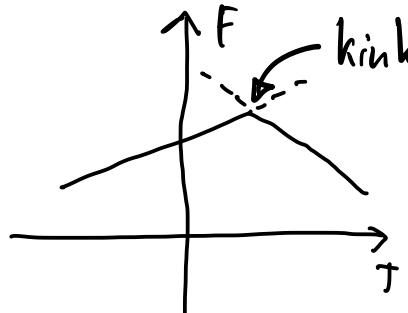
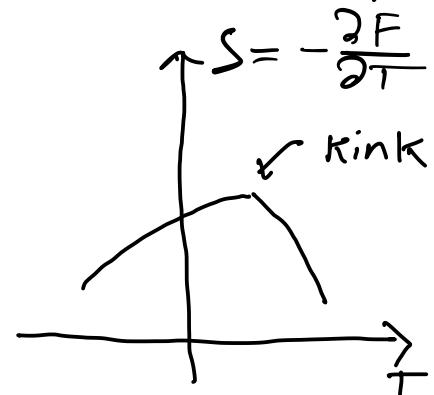


Landau Theory

A phase transition occurs when thermo-potential, e.g. $F(T, N, V)$, has non-analytic behavior in T, N, V, \dots



1st Order:
 $\frac{\partial^2 F}{\partial T^2}$ discontinuous

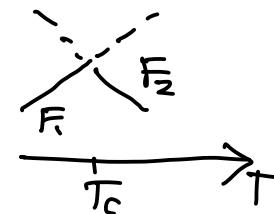


2nd Order
 $\frac{\partial^2 F}{\partial T^2}$ continuous,

Since $S = -\frac{\partial F}{\partial T}$, $P = -\frac{\partial F}{\partial V}$, $\chi = -\frac{1}{V} \frac{\partial^2 F}{\partial V^2}$ etc
 \hookrightarrow jumps / kinks in observables

= First order behaviour:

- metastability
- hysteresis
- Latent heat



$$S_1 = -\frac{\partial F_1}{\partial T} \leq S_2 = -\frac{\partial F_2}{\partial T}$$

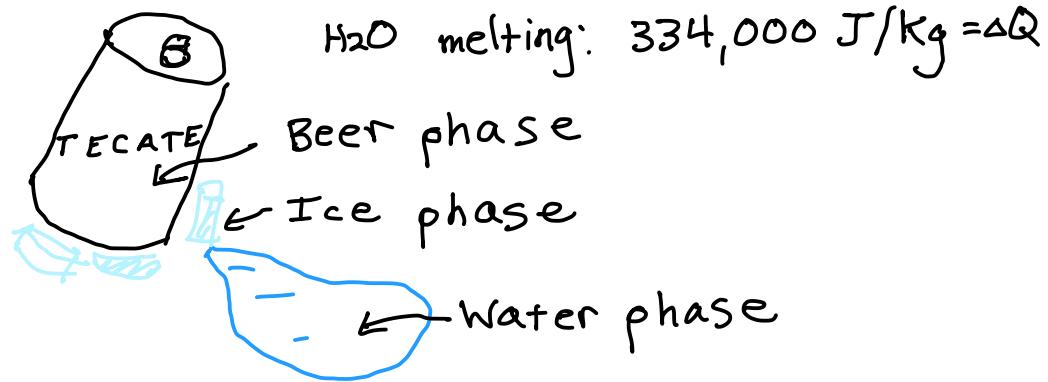
Since $F_1 = F_2$ at transition

$$F_1 = E_1 - S_1 T_c = E_2 - S_2 T_c$$

$$(S_2 - S_1) T_c = E_2 - E_1 \approx \Delta Q \geq 0$$

So if system heats up $T_c - \epsilon \rightarrow T_c + \epsilon$
it must absorb extensive "Latent Heat ΔQ "
like a $C = \frac{dE}{dT} = \infty$

Latent Heat is useful:



Liquid \rightarrow Steam: $2.2 \cdot 10^6 \text{ J/Kg} = \Delta Q$

Can also show SSB (rare).

