Project Euler.net

Problem 51: Prime Digit Replacements

Published on Friday, 29th August 2003, 06:00 pm / Solved by 35753 / Difficulty rating: 15%

By replacing the 1st digit of the 2-digit number *3, it turns out that six of the nine possible values: 13, 23, 43, 53, 73, and 83, are all prime.

By replacing the 3rd and 4th digits of 56**3 with the same digit, this 5-digit number is the first example having seven primes among the ten generated numbers, yielding the family: 56003, 56113, 56333, 56443, 56663, 56773, and 56993. Consequently 56003, being the first member of this family, is the smallest prime with this property.

Find the smallest prime which, by replacing part of the number (not necessarily adjacent digits) with the same digit, is part of an eight prime value family.

Problem 52: Permuted Multiples

Published on Friday, 12th September 2003, 06:00 pm / Solved by 67808 / Difficulty rating: 5%

It can be seen that the number, 125874, and its double, 251748, contain exactly the same digits, but in a different order.

Find the smallest positive integer, x, such that $2x$, $3x$, $4x$, $5x$, and $6x$, contain the same digits.

Problem 53: Combinatoric Selections

Published on Friday, 26th September 2003, 06:00 pm / Solved by 61322 / Difficulty rating: 5%

There are exactly ten ways of selecting three from five, 12345:

123, 124, 125, 134, 135, 145, 234, 235, 245, and 345

In combinatorics, we use the notation, $\displaystyle \binom 5 3 = 10$.

In general, $\displaystyle \binom n r = \dfrac{n!}{r!(n-r)!}$, where $r \le n$, $n! = n \times (n-1) \times ... \times 3 \times 2 \times 1$, and $0! = 1$.

It is not until $n = 23$, that a value exceeds one-million: $\displaystyle \binom {23} {10} = 1144066$.

How many, not necessarily distinct, values of $\displaystyle \binom n r$ for $1 \le n \le 100$, are greater than one-million?

Problem 54: Poker Hands

Published on Friday, 10th October 2003, 06:00 pm / Solved by 37881 / Difficulty rating: 10%

In the card game poker, a hand consists of five cards and are ranked, from lowest to highest, in the following way:

- **High Card**: Highest value card.
- **One Pair**: Two cards of the same value.
- **Two Pairs**: Two different pairs.
- **Three of a Kind**: Three cards of the same value.
- **Straight**: All cards are consecutive values.
- **Flush**: All cards of the same suit.
- **Full House**: Three of a kind and a pair.
- **Four of a Kind**: Four cards of the same value.
- **Straight Flush**: All cards are consecutive values of same suit.
- **Royal Flush**: Ten, Jack, Queen, King, Ace, in same suit.

The cards are valued in the order:
2, 3, 4, 5, 6, 7, 8, 9, 10, Jack, Queen, King, Ace.

If two players have the same ranked hands then the rank made up of the highest value wins; for example, a pair of eights beats a pair of fives (see example 1 below). But if two ranks tie, for example, both players have a pair of queens, then highest cards in each hand are compared (see example 4 below); if the highest cards tie then the next highest cards are compared, and so on.

Consider the following five hands dealt to two players:

Hand	Player 1	Player 2	Winner
1	5H 5C 6S 7S KD Pair of Fives	2C 3S 8S 8D TD Pair of Eights	Player 2
2	5D 8C 9S JS AC Highest card Ace	2C 5C 7D 8S QH Highest card Queen	Player 1
3	2D 9C AS AH AC Three Aces	3D 6D 7D TD QD Flush with Diamonds	Player 2
4	4D 6S 9H QH QC Pair of Queens Highest card Nine	3D 6D 7H QD QS Pair of Queens Highest card Seven	Player 1
5	2H 2D 4C 4D 4S Full House With Three Fours	3C 3D 3S 9S 9D Full House with Three Threes	Player 1

The file, **poker.txt**, contains one-thousand random hands dealt to two players. Each line of the file contains ten cards (separated by a single space): the first five are Player 1's cards and the last five are Player 2's cards. You can assume that all hands are valid (no invalid characters or repeated cards), each player's hand is in no specific order, and in each hand there is a clear winner.

How many hands does Player 1 win?

Problem 55: Lychrel Numbers

Published on Friday, 24th October 2003, 06:00 pm / Solved by 56064 / Difficulty rating: 5%

If we take 47, reverse and add, $47 + 74 = 121$, which is palindromic.

Not all numbers produce palindromes so quickly. For example,

\begin{align} 349 + 943 &= 1292\\ 1292 + 2921 &= 4213\\ 4213 + 3124 &= 7337 \end{align}

That is, 349 took three iterations to arrive at a palindrome.

