfies the k-**midway property** if

- $|P_{<z}| + |P_{>y}| > n - k$ for every $z \in P$ such that $x < z$ and $z \not> y$,
- $|P_{z<\cdot<y}| > k$ for every $z < x$.

Similarly, we say that (x, y) satisfies the **dual** k-**midway property** if:

- $|P_{>z}| + |P_{<x}| > n - k$ for every $z \in P$ such that $z < y$ and $z \not< x$, and
- $|P_{x<\cdot<z}| > k$ for every $z > y$.

For any $z \in P$, we say that z satisfies the k-midway property or satisfies the dual k-midway property if the relevant inequality for z in the corresponding property is satisfied. For example, if $z < x$, we would say that z satisfies the k-midway property if $|p_{z<\cdot<y}| > k$.

In Conjecture 8.7 in [15], Chan-Pak-Panova conjecture that when $N_k > 0$, the equality case $N_{k+1} = N_k = N_{k-1}$ occurs if and only if the midway property holds or the dual midway property holds. Using purely combinatorial methods, they prove this conjecture for width two posets where x and y are elements in the same chain in the two chain decomposition. By appealing to the extremals of the Alexandrov-Fenchel inequality, we are able to prove this conjecture, find all possible values of a for which $N_{i+1} = aN_i = a^2 N_{i-1}$, and give a combinatorial characterization for the equality cases in all possible values of a. We now state our main results.

Theorem 4.4. *If $N_k > 0$ and $N_k^2 = N_{k-1}N_{k+1}$, then we either have $N_{k+1} = N_k = N_{k-1}$ or $N_{k+1} = 2N_k = 4N_{k-1}$.*

Theorem 4.5. *If $N_k > 0$, then the following are equivalent:*

(a) $N_{k-1} = N_k = N_{k+1}$.

(b) (x, y) *satisfies the k-midway property or the dual k-midway property.*

Theorem 4.6. *If $N_k > 0$, then $N_{k+1} = 2N_k = 4N_{k-1}$ if and only if the following properties hold:*

(i) *k-midway holds for* END_x *and dual k-midway holds for* END_y.

(ii) *Every $z \in P$ is comparable to x and y.*

(iii) MID *is empty.*

(iv) *For every $z \in \mathsf{MID}_y$ and $z' \in \mathsf{MID}_x$ with $z < z'$, we have that $|P_{z<\cdot<y}| + |P_{x<\cdot<z'}| \geq k - 1$.*

Remark 4. Theorem 4.5 resolves Conjecture 8.7 in [15]. In their paper, Chan-Pak-Panova prove the important implication $(b) \implies (a)$. Since this result is central for the full characterization, we record it in Proposition 4.2.1. In Theorem 4.6, it turns out property (iii) will follow from the other conditions. However, we keep the condition in the theorem as it gives a better idea of the structure of the poset.

Proposition 4.2.1 (Theorem 8.9, Proposition 8.8 in [15])**.** *If (x, y) satisfies the k-midway property or the dual k-midway property, then $N_{k-1} = N_k = N_{k+1}$.*

The proof strategy of these results will be similar to characterization of the equality cases of the Stanley inequality in [53]. In this discussion, let us assume that $N_k^2 = N_{k-1}N_{k+1}$ and $N_k > 0$. Recall in the proof of Theorem 4.3, we showed the identity

$$N_k = (n-1)!\mathsf{V}_{n-1}\left(\underbrace{K_0, \ldots, K_0}_{n-k \text{ times}}, \underbrace{K_1, \ldots, K_1}_{k-1 \text{ times}}\right).$$

Then, we have equality in $N_k^2 = N_{k-1}N_{k+1}$ if and only if we have equality in the corresponding Alexandrov-Fenchel inequality:

$$\mathsf{V}_{n-1}(K_0[n-k], K_1[k-1])^2 \geq \mathsf{V}_{n-1}(K_0[n-k-1], K_1[k]) \cdot \mathsf{V}_{n-1}(K_0[n-k+1], K_1[k-2]).$$

Using Lemma 2.5.3, we have the characterization of positivity given in Proposition 4.2.2. We omit the proof since we have prove a similar positivity result in the Stanley case.

Proposition 4.2.2. *The following are equivalent.*

(a) $N_k = 0$,

(b) $\dim K_0 < n - k$ *or* $\dim K_1 < k - 1$,

(c) $|P_{<x}| + |P_{>y}| > n - k - 1$ *or* $|P_{x<\cdot<y}| > k - 1$.

Since $N_k^2 = N_{k-1}N_{k+1}$ and $N_k > 0$, we also have that $N_{k-1} > 0$ and $N_{k+1} > 0$. From Proposition 4.2.2 this gives us stronger inequalities on $\dim K_0$ and $\dim K_1$. From Proposition 6.2.3, we know that

$$\begin{aligned} \dim K_0 &= n - |P_{x<\cdot<y}| - 1 \\ \dim K_1 &= n - |P_{<x}| - |P_{>y}| - 2 \\ \dim(K_0 + K_1) &= n - 1. \end{aligned}$$

Let $V = (e_y - e_x)^{\perp} \cong \mathbb{R}^{n-1}$ be the orthogonal complement of the vector $e_y - e_x$. The polytopes K_0 and K_1 are contained in parallel copies of V. We can specialize Theorem 3.8 to our case and prove Theorem 4.7. This will be the main workhorse theorem to prove the combinatorial characterization of the equality case.

Theorem 4.7. *Let* $k \in \{2, \ldots, n-2\}$ *such that* $N_k > 0$ *and* $N_k^2 = N_{k-1}N_{k+1}$. *Then*

$$1 + |P_{x<\cdot<y}| < k < n - |P_{<x}| - |P_{>y}| - 1$$

and there exists a positive scalar $a > 0$ *and* $v \in V$ *such that* $h_{K_0}(u) = h_{aK_1+v}(u)$ *for all* $u \in \operatorname{supp} S_{B,K_0[n-k-1],K_1[k-2]}$ *where* $\mu := S_{B,K_0[n-k-1],K_1[k-2]}$ *is the mixed area measure on the copy of* $\mathbb{S}^{n-2}$ *in* V.

Proof. Since $N_k > 0$, we know that $N_{k-1}, N_{k+1} > 0$. Proposition 4.2.2 applied to N_{k-1} and N_{k+1} implies that $\dim K_0 \geq n-k+1$ and $\dim K_1 \geq k$. Thus, $\mathcal{P} := (K_0[n-k-1], K_1[k-2])$ is supercritical. The theorem then follows from Theorem 3.8. $\square$

The next lemma is a simple criterion for when a vector lies in the support of the mixed area measure.

Lemma 4.2.1. *For any vector $u \in V$, we have that $u \in \operatorname{supp}\mu$ if and only if*

$$\dim F(K_0, u) \geq n-k-1$$

$$\dim F(K_1, u) \geq k-2$$

$$\dim F(K_0+F_1, u) \geq n-3.$$

Proof. This is an application of Lemma 2.3 in [53]. □

4.2.3 Transition Vectors

In this section, we assume that the hypothesis of Theorem 4.7 is true. That is, we have $N_k > 0$ and $N_k^2 = N_{k-1}N_{k+1}$. We can label our poset $(P, \leq)$ as $P = \{z_1, \ldots, z_n\}$ where $x = z_{n-1}$ and $y = z_n$. Let $v = (v_1, \ldots, v_n)$ where v_i is the coordinate of v with respect to the coordinate indexed by z_i. When it is more convenient to label the coordinate by the actual poset element itself, we use the notation v_{z_i}. In Definition 4.2.2, we define the notion of transition vectors. These are vectors in $\operatorname{supp}\mu$ which, from Corollary 4.2.1, allows us to equate the two coordinates of v of the corresponding pair of vectors.

Definition 4.2.2. For $z_i, z_j \in P\backslash\{x, y\}$, we define the vector

$$e_{ij} := \frac{1}{\sqrt{2}}(e_i - e_j) \in V.$$

We call e_{ij} a **transition vector** if $z_i \lessdot z_j$ and $e_{ij} \in \operatorname{supp}\mu$. We also use the notation e_{z_i,z_j} in some cases.

The reason why transition vectors will equate coordinates of v is because the support function values of transition vectors will always be 0. Indeed, for $z_i \lessdot z_j$, we can compute

$$h_{K_0}(e_{ij}) = \frac{1}{\sqrt{2}} \sup_{t \in K_0} \langle t, e_i - e_j \rangle = \frac{1}{\sqrt{2}} \sup_{t \in K_0} t_i - t_j \leq 0$$
$$h_{K_1}(e_{ij}) = \frac{1}{\sqrt{2}} \sup_{t \in K_1} \langle t, e_i - e_j \rangle = \frac{1}{\sqrt{2}} \sup_{t \in K_1} t_i - t_j \leq 0$$

where the last inequalities follow from the fact that $z_i \lessdot z_j$ and the parameter t lies in the order polytope $\mathcal{O}_P$. Since we have $(0, \ldots, 0) \in K_0$ we have $h_{K_0}(e_{ij}) = 0$. To prove that $h_{K_1}(e_{ij}) = 0$, note that $z_i \lessdot z_j$ implies that it cannot be the case that $z_i < x$ and $z_j > y$. Since $\sum_{z_i \geq y} e_i, \sum_{z_i \not\leq x} e_i \in K_1$, we have $h_{K_1}(e_{ij}) = 0$.

Corollary 4.2.1. *Suppose that the hypothesis of Theorem 4.7 is true and let $v \in V$ be the vector in the theorem. If e_{ij} is a support vector, then $v_i = v_j$.*

Proof. From Theorem 4.7, we have that $h_{K_0}(e_{ij}) = h_{aK_1+v}(e_{ij}) = ah_{K_1}(e_{ij}) + \langle v, e_{ij}\rangle$. Since $h_{K_0}(e_{ij}) = h_{K_1}(e_{ij}) = 0$ and $\langle v, e_{ij}\rangle = v_i - v_j$, we have $v_i = v_j$. This proves the lemma. □

In Lemma 4.2.2, we compile a list of transition vectors.

Lemma 4.2.2. *Suppose that the hypothesis in Theorem 4.7 is true. Then, we have the following transition vectors:*

(a) Let $R = \mathsf{END}_x$ or $R = \mathsf{END}_y$. If $z_i, z_j \in R$ satisfy $z_i \lessdot z_j$, then e_{ij} is a transition vector.

(b) Let $R = \mathsf{MID}_x$ or $R = \mathsf{MID}_y$. If $z_i, z_j \in R$ satisfy $z_i \lessdot z_j$, then e_{ij} is a transition vector.

(c) If $z_i \in \mathsf{MID}_x$ and $z_j \in \mathsf{END}_y$ such that $z_i \lessdot z_j$ and there does not exist $z_l \in \mathsf{MID}_x$ with $z_i \lessdot z_l$, then e_{ij} is a transition vector.

(d) If $z_j \in \mathsf{MID}_y$ and $z_i \in \mathsf{END}_x$ such that $z_i \lessdot z_j$ and there does not exist $z_l \in \mathsf{MID}_y$ with $z_l \lessdot z_j$, then e_{ij} is a transition vector.

(e) If $z_i, z_j \in \mathsf{MID}$ such that $z_i \lessdot z_j$, then e_{ij} is a transition vector.

Proof. We only prove the result when $R = \mathsf{END}_x$ since the proofs of the other regions are relatively similar. A full proof of the result will be given in an upcoming paper. Before computing the dimensions of these faces, we first prove that there exists a linear extension $f : P \to [n]$ with $f(z_j) - f(z_i) = 1$ and $f(y) - f(x) = 1 + |P_{x<\cdot<y}|$. From Proposition 6.2.2, there is a linear extension $\tilde{f} : P \to [n]$ with $\tilde{f}(y) - \tilde{f}(x) = |P_{x<\cdot<y}| + 1$. Both z_i and z_j will be located to the left of x in the linear extension $\tilde{f}$. From Proposition 6.2.4, we can modify $\tilde{f}$ by only changing the elements between z_i and z_j in the linear extension $\tilde{f}$ to a linear extension f satisfying $f(z_j) - f(z_i) = 1$. Clearly, we have the following set inclusion

$$F_{K_0}(e_{ij}) \supseteq \Delta_f \cap \{t_i = t_j, t_x = t_y\}.$$

Taking the affine span of both sets, we have

$$\begin{aligned}\operatorname{aff} F_{K_0}(e_{ij}) &\supseteq \operatorname{aff} \Delta_f \cap \{t_i = t_j, t_x = t_y\}\\ &= \mathbb{R}\left[\sum_{\omega \in P_{x\leq\cdot\leq y}} e_\omega\right] \oplus \mathbb{R}[e_i + e_j] \oplus \bigoplus_{\omega \in P\setminus(P_{x\leq\cdot\leq y}\cup\{z_i,z_j\})} \mathbb{R}[e_\omega].\end{aligned}$$

This gives us the following bounds for the dimensions of $F_{K_0}(e_{ij})$ and $F_{K_1}(e_{ij})$:

$$\dim F_{K_0}(e_{ij}) \geq n - |P_{x<\cdot<y}| - 2 = \dim K_0 - 1$$
$$\dim F_{K_1}(e_{ij}) = \dim K_1.$$

From Lemma 6.2.6, we have $\dim F_{K_0+K_1}(e_{ij}) \geq \dim(K_1 + K_0) - 1 = n - 2$. From the bounds in Theorem 4.7, we have

$$\dim F_{K_0}(e_{ij}) = n - |P_{x<\cdot<y}| - 2 \geq n - k$$
$$\dim F_{K_1}(e_{ij}) = n - |P_{<x}| - |P_{>y}| - 2 \geq k.$$

From Lemma 4.2.1, we know that $e_{ij} \in \operatorname{supp} \mu$. This proves that e_{ij} is a transition vector. □

For any result about support vectors, the calculations needed to determine whether or not certain vectors are support vectors look very similar to the calculation performed in our proof of Lemma 4.2.2. The strategy is to find "extremal linear extensions" where the intersection between the corresponding simplex and the faces of our polytopes has large dimension. To avoid reptitiveness and bogging the thesis with technical details, we defer a large portion of the support vector calculations to a upcoming paper containing all of the details.

Lemma 4.2.3. *Suppose that the hypothesis of Theorem 4.7 holds. Let $z_i \in P\backslash\{x, y\}$ be an element of the poset. If z_i is minimal, then $v_i = 0$. If z_i is maximal, then $v_i = 1 - a$.*

Proof. If $z_i \in P$ is a minimal element, then we can show that $-e_i$ is in the support of the mixed area measure. If $z_i \in P$ is a maximal element, then we can show that e_i is in the support of the mixed area measure. These facts will imply via Theorem 4.7 that $v_i = 0$ when z_i is minimal and $v_i = 1 - a$ when z_i is maximal. □

Using Lemma 4.2.2 and Lemma 4.2.3, we can compute some of the coordinates of the vector v depending on the region containing the element poset corresponding to the coordinate.

Corollary 4.2.2. *Let $\omega \in P\backslash\{x, y\}$ be some element in our poset.*

(a) If $\omega \in \mathsf{END}_x \cup \mathsf{MID}_y$, then $v_\omega = 0$;

(b) If $\omega \in \mathsf{END}_y \cup \mathsf{MID}_x$, then $v_\omega = 1 - a$;

(c) If $\omega_1, \omega_2 \in \mathsf{MID}$ are comparable, then $v_{\omega_1} = v_{\omega_2}$.

Proof. Suppose that $\omega \in \mathsf{END}_x$. If ω is a minimal element, then we already know that $v_\omega = 0$ from Lemma 4.2.3. Suppose that ω is not a minimal element. Then there is a sequence of poset elements $\omega_1, \omega_2, \ldots, \omega_l$ for $l \geq 1$ such that $\omega_1 \lessdot \omega$, $\omega_{i+1} \lessdot \omega_i$ for all i, and ω_l is a minimal element. From Lemma 4.2.2(a), we know that $e_{\omega_1,\omega}$ and $e_{\omega_{i+1},\omega_i}$ are transition vectors for all i. From Corollary 4.2.1 and Corollary 4.2.3 applied to ω_l, we have $v_\omega = v_{\omega_1} = \ldots = v_{\omega_l} = 0$. A similar proof will work when $\omega \in \mathsf{END}_y$, but instead we built a maximal chain to a maximal element.

Now, suppose that $\omega \in \mathsf{MID}_x$. If ω is maximal, then we automatically know that $v_\omega = 1-a$. If ω is not maximal, we can build a chain $\omega \lessdot \omega_1 \lessdot \omega_2 \lessdot \ldots \lessdot \omega_l$ where ω_l is a maximal element by using a special procedure to guarentee that all adjacent elements correspond to transition vectors. To illustrate this procedure, suppose that we have picked $\omega \lessdot \omega_1 \lessdot \ldots \lessdot \omega_i$ up to some i. Then $\omega_i \in \mathsf{END}_y$ or $\omega_i \in \mathsf{MID}_x$. If $\omega_i \in \mathsf{END}_y$, then we can continue to a maximal element arbitrarily. Otherwise, if $\omega_i \in \mathsf{MID}_x$, then we pick ω_{i+1} to be an element of MID_x which covers ω_i. If none exists, we then pick ω_{i+1} to be an element of END_y which covers ω_i. If this doesn't exist again, then we know that ω_i is a maximal element. By our construction and Lemma 4.2.2, we know that the adjacent poset elements in the chain form transition vectors. This implies that $v_\omega = v_{\omega_1} = \ldots = v_{\omega_l} = 1-a$. A similar proof works for MID_y but instead we create the same type of chain to a minimal element.

For the result on MID, suppose that $\omega_1, \omega_2 \in \mathsf{MID}$ are comparable. Since any two comparable elements in MID can be connected by a chain completely contained in MID, it suffices to prove the result when $\omega_1 \lessdot \omega_2$. In this case, from Lemma 4.2.2(e), we know that v_{ω_1,ω_2} is a transition vector. Hence $v_{\omega_1} = v_{\omega_2}$. This suffices for the proof. $\square$

To conclude the section on transition vectors, we give conditions for when a vector of the form e_{ij} is a transition vector from MID to MID_x or MID_y.

Proposition 4.2.3. *Let $z_i \in P\backslash\{x,y\}$ be a poset element satisfying $x < z_i < y$.*

(a) If $z_j \in P$ satisfies $z_j \in \mathsf{MID}_y$ and $z_j \lessdot z_i$, then e_{ji} is a transition vector if $|\mathsf{MID}| + |S| + 1 \leq k$ where we define the set S to be

$$S = P_{z_i < \ldots < y} \backslash \mathsf{MID} = \{z : z_i < z < y, z \notin \mathsf{MID}\}.$$

(b) If $z_j \in P$ satisfies $z_j \in \mathsf{MID}_x$ and $z_j \gtrdot z_i$, then e_{ij} is a transition vector if $|\mathsf{MID}| + |S| + 1 \leq k$

where we define the set S to be

$$S = P_{x<\cdot<z_i} \setminus \mathsf{MID} = \{z : x < z < z_i, z \notin \mathsf{MID}\}.$$

Proof. We only prove (a) as the proof of (b) is similar. Note that

$$\begin{aligned}
\operatorname{aff} F_K(e_{ji}) &= \bigoplus_{\omega \notin \mathsf{MID} \sqcup S \sqcup \{x,y,z_i\}} \mathbb{R}[e_\omega] \oplus \mathbb{R}\left[\sum_{\omega \in \mathsf{MID} \sqcup S \sqcup \{x,y,z_i\}} e_\omega\right] \\
\operatorname{aff} F_L(e_{ji}) &= \bigoplus_{\omega \notin \{z_i,z_j\} \sqcup P_{\leq x} \sqcup P_{\geq y}} \mathbb{R}[e_\omega] \oplus \mathbb{R}[e_i + e_j] + \sum_{\omega \in P_{\geq y}} e_\omega \\
\operatorname{aff} F_{K+L}(e_{ji}) &= \bigoplus_{\omega \notin \{z_i,z_j,x,y\}} \mathbb{R}[e_\omega] \oplus \mathbb{R}\left[\sum_{\omega \in \mathsf{MID} \sqcup S \sqcup \{x,y,z_i\}} e_\omega\right] \oplus \mathbb{R}[e_i + e_j] \oplus \sum_{\omega \in P_{\geq y}} e_\omega.
\end{aligned}$$

The only obstruction to the vector e_{ji} being in the support is the dimension inequality corresponding to $\dim \operatorname{aff} F_K(e_{ji})$. Specifically, we need the inequality

$$n - |\mathsf{MID}| - |S| - 2 = \dim F_K(e_{ij}) \geq n - k - 1 \iff |\mathsf{MID}| + |S| + 1 \leq k.$$

This suffices for the proof. □

4.2.4 More Vectors

In this section, we introduce a new type of vector not found in the Stanley case that will allow us to extract the midway and dual-midway inequalities from our poset.

Definition 4.2.3. For $j \in \{1, \ldots, n-2\}$, define

$$\begin{aligned}
u_j^+ &:= \sqrt{\frac{2}{3}}\left(e_j - \frac{1}{2}(e_x + e_y)\right) \\
u_j^- &:= \sqrt{\frac{2}{3}}\left(\frac{1}{2}(e_x + e_y) - e_j\right)
\end{aligned}$$

where $e_i := e_{z_i}$ is the basis vector corresponding to the poset element $z_i \in P$.

We provide the following table which shows which vector we use in our analysis based on the location of each element in the poset. We also display the corresponding support values.

Region	Relation	Vector	$h_{K_0}(\cdot)$	$h_{K_1}(\cdot)$
$z_j \in \mathsf{MID}$,	$z_j \lessdot y$	u_j^+	0	$\sqrt{\frac{1}{6}}$
$z_j \in \mathsf{MID}$,	$z_j \gtrdot x$	u_j^-	0	$\sqrt{\frac{1}{6}}$
$z_j \in \mathsf{MID}_x$,	$z_j \gtrdot x$	u_j^-	0	$\sqrt{\frac{1}{6}}$
$z_j \in \mathsf{MID}_y$,	$z_j \lessdot y$	u_j^+	0	$\sqrt{\frac{1}{6}}$
$z_j \in \mathsf{END}_x$,	$z_j \lessdot x$	u_j^+	0	$-\sqrt{\frac{1}{6}}$
$z_j \in \mathsf{END}_y$,	$z_j \gtrdot y$	u_j^-	0	$-\sqrt{\frac{1}{6}}$

After finding the conditions for the vectors in the table to be in the support of the mixed area measure, we can prove Proposition 4.2.4. As discussed before, we will omit the proofs since they are all technical and the same style of computations.

Proposition 4.2.4. *Let $z_i \in P\backslash\{x,y\}$ be an element. Then*

(a) If $z_i \in \mathsf{MID}$, $z_i \lessdot y$, then $u_i^+ \in \operatorname{supp}\mu$ if and only if $|P_{>z_i}| \leq n-k-|P_{<x}|$.