Ex.: Potts model w/q > 4

$\langle \phi \rangle \neq 0$

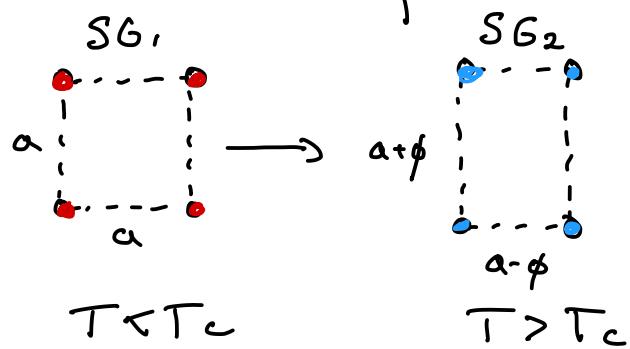
T_c T

but, maybe, it's not really spontaneous -
There must be an effective non-zero "field"

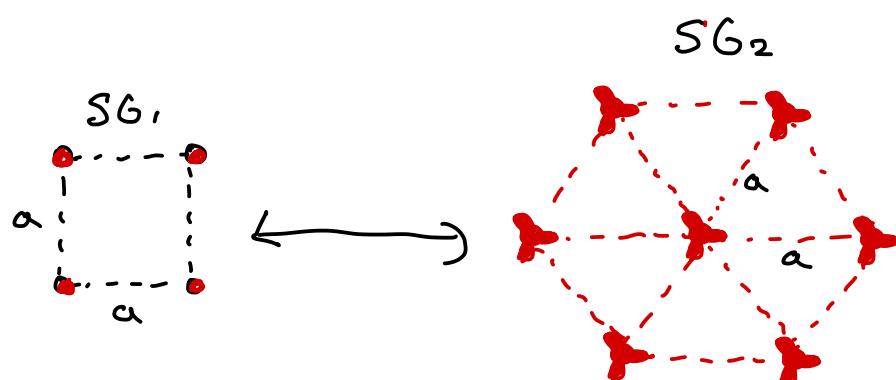
1st: Discontinuous change in order param.

2nd: cont.

Continuous Phase transitions, on the otherhand, are found to occur between certain "allowed" pairs of phases. For example, consider a transition between two structural phases of crystal:



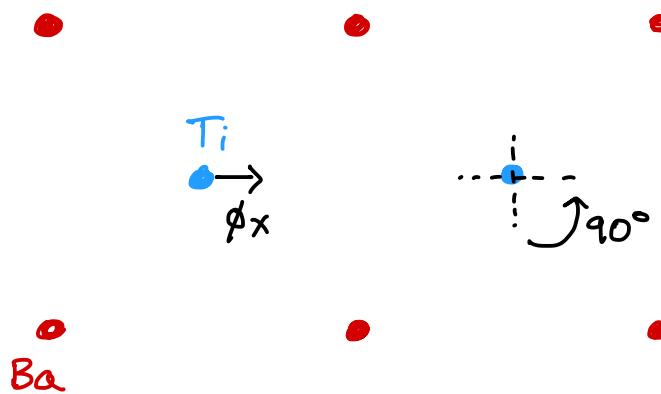
On both sides there is a "space group", SG_1 , SG_2 , describing symmetry of crystal (230 in 3D). In this case $SG_2 \subset SG_1$ and we will find continuous allowed. But for



no continuous critical point observed; "forbidden". Can be related to group-theoretic considerations, e.g. $SG_2 \not\subset SG_1$

Order parameters

Symmetry breaking always implies a phase transition. If $g \hat{H} g^{-1} = \hat{H}$ for $g \in G$ (e.g. $\tau \leftrightarrow -\tau$ for $H = \sum \tau_i \sigma_j$) and " ϕ " is an observable with $g\phi g^{-1} \neq \phi$, we call ϕ an "order parameter" for g . Ex:



The cubic crystal has various symmetries including $M_{x/y}$ and $C_4 (R_{90^\circ})$