Although no one has proved it yet, it is thought that some numbers, like 196, never produce a palindrome. A number that never forms a palindrome through the reverse and add process is called a Lychrel number. Due to the theoretical nature of these numbers, and for the purpose of this problem, we shall assume that a number is Lychrel until proven otherwise. In addition you are given that for every number below ten-thousand, it will either (i) become a palindrome in less than fifty iterations, or, (ii) no one, with all the computing power that exists, has managed so far to map it to a palindrome. In fact, 10677 is the first number to be shown to require over fifty iterations before producing a palindrome: $4668731596684224866951378664$ (53 iterations, 28-digits).

Surprisingly, there are palindromic numbers that are themselves Lychrel numbers; the first example is 4994.

How many Lychrel numbers are there below ten-thousand?

NOTE: Wording was modified slightly on 24 April 2007 to emphasise the theoretical nature of Lychrel numbers.

Problem 56: Powerful Digit Sum

Published on Friday, 7th November 2003, 06:00 pm / Solved by 60527 / Difficulty rating: 5%

A googol (10^{100}) is a massive number: one followed by one-hundred zeros; 100^{100} is almost unimaginably large: one followed by two-hundred zeros. Despite their size, the sum of the digits in each number is only 1.

Considering natural numbers of the form, a^b, where $a, b \lt 100$, what is the maximum digital sum?

Problem 57: Square Root Convergents

Published on Friday, 21st November 2003, 06:00 pm / Solved by 43593 / Difficulty rating: 5%

It is possible to show that the square root of two can be expressed as an infinite continued fraction.

$\sqrt 2 =1+ \frac 1 {2+ \frac 1 {2 +\frac 1 {2+ \dots}}}$

By expanding this for the first four iterations, we get:

$1 + \frac 1 2 = \frac 32 = 1.5$
$1 + \frac 1 {2 + \frac 1 2} = \frac 7 5 = 1.4$
$1 + \frac 1 {2 + \frac 1 {2+\frac 1 2}} = \frac {17}{12} = 1.41666 \dots$
$1 + \frac 1 {2 + \frac 1 {2+\frac 1 {2+\frac 1 2}}} = \frac {41}{29} = 1.41379 \dots$

The next three expansions are $\frac {99}{70}$, $\frac {239}{169}$, and $\frac {577}{408}$, but the eighth expansion, $\frac {1393}{985}$, is the first example where the number of digits in the numerator exceeds the number of digits in the denominator.

In the first one-thousand expansions, how many fractions contain a numerator with more digits than the denominator?

Problem 58: Spiral Primes

Published on Friday, 5th December 2003, 06:00 pm / Solved by 42078 / Difficulty rating: 5%

Starting with 1 and spiralling anticlockwise in the following way, a square spiral with side length 7 is formed.

37	36	35	34	33	32	**31**
38	**17**	16	15	14	**13**	30
39	18	**5**	4	**3**	12	29
40	19	6	1	2	11	28
41	20	**7**	8	9	10	27
42	21	22	23	24	25	26
43	44	45	46	47	48	49

It is interesting to note that the odd squares lie along the bottom right diagonal, but what is more interesting is that 8 out of the 13 numbers lying along both diagonals are prime; that is, a ratio of 8/13 ≈ 62%.

If one complete new layer is wrapped around the spiral above, a square spiral with side length 9 will be formed. If this process is continued, what is the side length of the square spiral for which the ratio of primes along both diagonals first falls below 10%?

Problem 59: XOR Decryption

Published on Friday, 19th December 2003, 06:00 pm / Solved by 43089 / Difficulty rating: 5%

Each character on a computer is assigned a unique code and the preferred standard is ASCII (American Standard Code for Information Interchange). For example, uppercase A = 65, asterisk (*) = 42, and lowercase k = 107.

A modern encryption method is to take a text file, convert the bytes to ASCII, then XOR each byte with a given value, taken from a secret key. The advantage with the XOR function is that using the same encryption key on the cipher text, restores the plain text; for example, 65 XOR 42 = 107, then 107 XOR 42 = 65.

For unbreakable encryption, the key is the same length as the plain text message, and the key is made up of random bytes. The user would keep the encrypted message and the encryption key in different locations, and without both "halves", it is impossible to decrypt the message.

Unfortunately, this method is impractical for most users, so the modified method is to use a password as a key. If the password is shorter than the message, which is likely, the key is repeated cyclically throughout the message. The balance for this method is using a sufficiently long password key for security, but short enough to be memorable.

Your task has been made easy, as the encryption key consists of three lower case characters. Using **0059_cipher.txt** (right click and 'Save Link/Target As...'), a file containing the encrypted ASCII codes, and the knowledge that the plain text must contain common English words, decrypt the message and find the sum of the ASCII values in the original text.