(b) If $z_i \in \mathsf{MID}$, $z_i \gtrdot x$, then $u_i^- \in \operatorname{supp}\mu$ if and only if $|P_{<z_i}| \leq n-k-|P_{>y}|$.

(c) If $z_i \in \mathsf{MID}_x^{min}$, then $u_i^- \in \operatorname{supp}\mu$ if and only if $|P_{<z_i}| \leq n-k-|P_{>y}|$.

(d) If $z_i \in \mathsf{END}_x$, $z_i \lessdot x$, then $u_i^+ \in \operatorname{supp}\mu$ if and only if $|P_{z_i<\cdot<y}| \leq k$.

(e) If $z_i \in \mathsf{MID}_y^{max}$, then $u_i^+ \in \operatorname{supp}\mu$ if and only if $|P_{>z_i}| \leq n-k-|P_{<x}|$.

(f) If $z_i \in \mathsf{END}_y$, $z_i \gtrdot y$, then $u_i^- \in \operatorname{supp}\mu$ if and only if $|P_{x<\cdot<z_i}| \leq k$.

Using Theorem 4.7 and the table of support value computations, we can translate Proposition 4.2.4 into information about v and a. The result of this application of Theorem 4.7 is given in Proposition 4.2.5.

Proposition 4.2.5. *Let $z_i \in P\backslash\{x,y\}$ be an element of our poset. Let $v_{xy} := \frac{1}{2}(v_x+v_y)$.*

(a) If $z_i \in \mathsf{MID}$ and $z_i \lessdot y$, then $v_i \neq v_{xy} - a/2 \implies |P_{>z_i}| > n-k-|P_{<x}|$.

(b) If $z_i \in \mathsf{MID}$ and $z_i \gtrdot x$, then $v_i \neq v_{xy} + a/2 \implies |P_{<z_i}| > n-k-|P_{>y}|$.

(c) If $z_i \in \mathsf{MID}_x^{min}$, then $v_i \neq v_{xy} + a/2 \implies |P_{<z_i}| > n-k-|P_{>y}|$.

(d) If $z_i \in \mathsf{MID}_y^{max}$, then $v_i \neq v_{xy} - a/2 \implies |P_{>z_i}| > n-k-|P_{<x}|$.

(e) If $z_i \in \mathsf{END}_x$ and $z_i \lessdot x$, then $v_i \neq v_{xy} + a/2 \implies |P_{z_i<\ldots<y}| > k$.

(f) *If* $z_i \in \mathsf{END}_y$ *and* $z_i \gtrdot y$*, then* $v_i \neq v_{xy} - a/2 \implies |P_{x<\cdot<z_i}| > k$.

In Proposition 4.2.5, the result was only stated for poset elements satisfying certain minimal/maximal relations. But, the result is still true for all poset elements in the corresponding region. This immediately gives Corollary 4.2.3.

Corollary 4.2.3. *Let* $z_i \in P\backslash\{x, y\}$ *be an element of our poset. Let* $v_{xy} := \frac{1}{2}(v_x + v_y)$*. Then, the following statements are true.*

(a) *If* $z_i \in \mathsf{MID}$*, then* $v_i \neq v_{xy} - a/2 \implies |P_{>z_i}| > n - k - |P_{<x}|$.

(b) *If* $z_i \in \mathsf{MID}$*, then* $v_i \neq v_{xy} + a/2 \implies |P_{<z_i}| > n - k - |P_{>y}|$.

(c) *If* $z_i \in \mathsf{MID}_x$*, then* $v_i \neq v_{xy} + a/2 \implies |P_{<z_i}| > n - k - |P_{>y}|$.

(d) *If* $z_i \in \mathsf{MID}_y$*, then* $v_i \neq v_{xy} - a/2 \implies |P_{>z_i}| > n - k - |P_{<x}|$.

(e) *If* $z_i \in \mathsf{END}_x$ *and* $z_i \lessdot x$*, then* $v_i \neq v_{xy} + a/2 \implies |P_{z_i<\ldots<y}| > k$.

(f) *If* $z_i \in \mathsf{END}_y$ *and* $z_i \gtrdot y$*, then* $v_i \neq v_{xy} - a/2 \implies |P_{x<\cdot<z_i}| > k$.

Proof. We only prove (a) since the rest are similar. Let $z_i \in \mathsf{MID}$. Then there is some $z_0 \in \mathsf{MID}$ so that $z_i \leq z_0 \lessdot y$. Then, since $v_{z_i} = v_{z_0}$, we have that

$$|P_{>z_i}| \geq |P_{>z_0}| > n - k - |P_{<x}|.$$

This suffices for the proof. □

From Corollary 4.2.2, we already know some of the values of v_i in Proposition 4.2.5. This immediately implies Corollary 4.2.4.

Corollary 4.2.4. *Let* $z_i \in P\backslash\{x, y\}$ *be an element of our poset. Let* $v_{xy} := \frac{1}{2}(v_x + v_y)$*. Then, the following statements are true.*

(a) *Suppose that* $z_i \in \mathsf{MID}_x$*. Then* $a \neq \frac{2}{3}(1 - v_{xy}) \implies |P_{<z_i}| > n - k - |P_{>y}|$.

(b) *Suppose that* $z_i \in \mathsf{MID}_y$*. Then* $a \neq 2v_{xy} \implies |P_{>z_i}| > n - k - |P_{<x}|$.

(c) *Suppose that* $z_i \in \mathsf{END}_x$*. Then* $a \neq -2v_{xy} \implies |P_{z_i<\cdot<y}| > k$.

(d) *Suppose that* $z_i \in \mathsf{END}_y$*. Then* $a \neq 2(1 - v_{xy}) \implies |P_{x<\cdot<z_i}| > k$.

Consider the four statements

$$
\begin{aligned}
M_x &:= \left\{a \neq \frac{2}{3}(1 - v_{xy})\right\},\\
M_y &:= \{a \neq 2v_{xy}\},\\
E_x &:= \{a \neq -2v_{xy}\},\\
E_y &:= \{a \neq 2(1 - v_{xy})\}.
\end{aligned}
$$

From Corollary 4.2.4, we know that M_x implies that dual-midway holds for MID_x, M_y implies that midway holds for MID_y, E_x implies that dual-midway holds for END_x, and E_y implies that midway holds for END_y.

Lemma 4.2.4. *If $a \neq 1/2$, then either dual-midway holds for END_y and MID_y or midway holds for END_x and MID_x.*

Proof. It suffices to prove that either M_x and E_x holds or M_y and E_y hold. Suppose that M_x and M_y do not hold simultaneously. Then, we have

$$2v_{xy} = \frac{2}{3}(1 - v_{xy}) \implies v_{xy} = \frac{1}{4}.$$

This would imply that $a = 1/2$, which contradicts the hypothesis. Thus M_x and M_y cannot be false simultaneously. Suppose that M_x and E_y do not hold simultaneously. Then,

$$2(1 - v_{xy}) = \frac{2}{3}(1 - v_{xy}) \implies v_{xy} = 1.$$

But then $a = 0$ which contradicts $N_i > 0$. Thus M_x and E_y cannot be false simultaneously. Suppose that E_x and M_y do not hold simultaneously. Then we run into the same contradiction $a = 0$. Now, suppose that E_x and E_y do not hold simultaneously. Then we get an obvious contradiction. This suffices for the proof. $\square$

4.2.5 Understanding MID

We now introduce some results that will give us a better understand of the region MID.

Lemma 4.2.5. *Suppose that the hypothesis in Theorem 4.7 holds. Let $z, w \in P$ be two comparable elements in MID. Then, the following are true.*

(a) *If z violates midway, then w cannot violate dual-midway. In other words, the inequality* $|P_{<x}|+|P_{>z}| \leq n-k$ *implies* $|P_{>y}|+|P_{<w}| > n-k$.

(b) *If z violates dual-midway, then w cannot violate midway. In other words, the inequality* $|P_{>y}|+|P_{<z}| \leq n-k$ *implies* $|P_{<x}|+|P_{>w}| > n-k$.

Proof. We only prove (a) as (b) is symmetric. Since $|P_{<x}|+|P_{>y}| \leq n-k$, we have that $v_z = v_{xy}+a/2$ from Proposition 4.2.5. Suppose for the sake of contradiction that $|P_{>y}|+|P_{<w}| \leq n-k$ as well. Then Proposition 4.2.5 would imply that $v_w = v_{xy}-a/2$. But, we know from (c) of Corollary 4.2.2 that $v_z = v_w$. Hence, we would have $a = 0$, which contradicts the hypothesis that $N_k > 0$. $\square$

Proposition 4.2.6. *Let $z \in P$ be an element satisfying $x < z < y$. Then, the following are true.*

(a) *If* $|P_{<x}|+|P_{>z}| \leq n-k$, *then there is a chain from z to a minimal element of $P_{x<\cdot<y}$ where the minimal element covers an element outside of $P_{x<\cdot<y}$.*

(b) *If* $|P_{>y}|+|P_{<z}| \leq n-k$, *then there is a chain from z to a maximal element of $P_{x<\cdot<y}$ which is covered by an element outside of $P_{x<\cdot<y}$.*

(c) *We cannot have both* $|P_{<x}|+|P_{>z}| \leq n-k$ *and* $|P_{>y}|+|P_{<z}| \leq n-k$.

Proof. Part (c) follows from Lemma 4.2.5. We only prove (a) since (b) is symmetric. There is some $z' \in \mathsf{MID}$ so that $x \lessdot z' \leq z < y$. Suppose that we cannot extend z' to a minimal element outside of $P_{x<\cdot<y}$. Then, we would have $|P_{<z'}| = 1+|P_{<x}|$. From (a) of Lemma 4.2.5, we have the inequality

$$n-k < |P_{>y}|+|P_{<z'}| = |P_{>y}|+|P_{<x}|+1 < n-k$$

where the last inequality follows from Theorem 4.7. This is a contradiction and suffices for the proof of the proposition. $\square$

Lemma 4.2.6. *Let $z_i \in \mathsf{MID}$ be an element in the poset. If z_i violates midway then M_x is false. If z_i violates dual midway, then M_y is false. In other words, we have the following implications:*

$$|P_{>y}|+|P_{<z_i}| \leq n-k \implies a = \frac{2}{3}(1-v_{xy})$$
$$|P_{<x}|+|P_{>z_i}| \leq n-k \implies a = 2v_{xy}.$$

Proof. From Lemma 4.2.4, we know that either END_x and MID_x both satisfy midway or END_y and MID_y satisfy dual-midway. Suppose that $z_j \in \mathsf{MID}$ violates midway. That is, we have that

$|P_{>y}| + |P_{<z_j}| \leq n-k$. Then, from Proposition 4.2.6, we know that there is an element z' satisfying $x < z_j \leq z' \lessdot y$ so that there exists $z'' \in \mathsf{MID}_x$ where $z' \lessdot z''$. From Proposition 4.2.5, we have the inequality $|P_{<x}| + |P_{>z'}| \geq n-k+1$. Let

$$S = P_{x<\cdot<z''} \backslash \mathsf{MID} = \{z : x < z < z'', z \notin \mathsf{MID}\}.$$

Then, we have that

$$|S| + |\mathsf{MID}| + |P_{<x}| + |P_{>z'}| \leq n$$

since these sets do not over lap. We can conclude that

$$|S| + |\mathsf{MID}| + 1 \leq n - (|P_{<x}| - |P_{>z'}|) + 1 \leq k.$$

From Proposition 4.2.3, this implies that $e_{z',z''}$ is a transition vector. Hence $v_{z_j} = 1-a$. Now consider $z_0 \in P$ satisfying $x \lessdot z_0 \leq z_j < y$. We have that

$$|P_{<z_0}| + |P_{>y}| \leq |P_{<z_j}| + |P_{>y}| \leq n-k.$$

This implies that $v_{z_0} = v_{xy} + a/2$ from Proposition 4.2.5. But then this implies that $1-a = v_{xy}+a/2$ or $a = \frac{2}{3}(1-v_{xy})$. This means that M_x is false. Similarly, if z_j violates dual-midway, then we would know that $a = 2v_{xy}$ or M_y is false. $\square$

Lemma 4.2.7. *If $a \neq 1/2$, then the hypothesis in Theorem 4.7 implies that (x, y) satisfies k-midway or dual k-midway.*

Proof. From Lemma 4.2.5, we know that every element either satisfies dual-midway or satisfies midway (or possibly both). Suppose that there is an element which violates midway and another element which violates dual-midway. Then we must have M_x and M_y are both false, which cannot happen since $a \neq 1/2$. Thus every element in MID either all satisfies midway or all satisfies dual-midway or all satisfies both. If it all satisfies both, then we are done. If there is at least one element in MID which violates midway, then all elements in MID satisfies dual-midway. Moreover, it also means that M_x is false from Lemma 4.2.6 and M_y, E_y are true. This means that MID, and MID_y and END_y satisfies dual-midway, i.e. P satisfies dual k-midway. By similar reasoning, if there is at least one element which violates dual-midway, then P satisfies k-midway. This suffices for the proof. $\square$

We can now prove Theorem 4.4 and Theorem 4.5.

Proof of Theorem 4.5. If $N_{k-1} = N_k = N_{k+1}$, that means that $a = 1$. From the previous theorem, this implies that P satisfies k-midway or dual k-midway. The reverse direction was proved in [15]. $\square$

Proof of Theorem 4.4. If $a \neq 1/2$, then we have that P satisfies k-midway or dual k-midway. But this implies that $N_k = N_{k-1} = N_{k+1}$, or $a = 1$. Hence the only two choices of a are $a = 1, 1/2$. This corresponds to $N_{k+1} = N_k = N_{k-1}$ and $N_{k+1} = 2N_k = 4N_{k-1}$. $\square$

The only case that is remaining is the case $N_{k+1} = 2N_k = 4N_{k-1}$. To show that this case actually occurs, we give an example in Figure 4.1. In this poset, we can compute explicitly compute $N_1 = 1, N_2 = 2$ and $N_3 = 4$. This satisfies $N_3 = 2N_2 = 4N_1$.

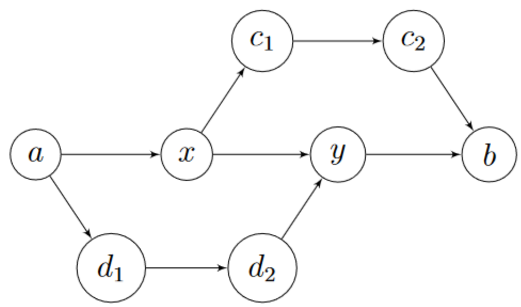

Figure 4.1: Example with $N_{i+1} = 2N_i = 4N_{i-1}$

Proof of Theorem 4.6. We first prove that if $N_{k+1} = 2N_k = 4N_{k-1}$ and $N_k > 0$, then conditions (i)-(iv) are true in Theorem 4.6. Since $a = 1/2$, we either have that both M_x and M_y are true or both E_x and E_y are true. Suppose that both M_x and M_y are true. Then every element in MID does not violate midway or dual midway. Moreover, at least one of E_x and E_y must be true as well. This proves that P satisfies midway and dual midway. But this would imply that $a = 1$, which contradicts $a = 1/2$. Hence, it must be the case that E_x and E_y are true. In other words, END_x and END_y both satisfy the midway and dual-midway property, respectively. This is exactly the property that (i) describes. The fact that MID is empty follows from the fact that END_x satisfies midway and END_y satisfies dual midway. Consider an arbitrary $z \in \mathsf{MID}$. Then, we can pick $z_1, z_2 \in P$ so that $x \lessdot z_1 \leq z \leq z_2 \lessdot y$. Since $v_{z_1} = v_{z_2} = v_z$, if both of the inequalities

$$|P_{>y}| + |P_{<z_1}| \leq n - k$$

$$|P_{<x}| + |P_{>z_2}| \leq n - k$$

are true, then Corollary 4.2.3 implies that $a = 0$, which cannot happen. This means that one of the inequalities must be wrong. Without loss of generality, suppose that $|P_{>y}| + |P_{<z_1}| > n - k$. Note that $\mathsf{MID}_x \sqcup \mathsf{MID} \sqcup \{y\} \sqcup P_{<z_1} \sqcup P_{>y} \subseteq P$ is a disjoint union of sets. Thus, we have that

$$|\mathsf{MID}_x| + |\mathsf{MID}| + |P_{<z_1}| + |P_{>y}| + 1 \leq n \implies |\mathsf{MID}_x| + |\mathsf{MID}| \leq k - 2.$$

Let $z'' \gtrdot y$. This element exists because P has a 1 element. Then, we have that

$$|P_{x<\cdot<z''}| \leq |\mathsf{MID}_x| + |\mathsf{MID}| \leq k - 2.$$

But this is a contradiction since $|P_{x<\cdot<z''}| > k$ from the fact that END_y satisfies dual-midway. This proves (iii). To prove (ii), suppose that $z \in P$ is incomparable to x and y. Then, we can build a chain from z to a maximal element by picking only elements incomparable to x and y until we are forced to pick elements in MID_x or END_y. We can do the same to reach a minimal element. It is not hard to prove that all of these vectors are transition vectors. Hence, we get that $0 = 1 - a$. But this cannot be true since $a = 1/2$. This proves (ii). To prove (iv), suppose that $z \in \mathsf{MID}_y$ and $z' \in \mathsf{MID}_x$ so that $z < z'$. Then, we can find $z_1 \in \mathsf{MID}_y$ and $z_2 \in \mathsf{MID}_x$ such that $z \leq z_1 \lessdot z_2 \leq z'$ since $\mathsf{MID} = \emptyset$. We analyze the conditions needed for e_{z_1,z_2} to be a transition vector. Note that

$$\begin{aligned}
\operatorname{aff} F_{K_0}(e_{z_1,z_2}) &= \bigoplus_{\omega \notin P_{z_1 \leq \cdot \leq y} \cup P_{x \leq \cdot \leq z_2}} \mathbb{R}[e_\omega] \oplus \mathbb{R}\left[\sum_{\omega \in P_{z_1 \leq \cdot \leq y} \cup P_{x \leq \cdot \leq z_2}} e_\omega\right] \\
\operatorname{aff} F_{K_1}(e_{z_1,z_2}) &= \sum_{\omega \geq y} e_\omega + \bigoplus_{\omega \notin P_{\leq x} \cup P_{\geq y} \cup \{z_1,z_2\}} \mathbb{R}[e_\omega] \\
\operatorname{aff} F_{K_0+K_1}(e_{z_1,z_2}) &= \sum_{\omega \geq y} e_\omega + \bigoplus_{\omega \notin \{z_1,z_2,x,y\}} \mathbb{R}[\omega] \oplus \mathbb{R}\left[\sum_{\omega \in P_{z_1 \leq \cdot \leq y} \cup P_{x \leq \cdot \leq z_2}} e_\omega\right].
\end{aligned}$$

The only obstruction to $e_{z_1,z_2} \in \operatorname{supp} \mu$ is the dimension of $F(K_0, e_{z_1,z_2})$. We have that $e_{z_1,z_2} \in \operatorname{supp} \mu$ if and only if

$$n - |P_{z_1<\cdot<y}| - |P_{x<\cdot<z_2}| - 3 \geq n - k - 1$$

or $|P_{z_1<\cdot<y}| + |P_{x<\cdot<z_2}| \leq k - 2$. But we know that e_{z_1,z_2} cannot be in the support because that would imply that $0 = 1 - a$. Hence, we have that

$$|P_{z<\cdot<y}| + |P_{x<\cdot<z'}| \geq |P_{z_1<\cdot<y}| + |P_{x<\cdot<z_2}| \geq k - 1.$$

To finish the proof, we must prove the reverse implication. Specifically, given the conditions that (0) $N_k > 0$, (1) END_x and END_y satisfy midway and dual-midway, (2) MID is empty, (3) every element is comparable to either x or y, and (4) for every $z \in \mathsf{MID}_y$ and $w \in \mathsf{MID}_x$ satifsying $z < w$, we have $|P_{z<\cdot<y}| + |P_{x<\cdot<w}| \geq k-1$, we want to prove that $N_{k+1} = 2N_k = 4N_{k-1}$. To prove this, we prove Claim 4.2.1.

Claim 4.2.1. *Let L_x be the number of linear extensions of the poset $\mathsf{MID}_y \sqcup \mathsf{END}_x$ and let L_y be the number of linear extensions of the poset $\mathsf{MID}_x \sqcup \mathsf{END}_y$. Then, we have that $N_m = 2^{m-1}L_xL_y$ for $m \in \{k-1, k, k+1\}$.*

Proof. Any linear extension f satisfying $f(y) - f(x) = k$ induces a total ordering on $\mathsf{MID}_x \sqcup \mathsf{END}_y$ and $\mathsf{MID}_y \sqcup \mathsf{END}_x$ as well as a binary string in $\{0,1\}^{m-1}$ where a 0 in the ith coordinate means that the ith element between x and y is in MID_x and a 1 in the ith coordinate menas that the ith element between x and y is in MID_y. Given this binary string and the two total orderings, we can reconstruct the linear extension f. This is because after specifying the positives between x and y that will be in MID_x and MID_y, the exact placement of the elements will be completely determined by the total orderings. In order to prove the claim, it suffices to prove that if we try to do this reconstruction process with any two linear extensions on $\mathsf{MID}_x \sqcup \mathsf{END}_y$ and $\mathsf{MID}_y \sqcup \mathsf{END}_x$ and any binary string in $\{0,1\}^{m-1}$, we will get a linear extension of P with $f(y) - f(x) = k$. There are three possibilities that may stop f from being a linear extension:

(1) MID_y may be too small. For example, we may not be able to accommodate the string of all zeros $0 \ldots 0$. To overcome this problem we must prove that $|\mathsf{MID}_y| \geq m-1$.

(2) MID_x may be too small for the same reason. Similarly, we must prove that $|\mathsf{MID}_x| \geq m-1$.

(3) Assuming that (1) and (2) are non-issues, the final issue is that the final ordering may not be a linear extension. This will happen when there is a $z \in \mathsf{MID}_x$ and $w \in \mathsf{MID}_y$ satisfying $w < z$ where z lies between x and w in the final ordering, and w lies between z and y in the final ordering.