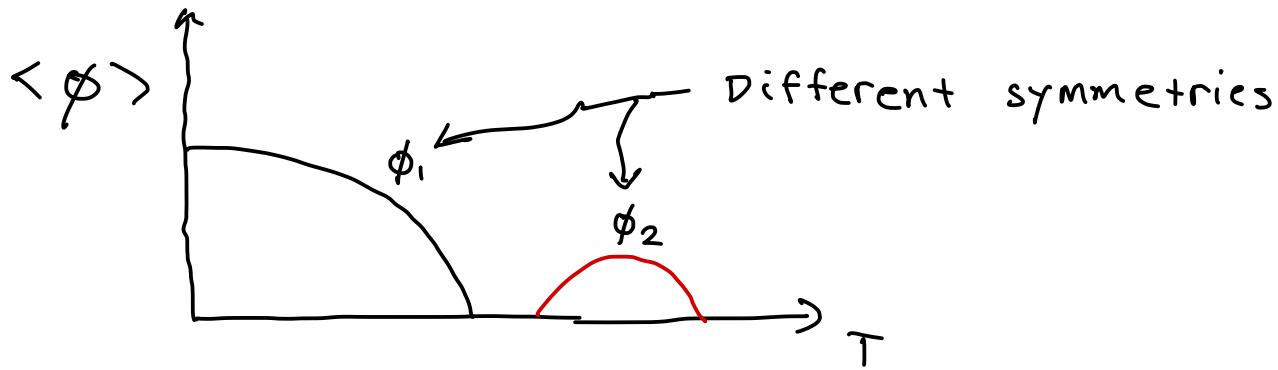
However in (say) $BaTiO_3$, above some T_c , the Ti displace by

$$\vec{\phi} = (\phi_x, \phi_y)$$

Since $R_{90^\circ} : (\phi_x, \phi_y) \rightarrow (-\phi_y, \phi_x) \neq \vec{\phi}$, $\vec{\phi}$ is "order parameter" for R_{90°

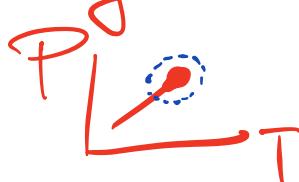
Formally, order parameters carry an "irreducible representation" (irrep) of the symmetry group.

If $\langle \phi \rangle \neq 0$, the symmetry is (spontaneously) broken. So $\langle \phi \rangle = 0$ vs $\langle \phi \rangle \neq 0$ is sharp distinction which must be separated by phase transition.



Landau theory is a phenomenological framework for deriving when it's possible to have ~~direct~~ continuous transitions between different orders (and there are some "beyond Landau" exceptions).

+ neighborhood of continuous transitions.



The "effective" Hamiltonian / Free energy

Suppose we have identified an order parameter/s $\{\phi_i\}$, e.g., $\phi = \frac{1}{N} \sum_{i=1}^N \sigma_i$.

How do we understand transition? Of course if we can compute

$\langle \phi \rangle = \frac{1}{Z} \sum_n e^{-\beta H(n)} \phi(n)$, we're good, but that's usually hard! Landau introduced idea of "effective" $H_{\text{eff}}(\phi)$. Key is trivial identity:

$$\int d\phi \delta(\phi - x) = 1$$

We can write

$$e^{-\beta F} = \sum_n e^{-\beta H(n)} = \int d\Phi \underbrace{\sum_n e^{-\beta H(n)}}_{\delta(\Phi - \phi(n))} \underbrace{e^{-\beta \cdot V \cdot H_{\text{eff}}(\Phi, \beta)}}_{= e^{-\beta \cdot V \cdot H_{\text{eff}}(\Phi, \beta)}}$$

$$e^{-\beta F} = \int d\Phi e^{-\beta \cdot V \cdot H_{\text{eff}}(\Phi, \beta)}$$

$$\langle \phi \rangle = \frac{\int d\Phi e^{-\beta \cdot V \cdot H_{\text{eff}}(\Phi, \beta)} \Phi}{\int d\Phi e^{-\beta \cdot V \cdot H_{\text{eff}}(\Phi, \beta)}}$$

As $V \rightarrow \infty$, dominated by max
 $\partial_{\Phi} H_{\text{eff}}(\Phi, \beta) = 0$

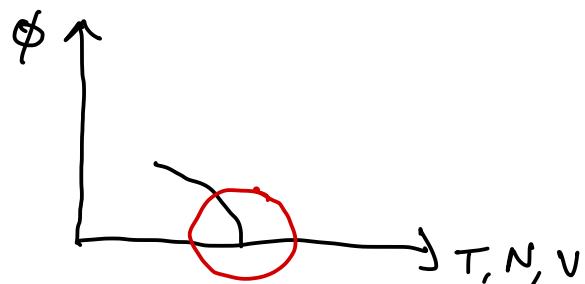
The solution then gives $\langle \phi \rangle$ and

$$\frac{F}{V} = H_{\text{eff}}(\phi, \beta)$$

So, if we could compute $H_{\text{eff}}(\Phi, \beta)$, we could calculate any property ϕ, β, F, C_V, S etc from one variable stat-mech! Of course we've just swept problem under the rug:

$$e^{-\beta \cdot V \cdot H_{\text{eff}}(\Phi, \beta)} = \sum_n e^{-\beta H[\nu]} \delta(\Phi - \phi(\nu))$$

is hard to compute. Here's where we go pheno! Near a (continuous) phase transition, ϕ is small:



So expand

key reason to look at 2nd order p.t.s!

$$H_{\text{eff}} = H_0 + \alpha \cdot \phi + A \phi^2 + C \phi^3 + B \phi^4 + \dots$$

where H_0, α, A etc depend on T, V, N_{etc} .