Problem 60: Prime Pair Sets

Published on Friday, 2nd January 2004, 06:00 pm / Solved by 28232 / Difficulty rating: 20%

The primes 3, 7, 109, and 673, are quite remarkable. By taking any two primes and concatenating them in any order the result will always be prime. For example, taking 7 and 109, both 7109 and 1097 are prime. The sum of these four primes, 792, represents the lowest sum for a set of four primes with this property.

Find the lowest sum for a set of five primes for which any two primes concatenate to produce another prime.

Problem 61: Cyclical Figurate Numbers

Published on Friday, 16th January 2004, 06:00 pm / Solved by 26861 / Difficulty rating: 20%

Triangle, square, pentagonal, hexagonal, heptagonal, and octagonal numbers are all figurate (polygonal) numbers and are generated by the following formulae:

Triangle	$P_{3,n}=n(n+1)/2$	$1, 3, 6, 10, 15, \dots$
Square	$P_{4,n}=n^2$	$1, 4, 9, 16, 25, \dots$
Pentagonal	$P_{5,n}=n(3n-1)/2$	$1, 5, 12, 22, 35, \dots$
Hexagonal	$P_{6,n}=n(2n-1)$	$1, 6, 15, 28, 45, \dots$
Heptagonal	$P_{7,n}=n(5n-3)/2$	$1, 7, 18, 34, 55, \dots$
Octagonal	$P_{8,n}=n(3n-2)$	$1, 8, 21, 40, 65, \dots$

The ordered set of three 4-digit numbers: 8128, 2882, 8281, has three interesting properties.

1. The set is cyclic, in that the last two digits of each number is the first two digits of the next number (including the last number with the first).
2. Each polygonal type: triangle ($P_{3,127}=8128$), square ($P_{4,91}=8281$), and pentagonal ($P_{5,44}=2882$), is represented by a different number in the set.
3. This is the only set of 4-digit numbers with this property.

Find the sum of the only ordered set of six cyclic 4-digit numbers for which each polygonal type: triangle, square, pentagonal, hexagonal, heptagonal, and octagonal, is represented by a different number in the set.

Problem 62: Cubic Permutations

Published on Friday, 30th January 2004, 06:00 pm / Solved by 33098 / Difficulty rating: 15%

The cube, 41063625 (345^3), can be permuted to produce two other cubes: 56623104 (384^3) and 66430125 (405^3). In fact, 41063625 is the smallest cube which has exactly three permutations of its digits which are also cube.

Find the smallest cube for which exactly five permutations of its digits are cube.

Problem 63: Powerful Digit Counts

Published on Friday, 13th February 2004, 06:00 pm / Solved by 45083 / Difficulty rating: 5%

The 5-digit number, $16807=7^5$, is also a fifth power. Similarly, the 9-digit number, $134217728=8^9$, is a ninth power.

How many n-digit positive integers exist which are also an nth power?

Problem 64: Odd Period Square Roots

Published on Friday, 27th February 2004, 06:00 pm / Solved by 23321 / Difficulty rating: 20%

All square roots are periodic when written as continued fractions and can be written in the form:

$\displaystyle \quad \quad \sqrt{N}=a_0+\frac 1 {a_1+\frac 1 {a_2+ \frac 1 {a3+ \dots}}}$

For example, let us consider $\sqrt{23}:$

$\quad \quad \sqrt{23}=4+\sqrt{23}-4=4+\frac 1 {\frac 1 {\sqrt{23}-4}}=4+\frac 1 {1+\frac{\sqrt{23}-3}7}$

If we continue we would get the following expansion:

$\displaystyle \quad \quad \sqrt{23}=4+\frac 1 {1+\frac 1 {3+ \frac 1 {1+\frac 1 {8+ \dots}}}}$

The process can be summarised as follows:

$\quad \quad a_0=4, \frac 1 {\sqrt{23}-4}=\frac {\sqrt{23}+4} 7=1+\frac {\sqrt{23}-3} 7$
$\quad \quad a_1=1, \frac 7 {\sqrt{23}-3}=\frac {7(\sqrt{23}+3)} {14}=3+\frac {\sqrt{23}-3} 2$
$\quad \quad a_2=3, \frac 2 {\sqrt{23}-3}=\frac {2(\sqrt{23}+3)} {14}=1+\frac {\sqrt{23}-4} 7$
$\quad \quad a_3=1, \frac 7 {\sqrt{23}-4}=\frac {7(\sqrt{23}+4)} 7=8+\sqrt{23}-4$
$\quad \quad a_4=8, \frac 1 {\sqrt{23}-4}=\frac {\sqrt{23}+4} 7=1+\frac {\sqrt{23}-3} 7$
$\quad \quad a_5=1, \frac 7 {\sqrt{23}-3}=\frac {7 (\sqrt{23}+3)} {14}=3+\frac {\sqrt{23}-3} 2$
$\quad \quad a_6=3, \frac 2 {\sqrt{23}-3}=\frac {2(\sqrt{23}+3)} {14}=1+\frac {\sqrt{23}-4} 7$
$\quad \quad a_7=1, \frac 7 {\sqrt{23}-4}=\frac {7(\sqrt{23}+4)} {7}=8+\sqrt{23}-4$

It can be seen that the sequence is repeating. For conciseness, we use the notation $\sqrt{23}=[4;(1,3,1,8)]$, to indicate that the block (1,3,1,8) repeats indefinitely.