Note that (1) and (2) are not issues after applying the midway property and dual midway property of END_x and END_y to $z_x \lessdot x$ and $z_y \gtrdot y$. These elements exists because of the assumption of a 0 element and 1 element. To prove that (3) is not an issue, suppose for the sake of contradiction that we had elements $z \in \mathsf{MID}_x$ and $w \in \mathsf{MID}_y$ that satisfy the problem in (3). Then, we would have

$$m - 1 \geq |P_{x<\cdot<z} \sqcup \{z\} \sqcup |P_{w<\cdot<y}| \sqcup \{w\}| = 2 + |P_{x<\cdot<z}| + |P_{w<\cdot<y}| \geq k+1.$$

But we know that $m \leq k+1$, which is a contradiction. This suffices for the proof of the claim. $\square$

From Claim 4.2.1, we have that

$$\frac{N_{k+1}}{4} = \frac{N_k}{2} = N_{k-1} = 2^{k-2} L_x L_y.$$

This completes the proof. $\square$

4.3 Stanley's Matroid Inequality

Stanley's poset inequality was proven by Richard Stanley in [55]. We have already written about this inequality extensively in Section 4.1. In the same paper, Stanley proves an inequality associated to the bases of a matroid. Let $M = (E, \mathcal{B})$ be a matroid of rank r with ground set E and bases $\mathcal{B}$. For any subsets $T_1, \ldots, T_r \subseteq E$, we can let $B(T_1, \ldots, T_r)$ denote the number of sequences $(y_1, \ldots, y_r)$ where $y_i \in T_i$ for $i \in [r]$ such that $\{y_1, \ldots, y_r\} \in \mathcal{B}$. Explicitly, we have that

$$B(T_1, \ldots, T_r) := \#\{(y_1, \ldots, y_r) \in T_1 \times \ldots \times T_r : \{y_1, \ldots, y_r\} \in \mathcal{B}(M)\}.$$

Stanley proves the inequality in Theorem 4.8. The inequality describes the log-concavity of the basis counting sequence $B(T_1, \ldots, T_r)$.

Theorem 4.8 (Theorem 2.1 in [55]). *Let $M = (E, \mathcal{B})$ be a unimodular matroid of rank r. Let $\mathcal{T} = (T_1, \ldots, T_{r-m})$ be a collection $r - m$ subsets of E and let $X, Y \subseteq E$. Then,*

$$B(\mathcal{T}, \underbrace{X, \ldots, X}_{k \text{ times}}, \underbrace{Y, \ldots, Y}_{m-k \text{ times}})^2 \geq B(\mathcal{T}, \underbrace{X, \ldots, X}_{k-1 \text{ times}}, \underbrace{Y, \ldots, Y}_{m-k+1 \text{ times}})B(\mathcal{T}, \underbrace{X, \ldots, X}_{k+1 \text{ times}}, \underbrace{Y, \ldots, Y}_{m-k-1 \text{ times}})$$

for all $1 \leq k \leq m-1$.

Before we present the proof of Theorem 4.8, we will first go over two different ways to view the basis counting numbers $B(T_1, \ldots, T_r)$. The first way to view these numbers is to associate them with a mixed volume of suitable polytopes. The second way to view these numbers is to associate them with a mixed discriminant of suitable discriminants. Both the mixed volume and mixed discriminant perspective will lead to immediate proofs of the theorem. When we study the equality case of the inequality, it is also useful to have both perspectives. If we wish to prove Theorem 4.8 with the regularity assumption removed, the mixed volume and mixed discriminant perspectives will prove to be insufficient. At the core of the mixed discriminant and mixed volume perspective is that the matroid has a unimodular coordinateization. In Section 4.3.5 we will remove the regularity assumption and prove Stanley's matroid inequality in its full generality using the technology of Lorentzian polynomials.

4.3.1 Mixed Volume Perspective of Basis Counting Number

Since our matroid $M = (E, \mathcal{B})$ is unimodular, there is a vector valued function $v : E \to \mathbb{R}^r$ such that the $r \times |E|$ matrix with $v(e)$ for $e \in E$ as columns is unimodular and the linear matroid generated by this matrix is isomorphic to M. We adopt the terminology of [55] and call this a **unimodular coordinatization** of M. Then, to every subset $T \subseteq E$, we can associate the zonotope

$$Z(T) := Z(v(e) : e \in T) = \sum_{e \in T} [0, v(e)].$$

Since $v : E \to \mathbb{R}^r$ is a unimodular coordination, from Example 2.5.5, we have that

$$\mathrm{Vol}_r(Z(T)) = \sum_{I \subseteq T : |I| = r} |\mathrm{Det}(v(e) : e \in I)| = \sum_{I \subseteq T : |I| = r} \mathbb{1}_{I \text{ is a basis}}. \tag{4.2}$$

Given subsets $T_1, \ldots, T_r \subseteq E$, consider the Minkowski sum $\sum_{i=1}^r \lambda_i Z(T_i)$. From Equation 4.2, we have that

$$\mathrm{Vol}_r \left(\sum_{i=1}^r \lambda_i Z(T_i) \right) = \mathrm{Vol}_n \left(\sum_{i=1}^r \sum_{e \in T_i} [0, \lambda_i v(e)] \right) = \sum_{a_1 + \ldots + a_r = r} C(a_1, \ldots, a_r) \lambda_1^{a_1} \ldots \lambda_r^{a_r}$$

where $C(a_1, \ldots, a_r)$ is the number of ways to pick subsets $Q_i \subseteq T_i$ and $|Q_i| = a_i$ for $i \in [r]$ where $Q_1 \cup \ldots \cup Q_r$ is a basis of M. According to Theorem 2.8, we have that

$$C(a_1, \ldots, a_r) = \binom{r}{a_1, \ldots, a_r} \mathsf{V}_r(Z(T_1)[a_1], \ldots, Z(T_r)[a_r])$$

and combinatorially we have $B(T_1, \ldots, T_r) = C(1, \ldots, 1)$. Thus, we have the equation

$$B(T_1, \ldots, T_r) = \binom{r}{1, \ldots, 1} \mathsf{V}_r(Z(T_1), \ldots, Z(T_r)) = r! \mathsf{V}_r(Z(T_1), \ldots, Z(T_r)). \tag{4.3}$$

Equation 4.3 immediately implies Theorem 4.8 via the Alexandrov-Fenchel inequality (Theorem 3.4).

4.3.2 Mixed Discriminant Perspective of the Basis Counting Number

From Subsection 4.3.1, we saw that the regularity condition on our matroid M implies that there exists a unimodular coordinatization $v : E \to \mathbb{R}^r$. From [9], this implies that we can view the basis counting sequence in terms of mixed discriminants. Now, consider $T_1, \ldots, T_s \subseteq E$. Recall that for $r_1, \ldots, r_s \geq 0$ satisfying $r_1 + \ldots + r_S = r$, we defined $B(T_1[r_1], \ldots, T_s[r_s])$ as the number of sequences in $T_1^{r_1} \times \ldots \times T_s^{r_s}$ which forms a basis of M. We define a different sequence $N(r_1, \ldots, r_s) := N^{T_1,\ldots,T_s}(r_1, \ldots, r_s)$ which is the number of ways to pick $Q_i \subseteq T_i$ with $|Q_i| = r_i$ such that $Q_1 \cup \ldots \cup Q_s$ is a basis of M. Then, we have that

$$B(T_1[r_1], \ldots, T_s[r_s]) = r_1! \ldots r_s! N^{T_1,\ldots,T_s}(r_1, \ldots, r_s). \tag{4.4}$$

For $1 \leq i \leq s$, let X_i be the matrix with colums $v(e)$ for $e \in T_i$. Let $A_i = X_i X_i^T$. Then, we know that

$$\mathsf{D}_r(A_1[r_1], \ldots, A_s[r_s]) = \frac{1}{r!} B(T_1[r_1], \ldots, T_s[r_s]) = \frac{N^{T_1,\ldots,T_s}(r_1, \ldots, r_s)}{\binom{r}{r_1,\ldots,r_s}} \tag{4.5}$$

from Lemma 2.4.1 and Equation 4.4. Equation 4.5 immediately implies Theorem 4.8 via Alexandrov's Inequality on mixed discriminants (Theorem 3.3).

4.3.3 Equality cases of Stanley's Matroid Inequality for Graphic Matroids

So far we have represented the Stanley's basis counting sequence in terms of mixed volumes and in terms of mixed discriminants. Based on our knowledge of the equality cases of the Alexandrov inequality for mixed discrminants and the Alexandrov-Fenchel inequality, we can hope to understand the combinatorial conditions on regular matroids M which give equality in the log-concavity inequality. To simplify the problem, we only consider the case where we have two subsets $T_1, T_2 \subseteq E$ which partition our ground set. In other words, we have some set $R \subseteq E$ where we define N_i to be the number of bases of M which intersect R at k points. From our analysis in previous sections, we know that

$$N_i = C^{R, E \backslash R}(i, r-i) = \binom{r}{i} \mathsf{V}_r(Z(R)[k], Z(E \backslash R)[r-k]).$$

Thus, the sequence N_i is ultra-log-concave. Let $\widetilde{N}_i = N_i / \binom{r}{i}$. We study the equality cases of $\widetilde{N}_i^2 = \widetilde{N}_{i-1} \widetilde{N}_{i+1}$.

From the mixed volume perspective, Stanley in [55] is able to give the equality cases for the

Minkowski inequality analog for his basis counting sequence. We write the contents of his theorem in Theorem 4.9.

Theorem 4.9 (Theorem 2.8 in [55]). *Let M be a loopless regular matroid of rank n on the finite set E, and let $R \subseteq S$. Then, the following two conditions are equivalent:*

(a) $\widetilde{N}_1^n = \widetilde{N}_0^{n-1}\widetilde{N}_n$.

(b) One of the following two conditions hold:

(i) $\widetilde{N}_1 = 0$.

(ii) There is some rational number $q \in (0,1)$ such that for every point $x \in E$, we have that $|\overline{x} \cap R| = q|\overline{x}|$ for all $x \in E$. Here $\overline{x}$ is the closure of the point x.

Before we present the proof of Theorem 4.9 in its full generality, we first present the proof of the theorem in the case where the matroid M is a graphic matroid using the mixed discriminant perspective. In general, the combinatorial meaning of the matrices which show up in the mixed discriminant perspective are difficult to decipher. However, in the case of graphic matroids, the matrices in the mixed discriminant are related to the Laplacian of the underlying graph. This makes Theorem 4.10 an interesting application of the mixed discriminant inequality to combinatorics. Suppose that M is a graphic matroid. Proposition 2.3.1 implies that there is a connected graph $G = (V, E)$ such that $M \cong M(G)$. Then, in the case of graphic matroids, Theorem 4.9 translates to Theorem 4.10.

Theorem 4.10. *Suppose that M is a graphic matroid corresponded to a connected graph $G = (V, E)$. Let $n = |V|$ and let $E = R \sqcup Q$ be a partition of the edge set so that both R and Q contain at least one spanning tree. Then, the following conditions are equivalent:*

(a) $\widetilde{N}_k^2 = \widetilde{N}_{k-1}\widetilde{N}_{k+1}$ *for some* $k \in \{1, \ldots, n-1\}$.

(b) $\widetilde{N}_k^2 = \widetilde{N}_{k-1}\widetilde{N}_{k+1}$ *for all* $k \in \{1, \ldots, n-1\}$.

(c) For any two distinct vertices $v, w \in V(G)$, the ratio between the number of edges between v and w that are in R to the number of edges between v and w that are in Q is some positive number which does not depend on our choice of $v, w \in V(G)$.

Proof. We first prove the theorem in the case where $n \geq 4$. Since spanning trees have at least two leaves, we can suppose without loss of generality that the vertices indexed by $n-1$ and n are such when we remove them from the graph there are still edges remaining. Let $B \in \mathbb{R}^{n \times |E|}$ be the incidence matrix of G. Then from Proposition 2.3.7, the matrix B and the reduced incidence matrix B_0 are totally unimodular matrices which represent M. We can then partition the columns in $B = [B_R|B_Q]$ where B_R are the columns corresponding to the edges in R and B_Q are the columns corresponding to the edges in Q. Let X_R, X_Q be the matrices so that

$$B_R = \begin{bmatrix} X_R \\ v_R^T \end{bmatrix}, \quad \text{and} \quad B_Q = \begin{bmatrix} X_Q \\ v_Q^T \end{bmatrix}$$

so that $B_0 = [X_R|X_Q]$ and $v_R, v_Q \in \{\pm 1, 0\}^{|E|}$. Since G is connected on n vertices, the rank of M will be $n-1$. This proves that B_0 is a unimodular coordinatization in the represents of the sequence $\widetilde{N}_k$ as mixed discriminants. Suppose that equality holds in $\widetilde{N}_k^2 \geq \widetilde{N}_{k-1}\widetilde{N}_{k+1}$. From the equality cases in Theorem 3.3, we know that $X_R X_R^T = \alpha X_Q X_Q^T$ for some $\alpha \in \mathbb{R}$. Let L be the Laplacian of the graph G. Then, from Proposition 2.2.2, we have that

$$L_{G|_R} = B_R B_R^T = \begin{bmatrix} X_R X_R^T & \bullet \\ \bullet & \bullet \end{bmatrix},$$
$$L_{G|_Q} = B_Q B_Q^T = \begin{bmatrix} X_Q X_Q^T & \bullet \\ \bullet & \bullet \end{bmatrix}.$$

where the first line is the Laplacian of the subgraph of G containing only edges from R, and the second line is the Laplacian of the subgrpah of G containig only edges from Q. From the equality case on Theorem 3.3, we know that there is some constant $\lambda \in \mathbb{R}$ such that $X_R X_R^T = \lambda X_Q X_Q^T$. This implies that for distinct $v, w \in [n-1]$, we have that $e_R(v, w) = \lambda e_Q(v, w)$ where $e_R(v, w)$ denotes the number of R edges between v, w, and $e_Q(v, w)$ denotes the number of Q edges between v, w. Since we are assuming that our sequence is strictly positive, we must have that $\lambda > 0$. If we repeat the argument except instead of deleting the last row of B to get B_0, we delete the second to last row, then we get that $e_R(v, w) = \lambda' e_Q(v, w)$ holds for λ' for $v, w \in [n] \backslash \{n-1\}$ where λ' may possibly be different from λ. Since $n \geq 4$ and we assumed that removing $n-1$ and n we still have an edge, this implies that the two constants are the same. This suffices for the proof of the theorem in the case where $n \geq 4$. The only non-trivial case to check is for $n = 3$. We can define $r_1 = e_R(v_2, v_3)$, $r_2 = e_R(v_1, v_3)$, and $r_3 = e_R(v_1, v_2)$. We can define q_1, q_2, and q_3 similarly. It is not difficult to conclude from our previous analysis that we must have the system of inequalities

$$q_3 r_2 - q_2 r_3 = 0$$

$$q_3 r_1 - q_1 r_3 = 0$$

$$q_2 r_1 - q_1 r_2 = 0.$$

If $r_1 = 0$, then we have that $q_1 r_2 = 0$ or $q_1 r_3 = 0$. We must have $r_2, r_3 > 0$ because of our assumption that the graph restricted to R contains a spanning tree. Thus we must have that $q_1 = 0$. Then we have $q_3 r_2 - q_2 r_3 = 0$. Since $r_2 > 0$ and $r_3 > 0$, we have that

$$\frac{q_3}{r_3} = \frac{q_2}{r_2}$$

which proves the result in this case. Now, we can suppose that $r_1, r_2, r_3 > 0$. This implies that

$$\frac{q_1}{r_1} = \frac{q_2}{r_2} = \frac{q_3}{r_3}.$$

This proves the result for $n = 3$. For $n < 3$, the result is vacuous. This suffices for the proof.

□

4.3.4 Equality cases of Stanley's Matroid Inequality for Regular Matroids

To prove Theorem 4.9 for the case of regular matroids, we prove Lemma 4.3.1 which gives necessary and sufficient conditions for when two zonotopes are homothetic to each other. Slightly weaker versions of Lemma 4.3.1 are proven in Lemma 2.7 of [55] and Lemma 5.3 of [35]. Strangely, for Stanley's application in [55], it is not immediately obvious why the version of Lemma 4.3.1 that he proves is enough to give equality cases. For this reason, we will write the prove a stronger version of the homothety characterization for zonotopes in Lemma 4.3.1.

Lemma 4.3.1 (Lemma 2.7 in [55]). *Let $v_1, \ldots, v_r$, $w_1, \ldots, w_s$ be vectors in $\mathbb{R}^n$ such that all of the vectors are contained in the same vector half-space of $\mathbb{R}^n$. Let $v_1', \ldots, v_l'$ be the vectors obtained from $v_1, \ldots, v_r$ from discarding any $v_i = 0$ and by adding together all remaining v_j's which are scalar multiples of each other. Similarly, define $w_1', \ldots, w_u'$. The zonotopes $Z(v_1, \ldots, v_r)$ and $Z(w_1, \ldots, w_s)$ are homothetic if and only if $t = u$ and after suitable indexing $v_i' = \gamma w_i'$ for $1 \leq i \leq t$ and some fixed $\gamma > 0$.*

Proof. Since the polytopes generated by $Z(v_1, \ldots, v_r) = Z(v_1', \ldots, v_l')$ and $Z(w_1, \ldots, w_s) = Z(w_1', \ldots, w_u')$, without loss of generality we can assume that the linear matroids generated by $v_1, \dots, v_r$ and $w_1, \dots, w_r$ are simple. Since all of our vectors lie in the same vector half-space, we know that 0 is a vertex of both zonotopes. Hence, the homothety must be of the form

$$Z(v_1, \dots, v_r) = \lambda Z(w_1, \dots, w_s)$$

for some $\lambda > 0$. By scaling each w_i by λ, we can further assume that $Z(v_1, \dots, v_r) = Z(w_1, \dots, w_s)$. At least one of the vectors v_i must be a vertex of $Z(v_1, \dots, v_r)$. Since this point is not the sum of other points in the zonotope, there must be a w_j such that $v_i = w_j$. Without loss of generality, let us say $v_1 = w_1$. Then, we have that

$$[0, v_1] + Z(v_2, \dots, v_r) = Z(v_1, \dots, v_r) = Z(w_1, \dots, w_s) = [0, w_1] + Z(w_2, \dots, w_s).$$

From Proposition 2.5.1, this gives us $Z(v_2, \dots, v_r) = Z(w_2, \dots, w_s)$. The lemma then follows from induction. $\square$

Remark 5. The version of Lemma 4.3.1 that Stanley proves requires that the vectors lie in the same orthant. However, it is not clear whether or not for any regular matroid we can find a unimodular coordinatization such that all of the vectors lie in the same orthant.

In order to apply Lemma 4.3.1, we want to show that in the unimodular coordinatization $v : E \to \mathbb{R}^r$, we can let $v(E)$ lie in the same vector half-space. If we negate any vector the coordinatization, this will not break the unimodularity since it will multiply the determinants of submatrices by 1 or -1. We can also negative all vectors so that the first coordinate is positive. This will guarentee that all vectors lie in the same closed half-space. Thus, we can apply Lemma 5.3.1 and prove Theorem 4.9.

Proof of Theorem 4.9. It is clear that the condition in (b) is enough to imply equality $\widetilde{N}_1^n = \widetilde{N}_0^{n-1}\widetilde{N}_n$. For the other direction, let $Q = E \backslash R$ and let $v : E \to \mathbb{R}^n$ be a unimodular coordinatization of M so that the vectors $v(E)$ lie in the same vector half-space. Recall that for all $0 \le k \le n$, we have that

$$N_k = \binom{n}{i} \mathsf{V}_n \left(\underbrace{Z(R), \dots, Z(R)}_{k \text{ times}}, \underbrace{Z(Q), \dots, Z(Q)}_{n-k \text{ times}} \right).$$

Thus the equality $\widetilde{N}_1^n = \widetilde{N}_0^{n-1}\widetilde{N}_n$ is equivalent to the equality in Minkowski's inequality (Theorem 3.7):

$$\mathsf{V}_n\left(Z(R),\underbrace{Z(Q),\dots,Z(Q)}_{n-1\text{ times}}\right)^n \geq \mathrm{Vol}_n(Z(R))^{n-1}\cdot \mathrm{Vol}_n(Z(Q)).$$

From the equality case of Theorem 3.7, we know that one of the following three things occur:

(1) $\dim Z(R) \leq n-2$.

(2) $\dim Z(R)$ and $\dim Z(Q)$ lie in parallel hyperplanes.

(3) $Z(R)$ and $Z(Q)$ are homothetic.

If $\dim Z(R) \leq n-2$ or $\dim Z(R)$ and $\dim Z(Q)$ lie in parallel hyperplanes, then we know that $\mathrm{Vol}_n(Z(R)) = \mathrm{Vol}_n(Z(Q)) = 0$. This implies that $\widetilde{N}_1 = 0$. Now suppose that $Z(R)$ and $Z(Q)$ are homothetic. Let $R = \{v_1,\dots,v_r\}$ and $Q = \{w_1,\dots,w_s\}$ with $r+s = |E|$ be the vectors in the coordinations. Let $R' = \{v_1',\dots,v_t'\}$ and $Q' = \{w_1',\dots,w_u'\}$ be the result of the modification in Lemma 4.3.1. Then $t = u$ and there must be a $\lambda > 0$ so that after a relabeling of indices we have $v_i' = \lambda w_i'$ for all $1 \leq i \leq t$. In the coordinatization, note that $v(e) = v(f)$ if and only if e and f lie in the same parallel class. Thus, it must be the case that equality holds if and only if there exists $\lambda > 0$ so that $|R \cap \overline{x}| = \lambda |Q \cap \overline{x}|$ for all $x \in E$. This suffices for the proof. □

With the mixed discriminant perspective, we can extend Theorem 4.9 slightly to the case where we are only guarenteed that one of the inequalities $\widetilde{N}_k^2 \geq \widetilde{N}_{k-1}\widetilde{N}_{k+1}$ is equality. Indeed, by viewing the sequence $\widetilde{N}_k$ as a sequence of mixed discriminants, we can prove Lemma 4.3.2.

Lemma 4.3.2. *Let M be a matroid of rank n. Suppose that $\widetilde{N}_0 > 0$ and $\widetilde{N}_n > 0$. Then $\widetilde{N}_1^n = \widetilde{N}_0^{n-1}\widetilde{N}_n$ if and only if $\widetilde{N}_k^2 = \widetilde{N}_{k-1}\widetilde{N}_{k+1}$ for some $k \in \{1,\dots,n-1\}$.*

Proof. Assuming that $\widetilde{N}_1^n = \widetilde{N}_0^{n-1}\widetilde{N}_n$, we know that $\widetilde{N}_k^2 = \widetilde{N}_{k-1}\widetilde{N}_{k+1}$ for all k from our proof of equality conditions in Theorem 3.7. For the converse, suppose that $\widetilde{N}_k^2 = \widetilde{N}_{k-1}\widetilde{N}_{k+1}$ for some k. Recall that we can view

$$\widetilde{N}_k = \mathsf{D}(A_R[k], A_Q[n-k])$$

where $A_R = X_R X_R^T$, $A_Q = X_Q X_Q^T$, and X_R, X_Q are the matrices which consist of the unimodular representations of the elements in R and Q as columns. From Corollary 3.3.2, we get that $\widetilde{N}_k^2 = \widetilde{N}_{k-1}\widetilde{N}_{k+1}$ holds for all $k \in \{1,\dots,n-1\}$. □

From Lemma 4.3.2, whenever $\widetilde{N}_0, \widetilde{N}_n > 0$ (which translate to R and Q both having full rank), we get that the necessary and sufficient conditions for equality in Theorem 4.9 are exactly the necessary and sufficient conditions to guarentee equality at a single instance of the log-concavity inequality. Thus, we make the following conjecture for all matroids (not necessarily regular).