If there are multiple order parameters, ϕ_i , then $\sum_i \alpha_i \cdot \phi_i + A_{ij} \phi_i \phi_j + \text{etc.}$

Since $H[g \cdot \mathcal{N}] = H[\mathcal{N}]$ for $g \in G$, we know

$$H_{\text{eff}}(g \cdot \Phi, T, N, v) = H_{\text{eff}}(\Phi, T, N, v)$$

This constrains the expansion. In particular, since $g \Phi \neq \Phi$ for some $g \in G$ (order parameter) $\boxed{\alpha = 0}$

$$H_{\text{eff}} = H_0 + A \phi^2 + C \phi^3 + B \phi^4 + \dots$$

In general, $A_{ij} \phi_i \phi_j$ (etc) can only have $A_{ij} \neq 0$ if decomposing the tensor product

$$\phi \otimes \phi = I_1 + I_2 + \dots$$

contains irrep $I_1 = 1L$ (+trivial)).

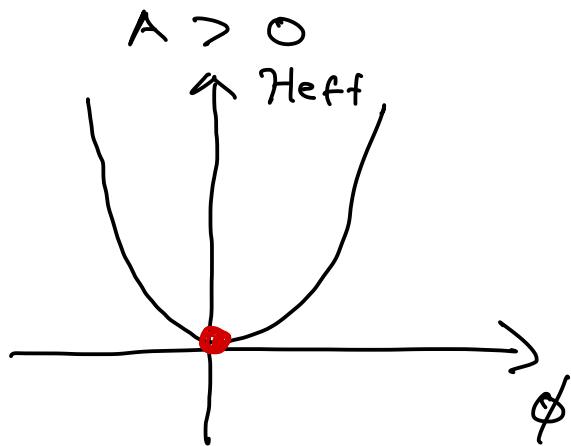
This is just complicated way of saying H_{eff} is symmetric under G .

Ising Example: $g(\sigma) = -\sigma$ $g^2 = 1$
 $G = \mathbb{Z}_2$ $\phi = \frac{1}{N} \sum \sigma_i$, $g(\phi) = -\phi$

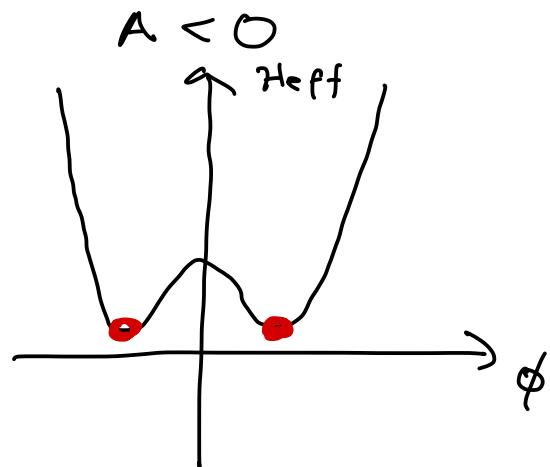
So ϕ^3 is forbidden:

$$H_{\text{eff}}(\beta) = H_0 + A \phi^2 + B \phi^4 + \dots$$

Let's assume (for now) $B > 0$, so $D\phi^6$ won't be important. Two cases:



$$\langle \phi \rangle = 0$$



$$\langle \phi \rangle \neq 0$$

$$0 = \frac{\partial H_{\text{eff}}}{\partial \phi} = \phi(2A + 4B\phi^2) = 0$$

$$\phi = \pm \sqrt{\frac{|A|}{2B}} \quad \text{for } A < 0$$

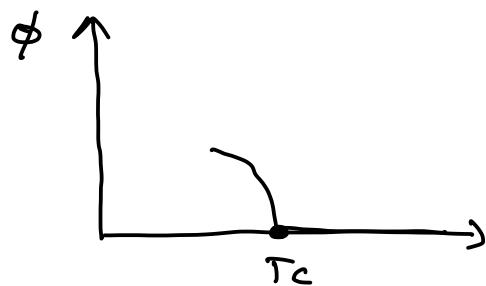
The critical point is $A=0$, so we can expand