The first ten continued fraction representations of (irrational) square roots are:

$\quad \quad \sqrt{2}=[1;(2)]$, period=$1$
$\quad \quad \sqrt{3}=[1;(1,2)]$, period=$2$
$\quad \quad \sqrt{5}=[2;(4)]$, period=$1$
$\quad \quad \sqrt{6}=[2;(2,4)]$, period=$2$
$\quad \quad \sqrt{7}=[2;(1,1,1,4)]$, period=$4$
$\quad \quad \sqrt{8}=[2;(1,4)]$, period=$2$
$\quad \quad \sqrt{10}=[3;(6)]$, period=$1$
$\quad \quad \sqrt{11}=[3;(3,6)]$, period=$2$
$\quad \quad \sqrt{12}=[3;(2,6)]$, period=$2$
$\quad \quad \sqrt{13}=[3;(1,1,1,1,6)]$, period=$5$

Exactly four continued fractions, for $N \le 13$, have an odd period.

How many continued fractions for $N \le 10\,000$ have an odd period?

Problem 65: Convergents of e

Published on Friday, 12th March 2004, 06:00 pm / Solved by 31313 / Difficulty rating: 15%

The square root of 2 can be written as an infinite continued fraction.

$\sqrt{2} = 1 + \dfrac{1}{2 + \dfrac{1}{2 + \dfrac{1}{2 + \dfrac{1}{2 + ...}}}}$

The infinite continued fraction can be written, $\sqrt{2} = [1; (2)]$, (2) indicates that 2 repeats *ad infinitum*. In a similar way, $\sqrt{23} = [4; (1, 3, 1, 8)]$.

It turns out that the sequence of partial values of continued fractions for square roots provide the best rational approximations. Let us consider the convergents for $\sqrt{2}$.

$\begin{align} &1 + \dfrac{1}{2} = \dfrac{3}{2} \\ &1 + \dfrac{1}{2 + \dfrac{1}{2}} = \dfrac{7}{5}\\ &1 + \dfrac{1}{2 + \dfrac{1}{2 + \dfrac{1}{2}}} = \dfrac{17}{12}\\ &1 + \dfrac{1}{2 + \dfrac{1}{2 + \dfrac{1}{2 + \dfrac{1}{2}}}} = \dfrac{41}{29} \end{align}$

Hence the sequence of the first ten convergents for $\sqrt{2}$ are:

$1, \dfrac{3}{2}, \dfrac{7}{5}, \dfrac{17}{12}, \dfrac{41}{29}, \dfrac{99}{70}, \dfrac{239}{169}, \dfrac{577}{408}, \dfrac{1393}{985}, \dfrac{3363}{2378}, ...$

What is most surprising is that the important mathematical constant,
$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, ... , 1, 2k, 1, ...]$.

The first ten terms in the sequence of convergents for *e* are:

$2, 3, \dfrac{8}{3}, \dfrac{11}{4}, \dfrac{19}{7}, \dfrac{87}{32}, \dfrac{106}{39}, \dfrac{193}{71}, \dfrac{1264}{465}, \dfrac{1457}{536}, ...$

The sum of digits in the numerator of the 10th convergent is $1 + 4 + 5 + 7 = 17$.

Find the sum of digits in the numerator of the 100th convergent of the continued fraction for e.

Problem 66: Diophantine Equation

Published on Friday, 26th March 2004, 06:00 pm / Solved by 20761 / Difficulty rating: 25%

Consider quadratic Diophantine equations of the form: $$x^2 - Dy^2 = 1$$

For example, when $D=13$, the minimal solution in x is $649^2 - 13 \times 180^2 = 1$.

It can be assumed that there are no solutions in positive integers when D is square.

By finding minimal solutions in x for $D = \{2, 3, 5, 6, 7\}$, we obtain the following:

\begin{align} 3^2 - 2 \times 2^2 &= 1\\ 2^2 - 3 \times 1^2 &= 1\\ {\color{red}{\mathbf 9}}^2 - 5 \times 4^2 &= 1\\ 5^2 - 6 \times 2^2 &= 1\\ 8^2 - 7 \times 3^2 &= 1 \end{align}

Hence, by considering minimal solutions in x for $D \le 7$, the largest x is obtained when $D=5$.