Conjecture 4.3.1. *Let M be a loopless matroid of rank of rank n on a set E. Let $R \subseteq E$ be a subset and let $Q = E \backslash R$. For k, $0 \leq k \leq n$, we define N_k to be the number of bases of M with k elements in R. Define $\widetilde{N}_k = \frac{N_k}{\binom{n}{k}}$. Suppose that R and Q both have rank n. Then, the following are equivalent.*

(a) $\widetilde{N}_k^2 = \widetilde{N}_{k-1}\widetilde{N}_{k+1}$ *for some* $k \in \{1, \dots, n-1\}$.

(b) $\widetilde{N}_k^2 = \widetilde{N}_{k-1}\widetilde{N}_{k+1}$ *for all* $k \in \{1, \dots, n-1\}$.

(c) There are positive integers $r, q \geq 1$ *such that* $|\overline{x} \cap R|r = |\overline{x} \cap Q|q$ *for all* $x \in E$.

To show some evidence of Conjecture 4.3.1, we will prove one of the directions of Conjecture 4.3.1. This result is written in the statement of Theorem 4.11.

Theorem 4.11. *Suppose that there are positive integers $r, s \geq 1$ such that $|\overline{x} \cap R|r = |\overline{x} \cap Q|q$. Then, $\widetilde{N}_k^2 = \widetilde{N}_{k-1}\widetilde{N}_{k+1}$ for all $k \in \{1, \dots, n-1\}$. In particular, we have that $\widetilde{N}_k = \frac{q}{r}\widetilde{N}_{k-1}$.*

Proof. For all x, we have that $|\overline{x} \cap R|r = |\overline{x} \cap Q|q$. This implies that for $x \in E$, we can define an isomorphism $\varphi_x : (\overline{x} \cap R) \times [r] \to (\overline{x} \cap Q) \times [q]$ where $\varphi_x = \left(\varphi_x^1, \varphi_x^2\right)$. Let the inverse map be $\psi_x = (\psi_x^1, \psi_x^2)$. For all $k \in \{1, \dots, n\}$, define the sets

$$\Omega_k := \{(a, i, U) : U \in \mathcal{B}(M), |U \cap R| = k, i \in [r], a \in U \cap R\}$$
$$\Omega_{k-1} := \{(b, j, V) : V \in \mathcal{B}(M), |V \cap R| = k-1, j \in [q], b \in V \cap Q\}.$$

To count the number of elements in Ω_k and Ω_{k+1}, we count the number of elements there are with fixed U and V. Then, we sum over all possible choices of U and V. This gives the identity

$$|\Omega_k| = N_k \cdot k \cdot r$$
$$|\Omega_{k-1}| = N_{k-1} \cdot (n-k+1) \cdot q.$$

I claim that there is one-to-one correspondence between Ω_k and Ω_{k-1}. We define maps $\varphi_\downarrow : \Omega_k \to \Omega_{k-1}$ and $\varphi_\uparrow : \Omega_{k-1} \to \Omega_k$ given by

$$\varphi_\downarrow(a, i, U) = (\varphi_1^x(a, i), \varphi_2^x(a, i), (U \backslash a) \cup \varphi_1^x(a, i))$$
$$\varphi_\uparrow(b, j, V) = (\psi_1^x(b, j), \psi_2^x(b, j), (V \backslash b) \cup \psi_1^x(b, j)).$$

From construction, the maps $\varphi_{\downarrow}$ and $\varphi_{\uparrow}$ are two-sided inverses of each other. This gives a one-to-one correspondence between Ω_k and Ω_{k-1}. From our computation of the cardinalities of these sets, we have that

$$N_k \cdot k \cdot r = |\Omega_k| = |\Omega_{k-1}| = N_{k-1}(n-k+1)q.$$

This implies that $\widetilde{N}_k = \frac{q}{r}\widetilde{N}_{k-1}$. This proves the theorem. $\square$

In the subsequent subsection, we prove Stanley's matroid inequality without the regularity condition using the technology of Lorentzian polynomials.

4.3.5 Lorentzian Perspective of the Basis Counting Number

One essential hypothesis in Stanley's matroid inequality was that the matroid $M = (E, \mathcal{B})$ had to be regular. This condition was needed in order to have a unimodular coordinatization which was at the heart of the mixed discriminant and mixed volume constructions. In this section, we generalize Stanley's matroid inequality by removing the hypothesis of unimodularity using the theory of Lorentzian polynomials. Our proof will be based on the fact that the basis generating polynomial of a matroid is Lorentzian. For simplicity, suppose that the ground set of $M = (E, \mathcal{I})$ is $E = [n]$ and $\operatorname{rank}(M) = r$. Recall that the basis generating polynomial of M is defined as

$$f_M(x_1, \ldots, x_n) = \sum_{B \in \mathcal{B}} x^B = \sum_{\substack{1 \le i_1 < \ldots < i_r \le n \\ \{i_1, \ldots, i_r\} \in \mathcal{B}(M)}} x_{i_1} \ldots x_{i_r}.$$

For any sequence of subsets $\mathcal{T} := (T_1, \ldots, T_m) \subseteq E$, we can define the following polynomial which lets us get a better handle on the basis counting numbers.

Definition 4.3.1. Let $\mathcal{T} := (T_1, \ldots, T_m)$ be a sequence of subsets of E. Then, consider the polynomial

$$g_M^{\mathcal{T}}(y_1, \ldots, y_m) := f_M\left(x_e = \sum_{i : e \in T_i} y_i\right)$$

where in the right hand side, we consider the basis generating polynomial where we replace each instance of the coordinate x_e with the linear form $\sum_{i : e \in T_i} y_i$.

For $a_1, \ldots, a_m \ge 0$ satisfying $a_1 + \ldots + a_m = r$ we define the number $N(a_1, \ldots, a_m)$ as the number of ways to pick subsets $Q_i \subset T_i$ with $|Q_i| = a_i$ such that $Q_1 \cup \ldots \cup Q_r$ is a basis of M.

Lemma 4.3.3. *For $T_1, \ldots, T_m \subseteq E$ and $a_1, \ldots, a_m \geq 0$ satisfying $a_1 + \ldots + a_m = r$, we have that*

$$N(a_1, \ldots, a_m) = \frac{B(T_1[a_1], \ldots, T_m[a_m])}{a_1! \ldots a_m!}.$$

Proof. Any choice of subsets $Q_i \subseteq T_i$ with $|Q_i| = a_i$ and $Q_1 \cup \ldots \cup Q_r$ a basis of M gives $a_1! \ldots a_m!$ sequences which are included in the count of $B(T_1[a_1], \ldots, T_m[a_m])$. Conversely, every sequence determines subsets $Q_i \subseteq T_i$ with $|Q_i| = a_i$ and $Q_1 \cup \ldots \cup Q_r$ a basis of M. It is clear that this is a $1 : a_1! \ldots a_m!$ correspondence. This suffices for the proof of the lemma. □

Proposition 4.3.1. *For $T_1, \ldots, T_m \subseteq E$, we have that*

$$g_M^{T_1,\ldots,T_m}(y_1, \ldots, y_m) = \sum_{a_1+\ldots+a_m=r} \frac{B(T_1[a_1], \ldots, T_m[a_m])}{a_1! \ldots a_m!} \cdot y_1^{a_1} \ldots y_m^{a_m}.$$

Proof. By substituting $x_e = \sum_{i:e\in\mathcal{T}_i} y_i$ in the formula for the basis generating polynomial, we get that

$$\begin{aligned} g_M^{\mathcal{T}}(y_1, \ldots, y_m) &= \sum_{\substack{1\leq i_1<\ldots<i_r\leq m \\ \{i_1,\ldots,i_r\}\in\mathcal{B}(M)}} \prod_{k=1}^{r} \left(\sum_{i_k\in T_j} y_j \right) \\ &= \sum_{\substack{1\leq i_1<\ldots<i_r\leq m \\ \{i_1,\ldots,i_r\}\in\mathcal{B}(M)}} \left(\sum_{a_1+\ldots+a_m=r} N^{\{i_1,\ldots,i_r\}}(a_1, \ldots, a_m) \right) y_1^{a_1} \ldots y_m^{a_m} \end{aligned}$$

where $N^B(a_1, \ldots, a_m)$ is the number of ways to pick subsets $Q_i \subseteq T_i$ with $|Q_i| = a_i$ and $Q_1 \cup \ldots \cup Q_r = B$. In particular, we have that

$$\sum_{B\in\mathcal{B}(M)} N^B(a_1, \ldots, a_m) = N(a_1, \ldots, a_m).$$

This allows us to simplify the equation after changing the order of summations:

$$\begin{aligned} g_M^{\mathcal{T}}(y_1, \ldots, y_m) &= \sum_{a_1+\ldots+a_m=r} \left(\sum_{B\in\mathcal{B}(M)} N^B(a_1, \ldots, a_m) \right) y_1^{a_1} \ldots y_m^{a_m} \\ &= \sum_{a_1+\ldots+a_m=r} N(a_1, \ldots, a_m) \cdot y_1^{a_1} \ldots y_m^{a_m}. \end{aligned}$$

From Lemma 4.3.3, we have proven the lemma. □

Lemma 4.3.4. *For $T_1, \ldots, T_m \subseteq E$, the polynomial $g_M^{T_1,\ldots,T_m}$ is Lorentzian.*

Proof. This follows from Theorem 3.9 and Proposition 3.5.1. □

We can now prove the main theorem for this section. This theorem will be a generalization of Stanley's matroid inequality in Theorem 4.8.

Theorem 4.12. *Let $M = (E, \mathcal{B})$ be any matroid of rank r. Let $\mathcal{T} = (T_1, \ldots, T_{r-m})$ be a sequence of subsets in E and let $Q, R \subseteq E$ be subsets. Define the sequence*

$$B_k(\mathcal{T}, Q, R) := B(T_1, \ldots, T_{r-m}, Q[k], R[m-k]).$$

Then the sequence $B_0(\mathcal{T}, Q, R), \ldots, B_m(\mathcal{T}, Q, R)$ is log-concave.

Proof. Consider the Lorentzian polynomial $g_M^{T_1,\ldots,T_{r-m},Q,R}$. The result then follows from Proposition 3.5.2. □

Using Theorem 4.12, we can also remove the regularity hypothesis from Corollary 2.4 in [55].

Corollary 4.3.1. *Let $M = (E, \mathcal{B})$ be any matroid of rank n and let $T_1, \ldots, T_r, Q, R$ be pairwise disjoint subsets of E whose union is E. Fix non-negative integers $a_1, \ldots, a_r$ such that $m = n - a_1 - \ldots - a_r \geq 0$, and for $0 \leq k \leq m$ define f_k to be the number of bases B of M such that $|B \cap T_i| = a_i$ for $1 \leq i \leq r$, and $|B \cap R| = k$ (so $|B \cap Q| = m - k$). Then the sequence $f_0, \ldots, f_m$ ultra-log-concave.*

Proof. Let $\mathcal{T} = (T_1[a_1], \ldots, T_r[a_r])$. Then we have $B_k(\mathcal{T}, Q, R) = a_1! \ldots a_r! k!(m-k)! f_k$ where we use the same notation as in Theorem 4.12. Thus, we have that

$$\frac{f_k}{\binom{m}{k}} = \frac{B_k(\mathcal{T}, Q, R)}{a_1! \ldots a_r!}.$$

The ultra-log-concavity of f_k then follows from Theorem 4.12. □

According to Stanley in [55], Theorem 4.12 would imply the first Mason conjecture. Indeed, this is the content of Theorem 2.9 in [55]. We rewrite the proof now for the sake of completeness.

Theorem 4.13 (Mason's Conjecture)**.** *Let $M = (E, \mathcal{I})$ be a matroid of rank n. For all $0 \leq k \leq n$, let I_k be the number of independent sets of M of rank k. Then $I_k^2 \geq I_{k-1} I_{k+1}$ for all $2 \leq k \leq n-1$.*

Proof. Let B_n be the boolean matroid on n elements. Consider $T^n(B_n + M)$ the level n truncation of the matroid sum $B_n + M$. Let f_k be the number of bases of $T^n(B_n + M)$ which shares k elements with $E(M)$. Then, we have that $f_k = I_k \binom{n}{n-k}$. From Corollary 4.3.1, we have that I_k is log-concave. This suffices for the proof. □

Remark 6. In [12], Bräden-Huh prove the strongest Mason conjecture using Lorentzian polynomials. Their proof involves proving that the homogeneous multivariate Tutte polynomial of a matroid is Lorentzian.

Remark 7. With the technology of Lorentzian polynomials, we can see that Stanley would have been unable to prove his matroid inequality without the hypothesis that the matroid is regular. Indeed, from Remark 4.3, the basis generating polynomial of a matroid on $[n]$ is the volume polynomial of n convex bodies precisely when the matroid is regular.

Chapter 5

Hodge Theory for Matroids

In this chapter we will study cohomology rings associated to matroids. We will also be interested in how to extract combinatorial information from these rings. In the general case, our cohomology ring will be a graded $\mathbb{R}$-algebra of the form $A^\bullet = \bigoplus_{i=0}^{d} A^i$. On the top degree, we will have an isomorphism $\mathsf{deg} : A^d \to \mathbb{R}$ called the degree map. In addition, the natural pairing $A^i \times A^{d-i} \to A^d \to \mathbb{R}$ is non-degenerate for all i, we will call our ring $A^\bullet$ a **Poincaré duality algebra**. In the finite dimensional case, this will automatically imply that $\dim A^k = \dim A^{d-k}$ for all k. Cohomology rings with these properties show up naturally in topology and algebraic geometry. For an overview of such examples, we refer the reader to [31]. Given a cohomology ring associated to a matroid, we will study when this graded object satisfies Poincaré duality, Hard Lefschetz, and Hodge-Riemann relations. If it satisfies all three properties in all degrees ($\leq \frac{d}{2}$) we say that it satisfies the Kähler package. For the development of the Hodge theory of matroids, we refer the reader to the papers [1, 11, 5, 20, 21] where the notions of the Chow ring, intersection cohomology, and conormal Chow ring are studied. Through this development, many difficult and long standing conjectures in combinatorics have been solved by proving that a suitable matroid cohomology satisfies the Kähler package. In the next three sections, we define the notions of the graded Möbius algebra, the Chow ring of a matroid, and the augmented Chow ring of a matroid. We discuss these notions at a surface level, and the sections primarily serve as a place to write down definitions and semi-small decomposition results for future sections (see Section 5.6).

5.1 Cohomology Rings for Matroids

In this section, we will review the definitions and properties of three cohomology rings associated to matroids: the graded Möbius algebra, the Chow ring, and the augmented Chow ring. The Chow ring and the augmented Chow ring of a matroid are automatically Poincaré duality algebras. The graded Möbius algebra is in general not a Poincaré duality algebra.

5.1.1 Graded Möbius Algebra

Let M be a matroid on ground set E and let $\mathcal{L}$ be its lattice of flats. For $k \geq 0$, let $\mathcal{L}^k(M)$ be the set of lattices of rank k. For every k, we can define the real vector space $H^k(M)$ given by

$$H^k(M) := \bigoplus_{F \in \mathcal{L}^k(M)} \mathbb{R} y_F$$

where we have a variable y_F for every flat $F \in \mathcal{L}^k(M)$. We can then define a graded multiplicative structure $H^k(M) \times H^l(M) \to H^{k+1}(M)$ where for $F_1 \in \mathcal{L}^k(M)$ and $F_2 \in \mathcal{L}^l(M)$ we have

$$y_{F_1} \cdot y_{F_2} = \begin{cases} y_{F_1 \vee F_2} & \text{if } \operatorname{rank}_M(F_1) + \operatorname{rank}_M(F_2) = \operatorname{rank}_M(F_1 \vee F_2), \\ 0 & \text{if } \operatorname{rank}_M(F_1) + \operatorname{rank}_M(F_2) > \operatorname{rank}_M(F_1 \vee F_2). \end{cases} \tag{5.1}$$

Definition 5.1.1. For every matroid M, we define the **graded Möbius algebra** to be the graded ring $H(M) = \bigoplus_{k \geq 0} H^k(M)$ equipped with the multiplicative structure defined in Equation 5.1.

For every $e \in E$ in the ground set, we can define a graded ring homomorphism $\theta_e : H(M \backslash e) \to H(M)$ where $\theta_e(y_F) = y_{\operatorname{clo}_M(F)}$ for all $F \in \mathcal{L}(M \backslash i)$. Explicitly, we have that $\operatorname{clo}_M(F)$ is F if $F \in \mathcal{L}(M)$ and $F \cup \{e\}$ if $F \notin \mathcal{L}(M)$. Since the flats of $M \backslash e$ are of the form $F - \{e\}$ where F is a flat of M, this map is well-defined. It is also a degree preserving map since $\operatorname{rank}(S) = \operatorname{rank}(\operatorname{clo}(S))$. It is also easy to see that θ_i is injective. We prove that θ_e is a ring homomorphism in Lemma 5.1.1.

Lemma 5.1.1. *For every $e \in M$, the map $\theta_e : H(M \backslash i) \to H(M)$ is a ring homomorphism.*

Proof. Let $F_1 \in \mathcal{L}^k(M \backslash e)$ and $F_2 \in \mathcal{L}^l(M \backslash e)$ be flats $M \backslash e$. We want to prove that $\theta_e(y_{F_1} \cdot y_{F_2}) = \theta_e(y_{F_1})\theta_e(y_{F_2})$. Since the multiplicative structure of our ring depends on the structure of our flats, we will prove the homomorphism property in separate cases.

(a) Suppose that $y_{F_1} \cdot y_{F_2} = 0$, then we have $\operatorname{rank}_M(F_1 \vee F_2) < \operatorname{rank}_M(F_1) + \operatorname{rank}_M(F_2)$.

(i) Additionally, suppose that $F_1, F_2 \notin \mathcal{L}(M)$. Then $\theta(y_{F_1}) = y_{F_1 \cup i}$ and $\theta(y_{F_2}) = y_{F_2 \cup i}$. To prove that $\theta_{y_{F_1}} \theta_{y_{F_2}} = 0$, it is enough to prove

$$\operatorname{rank}_M(F_1 \cup F_2 \cup e) < \operatorname{rank}_M(F_1 \cup e) + \operatorname{rank}_M(F_2 \cup e).$$

If $e \in \operatorname{clo}_M(F_1 \cup F_2)$, then this result follows from $\operatorname{rank}_M(F_1 \vee F_2) < \operatorname{rank}_M(F_1) + \operatorname{rank}_M(F_2)$. If $e \notin \operatorname{clo}_M(F_1 \cup F_2)$, then there is a basis of $F_1 \cup F_2$ such that $I \cup e$ is a basis of $F_1 \cup F_2 \cup e$. We then have

$$\operatorname{rank}_M(F_1 \cup F_2 \cup e) = 1 + |I| \leq 1 + |I \cap F_1| + |I \cap F_2| < \operatorname{rank}_M(F_1) + \operatorname{rank}_M(F_2).$$

This proves the homomorphism property for $F_1, F_2 \notin \mathcal{L}(M)$.

(ii) Now, suppose that $F_1 \notin \mathcal{L}(M)$ and $F_2 \in \mathcal{L}(M)$. Then $\operatorname{rank}(F_1) = \operatorname{rank}(F_1 \cup i)$ and $\operatorname{rank}(F_1 \cup F_2) < \operatorname{rank}(F_1) + \operatorname{rank}(F_2)$. We have that

$$\begin{aligned} \operatorname{rank}\left((F_1 \cup e) \cup F_2\right) &= \operatorname{rank}(F_1 \cup F_2) \\ &< \operatorname{rank}(F_1) + \operatorname{rank}(F_2) \\ &= \operatorname{rank}(F_1 \cup e) + \operatorname{rank}(F_2). \end{aligned}$$

This proves the homomorphism property for $F_1 \notin \mathcal{L}(M)$ and $F_2 \in \mathcal{L}(M)$.

(iii) Suppose that $F_1, F_2 \in \mathcal{L}(M)$. Then we automatically get the homomorphism property.

(b) Now, suppose that $y_{F_1} \cdot y_{F_2} = y_{F_1 \vee F_2}$. Then, we have $\operatorname{rank}(F_1 \vee F_2) = \operatorname{rank}(F_1) + \operatorname{rank}(F_2)$.

(i) Suppose that $F_1, F_2 \notin \mathcal{L}(M)$. Then, we have that $\operatorname{rank}(F_1) = \operatorname{rank}(F_1 \cup e)$ and $\operatorname{rank}(F_2) = \operatorname{rank}(F_2 \cup e)$. This implies that $\operatorname{rank}(F_1 \cup F_2 \cup e) = \operatorname{rank}(F_1 \cup F_2)$. Thus, we have that

$$\begin{aligned} \operatorname{rank}\left((F_1 \vee e) \vee (F_2 \vee e)\right) &= \operatorname{rank}(F_1 \cup F_2) \\ &= \operatorname{rank}(F_1) + \operatorname{rank}(F_2) \\ &= \operatorname{rank}(F_1 \vee e) + \operatorname{rank}(F_2 \vee e). \end{aligned}$$

This proves the homomorphism property for $F_1, F_2 \notin \mathcal{L}(M)$.

(ii) Suppose that $F_1 \notin \mathcal{L}(M)$ and $F_2 \in \mathcal{L}(M)$. Then we have that $\operatorname{rank}(F_1 \cup e) = \operatorname{rank}(F_1)$ and $\operatorname{rank}(F_2 \cup e) = 1 + \operatorname{rank}(F_2)$. We also have $\operatorname{rank}(F_1 \cup F_2 \cup i) = \operatorname{rank}(F_1 \cup F_2)$. Thus, we have

$$\begin{aligned}\operatorname{rank}((F_1 \cup e) \cup F_2) &= \operatorname{rank}(F_1 \cup F_2)\\ &= \operatorname{rank}(F_1) + \operatorname{rank}(F_2)\\ &= \operatorname{rank}(F_1 \cup e) + \operatorname{rank}(F_2).\end{aligned}$$

This proves the homomorphism property ofr $F_1 \notin \mathcal{L}(M)$ and $F_2 \in \mathcal{L}(M)$.

(iii) Finally, suppose that $F_1, F_2 \in \mathcal{L}(M)$. Then the homomorphism property follows automatically.