$$A = \alpha_0 (T - T_c) + \alpha_1 (T - T_c)^2 + \dots$$

To leading $T - T_c$,

$$\langle \phi \rangle = \pm \sqrt{\frac{\alpha_0 |T - T_c|}{2B}} \quad \propto \pm |T - T_c|^{1/2}$$

for $T < T_c$



Predicts critical exponent

$$\boxed{\text{"}\beta = 1/2\text{"}}$$

We can then compute F :

$$F(T) = F_0(T) + A \phi^2$$

$$= \begin{cases} F_0(T) - \frac{\alpha_0^2 (T - T_c)^2}{4B} + \dots, & T < T_c \\ F_0(T), & T > T_c \end{cases}$$

$$F_0(T) = f_0 + (T - T_c) \cdot f_1 + \dots \text{ is "smooth"}$$

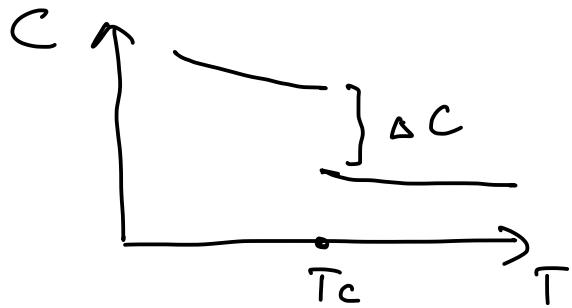
across transition, but $A\phi^2 + B\phi^4$

gives kink

$$S = -\frac{\partial F}{\partial T} = \begin{cases} S_0(T) + \frac{\alpha_0^2 (T - T_c)}{2B} + \dots, & T < T_c \\ S_0(T), & T > T_c \end{cases}$$

$$C = T \frac{\partial S}{\partial T} = \begin{cases} C_0(T) + T \cdot \frac{\alpha_0^2}{2B} \\ C_0(T) \end{cases}$$

so heat capacity (but not F, S, ϕ) jumps:



Field Dependence

If $H[\mu] \rightarrow H(\mu) - h \cdot \sum \sigma_i$, then we exactly have

$$H_{\text{eff}} = H_0 - h\phi + A\phi^2 + B\phi^4 + \dots$$

We can then use same analysis near T_c :

$$h = 2\alpha_0(T - T_c)\phi + 4B\phi^3$$

We now have $\phi \neq 0$ for any $t = T - T_c$:
 transition is "rounded out". The width
 of the rounding (in t) is

$$\alpha_0 t \phi^2 \sim h \phi$$

$$\phi t \sim \frac{h}{\alpha_0}$$

With $\phi \sim \sqrt{\alpha_0 t / 2B}$ ($h=0$ result)

Width of h -induced rounding:

$$\boxed{\Delta t \sim h^{2/3} B^{1/3} / \alpha_0}$$

Susceptibility

$$h = 2 \alpha_0 t \phi + 4 B \phi^3$$

For $t \geq 0$, to leading order

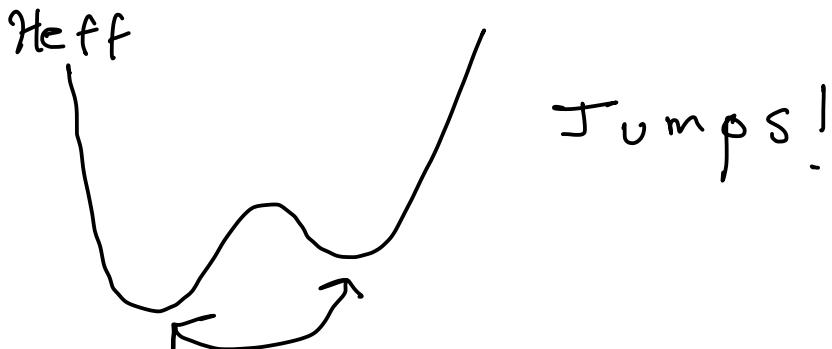
$$\phi = \frac{h}{2 \alpha_0 t} \Rightarrow \chi \Big|_{h=0} = \frac{1}{2 \alpha_0 t}$$

Diverges as $1/t$.