Find the value of $D \le 1000$ in minimal solutions of x for which the largest value of x is obtained.

Problem 67: Maximum Path Sum II

Published on Friday, 9th April 2004, 06:00 pm / Solved by 99045 / Difficulty rating: 5%

By starting at the top of the triangle below and moving to adjacent numbers on the row below, the maximum total from top to bottom is 23.

```
   3
  7 4
 2 4 6
8 5 9 3
```

That is, 3 + 7 + 4 + 9 = 23.

Find the maximum total from top to bottom in **triangle.txt** (right click and 'Save Link/Target As...'), a 15K text file containing a triangle with one-hundred rows.

NOTE: This is a much more difficult version of **Problem 18**. It is not possible to try every route to solve this problem, as there are 2^{99} altogether! If you could check one trillion (10^{12}) routes every second it would take over twenty billion years to check them all. There is an efficient algorithm to solve it. ;o)

Problem 68: Magic 5-gon Ring

Published on Friday, 23rd April 2004, 06:00 pm / Solved by 21870 / Difficulty rating: 25%

Consider the following "magic" 3-gon ring, filled with the numbers 1 to 6, and each line adding to nine.

Working **clockwise**, and starting from the group of three with the numerically lowest external node (4,3,2 in this example), each solution can be described uniquely. For example, the above solution can be described by the set: 4,3,2; 6,2,1; 5,1,3.

It is possible to complete the ring with four different totals: 9, 10, 11, and 12. There are eight solutions in total.

Total	**Solution Set**
9	4,2,3; 5,3,1; 6,1,2
9	4,3,2; 6,2,1; 5,1,3
10	2,3,5; 4,5,1; 6,1,3
10	2,5,3; 6,3,1; 4,1,5
11	1,4,6; 3,6,2; 5,2,4
11	1,6,4; 5,4,2; 3,2,6
12	1,5,6; 2,6,4; 3,4,5
12	1,6,5; 3,5,4; 2,4,6

By concatenating each group it is possible to form 9-digit strings; the maximum string for a 3-gon ring is 432621513.

Using the numbers 1 to 10, and depending on arrangements, it is possible to form 16- and 17-digit strings. What is the maximum **16-digit** string for a "magic" 5-gon ring?

Problem 69: Totient Maximum

Published on Friday, 7th May 2004, 06:00 pm / Solved by 36435 / Difficulty rating: 10%

Euler's totient function, $\phi(n)$ [sometimes called the phi function], is defined as the number of positive integers not exceeding n which are relatively prime to n. For example, as 1, 2, 4, 5, 7, and 8, are all less than or equal to nine and relatively prime to nine, $\phi(9)=6$.

n	Relatively Prime	$\phi(n)$	$n/\phi(n)$
2	1	1	2
3	1,2	2	1.5
4	1,3	2	2
5	1,2,3,4	4	1.25
6	1,5	2	3
7	1,2,3,4,5,6	6	1.1666...
8	1,3,5,7	4	2
9	1,2,4,5,7,8	6	1.5
10	1,3,7,9	4	2.5

It can be seen that $n = 6$ produces a maximum $n/\phi(n)$ for $n\leq 10$.

Find the value of $n\leq 1\,000\,000$ for which $n/\phi(n)$ is a maximum.

Problem 70: Totient Permutation

Published on Friday, 21st May 2004, 06:00 pm / Solved by 23443 / Difficulty rating: 20%

Euler's totient function, $\phi(n)$ [sometimes called the phi function], is used to determine the number of positive numbers less than or equal to n which are relatively prime to n. For example, as $1, 2, 4, 5, 7$, and 8, are all less than nine and relatively prime to nine, $\phi(9)=6$.
The number 1 is considered to be relatively prime to every positive number, so $\phi(1)=1$.

Interestingly, $\phi(87109)=79180$, and it can be seen that 87109 is a permutation of 79180.

Find the value of n, $1 \lt n \lt 10^7$, for which $\phi(n)$ is a permutation of n and the ratio $n/\phi(n)$ produces a minimum.

Problem 71: Ordered Fractions

Published on Friday, 4th June 2004, 06:00 pm / Solved by 30678 / Difficulty rating: 10%

Consider the fraction, $\dfrac n d$, where n and d are positive integers. If $n \lt d$ and $\operatorname{HCF}(n,d)=1$, it is called a reduced proper fraction.

If we list the set of reduced proper fractions for $d \le 8$ in ascending order of size, we get: $$\frac 1 8, \frac 1 7, \frac 1 6, \frac 1 5, \frac 1 4, \frac 2 7, \frac 1 3, \frac 3 8, \mathbf{\frac 2 5}, \frac 3 7, \frac 1 2, \frac 4 7, \frac 3 5, \frac 5 8, \frac 2 3, \frac 5 7, \frac 3 4, \frac 4 5, \frac 5 6, \frac 6 7, \frac 7 8$$

It can be seen that $\dfrac 2 5$ is the fraction immediately to the left of $\dfrac 3 7$.