This suffices for the proof. $\square$

5.1.2 Chow Ring of a Matroid

In this section, we give a surface level overview of the Chow ring associated to a matroid. We will not prove many of the claims we make, and refer the reader to [11] for the details. Let $M = (E, \mathcal{I})$ be a loopless matroid. According to [11], when M is realizable over a field k, then the Chow ring of M is isomorphic to the Chow ring of a smooth projective variety over k. The Chow ring of a matroid was originally introduced by Feichtner and Yuzvinsky in the paper [21]. It is defined as a quotient ring of the polynomial ring

$$\underline{\mathrm{S}}_M := \mathbb{R}\left[x_F : F \text{ is a nonempty proper flat of } M\right].$$

We can define the two ideals of $\underline{\mathrm{S}}_M$ defined by

$$\begin{aligned}\underline{\mathrm{I}}_M &:= \left\langle \sum_{i_1 \in F} x_F - \sum_{i_2 \in F} x_F : \text{ for all } i_1, i_2 \in E \right\rangle = \left\langle \sum_{i_1 \notin F} x_F - \sum_{i_2 \notin F} x_F : \text{ for all } i_1, i_2 \in E \right\rangle,\\ \underline{\mathrm{J}}_M &:= \langle x_{F_1} x_{F_2} : \ F_1 \text{ and } F_2 \text{ are incomparable} \rangle.\end{aligned}$$

Definition 5.1.2. For a matroid M, we define the **Chow ring** of M to be the quotient algebra

$$\underline{\mathrm{CH}}(M) := \frac{\underline{\mathrm{S}}_M}{\underline{\mathrm{I}}_M + \underline{\mathrm{J}}_M}.$$

The Chow ring of a matroid is a graded algebra where the grading is inherited from the grading on $\underline{\mathrm{S}}_M$. When M has rank d, the top dimension of $\underline{\mathrm{CH}}(M)$ is $d-1$. There exists an isomorphism

$$\underline{\deg} : \underline{\mathrm{CH}}(M)^{d-1} \longrightarrow \mathbb{R}, \quad \prod_{F \in \mathcal{F}} x_F \longmapsto 1$$

whenever $\mathcal{F}$ is any complete flag of nonempty proper flats. Similar to the graded Möbius algebra, there exists a natural embedding $\underline{\theta}_i : \underline{\mathrm{CH}}(M\backslash e) \to \underline{\mathrm{CH}}(M)$ which gives $\underline{\mathrm{CH}}(M)$ a $\underline{\mathrm{CH}}(M\backslash i)$-module structure. From [6], the category of graded $\underline{\mathrm{CH}}(M\backslash i)$ modules is a Krull-Schmidt category. Therefore, there is a unique way to decompose $\underline{\mathrm{CH}}(M)$ into indecomposable $\underline{\mathrm{CH}}(M\backslash i)$ modules. Theorem 1.2 in [11] gives the decomposition in terms of smaller matroids.

Theorem 5.1 (Theorem 1.2 in [11]). *If i is not a coloop of M, there is a decomposition of $\underline{\mathrm{CH}}(M)$ into indecomposable graded $\underline{\mathrm{CH}}(M\backslash i)$-modules such that*

$$\underline{\mathrm{CH}}(M) = \theta_i\left(\underline{\mathrm{CH}}(M\backslash i)\right) \oplus \bigoplus_{F \in \underline{\mathcal{S}}_i} x_{F\cup i} \cdot \theta_i\left(\underline{\mathrm{CH}}(M\backslash i)\right).$$

Here, $\underline{\mathcal{S}}_i$ consists of all non-empty proper flats F of $E\backslash i$ such that i is a coloop of $F \cup i$.

The semi-small decomposition in Theorem 5.1 provides an avenue to inductively prove that the Chow ring satisfies the Kähler package.

5.1.3 Augmented Chow Ring of a Matroid

The augmented Chow ring of a matroid is defined in [11]. This ring is intimately related to the Chow ring of matroid. The augmented Chow ring is defined to be a quotient ring of the polynomial ring

$$S_M := \mathbb{R}\left[x_F, y_i : i \in E, F \text{ a proper flat of } M\right].$$

Let $J_M^{(1)}$ be the ideal generated by $x_{F_1}x_{F_2}$ where F_1 and F_2 are incomparable proper flats. Let $J_M^{(2)}$ be the ideal generated by $y_i x_F$ where $i \in E$ and F is a proper flat not containing i. Then, we have the two ideals which are used to define the augmented Chow ring.

$$I_M := \left\langle y_i - \sum_{i \notin F} x_F : \text{ for all } i \in E \right\rangle,$$
$$J_M := J_M^{(1)} + J_M^{(2)}.$$

Definition 5.1.3. For a matroid M, we define the **augmented Chow ring** of M to be the quotient algebra

$$\mathrm{CH}(M) := \frac{S_M}{I_M + J_M}.$$

Like the Chow ring of a matroid, the augmented Chow ring of a matroid is a graded algebra with the grading inherited from the polynomial ring $\underline{S}_M$. The top dimension of $\mathrm{CH}(M)$ will be the rank of the matroid M, and there is an isomorphism $\deg : \mathrm{CH}^d(M) \longrightarrow \mathbb{R}$. Note that the subring generated by $\{y_i\}_{i\in E}$ is isomorphic to the graded Möbius algebra. Hence, there is a embedding $H(M) \to \mathrm{CH}(M)$ which also gives $\mathrm{CH}(M)$ a $H(M)$-module structure. There is also a natural injection $\theta_i : \mathrm{CH}(M\backslash i) \to \mathrm{CH}(M)$ for any $i \in E$ which makes $\mathrm{CH}(M)$ into a $\mathrm{CH}(M\backslash i)$ module. The Krull-Schmidt decomposition of $\mathrm{CH}(M)$ into $\mathrm{CH}(M\backslash i)$ modules when i is not a coloop is given by Theorem 1.5 in [11].

Theorem 5.2 (Theorem 1.5 in [11]). *If i is not a coloop of M, then we can decompose* $\mathrm{CH}(M)$ *into indecomposable graded* $\mathrm{CH}(M\backslash i)$*-modules such that*

$$\mathrm{CH}(M) = \theta_i(\mathrm{CH}(M\backslash i)) \oplus \bigoplus_{F\in\mathcal{S}_i} x_{F\cup i} \cdot \theta_i(\mathrm{CH}(M\backslash i)).$$

This decomposition is similar to that of Theorem 5.1. This allows us to hope that similar decompositions hold for other cohomology rings associated to matroids. We discuss this point in Section 5.6 when we discuss possible directions for future research.

5.2 Gorenstein Ring associated to a polynomial

In this section, we study a ring introduced in by Toshiaki Maeno and Yasuhide Numata in [37]. The Gorenstein ring associated to the basis generating polynomial of a matroid is intimately related to graded Möbius algebra associated to a matroid. In particular, we can realize the Gorenstein ring associated to the basis generating polynomial as a quotient ring of $H(M)$ by kernel of the Poincaré pairing. The Gorenstein stein ring associated to a polynomial is defined based on the action of differential forms on a fixed homogeneous polynomial.

Definition 5.2.1. Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a homomgeneous polynoimal and let $S := \mathbb{R}[\partial_1, \ldots, \partial_n]$ be the polynomial ring of differentials where $\partial_i := \partial_{x_i}$. Let $A_f^\bullet := S/\operatorname{Ann}_S(f)$. Then, we call $A_f^\bullet$ the **Gorenstein ring** associated to the polynomial f. Alternatively, we can view $\mathbb{R}[x_1, \ldots, x_n]$ as a

$\mathbb{R}[X_1, \ldots, X_n]$ modules where the action is defined by the relation

$$p(X_1, \ldots, X_n) \cdot q(x_1, \ldots, x_n) := p(\partial_1, \ldots, \partial_n)q(x_1, \ldots, x_n).$$

Then the Gorenstein ring is exactly $\mathbb{R}[X_1, \ldots, X_n]/\operatorname{Ann}(f)$ where the annihilator is with respect to the $\mathbb{R}[X_1, \ldots, X_n]$ action.

The Gorenstein ring associated to a polynomial has a natural grading with respect to the degree of the differential form. Before we prove that this actually gives $A_f^\bullet$ a graded ring structure, we first prove Lemma 5.2.1.

Lemma 5.2.1. *Let $\xi \in \mathbb{R}[\partial_1, \ldots, \partial_n]$ and $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial. We can decompose $\xi = \xi_0 + \xi_1 + \ldots$ into its homogeneous parts. If $\xi(f) = 0$, then $\xi_d(f) = 0$ for all $d \geq 0$.*

Proof. Let $d = \deg(f)$. If $\xi_i(f) \neq 0$, then $i \leq d$ and $\xi_i(f)$ is a homomgeneous polynomial of degree $d - i$. Thus, $\xi(f) = \xi_0(f) + \xi_1(f) + \ldots$ will be the homomgeneous decomposition of the polynomial $\xi(f)$. Since this is equal to 0, all components of the decomposition are equal to zero. This suffices for the proof. $\square$

Proposition 5.2.1. *The ring $A_f^\bullet$ is a graded $\mathbb{R}$-algebra where A_f^k consists of the forms of degree k.*

Proof. Let us define A_f^k as in the statement of the lemma. Let $d = \deg(f)$ be the degree of the homomgeneous polynomial. Whenver $k > d$, the ring A_f^k is clearly trivial. From Lemma 5.2.1, we have the direct sum decomposition

$$A_f^\bullet = \bigoplus_{k=0}^{d} A_f^k.$$

It is also clear that multiplication induces maps $A_f^r \times A_f^s \to A_f^{r+s}$ for all $r, s \geq 0$. $\square$

Proposition 5.2.2 states that the natural pairing in $A_f^\bullet$ equips the ring with a Poincaré duality algebra structure. The proposition follows from Theorem 2.1 in [38].

Proposition 5.2.2. *Let f be a homogeneous polynomial of degree d. Then, the ring $A_f^\bullet$ is a Poincaré-Duality algebra. That is, the ring satisfies the following two properties:*

(a) $A_f^d \simeq A_f^0 \simeq \mathbb{R}$;

(b) The pairing induced by multiplication $A_f^{d-k} \times A_f^k \to A_f^d \simeq \mathbb{R}$ is non-degenerate for all $0 \leq k \leq d$.

In Lemma 5.2.2, we give an characterization of non-degeneracy. From this characterization, it follows that the Hilbert polynomial of any Poincaré duality algebra is palindromic. In other words, $\dim A_f^k = \dim A_f^{d-k}$ for all k.

Lemma 5.2.2. *Let $B : V \times W \to k$ be a bilinear pairing between two finite-dimensional k-vector spaces V and W. Then, any two of the following three conditions imply the third.*

(i) The map $B_V : V \to W^*$ defined by $v \mapsto B(v, \cdot)$ has trivial kernel.

(ii) The map $B_W : W \to V^*$ defined by $w \mapsto B(\cdot, w)$ has trivial kernel.

(iii) $\dim V = \dim W$.

Proof. Condition (i) implies $\dim V \leq \dim W$ and Condition (ii) implies $\dim W \leq \dim V$. Thus (i) and (ii) both imply (iii). Now, suppose that (i) and (iii) are true. Then B_V is an isomorphism between V and W^* (see 3.69 in [7]). Let $v_1, \dots, v_n$ be a basis for V. Then $B_V(v_1), \dots, B_V(v_n)$ is a basis of W^*. Let $w_1, \dots, w_n$ be the dual basis in W with respect to this basis of W^*. Suppose that $\sum \lambda_i w_i \in \ker B_W$. Then for all $v \in V$, we have

$$\sum_{i=1}^{n} \lambda_i B_V(v)(w) = B\left(v, \sum_{i=1}^{n} \lambda_i w_i\right) = 0.$$

By letting $v = v_1, \dots, v_n$, we get $\lambda_i = 0$ for all i. □

We say a pairing between finite-dimensional vector spaces is non-degenerate whenever all three conditions in Lemma 5.2.2 hold. As a corollary, we have the following result.

Corollary 5.2.1. *Let f be a homogeneous polynomial of degree $d \geq 2$ and let k, $0 \leq k \leq d$, by a non-negative integer. Then* $\dim_{\mathbb{R}} A_f^k = \dim_{\mathbb{R}} A_f^{d-k}$.

Given a homogeneous polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ of degree d, we can define an isomorphism $\deg_f : A_f^d \to \mathbb{R}$ given by evaluation at f. This means that for any differential d-form $\xi \in A_f^d$, we have that $\deg_f(\xi) := \xi(f)$. Since ξ and f homogeneous of the same degree, the value of $\xi(f)$ will be a real number. Following the terminology in [1], we give the following definition.

Definition 5.2.2. Let f be a homogeneous polynomial of degree d and let $k \leq d/2$ be a non-negative integer. For an element $l \in A_f^1$, we define the following notions:

(a) The **Lefschetz operator** on A_f^k associated to l is the map $L_l^k : A_f^k \to A_f^{d-k}$ defined by $\xi \mapsto l^{d-2k} \cdot \xi$.

(b) The **Hodge-Riemann form** on A_f^k associated to l is the bilinear form $Q_l^k : A_f^k \times A_f^k \to \mathbb{R}$ defined by $Q_l^k(\xi_1, \xi_2) = (-1)^k \deg(\xi_1 \xi_2 l^{d-2k})$.

(c) The **primitive subspace** of A_f^k associated to l is the subspace

$$P_l^k := \{\xi \in A_f^k : l^{d-2k+1} \cdot \xi = 0\} \subseteq A_f^k.$$

Definition 5.2.3. Let f be a homogeneous polynomial of degree d, let $k \leq d/2$ be a non-negative integer, and let $l \in A_f^1$ be a linear differential form. We define the following notions:

(a) (Hard Lefschetz Property) We say A_f satisfies HL_k with respect to l if the Lefschetz operator L_l^k is an isomorphism.

(b) (Hodge-Riemann Relations) We say A_f satisfies HRR_k with respect to l if the Hodge-Riemann form Q_l^k is positive definite on the primitive subspace P_l^k.

We say that a homogeneous polynomial f satisifes HL or HRR if the associated ring $A_f^\bullet$ satisfies HL or HRR. For any $a \in \mathbb{R}^n$, we can define the linear differential form $l_a := a_1\partial_1 + \ldots + a_n\partial_n$. We say that f satisfies HL or HRR with respect to a if and only if it satisfies HL or HRR with respect to l_a. Since we have shown that $A_f^\bullet$ automatically satisfies Poincaré duality, we say that f satisfies the Kähler package for a linear form l if it satisfies HL_k and HRR_k with respect to l for all $k \leq \frac{d}{2}$.

Proposition 5.2.3 (Lemma 3.4 in [39]). *Let $f \in \mathbb{R}[x_1, \ldots, x_n]$ be a homogeneous polynomial of degree $d \geq 2$ and $a \in \mathbb{R}^n$. Assume that $f(a) > 0$. Then,*

(a) A_f has HL_1 with respect to l_a if and only $Q_{l_a}^1$ is non-degenerate.

(b) Suppose that A_f satisfies HL_1. Then A_f has HRR_1 with respect to l_a if and only if $-Q_{l_a}^1$ has signature $(+, -, \ldots, -)$.

Proof. We include a proof for completeness. We first prove the statement in (a). Suppose that A_f has HL_1 with respect to l_a. We have the following commutative diagram:

$$\begin{tikzcd} A_f^1 \times A_f^1 \arrow[rr, "\mathrm{id}\times L_{l_a}^1"] \arrow[dr, "-Q_{l_a}^1"'] & & A_f^1 \times A_f^{d-1} \arrow[dl] \\ & \mathbb{R} & \end{tikzcd}$$

where the missing mapping is multiplication. If A_f has HL_1 with respect to l_a, then the top map between $A_f^1 \times A_f^1 \to A_f^1 \times A_f^{d-1}$ is a bijection. Thus the non-degeneracy of $Q_{l_a}^1$ follows from the non-degeneracy of the multiplication pairing as stated in Proposition 5.2.2. Now, suppose that $Q_{l_a}^1$ is non-degenerate. Then, the map $B : A_f^1 \to (A_f^1)^*$ defined by $\xi \mapsto -Q_{l_a}^1(\xi, \cdot)$ is given by

$m(L^1_{l_a}\xi, \cdot)$ where $m : A^1_f \to A^{d-1}_f \to \mathbb{R}$ is the multiplication map. This is the composition of $A^1_f \to A^{d-1}_f \to (A^1_f)_*$ where the first map is $L^1_{l_a}$ and the second map is injective from the non-degeneracy of the multiplication map. This proves that $L^1_{l_a}$ is injective. From Corollary 5.2.1, the map $L^1_{l_a}$ is an isomorphism. This suffices for the proof of (a).

To prove (b), consider the commutative diagram

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\simeq"] & A^0_f \arrow[r, "\times l_a"] \arrow[rrr, bend right, "L^0_{l_a}"'] & A^1_f \arrow[r, "L^1_{l_a}"] & A^{d-1}_f \arrow[r, "\times l_a"] & A^d_f \arrow[r, "\simeq"] & \mathbb{R}
\end{tikzcd}$$

Note that $L^0_{l_a}$ is an isomorphism because

$$\deg L^0_{l_a}(1) = l^d_a(f) = d!f(a) \neq 0.$$

Thus, we have $A^1_f = \mathbb{R}l_a \oplus P^1_l$ where the direct sum is orthogonal over the Hodge-Riemann form by definition of the primitive subspace. Now, note that

$$-Q^1_{l_a}(l_a, l_a) = l^d_a(f) = -d!f(a) > 0.$$

Thus, the signature of $-Q^1_{l_a}$ is $(+, -, \ldots, -)$ if and only if $Q^1_{l_a}$ is positive definite over the primitive subspace if and only if A_f satisfies HRR_1 with respect to l_a. $\square$

Recall from our definition of Lorentzian polynomials, we know that non-zero Lorentzian polynomials are log-concave at any point in $a \in \mathbb{R}^n_{>0}$. When $f(a) > 0$, the polynomial f is log-concave at a if and only if its Hessian has exactly one positive eigenvalue. Hence, for any element in the non-negative orthant, we know that the Hessians of Lorentzian polynomials have at most one positive eigenvalue. This fact alongside Sylvester's Law of Intertia gives a proof of Lemma 5.2.3.

Lemma 5.2.3. *If $f \in \mathbb{R}[x_1, \ldots, x_n]$ is Lorentzian, then for any $a \in \mathbb{R}^n_{\geq 0}$ with $f(a) > 0$, A^1_f has HL_1 with respect to l_a if and only if f has the HRR_1 with respect to l_a.*

Proof. See Lemma 3.5 in [39]. $\square$

5.3 Local Hodge-Riemann Relations

This section illustrates an example of a more general inductive technique to prove Hodge-Riemann relations. We define a local version of the Hodge-Riemann relations. If this local version is satisfied, then this will imply that the Hard Lefschetz property will be satisfied. In some situations, this is enough to imply that the original Hodge-Riemann relations are satisfied. We give an example of this inductive process in Lemma 5.3.1. We take our definition of the local Hodge-Riemann relations from [39].

Definition 5.3.1. A homogeneous polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ of degree $d \geq 2k+1$ satisfies the **local** HRR_k with respect to a form $l \in A^1_f$ if for all $i \in [n]$, either $\partial_i f = 0$ or $\partial_i f$ satisfies HRR_k with respect to l.

Lemma 5.3.1 (Lemma 3.7 in [39]). *Let $f \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ be a homogeneous polynomial of degree d and k a positive integer with $d \geq 2k+1$, and $a = (a_1, \dots, a_n) \in \mathbb{R}^n$. Suppose that f has the local HRR_k with respect to l_a.*

(i) If $a \in \mathbb{R}^n_{>0}$, then A_f has the HL_k with respect to l_a.

(ii) If $a_1 = 0, a_2, \dots, a_n > 0$ and $\{\xi \in A^k_f : \partial_i \xi = 0$ for $i = 2, \dots, n\} = \{0\}$, then A_f has the HL_k with respect to l_a.

With Lemma 5.3.1, we can inductively prove that all Lorentzian polynomials satisfy HRR_1 with respect to l_a for all $a \in \mathbb{R}^n_{>0}$. We can prove this directly for all small Lorentzian polynomials. For an arbitrary Lorentzian polynomial, we know that all of its partial derivatives are Lorentzian. Thus, by induction, all of the partial derivatives satisfy HRR_1. This means that f satisfies the local HRR_1. From lemma 5.3.1, we know that f satisfies HL_1. But we have shown that HL_1 and HRR_1 are equivalent for Lorentzian polynomials. This gives a proof of Theorem 5.3 using the inductive procedure that we alluded to.

Theorem 5.3 (Theorem 3.8 in [39]). *If $f \in \mathbb{R}[x_1, \dots, x_n]$ is Lorentzian, then f has HRR_1 with respect to l_a for any $a \in \mathbb{R}^n_{>0}$.*

The next result in Corollary 5.3.1 gives a condition for general homogeneous polynomials to satisfy HRR_1 in terms of the signature of its Hessian. This result will hold for any $a \in \mathbb{R}^n$ satisfying the conditions in the statement.

Corollary 5.3.1. *Let f be a homogeneous polynomial of degree $d \geq 2$. If $\partial_1 f, \dots, \partial_n f$ are linearly independent in $\mathbb{R}[x_1, \dots, x_n]$ and $f(a) > 0$ for some $a \in \mathbb{R}^n$, then A_f satisfies HRR_1 with respect to l_a if and only if $\mathrm{Hess}_f |_{x=a}$ has signature $(+, -, \dots, -)$.*

Proof. Since $\partial_1 f_1, \dots, \partial_n f_n$ are linearly independent, the partials ∂_i form a basis for A^1_f. Thus, the signature of $Q^1_{l_a}$ is actually the signature of matrix of $Q^1_{l_a}$ with respect to the set $\{\partial_1, \dots, \partial_n\}$. We have

$$-Q^1_{l_a}(\partial_i, \partial_j) = \partial_i\partial_j l_a^{d-2} f = l_a^{d-2}\partial_i\partial_j f = (d-2)!\partial_i\partial_j f(a).$$

This proves that the signature of $-Q^1_{l_a}$ is the same as the signature of $\mathrm{Hess}_f\,|_{x=a}$. We are done from Lemma 5.2.3(b). $\square$

5.4 The Gorenstein Ring associated to the Basis Generating Polynomial of a Matroid

In this section, we specialize the Gorenstein ring associated to a polynomial to the Gorenstein ring associated to the basis generating polynomial of a matroid. This graded $\mathbb{R}$-algebra is another cohomology ring associated to a matroid. Thus, it is interesting study the Hard Lefschetz property and Hodge-Riemann relations on this algebra.

Definition 5.4.1. Let $M = (E, \mathcal{I})$ be a matroid. We let $\mathrm{A}(M)$ be the Gorenstein ring associated to the basis generating polynomial of M. Explicitly, we define the ring $\mathrm{S}_M := \mathbb{R}[X_e : e \in E]$ and the ideal $\mathrm{Ann}_M = \mathrm{Ann}_{\mathrm{S}_M}(f_M)$ where f_M is the basis generating polynomial of M. Then, the ring $\mathrm{A}(M)$ is equal to $\mathrm{A}(M) = \mathrm{S}_M \,/\, \mathrm{Ann}_M$.