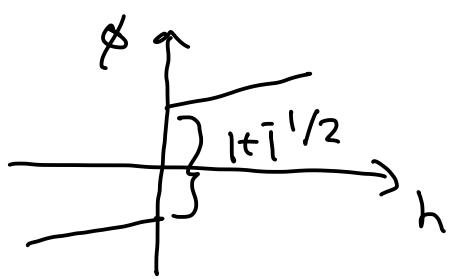
For $t = 0$:

$$\phi = \left(\frac{h}{4B} \right)^{1/3} \quad "S=3"$$

For $t < 0$:



For $t < 0$



However, we can calculate

$$\chi(t, h) = \partial_h \phi \text{ at } h = 0 \pm \epsilon;$$

From $h = 2 \alpha_0 t \phi + 4 B \phi^3$, we expand

$$\phi = \sqrt{-\frac{\alpha_0 t}{2B}} + \delta \phi$$

$$\hookrightarrow h = 2 \alpha_0 t \delta \phi + 12 B \delta \phi \left(-\frac{\alpha_0 t}{2B} \right) + O(\delta \phi^2)$$

$$\delta \phi = \frac{h}{-4 \alpha_0 \cdot t} \sim t^{-1}$$

Summary of "Exponents"

- "Heat capacity exponent" $\boxed{\alpha}$

$$C \sim \begin{cases} A_+ t^{-\alpha}, & t > 0 \\ A_- (-t)^{-\alpha}, & t < 0 \end{cases} : \alpha = 0$$

- "order parameter exponent" $\boxed{\beta}$

$$\langle \phi \rangle(t, h=0) \propto \begin{cases} 0, & t > 0 \\ \text{sgn}(h)(-t)^\beta, & t < 0 \end{cases} : \beta = \frac{1}{2}$$

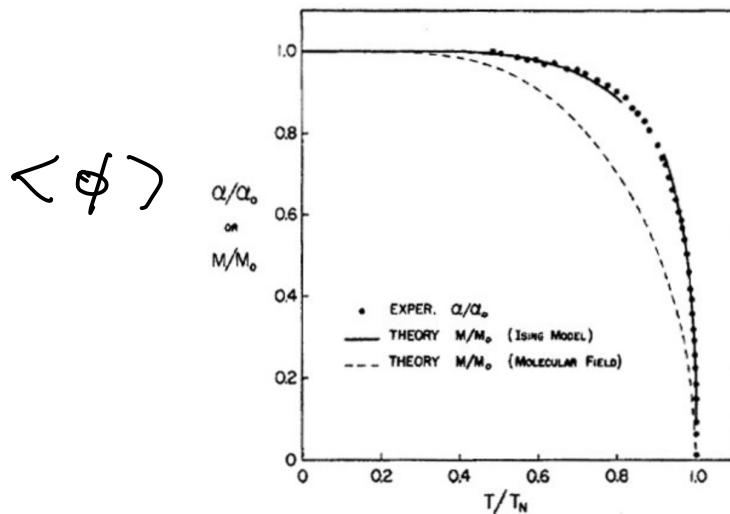
- "Susceptibility exponent" $\boxed{\gamma}$

$$\chi(t \neq 0, h=0) \propto \begin{cases} C_+ |t|^{-\gamma} \\ C_- |t|^{-\gamma} \end{cases}, \gamma = 1$$

- "Isotherm exponent" $\boxed{\delta}$

$$\langle \phi \rangle(t=0, h) \propto \text{sgn}(h) \cdot |h|^{1/\delta} \quad \delta = 3$$

Experiment: ferromagnetism in DyPO₄



As predicted, $\langle \phi \rangle \sim t^{-\beta}$ is a perfect power law near $t \sim 0$. But $\beta \approx 0.32$ vs prediction $\beta = 0.5$!