By listing the set of reduced proper fractions for $d \le 1\,000\,000$ in ascending order of size, find the numerator of the fraction immediately to the left of $\dfrac 3 7$.

Problem 72: Counting Fractions

Published on Friday, 18th June 2004, 06:00 pm / Solved by 23490 / Difficulty rating: 20%

Consider the fraction, $\dfrac n d$, where n and d are positive integers. If $n \lt d$ and $\operatorname{HCF}(n,d)=1$, it is called a reduced proper fraction.

If we list the set of reduced proper fractions for $d \le 8$ in ascending order of size, we get: $$\frac 1 8, \frac 1 7, \frac 1 6, \frac 1 5, \frac 1 4, \frac 2 7, \frac 1 3, \frac 3 8, \frac 2 5, \frac 3 7, \frac 1 2, \frac 4 7, \frac 3 5, \frac 5 8, \frac 2 3, \frac 5 7, \frac 3 4, \frac 4 5, \frac 5 6, \frac 6 7, \frac 7 8$$

It can be seen that there are 21 elements in this set.

How many elements would be contained in the set of reduced proper fractions for $d \le 1\,000\,000$?

Problem 73: Counting Fractions in a Range

Published on Friday, 2nd July 2004, 06:00 pm / Solved by 26216 / Difficulty rating: 15%

Consider the fraction, $\dfrac n d$, where n and d are positive integers. If $n \lt d$ and $\operatorname{HCF}(n, d)=1$, it is called a reduced proper fraction.

If we list the set of reduced proper fractions for $d \le 8$ in ascending order of size, we get: $$\frac 1 8, \frac 1 7, \frac 1 6, \frac 1 5, \frac 1 4, \frac 2 7, \frac 1 3, \mathbf{\frac 3 8, \frac 2 5, \frac 3 7}, \frac 1 2, \frac 4 7, \frac 3 5, \frac 5 8, \frac 2 3, \frac 5 7, \frac 3 4, \frac 4 5, \frac 5 6, \frac 6 7, \frac 7 8$$

It can be seen that there are 3 fractions between $\dfrac 1 3$ and $\dfrac 1 2$.

How many fractions lie between $\dfrac 1 3$ and $\dfrac 1 2$ in the sorted set of reduced proper fractions for $d \le 12\,000$?

Problem 74: Digit Factorial Chains

Published on Friday, 16th July 2004, 06:00 pm / Solved by 28016 / Difficulty rating: 15%

The number 145 is well known for the property that the sum of the factorial of its digits is equal to 145: $$1! + 4! + 5! = 1 + 24 + 120 = 145.$$

Perhaps less well known is 169, in that it produces the longest chain of numbers that link back to 169; it turns out that there are only three such loops that exist:

\begin{align} &169 \to 363601 \to 1454 \to 169\\ &871 \to 45361 \to 871\\ &872 \to 45362 \to 872 \end{align}

It is not difficult to prove that EVERY starting number will eventually get stuck in a loop. For example,

\begin{align} &69 \to 363600 \to 1454 \to 169 \to 363601 (\to 1454)\\ &78 \to 45360 \to 871 \to 45361 (\to 871)\\ &540 \to 145 (\to 145) \end{align}

Starting with 69 produces a chain of five non-repeating terms, but the longest non-repeating chain with a starting number below one million is sixty terms.

How many chains, with a starting number below one million, contain exactly sixty non-repeating terms?

Problem 75: Singular Integer Right Triangles

Published on Friday, 30th July 2004, 06:00 pm / Solved by 18956 / Difficulty rating: 25%

It turns out that $\pu{12 cm}$ is the smallest length of wire that can be bent to form an integer sided right angle triangle in exactly one way, but there are many more examples.

$\pu{\mathbf{12} \mathbf{cm}}$: $(3,4,5)$
$\pu{\mathbf{24} \mathbf{cm}}$: $(6,8,10)$
$\pu{\mathbf{30} \mathbf{cm}}$: $(5,12,13)$
$\pu{\mathbf{36} \mathbf{cm}}$: $(9,12,15)$
$\pu{\mathbf{40} \mathbf{cm}}$: $(8,15,17)$
$\pu{\mathbf{48} \mathbf{cm}}$: $(12,16,20)$

In contrast, some lengths of wire, like $\pu{20 cm}$, cannot be bent to form an integer sided right angle triangle, and other lengths allow more than one solution to be found; for example, using $\pu{120 cm}$ it is possible to form exactly three different integer sided right angle triangles.