For a matroid M, we will prove that the ring $\mathrm{A}(M)$ will depend only on its simplification. This is a reasonable claim because whenever we have two elements $e, f \in E$ which are parallel, the differential $\partial_e - \partial_f$ will be in the annihilator of f_M. Indeed, the elements e and f are interchangeable in any independent set. Moreover, if $e \in E(M)$ is a loop then ∂_e is clearly in the annihilator. We will prove this isomorphism in Theorem 5.4. Before proving this result, we first describe some common elements in the ideal Ann_M.

Proposition 5.4.1. *Let $M = (E, \mathcal{I})$ be a matroid. For any subsets $S, T \subseteq E$, we write $S \sim T$ if and only if $\overline{S} = \overline{T}$. Let $\Lambda^{(1)}_M, \Lambda^{(2)}_M$, $\Lambda^{(3)}_M$ be three subsets of S_M given by*

$$\begin{aligned}
\Lambda^{(1)}_M &:= \left\{X_e^2 : e \in E(M)\right\}, \\
\Lambda^{(2)}_M &:= \left\{X^S : S \notin \mathcal{I}(M)\right\}, \\
\Lambda^{(3)}_M &:= \left\{X^S - X^T : S, T \in \mathcal{I}(M) \textit{ such that } S \sim T\right\}.
\end{aligned}$$

Let $\Lambda_M = \Lambda^{(1)}_M \cup \Lambda^{(2)}_M \cup \Lambda^{(3)}_M$. Then $\Lambda_M \subseteq \mathrm{Ann}_M$.

Proof. See Proposition 3.1 in [37] $\square$

In general, it is not true that the elements in Λ_M generate the annihilator Ann_M. In fact, if we let $I(\Lambda_M)$ be the ideal generated by Λ_M, then $\mathrm{S}_M/I(\Lambda_M)$ is exactly the graded Möbius algebra $\mathrm{H}(M)$. The graded Möbius algebra is in general not even a Poincaré duality algebra, hence it cannot be $\mathrm{A}(M)$. We can find an explicit counterexample in [37]. Consider the matrix given by

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

The basis of the linear matroid generated by A are $\{123, 125, 134, 135, 145\}$. From Example 3.5 in [37], we have that

$$\mathrm{Ann}_M = I(\Lambda_M \cup (X_1X_3 + X_4X_5 - X_1X_5 - X_3X_4))$$

Let $M = (E, \mathcal{I})$ be a matroid and $\widetilde{M}$ be its simplification. We can define the maps $\phi : \mathrm{S}_M \to \mathrm{S}_{\widetilde{M}}$ and $\psi : \mathrm{S}_{\widetilde{M}} \to \mathrm{S}_M$ by

$$\phi(\partial_{x_e}) := \partial_{\overline{x_e}}, \quad \text{and} \quad \psi(\partial_{\overline{x}}) := \frac{1}{|\overline{x}|}\sum_{e\in\overline{x}} \partial_{x_e}.$$

and then extending to the whole polynomial ring using the universal property of polynomial rings.

Theorem 5.4. *The maps $\phi : S_M \to S_{\widetilde{M}}$ and $\psi : S_{\widetilde{M}} \to S_M$ induce isomorphisms between $A(M)$ and $A(\widetilde{M})$.*

Proof. We first prove that ϕ and ψ induce homomorphisms between the rings $\mathrm{A}(M)$ and $\mathrm{A}(\widetilde{M})$. To show that ϕ induces a homomorphism, consider the diagram in Equation 5.2.

$$\begin{tikzcd} S_M \arrow[r, "\phi"] \arrow[dr, "\pi_M"'] & S_{\widetilde{M}} \arrow[r, "\pi_{\widetilde{M}}"] & A(\widetilde{M}) \\ & A(M) \arrow[ur, dotted, "\exists!\Phi"'] & \end{tikzcd} \tag{5.2}$$

Let $\xi \in S_M$ be an element satisfying $\xi(f_M) = 0$. We will prove that $\phi(\xi)(f_{\widetilde{M}}) = 0$. In other words, we want to prove that $\mathrm{Ann}_M \subseteq \ker \pi_{\widetilde{M}} \circ \phi$. From Proposition 5.2.1, it suffices to consider the case where ξ is homogeneous. Let $e_1, \ldots, e_s$ be representatives of all parallel classes. Then, we have that $M = \overline{e_1} \sqcup \ldots \sqcup \overline{e_s} \cup E_0$ where E_0 denotes the loops in M. In terms of the basis generating polynomials, we have

$$f_{\widetilde{M}}(x_{\overline{e_1}}, \ldots, x_{\overline{e_s}}) = \sum_{\substack{1 \le i_1 < \ldots < i_r \le s \\ \{\overline{e_{i_1}}, \ldots, \overline{e_{i_r}}\} \in \mathcal{B}(\widetilde{M})}} x_{\overline{e_{i_1}}} \ldots x_{\overline{e_{i_r}}}$$

$$f_M(x_1, \ldots, x_n) = f_{\widetilde{M}}(y_1, \ldots, y_s)$$

where for $1 \le i \le s$, we define $y_i := \sum_{e \in \overline{e_i}} x_e$. Since ξ is homogeneous of degree k, we can write it in the form $\xi = \sum_{\substack{\alpha \subseteq [n] \\ |\alpha| = k}} c_\alpha \partial^\alpha$. Then, we have

$$\xi(f_M) = \sum_{\beta \in \mathcal{B}} \xi(x^\beta) = \sum_{\beta \in \mathcal{B}(M)} \sum_{\substack{\alpha \subseteq [n] \\ |\alpha| = k}} c_\alpha \partial^\alpha x^\beta = \sum_{\gamma \in \mathcal{I}_{r-k}(M)} \left(\sum_{\substack{\alpha \in \mathcal{I}_k \\ \alpha \cup \gamma \in \mathcal{I}_r(M)}} c_\alpha \right) x^\gamma. \tag{5.3}$$

Since $\xi(f_M) = 0$, we know that all of the coefficients on the right hand side of Equation 5.3 are equal to 0. Thus, we have that

$$\sum_{\substack{\alpha \in I_k \\ \alpha \cup \gamma \in \mathcal{I}_r(M)}} c_\alpha = 0, \text{ for all } \gamma \in \mathcal{I}_{r-k}(M).$$

On the other hand, we have

$$\phi(\xi) = \sum_{\substack{\alpha \subseteq [n] \\ |\alpha| = k}} c_\alpha \prod_{e \in \alpha} \partial_{\overline{e}} = \sum_{\beta \in \mathcal{I}_k(\widetilde{M})} \left(\sum_{\alpha \in \mathrm{fiber}(\beta)} c_\alpha \right) \partial^\beta.$$

We can let this differential act on $f_{\widetilde{M}}$ to get the expression

$$\phi(\xi)(f_{\widetilde{M}}) = \sum_{\gamma \in \mathcal{I}_{r-k}(\widetilde{M})} \left(\sum_{\substack{\beta \in \mathcal{I}_k(\widetilde{M}) \\ \beta \cup \gamma \in \mathcal{B}(\widetilde{M})}} \sum_{\alpha \in \mathrm{fiber}(\beta)} c_\alpha \right) x^\gamma = \sum_{\gamma \in \mathcal{I}_{r-k}(\widetilde{M})} \left(\sum_{\substack{\alpha \in \mathcal{I}_k(M) \\ \alpha \cup \gamma_0 \in \mathcal{B}(M)}} c_\alpha \right) x^\gamma = 0. \tag{5.4}$$

In Equation 5.4, the independent set $\gamma_0 \in \mathrm{fiber}(\gamma)$ is an arbitrary element in the fiber of γ. This proves that $\mathrm{Ann}_M \subseteq \ker \pi_{\widetilde{M}} \circ \phi$. Thus, there is a unique ring homomorphism $\Phi : A(M) \to A(\widetilde{M})$ which makes Equation 5.2 commute.

To prove that ψ induces a ring homomorphism, consider the diagram in Equation 5.5.

$$\begin{tikzcd} S_{\widetilde{M}} \arrow[r, "\psi"] \arrow[dr, "\pi_{\widetilde{M}}"'] & S_M \arrow[r, "\pi_M"'] & A(M) \\ & A(\widetilde{M}) \arrow[ur, dotted, "\exists!\Psi"'] & \end{tikzcd} \tag{5.5}$$

Consider a differential $\xi \in S_{\widetilde{M}}$ satisfying $\xi(f_{\widetilde{M}}) = 0$. We can write this as $\xi = \sum_{\alpha \in \mathcal{I}_k(\widetilde{M})} c_\alpha \partial^\alpha$ for some real constants c_α. Then, its image under ψ is equal to

$$\psi(\xi) = \sum_{\alpha \in \mathcal{I}_k(\widetilde{M})} \frac{c_\alpha}{\prod_{e \in \alpha} |e|} \sum_{\beta \in \text{fiber}(\alpha)} \partial^\beta.$$

Fix a $\alpha \in \mathcal{I}_k(\widetilde{M})$ and a $\beta \in \text{fiber}(\alpha)$. Since $\partial y_i / \partial x_e = \mathbb{1}_{e \in \overline{e_i}}$, we have

$$\partial^\beta f_M(x_1, \dots, x_n) = \partial^\beta f_{\widetilde{M}}(y_1, \dots, y_s) = \partial^\alpha f_{\widetilde{M}}(x_1, \dots, x_s)|_{x_1 = y_1, \dots, x_s = y_s} = 0.$$

Thus, we have that $\psi(\xi)(f_M) = 0$ and $\text{Ann}_{\widetilde{M}} \subseteq \ker \pi_M \circ \psi$. This proves that there is a unique ring homomrphism $\Psi : A(\widetilde{M}) \to A(M)$ which causes the diagram in Equation 5.5 to commute. Since the maps Ψ and Φ are inverses of each other, they are both isomorphisms. This suffices for the proof. $\square$

From Theorem 5.4, we get Corollary 5.4.1 and Corollary 5.4.2 immediately.

Corollary 5.4.1. *Let $M = (E, \mathcal{I})$ be a matroid. For any $a \in \mathbb{R}^E$, we can define the linear form $l_a := \sum_{e \in E} a_e \cdot \partial_{x_e} \in A^1(M)$. Let $\widetilde{l_a} := \Phi(l_a) = \sum_{e \in E} a_e \cdot \partial_{x_{\overline{e}}} \in A^1(\widetilde{M})$. Then, the following diagram commutes:*

$$\begin{tikzcd} \dots \arrow[r, "\times l_a"] & A^{i-1}(M) \arrow[r, "\times l_a"] \arrow[d, "\Phi"'] & A^i(M) \arrow[r, "\times l_a"] \arrow[d, "\Phi"'] & A^{i+1}(M) \arrow[r, "\times l_a"] \arrow[d, "\Phi"'] & \dots \\ \dots \arrow[r, "\times \widetilde{l_a}"] & A^{i-1}(\widetilde{M}) \arrow[r, "\times \widetilde{l_a}"] & A^i(\widetilde{M}) \arrow[r, "\times \widetilde{l_a}"] & A^{i+1}(\widetilde{M}) \arrow[r, "\times \widetilde{l_a}"] & \dots \end{tikzcd}$$

Corollary 5.4.2. *Let $M = (E, \mathcal{I})$ be a matroid. Then $A(M)$ satisfies HRR_k with respect to l if and only if $A(\widetilde{M})$ satisfies HRR_k with respect to $\Phi(l)$.*

From Theorem 5.4, Corollary 5.4.1, and Corollary 5.4.2, when we study Hodge-theoretic properties of A(M) it is enough to assume that $\widetilde{M}$ is simple. We may not always want to do this, but the translation from a matroid to its simplification is useful nonetheless. Not only do we know that the two rings are isomorphic, but we know how multiplication by 1-forms translate from one ring to the other. This means that we can relate the Hodge theoretic properties of the two rings to each other. We also understand some properties of $\mathrm{A}(M)$ better when we know that M is simple. For example, the vector space structure of $\mathrm{A}^1(M)$ is well-understood for simple M. Satoshi Murai, Takahiro Nagaoka, and Akiko Yazawa prove in [39] that if M is a simple matroid on $[n]$, then $\dim A^1(M) = n$. In other words, the spanning set $\partial_1, \ldots, \partial_n$ is a basis of $\mathrm{A}^1(M)$. We write this result in Lemma 5.4.1.

Lemma 5.4.1 (Theorem 2.5 in [39])**.** *If $M = ([n], \mathcal{I})$ is simple, then $\partial_1, \ldots, \partial_n$ is a basis of $A^1(M)$.*

From Corollary 5.3.1, this implies that when M is simple, the Hessian of the basis generating polynomial f_M has signature $(+, -, \ldots, -)$ at every point in the positive orthant. Thus, the basis generating polynomial for a simple matroid is strictly log-concave on the positive orthant, which was one of the main results in [39]. In the same paper, they formulate Conjecture 5.4.1. In the rest of the thesis, we would like to work towards a proof of Conjecture 5.4.2.

Conjecture 5.4.1. *Let $M = (E, \mathcal{I})$ be a matroid of rank d. The ring $\mathrm{A}(M)$ satisfies HRR_k for some $a \in \mathbb{R}^E_{>0}$ for all $k \leq \frac{d}{2}$.*

Conjecture 5.4.2. *Let $M = (E, \mathcal{I})$ be a matroid of rank d. The ring $\mathrm{A}(M)$ satisfies the Kähler package for all $a \in \mathbb{R}^E_{>0}$.*

Conjecture 5.4.2 is technically stronger than Conjecture 5.4.1, but it is hard to imagine a world where Conjecture 5.4.1 is proven without Conjecture 5.4.1 having been proven along the way. We believe that a crucial step in resolving Conjecture 5.4.2 is to analyze when $\mathrm{A}(M)$ satisfies HRR_1 on the facets of the positive orthant. We will resolve this question completely in Section 5.5. Before moving on to this question, we first give necessary and sufficient conditions on $\mathrm{A}(M)$ for HRR_1 with respect to $a \in \mathbb{R}^n$ in the case where $f(a) > 0$ and M is simple.

Corollary 5.4.3. *Let $M = (E, \mathcal{I})$ be a simple matroid of rank $r \geq 2$. If $a \in \mathbb{R}^E_{\geq 0}$ satisfies $f_M(a) > 0$, then $A(M)$ satisfies HRR_1 with respect to l_a if and only if $\mathrm{Hess}_{f_M}|_{x=a}$ is non-singular.*

Proof. This follows immediately from Lemma 5.2.3, Corollary 5.3.1, and Lemma 5.4.1. □

Lemma 5.4.2. *When D is invertible and A is a square matrix, we have*

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A - BD^{-1}C)\det(D).$$

Proof. See Section 5 in [54]. □

Theorem 5.5. *Let $M = (E, \mathcal{I})$ be a simple matroid of rank $r \geq 2$. If $a \in \mathbb{R}^E_{\geq 0}$ satisfies $f_M(a) > 0$ and $a_e = 0$ for some $e \in E$ which is not a co-loop, then $A(M)$ satisfies HRR_1 with respect to l_a if and only if*

$$\left(\nabla f_{M/e}^{\mathsf{T}} \cdot \mathrm{Hess}_{f_{M\backslash e}}^{-1} \cdot \nabla f_{M/e}\right)|_{x=a} \neq 0.$$

Proof. Without loss of generality, we can assume that $E(M) = [n]$ and $e = n$. In particular, this means that $a = (a_1, \ldots, a_{n-1}, 0) \in \mathbb{R}^n$ with $a_i \geq 0$ for all i. From Corollary 5.4.3, HRR_1 is satisfied if and only if the Hessian is non-singular. To compute the Hessian at $x = a$, note that because n is a co-loop, we can write the basis generating polynomial as $f_M = x_n f_{M/n} + f_{M\backslash n}$. Using this equation, we see that the Hessian of f_M at $(a_1, \ldots, a_{n-1}, 0)$ is equal to

$$\mathrm{Hess}_{f_M} = \begin{bmatrix} \mathrm{Hess}_{f_{M\backslash n}} & \nabla f_{M/n} \\ (\nabla f_{M/n})^{\mathsf{T}} & 0 \end{bmatrix}.$$

Since M is simple, we know that $M\backslash n$ is simple. Thus, the matrix $\mathrm{Hess}_{f_{M\backslash n}}$ is invertible as it has the same signature as the Hodge-Riemann form. From the Lemma 5.4.2, we have that

$$\det \mathrm{Hess}_{f_M} = \left(\nabla f_{M/n}^{\mathsf{T}} \cdot \mathrm{Hess}_{f_{M\backslash n}}^{-1} \cdot \nabla f_{M/n}\right).$$

This suffices for the proof. $\square$

5.5 Hodge-Riemann Relations on the Facets of the Positive Orthant

In this section, we will prove necessary and sufficient conditions for $\mathrm{A}(M)$ to satisfy HRR_1 on the facets of the positive orthant. From Lemma 5.3, we already know that $A(M)$ satisfies HRR_1 on $\mathbb{R}^n_{>0}$. For a matroid $M = (E, \mathcal{I})$, we can decompose the boundary set of $\mathbb{R}^E_{\geq 0}$ as

$$\mathrm{bd}\, \mathbb{R}^E_{\geq 0} = \bigcup_{e \in E} H_e$$

where $H_e := \{x \in \mathbb{R}^E_{\geq 0} : x_e = 0\}$. Before stating our main result in this section, we first prove a few technical lemmas that are necessary for us to apply the inductive procedure modeled in Lemma 5.3.1.

Lemma 5.5.1. *Let $M = (E, \mathcal{I})$ be a matroid satisfying $\mathrm{rank}(M) \geq 2$. If $e \in M$ is not a coloop of M, then e will not be a co-loop of M/i for any $i \in E\backslash e$.*

Proof. If e or i is a loop, then the statement is vacuously true. Now, suppose that e and i are both not loops. Suppose for the sake of contradiction that e is a coloop of M/i. This implies that any basis of M containing i must contain e. But, since e is not a coloop of M, the matroid $M\backslash e$ has the same rank of M. Moreover, since i is independent in $M\backslash e$, there is a basis of $M\backslash e$ which contains i. But this is automatically a basis of M that contains i but doesn't contain e. This is a contradiction, and suffices for the proof. □

Theorem 5.6 (Degree 1 Socles)**.** *Let* $M = (E, \mathcal{I})$ *be a matroid satisfying* $\operatorname{rank}(M) \geq 3$. *Let* $S \subseteq E(M)$ *be a subset with* $\operatorname{rank}(S) \leq \operatorname{rank}(M) - 2$. *Then*

$$\{\xi \in A_f^1 : \xi(\partial_i f) = 0 \text{ for } i \in E\backslash S\} = \{0\}.$$

Proof. Let $r = \operatorname{rank}(M) \geq 3$. We want to prove that if a linear form $\xi = \sum_{e\in E} c_e \cdot \partial_e$ satisfies $\xi(\partial_e f_M) = 0$ for all $e \in E\backslash S$, then we have $\xi(f_M) = 0$. For $i \in E\backslash S$, we have

$$0 = \xi(\partial_i f_M) = \sum_{e\in E} c_e \partial_e \partial_i f_M = \sum_{e\in E} c_e \sum_{\substack{\alpha\in\mathcal{I}_{r-2}(M)\\ \alpha\cup\{e,i\}\in\mathcal{I}_r(M)}} x^\alpha = \sum_{\alpha\in\mathcal{I}_{r-2}(M)} \left(\sum_{\substack{e\in E\\ \alpha\cup\{i,e\}\in\mathcal{I}_r(M)}} c_e \right) x^\alpha. \tag{5.6}$$

By setting all of the coefficients of the right hand side of Equation 5.6, we have that

$$\sum_{\substack{e\in E\\ \alpha\cup\{i,e\}\in\mathcal{I}_r(M)}} c_e = 0 \quad \text{for all } \alpha \in \mathcal{I}_{r-2}(M) \text{ and } i \in E\backslash S.$$

For any $\beta \in \mathcal{I}_{r-1}(M)$, we know that $\beta \not\subseteq S$ since $\operatorname{rank}(\beta) = r - 1 > \operatorname{rank}(S)$. There exists some $i \in \beta\backslash S$. Thus, we can write $\beta = \alpha \cup \{i\}$ where $\alpha \in \mathcal{I}_{r-2}(M)$ and $i \in E\backslash S$. This proves that for any $\beta \in \mathcal{I}_{r-1}(M)$, we have

$$\sum_{\substack{e\in E\\ \beta\cup\{e\}\in\mathcal{I}_r}} c_e = \sum_{\substack{e\in E\\ \alpha\cup\{i,e\}\in\mathcal{I}_r}} c_e = 0.$$

Finally, we have that

$$\xi(f_M) = \sum_{e\in E} c_e \partial_e f_M = \sum_{e\in E} c_e \sum_{\substack{\beta\in\mathcal{I}_{r-1}\\ \beta\cup\{e\}\in\mathcal{I}_r}} x^\beta = \sum_{\beta\in\mathcal{I}_{r-1}} \left(\sum_{\substack{e\in E\\ \beta\cup\{e\}\in\mathcal{I}_r}} c_e \right) x^\beta = 0.$$

This suffices for the proof. □

Theorem 5.7 (Higher Degree Socles)**.** *Let $M = (E, \mathcal{I})$ be a matroid. Let $S \subseteq E$ be a subset such that* $\operatorname{rank}(S) \leq \operatorname{rank}(M) - k - 1$. *Then,*

$$\{\xi \in A^k(M) : \xi(\partial_e f_M) = 0 \textit{ for all } e \in E \backslash S\} = \{0\}.$$

Proof. Any $\xi \in A^k(M)$ can be written as $\xi = \sum_{\alpha \in \mathcal{I}_k} c_\alpha \partial^\alpha$. For any $e \in E \backslash S$, we have

$$0 = \xi \partial_i f_M = \sum_{\alpha \in \mathcal{I}_k} c_\alpha \partial_e \partial^\alpha f_M = \sum_{\alpha \in \mathcal{I}_k} c_\alpha \sum_{\substack{\gamma \in \mathcal{I}_{r-k-1} \\ \gamma \cup \alpha \cup \{e\} \in \mathcal{I}_r}} x^\gamma = \sum_{\gamma \in \mathcal{I}_{r-k-1}} \left(\sum_{\substack{\alpha \in \mathcal{I}_k \\ \gamma \cup \alpha \cup \{e\} \in \mathcal{I}_r}} c_\alpha \right) x^\gamma.$$

This implies that for all $\gamma \in \mathcal{I}_{r-k-1}$ and $e \in E \backslash S$, we have

$$\sum_{\substack{\alpha \in \mathcal{I}_k \\ \gamma \cup \alpha \cup \{e\} \in \mathcal{I}_r}} c_\alpha = 0 \quad \text{for all } \gamma \in \mathcal{I}_{r-k-1} \text{ and } e \in E \backslash S.$$

Let $\beta \in \mathcal{I}_{r-k}$ be an arbitrary independent set of rank $r - k$. Note that $\operatorname{rank}(\beta) = r - k > \operatorname{rank}(S)$. Thus, we cannot have $\beta \subseteq S$. This implies that we can find $e \in \beta \backslash S$ such that $\beta = \alpha \cup \{e\}$ for $e \in E \backslash S$. Thus,

$$\sum_{\substack{\alpha \in \mathcal{I}_k \\ \beta \cup \alpha \in \mathcal{I}_r}} c_\alpha = 0 \quad \text{for all } \beta \in \mathcal{I}_{r-k}.$$

Then, we have

$$\xi(f_M) = \sum_{\alpha \in \mathcal{I}_k} c_\alpha \partial^\alpha f_M = \sum_{\alpha \in \mathcal{I}_k} c_\alpha \sum_{\substack{\beta \in \mathcal{I}_{r-k} \\ \beta \cup \alpha \in \mathcal{I}_r}} x^\beta = \sum_{\beta \in \mathcal{I}_{r-k}} \left(\sum_{\substack{\alpha \in \mathcal{I}_k \\ \beta \cup \alpha \in \mathcal{I}_r}} c_\alpha \right) x^\beta = 0,$$

which suffices for the proof. $\square$

We are now ready to state our main theorem about HRR_1 on the facets of $\mathbb{R}^E_{\geq 0}$. We study the properties of $\mathrm{A}(M)$ on the relative interiors of the facets H_e whenever e is not a coloop.