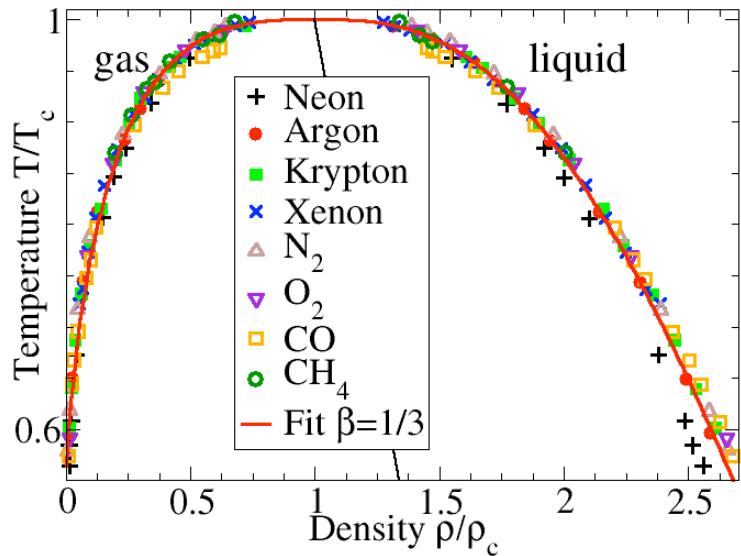
Remarkably, all 3D magnets with Ising-like order parameter are found to have the same exponents to within a couple digits:

	α	β	γ	δ
MFT:	0	0.5	1	3
Exp:	0.11	0.32	1.24	4.8

This surprising "universality" is subject of PH212.

Universality: Shared Critical Behavior

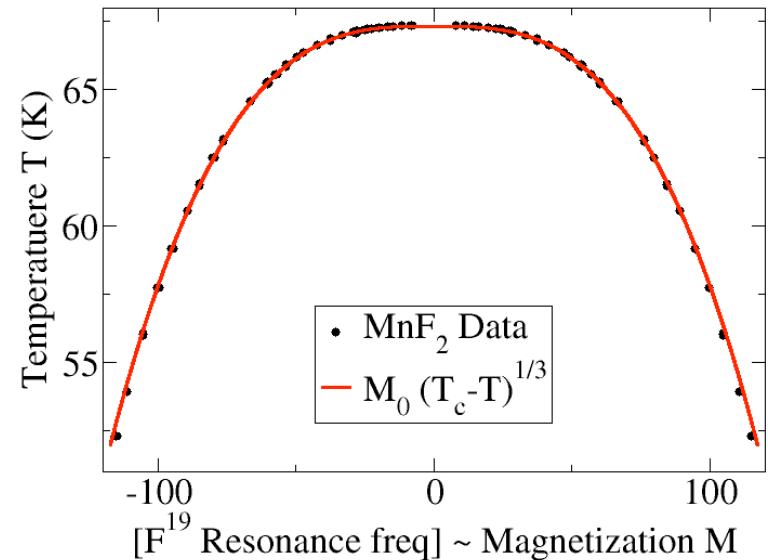
Ising Model and Liquid-Gas Critical Point



Liquid-Gas Critical Point

$$\begin{aligned} \rho - \rho_c &\sim (T_c - T)^\beta \\ \rho^{Ar}(T) &= A \rho^{CO}(BT) \end{aligned}$$

Same critical
exponent
 $\beta=0.332!$



Ising Critical Point

$$\begin{aligned} M(T) &\sim (T_c - T)^\beta \\ \rho^{Ar}(T) &= A(M(BT), T) \end{aligned}$$

Universality: Same Behavior up to Change in Coordinates

$$A(M, T) = a_1 M + a_2 + a_3 T + (\text{other singular terms})$$

Nonanalytic behavior at critical point (not parabolic top)

All power-law singularities (χ, c_ν, ξ) are shared by magnets, liquid/gas

(Non-symmetric Models: ϕ^3)

Appendix

The Ising symmetry allowed us to eliminate odd terms in H_{eff} , but that doesn't mean the formalism can't be applied more generally. Recall the liquid-gas transition:



Across the transition, the density "n" jumps. This suggests we consider

$$e^{-\beta V H_{\text{eff}}(n)} = \sum_n e^{-\beta H(n)} \delta(n - n(\omega))$$

Now n is not small, so we can't immediately Taylor expand in n . But define $n = n_{\text{c.p.}} + \delta n$, $\frac{\delta n}{n} \ll 1$, so

$$H_{\text{eff}}[\delta n] = E^* + \alpha_1 \delta n + \alpha_2 \delta n^2 + \alpha_3 \delta n^3 + \alpha_4 \delta n^4 \dots$$

with $\alpha_i(P, T)$, and $\alpha_1, \alpha_3 \neq 0$

Now let's make one more shift. For each P, T , we can shift by $\delta n^*(P, T)$