$\pu{\mathbf{120} \mathbf{cm}}$: $(30,40,50)$, $(20,48,52)$, $(24,45,51)$

Given that L is the length of the wire, for how many values of $L \le 1\,500\,000$ can exactly one integer sided right angle triangle be formed?

Problem 76: Counting Summations

Published on Friday, 13th August 2004, 06:00 pm / Solved by 29861 / Difficulty rating: 10%

It is possible to write five as a sum in exactly six different ways:

\begin{align} &4 + 1\\ &3 + 2\\ &3 + 1 + 1\\ &2 + 2 + 1\\ &2 + 1 + 1 + 1\\ &1 + 1 + 1 + 1 + 1 \end{align}

How many different ways can one hundred be written as a sum of at least two positive integers?

Problem 77: Prime Summations

Published on Friday, 27th August 2004, 06:00 pm / Solved by 20184 / Difficulty rating: 25%

It is possible to write ten as the sum of primes in exactly five different ways:

\begin{align} &7 + 3\\ &5 + 5\\ &5 + 3 + 2\\ &3 + 3 + 2 + 2\\ &2 + 2 + 2 + 2 + 2 \end{align}

What is the first value which can be written as the sum of primes in over five thousand different ways?

Problem 78: Coin Partitions

Published on Friday, 10th September 2004, 06:00 pm / Solved by 17825 / Difficulty rating: 30%

Let $p(n)$ represent the number of different ways in which n coins can be separated into piles. For example, five coins can be separated into piles in exactly seven different ways, so $p(5)=7$.

```
OOOOO
OOOO   O
OOO   OO
OOO   O   O
OO   OO   O
OO   O   O   O
O   O   O   O   O
```

Find the least value of n for which $p(n)$ is divisible by one million.

Problem 79: Passcode Derivation

Published on Friday, 17th September 2004, 06:00 pm / Solved by 42542 / Difficulty rating: 5%

A common security method used for online banking is to ask the user for three random characters from a passcode. For example, if the passcode was 531278, they may ask for the 2nd, 3rd, and 5th characters; the expected reply would be: 317.

The text file, **keylog.txt**, contains fifty successful login attempts.

Given that the three characters are always asked for in order, analyse the file so as to determine the shortest possible secret passcode of unknown length.

Problem 80: Square Root Digital Expansion

Published on Friday, 8th October 2004, 06:00 pm / Solved by 20814 / Difficulty rating: 20%

It is well known that if the square root of a natural number is not an integer, then it is irrational. The decimal expansion of such square roots is infinite without any repeating pattern at all.

The square root of two is $1.41421356237309504880\cdots$, and the digital sum of the first one hundred decimal digits is 475.

For the first one hundred natural numbers, find the total of the digital sums of the first one hundred decimal digits for all the irrational square roots.

Problem 81: Path sum: two ways

Published on Friday, 22nd October 2004, 06:00 pm / Solved by 35768 / Difficulty rating: 10%

In the 5 by 5 matrix below, the minimal path sum from the top left to the bottom right, by **only moving to the right and down**, is indicated in bold red and is equal to 2427.

$$ \begin{pmatrix} \color{red}{131} & 673 & 234 & 103 & 18\\ \color{red}{201} & \color{red}{96} & \color{red}{342} & 965 & 150\\ 630 & 803 & \color{red}{746} & \color{red}{422} & 111\\ 537 & 699 & 497 & \color{red}{121} & 956\\ 805 & 732 & 524 & \color{red}{37} & \color{red}{331} \end{pmatrix} $$

Find the minimal path sum from the top left to the bottom right by only moving right and down in **matrix.txt** (right click and "Save Link/Target As..."), a 31K text file containing an 80 by 80 matrix.

Problem 82: Path sum: three ways

Published on Friday, 5th November 2004, 06:00 pm / Solved by 22113 / Difficulty rating: 20%

NOTE: This problem is a more challenging version of **Problem 81**.

The minimal path sum in the 5 by 5 matrix below, by starting in any cell in the left column and finishing in any cell in the right column, and only moving up, down, and right, is indicated in red and bold; the sum is equal to 994.

$$ \begin{pmatrix} 131 & 673 & \color{red}{234} & \color{red}{103} & \color{red}{18}\\ \color{red}{201} & \color{red}{96} & \color{red}{342} & 965 & 150\\ 630 & 803 & 746 & 422 & 111\\ 537 & 699 & 497 & 121 & 956\\ 805 & 732 & 524 & 37 & 331 \end{pmatrix} $$

Find the minimal path sum from the left column to the right column in **matrix.txt** (right click and "Save Link/Target As..."), a 31K text file containing an 80 by 80 matrix.