Theorem 5.8. *Let $M = (E, \mathcal{I})$ be a matroid which satisfies* $\operatorname{rank}(M) \geq 2$. *For any $e \in E(M)$, the basis generating polynomial f_M satisfies* HRR_1 *on* $\operatorname{relint}(H_e)$ *if and only if e is not a co-loop.*

Proof. Without loss of generality, let M be a matroid on the set $[n]$ and let $e = 1$. Then, we want to prove that whenever $a_2, \ldots, a_n > 0$, the ring $A(M)$ satisfies HRR_1 on $a = (0, a_2, \ldots, a_n)$ if and only if e is not a co-loop. We first prove that if 1 is not a co-loop, then $A(M)$ satisfies HRR_1. To prove this, we induct on the rank of M. For the base case $\operatorname{rank}(M) = 2$, Corollary 5.4.2 implies that it suffices to prove that $A(\widetilde{M})$ satisfies HRR_1 on $\Phi(l_a)$. The only simple matroid of rank 2 is the uniform matroid of rank 2. From Corollary 5.4.3, it suffices to check that the signature of the Hessian is $(+,-,\ldots,-)$. But the Hessian of a rank 2 simple matroid at every point is the same matrix. If the matroid is on n elements, the Hessian matrix is

$$A(K_n) = \begin{bmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & 0 & 1 & \ldots & 1 \\ 1 & 1 & 0 & \ldots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \ldots & 0 \end{bmatrix}.$$

This happens to be the adjancency matrix of a complete graph. From Proposition 1.5 in [57], this matrix has an eigenvalue of -1 with multiplicity $n-1$ and an eigenvalue of $n-1$ with multiplicity 1. Hence, its signature is $(+,-,\ldots,-)$ which proves the base case. Now suppose that the claim is true for all matroids of rank less than r. Let M be a matroid of rank r. We want to prove that the claim is true for M. By the same reasoning as in the base case, we can assume that our matroid is simple. In the simplification, the codimension of our 1-form will not increase. For all $i \in E\backslash\{1\}$, we know that e is not a co-loop of M/i from Lemma 5.5.1. Thus, the simplified l_a will either be all positive (in which case the claim is known to be true) or l_a will be all positive except possibly at e. Since e is known not to be a coloop of M/i, the simplification of l_a will be a 1-form which satisfies the hypothesis in the inductive hypothesis. Since M is simple, we know that $\operatorname{rank}(M/i) = \operatorname{rank}(M) - 1 < \operatorname{rank}(M)$. Hence, from the inductive hypothesis, we know that $A(M/i) = A^{\bullet}_{\partial_i f_M}$ satisfies HRR_1 for l_a.

Now, we have enough information to directly prove that $A(M)$ satisfies HRR_1 with respect to l_a. From Lemma 5.2.3, it suffices to prove that $A(M)$ satisfies HL_1 with respect to l_a. Since $\dim_{\mathbb{R}} A^1(M) = \dim_{\mathbb{R}} A^{r-1}(M)$ from properties of Poincaré Duality algebras, it suffices to prove that the Lefschetz operator $L^1_{l_a} : A^1(M) \to A^{r-1}(M)$ is injective. Let $\Xi \in A^1(M)$ be the kernel of $L^1_{l_a}$. Note that this is equivalent to requiring $\Xi l_a^{r-2} = 0$ in $A(M)$. We want to prove that $\Xi = 0$ in $A^1(M)$. Since $\Xi l_a^{r-2} = 0$ in $A(M)$, we have

$$0 = -Q^1_{l_a}(\Xi,\Xi) = \deg_M(\Xi^2 \cdot l_a^{r-2}) = \sum_{i=2}^{n} a_i \deg_M(\Xi^2 \cdot l_a^{r-3} \cdot \partial_i).$$

Note that

$$\deg_M(\Xi^2 \cdot l_a^{r-3} \cdot \partial_i) = (\Xi^2 \cdot l_a^{r-3})(\partial_i f_M) = \deg_{M/i}(\Xi^2 \cdot l_a^{r-3}) = -Q_{M/i}(\Xi, \Xi).$$

where $Q_{M/i}$ is the Hodge-Riemann form of degree 1 with respect to l associated with $A(M/i)$. Thus,

$$\sum_{i=2}^{n} a_i Q_{M/i}(\Xi, \Xi) = 0.$$

In $A(M/i)$, the linear form Ξ is in the primitive subspace. Hence, since we know that $A(M/i)$ satisfies HRR_1 with respect to l, we know that the Hodge-Riemann form $Q_{M/i}$ is negative-definite on $\mathbb{R}\Xi$. Since $a_i > 0$ for $i = 2, \ldots, n$, this implies that $Q_{M/i}(\Xi, \Xi) = 0$ for all such i. Hence $\Xi = 0$ in $A(M/i)$ for all $i \in [2, n]$. In terms of polynomials, this means that $\Xi(\partial_i f_M) = 0$ for all $i \in [2, n]$. From Theorem 5.6, this implies that $\Xi(f_M) = 0$. Thus, $A(M)$ satisfies HRR_1 with respect to l.

For the other direction, we will prove that if e is a co-loop of M, then f_M does not satisfy HRR_1 on $\operatorname{relint}(H_e)$. Without loss of generality, we can suppose $E(M) = [n]$ and $e = 1$. The linear differential can be written as l_a where $a = (0, a_2, \ldots, a_n)$ for $a_2, \ldots, a_n \geq 0$. It suffices to consider the case when M is simple because the coefficient of ∂_1 in the simplification of l_a will remain 0. This is because co-loops have no parallel elements. In the simple case, the bottom $(n-1) \times (n-1)$ sub-matrix of Hess_{f_M} will be entirely 0. This means that the Hodge-Riemann form will singular and f_M cannot satisfy HRR_1 on $\operatorname{relint}(H_e)$. This suffices for the proof. □

With a same proof, we also have Theorem 5.9. Using our result about high degree socles in Theorem 5.7, we can prove an inductive procedure for local HRR_k in Theorem 5.10 for points $a \in \mathbb{R}^E_{\geq 0}$ whose support is large enough.

Theorem 5.9. *Let $M = (E, \mathcal{I})$ be a matroid which satisfies $\operatorname{rank}(M) \geq 2$. For any $S \subseteq E$, if $\operatorname{rank}(S) \leq \operatorname{rank}(M) - 2$ and S contains no co-loops, then f_M satisfies HRR_1 on $\operatorname{relint}(H_S)$.*

Theorem 5.10. *Let $M = (E, \mathcal{I})$ be a matroid and let $S \subseteq E$ be a subset such that $\operatorname{rank}(S) \leq \operatorname{rank}(M) - k - 1$. Let $l := \sum_{i \in E \setminus S} a_i \cdot e_i$ where $a_i > 0$. If $\partial_e f_M$ is 0 or satisfies HRR_k and with respect to l for all $e \in E \setminus S$, then $A(M)$ satisfies HL_k with respect to l.*

Proof. Note that $\partial_e f = 0$ if and only if e is a loop. In this case, the variable x_e doesn't appear in the basis generating polynomial. Hence, without loss of generality, we can suppose that M is loopless and $\partial_e f \neq 0$ for all $e \in E$. Since $A(M)$ is a Poincaré duality algebra, it suffices to prove for $\Xi \in A^k(M)$ that if $\Xi l^{d-2k} = 0$ in $A^{r-k}(M)$, then $\Xi = 0$ in $A^k(M)$. We know that Ξ is in the primitive subspaces of $\partial_e f$ for all $e \in E$ and we can compute the formula

$$0 = Q^k(\Xi, \Xi) = \sum_{i \in E \setminus S} a_i Q_{\partial_i f}(\Xi, \Xi).$$

From the positive definiteness on the primitive subspaces, we have $\Xi = 0$ in $A^k_{\partial_i f}$ for all $i \in E \setminus S$. From Theorem 5.7, we get $\Xi = 0$ in $A^k(M)$. This suffices for the proof. $\square$

From Theorem 5.10, it is enough to show that if A(M) satisfies HRR_i for $1 \leq i \leq k-1$ and HL_i for $1 \leq i \leq k$, then it satisfies HRR_k. Under these conditions, satisfying HRR_k is equivalent to a statement about the net signature of the Hodge-Riemann form. We define the notion of net signature in Definition 5.5.1.

Definition 5.5.1. Let $B : V \times V \to k$ be a symmetric bilinear form on a finite dimensional vector space V. Suppose that the signature of B has n_+ positive eigenvalues and n_- negative eigenvalues. Then, we define the **net signature** to be $\sigma(B) = n_+ - n_-$.

When our bilinear form $B : V \times V \to k$ is non-degenerate, then the net signature $\sigma(B)$ determines the exact signature of the form. Indeed, in the non-degenerate case, we have $n_+ - n_- = \sigma(B)$ and $n_+ + n_- = \dim V$.

Lemma 5.5.2. *Let $A(M)$ satisfies HRR_i and HL_i with respect to l for $1 \leq i \leq k$, then, we have*

$$\sigma\left((-1)^k Q_l^k\right) = \sum_{i=0}^{k} (-1)^i (\dim A^i(M) - \dim A^{i-1}(M)).$$

Proof. We induct on k. For the base case, we have $k = 1$ and the claim follows from Proposition 5.2.3. Suppose that the claim holds for $k-1$. Consider the composition of maps given by the following commutative diagram:

$$\begin{tikzcd} A^{k-1}(M) \arrow[r, "\times l_a"] & A^k(M) \arrow[r, "\times l_a^{d-2k}"] \arrow[rr, bend right, "\psi_a"'] & A^{d-k}(M) \arrow[r, "\times l_a"] & A^{d-k+1}(M) \end{tikzcd} \tag{5.7}$$

This diagram exhibits an isomorphism between $A^{k-1}(M)$ and $A^{d-k+1}(M)$ from HL_{k-1}. This implies that we can decompose $A^k(M)$ into

$$A^k(M) = l \cdot A^{k-1}(M) \oplus \ker \psi_a$$

where the two summands in the direct sum are orthogonal with respect to the Hodge-Riemann form Q_l^k. Let $u_1, \ldots, u_m \in A^{k-1}(M)$ be a basis for $A^{k-1}(M)$. Then, $l \cdot u_1, \ldots, l \cdot u_m$ is a basis for $A^k(M)$ and

$$(-1)^k Q_l^k(l \cdot u_i, l \cdot u_j) = (-1)^{k-1} Q_l^{k-1}(u_i, u_j).$$

Thus, the signature of Q_l^k on $l \cdot A^{k-1}(M)$ should be the negative of the signature of Q_l^{k-1} on $A^{k-1}(M)$. Since $A(M)$ satisfies HRR_k, we know that the signature of Q_l^k on $\ker \psi$ is $\dim \ker \psi$. This gives us the formula

$$\begin{aligned}\sigma((-1)^k Q_{l_a}^k) &= \sigma\left((-1)^{k-1} Q_{l_a}^{k-1}\right) + (-1)^k(\dim A^k(M) - \dim A^{k-1}(M)) \\ &= \sum_{i=0}^{k} (-1)^i (\dim A^i(M) - \dim A^{i-1}(M))\end{aligned}$$

from the inductive hypothesis. This suffices for the proof. □

Lemma 5.5.3. *Let $k \geq 2$ and $A(M)$ satisfies HRR_i and HL_i with respect to $l = l_a$ for some $a \in \mathbb{R}^n$ for all $1 \leq i \leq k-1$. Then*

(a) $A(M)$ satisfies HL_k with respect to l if and only if Q_l^k is non-degenerate on $A^k(M)$.

(b) Suppose that HL_k is satisfied. Then $A(M)$ satisfies HRR_k with respect to l if and only if

$$\sigma((-1)^k Q_l^k) = \sum_{i=0}^{k} (-1)^i (\dim A^i(M) - \dim A^{i-1}(M)).$$

Proof. The argument for (a) is exactly the same as the argument for (a) in Proposition 5.2.3. For part (b), consider the composition in Equation 5.7. Recall from the proof of Lemma 5.5.2, we have that

$$\sigma((-1)^k Q_{l_a}^k) = \sigma\left((-1)^{k-1} Q_{l_a}^{k-1}\right) + \sigma\left((-1)^k Q_{l_a}^k|_{\ker \psi_a}\right).$$

Since $A(M)$ satisfies HRR_k if and only if $Q_{l_a}^k$ is positive definition on $\ker \psi_a$, we know that $A(M)$ satisfies HRR_k if and only if $\sigma(Q_{l_a}^k) = \dim \ker \psi_a = \dim A^k(M) - \dim A^{k-1}(M)$. Then Lemma 5.5.2 completes the proof. □

In the situation where we know that A(M) satisfies HRR_i for $1 \leq i \leq k-1$ and HL_i for $1 \leq i \leq k$, we already know that HRR_k is equivalent to a statement about the signature of the Hodge Riemann form. Note that as a function of the point $a \in \mathbb{R}^E$ in l_a, the Hodge-Riemann form is continuous. In particular, the eigenvalues of the Hodge-Riemann form are continuous functions. This property will allow us to prove Lemma 5.5.4. This result tells us that it is enough to prove that HRR_k and HL_k holds on at least one boundary point.

Lemma 5.5.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a subset such that for all $x \in \Omega$, $A(M)$ satisfies HRR_i for $i \leq k-1$ and HL_i for $i \leq k$. Suppose that there is a continuous path $\gamma : [0,1] \to \mathbb{R}^n$ with image $\varphi([0,1]) \subseteq \Omega$, suct that $A(M)$ satisfies HRR_k with respect to $l_{\gamma(0)}$. Then $A(M)$ satisfies HRR_k with respect to $l_{\gamma(1)}$.*

Proof. Let $\lambda_i(a)$ be the ith largest eigenvalue of $Q^k_{l_a}$ as a function in a for $1 \leq i \leq \dim A^k(M)$. Then for all $1 \leq i \leq \dim A^k(M)$, the ith eigenvalue $\lambda_i(a)$ is a continuous function in a. Along the path γ, we know that $Q^k_{l_a}$ is non-degenerate. Hence, none of the functions $\lambda_i(a)$ cross zero. This implies that on the path, the signature $\sigma\left((-1)^k Q^k_{l_a}\right)$ remains constant. From Lemma 5.5.3(b), this completes the proof. □

5.6 Future Work

In our attempt to prove Conjecture 5.4.2, we were able to prove Theorem 5.9. This describes the exact conditions needed for $\mathrm{A}(M)$ to satisfy HRR_1 on the faces of the positive orthant. Theorem 5.10 tells us that for general k, we have a mechanism from going from local HRR_k to HL_k on positive orthant and possibly its boundary depending on some dimension conditions. This means that to prove the general conjecture, it suffices to prove that we can go from HRR_i for $1 \leq i \leq k-1$ and HL_i for $1 \leq i \leq k$ to HRR_k. In Lemma 5.5.4, we further reasoned that given HRR_i for $1 \leq i \leq k-1$ and HL_i for $1 \leq i \leq k$ as the inductive assumption, it suffices to prove that HRR_k is satisfied on a suitable boundary point. The reason why we might want to pick a boundary point even though it may seem more accessible is because we may hope for a semi-small decomposition for $\mathrm{A}(M)$ as in Theorem 5.1 and Theorem 5.2. Indeed, if we were to have such a decomposition, we could reduce HRR_k on the boundary for $\mathrm{A}(M)$ to HRR_k for $\mathrm{A}(M\backslash i)$. The conjecture would then follow from a more elaborate induction argument. Before we can contemplate such an argument, there are a few questions we must answer.

Question 1. Does there exist a graded ring homomorphism $\theta : \mathrm{A}(M\backslash i) \to \mathrm{A}(M)$ for every $i \in E$?

In order to have a decomposition from which we can extract inductive information about the Hodge structures, we must write $\mathrm{A}(M)$ as the direct sum of $\mathrm{A}(M\backslash i)$ submodules. Even before this, we would require that $\mathrm{A}(M)$ be a $\mathrm{A}(M\backslash i)$ itself. Note that there is a surjection $\Theta_M : \mathrm{H}(M) \to \mathrm{A}(M)$

between the graded Möbius algebra and the ring $\mathrm{A}(M)$ defined by

$$\Theta_M(y_F) = \begin{cases} y_F & \text{if } F \in \mathcal{L}(M) \\ y_{F\cup i} & \text{otherwise.} \end{cases}$$

A reasonable guess for an injection $\mathrm{A}(M\backslash i) \to \mathrm{A}(M)$ would be the map which makes the diagram in Equation 5.8 commute.

$$\begin{tikzcd} \mathrm{H}(M\backslash i) \arrow[r] \arrow[d] & \mathrm{H}(M) \arrow[d] \\ \mathrm{A}(M\backslash i) \arrow[r] & \mathrm{A}(M) \end{tikzcd} \tag{5.8}$$

This map is forced to send the equivalence class of $\mathrm{A}(M\backslash i)$ represented by a polynomial f to the equivalence class of $\mathrm{A}(M)$ represented by the same polynomial. Unfortunately, it is clear that this map is well-defined. In order to be well-defined, Question 2 must have an affirmative answer.

Question 2. Let M be a matroid on the ground set $[n]$. Let $\xi \in \mathbb{R}[\partial_1, \ldots, \partial_{n-1}]$ such that $\xi(f_{M\backslash n}) = 0$. Is it true that $\xi(f_M) = 0$?

In the case where n is a coloop, the answer to Question 2 is Yes. When n is not a coloop, we can write $f_M = x_n f_{M/n} + f_{M\backslash n}$. In this case, Question 2 becomes equivalent to Question 3.

Question 3. Let M be a matroid and suppose that $e \in E(M)$ is not a coloop. Is it the case that $\mathrm{Ann}(f_{M\backslash e}) \subseteq \mathrm{Ann}(f_{M/e})$?

At this time, we have not found a counter-example to Question 3. Note that the operation $M\backslash e \to M/e$ is called a **matroid quotient**. Assuming that the answers to all of these questions are Yes, then a decomposition of $\mathrm{A}(M)$ may possibly descend from a decomposition of $\mathrm{H}(M)$. Even if the mathematics doesn't work in this way, the question of having a semi-small decomposition of $\mathrm{H}(M)$ is an interesting question. We know that $\mathrm{H}(M)$ is a $\mathrm{H}(M\backslash i)$ module. From the Krull-Schmidt theorem, there is some unique $\mathrm{H}(M\backslash i)$-module decomposition of the form

$$\mathrm{H}(M) = S_1 \oplus \ldots \oplus S_m$$

where the S_i's are indecomposable $\mathrm{H}(M\backslash i)$-modules. If we look at the 0-degree parts of these modules, we know that $H^0(M) \cong \mathbb{R}$. Thus, all but one of the S_i's must have trivial 0-degree parts. This means that $1 \in S_i$ for some $i \in [m]$. Since they are $\mathrm{H}(M\backslash i)$-modules, this means that $\mathrm{H}(M\backslash i) \subseteq S_i$ for some $i \in [m]$. It turns how that $\mathrm{H}(M)$ does not necessary have a semi-small decomposition. Indeed, consider the matroid that comes from the affine configuration of $\{A, B, C, D, E\}$ in Figure 5.1. One can calculate that as $\mathrm{H}(M\backslash A)$ modules, the ring $\mathrm{H}(M)$ is indecomposable. This does not bode well for the possibility for a general semi-small decomposition. However, this may be a result of picking a bad point to delete. Perhaps, there always exists a point

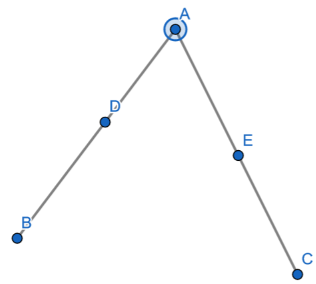

Figure 5.1: $\mathrm{H}(M)$ is indecomposable

that we can delete to give a nice decomposition. Another possibility is that many of the results that we want to be true are true in low dimensions (specifically $\leq \frac{d}{2}$). For example, we conjecture that Question 4, which asks about the equivalence of $\mathrm{A}(M)$ and $\mathrm{H}(M)$ in small dimensions, has an affirmative answer.

Question 4. Is it true that $\mathrm{H}^k(M) \to \mathrm{A}^k(M)$ is an isomorphism when $k \leq \frac{d}{2}$?

For any graded ring R of top dimension d, we let $D^k R$ be the graded ring of dimension k where we annihilate all elements of degree greater than k. Question 5 asks if there is a possibility of a semi-small decomposition of $\mathrm{H}(M)$ for a small enough k?

Question 5. Is $D^k\,\mathrm{H}(M\backslash i)$ a summand in the Krull-Schmidt decomposition of $D^k\,\mathrm{H}(M)$ for $k \leq \frac{d}{2}$?

Chapter 6

Appendix

6.1 Brunn-Minkowski and the Base Case of the Alexandrov-Fenchel Inequality

Proposition 6.1.1. *Let $K, L \subseteq \mathbb{R}^2$ be two convex bodies in the plane. Then* $\mathsf{V}_2(K,L)^2 \geq \mathsf{V}(K,K)\mathsf{V}(L,L)$.