$$\delta n = [\phi + \delta n^*(P, T)] \cdot \underbrace{\text{why possible?}}$$

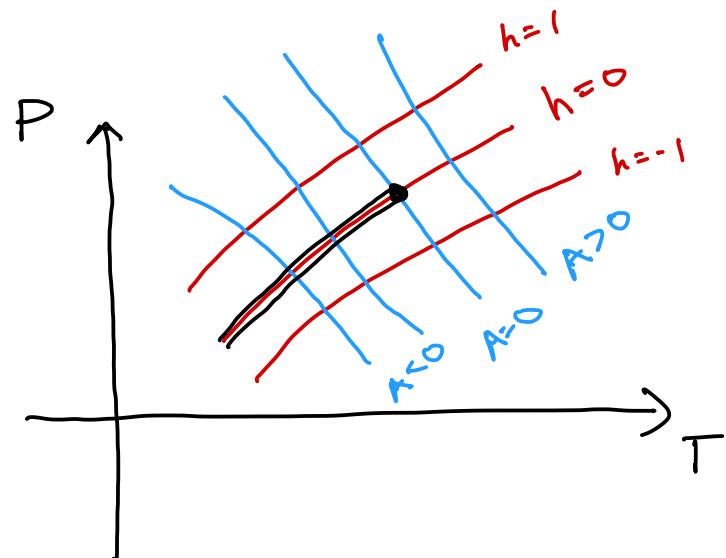
chosen such that $\alpha_3 \rightarrow 0$:

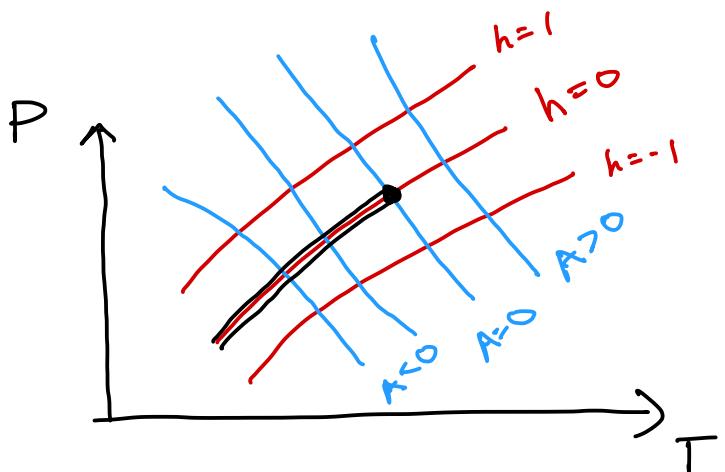
$$H_{\text{eff}}[\phi] = E^* + h(P, T) \phi + A(P, T) \phi^2 + B(P, T) \phi^4$$

However, we can't get rid of h .

$A(P, T), h(P, T)$ vary smoothly in P, T ,

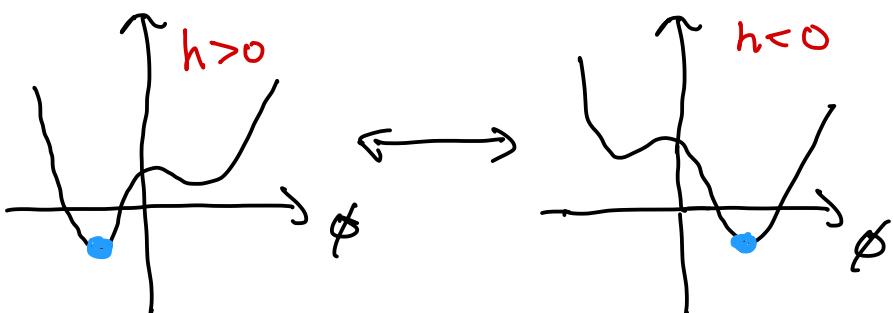
so they are like "change of coordinate" from $P, T \rightarrow A, h$ space:





In terms of ϕ , h , A , now no different than Ising; system minimizes $H_{\text{eff}}[\phi]$, which lead to phase transition

for all $A < 0$, $h = 0$:



However, this is first order. The continuous critical point occurs only at special $A = h = 0 \longleftrightarrow (T_*, P_*)$

This illustrates general principle in Landau theory. A critical transition requires both $A = h = 0$, so we need to control two knobs (P, T) to "tune" to this point. So generically transition is 1st order. With symmetry, however, $h = 0$ "for free", so only need 1 knob, $(T) \rightarrow$ generically continuous.