Problem 83: Path sum: four ways

Published on Friday, 19th November 2004, 06:00 pm / Solved by 19066 / Difficulty rating: 25%

NOTE: This problem is a significantly more challenging version of **Problem 81**.

In the 5 by 5 matrix below, the minimal path sum from the top left to the bottom right, by moving left, right, up, and down, is indicated in bold red and is equal to 2297.

$$ \begin{pmatrix} \color{red}{131} & 673 & \color{red}{234} & \color{red}{103} & \color{red}{18}\\ \color{red}{201} & \color{red}{96} & \color{red}{342} & 965 & \color{red}{150}\\ 630 & 803 & 746 & \color{red}{422} & \color{red}{111}\\ 537 & 699 & 497 & \color{red}{121} & 956\\ 805 & 732 & 524 & \color{red}{37} & \color{red}{331} \end{pmatrix} $$

Find the minimal path sum from the top left to the bottom right by moving left, right, up, and down in **matrix.txt** (right click and "Save Link/Target As..."), a 31K text file containing an 80 by 80 matrix.

Problem 84: Monopoly Odds

Published on Friday, 3rd December 2004, 06:00 pm / Solved by 13338 / Difficulty rating: 35%

In the game, **Monopoly**, the standard board is set up in the following way:

A player starts on the GO square and adds the scores on two 6-sided dice to determine the number of squares they advance in a clockwise direction. Without any further rules we would expect to visit each square with equal probability: 2.5%. However, landing on G2J (Go To Jail), CC (community chest), and CH (chance) changes this distribution.

In addition to G2J, and one card from each of CC and CH, that orders the player to go directly to jail, if a player rolls three consecutive doubles, they do not advance the result of their 3rd roll. Instead they proceed directly to jail.

At the beginning of the game, the CC and CH cards are shuffled. When a player lands on CC or CH they take a card from the top of the respective pile and, after following the instructions, it is returned to the bottom of the pile. There are sixteen cards in each pile, but for the purpose of this problem we are only concerned with cards that order a movement; any instruction not concerned with movement will be ignored and the player will remain on the CC/CH square.

- Community Chest (2/16 cards):
    1. Advance to GO
    2. Go to JAIL
- Chance (10/16 cards):
    1. Advance to GO
    2. Go to JAIL
    3. Go to C1
    4. Go to E3
    5. Go to H2
    6. Go to R1
    7. Go to next R (railway company)
    8. Go to next R
    9. Go to next U (utility company)
    10. Go back 3 squares.

The heart of this problem concerns the likelihood of visiting a particular square. That is, the probability of finishing at that square after a roll. For this reason it should be clear that, with the exception of G2J for which the probability of finishing on it is zero, the CH squares will have the lowest probabilities, as 5/8 request a movement to another square, and it is the final square that the player finishes at on each roll that we are interested in. We shall make no distinction between "Just Visiting" and being sent to JAIL, and we shall also ignore the rule about requiring a double to "get out of jail", assuming that they pay to get out on their next turn.

By starting at GO and numbering the squares sequentially from 00 to 39 we can concatenate these two-digit numbers to produce strings that correspond with sets of squares.

Statistically it can be shown that the three most popular squares, in order, are JAIL (6.24%) = Square 10, E3 (3.18%) = Square 24, and GO (3.09%) = Square 00. So these three most popular squares can be listed with the six-digit modal string: 102400.

If, instead of using two 6-sided dice, two 4-sided dice are used, find the six-digit modal string.

Problem 85: Counting Rectangles

Published on Friday, 17th December 2004, 06:00 pm / Solved by 25854 / Difficulty rating: 15%

By counting carefully it can be seen that a rectangular grid measuring 3 by 2 contains eighteen rectangles:

Although there exists no rectangular grid that contains exactly two million rectangles, find the area of the grid with the nearest solution.

Problem 86: Cuboid Route

Published on Friday, 7th January 2005, 06:00 pm / Solved by 13383 / Difficulty rating: 35%

A spider, S, sits in one corner of a cuboid room, measuring 6 by 5 by 3, and a fly, F, sits in the opposite corner. By travelling on the surfaces of the room the shortest "straight line" distance from S to F is 10 and the path is shown on the diagram.

However, there are up to three "shortest" path candidates for any given cuboid and the shortest route doesn't always have integer length.

It can be shown that there are exactly 2060 distinct cuboids, ignoring rotations, with integer dimensions, up to a maximum size of M by M by M, for which the shortest route has integer length when $M = 100$. This is the least value of M for which the number of solutions first exceeds two thousand; the number of solutions when $M = 99$ is 1975.

Find the least value of M such that the number of solutions first exceeds one million.

Problem 87: Prime Power Triples

Published on Friday, 21st January 2005, 06:00 pm / Solved by 21875 / Difficulty rating: 20%

The smallest number exp