Proof. From Theorem 3.6, we have $\sqrt{\mathrm{Vol}_2(K+L)} \geq \sqrt{\mathrm{Vol}_2(A)} + \sqrt{\mathrm{Vol}_2(B)}$. Squaring both sides, we get the equivalent inequality

$$\mathrm{Vol}_2(K+L) - \mathrm{Vol}_2(K) - \mathrm{Vol}_2(L) \geq 2\sqrt{\mathrm{Vol}_2(K)\,\mathrm{Vol}_2(L)} = 2\sqrt{\mathsf{V}_2(K,K)\mathsf{V}_2(L,L)}.$$

From Theorem 2.8, the left hand side is equal to

$$\mathrm{Vol}_2(K+L) - \mathrm{Vol}_2(K) - \mathrm{Vol}_2(L) = 2\mathsf{V}_2(K,L).$$

This suffices for the proof. $\square$

6.2 Computations in the Kahn-Saks Inequality

Let $(P, \leq)$ be a finite poset and fix two elements $x, y \in P$. For the sake of simplicity, we can assume that $x \leq y$. In the proof of Theorem 4.3, we defined special cross-sections of the order polytope for every $\lambda \in [0,1]$ given by

$$K_\lambda := \{t \in \mathcal{O}_P : t_y - t_x = \lambda\}.$$

We give a proof of Lemma 6.2.1. This result is important in the proof of Theorem 4.3 as it gives a way to interpret the volume of K_λ in terms of mixed volumes. The proof of Lemma 6.2.1 was stated in [34] but was not proven. We include the proof for the sake of completeness.

Lemma 6.2.1. *For $\lambda \in [0,1]$ we have that $K_\lambda = (1-\lambda)K_0 + \lambda K_1$.*

Proof. For all $\lambda \in \mathbb{R}$, we can define the hyperplanes $H_\lambda = \{t \in \mathbb{R}^n : t_y - t_x = \lambda\}$. Then $K_\lambda = H_\lambda \cap \mathcal{O}_P$. It is a simple computation to prove that $H_\lambda = (1-\lambda)H_0 + \lambda H_1$. For any linear extension $\sigma \in e(P)$, we define the polytopes:

$$\Delta_\sigma := \{0 \leq t_{\sigma^{-1}(1)} \leq \ldots \leq t_{\sigma^{-1}(n)} \leq 1\} \subseteq \mathcal{O}_P$$

$$\Delta_\sigma(\lambda) := \{t \in \Delta_\sigma : t_y - t_x = \lambda\} = K_\lambda \cap \Delta_\sigma = H_\lambda \cap \Delta_\sigma.$$

Then, we can decompose K_λ into

$$K_\lambda = H_\lambda \cap \mathcal{O}_P = H_\lambda \cap \bigcup_{\sigma \in e(P)} \Delta_e = \bigcup_{\sigma \in e(P)} \Delta_\sigma(\lambda).$$

where the polytopes in the unions are disjoint up to a set of measure zero. Now, note that we have

$$(1-\lambda)K_0 + \lambda K_1 \subseteq (1-\lambda)\mathcal{O}_P + \lambda \mathcal{O}_P = \mathcal{O}_P$$

$$(1-\lambda)K_0 + \lambda K_1 \subseteq (1-\lambda)H_0 + \lambda H_1 = H_\lambda.$$

Thus, we have that $(1-\lambda)K_0 + \lambda K_1 \subseteq \mathcal{O}_P \cap H_\lambda = K_\lambda$. It suffices to prove the opposite inclusion. Let $t \in K_\lambda$ be an arbitrary point. From our decomposition of K_λ into polytopes $\Delta_\sigma(\lambda)$, we know that there is some linear extension σ satisfying $\sigma(x) < \sigma(y)$ and $t \in \Delta_\sigma(\lambda)$. The linear extension σ induces a total order $<_\sigma$ on P defined by $z_1 <_\sigma z_2$ if and only if $\sigma(z_1) < \sigma(z_2)$. In particular, when $\omega_1 \geq_\sigma \omega_2$, we have that $t_{\omega_1} \geq t_{\omega_2}$. Now, we can define two points $P_0, P_1 \in \mathbb{R}^n$ so that for all $\omega \in P$, we have

$$(P_0)_\omega = \begin{cases} \frac{t_\omega}{1-\lambda} & \text{if } \omega \leq_\sigma x \\ \frac{t_x}{1-\lambda} & \text{if } x <_\sigma \omega <_\sigma y \\ \frac{t_\omega-\lambda}{1-\lambda} & \text{if } \omega \geq_\sigma y. \end{cases}$$

$$(P_1)_\omega = \begin{cases} 0 & \text{if } \omega \leq_\sigma x \\ \frac{t_\omega-t_x}{\lambda} & \text{if } x <_\sigma \omega <_\sigma y \\ 1 & \text{if } \omega \geq y. \end{cases}$$

Then $P_0 \in K_0$, $P_1 \in K_1$, and $t = (1-\lambda)P_0 + \lambda P_1$. Since t was arbitrary, this completes the proof that $K_\lambda = (1-\lambda)K_0 + \lambda K_1$. $\square$

Lemma 6.2.2. *For $i \in \{1, \dots, n-1\}$, we have that $N_i = (n-1)!\mathsf{V}_{n-1}(K_0[n-i], K_1[i-1])$.*

Proof. From Theorem 2.6 and Lemma 6.2.1, we have that

$$\begin{aligned} \mathrm{Vol}_{n-1}(K_\lambda) &= \mathrm{Vol}_{n-1}((1-\lambda)K_0 + \lambda K_1) \\ &= \sum_{j=0}^{n-1} \binom{n-1}{j} \mathsf{V}_{n-1}(K_0[n-j-1], K_1[j])(1-\lambda)^{n-j-1}\lambda^j. \end{aligned}$$

Alternatively, we can compute the volume $\mathrm{Vol}_{n-1}(K_\lambda)$ as

$$\begin{aligned} \mathrm{Vol}_{n-1}(K_\lambda) &= \mathrm{Vol}_{n-1}\left(\bigcup_{\substack{\sigma\in e(P) \\ \sigma(x)<\sigma(y)}} \Delta_\sigma(\lambda) \right) \\ &= \sum_{\substack{\sigma\in e(P) \\ \sigma(x)<\sigma(y)}} \mathrm{Vol}_{n-1}(\Delta_\sigma(\lambda)) \\ &= \sum_{k=1}^{n-1} \sum_{\substack{\sigma\in e(P) \\ \sigma(y)-\sigma(x)=k}} \mathrm{Vol}_{n-1}(\Delta_\sigma(\lambda)). \end{aligned}$$

To compute $\mathrm{Vol}_{n-1}(\Delta_\sigma(\lambda))$, let $P = \{z_1, \dots, z_n\}$ with $x = z_i$, $y = z_j$ with $\sigma(z_k) = k$ for $k \in [n]$. In this notation, we have the explicit description of $\Delta_\sigma(\lambda)$ as

$$\Delta_\sigma(\lambda) = \{t \in \mathbb{R}^n : 0 \leq t_1 \leq \ldots \leq t_n \leq 1 \text{ and } t_j - t_i = \lambda\}.$$

This set of inequalities is equivalently decribed by the inequalities

$$
\begin{aligned}
&0 \le t_{i+1} - t_i \le \ldots \le t_{j-1} - t_i \le \lambda \\
&0 \le t_1 \le \ldots \le t_{i-1} \le t_i \le t_{j+1} - \lambda \le \ldots \le t_n - \lambda \le 1 - \lambda.
\end{aligned}
$$

Thus, the volume of $\Delta_n(\lambda)$ ends up being the product of two simplices. Specifically, we have the equation

$$\mathrm{Vol}_{n-1}(\Delta_\sigma(\lambda)) = \frac{\lambda^{j-i-1}}{(j-i-1)!} \cdot \frac{(1-\lambda)^{n-(j-i)}}{(n-(j-i))!}.$$

We can finish the computation with

$$
\begin{aligned}
\mathrm{Vol}_{n-1}(K_\lambda) &= \sum_{k=1}^{n-1} \sum_{\substack{\sigma \in e(P) \\ \sigma(y)-\sigma(x)=k}} \frac{\lambda^{k-1}}{(k-1)!} \cdot \frac{(1-\lambda)^{n-k}}{(n-k)!} \\
&= \sum_{k=1}^{n-1} \binom{n-1}{k-1} \frac{N_k}{(n-1)!} \cdot \lambda^{k-1}(1-\lambda)^{n-k}.
\end{aligned}
$$

Thus, we have that

$$\frac{N_k}{(n-1)!} = \mathsf{V}_{n-1}(K_0[n-k], K_1[k-1]) \implies N_k = (n-1)!\mathsf{V}_{n-1}(K_0[n-k], K_1[k-1]).$$

This suffices for the proof of the Lemma. □

In

Lemma 6.2.3. *The affine hulls of K_0, K_1, and $K_0 + K_1$ are given by*

$$
\begin{aligned}
\mathrm{aff}(K_0) &= \mathbb{R}\left[\sum_{\omega \in P_{x \le \cdot \le y}} e_\omega\right] \oplus \bigoplus_{\omega \in P \setminus P_{x \le \cdot \le y}} \mathbb{R}[e_\omega] \\
\mathrm{aff}(K_1) &= \sum_{\omega \in P_{\ge y}} e_\omega + \bigoplus_{\omega \in P \setminus P_{\le x} \cup P_{\ge y}} \mathbb{R}[e_\omega] \\
\mathrm{aff}(K_0 + K_1) &= \sum_{\omega \in P_{\ge y}} e_\omega + \mathbb{R}\left[\sum_{\omega \in P_{x \le \cdot \le y}} e_\omega\right] \oplus \bigoplus_{\omega \in P \setminus \{x,y\}} \mathbb{R}[e_\omega]
\end{aligned}
$$

where $\mathbb{R}[v]$ denotes the linear span of the vector v. As an immediate corollary, we have

$$\begin{aligned} \dim K_0 &= n - |P_{x<\cdot<y}| - 1 \\ \dim K_1 &= n - |P_{<x}| - |P_{>y}| - 2 \\ \dim(K_0 + K_1) &= n - 1. \end{aligned}$$

Proof. From Proposition 6.2.2, there is a linear extension $f : P \to [n]$ such that $f(y) - f(x) = |P_{x<\cdot<y}| + 1$. Then, there are elements $\alpha, \beta_j, \gamma_k$ for $1 \leq i \leq a$, $1 \leq j \leq b$, and $1 \leq k \leq c$ such that

$$f(\alpha_1) < \ldots < f(\alpha_a) < f(x) < f(\beta_1) < \ldots < f(\beta_b) < f(y) < f(\gamma_1) < \ldots < f(\gamma_c).$$

This allows us to compute

$$K_0 \cap \Delta_f = \{t \in [0,1]^n : t_{\alpha_1} \leq \ldots \leq t_{\alpha_a} \leq t_x = \ldots = t_y \leq t_{\gamma_1} \leq \ldots \leq t_{\gamma_c}\}.$$

In particular, by taking affine spans, we have

$$\operatorname{aff} K_0 \supseteq \operatorname{aff}(K_0 \cap \Delta_f) = \mathbb{R}\left[\sum_{\omega \in P_{x\leq\cdot\leq y}} e_\omega\right] \oplus \bigoplus_{\omega \in P \setminus P_{x\leq\cdot\leq y}} \mathbb{R}[e_\omega].$$

To prove the other inclusion, let $v \in K_0$ be an arbitrary vector. Since we are working in a subset of the order polytope, it must be the case that $v_x \leq v_\omega \leq v_y$ for all $\omega \in P_{x\leq\cdot\leq y}$. But since $v_x = v_y$, this means that $v_\omega = v_x = v_y$ are all $\omega \in P_{x\leq\cdot\leq y}$. Thus

$$v \in \mathbb{R}\left[\sum_{\omega \in P_{x\leq\cdot\leq y}} e_\omega\right] \oplus \bigoplus_{\omega \in P \setminus P_{x\leq\cdot\leq y}} \mathbb{R}[e_\omega].$$

Since v is arbitrary, we have proved the given formula for $\operatorname{aff} K_0$.

To prove the formula for K_1, we first construct using Proposition 6.2.3 a linear extension $g : P \to [n]$ satisfying $g(x) = 1 + |P_{<x}$ and $g(y) = n - |P_{>y}|$. Note that

$$K_1 = \{t \in \mathcal{O}_P : t_y - t_x = 1\} = \{t \in \mathcal{O}_P : t_y = 1, t_x = 0\}.$$

In particular, for any $v \in K_1$, for all $\omega \leq x$ and $\eta \geq y$ we must have $v_\omega = 0$ and $v_\eta = 1$. Thus,

$$K_1 \subseteq \sum_{\omega \in P_{\geq y}} e_\omega + \bigoplus_{\omega \in P \setminus (P_{\leq x} \cup P_{\geq y})} \mathbb{R}[e_\omega].$$

On the other hand, If $P_{\leq x} = \{\alpha_1, \ldots, \alpha_a, x\}$ ordered by according to g and $P_{\geq y} = \{y, \beta_1, \ldots, \beta_b\}$ ordered according to g, then

$$K_1 \cap \Delta_g = \{t \in [0,1]^n : 0 = t_{\alpha_1} \leq \ldots \leq t_{\alpha_a} \leq t_x \leq \ldots \leq t_y = t_{\beta_1} = \ldots = t_{\beta_b} = 1\}.$$

Taking the affine span, we have

$$\operatorname{aff} K_1 \supseteq \operatorname{aff}(K_1 \cap \Delta_g) = \sum_{\omega \in P_{\geq y}} e_\omega + \bigoplus_{\omega \in P \setminus (P_{\leq x} \cup P_{\geq y}} \mathbb{R}[e_\omega].$$

This completes the proof for the given formula for $\operatorname{aff} K_1$. The third formula follows from the fact that $\operatorname{aff}(K + L) = \operatorname{aff} K + \operatorname{aff} L$ for any non-empty sets K and L. ☐

6.2.1 Modifications and linear extensions of posets

Recall that a linear extension of a poset is a way to extend the partial order to a total order. Formally, we say $f : P \to [n]$ is a linear extension if (and only if) for all relations $a < b$, we have $f(a) < f(b)$. In this section, we will define certain linear extensions and linear extension modifications to help with the analysis of the polytopes associated to the Kahn-Saks sequence. In particular, it will also be useful to have examples of "extremal linear extensions" which place x and y close to each or very far from each other. It will also be useful to modify an existing linear extension to make a covering relation adjacent in a linear extension with changing too many terms in the original extension.

Proposition 6.2.1. *The poset P as at least one linear extension.*

Proof. We induct on the size of P. If $|P| = 1$, then the partial order is already a total order. Now suppose the claim is true for posets of size $n - 1$ and let P be a poset of size n. Let $m \in P$ be a maximal element. Let P' be the poset which you get by deleting m. From the inductive hypothesis, there is a linear extension $g : P' \to [n]$. Then, the map $f : P \to [n]$ defined by

$$f(\omega) = \begin{cases} g(\omega) & \text{if } \omega \neq m, \\ n & \text{if } \omega = m \end{cases}$$

is a linear extension of P. $\square$

Proposition 6.2.2. *There exists a linear extension $f : P \to [n]$ satisfying $f(y) - f(x) = |P_{x<\cdot<y}| + 1$.*

Proof. Let $\mathscr{L}$ denote the set of linear extensions of P. From Proposition 6.2.1, there exists a linear extension $f_{\mathsf{min}} \in \mathscr{L}$ which minimizes $f(y) - f(x)$ out of all $f \in \mathscr{L}$. Since $x \leq y$, we know

$$f_{\mathsf{min}}(y) - f_{\mathsf{min}}(x) \geq 1 + |P_{x<\cdot<y}|.$$

If $f_{\mathsf{min}}(y) - f_{\mathsf{min}}(x) = 1 + |P_{x<\cdot<y}|$, then we proposition is proved. Otherwise, we can assume $f_{\mathsf{min}}(y) - f_{\mathsf{min}}(x) \geq 2 + |P_{x<\cdot<y}|$. Define the set

$$\begin{aligned} \mathsf{M}_{f_{\mathsf{min}}} &:= \{z \in P : f_{\mathsf{min}}(x) < f_{\mathsf{min}}(z) < f_{\mathsf{min}}(y)\} \\ &= f_{\mathsf{min}}^{-1}\{(f_{\mathsf{min}}(x), f_{\mathsf{min}}(y))\} \end{aligned}$$

which consist of the elements of the poset which appear between x and y in the linear extension f_{min}. From the assumption $f_{\mathsf{min}}(y) - f_{\mathsf{min}}(x) \geq 2 + |P_{x<\cdot<y}|$, we know there is at least one element in $\mathsf{M}_{f_{\mathsf{min}}}$ which is not comparable to both x and y. This is because any element that is comparable to both x and y must be in the set $P_{x<\cdot<y}$ which doesn't have enough elements to satisfy the inequality. Without loss of generality, suppose that there exists an element which is not comparable to x. Let $z \in \mathsf{M}_{f_{\mathsf{min}}}$ be such an element which minimizes $f_{\mathsf{min}}(z)$. In other words, in the linear extension, it is the leftmost element of this type. All elements between x and z in the linear extension must be greater than x from our choice of z. In particular, those elements cannot be less than z since that would imply the relation $x < z$. This means that we can move z to the immediate left of x and still have a working linear extension. But the resulting linear extension will have a smaller value of $f(y) - f(x)$. This contradicts the minimality of f_{min} and completes the proof of the proposition. $\square$

Proposition 6.2.3. *There exists a linear extension $f : P \to [n]$ satisfying $f(x) = |P_{<x}| + 1$ and $f(y) = n - |P_{>y}|$.*

Proof. Consider any linear extensions of the three subposets: $P_{\leq x}$, $P \backslash (P_{\leq x} \cup P_{\geq y})$, and $P_{\geq y}$. By appending the extensions in this order, we get an extension of P of the desired form. $\square$

Proposition 6.2.4. *Let $f : P \to [n]$ be a linear extension and let $a, b \in P$ be elements such that $a \lessdot b$. Then there is a linear extension $\tilde{f} : P \to [n]$ satisfying $\tilde{f}(b) - \tilde{f}(a) = 1$ and $\tilde{f}(z) = f(z)$ whenever $f(z) < f(a)$ or $f(z) > f(b)$.*

Proof. Consider the sub-poset consisting of a, b, and the elements in between a and b in the linear extension f. By Proposition 6.2.2, there is a linear extension of this sub-poset so that a lies to the immediate left of b. By replacing the portion of the extension f containing the elements of this sub-poset by this linear extension, we get the desired linear extension. □

Lemma 6.2.4. *Let $(P, \leq)$ be a poset and let $x, y \in P$ be two elements such that $x \not\geq y$. Suppose that x and y are incomparable with respect to the order $\leq$. Let $\leq_*$ be a binary relation defined by $z_1 \leq_* z_2$ if and only if $z_1 \leq z_2$ or $z_1 \leq x$ and $z_2 \geq y$. Then $\leq_*$ is a partial order on P.*

Proof. In the following, we separate the proofs of reflexivity, anti-symmetry, and transitivity for the binary relation $\leq_*$. Let $a, b, c \in P$ be arbitrary points in the poset.

(i) Since $a \leq a$, we must have $a \leq_* a$. This proves reflexivity of $\leq_*$.

(ii) If $a \leq_* b$ and $b \leq_* a$, then there are four possibilities. The first possibility is $a \leq b$ and $b \leq a$. In this case, we get $a = b$. The second possibility is $a \leq x, y \leq b$ and $b \leq x, y \leq a$. In this case, $y \leq a \leq x$ so $y \leq x$. But this contradicts our assumption that $x \not\geq y$. The third possibility is $a \leq b$ and $b \leq x$, $y \leq a$. In this case, we have $y \leq a \leq b \leq x$ which cannot happen for the same reason the second possibility cannot happen. Similarly, the fourth possibility cannot occur. Thus, in any case, $\leq_*$ satisfies the anti-symmetry property.

(iii) For the final property, suppose that $a \leq_* b$ and $b \leq_* c$. Again, there are four possibilities. The first possibility is $a \leq b$ and $b \leq c$. Then $a \leq c$ which implies $a \leq_* c$. The second possibility is $a \leq b$, $b \leq x$, and $y \leq c$. In this case, $a \leq x$ and $y \leq c$. Thus $a \leq_* c$ holds. The third possibility is $a \leq x$, $y \leq b$, and $b \leq c$. In this case $a \leq x$ and $y \leq c$. Thus $a \leq_* c$ holds. The fourth possibility is $a \leq x, y \leq b$ and $b \leq x, y \leq c$. But this implies that $y \leq x$ which cannot happen. In all cases, $\leq_*$ satisfies the transitivity property.

□

Lemma 6.2.5. *Let $(P, \leq)$ be a poset and $x, y \in P$ be elements such that $x \not\geq y$. Let $\leq_*$ be the partial order as defined in Lemma 6.2.4. Let $\{N_k\}$ be the Kahn-Saks sequence with respect to $\leq$ and let $\{\widetilde{N}_k\}$ be the Kahn-Saks sequence with respect to $\leq_*$. Then $N_k = \widetilde{N}_k$ for all k.*

Proof. We prove the stronger statement that a bijective map $f : P \to [n]$ is a linear extension of $\leq$ satisfying $f(y) - f(x) = k$ if and only if it is a linear extension of $\leq_*$ satisfying $f(y) - f(x) = k$. Clearly any linear extension of $\leq_*$ is a linear extension of $\leq$. Now suppose that f is a linear extension of $\leq$ satisfying $f(y) - f(x) = k$. Suppose for the sake of contradiction that it is not a linear extension of $\leq_*$. This means that there are two elements $a, b \in P$ such that $a <_* b$ and $f(b) < f(a)$. Since $a <_* b$, we either have $a < b$ or $a \leq x$ and $y \leq b$. In the first case, we would have $f(a) < f(b)$ since f is a linear extension of $\leq$. This would be give a contradiction. For the second case, we would have $f(a) \leq f(x) < f(y) \leq f(b)$, which is also a contradiction. This suffices for the proof. $\square$

Lemma 6.2.6 is a linear algebraic result which allows us skimp on calculating the exact affine span of our faces.

Lemma 6.2.6. *Let W_1 and W_2 be finite dimensional affine spaces. Let $V_1 \subseteq W_1$ and $V_2 \subseteq W_2$ be affine subspaces such that $\dim V_1 = \dim W_1 - a$ and $\dim V_2 = \dim W_2 - b$ for some $a, b \geq 0$. Then $\dim(V_1 + V_2) \geq \dim(W_1 + W_2) - a - b$.*

Proof. After suitable translations, it suffices to prove the result when all our spaces are vector spaces. Then, we have

$$\begin{aligned} \dim(V_1 + V_2) &= \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2) \\ &\geq \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) - a - b \\ &= \dim(W_1 + W_2) - a - b. \end{aligned}$$

This completes the proof of the lemma. $\square$

Our main application of Lemma 6.2.6 is when W_1 and W_2 are the affine spans of K_0 and K_1 while V_1 and V_2 are the affine spans of faces of K_0 and K_1.

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